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Approximation properties for groups and von Neumann algebras

PhD Thesis

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Abstract

The main topic of the thesis is approximation properties for locally compact groups with applications to operator algebras. In order to study the relationship between the Haagerup property and weak amenability, the weak Haagerup property and the weak Haagerup constant are introduced. The weak Haagerup property is (strictly) weaker than both weak amenability and the Haagerup property.

We establish a relation between the weak Haagerup property and semigroups of Herz-Schur multipliers. For free groups, we prove that a generator of a semigroup of radial, contractive Herz-Schur multipliers is linearly bounded by the word length function.

In joint work with Haagerup, we show that a connected simple Lie group has the weak Haagerup property if and only if its real rank is at most one. The result coincides with the characterization of connected simple Lie groups which are weakly amenable. Moreover, the weak Haagerup constants of all connected simple Lie groups are determined.

In order to determine the weak Haagerup constants of the rank one simple Lie groups, knowledge about the Fourier algebras of their minimal parabolic subgroups is needed. We prove that for these minimal parabolic subgroups, the Fourier algebra coincides with the elements of the Fourier-Stieltjes algebra vanishing at infinity.

In joint work with Li, we characterize the connected simple Lie groups all of whose countable subgroups have the weak Haagerup property. These groups are precisely the connected simple Lie groups locally isomorphic to either $\mathrm{SO}(3)$, $\mathrm{SL}(2,\mathbb{R})$, or $\mathrm{SL}(2,\mathbb{C})$.

Resumé

Hovedemnet for denne afhandling er approksimationsegenskaber for lokalkompakte grupper med anvendelser inden for operatoralgebra. Den svage Haagerupegenskab og den svage Haagerupkonstant introduceres med det formål at undersøge forholdet mellem Haagerupegenskaben og svag amenabilitet. Den svage Haagerupegenskab er (strengt) svagere end både svag amenabilitet og Haagerupegenskaben.

Der etableres en sammenhæng mellem den svage Haagerupegenskab og semigrupper af Herz-Schurmultiplikatorer. For frie grupper vises det, at en frembringer for en semigruppe af radiale Herz-Schurmultiplikatorer med norm højst én er lineært begrænset af ordlængdefunktionen.

I samarbejde med Haagerup vises det, at en sammenhængende simpel Liegruppe har den svage Haagerupegenskab, netop hvis dens reelle rang er højst én. Resultatet er sammenfaldende med klassifikationen af sammenhængende simple Liegrupper, som er svagt amenable. Desuden bestemmes den svage Haagerupkonstant af samtlige sammenhængende simple Liegrupper.

For at kunne bestemme den svage Haagerupkonstant for simple Liegrupper af rang én undersøges Fourieralgebraen af gruppernes minimale parabolske undergrupper. Det bevises, at for disse minimale parabolske undergrupper består Fourieralgebraen netop af de elementer i Fourier-Stieltjesalgebraen, som forsvinder i det uendeligt fjerne.

I samarbejde med Li bestemmes de sammenhængende simple Liegrupper, hvori alle tællelige undergrupper har den svage Haagerupegenskab. Disse grupper er netop de sammenhængende simple Liegrupper, som er lokalt isomorfe med enten $\mathrm{SO}(3)$, $\mathrm{SL}(2,\mathbb{R})$ eller $\mathrm{SL}(2,\mathbb{C})$.

Contents

Part I

Thesis overview

The material contained in this part is meant to provide an overview of the articles that follow. The main results of the articles are described, and some background is provided. No proofs are given here. Proofs can be found in the articles.

Sections 1–6 are introductory and describe previously known results. Sections 7–11 describe the results of the thesis. Finally, Section 12 discusses some open problems and possibilities for further research.

1. Introduction

The infinite is an unavoidable part of mathematics and has been so more or less since the beginning of mathematics. When dealing with the infinite, approximations and limits appear naturally as necessary and invaluable tools. Going back at least as early as Eudoxus' method of exhaustion around 400 BC, the idea of approximating intractable objects by manageable objects and then passing to a limit has played a crucial role in mathematics. Many concepts in mathematical analysis are defined as limits of some kind, e.g., integrals, derivatives, and even the real numbers. In short: approximation is everywhere in mathematical analysis.

In functional analysis, one often deals with infinite dimensional spaces, and the need for finite dimensional approximations is apparent. One way of formalizing this idea is due to Grothendieck [26]: a Banach space X is said to have the *approximation property* if every compact operator from a Banach space into X can be approximated in the norm topology by finite rank operators. Equivalently, X has the approximation property if the identity operator id_X can be approximated uniformly on compact subsets of X by bounded operators of finite rank (see [41, Theorem 1.e.4]). The approximation problem of Grothendieck asks whether every Banach space has the approximation property.

The approximation problem was open for a long time until Enflo [25] solved the problem in the negative by providing an example of a separable Banach space without the approximation property. As a consequence, Enflo also solved the famous basis problem posed by Banach, asking whether every separable Banach space has a Schauder basis. Any Banach space with a Schauder basis has the approximation property, and therefore Enflo had solved the basis problem with the same counterexample. Enflo's construction is quite technical, but later easier examples of Banach spaces without the approximation property have been found. For instance, Szankowski [51] proved that the space of bounded operators on an infinite dimensional Hilbert space does not have the approximation property.

2. Groups and operator algebras

Several other approximation properties of Banach spaces and more specifically operator algebras (C*-algebras, von Neumann algebras) have been introduced since then, and it turns out that seemingly unrelated properties of operator algebras have characterizations in terms of approximation properties. For instance, a C*-algebra is nuclear if and only if its identity map is a point-norm limit of finite rank, completely positive contractions [11], [38]. The book [7] gives a thorough treatment of approximation properties in the setting of operator algebras.

Since the early days of the subject of operator algebras, it has been known that groups give rise to interesting examples of operator algebras. One such construction is the group von Neumann algebra. Using this construction, Murray and von Neumann [43] produced the first example of two non-isomorphic II_1 factors, and groups continue to provide relevant examples of operator algebras. Today, much research goes into figuring out how forgetful the group von Neumann algebra construction and also the related reduced group C*-algebra construction are, and how well these constructions remember properties of the group.

In [27], Haagerup showed that the complete positivity in the above characterization of nuclearity is essential. He showed that the reduced group C*-algebra $C^*_\lambda(\mathbb{F}_2)$ of the free group $\mathbb{F}_2$ on two generators has the metric approximation property, that is, the identity map on $C^*_\lambda(\mathbb{F}_2)$ is a point-norm limit of finite rank contractions. However, by a result of Lance [39], as $\mathbb{F}_2$ is not amenable, the C*-algebra $C^*_\lambda(\mathbb{F}_2)$ is not nuclear.

A central idea in Haagerup's proof is that the free group has a certain approximation property: the constant function 1 on $\mathbb{F}_2$ can be approximated pointwise by positive definite functions vanishing at infinity. This fact initiated the study of groups with what is now called the Haagerup property (Definition 5.1).

In [27], Haagerup also showed that the Fourier algebra of the free group $\mathbb{F}_2$ admits an approximate unit which is bounded in multiplier norm. It was already known by a result of Leptin [40] that the Fourier algebra of a locally compact group admits an approximate unit bounded in norm, if and only if the group is amenable. The condition about boundedness in multiplier norm can be seen as a weak form of amenability. This fact initiated the study of weakly amenable groups (Definition 5.5).

3. Discrete groups and locally compact groups

As we have tried to argue above, groups are interesting from the viewpoint of operator algebras, because they provide interesting examples of operator algebras. Also, often it is easier to work with groups directly than with operator algebras. Many well-studied groups come naturally equipped with some extra structure, e.g., a topology or a manifold structure. Lie groups, in particular, are well suited for analysis.

On the other hand, discrete groups are the most interesting from the viewpoint of operator algebras. For instance, the reduced group C*-algebra $C^*_\lambda(G)$ of a discrete group G is nuclear if and only if G is amenable [39], whereas $C^*_\lambda(G)$ is nuclear for any separable locally compact group which is connected (see [12, Corollary 6.9] and [10, Theorem 3]). Discrete groups are, on the other hand, not so well suited for analysis. For instance, discrete groups rarely have a nice representation theory in the sense that they are rarely of type I [53].

The basic idea in the theory of approximation properties for groups and operator algebras is to remedy this by embedding discrete groups as lattices in Lie groups, do the analysis on the Lie groups and then transfer the results back to the lattice and in the end to the group C*-algebra or group von Neumann algebra.

Recall that a lattice in a locally compact group G is a discrete subgroup Γ such that the quotient space G/Γ admits G-invariant, regular probability measure. A standard example of a lattice is the integer lattice $\mathbb{Z}^n$ in the Euclidean space $\mathbb{R}^n$. A less straightforward example is the subgroup $\mathrm{SL}(n,\mathbb{Z})$ in $\mathrm{SL}(n,\mathbb{R})$ (see [6, Appendix B]).

The examples of groups in what follows are Lie groups and their lattices, but since many of the definitions make sense in the more general context of locally compact groups, we have chosen to work in that setting. Locally compact groups seem like a natural framework to work within.

4. Function algebras associated with groups

This section was supposed to contain descriptions of various function algebras associated with locally compact groups. However, since we feel that this is already done in enough detail in [B, Section 3] and [D, Section 2], we have chosen instead to refer the reader to those places and simply include a list of the relevant notation in Table 1.

Symbol	Name	Comment
$C_c(G)$	compactly supported, continuous functions	
$C_0(G)$	continuous functions vanishing at infinity	
$C^\infty(G)$	smooth functions	when G is a Lie group
$L^p(G)$	Lebesgue space	$p \in \{1, 2, \infty\}$
$A(G)$	Fourier algebra	predual of $L(G)$
$B(G)$	Fourier-Stieltjes algebra	dual of $C^*(G)$
$M_0A(G)$	completely bounded Fourier multipliers	dual of $Q(G)$
$C^*(G)$	full/universal group C*-algebra	predual of $B(G)$
$C^*_\lambda(G)$	reduced group C*-algebra	
$L(G)$	group von Neumann algebra	dual of $A(G)$
$B_2(G)$	Herz-Schur multipliers	$B_2(G) = M_0A(G)$
$Q(G)$		predual of $M_0A(G)$

Table 1. Function algebras related to a locally compact group G

5. The Haagerup property and weak amenability

The following definition is motivated by the results on free groups in [27].

Definition 5.1. A locally compact group G has the *Haagerup property* if there exists a net $(u_\alpha)_{\alpha\in A}$ of continuous positive definite functions on G vanishing at infinity such that u_α converges to 1 uniformly on compact sets.

The book [8] gives a nice and thorough treatment of groups with the Haagerup property. Apart from the free groups, there are many groups with the Haagerup property. For instance, it follows from (4) in the following well-known characterization of amenability that any amenable group has the Haagerup property. Amenable groups include all compact groups and all abelian groups.

Theorem 5.2 (Leptin, Hulanicki)**.** *Each of the following equivalent conditions characterize amenability of a locally compact group G.*

(1) *There is a left-invariant mean on $L^\infty(G)$.*
(2) *The Fourier algebra $A(G)$ admits a bounded approximate unit.*
(3) *The trivial representation of G is weakly contained in the regular representation of G.*
(4) *There is a net of continuous, compactly supported, positive definite functions on G converging to the constant function 1_G uniformly on compact sets.*

Many examples of groups with the Haagerup property are given in [8]. Here we only wish to include [8, Theorem 4.0.1] which characterizes the connected Lie groups with the Haagerup property.

Theorem 5.3 ([8])**.** *A connected Lie group has the Haagerup property if and only if it is locally isomorphic to a direct product*

$$M \times \mathrm{SO}(n_1, 1) \times \cdots \times \mathrm{SO}(n_k, 1) \times \mathrm{SU}(m_1, 1) \times \cdots \times \mathrm{SU}(m_l, 1),$$

where M is an amenable Lie group.

Before we leave the Haagerup property and turn to weak amenability, we record the following characterization of the Haagerup property.

Theorem 5.4 ([1])**.** *Let G be a locally compact, σ-compact group. The following are equivalent.*

(1) *G has the Haagerup property, that is, there exists a sequence of positive definite functions in $C_0(G)$ converging to 1_G uniformly on compact subsets of G.*
(2) *There is a continuous, proper function $\psi\colon G \to [0, \infty[$ which is conditionally negative definite.*

The definition of weak amenability introduced by Cowling and Haagerup [16] is motivated by the results from [14],[17],[27].

Definition 5.5. A locally compact group G is *weakly amenable* if there exist a constant $C > 0$ and a net $(u_\alpha)_{\alpha\in A}$ in $A(G)$ such that

$$\|u_\alpha\|_{M_0A} \leq C \quad \text{for every } \alpha \in A, \tag{5.6}$$

$$u_\alpha \to 1 \text{ uniformly on compacts.} \tag{5.7}$$

The norm $\| \ \|_{M_0A}$ is the completely bounded multiplier norm. The best (i.e. lowest) possible constant C in (5.6) is called the *weak amenability constant* and denoted $\Lambda_{\mathrm{WA}}(G)$. If G is not weakly amenable, then we put $\Lambda_{\mathrm{WA}}(G) = \infty$. The weak amenability constant $\Lambda_{\mathrm{WA}}(G)$ is also called the Cowling-Haagerup constant and denoted $\Lambda_{\mathrm{cb}}(G)$ or Λ_G in the literature.

Remark 5.8. In the definition of weak amenability one could replace (5.7) by the following condition (see [16, Proposition 1.1]):

$$\|u_\alpha v - v\|_A \to 0 \text{ for every } v \in A(G). \tag{5.9}$$

Moreover, one could also replace the requirement $u_\alpha \in A(G)$ with the requirement $u_\alpha \in B_2(G) \cap C_c(G)$ (see [B, Appendix B]).

Some of the results on free groups in [27] were improved in [17, Corollary 3.9], where it was shown that $A(\mathbb{F}_2)$ admits an approximate unit which is bounded in the completely bounded multiplier norm. In short: $\mathbb{F}_2$ is weakly amenable. A list of other weakly amenable groups can be found in [8, p. 7]. It is worth noting the big overlap between groups with the Haagerup property and groups that are weakly amenable with constant one.

Although the description of weak amenability for connected Lie groups is not completely finished (yet), a lot is known (see [15]). Here we record the case of simple Lie groups which is the combined work of [14], [16], [17], [23], [24], [28], [32].

Theorem 5.10. *A connected simple Lie group G is weakly amenable if and only if the real rank of G is zero or one. In that case, the weak amenability constant is*

$$\Lambda_{\mathrm{WA}}(G) = \begin{cases} 1 & \text{when } G \text{ has real rank zero} \\ 1 & \text{when } G \approx \mathrm{SO}_0(1,n) \\ 1 & \text{when } G \approx \mathrm{SU}(1,n) \\ 2n-1 & \text{when } G \approx \mathrm{Sp}(1,n) \\ 21 & \text{when } G \approx \mathrm{F}_{4(-20)}. \end{cases} \tag{5.11}$$

Above, $G \approx H$ means that G is locally isomorphic to H.

Several permanence results are known for the class of groups with the Haagerup property and the class of weakly amenable groups [8], [16], [35], [37], [28]. We list some of them here. Some of the statements hold in greater generality, but here we wish to emphasize the mostly used cases.

The class of groups with the Haagerup property is closed under forming (finite) direct products, passing to closed subgroups and passing to a directed union of open subgroups. Let G be a locally compact, second countable group. If G has a closed normal subgroup N with the Haagerup property such that the quotient is amenable, then G has the Haagerup property. If a lattice in G has the Haagerup property then so does G.

The same permanence results as for the Haagerup property hold for the class of groups G satisfying $\Lambda_{\mathrm{WA}}(G) = 1$. Moreover, Λ_{WA} is multiplicative [16, Corollary 1.5].

The group $\mathbb{R}^2 \rtimes \mathrm{SL}(2,\mathbb{R})$ does not have the Haagerup property as the non-compact subgroup $\mathbb{R}^2$ has relative property (T) in $\mathbb{R}^2 \rtimes \mathrm{SL}(2,\mathbb{R})$ (see [6]). In particular, extensions of groups with the Haagerup property need not have the Haagerup property, and the assumption above about amenability of the quotient cannot be removed. Similarly, for weak amenability the following holds.

Theorem 5.12 ([28],[23])**.** *The group $\mathbb{R}^2 \rtimes \mathrm{SL}(2,\mathbb{R})$ is not weakly amenable.*

Combining the results on weak amenability and the Haagerup property for simple Lie groups (Theorem 5.3 and Theorem 5.10) one obtains the following theorem.

Theorem 5.13. *Let G be a connected simple Lie group. The following are equivalent.*

(1) *G is compact or locally isomorphic to $\mathrm{SO}(n,1)$ or $\mathrm{SU}(n,1)$ for some $n \geq 2$.*
(2) *G has the Haagerup property.*
(3) *G is weakly amenable with constant 1.*

Looking at Theorem 5.13 and comparing the lists of weakly amenable groups and groups with the Haagerup property from [8], it is tempting to assume some sort of relation between the two classes of groups. The following question was posed by Cowling [8, p. 7].

Question 5.14 (Cowling)**.** *Let G be a locally compact group. Is G weakly amenable with $\Lambda_{\mathrm{WA}}(G) = 1$ if and only if G has the Haagerup property?*

Question 5.14, or more precisely Question 7.1 below, has been the main problem in my thesis.

6. A counterexample in one direction: wreath products

Around 2008, two results on wreath products appeared which together solve one direction of Question 5.14 in the negative. We recall that the (standard or restricted) wreath product $H \wr G$ of two groups H and G is the semidirect product

$$H \wr G = \left(\bigoplus_G H \right) \rtimes G,$$

where G acts by shifting the direct sum $\bigoplus_G H$ of copies of H.

The first result, proved by de Cornulier, Stalder and Valette [19], is that if H is a finite group and $\mathbb{F}_n$ is a free group on n generators, then the wreath product group $H \wr \mathbb{F}_n$ has the Haagerup property. They later generalized their result to include all wreath products $H \wr G$ where H and G are countable groups with the Haagerup property [20].

The second result on wreath products, obtained by Ozawa and Popa [46, Corollary 2.12], is that $\Lambda_{\mathrm{WA}}(H \wr G) > 1$ whenever H is a non-trivial group and G is non-amenable. Combining the two results, one obtains a counterexample to Question 5.14: the wreath product group $W = \mathbb{Z}/2 \wr \mathbb{F}_2$ has the Haagerup property, but $\Lambda_{\mathrm{WA}}(W) > 1$. Here, $\mathbb{Z}/2$ denotes the group with two elements.

As the weak amenability constant Λ_{WA} is multiplicative [16, Corollary 1.5], one can produce an even stronger counterexample to Question 5.14: the infinite direct sum $\bigoplus_{\mathbb{Z}} W$ has the Haagerup property, but is not weakly amenable.

Subsequently, Ozawa proved [45, Corollary 4] that a wreath product $H \wr G$ is not weakly amenable, whenever H is non-trivial and G is non-amenable.

Proposition 6.1 ([20],[45])**.** *The wreath product $\mathbb{Z}/2 \wr \mathbb{F}_2$ of the group on two elements with the free group on two generators has the Haagerup property, but is not weakly amenable.*

More generally, if H is any non-trivial countable group with the Haagerup property, and G is any countable non-amenable group with the Haagerup property, then $H \wr G$ has the Haagerup property, but is not weakly amenable.

Combining the results of [20],[45] with the permanence results (see Section 5), one obtains the following complete classification of wreath products with the Haagerup property and wreath products that are weakly amenable. Proposition 6.2 does not appear explicitly in [20],[45].

Proposition 6.2 ([20],[45])**.** *Let G and H be non-trivial, countable groups. The following are equivalent.*

(1) *The wreath product $H \wr G$ has the Haagerup property.*

(2) *Both H and G have the Haagerup property.*

If G is infinite the following are equivalent.

(1') *The wreath product $H \wr G$ is weakly amenable, that is, $\Lambda_{\mathrm{WA}}(H \wr G) < \infty$.*
(2') $\Lambda_{\mathrm{WA}}(H \wr G) = 1$.
(3') *G is amenable and $\Lambda_{\mathrm{WA}}(H) = 1$.*

If G is finite, then $\Lambda_{\mathrm{WA}}(H \wr G) = \Lambda_{\mathrm{WA}}(H)^{|G|}$ so the following are equivalent.

(1") *The wreath product $H \wr G$ is weakly amenable.*
(2") *H is weakly amenable.*

7. An attempt in the other direction

The results on wreath products described in the previous section solved half of Question 5.14 in the negative, but left open the remaining half:

Question 7.1. *Let G be a locally compact group with $\Lambda_{\mathrm{WA}}(G) = 1$. Does G have the Haagerup property?*

The article [A] dealt with Question 7.1, but no conclusive results were obtained. As of today, Question 7.1 is still open.

It is clear that if a locally compact group G satisfies $\Lambda_{\mathrm{WA}}(G) = 1$, then there is a net $(u_i)_{i\in I}$ of contractive Herz-Schur multipliers in $C_0(G)$ tending to 1 uniformly on compact subsets of G. The point of this trivial observation is that the latter condition is equivalent to condition (2) in Theorem 7.2 below which is reminiscent of condition (2) in Theorem 5.4 characterizing the Haagerup property.

Theorem 7.2 ([A],[B]). *Let G be a locally compact, σ-compact group. The following are equivalent.*

(1) *There is a sequence of contractive Herz-Schur multipliers in $C_0(G)$ converging to 1 uniformly on compact subsets of G.*
(2) *There is a continuous, proper function $\psi\colon G \to [0,\infty[$ such that $\|e^{-t\psi}\|_{B_2} \leq 1$ for every $t > 0$.*

Instead of attacking Question 7.1 directly, the idea in [A] was to take condition (2) of Theorem 7.2 as a starting point and consider the following related problem.

Problem 7.3 ([A]). *Let G be a countable, discrete group, and let $\varphi\colon G \to \mathbb{R}$ be a symmetric function satisfying $\|e^{-t\varphi}\|_{B_2} \leq 1$ for all $t > 0$. Does there exist a conditionally negative definite function ψ on G such that $\varphi \leq \psi$?*

A positive solution to this problem would provide a positive solution to Question 7.1. Problem 7.3 was not solved, but the following related result was obtained. Note that φ is not required to be proper in the theorem below.

Theorem 7.4 ([A]). *Let G be a countable, discrete group with a symmetric function $\varphi : G \to \mathbb{R}$. Then $\|e^{-t\varphi}\|_{B_2} \leq 1$ for every $t > 0$ if and only if φ splits as*

$$\varphi(y^{-1}x) = \psi(x,y) + \theta(x,y) + \theta(e,e) \qquad (x, y \in G),$$

where

· *ψ is a conditionally negative definite kernel on G vanishing on the diagonal,*
· *and θ is a bounded, positive definite kernel on G.*

The downside of the above theorem is that the ψ and θ are kernels on G, i.e., defined on $G \times G$, and not necessarily functions on G. One can, of course, hope to be able to strengthen Theorem 7.4 and produce functions ψ and θ defined on the group G itself, but as the following theorem shows, this is not always possible unless G is amenable.

Theorem 7.5 ([A]). *Let G be a countable, discrete group. Then G is amenable if and only if the following condition holds. Whenever $\varphi : G \to \mathbb{R}$ is a symmetric function such that $\|e^{-t\varphi}\|_{B_2} \leq 1$ for every $t > 0$, then φ splits as*

$$\varphi(x) = \psi(x) + ||\xi||^2 + \langle \pi(x)\xi, \xi \rangle \qquad (x \in G)$$

where

· *ψ is a conditionally negative definite function on G with $\psi(e) = 0$,*
· *π is an orthogonal representation of G on some real Hilbert space H,*
· *and ξ is a vector in H.*

Note that the function $x \mapsto \langle \pi(x)\xi, \xi \rangle$ is positive definite, and every positive definite function has this form.

Problem 7.3 was solved in the special case where G is a free group and the function φ is radial. Indeed, as the word length function on a free group is conditionally negative definite [27], the following solves Problem 7.3 in the special case.

Theorem 7.6 ([A]). *Let $\mathbb{F}_n$ be the free group on n generators ($2 \leq n \leq \infty$), and let $\varphi : \mathbb{F}_n \to \mathbb{R}$ be a radial function, i.e., $\varphi(x)$ depends only on the word length $|x|$. If $\|e^{-t\varphi}\|_{B_2} \leq 1$ for every $t > 0$, then there are constants $a, b \geq 0$ such that*

$$\varphi(x) \leq b + a|x| \qquad \text{for all } x \in \mathbb{F}_n.$$

8. The weak Haagerup property

The equivalent conditions in Theorem 7.2 motivated the definition of the weak Haagerup property introduced in [A], [B].

Definition 8.1 (Weak Haagerup property [A], [B]). A locally compact group G has the *weak Haagerup property* if there are a constant $C > 0$ and a net $(u_\alpha)_{\alpha\in A}$ in $B_2(G) \cap C_0(G)$ such that

$$\|u_\alpha\|_{B_2} \leq C \quad \text{for every } \alpha \in A,$$

$$u_\alpha \to 1 \text{ uniformly on compacts as } \alpha \to \infty.$$

The weak Haagerup constant $\Lambda_{\mathrm{WH}}(G)$ is defined as the infimum of those C for which such a net (u_α) exists, and if no such net exists we write $\Lambda_{\mathrm{WH}}(G) = \infty$. It is not difficult to see that the infimum is actually a minimum.

It is natural to compare the weak Haagerup property with the previously introduced approximation properties, weak amenability and the Haagerup property. Obviously, the weak Haagerup property is weaker than both of them. It turns out that the weak Haagerup property is in fact strictly weaker:

Example 8.2 ([B])**.** The class of groups with the weak Haagerup property contains groups that are neither weakly amenable nor have the Haagerup property. One such example can be manufactured in the following way: Let Γ be the quaternion integer lattice in the simple Lie group $\mathrm{Sp}(1,n)$. In other words, Γ consists of the matrices in $\mathrm{Sp}(1,n)$ whose entries belong to $\mathbb{Z}+i\mathbb{Z}+j\mathbb{Z}+k\mathbb{Z}$ (with standard notation). It is known that Γ does not have the Haagerup property. In fact, Γ has property (T). However, Γ is weakly amenable. The group

$$\Gamma \times (\mathbb{Z}/2 \wr \mathbb{F}_2)$$

then has the weak Haagerup property, but does not have the Haagerup property and is not weakly amenable.

In [B], the weak Haagerup property for groups is systematically studied. Also, [B] treated the weak Haagerup property for von Neumann algebras (see Section 11). The main results of [B] on groups are the permanence results summarized in the following theorem. Some of these results were generalized by Jolissaint [37] and by Deprez, Li [21].

Theorem 8.3 ([B])**.** *Let G be a locally compact group.*

(1) *If H is a closed subgroup of G, and G has the weak Haagerup property, then H has the weak Haagerup property. More precisely,*
$$\Lambda_{\mathrm{WH}}(H) \leq \Lambda_{\mathrm{WH}}(G).$$
(2) *If K is a compact normal subgroup of G, then G has the weak Haagerup property if and only if G/K has the weak Haagerup property. More precisely,*
$$\Lambda_{\mathrm{WH}}(G) = \Lambda_{\mathrm{WH}}(G/K).$$
(3) *The weak Haagerup property is preserved under finite direct products. More precisely, if G' is a locally compact group, then*
$$\Lambda_{\mathrm{WH}}(G \times G') \leq \Lambda_{\mathrm{WH}}(G)\Lambda_{\mathrm{WH}}(G').$$
(4) *If $(G_i)_{i\in I}$ is a directed set of open subgroups of G, then*
$$\Lambda_{\mathrm{WH}}(\bigcup_i G_i) = \lim_i \Lambda_{\mathrm{WH}}(G_i).$$
(5) *If $1 \longrightarrow N \hookrightarrow G \longrightarrow G/N \longrightarrow 1$ is a short exact sequence of locally compact groups, where G is second countable or discrete, and if G/N is amenable, then G has the weak Haagerup property if and only if N has the weak Haagerup property. More precisely,*
$$\Lambda_{\mathrm{WH}}(G) = \Lambda_{\mathrm{WH}}(N).$$
(6) *If Γ is a lattice in G and if G is second countable, then G has the weak Haagerup property if and only if Γ has the weak Haagerup property. More precisely,*
$$\Lambda_{\mathrm{WH}}(\Gamma) = \Lambda_{\mathrm{WH}}(G).$$

In subsequent work joint with Haagerup [C], the weak Haagerup property was studied in connection with simple Lie groups. Since everything is known about weak amenability and the Haagerup property for simple Lie groups, it is reasonable to expect that similar results can be obtained for the weak Haagerup property. This is indeed the case, but the proofs of the following results use several deep results from [31], [29], [30], [D] that were not available when similar results were established for weak amenability.

The following result extends a result from [28], [23] (see Theorem 5.12).

Theorem 8.4 ([C]). *The group* $\mathbb{R}^2 \rtimes \mathrm{SL}(2,\mathbb{R})$ *does not have the weak Haagerup property.*

The following result completely characterizes connected simple Lie groups with the weak Haagerup property.

Theorem 8.5 ([C]). *Let G be a connected simple Lie group. Then G has the weak Haagerup property if and only if the real rank of G is at most one.*

Theorem 8.5 follows essentially from structure theory and the following theorem.

Theorem 8.6 ([C]). *The groups* $\mathrm{SL}(3,\mathbb{R})$, $\mathrm{Sp}(2,\mathbb{R})$, *and* $\widetilde{\mathrm{Sp}}(2,\mathbb{R})$ *do not have the weak Haagerup property.*

Theorem 8.6 is a fairly easy consequence of recent results of Haagerup and de Laat [30], [29]. In the real rank one case, it is interesting to compute the weak Haagerup constants. Comparing Theorem 8.7 with (5.11) one sees that $\Lambda_{\mathrm{WA}}(G) = \Lambda_{\mathrm{WH}}(G)$ for every connected simple Lie group G.

Theorem 8.7 ([C]). *Let G be a connected simple Lie group of real rank one. Then*

$$\Lambda_{\mathrm{WH}}(G) = \begin{cases} 1 & \text{for } G \approx \mathrm{SO}_0(1,n) \\ 1 & \text{for } G \approx \mathrm{SU}(1,n) \\ 2n-1 & \text{for } G \approx \mathrm{Sp}(1,n) \\ 21 & \text{for } G \approx \mathrm{F}_{4(-20)} \end{cases}$$

where $G \approx H$ means that G is locally isomorphic to H.

The proof of Theorem 8.7 is based on the work of Cowling and Haagerup [16], who computed the weak amenability constants of most of the rank one simple Lie groups (5.11).

9. An aside on Fourier algebras

Recall that $A(G)$ and $B(G)$ denote the Fourier algebra and the Fourier-Stieltjes algebra of a locally compact group G, respectively. A key ingredient in the proof of Theorem 8.7 is the fact that the simple Lie groups of real rank one contain certain parabolic subgroups P with the property that every element of $B_2(P) \cap C_0(P)$ is a matrix coefficient of the regular representation of P, i.e., belongs to $A(P)$.

Let $n \geq 2$, let G be one of the classical simple Lie groups $\mathrm{SO}_0(n,1)$, $\mathrm{SU}(n,1)$, $\mathrm{Sp}(n,1)$, or the exceptional group $\mathrm{F}_{4(-20)}$, and let $G = KAN$ be the Iwasawa decomposition. If M is the centralizer of A in K, then $P = MAN$ is the minimal parabolic subgroup of G. As the parabolic subgroups are amenable, $B_2(P) = B(P)$, and the result we are after is the following.

Theorem 9.1 ([D]). *Let P be the minimal parabolic subgroup in one of the simple Lie groups* $\mathrm{SO}_0(n,1)$, $\mathrm{SU}(n,1)$, $\mathrm{Sp}(n,1)$, *or* $\mathrm{F}_{4(-20)}$. *Then* $A(P) = B(P) \cap C_0(P)$.

Similarly, the proof of Theorem 8.4 is based on the following result.

Theorem 9.2 ([D]). *For the group*

$$P = \left\{ \begin{pmatrix} \lambda & a & c \\ 0 & \lambda^{-1} & b \\ 0 & 0 & 1 \end{pmatrix} \middle| a, b, c \in \mathbb{R},\ \lambda > 0 \right\}$$

we have $A(P) = B(P) \cap C_0(P)$.

There is a natural way to think of P as a subgroup of $\mathbb{R}^2 \rtimes \mathrm{SL}_2(\mathbb{R})$, and this is how Theorem 9.2 is used in the proof of Theorem 8.4.

The proofs of Theorem 9.1 and Theorem 9.2 are based on [D, Theorem 4] which gives sufficient conditions on a locally compact group G that allows one to conclude

$$A(G) = B(G) \cap C_0(G). \tag{9.3}$$

The paper [D] starts with a general discussion of groups G with/without the property (9.3).

10. Countable subgroups in simple Lie groups

Many amenable groups such as the compact groups $\mathrm{SO}(n)$, $n \geq 3$, contain non-abelian free subgroups, and such subgroups are non-amenable when viewed as discrete groups. Arguably, the existence of free subgroups in SO(3), going back to Hausdorff [33], [34, p. 469], was the starting point of the study of amenability which eventually led to the study of the approximation properties described in this thesis. One can ask if the rotation groups $\mathrm{SO}(n)$ contain countable subgroups which are not weakly amenable or do not have the Haagerup property. More generally, one could ask which simple Lie groups contain countable subgroups which are not weakly amenable or do not have the Haagerup property. In [18, Theorem 1.14], de Cornulier answered this question for the Haagerup property.

Inspired by de Cornulier's result on the Haagerup property, a similar study was carried out for weak amenability and the weak Haagerup property in a joint paper with Li [E]. The main result of that paper is (a reformulation of) the following theorem.

Theorem 10.1 ([E]). *Let G be a connected simple Lie group. The following are equivalent.*

(1) *G is locally isomorphic to* SO(3)*,* $\mathrm{SL}(2,\mathbb{R})$*, or* $\mathrm{SL}(2,\mathbb{C})$.
(2) *Every countable subgroup of G has the Haagerup property.*
(3) *Every countable subgroup of G is weakly amenable with constant 1.*
(4) *Every countable subgroup of G is weakly amenable.*
(5) *Every countable subgroup of G has the weak Haagerup property with constant 1.*
(6) *Every countable subgroup of G has the weak Haagerup property.*

The equivalence of (1) and (2) in Theorem 10.1 is due to de Cornulier [18, Theorem 1.14] and holds in greater generality. Note that Theorem 10.1 answers Question 5.14 for the class of connected simple Lie groups *equipped with the discrete topology* similar to how Theorem 5.13 answers Question 5.14 for connected simple Lie groups with their usual (Lie group) topology. The methods used to prove Theorem 10.1 are similar to those in [18] but use also the recent classification of simple Lie groups with the weak Haagerup property (Theorem 8.5).

11. Von Neumann algebras

As mentioned in Section 2, a major motivation for studying approximation properties of groups is to obtain results about operator algebras. Approximation properties of groups that have operator algebraic analogues are of particular interest. Here we will focus on the case of finite von Neumann algebras, but C*-algebraic versions of some of the approximation properties discussed earlier also exist. Throughout this section, M denotes a von Neumann algebra equipped with a faithful normal tracial state τ.

The weak* completely bounded approximation property. In [28], the *weak* completely bounded approximation property* (in short: W*CBAP) was introduced. The von Neumann algebra M has the W*CBAP if there is a net T_α of ultraweakly continuous, finite rank operators on M such that $T_\alpha \to 1_M$ point-ultraweakly and $\|T_\alpha\|_{\mathrm{cb}} \leq C$ for some $C > 0$ and every α. Here $\| \ \|_{\mathrm{cb}}$ denotes the completely bounded operator norm. The infimum of all C for which such a net exists is denoted $\Lambda_{\mathrm{cb}}(M)$. It was shown in [28] that a discrete group Γ is weakly amenable if and only if its group von Neumann algebra $L(\Gamma)$ has the W*CBAP, and moreover

$$\Lambda_{\mathrm{cb}}(L(\Gamma)) = \Lambda_{\mathrm{WA}}(\Gamma). \tag{11.1}$$

In [16, Corollary 6.7], the W*CBAP was used to give an example of a II_1 factor M such that all its tensor powers M, $M\overline{\otimes}M$, $M\overline{\otimes}M\overline{\otimes}M$, ..., are non-isomorphic. Apart from (11.1), the example was, among other things, based on the fact that the weak amenability constant Λ_{WA} is multiplicative with respect to direct products [16, Corollary 1.5] (see also [50, Theorem 4.1]). We refer the reader to [7, Section 12.3] and [2, Section 4] for more on the W*CBAP.

The Haagerup property. The Haagerup property for II_1 factors was introduced by Connes [13] and Choda [9] and later generalized to finite von Neumann algebras by Anantharaman-Delaroche [2, Definition 4.15]. See also [36] and the references therein for more on the Haagerup property. The von Neumann algebra M (with trace τ) has the *Haagerup property* if there is a net T_α of ultraweakly continuous, completely positive operators on M such that

(1) $\tau \circ T_\alpha \leq \tau$ for every α,
(2) T_α extends to a compact operator on $L^2(M, \tau)$,
(3) $\|T_\alpha x - x\|_\tau \to 0$ for every $x \in M$.

As proved by Jolissaint [36], one can arrange that $\tau \circ T_\alpha = \tau$ and that T_α is unital. Moreover, the Haagerup property of M does not depend on the choice of τ.

Choda [9] proved that a discrete group Γ has the Haagerup property if and only if its group von Neumann algebra $L(\Gamma)$ has the Haagerup property.

The weak Haagerup property. In [B], the weak Haagerup property was introduced for von Neumann algebras. As the name suggests, it is weaker than the Haagerup property.

Definition 11.2 ([B])**.** Let M be a von Neumann algebra with a faithful normal tracial state τ. Then (M, τ) has the *weak Haagerup property*, if there are a constant $C > 0$ and a net T_α of ultraweakly continuous, completely bounded maps on M such that

(1) $\|T_\alpha\|_{\mathrm{cb}} \le C$ for every α,
(2) $\langle T_\alpha x, y\rangle_\tau = \langle x, T_\alpha y\rangle_\tau$ for every $x, y \in M$,
(3) each T_α extends to a *compact* operator on $L^2(M,\tau)$,
(4) $T_\alpha x \to x$ ultraweakly for every $x \in M$.

The weak Haagerup constant $\Lambda_{\mathrm{WH}}(M,\tau)$ is defined as the infimum of those C for which such a net T_α exists, and if no such net exists we write $\Lambda_{\mathrm{WH}}(M,\tau) = \infty$.

The weak Haagerup constant of M is independent of the choice of trace on M, that is, $\Lambda_{\mathrm{WH}}(M,\tau) = \Lambda_{\mathrm{WH}}(M,\tau')$ for any two faithful normal tracial states τ and τ' on M [B, Proposition 8.4]. Thus, one often writes $\Lambda_{\mathrm{WH}}(M)$ instead of $\Lambda_{\mathrm{WH}}(M,\tau)$.

It is not immediately clear that the weak Haagerup property is weaker than the Haagerup property, but this follows basically from Jolissaint's characterization and [3, Lemma 2.5] (see [B, Proposition 7.6]). As should be expected, the following holds.

Theorem 11.3 ([B])**.** *Let Γ be a discrete group. The following conditions are equivalent.*

(1) *The group Γ has the weak Haagerup property.*
(2) *The group von Neumann algebra $L(\Gamma)$ has the weak Haagerup property.*

More precisely, $\Lambda_{\mathrm{WH}}(\Gamma) = \Lambda_{\mathrm{WH}}(L(\Gamma))$.

One can also prove several permanence results for the class of von Neumann algebras with the weak Haagerup property.

Theorem 11.4 ([B])**.** *Let $M, M_1, M_2, \dots$ be von Neumann algebras which admit faithful normal tracial states.*

(1) *If $M_2 \subseteq M_1$ is a von Neumann subalgebra, then $\Lambda_{\mathrm{WH}}(M_2) \le \Lambda_{\mathrm{WH}}(M_1)$.*
(2) *If $p \in M$ is a non-zero projection, then $\Lambda_{\mathrm{WH}}(pMp) \le \Lambda_{\mathrm{WH}}(M)$.*
(3) *Suppose that $1 \in M_1 \subseteq M_2 \subseteq \cdots$ are von Neumann subalgebras of M generating all of M, and there is an increasing sequence of non-zero projections $p_n \in M_n$ with strong limit 1. Then $\Lambda_{\mathrm{WH}}(M) = \sup_n \Lambda_{\mathrm{WH}}(p_n M_n p_n)$.*
(4)
$$\Lambda_{\mathrm{WH}}\left(\bigoplus_n M_n\right) = \sup_n \Lambda_{\mathrm{WH}}(M_n).$$
(5)
$$\Lambda_{\mathrm{WH}}(M_1 \otimes M_2) \le \Lambda_{\mathrm{WH}}(M_1)\Lambda_{\mathrm{WH}}(M_2).$$

Example 11.5 ([B])**.** Using Theorem 11.3 and the results on groups with the weak Haagerup property (mostly Theorem 8.4 and Theorem 8.7) it is possible to give examples of two II_1 factors whose weak Haagerup constants differ. Moreover, an example can be constructed such that the other approximation properties (W*CBAP and the Haagerup property) do not distinguish the two factors. We describe the factors here and refer to [B, Section 9] for more details.

Let Γ_0 be the quaternion integer lattice in $\mathrm{Sp}(1,n)$ modulo its center $\{\pm I\}$, and let $W = \mathbb{Z}/2 \wr \mathbb{F}_2$ be the wreath product group (Section 6). We let $\Gamma_1 = \Gamma_0 \times W$ and $\Gamma_2 = \mathbb{Z}^2 \rtimes \mathrm{SL}(2,\mathbb{Z})$.

For $i = 1, 2$, let M_i be the group von Neumann algebra $L(\Gamma_i)$ of Γ_i. Then M_1 and M_2 are II_1 factors and
$$\Lambda_{\mathrm{WH}}(M_1) = 2n - 1, \quad \Lambda_{\mathrm{WH}}(M_2) = \infty.$$

Another example could have been obtained using only M_1 by simply choosing two different values for n.

12. What now?

Although the thesis has provided some partial answers to Question 7.1, no definite answer was obtained. We list the question here together with a series of related problems.

Problem 12.1. *Let G be a locally compact group with $\Lambda_{\mathrm{WA}}(G) = 1$. Does G have the Haagerup property?*

Non-simple Lie groups. One of the main results of the thesis is the computation of the weak Haagerup constants $\Lambda_{\mathrm{WH}}(G)$ for all connected simple Lie groups G (Section 8). It is natural to ask what role the assumptions of connectedness and simplicity play. The assumption of connectedness cannot be avoided since otherwise we would also need to deal with all discrete groups, and this is clearly a very different (and inaccessible) problem.

Removing the assumption of simplicity, on the other hand, is more interesting. First of all, it is easy to classify connected semisimple Lie groups with the weak Haagerup property, if they are direct products of simple Lie groups. This is because a finite direct product of groups has the weak Haagerup property if and only if each factor has the property. The computation of the weak Haagerup constant Λ_{WH}, however, is not obvious. The weak Haagerup constant is sub-multiplicative with respect to direct products [B, Proposition 5.5], but the following is unsolved.

Problem 12.2. *Let G and H be locally compact groups. Is*

$$\Lambda_{\mathrm{WH}}(G \times H) = \Lambda_{\mathrm{WH}}(G)\Lambda_{\mathrm{WH}}(H)?$$

We remark that the similar problem for the weak amenability constant Λ_{WA} is known to be true [16, Corollary 1.5].

In [23], Dorofaeff showed that the groups $\mathbb{R}^n \rtimes \mathrm{SL}(2,\mathbb{R})$, $n \geq 2$, are not weakly amenable. Here, the action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{R}^n$ is by the unique irreducible representation. Subsequently, in [24], Dorofaeff used this fact to complete the classification of connected simple Lie groups that are weakly amenable. The paper [15] deals with the non-simple case and almost completes the classification of connected Lie groups that are weakly amenable. This is based on Dorofaeff's results combined with the fact (proved in [15]) that the groups $H_n \rtimes \mathrm{SL}(2,\mathbb{R})$, $n \geq 1$, are not weakly amenable. Here, $\mathrm{SL}(2,\mathbb{R})$ acts on the Heisenberg group H_n of dimension $2n+1$ by fixing the center of H_n and acting on the vector space $\mathbb{R}^{2n}$ by the unique irreducible representation of dimension $2n$. It is plausible that these groups also fail to have the weak Haagerup property.

Problem 12.3. *Do the groups $\mathbb{R}^n \rtimes \mathrm{SL}(2,\mathbb{R})$, $n \geq 3$, fail to have the weak Haagerup property? Do the groups $H_n \rtimes \mathrm{SL}(2,\mathbb{R})$, $n \geq 1$, fail to have the weak Haagerup property?*

If true, a strategy for solving Problem 12.3 could be to locate a suitable solvable group P inside $\mathbb{R}^n \rtimes \mathrm{SL}(2,\mathbb{R})$ or $H_n \rtimes \mathrm{SL}(2,\mathbb{R})$ and then mimic the proof for $\mathbb{R}^2 \rtimes \mathrm{SL}(2,\mathbb{R})$ (see [C]). The meaning of "suitable" is that P contains a certain nilpotent subgroup of $\mathbb{R}^n \rtimes \mathrm{SL}(2,\mathbb{R})$ or $H_n \rtimes \mathrm{SL}(2,\mathbb{R})$ on which many estimates are made in [23] and [15], and at the same time P should satisfy $A(P) = B(P) \cap C_0(P)$. This is one motivation for studying Problem 12.10 below.

Generalizing Problem 12.3, one can ask the following:

Problem 12.4. *Is $\Lambda_{\mathrm{WA}}(G) = \Lambda_{\mathrm{WH}}(G)$, when G is a connected Lie group?*

Note that Problem 12.4 is true for connected simple Lie groups [C], although at the moment the proof is not intrinsic but uses instead the computation of Λ_{WA} and Λ_{WH} for every connected simple Lie group. As mentioned in Example 8.2, Problem 12.4 is not true for discrete groups in general.

Locally compact groups made discrete. As mentioned in Section 10, the equivalence of (1) and (2) in Theorem 10.1 holds more generally (see [18, Theorem 1.14]). Thus, it is natural to ask the following.

Problem 12.5. *Are statements (2)–(6) in Theorem 10.1 equivalent for all connected Lie groups G?*

It is a curious fact that if G is a locally compact group which *as a discrete group* is amenable, then G is also amenable *as a locally compact group* [47, Proposition 4.21]. One can ask whether the same thing holds with weak amenability:

Problem 12.6 ([E]). *If G is a locally compact group and G_{d} denotes the group G equipped with the discrete topology, is it then true that if G_{d} is weakly amenable, then G is weakly amenable?*

Or more specifically, does the inequality $\Lambda_{\mathrm{WA}}(G) \leq \Lambda_{\mathrm{WA}}(G_{\mathrm{d}})$ hold? Similar questions can of course be asked for the Haagerup property and the weak Haagerup property.

It is not reasonable to expect an implication in the other direction. For instance, many (amenable) compact groups such as $\mathrm{SO}(n)$, $n \geq 3$, are non-amenable as discrete groups. It follows from Theorem 10.1 that when $n \geq 5$, then $\mathrm{SO}(n)$ as a discrete group does not even have the weak Haagerup property.

Free products. It was shown by Jolissaint that free products of discrete groups with the Haagerup property have the Haagerup property (see [35, Proposition 2.5] or [8, Proposition 6.2.3]). By a result of Ricard and Xu [49, Theorem 4.13], it is also known that the class of discrete groups with weak amenability constant 1 is closed under free products. The corresponding result for weakly amenable groups (no assumption on the constant) is unknown.

Problem 12.7. *Is the class of discrete groups with weak Haagerup constant 1 closed under free products?*

Von Neumann algebras. For groups it is clear that both weak amenability and the Haagerup property imply the weak Haagerup property. For von Neumann algebras, the Haagerup property still implies the weak Haagerup property [B, Proposition 7.6], but the corresponding result for the W*CBAP is not known.

Problem 12.8. *Let M be a von Neumann algebra which admits a faithful normal tracial state. If M has the W*CBAP, does M have the weak Haagerup property?*

Linear groups. Recall that a group is linear if it has a faithful representation in a general linear group $\mathrm{GL}(n, K)$ for some natural number n and some field K. Linear groups are much studied, see e.g. the book [55] by Wehrfritz. Finitely generated, linear groups are known to be residually finite by a theorem of Mal'cev [42], and linear groups satisfy Tits' alternative [54]: Over a field of characteristic zero, a linear group either has a non-abelian free subgroup or possesses a solvable subgroup of finite index. Over a field of non-zero characteristic, a linear group either has a non-abelian free subgroup or possesses a normal solvable subgroup such that the quotient is locally finite (i.e. every finite subset generates a finite subgroup).

A longstanding problem, going back to von Neumann and the origin of amenability, asks whether every non-amenable group contains a non-abelian free subgroup. The problem was solved in the negative by Ol'šanskiĭ [44], but Tits' alternative shows that any counterexample to the problem cannot be linear.

Similarly, it follows from Tits' alternative that any (discrete) counterexample to the unsolved Dixmier problem [22], asking whether unitarizable groups are amenable, cannot be linear. See [48] for more on the Dixmier problem.

The group $\mathbb{Z}/2 \wr \mathbb{F}_2$, which has played an important role in relation to Question 5.14, is not linear, as we will now see. It is finitely generated.

Linear wreath products were completely classified by Vapne and Wehrfritz, independently (see e.g. [55, Theorem 10.21]). It follows from their classification that if $G \wr H$ is linear for some non-trivial group G, then H is a finite extension of an abelian group. In particular, H must be amenable in order for the wreath product $G \wr H$ to be linear. As free groups are not amenable, the group $\mathbb{Z}/2 \wr \mathbb{F}_2$ is not linear.

Since wreath product groups are the only known counterexamples to Question 5.14, the following problem is still open.

Problem 12.9. *Is it true that a discrete, linear group* Γ *is weakly amenable with* $\Lambda_{\mathrm{WA}}(\Gamma) = 1$ *if and only if* Γ *has the Haagerup property?*

Fourier algebras. Recall that $A(G)$ and $B(G)$ denote the Fourier algebra and the Fourier-Stieltjes algebra of a locally compact group G, respectively. It is always the case that $A(G) \subseteq B(G) \cap C_0(G)$ whereas $B(G)$ contains the constant functions. In particular, when G is non-compact $A(G) \neq B(G)$. Consider the following problem.

Problem 12.10 ([D])**.** *Let* G *be a second countable, locally compact group. When does the equality*

$$A(G) = B(G) \cap C_0(G) \tag{$\star$}$$

hold?

A great deal is already known about this problem (see e.g. [52], [D]). It is known that if ($\star$) holds, then the regular representation of G is completely reducible [5], that is, the regular representation decomposes as a direct sum of irreducible unitary representations. Such groups are usually called [AR]-groups (atomic regular). For a while, ($\star$) was thought to characterize [AR]-groups, but this turned out not to be the case [4]. However, the relation between [AR]-groups and groups satisfying ($\star$) is still quite unclear. For instance the following problem is still unsettled.

Problem 12.11 ([4])**.** *Let* G *be a second countable, locally compact* [AR]*-group. Suppose moreover that* G *is unimodular. Does* ($\star$) *hold?*

Part II

Articles

ARTICLE A

Semigroups of Herz-Schur multipliers

This chapter contains the published version of the following article:

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Semigroups of Herz–Schur multipliers

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Abstract

In order to investigate the relationship between weak amenability and the Haagerup property for groups, we introduce the weak Haagerup property, and we prove that having this approximation property is equivalent to the existence of a semigroup of Herz–Schur multipliers generated by a proper function (see Theorem 1.2). It is then shown that a (not necessarily proper) generator of a semigroup of Herz–Schur multipliers splits into a positive definite kernel and a conditionally negative definite kernel. We also show that the generator has a particularly pleasant form if and only if the group is amenable. In the second half of the paper we study semigroups of radial Herz–Schur multipliers on free groups. We prove that a generator of such a semigroup is linearly bounded by the word length function (see Theorem 1.6).

Keywords: Herz–Schur multipliers; Approximation properties

1. Introduction

The notion of amenability for groups was introduced by von Neumann [17] and has played an important role in the field of operator algebras for many years. It is well-known that amenability of a group is reflected by approximation properties of the C^*-algebra and von Neumann algebra associated with the group. More precisely, a discrete group is amenable if and only if its (reduced or universal) group C^*-algebra is nuclear if and only if its group von Neumann algebra is semidiscrete.

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Amenability may be seen as a rather strong condition to impose on a group, and several weakened forms have appeared, two of which are *weak amenability* and the *Haagerup property*. Recall that a discrete group G is amenable if and only if there is a net $(\varphi_i)_{i\in I}$ of finitely supported, positive definite functions on G such that $\varphi_i \to 1$ pointwise. When the discrete group is countable, which will always be our assumption in this paper, we can of course assume that the net is actually a sequence. We have included a few well-known alternative characterizations of amenability in Theorem 5.1.

A countable, discrete group G is called *weakly amenable* if there exist $C > 0$ and a net $(\varphi_i)_{i\in I}$ of finitely supported Herz–Schur multipliers on G converging pointwise to 1 and $\|\varphi_i\|_{B_2} \leqslant C$ for all $i \in I$ where $\|\cdot\|_{B_2}$ denotes the Herz–Schur norm. The infimum of all C for which such a net exists, is called the *Cowling–Haagerup constant* of G, usually denoted $\Lambda_{\mathrm{cb}}(G)$.

The countable, discrete group G has the *Haagerup property* if there is a net $(\varphi_i)_{i\in I}$ of positive definite functions on G converging pointwise to 1 such that each φ_i vanishes at infinity. An equivalent condition is the existence of a conditionally negative definite function $\psi : G \to \mathbb{R}$ such that ψ is proper, i.e. $\{g \in G \mid |\psi(g)| < n\}$ is finite for each $n \in \mathbb{N}$ (see for instance [5, Theorem 2.1.1]). It follows from Schoenberg's Theorem that given such a ψ, the family $(e^{-t\psi})_{t>0}$ witnesses the Haagerup property.

For a general treatment of weak amenability and the Haagerup property, including examples of groups with and without these properties, we refer the reader to [4].

Since positive definite functions are also Herz–Schur multipliers with norm 1, it is clear that amenability is stronger than both weak amenability with (Cowling–Haagerup) constant 1 and the Haagerup property. A natural question to ask is how weak amenability and the Haagerup property are related. For a long time the known examples of weakly amenable groups with constant 1 also had the Haagerup property and vice versa. Also, the groups that were known to not be weakly amenable also failed the Haagerup property. So it seemed natural to ask if the Haagerup property is equivalent to weak amenability with constant 1. This turned out to be false, and the first counterexample was the wreath product $\mathbb{Z}/2 \wr \mathbb{F}_2$. This group is defined as the semidirect product $(\bigoplus_{\mathbb{F}_2} \mathbb{Z}/2) \rtimes \mathbb{F}_2$, where the action $\mathbb{F}_2 \curvearrowright \bigoplus_{\mathbb{F}_2} \mathbb{Z}/2$ is the shift. In [6] it is shown that the group $\mathbb{Z}/2 \wr \mathbb{F}_2$ has the Haagerup property, and in [12, Corollary 2.12] it was proved that $\mathbb{Z}/2 \wr \mathbb{F}_2$ is not weakly amenable with constant 1. In fact, the group is not even weakly amenable [11, Corollary 4].

It is still an open question if groups that are weakly amenable with constant 1 have the Haagerup property. It may be formulated as follows. Given a net $(\varphi_i)_{i\in I}$ of finitely supported functions on G such that $\|\varphi_i\|_{B_2} \leqslant 1$ and $\varphi_i \to 1$ pointwise, does there exist a proper, conditionally negative definite function on G? We do not answer this question here, but we consider the following related problem. If we replace the condition that each φ_i is finitely supported with the condition that φ_i vanishes at infinity, what can then be said? We make the following definition.

Definition 1.1. A discrete group G has the *weak Haagerup property* if there exist $C > 0$ and a net $(\varphi_i)_{i\in I}$ of Herz–Schur multipliers on G converging pointwise to 1 such that each φ_i vanishes at infinity and satisfies $\|\varphi_i\|_{B_2} \leqslant C$. If we may take $C = 1$, then G has the weak Haagerup property *with constant* 1.

A priori the weak Haagerup property is even less tangible than weak amenability, but the point is that with the weak Haagerup property with constant 1, we can assume that the net in question is a semigroup of the form $(e^{-t\varphi})_{t>0}$, as the following holds.

Theorem 1.2. *For a countable, discrete group G, the following are equivalent.*

(1) *There is a sequence (φ_n) of functions vanishing at infinity such that $\varphi_n \to 1$ pointwise and $\|\varphi_n\|_{B_2} \leqslant 1$ for all n.*
(2) *There is $\varphi : G \to \mathbb{R}$ such that φ is proper and $\|e^{-t\varphi}\|_{B_2} \leqslant 1$ for every $t > 0$.*

The proof of the above is reminiscent of the proof concerning the equivalent formulations of the Haagerup property (see Theorem 2.1.1 in the book [5]). We provide a proof in Section 2 (see Theorem 3.1).

Clearly, weak amenability with constant 1 implies the weak Haagerup property with constant 1, and the converse is false by the example $\mathbb{Z}/2 \wr \mathbb{F}_2$ from before. It is also obvious that the Haagerup property implies the weak Haagerup property with constant 1. It is not clear, however, if they are in fact equivalent.

In the light of the previous theorem we consider the following problem.

Problem 1.3. Let G be a countable, discrete group, and let $\varphi : G \to \mathbb{R}$ be a symmetric function satisfying $\|e^{-t\varphi}\|_{B_2} \leqslant 1$ for all $t > 0$. Does there exist a conditionally negative definite function ψ on G such that $\varphi \leqslant \psi$?

Note that φ is proper if and only if each $e^{-t\varphi}$ vanishes at infinity. A positive solution to the problem would prove that the Haagerup property is equivalent to the weak Haagerup property with constant 1. So in particular, a solution to Problem 1.3 would prove that weak amenability with constant 1 implies the Haagerup property.

We will prove the following theorem.

Theorem 1.4. *Let G be a countable, discrete group with a symmetric function $\varphi : G \to \mathbb{R}$. Then $\|e^{-t\varphi}\|_{B_2} \leqslant 1$ for every $t > 0$ if and only if φ splits as*

$$\varphi\big(y^{-1}x\big) = \psi(x, y) + \theta(x, y) + \theta(e, e) \quad (x, y \in G),$$

where

- *ψ is a conditionally negative definite kernel on G vanishing on the diagonal,*
- *and θ is a bounded, positive definite kernel on G.*

The downside of the above theorem is that the functions ψ and θ are defined on $G \times G$ instead of simply G. A natural question to ask is in which situations we may strengthen Theorem 1.4 to produce functions ψ and θ defined on the group G itself. It is not so hard to prove that this happens if G is amenable. Moreover, this actually characterizes amenability. We thus have following theorem.

Theorem 1.5. *Let G be a countable, discrete group. Then G is amenable if and only if the following condition holds. Whenever $\varphi : G \to \mathbb{R}$ is a symmetric function such that $\|e^{-t\varphi}\|_{B_2} \leqslant 1$ for every $t > 0$, then φ splits as*

$$\varphi(x) = \psi(x) + \|\xi\|^2 + \big\langle \pi(x)\xi, \xi\big\rangle \quad (x \in G)$$

where

- ψ *is a conditionally negative definite function on* G *with* $\psi(e) = 0$,
- π *is an orthogonal representation of* G *on some real Hilbert space* H,
- *and* ξ *is a vector in* H.

Note that the function $x \mapsto \langle \pi(x)\xi, \xi \rangle$ is positive definite, and every positive definite function has this form.

We solve Problem 1.3 in the special case where G is a free group and the function φ is radial. The result is the following theorem, which generalizes Corollary 5.5 from [7].

Theorem 1.6. *Let* $\mathbb{F}_n$ *be the free group on* n *generators* $(2 \leqslant n \leqslant \infty)$, *and let* $\varphi : \mathbb{F}_n \to \mathbb{R}$ *be a radial function, i.e.,* $\varphi(x)$ *depends only on the word length* $|x|$. *If* $\|e^{-t\varphi}\|_{B_2} \leqslant 1$ *for every* $t > 0$, *then there are constants* $a, b \geqslant 0$ *such that*

$$\varphi(x) \leqslant b + a|x| \quad \textit{for all } x \in \mathbb{F}_n.$$

The paper is organized as follows. In Section 2 we introduce many of the relevant notions needed in the rest of the paper. Section 3 contains the proof of Theorem 1.2, and Section 4 contains the proof of Theorem 1.4. Section 5 considers the case of amenable groups. Here we prove Theorem 1.5.

The proof of Theorem 1.6 concerning $\mathbb{F}_n$ takes up the second half of the paper. The proof is divided into two cases depending on whether n is finite or infinite. In Section 6 we deal with the infinite case, and Section 7 contains the finite case. The strategy of the proof is to compare the Herz–Schur norm of $e^{-t\varphi}$ with the norm of certain functionals on the Toeplitz algebra. This is accomplished in Propositions 6.12 and 6.13. It turns out that a certain norm bound on the functionals produces a splitting of our function φ into a positive definite and a conditionally negative definite part (Theorem 6.9). Characterizing the positive and conditionally negative parts (Corollary 6.4 and Proposition 6.6) then leads to Theorem 1.6 in the case of $\mathbb{F}_\infty$.

When $n < \infty$, Theorem 1.6 is deduced in basically the same way as the case $n = \infty$, but the details are more complicated. The transformations introduced in Section 7.1 allow us to reduce many of the arguments for $\mathbb{F}_n$ with n finite to the case of $\mathbb{F}_\infty$.

2. Preliminaries

Let X be a non-empty set. A *kernel* on X is a function $k : X \times X \to \mathbb{C}$. The kernel is called *symmetric* if $k(x, y) = k(y, x)$ for all $x, y \in X$, and *hermitian* if $k(y, x) = \overline{k(x, y)}$.

The kernel k is said to be *positive definite*, if

$$\sum_{i,j=0}^{n} c_i \overline{c_j} k(x_i, x_j) \geqslant 0$$

for all $n \in \mathbb{N}$, $x_1, \dots, x_n \in X$ and $c_1, \dots, c_n \in \mathbb{C}$. It is called *conditionally negative definite* if it is hermitian and

$$\sum_{i,j=0}^{n} c_i \overline{c_j} k(x_i, x_j) \leqslant 0$$

for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$ and $c_1, \ldots, c_n \in \mathbb{C}$ such that $\sum_{i=0}^{n} c_i = 0$.

Recall Schoenberg's Theorem which asserts that k is conditionally negative definite if and only if e^{-tk} is positive definite for all $t > 0$.

Let H be a Hilbert space, and let $a : X \to H$ be a map. Then the kernel $\varphi : X \times X \to \mathbb{C}$ defined by

$$\varphi(x, y) = \langle a(x), a(y) \rangle$$

is positive definite. Conversely, every positive definite kernel is of this form for some suitable Hilbert space H and map a. On the other hand, the kernel $\psi : X \times X \to \mathbb{C}$ defined by

$$\psi(x, y) = \| a(x) - a(y) \|^2$$

is conditionally negative definite, and every real-valued, conditionally negative definite kernel that vanishes on the diagonal $\{(x, x) \mid x \in X\}$ is of this form.

It is well-known that the set of positive definite kernels on X is closed under pointwise products and pointwise convergence. Also, the set of conditionally negative definite kernels is closed under adding constants and under pointwise convergence. We refer to [1, Chapter 3] for details.

A kernel $k : X \times X \to \mathbb{C}$ is called a *Schur multiplier* if for every operator $A = [a_{xy}]_{x,y \in X} \in B(\ell^2(X))$ the matrix $[k(x, y)a_{xy}]_{x,y \in X}$ represents an operator in $B(\ell^2(X))$, denoted $m_k(A)$. If k is a Schur multiplier, it is a consequence of the closed graph theorem that m_k defines a *bounded* operator on $B(\ell^2(X))$. We define the *Schur norm* $\|k\|_S$ to be $\|m_k\|$. The following characterization of Schur multipliers is well-known (see [4, Appendix D]).

Proposition 2.1. *Let $k : X \times X \to \mathbb{C}$ be a kernel, and let $C \geqslant 0$ be given. The following are equivalent.*

(1) *The kernel k is a Schur multiplier with $\|k\|_S \leqslant C$.*
(2) *There exist a Hilbert space H and two bounded maps $a, b : X \to H$ such that*

$$k(x, y) = \langle a(x), b(y) \rangle, \quad \text{for all } x, y \in X,$$

and $\|a(x)\| \|b(y)\| \leqslant C$ for all $x, y \in X$.

Let G be a discrete group, and let $\varphi : G \to \mathbb{C}$ be a function. Let $\hat{\varphi} : G \times G \to \mathbb{C}$ be defined by $\hat{\varphi}(x, y) = \varphi(y^{-1}x)$. All the terminology introduced above is inherited to functions $\varphi : G \to \mathbb{C}$ by saying, for instance, that φ is positive definite if the kernel $\hat{\varphi}$ is positive definite. The only exception is that a function $\varphi : G \to \mathbb{C}$ is called a *Herz–Schur multiplier* if $\hat{\varphi}$ is a Schur multiplier.

All positive definite functions on G are of the form $\varphi(x) = \langle \pi(x)\xi, \xi \rangle$ for a unitary representation π on some Hilbert space H and a vector $\xi \in H$. If φ is real, then π may be taken as an orthogonal representation on a real Hilbert space.

The set of Herz–Schur multipliers on G is denoted $B_2(G)$. It is a Banach space, in fact a Banach algebra, when equipped with the norm $\|\varphi\|_{B_2} = \|\hat{\varphi}\|_S = \|m_{\hat{\varphi}}\|$. The unit ball $B_2(G)_1$

is closed in the topology of pointwise convergence. It was proved in [3] that the space of Herz–Schur multipliers coincides isometrically with the space of completely bounded Fourier multipliers.

Another useful algebra of functions on G is the Fourier–Stieltjes algebra, denoted $B(G)$. It may be defined as the linear span of the positive definite functions on G. It is isometrically isomorphic to the dual of the full group C^*-algebra of G, i.e., $B(G) \simeq C^*(G)^*$. Since any positive definite function is a Herz–Schur multiplier, it follows that $B(G) \subseteq B_2(G)$. Equality holds, if and only if G is amenable (see [2] or Proposition 5.6 below).

Given C^*-algebras A and B and a linear map $\varphi : A \to B$ we denote by $\varphi^{(n)}$ the map $\varphi^{(n)} = \varphi \otimes \mathrm{id}_n : A \otimes M_n(\mathbb{C}) \to B \otimes M_n(\mathbb{C})$, where $\mathrm{id}_n : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is the identity. The map φ is called *completely bounded*, if

$$\|\varphi\|_{\mathrm{cb}} = \sup_n \big\|\varphi^{(n)}\big\| < \infty.$$

We say that φ is *completely positive*, if each $\varphi^{(n)}$ is positive between the C^*-algebras $M_n(A)$ and $M_n(B)$. We abbreviate unital, completely positive as u.c.p. It is well-known that bounded functionals $\varphi : A \to \mathbb{C}$ are completely bounded with $\|\varphi\|_{\mathrm{cb}} = \|\varphi\|$. States on C^*-algebras as well as $*$-homomorphism are completely positive.

3. Characterization of the weak Haagerup property with constant 1

The following theorem gives the promised alternative characterization of the weak Haagerup property with constant 1.

Theorem 3.1. *Let G be a countable, discrete group. The following are equivalent.*

(1) *There is a sequence (φ_n) of functions vanishing at infinity such that $\varphi_n \to 1$ pointwise and $\|\varphi_n\|_{B_2} \leqslant 1$ for all n.*
(2) *There is $\varphi : G \to \mathbb{R}$ such that φ is proper and $\|e^{-t\varphi}\|_{B_2} \leqslant 1$ for every $t > 0$.*

Proof. (2) $\Longrightarrow$ (1): This is trivial: put $\varphi_n = e^{-\varphi/n}$.

(1) $\Longrightarrow$ (2): Choose an increasing, unbounded sequence (α_n) of positive real numbers and a decreasing sequence (ε_n) tending to zero such that $\sum_n \alpha_n \varepsilon_n$ converges. We enumerate the elements in G as $G = \{g_1, g_2, \ldots\}$. For each n we may choose a function φ_n in $C_0(G)$ with $\|\varphi_n\|_{B_2} \leqslant 1$ such that

$$\max\big\{\big|1 - \varphi_n(g_i)\big| \;\big|\; i = 1, \ldots, n\big\} \leqslant \varepsilon_n.$$

We may replace φ_n by $|\varphi_n|^2$ to ensure that $0 \leqslant \varphi_n \leqslant 1$. Now, let $\varphi : G \to \mathbb{R}_+$ be given by

$$\varphi(g) = \sum_{n=1}^{\infty} \alpha_n \big(1 - \varphi_n(g)\big).$$

Note that this sum converges. We claim that φ is also proper. Let $R > 0$ be given, and choose k such that $\alpha_k \geqslant 2R$. Since $\varphi_k \in C_0(G)$, there is a finite set $F \subseteq G$ such that $|\varphi_k(g)| < 1/2$

whenever $g \in G \setminus F$. Now if $\varphi(g) \leqslant R$, then $\varphi(g) \leqslant \alpha_k/2$, and in particular $\alpha_k(1-\varphi_k(g)) \leqslant \alpha_k/2$, which implies that $1-\varphi_k(g) \leqslant 1/2$. Hence, we have argued that

$$\{g \in G \mid \varphi(g) \leqslant R\} \subseteq \{g \in G \mid 1-\varphi_k(g) \leqslant 1/2\} \subseteq F.$$

This proves that φ is proper.

Now let $t > 0$ be fixed. We need to show that $\|e^{-t\varphi}\|_{B_2} \leqslant 1$. Define

$$\psi_i = \sum_{n=1}^{i} \alpha_n(1-\varphi_n).$$

Since ψ_i converges pointwise to φ, it will suffice to prove that $\|e^{-t\psi_i}\|_{B_2} \leqslant 1$ eventually (as $i \to \infty$), because the unit ball of $B_2(G)$ is closed under pointwise limits. Observe that

$$e^{-t\psi_i} = \prod_{n=1}^{i} e^{-t\alpha_n(1-\varphi_n)},$$

and so it suffices to show that $e^{-t\alpha_n(1-\varphi_n)}$ belongs to the unit ball of $B_2(G)$ for each n. And this is clear, since

$$\left\|e^{-t\alpha_n(1-\varphi_n)}\right\|_{B_2} = e^{-t\alpha_n}\left\|e^{t\alpha_n\varphi_n}\right\|_{B_2} \leqslant e^{-t\alpha_n}e^{t\alpha_n\|\varphi_n\|_{B_2}} \leqslant 1. \qquad \square$$

4. Splitting a semigroup generator into positive and negative parts

The key idea in the proof of Theorem 1.4 is that a Schur multiplier is a corner in a positive definite matrix (Lemma 4.2). Together with an ultraproduct argument this will give the proof of Theorem 1.4.

We consider the following as well-known.

Lemma 4.1. *Let $\varphi : X \times X \to \mathbb{C}$ be a kernel. Then*

$$\|\varphi\|_S = \sup\{\|\varphi|_{F\times F}\|_S \mid F \subseteq X \text{ finite}\}.$$

The following follows from Proposition 2.1.

Lemma 4.2. *Let $a \in M_n(\mathbb{C})$. The following are equivalent.*

(1) $\|a\|_S \leqslant 1$.
(2) *There exist $b, c \in M_n(\mathbb{C})_+$ with $b_{ii} \leqslant 1$, $c_{ii} \leqslant 1$, $i = 1, \dots, n$ such that*

$$\begin{pmatrix} b & a \\ a^* & c \end{pmatrix} \geqslant 0.$$

Proof. Let $X = \{1, \dots, n\}$ and consider $a \in M_n(\mathbb{C})$ as a kernel $a : X \times X \to \mathbb{C}$.

Suppose first $\|a\|_S \leqslant 1$. By Proposition 2.1 there is a Hilbert space H and two maps $p, q : X \to H$ such that $a_{ij} = \langle p(i), q(j)\rangle$ and $\|p\|_\infty \|q\|_\infty \leqslant 1$ for all i, j. After replacing $p(i)$ and $q(j)$ by

$$p'(i) = \frac{\sqrt{\|p\|_\infty \|q\|_\infty}}{\|p\|_\infty} p(i), \qquad q'(j) = \frac{\sqrt{\|p\|_\infty \|q\|_\infty}}{\|q\|_\infty} q(j)$$

respectively, we may assume that $\|p\|_\infty \leqslant 1$ and $\|q\|_\infty \leqslant 1$. Let

$$b_{ij} = \big\langle p(i), p(j)\big\rangle, \qquad c_{ij} = \big\langle q(i), q(j)\big\rangle.$$

Then b and c are positive matrices with diagonal below 1 and the matrix

$$M = \begin{pmatrix} b & a \\ a^* & c \end{pmatrix} = \big(\big\langle r(i), r(j)\big\rangle\big)_{i,j=1}^{2n}$$

is positive where

$$r(i) = \begin{cases} p(i), & 1 \leqslant i \leqslant n, \\ q(i-n), & n < i \leqslant 2n. \end{cases}$$

Conversely, suppose that

$$M = \begin{pmatrix} b & a \\ a^* & c \end{pmatrix} \geqslant 0,$$

for some $b, c \in M_n(\mathbb{C})_+$ with $b_{ii} \leqslant 1$, $c_{ii} \leqslant 1$. Then there is a Hilbert space H and map $r : \{1, \dots, 2n\} \to H$ such that $M_{ij} = \langle r(i), r(j)\rangle$ for $i, j = 1, \dots, 2n$. Put $p(i) = r(i)$ and $q(i) = r(i+n)$, $i = 1, \dots, n$. Then $a_{ij} = \langle p(i), q(j)\rangle$ and

$$\big\|p(i)\big\|^2 = b_{ii} \leqslant 1, \qquad \big\|q(j)\big\|^2 = c_{jj} \leqslant 1.$$

It now follows from Proposition 2.1 that $\|a\|_S \leqslant 1$. □

Theorem 1.4 is an immediate consequence of the following.

Proposition 4.3. *Let G be a countable, discrete group with a symmetric function $\varphi : G \to \mathbb{R}$. The following are equivalent.*

(1) $\|e^{-t\varphi}\|_{B_2} \leqslant 1$ *for every* $t > 0$.
(2) *There exist a real Hilbert space H and maps $R, S : G \to H$ such that*

$$\varphi\big(y^{-1}x\big) = \big\|R(x) - R(y)\big\|^2 + \big\|S(x) + S(y)\big\|^2 \quad \text{for all } x, y \in G.$$

In particular, $\|S(x)\|^2$ is constant.

Proof. We will need to work with two disjoint copies of G, so let $\overline{G}$ denote another copy of G. We denote the elements of $\overline{G}$ by $\overline{g}$, when $g \in G$.

(2) $\Longrightarrow$ (1): It suffices to prove the case when $t = 1$. After replacing the maps R and S by the maps $R', S' : G \to H \oplus H$ given by $R'(x) = (R(x), 0)$ and $S'(x) = (0, S(x))$, we may assume that R and S have orthogonal ranges. Then

$$\varphi\big(y^{-1}x\big) = \big\| R(x) + S(x) - \big(R(y) - S(y)\big)\big\|^2.$$

Let $P = R + S$ and $Q = R - S$. Define a map $T : G \sqcup \overline{G} \to H$ by

$$T(x) = \begin{cases} P(x), & x \in G, \\ Q(x), & x \in \overline{G}. \end{cases}$$

Then the function $\rho(x, y) = \|T(x) - T(y)\|^2$ is a conditionally negative definite kernel on the set $G \sqcup \overline{G}$, and by Schoenberg's Theorem the function $e^{-\rho}$ is positive definite, and we notice that $e^{-\rho}$ takes the value 1 on the diagonal.

Given any finite subset $F = \{g_1, \dots, g_n\}$ of G we let $\overline{F}$ denote its copy inside $\overline{G}$. We see that the $2n \times 2n$ matrix $(e^{-\rho(x,y)})_{x,y \in F \sqcup \overline{F}}$ in $B(\ell^2(F \sqcup \overline{F}))$ is

$$A = \left(\begin{array}{c|c} e^{-\|P(g_i)-P(g_j)\|^2} & e^{-\|P(g_i)-Q(g_j)\|^2} \\ \hline e^{-\|Q(g_i)-P(g_j)\|^2} & e^{-\|Q(g_i)-Q(g_j)\|^2} \end{array}\right).$$

Since $e^{-\rho}$ is positive definite, A is positive. Now, Lemma 4.2 implies that the upper right block of A has Schur norm at most 1. And this precisely means that $\|e^{-\varphi}|_F\|_S \leqslant 1$. An application of Lemma 4.1 now shows that $\|e^{-\varphi}\|_S \leqslant 1$.

(1) $\Longrightarrow$ (2): We list the elements of G as $G = \{g_1, g_2, \dots\}$ and we let $G_n = \{g_1, \dots, g_n\}$ when $n \in \mathbb{N}$. Since $\|e^{-\varphi/n}\|_{B_2} \leqslant 1$ by assumption, we invoke Lemma 4.2 to get matrices $b_n, c_n \in M_n(\mathbb{C})_+$ with diagonal entries at most one, and so that

$$A_n = \left(\begin{array}{c|c} b_n & e^{-\varphi/n} \\ \hline e^{-\varphi/n} & c_n \end{array}\right) \geqslant 0.$$

Here $e^{-\varphi/n}$ denotes the $n \times n$ matrix whose (i, j) entry is $e^{-\varphi(g_j^{-1}g_i)/n}$. After adding the appropriate diagonal matrix we may assume that the diagonal entries of b_n and c_n are 1, and A_n is still positive.

Let $k_n : (G_n \sqcup \overline{G}_n)^2 \to \mathbb{C}$ be the kernel that represents A_n in the sense that

$$k_n(g_i, g_j) = (b_n)_{i,j}, \qquad k_n(\overline{g}_i, \overline{g}_j) = (c_n)_{i,j},$$
$$k_n(g_i, \overline{g}_j) = e^{-\varphi(g_j^{-1}g_i)}, \qquad k_n(\overline{g}_i, g_j) = e^{-\varphi(g_j^{-1}g_i)}.$$

Since A_n is positive, k_n is a positive definite kernel. We define k_n to be zero outside $(G_n \sqcup \overline{G}_n)^2$, which gives us a positive definite kernel on $G \sqcup \overline{G}$. Then the function $n(1 - k_n)$ is a conditionally negative definite kernel with zero in the diagonal, and hence there is a map $T_n : G \sqcup \overline{G} \to H_n$ such that

$$\left\|T_n(x) - T_n(y)\right\|^2 = n\bigl(1 - k_n(x, y)\bigr), \quad x, y \in G \sqcup \overline{G}$$

for some real Hilbert space H_n. We may assume that $T_n(\bar{e}) = 0$.

Now, as we let n vary over $\mathbb{N}$ we obtain a sequence of maps $(T_n)_{n\geqslant 1}$. Because $\lim_{t\to 0}(1 - e^{-ta})/t = a$, we see that for $(x, \bar{y}) \in G_N \times \overline{G}_N$ and $n \geqslant N$

$$\left\|T_n(x) - T_n(\bar{y})\right\|^2 = n\bigl(1 - k_n(x, \bar{y})\bigr) = n\bigl(1 - e^{-\varphi(y^{-1}x)/n}\bigr) \to \varphi\bigl(y^{-1}x\bigr) \quad \text{as } n \to \infty.$$

Since $T_n(\bar{e}) = 0$, this shows in particular that $(\|T_n(x)\|)_{n\geqslant 1}$ is a bounded sequence for each $x \in G$ and hence also for each $x \in \overline{G}$.

Consider the ultraproduct of the Hilbert spaces H_n with respect to some free ultrafilter ω. We denote this space by H. Let $T(x)$ denote the vector corresponding to the sequence $(T_n(x))_{n\geqslant 1}$, i.e., the equivalence class of that sequence. Then

$$\varphi\bigl(y^{-1}x\bigr) = \left\|T(x) - T(\bar{y})\right\|^2 \quad \text{for every } (x, \bar{y}) \in G \times \overline{G}. \tag{4.1}$$

Let $P = T|_G$ and let Q be defined on G by $Q(x) = T(\bar{x})$. We think of Q as the restriction of T to $\overline{G}$ but defined on G. Then Eq. (4.1) translates to

$$\varphi\bigl(y^{-1}x\bigr) = \left\|P(x) - Q(y)\right\|^2 \quad \text{for every } x, y \in G.$$

Let $R = (P + Q)/2$ and $S = (P - Q)/2$. The rest of the proof is simply to apply the parallelogram law. We have

$$\frac{1}{2}\bigl(\left\|P(x) - Q(y)\right\|^2 + \left\|P(y) - Q(x)\right\|^2\bigr) = \left\|S(x) + S(y)\right\|^2 + \left\|R(x) - R(y)\right\|^2.$$

Since φ is symmetric, the left-hand side equals $\varphi(y^{-1}x)$, and the proof is complete. □

5. The amenable case

In this section we prove Theorem 1.5. Theorem 5.3 and Theorem 5.8 combine to give Theorem 1.5.

We will need a few characterizations of amenability. The following theorem is well-known (for a proof, see [4, Theorem 2.6.8]).

Theorem 5.1. *Let G be a discrete group. The following are equivalent.*

(1) *G is amenable, i.e., there is a left-invariant, finitely additive probability measure defined on all subsets of G.*
(2) *There is a net of finitely supported, positive definite functions on G converging pointwise to 1.*
(3) *For any finite, symmetric subset $E \subseteq G$ we have $\|\lambda(1_E)\| = |E|$. Here λ denotes the left regular representation, and 1_E denotes the characteristic function of the subset E.*

Corollary 5.2. *If G is discrete and non-amenable, then for each $\varepsilon > 0$ there exists a positive, finitely supported, symmetric function $g \in C_c(G)$ such that*

$$\|\lambda(g)\| < \varepsilon\|g\|_1,$$

where λ denotes the left regular representation.

Proof. If G is non-amenable, there is a finite, symmetric set $S \subseteq G$ such that $\|\lambda(1_S)\| < |S|$. Let $g = 1_S * \cdots * 1_S$ be the n-fold convolution of 1_S with itself, where n is to be determined later. Then g is positive, finitely supported and symmetric. Observe that

$$\|g\|_1 = |S|^n.$$

Now, given any $0 < \varepsilon < 1$, choose n so large that $\frac{\|\lambda(1_S)\|}{|S|} < \sqrt[n]{\varepsilon} < 1$. Then

$$\|\lambda(g)\| \leqslant \|\lambda(1_S)\|^n < \varepsilon|S|^n = \varepsilon\|g\|_1. \qquad \square$$

The following theorem proves one direction in Theorem 1.5.

Theorem 5.3. *Let G be a countable, discrete amenable group with a symmetric function $\varphi : G \to \mathbb{R}$. If $\|e^{-t\varphi}\|_{B_2} \leqslant 1$ for every $t > 0$, then φ splits as*

$$\varphi(x) = \psi(x) + \|\xi\|^2 + \langle \pi(x)\xi, \xi\rangle \quad (x \in G),$$

where

- *ψ is a conditionally negative definite function on G with $\psi(e) = 0$,*
- *π is an orthogonal representation of G on some real Hilbert space H,*
- *and ξ is a vector in H.*

Proof. The idea of the proof is to use the characterization given in Proposition 4.3 and then average the two parts of φ by using an invariant mean on G.

Suppose we are given a function φ as in the statement of the proposition. By Proposition 4.3 we may write φ in the form

$$\varphi(y^{-1}x) = \|R(x) - R(y)\|^2 + \|S(x) + S(y)\|^2 \quad \text{for all } x, y \in G,$$

where R, S are maps from G with values in some real Hilbert space H. We define kernels φ_1 and φ_2 on G by

$$\varphi_1(x, y) = \|R(x) - R(y)\|^2, \qquad \varphi_2(x, y) = \|S(x) + S(y)\|^2 \quad \text{for all } x, y \in G.$$

Then $\varphi(y^{-1}x) = \varphi_1(x, y) + \varphi_2(x, y)$. Note that φ_2 is a bounded function, since $\|2S(x)\|^2 = \varphi(e)$ for every $x \in G$. In general, φ_1 is not bounded, but it is bounded on the diagonals, i.e., for each

$x, y \in G$ the function $z \mapsto \varphi_1(zx, zy)$ is bounded. To see this, simply observe that

$$\varphi_1(zx, zy) = \varphi\big((zy)^{-1}(zx)\big) - \varphi_2(zx, zy) = \varphi\big(y^{-1}x\big) - \varphi_2(zx, zy).$$

Since φ_2 is bounded, it follows that φ_1 is bounded on diagonals.

As we assumed G to be amenable, there is a left-invariant mean μ on G. Let

$$\chi_i(x, y) = \int_G \varphi_i(zx, zy)\, d\mu(z), \quad x, y \in G,\ i = 1, 2.$$

The left-invariance of μ implies that $\chi_i(wx, wy) = \chi_i(x, y)$ for every $x, y, w \in G$, so χ_i induces a function $\overline{\varphi}_i$ defined on G by

$$\overline{\varphi}_i\big(y^{-1}x\big) = \chi_i(x, y).$$

An easy computation will show that $\varphi = \overline{\varphi}_1 + \overline{\varphi}_2$.

Since φ_1 is a conditionally negative definite kernel on G, it follows that χ_1 is conditionally negative definite. So $\overline{\varphi}_1$ is conditionally negative definite. The same argument shows that $\overline{\varphi}_2$ is positive definite, because φ_2 is. More precisely we have

$$\overline{\varphi}_2\big(y^{-1}x\big) = \int_G \big\|S(zx) + S(zy)\big\|^2\, d\mu(z) = \frac{\varphi(e)}{2} + 2\int_G \big\langle S(zx), S(zy)\big\rangle\, d\mu(z),$$

where each function $(x, y) \mapsto \langle S(zx), S(zy)\rangle$ is a positive definite kernel. So the function on G given by

$$y^{-1}x \mapsto \int_G \big\langle S(zx), S(zy)\big\rangle\, d\mu(z)$$

is positive definite, and so it has the form

$$g \mapsto \big\langle \pi(g)\xi', \xi'\big\rangle$$

for some orthogonal representation π. Since $\overline{\varphi}_1(e) = 0$, we must have

$$\varphi(e) = \overline{\varphi}_2(e) = \frac{\varphi(e)}{2} + 2\big\|\xi'\big\|^2,$$

and so $\frac{\varphi(e)}{2} = 2\|\xi'\|^2$. The proof is now complete if we let $\psi = \overline{\varphi}_1$ and $\xi = \sqrt{2}\xi'$. □

We now turn to prove that the amenability assumption is essential in the theorem above. This will be accomplished in Theorem 5.8.

In [2] Bożejko proved that a countable, discrete group G is amenable if and only if its Fourier–Stieltjes algebra $B(G)$ (the linear span of positive definite functions) coincides with the Herz–Schur multiplier algebra $B_2(G)$. In Proposition 5.6 we will strengthen this result slightly to fit our needs. Our proof of Proposition 5.6 is merely an adaption of Bożejko's proof.

In the following we will introduce the *Littlewood kernels* and *Littlewood functions*. Let X be a non-empty set. A bounded operator $T : \ell^1(X) \to \ell^2(X)$ is identified with its matrix $T = [T_{xy}]$ given by $T_{xy} = \langle T\delta_y, \delta_x\rangle$. We also identify the matrix with the corresponding kernel t on X given by $t(x, y) = T_{xy}$. Similarly, the Banach space adjoint $T^* : \ell^2(X)^* \to \ell^1(X)^*$ has matrix $T^*_{xy} = \overline{\langle T\delta_x, \delta_y\rangle}$ and may be identified with a kernel on X.

We shall identify $\ell^1(X)^* = \ell^\infty(X)$ and $\ell^2(X)^* = \ell^2(X)$. It is known that every bounded operator $\ell^2(X) \to \ell^\infty(X)$ arises as the adjoint of a (unique) bounded operator $\ell^1(X) \to \ell^2(X)$. We note that a kernel $b : X \times X \to \mathbb{C}$ is the matrix of a bounded operator $\ell^1(X) \to \ell^2(X)$ if and only if

$$\|b\|^2_{\ell^1 \to \ell^2} = \sup_{y \in X} \sum_{x \in X} |b_{xy}|^2$$

is finite. In the same way, $c : X \times X \to \mathbb{C}$ is the matrix of a bounded operator $\ell^2(X) \to \ell^\infty(X)$ if and only if

$$\|c\|^2_{\ell^2 \to \ell^\infty} = \sup_{x \in X} \sum_{y \in X} |c_{xy}|^2$$

is finite.

We define the *Littlewood kernels* on X to be

$$t_2(X) = \bigl\{b + c \bigm| b \in B\bigl(\ell^1(X), \ell^2(X)\bigr),\ c \in B\bigl(\ell^2(X), \ell^\infty(X)\bigr)\bigr\}.$$

The space $t_2(X)$ is naturally equipped with the (complete) norm

$$\|a\|_L = \inf\bigl\{\max\bigl(\|b\|_{\ell^1 \to \ell^2}, \|c\|_{\ell^2 \to \ell^\infty}\bigr) \bigm| a = b + c\bigr\}.$$

The following characterization of Littlewood kernels is due to Varopoulos and is a special case of [16, Lemma 5.1]. For completeness, we include a proof of our special case.

Lemma 5.4. *Let X be a countable set, and let $a : X \times X \to \mathbb{C}$ be a kernel. Then a is a Littlewood kernel if and only if the norm*

$$\|a\|_{t_2} = \sup\left\{\frac{1}{|F_1|} \sum_{\substack{i \in F_1 \\ j \in F_2}} |a_{ij}|^2 \;\middle|\; F_1, F_2 \subseteq X \text{ finite},\ |F_1| = |F_2|\right\}^{1/2}$$

is finite. The norms $\|\ \|_{t_2}$ and $\|\ \|_L$ are equivalent.

It is implicit in the statement that $\|\ \|_{t_2}$ in fact defines a norm on $t_2(X)$. This is not hard to check, and moreover $t_2(X)$ is a Banach space with this norm. The lemma is also true when X is uncountable, but we have no need for this generality.

Proof of Lemma 5.4. Suppose first that a is a Littlewood kernel, and write $a = b + c$ as in the definition. Given finite subsets $F_1, F_2 \subseteq X$ of the same size we have

$$\frac{1}{|F_1|}\sum_{\substack{i\in F_1\\ j\in F_2}}|a_{ij}|^2\leqslant\frac{2}{|F_1|}\sum_{\substack{i\in F_1\\ j\in F_2}}|b_{ij}|^2+|c_{ij}|^2\leqslant 4\max\bigl(\|b\|^2,\|c\|^2\bigr),$$

and so

$$\|a\|_{t_2}\leqslant 2\max\bigl(\|b\|,\|c\|\bigr)<\infty.$$

We have actually shown that $\|a\|_{t_2}\leqslant 2\|a\|_L$.

Suppose conversely that a is a kernel such that $C=\|a\|_{t_2}$ is finite. We will show that a is a Littlewood kernel of the form $b+c$, where b and c have disjoint supports, and $\|b\|\leqslant C$ and $\|c\|\leqslant C$. We finish the proof of the lemma first in the case where X is finite and proceed by induction on $|X|$. The case $|X|=1$ is trivial. Assume then $n=|X|\geqslant 2$ and write

$$a=\begin{pmatrix}a_{11}&\cdots&a_{1n}\\ \vdots&\ddots&\vdots\\ a_{n1}&\cdots&a_{nn}\end{pmatrix}.$$

Choose an index i such that $\sum_j|a_{ij}|^2$ is as small as possible. In particular, our assumption implies that $\sum_j|a_{ij}|^2\leqslant C$. Similarly, choose an index j such that $\sum_i|a_{ij}|^2$ is as small as possible. Consider then the submatrix a' of a with i'th row and j'th column removed. To simplify the notation we assume that $i=j=1$. Then

$$a'=\begin{pmatrix}a_{22}&\cdots&a_{2n}\\ \vdots&\ddots&\vdots\\ a_{n2}&\cdots&a_{nn}\end{pmatrix}.$$

By our induction hypothesis a' is a Littlewood kernel with $\|a'\|_L\leqslant C$, and so we may write $a'=b+c$, that is

$$\begin{pmatrix}a_{22}&\cdots&a_{2n}\\ \vdots&\ddots&\vdots\\ a_{n2}&\cdots&a_{nn}\end{pmatrix}=\begin{pmatrix}b_{22}&\cdots&b_{2n}\\ \vdots&\ddots&\vdots\\ b_{n2}&\cdots&b_{nn}\end{pmatrix}+\begin{pmatrix}c_{22}&\cdots&c_{2n}\\ \vdots&\ddots&\vdots\\ c_{n2}&\cdots&c_{nn}\end{pmatrix},$$

where b and c have disjoint supports and $\max(\|b\|,\|c\|)\leqslant C$. We then obtain the desired splitting for a by putting the removed rows back (we do not care whether $a_{ij}=a_{11}$ is put in the first or second matrix, so simply put it in the first),

$$\left(\begin{array}{c|ccc}a_{11}&a_{12}&\cdots&a_{1n}\\ \hline a_{21}&a_{22}&\cdots&a_{2n}\\ \vdots&\vdots&\ddots&\vdots\\ a_{n1}&a_{n2}&\cdots&a_{nn}\end{array}\right)=\left(\begin{array}{c|ccc}a_{11}&a_{12}&\cdots&a_{1n}\\ \hline 0&b_{22}&\cdots&b_{2n}\\ \vdots&\vdots&\ddots&\vdots\\ 0&b_{n2}&\cdots&b_{nn}\end{array}\right)+\left(\begin{array}{c|ccc}0&0&\cdots&0\\ \hline a_{21}&c_{22}&\cdots&c_{2n}\\ \vdots&\vdots&\ddots&\vdots\\ a_{n1}&c_{n2}&\cdots&c_{nn}\end{array}\right).$$

This completes the induction step.

We now turn to the general case, where X is countably infinite. We may assume $X=\mathbb{N}$. Let ω be a free ultrafilter on $\mathbb{N}$. For each $k\in\mathbb{N}$ we let $a^{(k)}$ be the restriction of a to $\{1,\dots,k\}^2$, and choose a splitting $a^{(k)} = b^{(k)} + c^{(k)}$. For each $i, j \in \mathbb{N}$ and $k \geqslant i, j$ we have $|b_{ij}^{(k)}|^2 \leqslant C$, so each sequence $(b_{ij}^{(k)})_k$ is bounded. Similarly, $(c_{ij}^{(k)})_k$ is bounded. Let

$$b_{ij} = \lim_{k\to\omega} b_{ij}^{(k)}, \qquad c_{ij} = \lim_{k\to\omega} c_{ij}^{(k)}.$$

Since $a_{ij} = b_{ij}^{(k)} + c_{ij}^{(k)}$ for every $k \geqslant i, j$, it follows that $a_{ij} = b_{ij} + c_{ij}$. Also, since $b_{ij}^{(k)} \in \{a_{ij}, 0\}$ for every k, we must have $b_{ij} \in \{a_{ij}, 0\}$. Similarly with c_{ij}. This shows that b and c have disjoint supports. The sum conditions

$$\sup_j \sum_i |b_{ij}|^2 \leqslant C, \qquad \sup_i \sum_j |c_{ij}|^2 \leqslant C$$

are also satisfied. In particular we have $\|a\|_L \leqslant \|a\|_{t_2}$. □

If $X = G$ is a group, and $a : G \to \mathbb{C}$ is a function, we say that a is a *Littlewood function*, if $\hat{a}(x, y) = a(y^{-1}x)$ is a Littlewood kernel. We denote the set of Littlewood functions on G by $T_2(G)$ and equip it with the norm $\|a\|_{T_2} = \|\hat{a}\|_{t_2}$. It is easy to see that $\|a\|_{T_2} \leqslant \|a\|_{\ell^2}$, so $\ell^2(G) \subseteq T_2(G)$.

Let $M(\ell^\infty(G), B_2(G)) = M(\ell^\infty, B_2)$ be the set of functions $a : G \to \mathbb{C}$ such that the pointwise product $a \cdot f$ is a Herz–Schur multiplier for every $f \in \ell^\infty(G)$. It is a Banach space when equipped with the norm

$$\|g\|_{M(\ell^\infty, B_2)} = \sup\bigl\{\|a \cdot f\|_{B_2} \bigm| \|f\|_\infty \leqslant 1\bigr\}.$$

Lemma 5.5. *The following inclusion holds.*

$$T_2(G) \subseteq M\bigl(\ell^\infty(G), B_2(G)\bigr).$$

Proof. Note first that $T_2(G) \subseteq B_2(G)$, since if $a \in T_2(G)$ is given, and $\hat{a} = b + c$ is a splitting as in the definition of Littlewood kernels, then

$$a\bigl(y^{-1}x\bigr) = \langle b\delta_y, \delta_x\rangle + \overline{\langle c\delta_x, \delta_y\rangle}.$$

Now use Proposition 2.1.

Secondly, it is easy to see that $t_2(X) \cdot \ell^\infty(X \times X) \subseteq t_2(X)$, and we conclude that

$$T_2(G) \cdot \ell^\infty(G) \subseteq T_2(G) \subseteq B_2(G). \qquad \square$$

In the proof of Proposition 5.6 we will need the notion of a cotype 2 Banach space. A Banach space X is of *cotype* 2 if there is a constant $C > 0$ such that for any finite subset $\{x_1, \ldots, x_n\}$ of X we have

$$C\left(\sum_{k=1}^n \|x_k\|^2\right)^{1/2} \leqslant \int_0^1 \left\|\sum_{k=1}^n r_k(t)x_k\right\| dt.$$

Here r_n are the Rademacher functions on [0, 1]. It is well-known that L^p-spaces are of cotype 2 when $1 \leqslant p \leqslant 2$. Also, the dual of a C^*-algebra is of cotype 2 (see [15]). (See also [13] for a simple proof of this fact.)

Whenever A is a set of functions $G \to \mathbb{C}$ defined on a group G, we denote by A_{sym} the symmetric functions in A, i.e., $A_{\text{sym}} = \{\varphi \in A \mid \varphi(x) = \varphi(x^{-1}) \text{ for all } x \in G\}$.

Proposition 5.6. *Let G be a discrete group. The following are equivalent.*

(1) *G is amenable.*
(2) $B_2(G) = B(G)$.
(3) $B_2(G)_{\text{sym}} = B(G)_{\text{sym}}$.

Proof. For the implication (1) $\Longrightarrow$ (2) we refer to Theorem 1 in [14]. The implication (2) $\Longrightarrow$ (3) is trivial. So we prove (3) $\Longrightarrow$ (1), and we do this by adapting Bożejko's proof of (2) $\Longrightarrow$ (1).

Since $B(G)$ may be identified with the dual of the full group C^*-algebra of G, it is of cotype 2. Being of cotype 2 obviously passes to (closed) subspaces, so $B(G)_{\text{sym}}$ is of cotype 2. By assumption $B_2(G)_{\text{sym}} = B(G)_{\text{sym}}$, and because the two spaces have equivalent norms, $B_2(G)_{\text{sym}}$ is also of cotype 2.

Now we show that

$$M\big(\ell^\infty(G), B_2(G)\big)_{\text{sym}} \subseteq \ell^2(G).$$

Suppose $g \in M(\ell^\infty(G), B_2(G))_{\text{sym}}$ and write g in the form

$$g = \sum_{n=1}^{\infty} a_n(\delta_{x_n} + \delta_{x_n^{-1}})$$

with no repetitions among the sets $\{x_n, x_n^{-1}\}_{n=1}^\infty$. For each $t \in [0, 1]$ and $N \in \mathbb{N}$ the function

$$g_{t,N} = \sum_{n=1}^{N} a_n r_n(t)(\delta_{x_n} + \delta_{x_n^{-1}})$$

lies in $B_2(G)_{\text{sym}}$ and $\|g_{t,N}\|_{B_2} \leqslant \|g\|_{M(\ell^\infty,B_2)}$. Using that $B_2(G)_{\text{sym}}$ has cotype 2 we get

$$C\left(\sum_{n=1}^{N} |a_n|^2\right)^{1/2} \leqslant \int_0^1 \left\| \sum_{n=1}^{N} r_n(t) a_n(\delta_{x_n} + \delta_{x_n^{-1}}) \right\|_{B_2} dt \leqslant \|g\|_{M(\ell^\infty,B_2)}$$

for any $N \in \mathbb{N}$, so $g \in \ell^2(G)$.

Now, consider the set $T^2(G)$ of Littlewood functions. As noted in Lemma 5.5,

$$T_2(G) \subseteq M\big(\ell^\infty(G), B_2(G)\big),$$

so it follows that $T_2(G)_{\text{sym}} \subseteq \ell^2(G)$. Conversely, the inclusion $\ell^2(G) \subseteq T_2(G)$ is trivial, so we must have $T_2(G)_{\text{sym}} = \ell^2(G)_{\text{sym}}$.

Let $\widetilde{f}(x) = \overline{f(x^{-1})}$. It is easy to check that

$$\langle \lambda(f)x, y\rangle = \langle f, y * \widetilde{x}\rangle \quad \text{for all } f, x, y \in \mathbb{C}[G].$$

Hence for any symmetric $f \in \mathbb{C}[G]$ we have

$$\begin{aligned}\|f\|_{T_2}^2 &= \sup_{|F_1|=|F_2|<\infty} \left\{ \frac{1}{|F_1|} \langle |f|^2, \chi_{F_1} * \widetilde{\chi}_{F_2} \rangle \right\} \\ &= \sup_{|F_1|=|F_2|<\infty} \left\{ \frac{1}{|F_1|} \langle \lambda(|f|^2)\chi_{F_2}, \chi_{F_1} \rangle \right\} \leqslant \|\lambda(|f|^2)\|.\end{aligned}$$

Since $T_2(G)_{\mathrm{sym}} = \ell^2(G)_{\mathrm{sym}}$, and these spaces have equivalent norms, we get

$$\big\| |f| \big\|_{\ell^2}^2 \leqslant C' \big\| \lambda(|f|^2) \big\| \quad \text{for all } f \in \mathbb{C}[G]_{\mathrm{sym}}$$

for some constant C'. This implies that

$$\|g\|_{\ell^1} \leqslant C'' \big\| \lambda(g) \big\|$$

for any positive, symmetric function $g \in \mathbb{C}[G]$ and some constant C''. Corollary 5.2 yields that G must be amenable. □

Lemma 5.7. *Let G be a group, and $\psi : G \to \mathbb{R}$ a conditionally negative definite function. If ψ is bounded, then $\psi = c - \varphi$ for some constant $c \in \mathbb{R}$ and some positive definite function $\varphi : G \to \mathbb{R}$.*

Proof. Without loss of generality we may assume $\psi(e) = 0$. It is then well-known that ψ has the form $\psi(y^{-1}x) = \|\sigma(x) - \sigma(y)\|^2$ for some 1-cocycle $\sigma : G \to H$ with coefficients in an orthogonal representation $\pi : G \to O(H)$, where H is a real Hilbert space. Since ψ is bounded, so is σ. Any bounded 1-cocycle is a 1-coboundary (see [4, Lemma D.10]). Hence there is $\xi \in H$ such that $\sigma(x) = \xi - \pi(x)\xi$ for every $x \in G$. Then

$$\psi(y^{-1}x) = \big\|\sigma(x) - \sigma(y)\big\|^2 = 2\|\xi\|^2 - 2\langle \pi(y^{-1}x)\xi, \xi\rangle.$$

Now, put $c = 2\|\xi\|^2$ and $\varphi(x) = 2\langle \pi(x)\xi, \xi\rangle$. □

We are now ready to prove the other direction of Theorem 1.5.

Theorem 5.8. *Let G be a countable, discrete group. Suppose every symmetric function $\varphi : G \to \mathbb{R}$ such that $\|e^{-t\varphi}\|_{B_2} \leqslant 1$ for every $t > 0$ splits as*

$$\varphi(x) = \psi(x) + \|\xi\|^2 + \langle \pi(x)\xi, \xi\rangle \quad (x \in G),$$

where

- *ψ is a conditionally negative definite function on G,*
- *π is an orthogonal representation of G on some real Hilbert space H,*
- *and ξ is a vector in H.*

Then G is amenable.

Proof. It is always the case that $B(G) \subseteq B_2(G)$. Suppose $\rho \in B_2(G)$ is real, symmetric, with $\|\rho\|_{B_2} = 1$. If we put $\varphi = 1 - \rho$, then

$$\left\|e^{-t\varphi}\right\|_{B_2} = e^{-t}\left\|e^{t\rho}\right\|_{B_2} \leqslant e^{-t}e^{t\|\rho\|_{B_2}} = 1.$$

By our assumption we have a splitting

$$\varphi(x) = \psi(x) + \|\xi\|^2 + \langle \pi(x)\xi, \xi\rangle \quad (x \in G).$$

Obviously, ρ is bounded, and it follows that ψ is bounded. By the previous lemma there is some positive definite function φ' on G and a constant $c \in \mathbb{R}$ such that $\psi = c - \varphi'$. Hence $\psi \in B(G)$. From this we get that $\varphi \in B(G)$, so $\rho \in B(G)$.

It now follows that $B_2(G)_{\mathrm{sym}} \subseteq B(G)$, so $B_2(G)_{\mathrm{sym}} = B(G)_{\mathrm{sym}}$. From Proposition 5.6 we conclude that G is amenable. □

6. Radial semigroups of Herz–Schur multipliers on $\mathbb{F}_\infty$

We now change the focus of Problem 1.3 to the particular case where the group in question is a free group. We briefly recall Problem 1.3. Suppose $\varphi : G \to \mathbb{R}$ is symmetric and $\|e^{-t\varphi}\|_{B_2} \leqslant 1$ for every $t > 0$. Is it then possible to find a conditionally negative definite function $\psi : G \to \mathbb{R}$ such that $\varphi \leqslant \psi$. In the case of radial functions on free groups we provide a positive solution to the problem (see Theorem 6.15).

Let $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, and let $\sigma : \mathbb{N}_0^2 \to \mathbb{N}_0^2$ denote the *shift map* given as $\sigma(m, n) = (m + 1, n + 1)$. Let $\mathbb{F}_n$ denote the free group on n generators, where $2 \leqslant n \leqslant \infty$. We use $|x|$ to denote the word length of $x \in \mathbb{F}_n$.

Definition 6.1. A function $\varphi : \mathbb{F}_n \to \mathbb{C}$ is called *radial* if there is a (necessarily unique) function $\dot\varphi : \mathbb{N}_0 \to \mathbb{C}$ such that $\varphi(x) = \dot\varphi(|x|)$ for all $x \in \mathbb{F}_n$, i.e., if the value $\varphi(x)$ only depends on $|x|$.

A function $\varphi : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{C}$ is called a *Hankel function* if the value $\varphi(m, n)$ only depends on $m + n$.

Given a radial function φ on $\mathbb{F}_n$, we let $\widetilde\varphi$ be the kernel on $\mathbb{N}_0$ defined by $\widetilde\varphi(m, n) = \dot\varphi(m + n)$. Note that $\widetilde\varphi$ is a Hankel function.

Actually, the free groups will not enter the picture before Theorem 6.11. Until then we will simply study properties of kernels on $\mathbb{N}_0$.

6.1. Functionals on the Toeplitz algebra

Let S be the unilateral shift operator on $\ell^2(\mathbb{N}_0)$. The C^*-algebra $C^*(S)$ generated by S is the Toeplitz algebra. Since $S^*S = I$, the set

$$D = \operatorname{span}\{S^k (S^*)^l \mid k, l \in \mathbb{N}_0\} \tag{6.1}$$

is a $*$-algebra, and its closure is $C^*(S)$. The Toeplitz algebra fits in the exact sequence

$$0 \to \mathcal{K} \to C^*(S) \xrightarrow{\pi} C(\mathbb{T}) \to 0, \tag{6.2}$$

where $\mathcal{K}$ denotes the C^*-algebra of compact operators (on $\ell^2(\mathbb{N}_0)$), $C(\mathbb{T})$ is the C^*-algebra of continuous functions on the unit circle $\mathbb{T}$, and π is the quotient map that maps S to the generating unitary $\mathrm{id}_{\mathbb{T}}$.

When $\varphi : \mathbb{N}_0^2 \to \mathbb{C}$ is a kernel we let ω_φ denote the linear functional defined on D by

$$\omega_\varphi(S^m (S^*)^n) = \varphi(m, n). \tag{6.3}$$

It may or may not happen that ω_φ extends to a bounded functional on $C^*(S)$. If it does, we also denote the extension by ω_φ. Along the same lines we consider the linear map M_φ defined on D by

$$M_\varphi(S^m (S^*)^n) = \varphi(m, n) S^m (S^*)^n,$$

and if it extends to a bounded linear map on $C^*(S)$, we also denote the extension by M_φ. We call it the *multiplier* of φ.

Remark 6.2. Consider the C^*-algebra $C^*(S \otimes S)$ generated by the operator $S \otimes S$ inside $B(\ell^2(\mathbb{N}_0) \otimes \ell^2(\mathbb{N}_0))$. Since the operator $S \otimes S$ is a proper isometry, it follows from Coburn's Theorem (see [10, Theorem 3.5.18]) that there is a $*$-isomorphism $\alpha : C^*(S) \to C^*(S \otimes S)$ such that $\alpha(S) = S \otimes S$. Let π be the quotient map $C^*(S) \to C(\mathbb{T})$ from before and let $\mathrm{ev}_1 : C(\mathbb{T}) \to \mathbb{C}$ be evaluation at $1 \in \mathbb{T}$. Then we note that $\omega_\varphi = \mathrm{ev}_1 \circ \pi \circ M_\varphi$, while $M_\varphi = (\mathrm{id}_{C^*(S)} \otimes \omega_\varphi) \circ \alpha$, where we have identified $C^*(S) \otimes \mathbb{C}$ with $C^*(S)$.

It follows from the mentioned relation between ω_φ and M_φ that ω_φ extends to $C^*(S)$ if and only if M_φ extends to $C^*(S)$. If this is the case, then M_φ is even completely bounded, since bounded functionals and $*$-homomorphisms are always completely bounded. Similarly, ω_φ is positive if and only if M_φ is positive if and only if M_φ is *completely* positive. Finally, $\|M_\varphi\| = \|\omega_\varphi\|$.

6.2. Positive and conditionally negative functions

The following proposition characterizes the functions φ that induce states on the Toeplitz algebra.

Proposition 6.3. *Let $\varphi : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{C}$ be a function. The following are equivalent.*

(1) *The functional ω_φ extends to a state on the Toeplitz algebra $C^*(S)$.*
(2) *The multiplier M_φ extends to a u.c.p. map on the Toeplitz algebra $C^*(S)$.*

(3) *There exist a positive trace class operator* $h = [h(i,j)]_{i,j=0}^{\infty}$ *on* $\ell^2(\mathbb{N}_0)$ *and a positive definite function* $\varphi_0 : \mathbb{Z} \to \mathbb{C}$ *such that*

$$\varphi(k,l) = \sum_{i=0}^{\infty} h(k+i, l+i) + \varphi_0(k-l)$$

for all $(k,l) \in \mathbb{N}_0 \times \mathbb{N}_0$ *and* $\varphi(0,0) = 1$.

(4) φ *is a positive definite kernel with* $\varphi(0,0) = 1$ *and* $\varphi - \varphi \circ \sigma$ *is a positive definite kernel.*

Proof. The equivalence (1) $\Longleftrightarrow$ (2) follows from Remark 6.2.

The rest of the proof goes as follows: (1) $\Longrightarrow$ (4) $\Longrightarrow$ (3) $\Longrightarrow$ (1).

(1) $\Longrightarrow$ (4): Given complex numbers $c_0, \ldots, c_n$, we see that

$$\omega_\varphi\left(\left(\sum_{k=0}^{n} c_k S^k\right)\left(\sum_{l=0}^{n} c_l S^l\right)^*\right) = \sum_{k,l=0}^{n} c_k \bar{c}_l \varphi(k,l),$$

so φ is positive definite, since ω_φ is a positive functional. If we let $(e_{kl})_{k,l=0}^{\infty}$ denote the standard matrix units in $B(\ell^2(\mathbb{N}_0))$, then

$$e_{kl} = S^k\left(S^*\right)^l - S^{k+1}\left(S^*\right)^{l+1},$$

and so

$$\varphi(k,l) - \varphi(k+1,l+1) = \omega_\varphi\left(S^k\left(S^*\right)^l - S^{k+1}\left(S^*\right)^{l+1}\right) = \omega_\varphi(e_{kl}).$$

It follows that

$$0 \leqslant \omega_\varphi\left(\left(\sum_{k=0}^{n} c_k e_{k0}\right)\left(\sum_{l=0}^{n} c_l e_{l0}\right)^*\right) = \sum_{k,l=0}^{n} c_k \bar{c}_l \big(\varphi(k,l) - \varphi(k+1,l+1)\big),$$

so $\varphi - \varphi \circ \sigma$ is positive definite. Finally, $\varphi(0,0) = \omega_\varphi(1) = 1$.

(4) $\Longrightarrow$ (3): Since $\varphi - \varphi \circ \sigma$ is positive definite, $\varphi(0,0) \geqslant \varphi(1,1) \geqslant \cdots$, and since φ is positive definite, $\varphi(k,k) \geqslant 0$ for every k. Hence $\lim_k \varphi(k,k)$ exists. Let $h = \varphi - \varphi \circ \sigma$. Then h is positive definite, and

$$\sum_{k=0}^{\infty} h(k,k) = \sum_{k=0}^{\infty} (\varphi - \varphi \circ \sigma)(k,k) = \varphi(0,0) - \lim_{k \to \infty} \varphi(k,k) < \infty.$$

By the Cauchy–Schwarz inequality,

$$\sum_{i=0}^{\infty} \left|h(k+i, l+i)\right| \leqslant \sum_{i=0}^{\infty} \sqrt{h(k+i, k+i)}\sqrt{h(l+i, l+i)} < \infty,$$

so $\lim_i \varphi(k+i, l+i)$ exists for every k, l and depends of course only on $k-l$. Define $\varphi_0(k-l) = \lim_i \varphi(k+i, l+i)$. Since φ is positive definite, it follows that $\varphi \circ \sigma^i$ is positive definite, so the limit φ_0 is as well. Finally note that

$$\varphi(k,l) = \sum_{i=0}^{\infty} h(k+i, l+i) + \varphi_0(k-l).$$

(3) $\Longrightarrow$ (1): Let ω_1 be the functional on $B(\ell^2(\mathbb{N}_0))$ given by $\omega_1(x) = \mathrm{Tr}(h^t x)$, where h^t denotes the transpose of h. Since h is positive, this is a positive, normal, linear functional. Note that $\omega_1(e_{kl}) = \mathrm{Tr}(e_{0l} h^t e_{k0}) = h^t(l,k) = h(k,l)$, so that

$$\omega_1\bigl(S^k\bigl(S^*\bigr)^l\bigr) = \omega_1(e_{kl} + e_{k+1,l+1} + \cdots) = \sum_{i=0}^{\infty} h(k+i, l+i).$$

The positive definite function $\varphi_0 : \mathbb{Z} \to \mathbb{R}$ corresponds to a positive functional ω_0 on $C(\mathbb{T})$ given by $\omega_0(z^{k-l}) = \varphi_0(k-l)$, where z denotes the standard unitary generator of $C(\mathbb{T})$. Letting $\pi : C^*(S) \to C(\mathbb{T})$ be the quotient map as usual, we see that $\omega = \omega_1 + \omega_0 \circ \pi$ is a positive linear functional on $C^*(S)$ with

$$\omega\bigl(S^k\bigl(S^*\bigr)^l\bigr) = \sum_{i=0}^{\infty} h(k+i, l+i) + \varphi_0(k-l) = \varphi(k,l).$$

Hence $\omega_\varphi = \omega$ is the desired positive functional on $C^*(S)$. Since $\omega(1) = \varphi(0,0) = 1$, it is a state. □

Corollary 6.4. *Let $\theta : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{C}$ be a function. The conditions* (1) *and* (2) *below are equivalent. Moreover,* (1) *implies* (3)*.*

(1) *The function $\theta - \frac{1}{2}\theta(0,0)$ is positive definite, and $\theta - \theta \circ \sigma$ is positive definite.*
(2) *There exist a Hilbert space H with vectors $\xi_i \in H$ such that $\sum_{i=0}^{\infty} \|\xi_i\|^2 < \infty$ and a positive definite function $\theta_0 : \mathbb{Z} \to \mathbb{C}$ such that θ is given as*

$$\theta(k,l) = \sum_{i=0}^{\infty} \|\xi_i\|^2 + \sum_{i=0}^{\infty} \langle \xi_{k+i}, \xi_{l+i} \rangle + \theta_0(0) + \theta_0(k-l), \quad k, l \in \mathbb{N}_0.$$

(3) *For all $t > 0$ we have $\|M_{e^{-t\theta}}\| \leqslant 1$.*

Proof. Let $\varphi = \theta - \frac{1}{2}\theta(0,0)$, and observe that $\theta - \theta \circ \sigma = \varphi - \varphi \circ \sigma$. The equivalence (1) $\Longleftrightarrow$ (2) follows easily from Proposition 6.3 applied to φ. For the norm estimate in (3) (assuming (1)) we use

$$\|M_{e^{-t\theta}}\| = e^{-t\varphi(0,0)} \|M_{e^{-t\varphi}}\| \leqslant e^{-t\varphi(0,0)} e^{t\|M_\varphi\|} = 1,$$

where we used Proposition 6.3 to get $\|M_\varphi\| = \varphi(0,0)$. □

Our next goal is to characterize the functions $\psi : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{C}$ that generate semigroups $(e^{-t\psi})_{t>0}$ so that each $e^{-t\psi}$ induces a state on the Toeplitz algebra. With Schoenberg's Theorem in mind, the result in Proposition 6.6 is not surprising. But first we prove a lemma.

Lemma 6.5. *Let $\psi : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{C}$ be a function. Suppose ψ is a conditionally negative definite kernel, and $\psi \circ \sigma - \psi$ is a positive definite kernel. Then there exist a Hilbert space H, a sequence of vectors $(\eta_i)_{i=0}^\infty$ in H such that for every $m, n \in \mathbb{N}_0$*

$$\sum_{k=0}^\infty \|\eta_{m+k} - \eta_{n+k}\|^2 < \infty,$$

and $(\psi \circ \sigma - \psi)(m, n) = \langle \eta_m, \eta_n \rangle$.

Proof. Let $\varphi = \psi \circ \sigma - \psi$. Since by assumption φ is positive definite, there are vectors $\eta_i \in H$, where H is a Hilbert space, such that $\varphi(k, l) = \langle \eta_k, \eta_l \rangle$ for every $k, l \in \mathbb{N}_0$. Define

$$\rho(k, l) = \psi(k + 1, l) + \psi(k, l + 1) - \psi(k, l) - \psi(k + 1, l + 1).$$

Then

$$\begin{aligned}(\rho - \rho \circ \sigma)(k, l) &= -\varphi(k + 1, l) - \varphi(k, l + 1) + \varphi(k, l) + \varphi(k + 1, l + 1)\\ &= \langle \eta_k - \eta_{k+1}, \eta_l - \eta_{l+1} \rangle,\end{aligned}$$

so $\rho - \rho \circ \sigma$ is a positive definite kernel. In particular,

$$\rho(0, 0) \geqslant \rho(1, 1) \geqslant \rho(2, 2) \geqslant \cdots. \tag{6.4}$$

Since ψ is conditionally negative definite,

$$-\rho(k, k) = (1, -1) \begin{pmatrix} \psi(k, k) & \psi(k, k + 1) \\ \psi(k + 1, k) & \psi(k + 1, k + 1) \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \leqslant 0,$$

so $\rho(k, k) \geqslant 0$ for every k. Combining this with (6.4) we see that $\lim_k \rho(k, k)$ exists. Hence

$$\sum_{k=0}^\infty \|\eta_k - \eta_{k+1}\|^2 = \sum_{k=0}^\infty (\rho - \rho \circ \sigma)(k, k) = \rho(0, 0) - \lim_k \rho(k, k) < \infty.$$

Let $C = (\sum_{k=0}^\infty \|\eta_k - \eta_{k+1}\|^2)^{1/2}$. The triangle inequality (for the Hilbert space $H \oplus H \oplus \cdots$) yields that

$$\left(\sum_{k=0}^\infty \|\eta_k - \eta_{k+2}\|^2\right)^{1/2} \leqslant C + \left(\sum_{k=0}^\infty \|\eta_{k+1} - \eta_{k+2}\|^2\right)^{1/2} \leqslant 2C,$$

and similarly

$$\left(\sum_{i=0}^{\infty}\|\eta_{k+i}-\eta_{l+i}\|^2\right)^{1/2}\leqslant|k-l|C<\infty$$

for all $k, l \in \mathbb{N}_0$. □

Proposition 6.6. *Let $\psi : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{C}$ be a function. The following are equivalent.*

(1) *For all $t > 0$ the function $e^{-t\psi}$ satisfies the equivalent conditions in Proposition 6.3.*
(2) *ψ is a conditionally negative definite kernel with $\psi(0,0) = 0$, and $\psi \circ \sigma - \psi$ is a positive definite kernel.*

Moreover, if ψ takes only real values, this is equivalent to the following assertion.

(3) *There exist a Hilbert space H, a sequence of vectors $(\eta_i)_{i=0}^{\infty}$ in H and a conditionally negative definite function $\psi_0 : \mathbb{Z} \to \mathbb{R}$ with $\psi_0(0) = 0$ such that*

$$\psi(k,l)=\frac{1}{2}\left(\sum_{i=0}^{k-1}\|\eta_i\|^2+\sum_{i=0}^{l-1}\|\eta_i\|^2+\sum_{i=0}^{\infty}\|\eta_{k+i}-\eta_{l+i}\|^2\right)+\psi_0(k-l)$$

for all $(k,l) \in \mathbb{N}_0 \times \mathbb{N}_0$ (and the infinite sum is convergent).

Proof. (1) $\Longrightarrow$ (2): We assume that condition (4) of Proposition 6.3 holds for $e^{-t\psi}$ for each $t > 0$. Then the function $e^{-t\psi}$ is a positive definite kernel for each $t > 0$, and so ψ is conditionally negative definite by Schoenberg's Theorem. Since $e^{-t\psi(0,0)} = 1$ for all $t > 0$, we must have $\psi(0,0) = 0$. Moreover, $e^{-t\psi} - e^{-t\psi\circ\sigma}$ is positive definite for each $t > 0$, and hence is

$$\psi\circ\sigma-\psi=\lim_{t\to 0}\frac{e^{-t\psi}-e^{-t\psi\circ\sigma}}{t},$$

where the limit is pointwise.

(2) $\Longrightarrow$ (1): For obvious reasons it suffices to prove the case $t = 1$. We verify condition (4) of Proposition 6.3. An application of Schoenberg's Theorem shows that $e^{-\psi}$ is positive definite, and of course $e^{-\psi(0,0)} = 1$. Consider

$$e^{-\psi}-e^{-\psi\circ\sigma}=e^{-\psi\circ\sigma}\left(e^{(\psi\circ\sigma-\psi)}-1\right). \tag{6.5}$$

The function $e^{-\psi\circ\sigma}$ is positive definite by Schoenberg's Theorem. Expanding the exponential function in the parenthesis as a power series we get

$$e^{\psi\circ\sigma-\psi}-1=\sum_{n=1}^{\infty}\frac{(\psi\circ\sigma-\psi)^n}{n!},$$

and since $\psi \circ \sigma - \psi$ is positive definite, so is each power $(\psi \circ \sigma - \psi)^n$, and so is the sum, and hence also the product in (6.5). The conditions in (4) of Proposition 6.3 have now been verified.

Suppose ψ takes only real values.

(2) $\Longrightarrow$ (3): By Lemma 6.5 there are vectors $(\eta_i)_{i=0}^\infty$ in a Hilbert space H, such that $(\psi \circ \sigma - \psi)(m,n) = \langle \eta_m, \eta_n \rangle$ and

$$\sum_{k=0}^\infty \|\eta_{m+k} - \eta_{n+k}\|^2 < \infty$$

for every $m, n \in \mathbb{N}_0$. Since ψ is hermitian and real, it is symmetric. Hence $\langle \eta_m, \eta_n \rangle = \langle \eta_n, \eta_m \rangle$.

Let $f(k) = \frac{1}{2}\sum_{i=0}^{k-1} \|\eta_i\|^2$, and set

$$\psi_2(k,l) = \psi(k,l) - f(k) - f(l), \qquad \psi_1(k,l) = \psi_2(k,l) - \frac{1}{2}\sum_{i=0}^\infty \|\eta_{k+i} - \eta_{l+i}\|^2.$$

We claim that ψ_1 is conditionally negative definite, $\psi_1(0,0) = 0$, and that $\psi_1(k,l)$ only depends on $k - l$. These claims will finish the proof of (2) $\Longrightarrow$ (3). We find

$$\begin{aligned}(\psi_1 \circ \sigma - \psi_1)(k,l) &= (\psi \circ \sigma - \psi)(k,l) - \frac{1}{2}\|\eta_k\|^2 - \frac{1}{2}\|\eta_l\|^2 + \frac{1}{2}\|\eta_k - \eta_l\|^2 \\ &= \langle \eta_k, \eta_l \rangle - \frac{1}{2}\|\eta_k\|^2 - \frac{1}{2}\|\eta_l\|^2 + \frac{1}{2}\|\eta_k - \eta_l\|^2 = 0,\end{aligned}$$

and hence $\psi_1(k,l)$ only depends on $k - l$. Letting $\psi_0(k - l) = \psi_1(k,l)$ gives a well-defined function $\psi_0 : \mathbb{Z} \to \mathbb{R}$. Note that

$$\psi_0(0) = \psi_1(0,0) = \psi_2(0,0) = \psi(0,0) = 0.$$

It remains to be seen that ψ_0 is conditionally negative definite. Observe that ψ_2 is conditionally negative definite, because ψ is. Also,

$$\psi_2(k,l) = \psi_0(k-l) + \frac{1}{2}\sum_{i=0}^\infty \|\eta_{k+i} - \eta_{l+i}\|^2.$$

Replacing (k,l) by $(k+n, l+n)$ we see that

$$\psi_2(k+n, l+n) = \psi_0(k-l) + \frac{1}{2}\sum_{i=n}^\infty \|\eta_{k+i} - \eta_{l+i}\|^2,$$

and so

$$\lim_{n\to\infty} \psi_2(k+n, l+n) = \psi_0(k-l).$$

Since ψ_2 was conditionally negative definite (and hence also $\psi_2 \circ \sigma^n$), it follows that ψ_0 is conditionally negative definite being the pointwise limit of conditionally negative definite kernels.

(3) $\Longrightarrow$ (2): Since each of the functions

$$(k,l) \mapsto \sum_{i=0}^{k-1} \|\eta_i\|^2, \quad (k,l) \mapsto \|\eta_{k+i} - \eta_{l+i}\|^2, \quad \psi_0$$

is a conditionally negative definite kernel, so is ψ. Also $\psi(0,0) = \psi_0(0) = 0$. Finally, $(\psi \circ \sigma - \psi)(k,l) = \langle \eta_k, \eta_l \rangle$, and hence $\psi \circ \sigma - \psi$ is positive definite. □

6.3. Decomposition into positive and negative parts

In the following section we will describe the kernels φ such that $\|\omega_{e^{-t\varphi}}\| \leqslant 1$ for every $t > 0$. The main result here is contained in Theorem 6.9.

Definition 6.7. Denote by $\mathcal{S}$ the set of hermitian functions $\varphi : \mathbb{N}_0^2 \to \mathbb{C}$ that split as $\varphi = \psi + \theta$, where

- ψ is a conditionally negative definite kernel with $\psi(0,0) = 0$,
- $\psi \circ \sigma - \psi$ is a positive definite kernel,
- $\theta - \frac{1}{2}\theta(0,0)$ is a positive definite kernel,
- $\theta - \theta \circ \sigma$ is a positive definite kernel.

Observe that $\mathcal{S}$ is stable under addition, multiplication by positive numbers and addition by positive constant functions.

Lemma 6.8. *The set $\mathcal{S}$ is closed in the topology of pointwise convergence.*

Proof. Let $(\varphi_i)_{i \in I}$ be a net in $\mathcal{S}$ converging pointwise to φ, and let $\varphi_i = \psi_i + \theta_i$ be a splitting guaranteed by the assumption $\varphi_i \in \mathcal{S}$.

An application of the Cauchy–Schwarz inequality to the positive definite kernel $\theta_i - \frac{1}{2}\theta_i(0,0)$ gives

$$\left|\theta_i(k,l) - \frac{1}{2}\theta_i(0,0)\right| \leqslant \left(\theta_i(k,k) - \frac{1}{2}\theta_i(0,0)\right)^{1/2} \left(\theta_i(l,l) - \frac{1}{2}\theta_i(0,0)\right)^{1/2}$$

and using positive definiteness of $\theta_i - \theta_i \circ \sigma$ then gives

$$\left|\theta_i(k,l) - \frac{1}{2}\theta_i(0,0)\right| \leqslant \theta_i(0,0) - \frac{1}{2}\theta_i(0,0) = \frac{1}{2}\theta_i(0,0) = \frac{1}{2}\varphi_i(0,0).$$

Since $\theta_i(0,0) = \varphi_i(0,0) \to \varphi(0,0)$, this shows that the net $(\theta_i(k,l))_{i \in I}$ is eventually bounded for each pair (k,l). It follows that for each pair (k,l) the net $\psi_i(k,l)$ is also eventually bounded.

Let $(\psi_j)_{j \in J}$ and $(\theta_j)_{j \in J}$ be universal subnets of $(\psi_i)_{i \in I}$ and $(\theta_i)_{i \in I}$ (we can assume, as we have done, that they have the same index set J). Since the net $(\psi_j)_{j \in J}$ is pointwise eventually bounded, it converges to some limit ψ. Similarly let $\theta = \lim_j \theta_j$. Since the defining properties of the splitting $\varphi_j = \psi_j + \theta_j$ pass to the limits ψ and θ, we have the desired splitting $\varphi = \psi + \theta$, and the proof is done. □

We have the following alternative characterization of the set $\mathcal{S}$. This should be compared with the result in Theorem 1.4.

Theorem 6.9. *Let $\varphi : \mathbb{N}_0^2 \to \mathbb{C}$ be a hermitian function. Then $\varphi \in \mathcal{S}$ if and only if*

$$\|\omega_{e^{-t\varphi}}\| \leqslant 1 \quad \text{for every } t > 0,$$

where $\omega_{e^{-t\varphi}}$ is the functional associated with $e^{-t\varphi}$ as in (6.3).

For the proof we need the following lemma.

Lemma 6.10. *Let $\varphi : \mathbb{N}_0^2 \to \mathbb{C}$ be a hermitian function. If $\|\omega_\varphi\| \leqslant 1$, then $1 - \varphi \in \mathcal{S}$.*

Proof. Since φ is hermitian, ω_φ is hermitian. Now, use the Hahn–Jordan Decomposition Theorem to write $\omega_\varphi = \omega^+ - \omega^-$, where $\omega^+, \omega^- \in C^*(S)^*$ are positive functionals. If we define functions φ^+, φ^- by

$$\varphi^\pm(m,n) = \omega^\pm\big(S^m\big(S^*\big)^n\big),$$

then φ^+ and φ^- satisfy the second condition of Proposition 6.3 (up to a scaling factor). Also, it is clear that $\varphi = \varphi^+ - \varphi^-$.

Let $c = \varphi^+(0,0)$, and put

$$\psi = c - \varphi^+, \qquad \theta = 1 - c + \varphi^-.$$

Obviously, $\psi + \theta = 1 - \varphi$. It remains to show that ψ and θ have the desired properties used in the definition of $\mathcal{S}$.

Since φ^+ is positive definite, ψ is conditionally negative definite with $\psi(0,0) = 0$. Also, $\psi \circ \sigma - \psi = \varphi^+ - \varphi^+ \circ \sigma$, which is positive definite by Proposition 6.3.

Moreover, $\theta - \theta \circ \sigma = \varphi^- - \varphi^- \circ \sigma$ is positive definite. Finally,

$$\theta - \frac{1}{2}\theta(0,0) = \frac{1}{2}\big(1 - \varphi(0,0)\big) + \varphi^-,$$

and since $|\varphi(0,0)| \leqslant \|\omega_\varphi\| \leqslant 1$ and $\varphi(0,0)$ is real, it follows that $1 - \varphi(0,0) \geqslant 0$, so $\theta - \frac{1}{2}\theta(0,0)$ is positive definite (using that φ^- is positive definite). □

Proof of Theorem 6.9. Suppose first $\|\omega_{e^{-t\varphi}}\| \leqslant 1$ for all $t > 0$. It follows from the previous lemma that $1 - e^{-t\varphi} \in \mathcal{S}$ for every $t > 0$. Hence the functions $(1 - e^{-t\varphi})/t$ are in $\mathcal{S}$, and they converge pointwise to φ as $t \to 0$. Since $\mathcal{S}$ is closed under pointwise convergence, we conclude that $\varphi \in \mathcal{S}$.

Conversely, suppose $\varphi \in \mathcal{S}$. Write $\varphi = \psi + \theta$ as in the definition of $\mathcal{S}$. From Proposition 6.6 we get that $M_{e^{-t\psi}}$ is a u.c.p. map for every $t > 0$, and hence $\|M_{e^{-t\psi}}\| = 1$. Also, from Corollary 6.4 we get that $\|M_{e^{-t\theta}}\| \leqslant 1$ for every $t > 0$. This combines to show

$$\|\omega_{e^{-t\varphi}}\| = \|M_{e^{-t\varphi}}\| \leqslant \|M_{e^{-t\psi}}\|\|M_{e^{-t\theta}}\| \leqslant 1. \qquad \square$$

6.4. Comparison of norms

In this section we establish the connection between norms of radial Herz–Schur multipliers on $\mathbb{F}_\infty$ and functionals on the Toeplitz algebra. This will be accomplished in Proposition 6.13.

In [8] the following theorem is proved (see Theorem 5.2 therein).

Theorem 6.11. *Let $\mathbb{F}_\infty$ be the free group on (countably) infinitely many generators, let $\varphi : \mathbb{F}_\infty \to \mathbb{C}$ be a radial function, and let $\dot{\varphi} : \mathbb{N}_0 \to \mathbb{C}$ be as in Definition 6.1. Finally, let $h = (h_{ij})_{i,j\in\mathbb{N}_0}$ be the Hankel matrix given by $h_{ij} = \dot{\varphi}(i+j) - \dot{\varphi}(i+j+2)$ for $i, j \in \mathbb{N}_0$. Then the following are equivalent:*

(i) *φ is a Herz–Schur multiplier on $\mathbb{F}_\infty$.*
(ii) *h is of trace class.*

If these two equivalent conditions are satisfied, then there exist unique constants $c_\pm \in \mathbb{C}$ and a unique $\dot{\psi} : \mathbb{N}_0 \to \mathbb{C}$ such that

$$\dot{\varphi}(n) = c_+ + c_-(-1)^n + \dot{\psi}(n) \quad (n \in \mathbb{N}_0) \tag{6.6}$$

and

$$\lim_{n\to\infty} \dot{\psi}(n) = 0.$$

Moreover,

$$\|\varphi\|_{B_2} = |c_+| + |c_-| + \|h\|_1.$$

The Fourier–Stieltjes algebra $B(\mathbb{Z})$ of the group of integers is the linear span of positive definite functions on $\mathbb{Z}$. It is naturally identified with dual space of $C^*(\mathbb{Z}) \simeq C(\mathbb{T})$, i.e., with the set $M(\mathbb{T})$ of complex Radon measures on the circle, where $\varphi \in B(\mathbb{Z})$ corresponds to $\mu \in M(\mathbb{T})$, if and only if

$$\varphi(n) = \int_{\mathbb{T}} z^n \, d\mu(z) \quad \text{for all } n \in \mathbb{Z}. \tag{6.7}$$

Under this identification $B(\mathbb{Z})$ becomes a Banach space when the norm $\|\varphi\|_{B(\mathbb{Z})}$ is defined to be $\|\mu\|$, the total variation of μ.

Proposition 6.12. *Let $\varphi : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{C}$ be a function, and let $h = \varphi - \varphi \circ \sigma$. The functional ω_φ extends to a bounded functional on $C^*(S)$ if and only if h is of trace class, and the function $\varphi_0 : \mathbb{Z} \to \mathbb{C}$ given by $\varphi_0(m-n) = \lim_k \varphi(m+k, n+k)$ (which is then well-defined) lies in $B(\mathbb{Z})$. If this is the case, then*

$$\|\omega_\varphi\| = \|h\|_1 + \|\varphi_0\|_{B(\mathbb{Z})}.$$

Proof. The proposition is actually a special case of a general phenomenon. Given an extension $0 \to I \to A \to A/I \to 0$ of C^*-algebras, then $A^* \simeq I^* \oplus_1 (A/I)^*$ isometrically. The extension under consideration in our proposition is (6.2). The general theory is described in the book [9]. We have included a more direct proof.

Suppose first h is of trace class, and $\varphi_0 \in B(\mathbb{Z})$. Let $\mu \in M(\mathbb{T})$ be given by (6.7), and define $\omega_0 \in C(\mathbb{T})^*$ by

$$\omega_0(f) = \int_{\mathbb{T}} f \, d\mu \quad \text{for all } f \in C(\mathbb{T}).$$

Define a functional ω_1 on $C^*(S)$ by $\omega_1(x) = \mathrm{Tr}(h^t x)$ for $x \in C^*(S)$, and also let $\omega = \omega_1 + \omega_0 \circ \pi$. Observe that

$$\omega_1\big(S^m \big(S^*\big)^n\big) = \sum_{k=0}^{\infty} h(m+k, n+k).$$

It follows that

$$\omega\big(S^m \big(S^*\big)^n\big) = \sum_{k=0}^{\infty} h(m+k, n+k) + \varphi_0(m-n) = \varphi(m,n),$$

so that $\omega = \omega_\varphi$. Hence ω_φ extends to a bounded functional on $C^*(S)$.

Suppose instead that ω_φ extends to a bounded functional on $C^*(S)$. Proposition 2.8 in [8] ensures the existence of a complex Borel measure μ on $M(\mathbb{T})$ and a trace class operator T on $\ell^2(\mathbb{N}_0)$ such that

$$\omega_\varphi\big(S^m \big(S^*\big)^n\big) = \int_{\mathbb{T}} z^{m-n} \, d\mu(z) + \mathrm{Tr}\big(S^m \big(S^*\big)^n T\big) \quad \text{for all } m, n \in \mathbb{N}_0.$$

From this we get that $T^t_{mn} = h(m,n)$, where T^t is the transpose of T, and

$$\varphi_0(m-n) = \int_{\mathbb{T}} z^{m-n} \, d\mu(z).$$

Hence h is of trace class and $\varphi_0 \in B(\mathbb{Z})$. From [8] we also have $\|\omega_\varphi\| = \|\mu\| + \|T\|_1$, which concludes our proof, since

$$\|\mu\| = \|\varphi_0\| \quad \text{and} \quad \big\|T^t\big\|_1 = \|h\|_1. \qquad \square$$

Proposition 6.13. *Let $\varphi : \mathbb{F}_\infty \to \mathbb{C}$ be a radial function, and let $\widetilde{\varphi} : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{C}$ be as in Definition 6.1. Then $\|\omega_{\widetilde{\varphi}}\| = \|\varphi\|_{B_2}$.*

Proof. Let $h_{ij} = \widetilde{\varphi}(i, j) - \widetilde{\varphi}(i+1, j+1)$. From Theorem 6.11 and Proposition 6.12 we see that it suffices to consider the case where h is the matrix of a trace class operator, since otherwise $\|\omega_{\widetilde{\varphi}}\| = \|\varphi\|_{B_2} = \infty$. If h is of trace class, then we let $\widetilde{\varphi}_0(n) = \lim_k \widetilde{\varphi}(k+n, k)$. From Theorem 6.11 and Proposition 6.12 it follows that

$$\|\omega_{\widetilde{\varphi}}\| = \|h\|_1 + \|\widetilde{\varphi}_0\|_{B(\mathbb{Z})},$$

$$\|\varphi\|_{B_2} = \|h\|_1 + |c_+| + |c_-|,$$

where $c_\pm$ are the constants obtained in Theorem 6.11. It follows from (6.6) that

$$\widetilde{\varphi}_0(n) = c_+ + (-1)^n c_-.$$

Now we only need to see why $|c_+| + |c_-| = \|\widetilde{\varphi}_0\|_{B(\mathbb{Z})}$. Let $\nu \in C(\mathbb{T})^*$ be the functional given by $\nu(f) = c_+ f(1) + c_- f(-1)$ for all $f \in C(\mathbb{T})$. Observe that $\nu(z \mapsto z^n) = c_+ + (-1)^n c_-$. Hence ν corresponds to $\widetilde{\varphi}_0$ under the isometric isomorphism $B(\mathbb{Z}) \simeq C(\mathbb{T})^*$. It is easily seen that $\|\nu\| = |c_+| + |c_-|$. So

$$\|\widetilde{\varphi}_0\|_{B(\mathbb{Z})} = |c_+| + |c_-|.$$

This completes the proof. □

6.5. *The linear bound*

We now restrict our attention to functions $\varphi : \mathbb{N}_0^2 \to \mathbb{C}$ of the form $\varphi(m, n) = \dot{\varphi}(m+n)$ for some function $\dot{\varphi} : \mathbb{N}_0 \to \mathbb{C}$. Recall that such functions are called *Hankel functions*.

A function $\varphi : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{C}$ is called *linearly bounded* if there are constants $a, b \geqslant 0$ such that $|\varphi(m, n)| \leqslant b + a(m+n)$ for all $m, n \in \mathbb{N}_0$.

Proposition 6.14. *If φ is a Hankel function and $\varphi \in \mathcal{S}$, then φ is linearly bounded.*

Proof. Write $\varphi = \psi + \theta$ as in Definition 6.7. Note that

$$\varphi \circ \sigma - \varphi = (\psi \circ \sigma - \psi) - (\theta - \theta \circ \sigma),$$

where $h_1 = \psi \circ \sigma - \psi$ and $h_2 = \theta - \theta \circ \sigma$ are positive definite. As in the definition of a Hankel function, write $\varphi(m, n) = \dot{\varphi}(m+n)$, and let $\dot{h}(m) = \dot{\varphi}(m+2) - \dot{\varphi}(m)$, so that

$$\dot{h}(m+n) = (\varphi \circ \sigma - \varphi)(m, n) = h_1(m, n) - h_2(m, n).$$

We will now prove that $\dot{h}$ is bounded, and this will lead to the conclusion of the proposition.

From Corollary 6.4 and Lemma 6.5 we see that there are vectors ξ_k, η_k in a Hilbert space such that

$$h_1(m, n) = \langle \eta_m, \eta_n \rangle, \qquad h_2(m, n) = \langle \xi_m, \xi_n \rangle$$

for all $m, n \in \mathbb{N}_0$, and

$$\sum_{k=0}^{\infty} \|\eta_k - \eta_{k+1}\|^2 < \infty, \qquad \sum_{k=0}^{\infty} \|\xi_k\|^2 < \infty.$$

From this we see that h_2 is the matrix of a positive trace class operator. Also, there is $c > 0$ such that $\|\eta_k - \eta_{k+1}\| \leqslant c$ for every k and $\|\eta_0\| \leqslant c$, so we get the linear bound $\|\eta_k\| \leqslant c(k+1)$. From the Cauchy–Schwarz inequality we get

$$\left|h_1(m,n)\right| \leqslant c^2(m+1)(n+1). \tag{6.8}$$

Since $h_2 \leqslant \|h_2\| I \leqslant \|h_2\|_1 I$ (as positive definite matrices, where I is the identity operator), we deduce that the function

$$(m,n) \mapsto \dot{h}(m+n) + \|h_2\|_1 \delta_{mn} = h_1(m,n) + \left(\|h_2\|_1 \delta_{mn} - h_2(m,n)\right)$$

is a positive definite kernel. By the Cauchy–Schwarz inequality we have

$$\left|\dot{h}(k)\right|^2 \leqslant \left(\dot{h}(0) + d\right)\left(\dot{h}(2k) + d\right) \tag{6.9}$$

for every $k \geqslant 1$, where $d = \|h_2\|_1$. If $e = \dot{h}(0) + d$ is zero, then clearly $\dot{h}(k) = 0$ when $k \geqslant 1$. Suppose $e > 0$. Then we may rewrite (6.9) as

$$\dot{h}(2k) \geqslant \frac{|\dot{h}(k)|^2}{e} - d.$$

We claim that $\dot{h}$ is bounded by $2e + d$. Suppose by contradiction that $|\dot{h}(k_0)| > 2e + d$ for some $k_0 \geqslant 1$. Then by induction over n we may prove that for any $n \in \mathbb{N}_0$

$$\dot{h}\left(k_0 2^{n+1}\right) \geqslant 2^{2^n}(2e+d). \tag{6.10}$$

For $n = 0$ we have

$$\dot{h}(2k_0) \geqslant \frac{|\dot{h}(k_0)|^2}{e} - d \geqslant \frac{(2e+d)^2}{e} - d \geqslant 4e + 3d \geqslant 2(2e+d).$$

For $n \geqslant 1$ we get (using our induction hypothesis)

$$\begin{aligned} \dot{h}\left(k_0 2^{n+1}\right) &\geqslant \frac{|\dot{h}(k_0 2^n)|^2}{e} - d \geqslant \left(2^{2^{n-1}}\right)^2 \frac{(2e+d)^2}{e} - d \\ &\geqslant 2^{2^n}(4e+4d) - d \geqslant 2^{2^n}(2e+d). \end{aligned}$$

Using (6.8) we observe that for every $m \in \mathbb{N}_0$ we have

$$\left|\dot{h}(m)\right| \leqslant \left|h_1(m,0)\right| + \left|h_2(m,0)\right| \leqslant c^2(m+1) + d.$$

Since $e > 0$, this contradicts (6.10). This proves the claim. It follows that

$$\left|\dot{\varphi}(2k)\right| \leqslant \left|\dot{\varphi}(0)\right| + (2e+d)k$$

and

$$\left|\dot{\varphi}(2k+1)\right| \leqslant \left|\dot{\varphi}(1)\right| + (2e+d)k.$$

With $b = \max\{|\dot{\varphi}(0)|, |\dot{\varphi}(1)|\}$ and $a = 2e + d$ this shows that

$$\left|\dot{\varphi}(m)\right| \leqslant b + am,$$

and thus $|\varphi(m,n)| \leqslant b + a(m+n)$, which proves the proposition. □

Theorem 6.15. *If $\varphi : \mathbb{F}_\infty \to \mathbb{R}$ is a radial function such that $\|e^{-t\varphi}\|_{B_2} \leqslant 1$ for each $t > 0$, then there are constants $a, b \geqslant 0$ such that $\varphi(x) \leqslant b + a|x|$ for all $x \in \mathbb{F}_\infty$. Here $|x|$ denotes the word length function on $\mathbb{F}_\infty$.*

Proof. Suppose $\varphi : \mathbb{F}_\infty \to \mathbb{R}$ is a radial function such that $\|e^{-t\varphi}\|_{B_2} \leqslant 1$ for each $t > 0$, and let $\widetilde{\varphi}$ be as in Definition 6.1. First observe that $\widetilde{\varphi}$ is real, symmetric, and thus hermitian. From Proposition 6.13 we get that $\|\omega_{e^{-t\widetilde{\varphi}}}\|_{B_2} \leqslant 1$ for every $t > 0$, so Theorem 6.9 implies that $\widetilde{\varphi} \in \mathcal{S}$. Since $\widetilde{\varphi}$ is a Hankel function, Proposition 6.14 ensures that

$$\left|\widetilde{\varphi}(m,n)\right| \leqslant b + a(m+n) \quad \text{for all } m, n \in \mathbb{N}_0$$

for some constants a and b. This shows that

$$\varphi(x) \leqslant b + a|x| \quad \text{for all } x \in \mathbb{F}_\infty. \qquad \square$$

This finishes the proof of Theorem 1.6 in the case of the free group on infinitely many generators.

7. Radial semigroups of Herz–Schur multipliers on $\mathbb{F}_n$

The proof of Theorem 1.6 for the finitely generated free groups is more technical than the proof concerning $\mathbb{F}_\infty$, but the general approach is the same, and most of the steps in the proof can be deduced from what we have already done for $\mathbb{F}_\infty$. In order to do so we introduce the transformations F and G that, loosely speaking, translate between the two cases, the finite and the infinite.

7.1. The transformations F and G

From now on we fix a natural number q with $2 \leqslant q < \infty$. If the number of generators of the free group under consideration is n, we will let $q = 2n - 1$. The parametrization using q instead of n is adapted from [8]. As before, the free groups will enter the picture quite late (Theorem 7.19), and we will mainly focus on functions $\varphi : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{C}$.

We still denote the unilateral shift operator on $\ell^2(\mathbb{N}_0)$ by S. For each $m, n \in \mathbb{N}_0$ we let $S_{m,n}$ denote the operator

$$S_{m,n} = \left(1 - \frac{1}{q}\right)^{-1}\left(S^m\left(S^*\right)^n - \frac{1}{q}S^*S^m\left(S^*\right)^n S\right).$$

Observe that

$$S_{m,n} = \begin{cases} S^m(S^*)^n & \text{if } \min\{m,n\} = 0, \\ (1-\frac{1}{q})^{-1}(S^m(S^*)^n - \frac{1}{q}S^{m-1}(S^*)^{n-1}) & \text{if } m,n \geqslant 1, \end{cases} \tag{7.1}$$

and

$$S^m\left(S^*\right)^n = \left(1 - \frac{1}{q}\right)S_{m,n} + \frac{1}{q}S^{m-1}\left(S^*\right)^{n-1}, \quad \text{when } m,n \geqslant 1. \tag{7.2}$$

It follows by induction over $\min\{m,n\}$ that $S^m(S^*)^n \in \operatorname{span}\{S_{k,l} \mid k,l \in \mathbb{N}_0\}$ for all $m,n \in \mathbb{N}_0$, so $\operatorname{span}\{S_{k,l} \mid k,l \in \mathbb{N}_0\} = D$, where D is given by (6.1).

When $\varphi : \mathbb{N}_0^2 \to \mathbb{C}$ is a function we let χ_φ denote the linear functional defined on D by

$$\chi_\varphi(S_{m,n}) = \varphi(m,n),$$

and if it extends to a bounded functional on $C^*(S)$, we also denote the extension by χ_φ.

Let $\mathcal{V}$ be the set of kernels on $\mathbb{N}_0$, that is, $\mathcal{V} = \mathbb{C}^{\mathbb{N}_0 \times \mathbb{N}_0} = \{\varphi : \mathbb{N}_0^2 \to \mathbb{C}\}$. Then $\mathcal{V}$ is a vector space over $\mathbb{C}$ under pointwise addition and scalar multiplication. We equip $\mathcal{V}$ with the topology of pointwise convergence.

Recall that $\sigma : \mathbb{N}_0^2 \to \mathbb{N}_0^2$ is the shift map $\sigma(k,l) = (k+1, l+1)$. We now define operators τ and τ^* on $\mathcal{V}$. For $\varphi \in \mathcal{V}$ the operator τ^* is given by $\tau^*(\varphi) = \varphi \circ \sigma$, and $\tau(\varphi)$ given by

$$\tau(\varphi)(k,l) = \begin{cases} \varphi(k-1, l-1), & k,l \geqslant 1, \\ 0, & \min\{k,l\} = 0. \end{cases}$$

Then $\tau^* \circ \tau = \mathrm{id}$, and $(\tau \circ \tau^*)(\varphi) = 1_{\mathbb{N}\times\mathbb{N}}\varphi$. We have the following rules

$$\left(\tau \circ \tau^*\right)^2 = \tau \circ \tau^*, \qquad \tau^* \circ \tau \circ \tau^* = \tau^*, \qquad \tau \circ \tau^* \circ \tau = \tau.$$

Each element of $B(\ell^2(\mathbb{N}_0))$ may be identified with its matrix representation (with respect to the canonical orthonormal basis) and may in this way be considered as an element of $\mathcal{V}$. Under this identification τ and τ^* restrict to maps on $B(\ell^2(\mathbb{N}_0))$ given by $\tau(A) = SAS^*$ and $\tau^*(A) = S^*AS$. Clearly, τ is an isometry on the bounded operators. As noted in [8], it is also an isometry on the trace class operators $B_1(\ell^2(\mathbb{N}_0))$ (with respect to the trace norm). The operator

$$\left(1 - \frac{\tau}{\alpha}\right)^{-1} = \sum_{n=0}^{\infty} \frac{\tau^n}{\alpha^n}$$

on $B_1(\ell^2(\mathbb{N}_0))$ is therefore well-defined when $\alpha > 1$, and its norm is bounded by $(1-\frac{1}{\alpha})^{-1}$. To shorten notation we let

$$F = \left(1-\frac{1}{q}\right)\left(\mathrm{id}-\frac{\tau}{q}\right)^{-1} = \left(1-\frac{1}{q}\right)\sum_{n=0}^{\infty}\frac{\tau^n}{q^n}. \tag{7.3}$$

We note that F defined by (7.3) also makes sense as an invertible operator on $\mathcal{V}$ as $(\tau^n\varphi)(k,l) = 0$ for $n \gg 0$.

Let G be the operator on $\mathcal{V}$ defined recursively by

$$G\varphi(m,n) = \varphi(m,n) \quad \text{if } \min\{m,n\} = 0$$

and

$$G\varphi(m,n) = \left(1-\frac{1}{q}\right)\varphi(m,n) + \frac{1}{q}G\varphi(m-1,n-1), \quad \text{if } m,n \geqslant 1,$$

when $\varphi \in \mathcal{V}$. We can reconstruct φ from $G\varphi$. In fact $G^{-1}\varphi$ is given by

$$G^{-1}\varphi(k,l) = \begin{cases} \varphi(k,l), & \min\{k,l\} = 0, \\ (1-\frac{1}{q})^{-1}(\varphi(k,l) - \frac{1}{q}\varphi(k-1,l-1)), & k,l \geqslant 1. \end{cases} \tag{7.4}$$

We may express G^{-1} in terms of τ and τ^*. We have

$$G^{-1} = \mathrm{id} + \frac{1}{q-1}\left(\tau\circ\tau^* - \tau\right) = \left(\mathrm{id} + \frac{1}{q-1}\tau\circ\tau^*\right)\circ\left(\mathrm{id} - \frac{1}{q}\tau\right). \tag{7.5}$$

Using $(\tau\circ\tau^*)^n = \tau\circ\tau^*$ we obtain

$$\left(\mathrm{id} - \frac{1}{q}\tau\circ\tau^*\right)^{-1} = \mathrm{id} + \left(\frac{1}{q} + \frac{1}{q^2} + \cdots\right)\tau\circ\tau^* = \mathrm{id} + \frac{1}{q-1}\tau\circ\tau^*,$$

so it follows that

$$G = \left(\mathrm{id} - \frac{1}{q}\tau\right)^{-1}\circ\left(\mathrm{id} + \frac{1}{q-1}\tau\circ\tau^*\right)^{-1} = \left(\mathrm{id} - \frac{1}{q}\tau\right)^{-1}\circ\left(\mathrm{id} - \frac{1}{q}\tau\circ\tau^*\right).$$

We now record some elementary facts about F and G, which will be used later on without reference.

Lemma 7.1. *The transformations F and G are linear and continuous on $\mathcal{V}$. Furthermore, G takes the constant function* 1 *to itself. Also, $G\varphi(0,0) = \varphi(0,0)$ for every $\varphi \in \mathcal{V}$.*

Lemma 7.2. *Let $\varphi \in \mathcal{V}$, and fix $m,n \in \mathbb{N}$. If $\lim_{k\to\infty} G\varphi(m+k,n+k)$ exists, then $\lim_{k\to\infty}\varphi(m+k,n+k)$ exists, and the limits are equal.*

Proof. We may assume that $m, n \geqslant 1$. Suppose $\lim_k G\varphi(m+k, n+k)$ exists. Since

$$\varphi(m+k, n+k) = \left(1 - \frac{1}{q}\right)^{-1} \left(G\varphi(m+k, n+k) - \frac{1}{q} G\varphi(m+k-1, m+k-1)\right)$$

when $k \geqslant 1$, we see that the limit $\lim_k \varphi(m+k, n+k)$ exists and is equal to $\lim_k G\varphi(m+k, n+k)$. $\square$

Lemma 7.3. *The transformations F and G both take hermitian functions to hermitian functions. So do their inverses F^{-1} and G^{-1}.*

Proof. For G this easily follows from inspecting the definition. For G^{-1} simply look at (7.4). For F it suffices to note that τ^n preserves hermitian functions for each n, and hence does the sum in (7.3). For F^{-1} it suffices to note that τ preserves hermitian functions. $\square$

Lemma 7.4. *There is the following relationship between F and G.*

$$\left(1 - \tau^*\right) \circ G = F \circ \left(1 - \tau^*\right).$$

In other words, $G\varphi - G\varphi \circ \sigma = F(\varphi - \varphi \circ \sigma)$ for every $\varphi \in \mathcal{V}$.

Proof. It is equivalent to show $F^{-1} \circ (1 - \tau^*) = (1 - \tau^*) \circ G^{-1}$, and this is easy using (7.3) and (7.5). $\square$

Proposition 7.5. *For any $\varphi : \mathbb{N}_0^2 \to \mathbb{C}$ we have $\chi_\varphi = \omega_{G\varphi}$, where $\omega_{G\varphi}$ is as in (6.3).*

Proof. It is enough to show that χ_φ attains the value $G\varphi(m, n)$ at $S^m(S^*)^n$ for every $m, n \in \mathbb{N}_0$. Observe that if $\min\{m, n\} = 0$ we obviously have

$$\chi_\varphi\left(S^m\left(S^*\right)^n\right) = \chi_\varphi(S_{m,n}) = \varphi(m, n) = G\varphi(m, n).$$

Inductively, for $m, n \geqslant 1$ we get using (7.2) that

$$\begin{aligned}
\chi_\varphi\left(S^m\left(S^*\right)^n\right) &= \left(1 - \frac{1}{q}\right)\chi_\varphi(S_{m,n}) + \frac{1}{q}\chi_\varphi\left(S^{m-1}\left(S^*\right)^{n-1}\right) \\
&= \left(1 - \frac{1}{q}\right)\varphi(m, n) + \frac{1}{q} G\varphi(m-1, n-1) \\
&= G\varphi(m, n). \qquad \square
\end{aligned}$$

The following proposition is analogous to Proposition 6.12, and the proof is to deduce it from Proposition 6.12 by using transformations F and G.

Proposition 7.6. *Let $\varphi : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{C}$ be a function, and let $h = \varphi - \varphi \circ \sigma$. The functional χ_φ extends to a bounded functional on $C^*(S)$ if and only if h is of trace class, and the function $\varphi_0 : \mathbb{Z} \to \mathbb{C}$ given by $\varphi_0(m - n) = \lim_k \varphi(m + k, n + k)$ (which is then well-defined) lies in $B(\mathbb{Z})$. If this is the case, then*

$$\|\chi_\varphi\| = \|Fh\|_1 + \|\varphi_0\|_{B(\mathbb{Z})},$$

where F is the operator defined in (7.3).

Proof. From Proposition 7.5 we know that $\chi_\varphi = \omega_{G\varphi}$. From the characterization in Proposition 6.12 we deduce that χ_φ extends if and only if $G\varphi - G\varphi \circ \sigma$ is of trace class and the function φ_0' given by

$$\varphi_0'(m-n) = \lim_{k\to\infty} G\varphi(m+k, n+k) \quad (m, n \in \mathbb{N}_0)$$

lies in $B(\mathbb{Z})$. Recall (Lemma 7.4) that $G\varphi - G\varphi \circ \sigma = F(\varphi - \varphi \circ \sigma) = Fh$, and h is of trace class if and only if $F(h)$ is of trace class. Also from Lemma 7.2 we see that $\varphi_0' = \varphi_0$.

It remains to show the norm equality. We have from Proposition 6.12

$$\|\chi_\varphi\| = \|\omega_{G\varphi}\| = \|Fh\|_1 + \left\|\varphi_0'\right\|_{B(\mathbb{Z})},$$

and this completes to proof. □

7.2. Relation between multipliers and functionals

Similarly to how we defined χ_φ as the analogue of ω_φ we will now define N_φ as the analogue of M_φ. More precisely, let N_φ be the linear map defined on D by

$$N_\varphi(S_{m,n}) = \varphi(m, n) S_{m,n}.$$

It may or may not happen that N_φ extends to $C^*(S)$, and if it does we will also denote the extension by N_φ. It turns out that this happens exactly when χ_φ extends (see Proposition 7.10), but the proof is not as easy as the case with M_φ and ω_φ. The reason is that there is no $*$-homomorphism $\alpha : C^*(S) \to C^*(S \otimes S)$ that maps $S_{m,n}$ to $S_{m,n} \otimes S_{m,n}$. So we cannot directly follow the approach of Remark 6.2.

Observe that $\chi_\varphi = \mathrm{ev}_1 \circ \pi \circ N_\varphi$, where ev_1 and π are as in Remark 6.2. Hence $\|\chi_\varphi\| \leqslant \|N_\varphi\|$. We will now prove the reverse inequality (Proposition 7.10). The proof is partly contained in the proof of Theorem 2.3 in [8], so we will refer to that proof and emphasize the differences.

The overall strategy of our proof is the following. We find an isometry U on a Hilbert space $\ell^2(X)$ and a function $\widetilde{\varphi} : X \times X \to \mathbb{C}$ such that $\widetilde{\varphi}$ is a Schur multiplier. We construct them in such a way that we may find a $*$-isomorphism β between $C^*(S)$ and $C^*(U)$ such that $N_\varphi = \beta^{-1} \circ m_{\widetilde{\varphi}} \circ \beta$, where $m_{\widetilde{\varphi}}$ is the multiplier corresponding to $\widetilde{\varphi}$. The construction is similar to the one in [8].

Let X be a homogeneous tree of degree $q + 1$, i.e., each vertex has degree $q + 1$. We will identify the vertex set with X. We fix an infinite, non-returning path $\omega = (x_0, x_1, x_2, \dots)$ in X, i.e., $x_i \neq x_j$ when $i \neq j$. Define the map $c : X \to X$ such that for any $x \in X$ the sequence $x, c(x), c^2(x), \dots$ is the unique infinite, non-returning path eventually following ω. This path is denoted $[x, \omega[$. Visually, c is the “contraction” of the tree towards the boundary point ω.

Definition 7.7. Observe that for each pair of vertices $x, y \in X$ there are smallest numbers $m, n \in \mathbb{N}_0$ such that $c^m(x) \in [y, \omega[$ and $c^n(y) \in [x, \omega[$. When we need to keep track of more than two points at a time, we denote m and n by $m(x, y)$ and $n(x, y)$ respectively.

Given a function $\varphi : \mathbb{N}_0^2 \to \mathbb{C}$ we define the function $\widetilde{\varphi} : X \times X \to \mathbb{C}$ by

$$\widetilde{\varphi}(x, y) = \varphi(m, n).$$

Lemma 7.8. *Using the notation of Definition 7.7 we have*

$$m(x, y) - n(x, y) = m(x, z) - n(x, z) + m(z, y) - n(z, y)$$

for every $x, y, z \in X$.

Proof. Let $v \in X$ be a point sufficiently far out in ω such that v lies beyond the following three points:

$$c^{m(x,y)}(x) = c^{n(x,y)}(y), \qquad c^{m(x,z)}(x) = c^{n(x,z)}(z), \qquad c^{m(z,y)}(z) = c^{n(z,y)}(y).$$

If we let d denote the graph distance, then

$$\begin{aligned} m(x, y) - n(x, y) &= m(x, y) + d\big(c^{m(x,y)}(x), v\big) - d\big(c^{n(x,y)}(y), v\big) - n(x, y) \\ &= d(x, v) - d(y, v), \end{aligned} \tag{7.6}$$

and similarly

$$m(x, z) - n(x, z) = d(x, v) - d(z, v), \qquad m(z, y) - n(z, y) = d(z, v) - d(y, v). \tag{7.7}$$

The lemma now follows by combining (7.6) and (7.7). □

Lemma 7.9. *Let $\varphi_0 \in B(\mathbb{Z})$ be given, and let $\varphi(m, n) = \varphi_0(m - n)$. The function $\widetilde{\varphi}$ from Definition 7.7 is a Schur multiplier, and*

$$\|\widetilde{\varphi}\|_S \leqslant \|\varphi_0\|_{B(\mathbb{Z})}.$$

Proof. It is enough to prove the lemma when $\|\varphi_0\|_{B(\mathbb{Z})} \leqslant 1$. Write φ_0 in the form

$$\varphi_0(n) = \int_{\mathbb{T}} z^n \, d\mu(z), \quad n \in \mathbb{Z},$$

for some complex Radon measure μ on $\mathbb{T}$. First assume that $\mu = \delta_s$ for some $s \in \mathbb{T}$, so that φ_0 is of the form $\varphi_0(n) = s^n$. Then φ_0 is a group homomorphism, so from Lemma 7.8 we get

$$\widetilde{\varphi}(x, y) = \widetilde{\varphi}(x, z)\widetilde{\varphi}(z, y), \quad x, y, z \in X.$$

In particular, if we fix some vertex, say x_0 from before, we have

$$\widetilde{\varphi}(x, y) = \widetilde{\varphi}(x, x_0)\overline{\widetilde{\varphi}(y, x_0)},$$

so $\widetilde{\varphi}$ is a positive definite kernel on X. Since also $\widetilde{\varphi}(x,x) = 1$ for every $x \in X$, it follows that $\widetilde{\varphi}$ is a Schur multiplier with norm at most 1 (see [4, Theorem D.3]).

It follows that if μ lies in the set $C = \operatorname{conv}\{c\delta_s \mid c, s \in \mathbb{T}\}$, then $\widetilde{\varphi}$ is a Schur multiplier of norm at most 1.

Now, let μ in $M(\mathbb{T})_1$ be arbitrary. It follows from the Hahn–Banach Theorem that the vague closure of the set C is $M(\mathbb{T})_1$, so there is a net $(\mu_\alpha)_{\alpha\in A}$ in C such that $\mu_\alpha \to \mu$ vaguely, that is,

$$\int_{\mathbb{T}} f\, d\mu_\alpha \to \int_{\mathbb{T}} f\, d\mu \quad \text{as } \alpha \to \infty$$

for each $f \in C(\mathbb{T})$. In particular,

$$\int_{\mathbb{T}} z^n\, d\mu_\alpha(z) \to \varphi_0(n) \quad \text{as } \alpha \to \infty,$$

so $\widetilde{\varphi}$ is the pointwise limit of Schur multipliers with norm at most 1. The proof is now complete, since the Schur multipliers of norm at most 1 are closed under pointwise convergence. □

Let U be the operator on $\ell^2(X)$ defined by

$$U\delta_x = \frac{1}{\sqrt{q}} \sum_{c(z)=x} \delta_z,$$

where $(\delta_x)_{x\in X}$ are the standard basis vectors in $\ell^2(X)$, and let $U_{m,n}$ be defined similarly to how we defined $S_{m,n}$:

$$U_{m,n} = \left(1 - \frac{1}{q}\right)^{-1} \left(U^m (U^*)^n - \frac{1}{q} U^* U^m (U^*)^n U\right).$$

It is shown in [8] that U is a proper isometry. Also if $\widetilde{\varphi}$ is a Schur multiplier, then $C^*(U)$ is invariant under $m_{\widetilde{\varphi}}$, and

$$m_{\widetilde{\varphi}}(U_{m,n}) = \varphi(m,n) U_{m,n}.$$

Actually the authors only state the mentioned result under the assumption that $\varphi(m,n)$ depends only on $m+n$, but the proof without this assumption is exactly the same (see Lemma 2.6 and Corollary 2.7 in [8]).

Since U is a proper isometry, there is a $*$-isomorphism $\beta : C^*(S) \to C^*(U)$ such that $\beta(S) = U$. It follows that $\beta(S_{m,n}) = U_{m,n}$ for all $m, n \in \mathbb{N}_0$.

Proposition 7.10. *Let $\varphi : \mathbb{N}_0^2 \to \mathbb{C}$ be a function. Then χ_φ extends to a bounded functional on $C^*(S)$ if and only if N_φ extends to a (completely) bounded map on $C^*(S)$, and in this case*

$$\|N_\varphi\| = \|\chi_\varphi\|.$$

Proof. As mentioned earlier it suffices to prove that if χ_φ extends to a bounded functional on $C^*(S)$, then N_φ extends to a bounded map on $C^*(S)$ as well, and $\|N_\varphi\| \leqslant \|\chi_\varphi\|$, since the other direction has already been taken care of.

Suppose χ_φ extends to $C^*(S)$. From Proposition 7.6 we know that $h = \varphi - \varphi \circ \sigma$ is of trace class, and the function $\varphi_0 : \mathbb{Z} \to \mathbb{C}$ given by $\varphi_0(m-n) = \lim_k \varphi(m+k, n+k)$ is well-defined and lies in $B(\mathbb{Z})$. Also

$$\|\chi_\varphi\| = \|Fh\|_1 + \|\varphi_0\|_{B(\mathbb{Z})}.$$

Let $\psi(m,n) = \varphi(m,n) - \varphi_0(m-n)$, and notice that

$$\psi(m,n) = \sum_{k=0}^{\infty} h(m+k, n+k) = \operatorname{Tr}\bigl(S_{m,n}(Fh)\bigr) \tag{7.8}$$

by [8, Lemma 2.2]. In the proof of [8, Theorem 2.3] it is shown that there are maps $P_k, Q_k : X \to \ell^2(X)$ such that if m, n are chosen as in Definition 7.7, then

$$\sum_{k=0}^{\infty} \bigl\langle P_k(x), Q_k(y)\bigr\rangle = \operatorname{Tr}\bigl(S_{n,m}(Fh)\bigr) \quad (x, y \in X), \tag{7.9}$$

and

$$\sum_{k=0}^{\infty} \|P_k\|_\infty \|Q_k\|_\infty = \|Fh\|_1. \tag{7.10}$$

We set

$$\widetilde{\varphi}(x,y) = \varphi(m,n) \quad \text{and} \quad \widetilde{\psi}(x,y) = \psi(m,n)$$

as in Definition 7.7. Combining (7.8), (7.9) and (7.10) we see that the function $(x,y) \mapsto \widetilde{\psi}(y,x) = \psi(n,m)$ is a Schur multiplier on X with norm at most $\|Fh\|_1$. Hence $\widetilde{\psi}$ is also a Schur multiplier with norm at most $\|Fh\|_1$.

We have

$$\widetilde{\varphi}(x,y) = \widetilde{\psi}(x,y) + \varphi_0(m-n),$$

so by using Lemma 7.9 we see that $\widetilde{\varphi}$ is a Schur multiplier with

$$\|\widetilde{\varphi}\|_S \leqslant \|Fh\|_1 + \|\varphi_0\|_{B(\mathbb{Z})}.$$

By definition, $\|m_{\widetilde{\varphi}}\| = \|\widetilde{\varphi}\|_S$, and since $N_\varphi = \beta^{-1} \circ m_{\widetilde{\varphi}} \circ \beta$, we conclude that N_φ is completely bounded with

$$\|N_\varphi\| \leqslant \|m_{\widetilde{\varphi}}\| \leqslant \|Fh\|_1 + \|\varphi_0\|_{B(\mathbb{Z})} = \|\chi_\varphi\|. \qquad \square$$

7.3. Positive and conditionally negative functions

Our next goal is to prove analogues of Propositions 6.3 and 6.6. The following proposition characterizes the functions φ that induce states on the Toeplitz algebra.

Proposition 7.11. *Let $\varphi : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{C}$ be a function. Then the following are equivalent.*

(1) *The functional χ_φ extends to a state on the Toeplitz algebra $C^*(S)$.*
(2) *The multiplier N_φ extends to a u.c.p. map on the Toeplitz algebra $C^*(S)$.*
(3) *The functions $F(\varphi - \varphi \circ \sigma)$ and $G\varphi$ are positive definite, and $\varphi(0,0) = 1$.*

Proof. The equivalence (1) $\Longleftrightarrow$ (3) follows from Proposition 6.3 and Proposition 7.5 together with Lemma 7.4. The implication (2) $\Longrightarrow$ (1) follows from the equality $\chi_\varphi = \mathrm{ev}_1 \circ \pi \circ N_\varphi$. So we will only be concerned with (1) $\Longrightarrow$ (2).

Assume χ_φ extends to a state on $C^*(S)$. Following the proof of Proposition 7.10 we see that $F(\varphi - \varphi \circ \sigma)$ is the matrix of a trace class operator, and it is also positive definite. Going through the proof of [8, Theorem 2.3] we make the following observation. When $Fh = F(\varphi - \varphi \circ \sigma)$ is positive definite, we may choose $P_k = Q_k$, and $\widetilde{\varphi}$ becomes a positive definite kernel on the tree X. Since $\widetilde{\varphi}(x,x) = \varphi(0) = 1$ for every $x \in X$, we deduce that $m_{\widetilde{\varphi}}$ is u.c.p. Finally, $N_\varphi = \beta^{-1} \circ m_{\widetilde{\varphi}} \circ \beta$ is also u.c.p., where β is the $*$-isomorphism from $C^*(S) \to C^*(U)$ from before. □

Corollary 7.12. *Let $\theta : \mathbb{N}_0^2 \to \mathbb{C}$ be a function. If $G(\theta - \frac{1}{2}\theta(0,0))$ is positive definite, and $F(\theta - \theta \circ \sigma)$ is positive definite, then*

$$\|N_{e^{-t\theta}}\| \leqslant 1 \quad \text{for every } t > 0.$$

Proof. Observe first that for any $\varphi : \mathbb{N}_0^2 \to \mathbb{C}$ we have $\|N_{e^{-t\varphi}}\| \leqslant e^{t\|N_\varphi\|}$.

Let $\varphi = \theta - \frac{1}{2}\theta(0,0)$. From Proposition 7.11 we deduce that $\|N_\varphi\| = \varphi(0,0)$. Since $\theta = \varphi + \varphi(0,0)$ we find

$$\|N_{e^{-t\theta}}\| = e^{-t\varphi(0,0)} \|N_{e^{-t\varphi}}\| \leqslant e^{-t\varphi(0,0)} e^{t\|N_\varphi\|} = 1. \qquad \square$$

Proposition 7.13. *Let $\psi : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{C}$ be a function. Then the following are equivalent.*

(1) *For all $t > 0$ the function $e^{-t\psi}$ satisfies the equivalent conditions in Proposition 7.11.*
(2) *$G\psi$ is a conditionally negative definite kernel, $F(\psi \circ \sigma - \psi)$ is a positive definite kernel, and $\psi(0,0) = 0$.*

Proof. (1) $\Longrightarrow$ (2): Assume that condition (3) of Proposition 7.11 holds for $e^{-t\psi}$ for each $t > 0$. Then $Ge^{-t\psi}$ is positive definite. It follows that $1 - Ge^{-t\psi}$ is a conditionally negative definite kernel, and therefore so is the pointwise limit

$$\lim_{t \to 0} \frac{1 - Ge^{-t\psi}}{t} = \lim_{t \to 0} G\left(\frac{1 - e^{-t\psi}}{t}\right) = G\psi.$$

Since $e^{-t\psi(0,0)} = 1$, we get $\psi(0,0) = 0$. Moreover, $F(e^{-t\psi} - e^{-t\psi \circ \sigma})$ is positive definite by assumption. It follows that the pointwise limit

$$\lim_{t\to 0} F\left(\frac{e^{-t\psi} - e^{-t\psi\circ\sigma}}{t}\right) = F(\psi\circ\sigma - \psi)$$

is positive definite.

(2) $\Longrightarrow$ (1): First we note that the set of functions φ satisfying the equivalent conditions of Proposition 7.11 is closed under products and pointwise limits. Stability under products is most easily established using condition (2), while closure under pointwise limits is most easily seen in condition (3).

By assumption $G\psi$ is conditionally negative definite, so the function $e^{-sG\psi}$ is positive definite for each $s > 0$. We let

$$\rho_s = G^{-1}\left(\frac{e^{-sG\psi}}{s}\right),$$

so that $G\rho_s$ is positive definite, and

$$\frac{1}{s} - \rho_s \to G^{-1}G\psi = \psi \tag{7.11}$$

pointwise as $s \to 0$. Using Lemma 7.4 we see that $F(\rho_s - \rho_s\circ\sigma) = G\rho_s - (G\rho_s)\circ\sigma$, so

$$F(\rho_s - \rho_s\circ\sigma) = \frac{(e^{-sG\psi} - e^{-s(G\psi)\circ\sigma})}{s} = \frac{e^{-s(G\psi)\circ\sigma}(e^{s((G\psi)\circ\sigma - G\psi)} - 1)}{s}.$$

Since $G\psi$ is conditionally negative definite, so is $(G\psi)\circ\sigma$, and hence $e^{-s(G\psi)\circ\sigma}$ is positive definite. Expanding the exponential function gives

$$e^{s((G\psi)\circ\sigma - G\psi)} - 1 = \sum_{n=1}^{\infty} \frac{(sF(\psi\circ\sigma - \psi))^n}{n!}.$$

Since $F(\psi\circ\sigma - \psi)$ is positive definite, so are its powers and hence the sum in the above equation. It follows that $F(\rho_s - \rho_s\circ\sigma)$ is a product of two positive definite functions and hence itself positive definite.

Looking at Proposition 7.11 we see that N_{ρ_s} is completely positive. It follows that $N_{e^{t\rho_s}}$ is completely positive, so $e^{-t(\frac{1}{s}-\rho_s)}$ satisfies the conditions of Proposition 7.11. Finally, since

$$e^{-t\psi} = \lim_{s\to 0} e^{-t(\frac{1}{s}-\rho_s)},$$

we conclude that $e^{-t\psi}$ satisfies the conditions of Proposition 7.11. □

7.4. The linear bound

As in the case of $\mathbb{F}_\infty$ we prove that if a kernel φ satisfies $\|\chi_{e^{-t\varphi}}\| \leqslant 1$ for every $t > 0$, then it splits in a useful way. We are also able to compare norms of radial Herz–Schur multipliers on $\mathbb{F}_n$ with norms of functionals on the Toeplitz algebra. These are Theorem 7.16 and Proposition 7.20.

Recall the definition of the set $\mathcal{S}$ from Definition 6.7.

Lemma 7.14. *Let $\varphi : \mathbb{N}_0^2 \to \mathbb{C}$ be a function. If $G\varphi \in \mathcal{S}$, then*

$$\|N_{e^{-t\varphi}}\| \leqslant 1 \quad \text{for every } t > 0.$$

Proof. Suppose $G\varphi \in \mathcal{S}$ and write $G\varphi = G\psi + G\theta$, where

- $G\psi$ is a conditionally negative definite kernel with $G\psi(0,0) = 0$,
- $F(\psi \circ \sigma - \psi)$ is a positive definite kernel,
- $G(\theta - \frac{1}{2}\theta(0,0))$ is a positive definite kernel,
- $F(\theta - \theta \circ \sigma)$ is a positive definite kernel.

Then also $\varphi = \psi + \theta$. From Proposition 7.13 we get that $N_{e^{-t\psi}}$ is a u.c.p. map for every $t > 0$, and hence $\|N_{e^{-t\psi}}\| = 1$. Also, from Corollary 7.12 we get that $\|N_{e^{-t\theta}}\| \leqslant 1$ for every $t > 0$. This combines to show

$$\|N_{e^{-t\varphi}}\| \leqslant \|N_{e^{-t\psi}}\| \|N_{e^{-t\theta}}\| \leqslant 1. \qquad \square$$

Lemma 7.15. *Let $\varphi : \mathbb{N}_0^2 \to \mathbb{C}$ be a hermitian function. If $\|\chi_\varphi\| \leqslant 1$, then $1 - G\varphi \in \mathcal{S}$.*

Proof. Use Proposition 7.5 together with Lemma 6.10 and Lemma 7.3. □

Theorem 7.16. *Let $\varphi : \mathbb{N}_0^2 \to \mathbb{C}$ be a hermitian function. Then $G\varphi \in \mathcal{S}$ if and only if $\|\chi_{e^{-t\varphi}}\| \leqslant 1$ for every $t > 0$.*

Proof. Suppose $\|\chi_{e^{-t\varphi}}\| \leqslant 1$ for every $t > 0$. From the previous lemma we see that $G(1 - e^{-t\chi})$ lies in $\mathcal{S}$ for every $t > 0$. Hence, so does $G(1 - e^{-t\chi})/t$ which converges pointwise to $G\chi$ as $t \to 0$. It now follows from Lemma 6.8 that $G\chi \in \mathcal{S}$.

The converse direction is Lemma 7.14 combined with Proposition 7.10. □

Lemma 7.17. *Let $h \in \mathcal{V}$. Then h is bounded if and only if $F(h)$ is bounded.*

Proof. Let $h \in \mathcal{V}$ be bounded. We prove that $F(h)$ and $F^{-1}(h)$ are bounded. This will complete the proof.

Observe that $\tau(h)$ is bounded with the same bound as h. Then $(\mathrm{id} - \tau/q)(h)$ is bounded, so

$$F^{-1}(h) = \left(1 - \frac{1}{q}\right)^{-1} \left(\mathrm{id} - \frac{\tau}{q}\right)(h)$$

is also bounded.

Suppose $c \geqslant 0$ is a bound for h. Using (7.3) we find

$$\left|Fh(m,n)\right| \leqslant \left(1 - \frac{1}{q}\right) \sum_{k=0}^{\infty} \frac{|\tau^k(h)(m,n)|}{q^k} \leqslant \left(1 - \frac{1}{q}\right) \sum_{k=0}^{\infty} \frac{c}{q^k} = c.$$

This proves that $F(h)$ is bounded as well. □

Proposition 7.18. *If φ is a Hankel function, and $G\varphi \in \mathcal{S}$, then φ is linearly bounded.*

Proof. Suppose $\varphi \in \mathcal{V}$ is a Hankel function, and $G\varphi \in \mathcal{S}$. Let $h = \varphi \circ \sigma - \varphi$. We wish to prove that h is bounded, since this will give the desired bound on φ. Observe that h is also a Hankel function. We write $h(m,n) = \dot{h}(m+n)$ for some function $\dot{h} : \mathbb{N}_0 \to \mathbb{C}$.

Since $G\varphi$ lies in $\mathcal{S}$, there is a splitting of the form $G\varphi = G\psi + G\theta$, where

- $G\psi$ is a conditionally negative definite kernel with $G\psi(0,0) = 0$,
- $F(\psi \circ \sigma - \psi)$ is a positive definite kernel,
- $G(\theta - \frac{1}{2}\theta(0,0))$ is a positive definite kernel,
- $F(\theta - \theta \circ \sigma)$ is a positive definite kernel.

Write $h_1 = \psi \circ \sigma - \psi$ and $h_2 = \theta - \theta \circ \sigma$, and observe that $h = h_1 - h_2$. By the above, Corollary 6.4 and Proposition 6.6 we see that there are vectors ξ_k, η_k in a Hilbert space such that

$$Fh_1(m,n) = \langle \eta_m, \eta_n \rangle, \qquad Fh_2(m,n) = \langle \xi_m, \xi_n \rangle,$$

for all $m, n \in \mathbb{N}_0$ and

$$\sum_{k=0}^{\infty} \|\eta_k - \eta_{k+1}\|^2 < \infty, \qquad \sum_{k=0}^{\infty} \|\xi_k\|^2 < \infty.$$

From this we see that Fh_2 is the matrix of a positive trace class operator, and from Lemma 7.17 we see that h_2 is a bounded function. Also, there is $c > 0$ such that $\|\eta_k - \eta_{k+1}\| \leqslant c$ for every k and $\|\eta_0\| \leqslant c$, and so we get the linear bound $\|\eta_k\| \leqslant c(k+1)$. We may even choose c such that $|h_2(m,n)| \leqslant c^2$ for every $m, n \in \mathbb{N}_0$. From the Cauchy–Schwarz inequality we get

$$\left|Fh_1(m,n)\right| \leqslant c^2(m+1)(n+1).$$

We remark that $Fh(0,n) = (1 - \frac{1}{q})h(0,n)$, so the above with $m = 0$ gives us

$$\left|h_1(0,n)\right| = \left(1 - \frac{1}{q}\right)^{-1} \left|Fh_1(0,n)\right| \leqslant 2c^2(n+1).$$

Putting all this together gives the linear bound

$$\left|\dot{h}(n)\right| = \left|h_1(0,n) - h_2(0,n)\right| \leqslant 2c^2(n+1) + c^2 \leqslant 2c^2(n+2). \tag{7.12}$$

Since $Fh_2 \leqslant \|Fh_2\| I \leqslant \|Fh_2\|_1 I = \|h_2\|_1 I$ (as positive definite matrices, where I is the identity operator), we deduce that the function

$$Fh(m,n) + \|h_2\|_1 \delta_{mn} = Fh_1(m,n) + \left(\|h_2\|_1 \delta_{mn} - Fh_2(m,n)\right)$$

is positive definite. By the Cauchy–Schwarz inequality we have

$$\left|Fh(0,n)\right|^2 \leqslant \left(Fh(0,0) + \|h_2\|_1\right)\left(Fh(n,n) + \|h_2\|_1\right)$$

for every $n \geqslant 1$, and hence

$$\left|\dot{h}(n)\right|^2 \leqslant e\bigl(Fh(n,n) + \|h_2\|_1\bigr), \tag{7.13}$$

where we, in order to shorten notation, have put

$$e = \left(1 - \frac{1}{q}\right)^{-2}\bigl(Fh(0,0) + \|h_2\|_1\bigr).$$

If e is zero, then clearly $\dot{h}(k) = 0$ when $k \geqslant 1$. Suppose $e > 0$. Then from (7.13) we get

$$\begin{aligned}
\frac{|\dot{h}(n)|^2}{e} - \|h_2\|_1 &\leqslant \sum_{k=0}^{n} \frac{h(n-k, n-k)}{q^k} \\
&= \sum_{k=0}^{n} \frac{\dot{h}(2n-2k)}{q^k} \\
&\leqslant \frac{q}{q-1} \max\bigl\{\bigl|\dot{h}(2n-2k)\bigr| \bigm| 0 \leqslant k \leqslant n\bigr\} \\
&\leqslant 2\max\bigl\{\bigl|\dot{h}(2k)\bigr| \bigm| 0 \leqslant k \leqslant n\bigr\}.
\end{aligned}$$

In particular we have the following useful observation. Let $a = 2e$ and $b = \|h_2\|_1/2$. Then for each $n \in \mathbb{N}$ there is a $k \leqslant 2n$ such that

$$\bigl|\dot{h}(k)\bigr| \geqslant \frac{|\dot{h}(n)|^2}{a} - b.$$

We will now show that $|\dot{h}(n)| \leqslant 2a + b$ for every n. Suppose by contradiction that $|\dot{h}(n_0)| > 2a + b$ for some n_0. We claim that this assumption will lead to the following. For each $m \in \mathbb{N}_0$ there is a $k_m \leqslant 2^{m+1} n_0$ such that

$$\bigl|\dot{h}(k_m)\bigr| \geqslant 2^{2^m}(2a + b). \tag{7.14}$$

We prove this by induction over m. From our observation above we get that there is a $k_0 \leqslant 2n_0$ such that

$$\bigl|\dot{h}(k_0)\bigr| \geqslant \frac{|\dot{h}(n_0)|^2}{a} - b \geqslant 4a + \frac{b^2}{a} + 4b - b \geqslant 2(2a + b),$$

and so (7.14) holds for $m = 0$.

Assume that we have found $k_0, \dots, k_m$. Using our observation we may find $k_{m+1} \leqslant 2k_m$ such that

$$
\begin{aligned}
\left|\dot h(k_{m+1})\right| &\geqslant \frac{|\dot h(k_m)|^2}{a} - b \geqslant \frac{(2^{2^m}(2a+b))^2}{a} - b \\
&= 2^{2^{m+1}} \frac{4a^2+b^2+4ab}{a} - b \geqslant 2^{2^{m+1}}(4a+4b) - b \\
&\geqslant 2^{2^{m+1}}(2a+b).
\end{aligned}
$$

Finally, note that $k_{m+1} \leqslant 2k_m \leqslant 2(2^{m+1}n_0) = 2^{m+2}n_0$ as desired. This proves (7.14). But clearly (7.14) is in contradiction with (7.12), and so we conclude that $|\dot h(n)| \leqslant 2a+b$ for all $n \in \mathbb{N}_0$. This proves that $\dot h$ is bounded, and hence φ is linearly bounded. □

In [8] the following theorem is proved (Theorem 5.2).

Theorem 7.19. *Let $\mathbb{F}_n$ be the free group on n generators ($2 \leqslant n < \infty$), let $\varphi : \mathbb{F}_n \to \mathbb{C}$ be a radial function, and let $\dot\varphi : \mathbb{N}_0 \to \mathbb{C}$ be as in Definition 6.1. Finally, let $h = (h_{ij})_{i,j\in\mathbb{N}_0}$ be the Hankel matrix given by $h_{ij} = \dot\varphi(i+j) - \dot\varphi(i+j+2)$ for $i, j \in \mathbb{N}_0$. Then the following are equivalent:*

(i) *φ is a Herz–Schur multiplier on $\mathbb{F}_n$,*
(ii) *h is of trace class.*

If these two equivalent conditions are satisfied, then there exist unique constants $c_\pm \in \mathbb{C}$ and a unique $\dot\psi : \mathbb{N}_0 \to \mathbb{C}$ such that

$$\dot\varphi(k) = c_+ + c_-(-1)^k + \dot\psi(k) \quad (k \in \mathbb{N}_0) \tag{7.15}$$

and

$$\lim_{k\to\infty} \dot\psi(k) = 0.$$

Moreover, with $q = 2n-1$

$$\|\varphi\|_{B_2} = |c_+| + |c_-| + \|Fh\|_1,$$

where F is the operator defined by (7.3).

Proposition 7.20. *If $\varphi : \mathbb{F}_n \to \mathbb{C}$ is a radial function, and $\widetilde\varphi$ is as in Definition 6.1, then $\|\chi_{\widetilde\varphi}\| = \|\varphi\|_{B_2}$.*

Proof. Let $h_{ij} = \widetilde\varphi(i,j) - \widetilde\varphi(i+1,j+1)$. From Theorem 7.19 and Proposition 7.6 we see that it suffices to consider the case where h is the matrix of a trace class operator, since otherwise $\|\chi_{\widetilde\varphi}\| = \|\varphi\|_{B_2} = \infty$. If h is of trace class, then we let $\widetilde\varphi_0(n) = \lim_k \widetilde\varphi(k+n,k)$. From Theorem 7.19 and Proposition 7.6 it follows that

$$
\begin{aligned}
\|\chi_{\widetilde\varphi}\| &= \|Fh\|_1 + \|\widetilde\varphi_0\|_{B(\mathbb{Z})}, \\
\|\varphi\|_{B_2} &= \|Fh\|_1 + |c_+| + |c_-|,
\end{aligned}
$$

where $c_\pm$ are the constants obtained in Theorem 7.19. It follows from (7.15) that

$$\widetilde{\varphi}_0(n) = c_+ + (-1)^n c_-.$$

Now we only need to see why $|c_+| + |c_-| = \|\widetilde{\varphi}_0\|_{B(\mathbb{Z})}$.

Let $\nu \in C(\mathbb{T})^*$ be given by $\nu(f) = c_+ f(1) + c_- f(-1)$ for all $f \in C(\mathbb{T})$. Observe that $\nu(z \mapsto z^n) = c_+ + (-1)^n c_-$. Hence ν corresponds to $\widetilde{\varphi}_0$ under the isometric isomorphism $B(\mathbb{Z}) \simeq C(\mathbb{T})^*$. Hence,

$$\|\widetilde{\varphi}_0\|_{B(\mathbb{Z})} = \|\nu\| = |c_+| + |c_-|.$$

This completes the proof. □

Theorem 7.21. *If $\varphi : \mathbb{F}_n \to \mathbb{R}$ is a radial function such that $\|e^{-t\varphi}\|_{B_2} \leqslant 1$ for each $t > 0$, then there are constants $a, b \geqslant 0$ such that $\varphi(x) \leqslant b + a|x|$ for all $x \in \mathbb{F}_n$. Here $|x|$ denotes the word length function on $\mathbb{F}_n$.*

Proof. Suppose $\varphi : \mathbb{F}_n \to \mathbb{R}$ is a radial function such that $\|e^{-t\varphi}\|_{B_2} \leqslant 1$ for each $t > 0$, and let $\widetilde{\varphi}$ be as in Definition 6.1. First observe that $\widetilde{\varphi}$ is real and symmetric, and hence hermitian. From Proposition 7.20 we get that $\|\chi_{e^{-t\widetilde{\varphi}}}\| \leqslant 1$ for every $t > 0$, so Theorem 7.16 implies that $G(\widetilde{\varphi}) \in \mathcal{S}$. Since $\widetilde{\varphi}$ is a Hankel function, Proposition 7.18 ensures that

$$\left|\widetilde{\varphi}(m,n)\right| \leqslant b + a(m+n) \quad \text{for all } m, n \in \mathbb{N}_0$$

for some constants a and b. This implies that

$$\varphi(x) \leqslant b + a|x| \quad \text{for all } x \in \mathbb{F}_n. \qquad \square$$

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ARTICLE B

The weak Haagerup property

This chapter contains the preprint version of the following article:

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THE WEAK HAAGERUP PROPERTY

SØREN KNUDBY

ABSTRACT. We introduce the weak Haagerup property for locally compact groups and prove several hereditary results for the class of groups with this approximation property. The class contains a priori all weakly amenable groups and groups with the usual Haagerup property, but examples are given of groups with the weak Haagerup property which are not weakly amenable and do not have the Haagerup property.

In the second part of the paper we introduce the weak Haagerup property for finite von Neumann algebras, and we prove several hereditary results here as well. Also, a discrete group has the weak Haagerup property if and only if its group von Neumann algebra does.

Finally, we give an example of two II_1 factors with different weak Haagerup constants.

CONTENTS

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1. INTRODUCTION

In connection with the famous Banach-Tarski paradox, the notion of amenability was introduced by von Neumann [57], and since then the theory of amenable groups has grown into a huge research area in itself (see the book [49]). Today, we know that amenable groups can be characterized in many different ways, one of which is the following. A locally compact group G is amenable if and only if there is a net $(u_\alpha)_{\alpha\in A}$ of continuous compactly supported positive definite functions on G such that $u_\alpha \to 1$ uniformly on compact subsets of G (see [49, Chap. 2, Sec. 8]). When formulated like this, amenability is viewed as an approximation property, and over the years several other (weaker) approximation properties resembling amenability have been studied. For a combined treatment of the study of such approximation properties we refer to [7, Chapter 12]. We mention some approximation properties below and relate them to each other (see Figure 1).

Recall that a locally compact group G is *weakly amenable*, if there is a net (u_α) of compactly supported Herz-Schur multipliers on G, uniformly bounded in Herz-Schur norm, such that $u_\alpha \to 1$ uniformly on compacts. The least uniform bound on the norms of such nets (if such a bound exists at all) is the weak amenability constant of G. We denote the weak amenability constant (also called the Cowling-Haagerup constant) by $\Lambda_{\mathrm{WA}}(G)$. The notation Λ_G and $\Lambda_{\mathrm{cb}}(G)$ for the weak amenability constant is also found in the literature. For the definition of Herz-Schur multipliers and the Herz-Schur norm we refer to Section 3, but let us mention here that any (normalized) positive definite function on the group G is a Herz-Schur multiplier (of norm 1). Hence all amenable groups are also weakly amenable (how lucky?) and their weak amenability constant is 1. If a group is not weakly amenable we write $\Lambda_{\mathrm{WA}}(G) = \infty$.

If, in the definition of weak amenability, no condition were put on the boundedness of the norms, then any G group would admit such a net of functions approximating 1 uniformly on compacts: It follows from Lemma 3.2 in [23] that given any compact subset K of a locally compact group G, there is a compactly supported Herz-Schur multiplier u taking the value 1 on all of K. The lemma in fact states something much stronger, namely that one can even arrange for u to be in the linear span of the set of continuous compactly supported positive definite functions. But the Herz-Schur norm of u will in general not stay bounded when the compact set K grows.

Weak amenability of groups has been extensively studied. Papers studying weak amenability include [14], [15], [16], [17], [20], [21], [26], [27].

The *Haagerup property* is another much studied approximation property (see the book [8]). It appeared in connection with the study of approximation properties for operator algebras (see e.g. [26] and [10]). It is known that groups with Haagerup property satisfy the Baum-Connes conjecture [33], [34]. The definition is as follows.

A locally compact group G has the Haagerup property, if there is a net (u_α) of continuous positive definite functions on G vanishing at infinity such that $u_\alpha \to 1$ uniformly on compacts. It is clear that amenability implies the Haagerup property, but the free groups demonstrate that the converse is not true (see [26]). It is however not clear what the relation between weak amenability and the Haagerup property is. When Cowling and Haagerup proved that the simple Lie groups $\mathrm{Sp}(1, n)$ are weakly amenable [16], it became clear that weak amenability does not imply the Haagerup property, because these groups also have Property (T) when $n \geq 2$ (see [41],[42],[3]), and Property (T) is a strong negation of the

Haagerup property. However, since the weak amenability constant of $\mathrm{Sp}(1,n)$ is $2n-1$, it does not reveal if having $\Lambda_{\mathrm{WA}}(G)=1$ implies having the Haagerup property.

In the light of the approximation properties described so far, and in order to study the relation between weak amenability and the Haagerup property, the *weak Haagerup property* was introduced (for discrete groups) in [40]. The class of groups with the weak Haagerup property encompasses in a natural way all the weakly amenable groups and groups with the Haagerup property. The definition goes as follows (see also Definition 4.1).

A locally compact group G has the *weak Haagerup property*, if there is a net (u_α) of Herz-Schur multipliers on G vanishing at infinity and uniformly bounded in Herz-Schur norm such that $u_\alpha \to 1$ uniformly on compacts. The least uniform bound on the norms of such nets (if such a bound exists at all) is the *weak Haagerup constant* of G, denoted $\Lambda_{\mathrm{WH}}(G)$.

In the same way that one deduces that amenable groups are weakly amenable, one sees that groups with the Haagerup property also have the weak Haagerup property. Also, it is trivial that $1 \leq \Lambda_{\mathrm{WH}}(G) \leq \Lambda_{\mathrm{WA}}(G)$ for every locally compact group G, and in particular all weakly amenable groups have the weak Haagerup property.

It is not immediately clear if the potentially larger class of groups with the weak Haagerup property actually contains groups which are not weakly amenable and at the same time without the Haagerup property. In Corollary 5.7 we will demonstrate that this is the case.

There are many examples of groups G where $\Lambda_{\mathrm{WH}}(G) = \Lambda_{\mathrm{WA}}(G)$, e.g. all amenable groups and more generally all groups G with $\Lambda_{\mathrm{WA}}(G) = 1$. There are also examples where the two constants differ. In fact, the wreath product group $H = \mathbb{Z}/2 \wr \mathbb{F}_2$ of the cyclic group of order two with the non-abelian free group of rank two is such an example. The group $H = \mathbb{Z}/2 \wr \mathbb{F}_2$ is defined as the semidirect product of $\bigoplus_{\mathbb{F}_2} \mathbb{Z}/2$ by $\mathbb{F}_2$ where $\mathbb{F}_2$ acts on $\bigoplus_{\mathbb{F}_2} \mathbb{Z}/2$ by the shift action. It is known that H has the Haagerup property (see [18]), and hence $\Lambda_{\mathrm{WH}}(H) = 1$. But in [47, Corollary 2.12] it was shown that $\Lambda_{\mathrm{WA}}(H) \neq 1$. It was later shown in [46, Corollary 4] that in fact $\Lambda_{\mathrm{WA}}(H) = \infty$.

There is another approximation property of locally compact groups that we would like to briefly mention. It is called the *Approximation Property* or simply AP and was introduced in [31] (see the end of Section 3 for the definition). It is known that all weakly amenable groups have AP, and there are non-weakly amenable groups with the AP as well (see [31]).

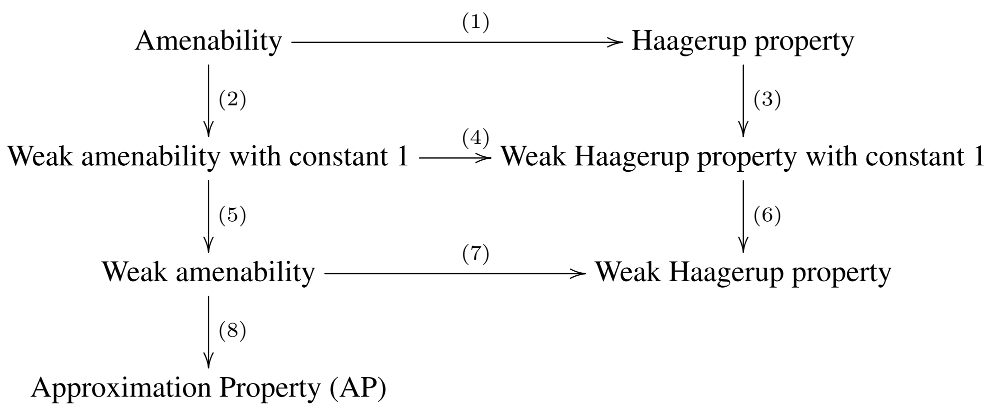

FIGURE 1. Approximation properties

Figure 1 displays the relations between the approximation properties mentioned so far. At the moment, all implications are known to be strict except for (3) and (6). In a forthcoming paper [30] by Haagerup and the author, implication (6) will be shown to be strict as well.

The study of approximation properties of groups has important applications in the theory of operator algebras due to the fact that the approximation properties have operator algebraic counterparts. The standard examples are nuclearity of C^*-algebras and semidiscreteness of von Neumann algebras which correspond to amenability of groups in the sense that a discrete group is amenable if and only if its reduced group C^*-algebra is nuclear if and only if its group von Neumann algebra is semidiscrete (see [7, Theorem 2.6.8]). Also weak amenability and the Haagerup property have operator algebra analogues (see [7, Chapter 12]). In the second part of the present paper we introduce a von Neumann algebraic analogue of the weak Haagerup property and the weak Haagerup constant (see Definition 7.2).

2. Main results

The main results of this paper concern hereditary properties of the weak Haagerup property for locally compact groups and von Neumann algebras. As applications we are able to provide many examples of groups and von Neumann algebras with the weak Haagerup property. We additionally provide some reformulations of the weak Haagerup property (see Proposition 4.3 and Proposition 4.4).

See Definition 4.1 for the definition of the weak Haagerup property for locally compact groups. Concerning the weak Haagerup property for locally compact groups we prove the following collection of hereditary results in Section 5.

Theorem A. *Let G be a locally compact group.*

(1) *If H is a closed subgroup of G, and G has the weak Haagerup property, then H has the weak Haagerup property. More precisely,*
$$\Lambda_{\mathrm{WH}}(H) \leq \Lambda_{\mathrm{WH}}(G).$$
(2) *If K is a compact normal subgroup of G, then G has the weak Haagerup property if and only if G/K has the weak Haagerup property. More precisely,*
$$\Lambda_{\mathrm{WH}}(G) = \Lambda_{\mathrm{WH}}(G/K).$$
(3) *The weak Haagerup property is preserved under finite direct products. More precisely, if G' is a locally compact group, then*
$$\Lambda_{\mathrm{WH}}(G \times G') \leq \Lambda_{\mathrm{WH}}(G)\Lambda_{\mathrm{WH}}(G').$$
(4) *If $(G_i)_{i \in I}$ is a directed set of open subgroups of G, then*
$$\Lambda_{\mathrm{WH}}(\bigcup_i G_i) = \lim_i \Lambda_{\mathrm{WH}}(G_i).$$
(5) *If $1 \longrightarrow N \hookrightarrow G \longrightarrow G/N \longrightarrow 1$ is a short exact sequence of locally compact groups, where G is second countable or discrete, and if G/N is amenable, then G has the weak Haagerup property if and only if N has the weak Haagerup property. More precisely,*
$$\Lambda_{\mathrm{WH}}(G) = \Lambda_{\mathrm{WH}}(N).$$

(6) *If Γ is a lattice in G and if G is second countable, then G has the weak Haagerup property if and only if Γ has the weak Haagerup property. More precisely,*

$$\Lambda_{\mathrm{WH}}(\Gamma) = \Lambda_{\mathrm{WH}}(G).$$

As mentioned, examples of groups with the weak Haagerup property trivially include all weakly amenable groups and groups with the Haagerup property. Apart from all these examples, we provide an additional example in Corollary 5.7 to show that the class of weakly Haagerup groups is strictly larger than the class of weakly amenable groups and groups with the Haagerup property combined. Examples of groups without the weak Haagerup property will be one of the subjects of another paper [30] by Haagerup and the author.

See Definition 7.2 and Remark 7.3 for the definition of the weak Haagerup property for finite von Neumann algebras. Concerning the weak Haagerup property for finite von Neumann algebras we will prove the following theorems.

Theorem B. *Let Γ be a discrete group. The following conditions are equivalent.*

(1) *The group Γ has the weak Haagerup property.*
(2) *The group von Neumann algebra $L(\Gamma)$ has the weak Haagerup property.*

More precisely, $\Lambda_{\mathrm{WH}}(\Gamma) = \Lambda_{\mathrm{WH}}(L(\Gamma))$.

Theorem C. *Let $M, M_1, M_2, \dots$ be finite von Neumann algebras which admit faithful normal traces.*

(1) *If $M_2 \subseteq M_1$ is a von Neumann subalgebra, then $\Lambda_{\mathrm{WH}}(M_2) \leq \Lambda_{\mathrm{WH}}(M_1)$.*
(2) *If $p \in M$ is a non-zero projection, then $\Lambda_{\mathrm{WH}}(pMp) \leq \Lambda_{\mathrm{WH}}(M)$.*
(3) *Suppose that $1 \in M_1 \subseteq M_2 \subseteq \cdots$ are von Neumann subalgebras of M generating all of M, and there is an increasing sequence of non-zero projections $p_n \in M_n$ with strong limit 1. Then $\Lambda_{\mathrm{WH}}(M) = \sup_n \Lambda_{\mathrm{WH}}(p_n M_n p_n)$.*
(4)
$$\Lambda_{\mathrm{WH}}\left(\bigoplus_n M_n\right) = \sup_n \Lambda_{\mathrm{WH}}(M_n).$$
(5)
$$\Lambda_{\mathrm{WH}}(M_1 \otimes M_2) \leq \Lambda_{\mathrm{WH}}(M_1)\Lambda_{\mathrm{WH}}(M_2).$$

As an application of the theorems above, in Section 9 we give an example of two von Neumann algebras, in fact II_1 factors, which are distinguished by the weak Haagerup property, i.e. the two von Neumann algebras do not have the same weak Haagerup constant. None of the other approximation properties mentioned in the introduction (see Figure 1), or more precisely the corresponding operator algebraic approximation properties, can distinguish the two factors (see Remark 9.1).

As another application of Theorem C (or rather Theorem C' in Section 8) we are able to prove that the weak Haagerup constant of a von Neumann algebra with a faithful normal trace does not depend on the choice of trace (see Proposition 8.4).

Although the following result is not proved in this paper, we would like to mention it here, because it gives a complete description of the weak Haagerup property for connected simple Lie groups.

Theorem ([30]). *A connected simple Lie group has the weak Haagerup property if and only if it has real rank zero or one.*

3. Preliminaries

We always let G denote a locally compact group equipped with left Haar measure. We always include the Hausdorff requirement whenever we discuss topological groups and spaces.

The space of continuous functions on G (with complex values) is denoted $C(G)$. It contains the subspace $C_0(G)$ of continuous functions vanishing at infinity and the subspace $C_c(G)$ of compactly supported continuous functions. When G is a Lie group, $C^\infty(G)$ denotes the space of smooth functions on G.

In the following we introduce the Fourier algebra $A(G)$, the group von Neumann algebra $L(G)$, the completely bounded Fourier multipliers $M_0A(G)$, the algebra of Herz-Schur multipliers $B_2(G)$ and its predual $Q(G)$. This is quite a mouthful, so we encourage you to take a deep breath before you read any further. The most important of these spaces in the present context is the space of Herz-Schur multipliers $B_2(G)$ which occurs also in the definition of the weak Haagerup property, Definition 4.1.

When π is a continuous unitary representation of G on some Hilbert space $\mathcal{H}$, and when $h, k \in \mathcal{H}$, then the continuous function u defined by

$$u(x) = \langle \pi(x)h, k\rangle \quad \text{for all } x \in G \tag{3.1}$$

is a *matrix coefficient* of π. The *Fourier algebra* $A(G)$ is the space of matrix coefficients of the left regular representation $\lambda : G \to L^2(G)$. That is, $u \in A(G)$ if and only if there are $h, k \in L^2(G)$ such that

$$u(x) = \langle \lambda(x)h, k\rangle, \quad \text{for all } x \in G. \tag{3.2}$$

With pointwise operations, $A(G)$ becomes an algebra, and when equipped with the norm

$$\|u\|_A = \inf\{\|h\|_2\|k\|_2 \mid (3.2) \text{ holds}\}.$$

$A(G)$ is in fact a Banach algebra.

Given $u \in A(G)$ there are $f, g \in L^2(G)$ such that $u = f * \check{g}$ and $\|u\| = \|f\|_2\|g\|_2$, where $\check{g}(x) = g(x^{-1})$ and $*$ denotes convolution. This is often written as

$$A(G) = L^2(G) * L^2(G).$$

It is known that $\|u\|_\infty \leq \|u\|_A$ for any $u \in A(G)$, and $A(G) \subseteq C_0(G)$.

The Fourier algebra was introduced and studied in Eymard's excellent paper [23] to which we refer to details about the Fourier algebra. When G is not compact, the Fourier algebra $A(G)$ contains no unit. But it was shown in [44] that $A(G)$ has a bounded approximate unit if and only if G is amenable (see also [49, Theorem 10.4]).

The von Neumann algebra generated by the image of the left regular representation $\lambda : G \to B(L^2(G))$ is the *group von Neumann algebra*, $L(G)$. The Fourier algebra $A(G)$ can be identified isometrically with the (unique) predual of $L(G)$, where the duality is given by

$$\langle u, \lambda(x)\rangle = u(x), \quad x \in G,\ u \in A(G).$$

A function $v : G \to \mathbb{C}$ is called a *Fourier multiplier*, if $vu \in A(G)$ for every $u \in A(G)$. A Fourier multiplier v is continuous and bounded, and it defines bounded multiplication operator $m_v : A(G) \to A(G)$. The dual operator of m_v is a normal (i.e. ultraweakly continuous) bounded operator $M_v : L(G) \to L(G)$ such that

$$M_v \lambda(x) = v(x)\lambda(x).$$

In [17, Proposition 1.2] it is shown that Fourier multipliers can actually be characterized as the continuous functions $v : G \to \mathbb{C}$ such that

$$\lambda(x) \mapsto v(x)\lambda(x)$$

extends to a normal, bounded operator on the group von Neumann algebra $L(G)$. If M_v is not only bounded but a completely bounded operator on $L(G)$, we say that v is a *completely bounded Fourier multiplier*. We denote the space of completely bounded Fourier multipliers by $M_0A(G)$. When equipped with the norm $\|v\|_{M_0A} = \|M_v\|_{\mathrm{cb}}$, where $\| \|_{\mathrm{cb}}$ denotes the completely bounded norm, $M_0A(G)$ is a Banach algebra. It is clear that

$$\|vu\|_A \leq \|v\|_{M_0A}\|u\|_A \qquad \text{for every } v \in M_0A(G),\ u \in A(G). \tag{3.3}$$

One of the key notions of this paper is the notion of a Herz-Schur multiplier, which we now recall. Let X be a non-empty set. A function $k : X \times X \to \mathbb{C}$ is called a *Schur multiplier* on X if for every bounded operator $A = [a_{xy}]_{x,y\in X} \in B(\ell^2(X))$ the matrix $[k(x,y)a_{xy}]_{x,y\in X}$ represents a bounded operator on $\ell^2(X)$, denoted $m_k(A)$. If k is a Schur multiplier, it is a consequence of the closed graph theorem that m_k defines a *bounded* operator on $B(\ell^2(X))$. We define the *Schur norm* $\|k\|_S$ to be the operator norm $\|m_k\|$ of m_k.

Let $u : G \to \mathbb{C}$ be a continuous function. Then u is as *Herz-Schur multiplier* if and only if the function $\widehat{u} : G \times G \to \mathbb{C}$ defined by

$$\widehat{u}(x,y) = u(y^{-1}x), \quad x, y \in G,$$

is a Schur multiplier on G. The set of Herz-Schur multipliers on G is denoted $B_2(G)$. It is a Banach space, in fact a unital Banach algebra, when equipped with the *Herz-Schur norm* $\|u\|_{B_2} = \|\widehat{u}\|_S = \|m_{\widehat{u}}\|$.

It is known that $B_2(G) = M_0A(G)$ isometrically (see [5], [35], [50, Theorem 5.1]). We include several well-known characterizations of the Herz-Schur multipliers $B_2(G)$ below.

Proposition 3.1. *Let G be a locally compact group, let $u : G \to \mathbb{C}$ be a function, and let $k \geq 0$ be given. The following are equivalent.*

(1) *u is a Herz-Schur multiplier with $\|u\|_{B_2} \leq k$.*
(2) *u is continuous, and for every $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in G$*

$$\|u(x_j^{-1}x_i)_{i,j=1}^n\|_S \leq k.$$

(3) *u is a completely bounded Fourier multiplier with $\|u\|_{M_0A(G)} \leq k$.*
(4) *There exist a Hilbert space $\mathcal{H}$ and two bounded, continuous maps $P, Q : G \to \mathcal{H}$ such that*

$$u(y^{-1}x) = \langle P(x), Q(y)\rangle \qquad \text{for all } x, y \in G$$

and

$$(\sup_{x\in G}\|P(x)\|)(\sup_{y\in G}\|Q(y)\|) \leq k.$$

If G is second countable, then the above conditions are equivalent to

(5) *There exist a Hilbert space $\mathcal{H}$ and two bounded, Borel maps $P, Q : G \to \mathcal{H}$ such that*
$$u(y^{-1}x) = \langle P(x), Q(y) \rangle \qquad \text{for all } x, y \in G$$
and
$$(\sup_{x \in G} \|P(x)\|)(\sup_{y \in G} \|Q(y)\|) \leq k.$$

A proof taken from the unpublished manuscript [27] of the equivalence of (4) and (5) is included in the appendix (see Lemma C.1).

The space $B_2(G)$ of Herz-Schur multipliers has a Banach space predual. More precisely, let $Q(G)$ denote the completion of $L^1(G)$ in the norm
$$\|f\|_Q = \sup\left\{\left|\int_G f(x)u(x)\, dx\right| \mid u \in B_2(G),\ \|u\|_{B_2} \leq 1\}\right\}.$$
In [17] it is proved that the dual Banach space of $Q(G)$ may be identified isometrically with $B_2(G)$, where the duality is given by
$$\langle f, u \rangle = \int_G f(x)u(x)\, dx, \quad f \in L^1(G),\ u \in B_2(G).$$
Thus, $B_2(G)$ may be equipped with the weak*-topology arising from its predual $Q(G)$. This topology will also be denoted the $\sigma(B_2, Q)$-topology.

We note that since $\|u\|_\infty \leq \|u\|_{B_2}$ for any $u \in B_2(G)$, then $\|f\|_Q \leq \|f\|_1$ for every $f \in L^1(G)$. In particular, $C_c(G)$ is dense in $Q(G)$ with respect to the Q-norm, because $C_c(G)$ is dense in $L^1(G)$ with respect to the 1-norm.

The Approximation Property (AP) briefly mentioned in the introduction is defined as follows. A locally compact group G has AP if there is a net (u_α) in $A(G)$ such that $u_\alpha \to 1$ in the $\sigma(B_2, Q)$-topology. It was shown in [31, Theorem 1.12] that weakly amenable groups have AP. Only recently (in [28], [29], [43]) it was proved that there are (m)any groups without AP. Examples of groups without AP include the special linear groups $\mathrm{SL}_n(\mathbb{R})$ when $n \geq 3$ and their lattices $\mathrm{SL}_n(\mathbb{Z})$.

4. The weak Haagerup property for locally compact groups

The following definition is the main focus of the present paper.

Definition 4.1. Let G be a locally compact group. Then G has the *weak Haagerup property*, if there are a constant $C > 0$ and a net $(u_\alpha)_{\alpha \in A}$ in $B_2(G) \cap C_0(G)$ such that
$$\|u_\alpha\|_{B_2} \leq C \quad \text{for every } \alpha \in A,$$
$$u_\alpha \to 1 \text{ uniformly on compacts as } \alpha \to \infty.$$

The weak Haagerup constant $\Lambda_{\mathrm{WH}}(G)$ is defined as the infimum of those C for which such a net (u_α) exists, and if no such net exists we write $\Lambda_{\mathrm{WH}}(G) = \infty$. It is not hard to see that the infimum is actually a minimum. If a group G has the weak Haagerup property, we will also sometimes say that G *is weakly Haagerup*.

If, in the above definition, ones replaces the requirement $u_\alpha \in C_0(G)$ with the stronger requirement $u_\alpha \in C_c(G)$, one obtains the definition of weak amenability.

Apart from the norm topology, there are (at least) three interesting topologies one can put on the norm bounded sets in $B_2(G)$ one of which is the locally uniform topology used in Definition 4.1 and the others being the $\sigma(B_2, Q)$-topology and the point-norm topology (see Appendix A). Proposition 4.2 and 4.3 below show that any of these three topologies could have been used in Definition 4.1. More precisely, we have the following characterizations of the weak Haagerup property.

Proposition 4.2. *Let G be a locally compact group. Then $\Lambda_{\mathrm{WH}}(G) \leq C$ if and only if there is a net (u_α) in $B_2(G) \cap C_0(G)$ such that*

$$\|u_\alpha\|_{B_2} \leq C \quad \textit{for every } \alpha,$$
$$u_\alpha \to 1 \textit{ in the } \sigma(B_2, Q)\textit{-topology}.$$

Proof. Suppose first $\Lambda_{\mathrm{WH}}(G) \leq C$. Then by Lemma A.1 (2), the conditions in our proposition are satisfied.

Conversely, suppose we are given a net (u_α) in $B_2(G) \cap C_0(G)$ such that

$$\|u_\alpha\|_{B_2} \leq C \quad \text{for every } \alpha,$$
$$u_\alpha \to 1 \text{ in the } \sigma(B_2, Q)\text{-topology}.$$

Let $v_\alpha = h * u_\alpha$, where h is a continuous, non-negative, compactly supported function on G such that $\int h(x)\,dx = 1$. Then using the convolution trick (see Lemma B.1, Lemma B.2 and Remark B.3) we see that the net (v_α) witnesses $\Lambda_{\mathrm{WH}}(G) \leq C$. □

The following Proposition (and its proof) is inspired by [16, Proposition 1.1].

Proposition 4.3. *Let G be a locally compact group and suppose $\Lambda_{\mathrm{WH}}(G) \leq C$. Then there exists a net $(v_\alpha)_{\alpha \in A}$ in $B_2(G) \cap C_0(G)$ such that*

$$\|v_\alpha\|_{B_2} \leq C \quad \textit{for every } \alpha,$$
$$\|v_\alpha u - u\|_A \to 0 \quad \textit{for every } u \in A(G),$$
$$v_\alpha \to 1 \textit{ uniformly on compacts.}$$

If L is any compact subset of G and $\varepsilon > 0$, then there exists $w \in B_2(G) \cap C_0(G)$ so that

$$\|w\|_{B_2} \leq C + \varepsilon,$$
$$w = 1 \quad \textit{for every } x \in L.$$

Moreover, if K is a compact subgroup of G, then the net (v_α) can be chosen to consist of K-bi-invariant functions. Finally, if G is a Lie group, the net (v_α) can additionally be chosen to consist of smooth functions.

Proof. Let (u_α) be a net witnessing $\Lambda_{\mathrm{WH}}(G) \leq C$. Using the bi-invariance trick (see Appendix B) we see that the net (u_α^K) obtained by averaging each u_α from left and right over the compact subgroup K is a net of K-bi-invariant functions witnessing $\Lambda_{\mathrm{WH}}(G) \leq C$. We let $v_\alpha = h^K * u_\alpha^K$, where $h \in C_c(G)$ is a non-negative, continuous function with compact support and integral 1. Using the convolution trick (see Lemma B.1 and Lemma B.2) we see that the net (v_α) has the desired properties (that $v_\alpha \to 1$ uniformly on compacts follows from Lemma A.1).

Let $L \subseteq G$ be compact and $\varepsilon > 0$ be arbitrary. By [23, Lemma 3.2] there is $u \in A(G)$ such that $u(x) = 1$ for all $x \in K$. According to the first part of our proposition, there is $v \in B_2(G) \cap C_0(G)$ such that $\|v\|_{B_2} \leq C$ and $\|vu - u\|_A \leq \varepsilon$. Let $w = v - (vu - u)$. Then w has the desired properties.

If G is a Lie group, we let h be as before with the extra condition that $C^\infty(G)$ and use the arguments above. □

Proposition 4.4 gives an equivalent formulation of the weak Haagerup property with constant 1. Recall that a continuous map is *proper* if the preimage of a compact set is compact.

Proposition 4.4. *Let G be a locally compact and σ-compact group. Then G is weakly Haagerup with constant 1, if and only if there is a continuous, proper function $\psi : G \to [0,\infty[$ such that $\|e^{-t\psi}\|_{B_2} \leq 1$ for every $t > 0$.*

Moreover, we can take ψ to be symmetric.

The idea of the proof of the proposition is taken from the proof of Proposition 2.1.1 in [8]. A proof in the case where G is discrete can be found in [40].

Proof. Suppose first such a map ψ exists, and let $u_t = e^{-t\psi}$. The fact that ψ is proper implies that $u_t \in C_0(G)$ for every $t > 0$. If $K \subseteq G$ is compact, then $\psi(K) \subseteq [0,r]$ for some $r > 0$. Hence $u_t(K) \subseteq [e^{-tr}, 1]$. This shows that $u_t \to 1$ uniformly on K as $t \to 0$. It follows that G is weakly Haagerup with constant 1.

Conversely, suppose G is weakly Haagerup with constant 1. Since G is locally compact and σ-compact, it is the union of an increasing sequence $(U_n)_{n=1}^\infty$ of open sets such that the closure $\overline{U_n}$ of U_n is compact and contained in U_{n+1} (see [25, Proposition 4.39]). Choose an increasing, unbounded sequence (α_n) of positive real numbers and a decreasing sequence (ε_n) tending to zero such that $\sum_n \alpha_n \varepsilon_n$ converges. For every n choose a function $u_n \in B_2(G) \cap C_0(G)$ with $\|u_n\|_{B_2} \leq 1$ such that

$$\sup_{g \in \overline{U_n}} |u_n(g) - 1| \leq \varepsilon_n / 2.$$

Replace u_n by $|u_n|^2$, if necessary, to ensure $0 \leq u_n \leq 1$ and

$$\sup_{g \in \overline{U_n}} |u_n(g) - 1| \leq \varepsilon_n.$$

Define $\psi_i : G \to [0,\infty[$ and $\psi : G \to [0,\infty[$ by

$$\psi_i(g) = \sum_{n=1}^{i} \alpha_n (1 - u_n(g)), \qquad \psi(g) = \sum_{n=1}^{\infty} \alpha_n (1 - u_n(g)).$$

It is easy to see that ψ is well-defined. We claim that $\psi_i \to \psi$ uniformly on compacts. For this, let $K \subseteq G$ be compact. By compactness, $K \subseteq U_N$ for some N, and hence if $g \in K$ and $i \geq N$,

$$|\psi(g) - \psi_i(g)| = |\sum_{n=i+1}^{\infty} \alpha_n (1 - u_n(g))| \leq \sum_{n=i+1}^{\infty} \alpha_n \varepsilon_n.$$

Since $\sum_n \alpha_n \varepsilon_n$ converges, this proves that $\psi_i \to \psi$ uniformly on K. In particular, since each ψ_i is continuous, ψ is continuous.

We claim that ψ is proper. Let $R > 0$ be given, and choose n such that $\alpha_n \geq 2R$. Since $u_n \in C_0(G)$, there is a compact set $K \subseteq G$ such that $|u_n(g)| < 1/2$ whenever $g \in G \setminus K$. Now if $\psi(g) \leq R$, then $\psi(g) \leq \alpha_n/2$, and in particular $\alpha_n(1 - u_n(g)) \leq \alpha_n/2$, which implies that $1 - u_n(g) \leq 1/2$. Hence we have argued that

$$\{g \in G \mid \psi(g) \leq R\} \subseteq \{g \in G \mid 1 - u_n(g) \leq 1/2\} \subseteq K.$$

This proves that ψ is proper.

Now let $t > 0$ be fixed. We must show that $\|e^{-t\psi}\|_{B_2} \leq 1$. Since ψ_i converges locally uniformly to ψ, it will suffice to prove that $\|e^{-t\psi_i}\|_{B_2} \leq 1$, because the unit ball of $B_2(G)$ is closed under locally uniform limits (see Lemma A.3). Observe that

$$e^{-t\psi_i} = \prod_{n=1}^{i} e^{-t\alpha_n(1-u_n)},$$

and so it suffices to show that $e^{-t\alpha_n(1-u_n)}$ belongs to the unit ball of $B_2(G)$ for each n. And this is clear, since

$$\|e^{-t\alpha_n(1-u_n)}\|_{B_2} = e^{-t\alpha_n}\|e^{t\alpha_n u_n}\|_{B_2} \leq e^{-t\alpha_n}e^{t\alpha_n\|u_n\|_{B_2}} \leq 1.$$

To prove the last assertion, put $\bar{\psi} = \psi + \check{\psi}$, where $\check{\psi}(g) = \psi(g^{-1})$. Clearly, $\bar{\psi}$ is continuous and proper. Finally, for every $t > 0$

$$\|e^{-t\bar{\psi}}\|_{B_2} \leq \|e^{-t\psi}\|_{B_2}\|e^{-t\check{\psi}}\|_{B_2} \leq 1,$$

since $\|\check{u}\|_{B_2} = \|u\|_{B_2}$ for every Herz-Schur multiplier $u \in B_2(G)$.

□

Having settled the definition of the weak Haagerup property for locally compact groups and various reformulations of the property, we move on to prove hereditary results for the class of groups with the weak Haagerup property.

5. Hereditary properties I

In this section we prove hereditary results for the weak Haagerup property of locally compact groups. The hereditary properties under consideration involve passing to closed subgroups, taking quotients by compact normal subgroups, taking finite direct products, taking direct unions of open subgroups and extending from co-Følner subgroups and lattices to the whole group.

We begin this section with an easy lemma.

Lemma 5.1. *Suppose G is a locally compact group with a closed subgroup H.*

(1) *If $u \in C_0(G)$, then $u|_H \in C_0(H)$.*
(2) *If $u \in B_2(G)$, then $u|_H \in B_2(H)$ and*

$$\|u|_H\|_{B_2(H)} \leq \|u\|_{B_2(G)}.$$

Proof. (1) is obvious, and (2) is obvious from the characterization in Proposition 3.1. □

An immediate consequence of the previous lemma is the following.

Proposition 5.2. *The class of weakly Haagerup groups is stable under taking subgroups. More precisely, if G is a locally compact group with a closed subgroup H, then $\Lambda_{\mathrm{WH}}(H) \leq \Lambda_{\mathrm{WH}}(G)$.*

Lemma 5.3. *If $K \subseteq G$ is a compact, normal subgroup, then*

1. *$C(G/K)$ may be canonically and isometrically identified with the subspace of $C(G)$ of functions constant on the cosets of K in G.*
2. *Under the canonical identification from (1), $C_0(G/K)$ is isometrically identified with the subspace of $C_0(G)$ of functions constant on the cosets of K in G.*
3. *Under the canonical identification from (1), $B_2(G/K)$ is isometrically identified with the subspace of $B_2(G)$ of functions constant on the cosets of K in G.*
4. *Moreover, the canonical identification preserves the topology of locally uniform convergence.*

Proof.

(1) Let $q : G \to G/K$ denote the quotient map. If $f \in C(G)$ is constant on K-cosets, it is easy to see that the induced map $\bar{f}$ defined by $\bar{f}([x]_K) = f(x)$ is continuous. Conversely, if $g \in C(G/K)$ is given, then the composite $g \circ q$ is continuous on G and constant on cosets.

(2) One must check that $g \in C_0(G/K)$ if and only if $g \circ q \in C_0(G)$. Note first that a subset $L \subseteq G/K$ is compact if and only if $q^{-1}(L)$ is compact. In other words, q is proper. The rest is elementary. It is also clear, that the correspondence is isometric with respect to the uniform norm. This completes (2).

(3) This is Proposition 1.3 in [16].

(4) One must check that if (g_n) is a net in $C(G/K)$ and $g \in C(G/K)$, then $g_n \to g$ uniformly on compacts if and only if $g_n \circ q \to g \circ q$ uniformly on compacts. This is elementary using properness of q. □

Proposition 5.4. *If G is a locally compact group with a compact, normal subgroup $K \triangleleft G$, then $\Lambda_{\mathrm{WH}}(G/K) = \Lambda_{\mathrm{WH}}(G)$.*

Proof. Apply the last part of Proposition 4.3 and Lemma 5.3. □

Concerning direct products of groups we have the following proposition.

Proposition 5.5. *The class of weakly Haagerup groups is stable under finite direct products. More precisely, we have*

$$\Lambda_{\mathrm{WH}}(G \times H) \leq \Lambda_{\mathrm{WH}}(G)\Lambda_{\mathrm{WH}}(H) \tag{5.1}$$

for locally compact groups G and H.

Proof. From the characterization in Proposition 3.1, it easily follows that if $u \in B_2(G)$ and $v \in B_2(H)$, then $u \times v \in B_2(G \times H)$ and $\|u \times v\|_{B_2} \leq \|u\|_{B_2}\|v\|_{B_2}$. Also, if $u \in C_0(G)$ and $v \in C_0(H)$, then clearly $u \times v \in C_0(G \times H)$. It is now clear that if (u_α) and (v_β) are bounded nets in $B_2(G) \cap C_0(G)$ and $B_2(H) \cap C_0(H)$, respectively, converging locally uniformly to 1, then the product net $(u_\alpha \times v_\beta)$ (with the product order) belongs to $B_2(G \times H) \cap C_0(G \times H)$ and converges locally uniformly to 1. This proves that

$$\Lambda_{\mathrm{WH}}(G \times H) \leq \Lambda_{\mathrm{WH}}(G)\Lambda_{\mathrm{WH}}(H).$$

□

Remark 5.6. It would of course be interesting to know if equality actually holds in (5.1). The corresponding result for weak amenability is known to be true (see [16, Corollary 1.5]). It is not hard to see that if either $\Lambda_{\mathrm{WH}}(G) = 1$ or $\Lambda_{\mathrm{WH}}(H) = 1$, then (5.1) is an equality.

With Proposition 5.5 at our disposal, we can show the following.

Corollary 5.7. *The class of weakly Haagerup groups contains groups that are neither weakly amenable nor have the Haagerup property.*

Proof. It is known that the Lie group $G = \mathrm{Sp}(1,n)$ is weakly amenable with $\Lambda_{\mathrm{WA}}(G) = 2n-1$ (see [16]). It is also known that G has Property (T) when $n \geq 2$ (see [3, Section 3.3]), and hence G does not have the Haagerup property (since G is not compact). As we mentioned earlier, the group $H = \mathbb{Z}/2 \wr \mathbb{F}_2$ has the Haagerup property, but is not weakly amenable. Hence both G and H have the weak Haagerup property. It now follows from the previous proposition that the group $G \times H$ has the weak Haagerup property.

Both the Haagerup property and weak amenability passes to subgroups, so it also follows that $G \times H$ has neither of these properties. □

Remark 5.8. An example of a *discrete* group with the weak Haagerup property outside the class of weakly amenable groups and the Haagerup groups is given by taking Γ to be a lattice in $\mathrm{Sp}(1,n)$ and considering the group $\Gamma \times H$, where again $H = \mathbb{Z}/2 \wr \mathbb{F}_2$.

The group constructed in the proof of Corollary 5.7 is of course tailored exactly to prove the corollary, and one might argue that it is not a natural example. It would be interesting to find more natural examples, for instance a simple group.

Using the characterization of Herz-Schur multipliers given in Proposition 3.1, it is not hard to prove the following (see [59, Lemma 4.2]).

Lemma 5.9. *Let H be an open subgroup of a locally compact group G. Extend $u \in B_2(H)$ to $\widetilde{u} : G \to \mathbb{C}$ by letting $\widetilde{u}(x) = 0$ when $x \notin H$. Then $\widetilde{u} \in B_2(G)$ and $\|u\|_{B_2} = \|\widetilde{u}\|_{B_2}$.*

Moreover, if $u \in C_0(H)$, then $\widetilde{u} \in C_0(G)$.

We note that there are examples of groups $H \leq G$, where some $u \in B_2(H)$ has no extension to $B_2(G)$ (see [6, Theorem 4.4]). In these examples, H is of course not open.

Proposition 5.10. *If $(G_i)_{i \in I}$ is a directed set of open subgroups in a locally compact group G, and $G = \bigcup_i G_i$, then*

$$\Lambda_{\mathrm{WH}}(G) = \sup_i \Lambda_{\mathrm{WH}}(G_i).$$

Proof. From Proposition 5.2 we already know that $\Lambda_{\mathrm{WH}}(G) \geq \sup_i \Lambda_{\mathrm{WH}}(G_i)$. We will now show the other inequality. We may assume that $\sup_i \Lambda_{\mathrm{WH}}(G_i) < \infty$ since otherwise there is nothing to prove.

Let $L \subseteq G$ be a compact set and let $\varepsilon > 0$ be given. By compactness and directedness there is $j \in I$ such that $L \subseteq G_j$. Using Proposition 4.3 we may find $w \in B_2(G_j) \cap C_0(G_j)$ so that

$$\|w\|_{B_2} \leq \Lambda_{\mathrm{WH}}(G_j) + \varepsilon \leq \sup_i \Lambda_{\mathrm{WH}}(G_i) + \varepsilon,$$
$$w(x) = 1 \quad \text{for every } x \in L.$$

By Lemma 5.9, there is $\widetilde{w} \in B_2(G) \cap C_0(G)$ such that

$$\|\widetilde{w}\|_{B_2} \leq \sup_i \Lambda_{\mathrm{WH}}(G_i) + \varepsilon,$$
$$\widetilde{w}(x) = 1 \quad \text{for every } x \in L.$$

Since L and ε were arbitrary, it now follows that

$$\Lambda_{\mathrm{WH}}(G) \leq \sup_i \Lambda_{\mathrm{WH}}(G_i),$$

and the proof is complete. □

The next result, Proposition 5.15, is inspired by [36]. Let G be a locally compact, second countable group, and let (X, μ) be a standard measure space with a Borel action of G. We assume that the measure μ is a probability measure which is invariant under the action. In [36], quasi-invariant measures are considered as well, but we will stick to invariant measures all the time, because the invariance is needed in the proof of Lemma 5.13 (1) and (3).

Further, let H be a locally compact, second countable group, and let $\alpha : G \times X \to H$ be a Borel cocycle, i.e. α is a Borel map and for all $g, h \in G$ we have

$$\alpha(gh, x) = \alpha(g, hx)\alpha(h, x) \qquad \text{for } \mu\text{-almost all } x \in X.$$

The following definition of a proper cocycle is taken from [36], although we have modified it slightly.

Definition 5.11. Let $\alpha : G \times X \to H$ be as above. We say that α is *proper*, if there is a generating family $\mathcal{A}$ of Borel subsets of X such that the following three conditions hold.

(1) X is the union of an increasing sequence of elements in $\mathcal{A}$.
(2) For every $A \in \mathcal{A}$ and every compact subset L of G the set $\alpha(L \times A)$ is pre-compact.
(3) For every $A \in \mathcal{A}$ and every compact subset L of H, the set $K(A, L)$ of elements $g \in G$ such that $\{x \in A \cap g^{-1}A \mid \alpha(g, x) \in L\}$ has positive μ-measure is pre-compact.

We mention the following examples of proper cocycles. All examples are taken from [36, p. 490].

Example 5.12.

(a) Suppose H is a closed subgroup of G and that $X = G/H$ has an invariant probability measure μ for the action by left translation. Let $\sigma : G/H \to G$ be a regular Borel cross section of the projection map $p : G \to G/H$, i.e. a Borel map such that $p \circ \sigma = \mathrm{id}_{G/H}$ and $\sigma(L)$ has compact closure for each compact $L \subseteq G/H$ (see [45, Lemma 1.1]). We define $\alpha : G \times X \to H$ by

$$\alpha(g, x) = \sigma(gx)^{-1} g\sigma(x).$$

With $\mathcal{A}$ the family of all compact subsets of X, we verify the three conditions in Definition 5.11. Since G is second countable, it is also σ-compact. Then X is also σ-compact, and condition (1) is satisfied.

Let $A \in \mathcal{A}$ and let $L \subseteq G$ be compact. By regularity of σ,

$$\alpha(L \times A) \subseteq \sigma(LA)^{-1} L \sigma(A)$$

is pre-compact, and condition (2) is satisfied.

Let $A \in \mathcal{A}$ and let $L \subseteq H$ be compact. It is easy to see that

$$K(A, L) \subseteq \sigma(A) L \sigma(A)^{-1}.$$

Again by regularity of σ, it follows that $K(A, L)$ is pre-compact. Thus, condition (3) is satisfied.

(b) Suppose $K \triangleleft G$ is normal and compact. Let $H = G/K$, let $X = K$ and let μ be the normalized Haar measure on K. Then G acts on K by conjugation, and μ is invariant under this action. We let $\mathcal{A}$ be the collection of all Borel subsets of K, and we define $\alpha : G \times X \to H$ by

$$\alpha(g, x) = p(g),$$

where $p : G \to H$ is the quotient map. Conditions (1) and (2) of Definition 5.11 are immediate. For condition (3) we first note that if $L \subseteq H$ is compact, then $\alpha^{-1}(L) = p^{-1}(L) \times K$. Since p is a quotient homomorphism with compact kernel, it is proper. Hence $p^{-1}(L)$ is compact, and $K(A, L) \subseteq p^{-1}(L)$.

We emphasize the following special case of (a).

(c) Recall that a subgroup $\Gamma \subseteq G$ is a *lattice*, if Γ is discrete and the quotient space G/Γ admits a finite G-invariant measure. Hence, when $H = \Gamma$ is a lattice in G, we are in the situation mentioned in (a).

Let G and H be locally compact, second countable groups, and let (X, μ) be a standard G-space with a G-invariant probability measure. Let $\alpha : G \times X \to H$ be a proper Borel cocycle. When $u \in B_2(H)$ we define $\widehat{u} : G \to \mathbb{C}$ by

$$\widehat{u}(g) = \int_X u(\alpha(g, x))\, d\mu(x), \qquad g \in G. \tag{5.2}$$

The construction is taken from [36], where it is shown in Lemma 2.11 that $\widehat{u} \in B_2(G)$ and also $\|\widehat{u}\|_{B_2} \leq \|u\|_{B_2}$. We refer to Lemma C.1 for the continuity of $\widehat{u}$.

Lemma 5.13. *Let $\alpha : G \times X \to H$ be a proper cocycle as above, and let $u \in B_2(H)$ be given.*

(1) *$\widehat{u} \in B_2(G)$ and $\|\widehat{u}\|_{B_2} \leq \|u\|_{B_2}$.*
(2) *$\|\widehat{u}\|_\infty \leq \|u\|_\infty$.*
(3) *If $u \in C_0(H)$, then $\widehat{u} \in C_0(G)$.*

Proof.

(1) This is [36, Lemma 2.11].

(2) This is obvious.

(3) Given $\varepsilon > 0$ there is $L \subseteq H$ compact such that $h \notin L$ implies $|u(h)| \leq \varepsilon$. Since X is the union of an increasing sequence of sets in $\mathcal{A}$, we may take $A \in \mathcal{A}$ such that $\mu(X \setminus A) \leq \varepsilon$. The set $K = \overline{K(A, L)}$ is compact in G, and if $g \notin K$ then

$$X_g = \{x \in A \cap g^{-1}A \mid \alpha(g, x) \in L\}$$

is a null set. Hence for $g \notin K$

$$\begin{aligned} |\widehat{u}(g)| &\leq \int_{X \setminus X_g} |u(\alpha(g, x))| \, d\mu(x) \\ &\leq \int_{X \setminus (A \cap g^{-1}A)} \|u\|_\infty \, d\mu(x) + \int_{(A \cap g^{-1}A) \setminus X_g} \varepsilon \, d\mu(x) \\ &\leq 2\varepsilon \|u\|_\infty + \varepsilon. \end{aligned}$$

This shows that $\widehat{u} \in C_0(G)$. □

Lemma 5.14. *Let $\alpha : G \times X \to H$ be a proper cocycle as above. The contractive linear map $B_2(H) \to B_2(G)$ defined by $u \mapsto \widehat{u}$, where $\widehat{u}$ is given by* (5.2)*, is continuous on norm bounded sets with respect to the topology of locally uniform convergence.*

Proof. Suppose $u_n \to 0$ in $B_2(H)$ uniformly on compacts, and $\|u_n\|_{B_2} < c$ for every n. In particular, $\|u_n\|_\infty < c$ for every n. Let $K \subseteq G$ be compact, and let $\varepsilon > 0$ be given. Choose $A \in \mathcal{A}$ such that $\mu(X \setminus A) \leq \varepsilon/2c$, and let $L = \overline{\alpha(K \times A)}$. Since L is compact, we have eventually that $|u_n(h)| < \varepsilon/2$ for every $h \in L$. Then for $g \in K$ we have

$$|\widehat{u}_n(g)| = \left| \int_X u_n(\alpha(g, x)) \, d\mu(x) \right| \leq \int_A \varepsilon/2 \, d\mu(x) + \int_{X \setminus A} c \, d\mu(x) \leq \varepsilon.$$

This completes the proof. □

Proposition 5.15. *Let G and H be locally compact, second countable group, and let (X, μ) be a standard Borel G-space with a G-invariant probability measure. If there is a proper Borel cocycle $\alpha : G \times X \to H$, then $\Lambda_{\mathrm{WH}}(G) \leq \Lambda_{\mathrm{WH}}(H)$.*

Proof. Suppose $\Lambda_{\mathrm{WH}}(H) \leq C$, and choose a net (u_i) in $B_2(H) \cap C_0(H)$ such that

$$\|u_i\|_{B_2} \leq C \quad \text{for every } i,$$
$$u_i \to 1 \text{ uniformly on compacts .}$$

It follows from Lemma 5.13 that $\widehat{u}_i \in B_2(G) \cap C_0(G)$ and

$$\|\widehat{u}_i\|_{B_2} \leq C \quad \text{for every } i.$$

From Lemma 5.14 we also see that

$$\widehat{u}_i \to 1 \text{ uniformly on compacts .}$$

This shows that $\Lambda_{\mathrm{WH}}(G) \leq C$, and the proof is complete. □

In view of Example 5.12 (a) we get the following corollary.

Corollary 5.16. *Let G be a locally compact, second countable group with a closed subgroup H such that G/H admits a G-invariant probability measure. Then G is weakly Haagerup if and only if H is weakly Haagerup. More precisely, $\Lambda_{\mathrm{WH}}(G) = \Lambda_{\mathrm{WH}}(H)$.*

Proof. From Proposition 5.2 we know that $\Lambda_{\mathrm{WH}}(H) \leq \Lambda_{\mathrm{WH}}(G)$. The other inequality follows from Proposition 5.15 in view of Example 5.12. □

Corollary 5.17. *Let G be a locally compact, second countable group with a lattice $\Gamma \subseteq G$. Then G is weakly Haagerup if and only if Γ is weakly Haagerup. More precisely, $\Lambda_{\mathrm{WH}}(G) = \Lambda_{\mathrm{WH}}(\Gamma)$.*

Inspired by the proof of Proposition 5.15 we now set out to prove that the weak Haagerup property can be lifted from a co-Følner subgroup to the whole group. In particular, extensions of amenable groups by weakly Haagerup groups yield weakly Haagerup groups.

Recall that a closed subgroup H in a locally compact group G is *co-Følner* if there is a G-invariant Borel measure μ on the coset space G/H and if for each $\varepsilon > 0$ and compact set $L \subseteq G$ there is a compact set $F \subseteq G/H$ such that $0 < \mu(F) < \infty$ and

$$\frac{\mu(gF \triangle F)}{\mu(F)} < \varepsilon \quad \text{for all } g \in L.$$

Here $\triangle$ denotes symmetric difference of sets. The most natural examples of co-Følner subgroups are closed normal subgroups with amenable quotients. Indeed, it follows from the Følner characterization of amenability (see [49, Theorem 7.3] and [49, Proposition 7.4]) that such groups are co-Følner.

Proposition 5.18. *Let G be a locally compact group with a closed subgroup H. Assume that G is second countable or discrete. If H is weakly Haagerup and co-Følner, then G is weakly Haagerup. More precisely, $\Lambda_{\mathrm{WH}}(G) = \Lambda_{\mathrm{WH}}(H)$.*

Proof. Let $C = \Lambda_{\mathrm{WH}}(H)$. We already know from Proposition 5.2 that $\Lambda_{\mathrm{WH}}(G) \geq C$, so it suffices to prove the other inequality. For this it is enough prove that for each compact $L \subseteq G$ and $\varepsilon > 0$ there is $v \in B_2(G) \cap C_0(G)$ with $\|v\|_{B_2} \leq C$ such that

$$|v(g) - 1| \leq 2\varepsilon \quad \text{for all } g \in L.$$

Thus, suppose that $L \subseteq G$ is compact and $\varepsilon > 0$. Let $\sigma : G/H \to G$ be a regular Borel cross section. If G is discrete the existence of σ is trivial, and if G is second countable then the existence of σ is a standard result (see [45, Lemma 1.1]). Define the corresponding cocycle $\alpha : G \times G/H \to H$ by

$$\alpha(g, x) = \sigma(gx)^{-1} g \sigma(x) \quad \text{for all } g \in G,\ x \in G/H.$$

Choose an invariant Borel measure μ on G/H and a compact set $F \subseteq G/H$ such that $0 < \mu(F) < \infty$ and

$$\frac{\mu(gF \triangle F)}{\mu(F)} < \varepsilon \quad \text{for all } g \in L.$$

By regularity of σ, the set $K = \overline{\alpha(L \times F)}$ is compact, because

$$\alpha(L \times F) \subseteq \sigma(LF)^{-1} L \sigma(F).$$

Since $\Lambda_{\mathrm{WH}}(H) \leq C$ there is a Herz-Schur multiplier $u \in B_2(H) \cap C_0(H)$ such that $\|u\|_{B_2} \leq C$ and

$$|u(h) - 1| \leq \varepsilon \quad \text{for all } h \in K.$$

Define $v : G \to \mathbb{C}$ by

$$v(g) = \frac{1}{\mu(F)} \int_{G/H} 1_{F \cap g^{-1}F}(x) u(\alpha(g, x))\, d\mu(x).$$

We claim that v has the desired properties. First we check that $v \in B_2(G)$ with $\|v\|_{B_2} \leq C$. Since $u \in B_2(H)$ there are a Hilbert space $\mathcal{H}$ and bounded, continuous maps $P, Q : H \to \mathcal{H}$ such that

$$u(ab^{-1}) = \langle P(a), Q(b) \rangle \quad \text{for all } a, b \in H.$$

If G is second countable, then so is H and we can (and will) assume that $\mathcal{H}$ is separable. Consider the Hilbert space $L^2(G/H, \mathcal{H})$, and define Borel maps $\widetilde{P}, \widetilde{Q} : G \to L^2(G/H, \mathcal{H})$ by

$$\begin{aligned}
\widetilde{P}(g)(x) &= \frac{1}{\mu(F)^{1/2}} 1_{g^{-1}F}(x) P(\alpha(g, x)) \\
\widetilde{Q}(g)(x) &= \frac{1}{\mu(F)^{1/2}} 1_{g^{-1}F}(x) Q(\alpha(g, x))
\end{aligned}$$

for all $g \in G$, $x \in G/H$. We note that $\|\widetilde{P}(g)\|_2 \leq \|P\|_\infty$ and $\|\widetilde{Q}(g)\|_2 \leq \|Q\|_\infty$ for every $g \in G$. Using the cocycle identity and the invariance of μ under the action of G, we find that

$$\begin{aligned}
\langle \widetilde{P}(g), \widetilde{Q}(h) \rangle &= \frac{1}{\mu(F)} \int 1_{g^{-1}F \cap h^{-1}F}(x) \; \langle P(\alpha(g, x)), Q(\alpha(h, x)) \rangle \; d\mu(x) \\
&= \frac{1}{\mu(F)} \int 1_{g^{-1}F \cap h^{-1}F}(x) \; u(\alpha(g, x)\alpha(h, x)^{-1}) \; d\mu(x) \\
&= \frac{1}{\mu(F)} \int 1_{g^{-1}F \cap h^{-1}F}(x) \; u(\alpha(gh^{-1}, hx)) \; d\mu(x) \\
&= \frac{1}{\mu(F)} \int 1_{F \cap (gh^{-1})^{-1}F}(x) \; u(\alpha(gh^{-1}, x)) \; d\mu(x) \\
&= v(gh^{-1}).
\end{aligned}$$

Thus, $v \in B_2(G)$ by Proposition 3.1 and $\|v\|_{B_2} \leq \|u\|_{B_2} \leq C$.

To see that $v \in C_0(G)$ we let $\delta > 0$ be given. Since $u \in C_0(H)$ there is a compact set $M \subseteq H$ such that $h \notin M$ implies $|u(h)| \leq \delta$.

If $x \in G/H$ and $g \in G$ is such that $x \in F \cap g^{-1}F$ and $\alpha(g, x) \in M$, then $g \in \sigma(F)M\sigma(F)^{-1}$, which is pre-compact since σ is regular. Then it is not hard to see that if $g \notin \sigma(F)M\sigma(F)^{-1}$ then

$$|v(g)| \leq \frac{1}{\mu(F)} \int_{F \cap g^{-1}F} |u(\alpha(g, x))| \; d\mu(x) \leq \delta.$$

This proves that $v \in C_0(G)$.

Finally, suppose $g \in L$. We show that $|v(g) - 1| \leq 2\varepsilon$. If $x \in F$, then $\alpha(g, x) \in K$ and $|u(\alpha(g, x)) - 1| \leq \varepsilon$. Hence

$$\begin{aligned}
|v(g) - 1| &= \frac{1}{\mu(F)} \left| \int 1_{F \cap g^{-1}F}(x) u(\alpha(g, x)) - 1_{F \cap g^{-1}F}(x) - 1_{F \setminus g^{-1}F}(x) \; d\mu(x) \right| \\
&\leq \frac{1}{\mu(F)} \int 1_{F \cap g^{-1}F}(x) |u(\alpha(g, x)) - 1| + 1_{F \setminus g^{-1}F}(x) \; d\mu(x) \\
&\leq \varepsilon + \frac{\mu(F \setminus g^{-1}F)}{\mu(F)} \\
&\leq 2\varepsilon.
\end{aligned}$$

□

Corollary 5.19. *Let N be a closed normal subgroup in a locally compact group G. Assume that G is either second countable or discrete. If N has the weak Haagerup property and G/N is amenable, then G has the weak Haagerup property. In fact, $\Lambda_{\mathrm{WH}}(G) = \Lambda_{\mathrm{WH}}(N)$.*

Remark 5.20. Proposition 5.15 has recently been generalized by Jolissaint [38] and Deprez, Li [19]. The more general version allows probability measures μ on X which are not invariant, but only quasi-invariant. It is then assumed that the pair (G, X) is *amenable* in the sense described in [38]. This includes also Proposition 5.18 as a special case.

6. The weak Haagerup property for simple Lie groups

This section contains results from [30] about the weak Haagerup property for connected simple Lie groups. The results are merely included here for completeness. The results are consequences of some of the hereditary properties proved here in Section 5 combined with work of de Laat and Haagerup [28], [29]. But before we mention the results, we summarize the situation concerning connected simple Lie groups, the Haagerup property and weak amenability.

Since compact groups are amenable, they also possess the Haagerup property, and they are weakly amenable. So only the non-compact case is of interest. It is known which connected simple Lie groups have the Haagerup property (see [8, p. 12]). We summarize the result.

Theorem 6.1 ([8])**.** *Let G be a non-compact connected simple Lie group. Then G has the Haagerup property if and only if G is locally isomorphic to either $\mathrm{SO}_0(1,n)$ or $\mathrm{SU}(1,n)$. Otherwise, G has property (T).*

Concerning weak amenability the situation is more subtle, if one wants to include the weak amenability constant, but still the full answer is known.

Theorem 6.2 ([14],[16],[17],[21],[27],[32])**.** *Let G be a non-compact connected simple Lie group. Then*

$$\Lambda_{\mathrm{WA}}(G) = \begin{cases} 1 & \textit{for } G \approx \mathrm{SO}(1,n) \\ 1 & \textit{for } G \approx \mathrm{SU}(1,n) \\ 2n-1 & \textit{for } G \approx \mathrm{Sp}(1,n) \\ 21 & \textit{for } G \approx \mathrm{F}_{4(-20)}. \\ \infty & \textit{otherwise }. \end{cases}$$

Here $\approx$ denotes local isomorphism. We remark that in the above situation $\Lambda_{\mathrm{WA}}(G) = 1$ in exactly the same cases as where G has the Haagerup property.

If the only concern is whether or not $\Lambda_{\mathrm{WA}}(G) < \infty$, i.e., whether or not G is weakly amenable, then the result can be rephrased as follows.

Corollary 6.3 ([14], [16], [17], [27],[32])**.** *A connected simple Lie group is weakly amenable if and only if it has real rank zero or one.*

As mentioned earlier, $\Lambda_{\mathrm{WH}}(G) \leq \Lambda_{\mathrm{WA}}(G)$ for every locally compact group G, and there are examples to show that the inequality can be strict in the most extreme sense: $\Lambda_{\mathrm{WA}}(H) = \infty$ and $\Lambda_{\mathrm{WH}}(H) = 1$, when $H = \mathbb{Z}/2 \wr \mathbb{F}_2$. For connected simple Lie groups, however, it turns out that the weak Haagerup property behaves like weak amenability. The following is proved in [30] using results of [28], [29].

Theorem 6.4 ([30]). *A connected simple Lie group has the weak Haagerup property if and only if it has real rank zero or one.*

7. The weak Haagerup property for von Neumann algebras

In this section we introduce the weak Haagerup property for finite von Neumann algebras, and we prove that a group von Neumann algebra has this property, if and only if the group has the weak Haagerup property.

In the following, let M be a (finite) von Neumann algebra with a faithful normal trace τ. By a trace we always mean a tracial state. We denote the induced inner product on M by $\langle\ ,\ \rangle_\tau$. In other words, $\langle x, y\rangle_\tau = \tau(y^*x)$ for $x, y \in M$. The completion of M with respect to this inner product is a Hilbert space, denoted $L^2(M,\tau)$ or simply $L^2(M)$. The norm on $L^2(M)$ is denoted $\|\ \|_2$ or $\|\ \|_\tau$ and satisfies $\|x\|_2 \leq \|x\|$ for every $x \in M$, where $\|\ \|$ denotes the operator norm on M.

When $T : M \to M$ is a bounded operator on M, it will be relevant to know sufficient conditions for T to extend to a bounded operator on $L^2(M)$. The following result uses a standard interpolation technique.

Proposition 7.1. *Let (M,τ) be a finite von Neumann algebra with faithful normal trace, and let $S : M \to M$ and $T : M \to M$ be bounded operators on M. Suppose $\langle Tx, y\rangle_\tau = \langle x, Sy\rangle_\tau$ for every $x, y \in M$. Then T extends to a bounded operator $\widetilde{T}$ on $L^2(M)$, and $\|\widetilde{T}\| \leq \max\{\|T\|, \|S\|\}$.*

Proof. After scaling both T and S with $\max\{1, \|T\|, \|S\|\}^{-1}$, we may assume that $\|S\| \leq 1$ and $\|T\| \leq 1$. By [9, Theorem 5] the set of invertible elements in M is norm dense, since M is finite. Hence it suffices to prove that $\|Tx\|_2 \leq \|x\|_2$ for every invertible $x \in M$. We prove first that $\|Tx\|_1 \leq \|x\|_1$, and an interpolation technique will then give the result.

Let M_1 denote the unit ball of M. Recall that $\|x\|_1 = \tau(|x|) = \sup\{|\tau(y^*x)| \mid y \in M_1\}$. Hence

$$\|Tx\|_1 = \sup_{y\in M_1} |\tau(y^*Tx)| = \sup_{y\in M_1} |\tau((Sy)^*x)| \leq \sup_{z\in M_1} |\tau(z^*x)| = \|x\|_1.$$

Since also $\|Tx\| \leq \|x\|$, it follows by an interpolation argument that $\|Tx\|_2 \leq \|x\|_2$. The interpolation argument goes as follows.

Assume for simplicity that $\|x\|_2 \leq 1$. We will show that $\|Tx\|_2 \leq 1$. Since x is invertible, it has polar decomposition $x = uh$, where u is unitary, and $h \geq 0$ is invertible. For $s \in \mathbb{C}$ define

$$F(s) = uh^{2s}, \quad G(s) = T(F(s)), \quad g(s) = \tau(G(s)G(1-\bar{s})^*).$$

Since h is positive and invertible, F is well-defined and analytic. It follows that G and g are analytic as well.

Next we show that g is bounded on the vertical strip $\Omega = \{s \in \mathbb{C} \mid 0 \leq \mathrm{Re}(s) \leq 1\}$. Since τ and T are bounded, it suffices to see that F is bounded on Ω. We have

$$\|F(s)\| = \|uh^{2s}\| \leq \|h^{2\mathrm{Re}(s)}\| \leq \sup_{0\leq t\leq 1} \|h^{2t}\| < \infty.$$

Observe that if v and w are unitaries in M, and w commutes with $y \in M$, then $|vyw| = |y|$, and hence $\|vyw\|_1 = \|y\|_1$. On the boundary of Ω we have the following estimates.

$$\|G(it)\| = \|T(uh^{2it})\| \leq 1,$$

since $\|T\| \leq 1$, and u and h^{2it} are unitaries. Also

$$\|G(1+it)\|_1 = \|T(uh^2h^{2it})\|_1 \leq \|uh^2h^{2it}\|_1 = \|h^2\|_1 = \|x\|_2^2 \leq 1,$$

It follows that

$$|g(it)| = |\tau(G(it)G(1+it)^*)| \leq \|G(it)\| \, \|G(1+it)\|_1 \leq 1$$

and

$$\|g(1+it)\| = |\tau(G(1+it)G(it)^*)| \leq \|G(it)\| \, \|G(1+it)\|_1 \leq 1.$$

In conclusion, g is an entire function, bounded on the strip Ω and bounded by 1 on the boundary of Ω. It follows from the Three Lines Theorem that $|g(s)| \leq 1$ whenever $s \in \Omega$.

Finally, observe that $g(\frac{1}{2}) = \tau(Tx(Tx)^*) = \|Tx\|_2^2$. This proves $\|Tx\|_2 \leq 1$. Hence T extends to a bounded operator on $L^2(M)$ of norm at most one. □

Definition 7.2. Let M be a von Neumann algebra with a faithful normal trace τ. Then (M, τ) has the *weak Haagerup property*, if there is a constant $C > 0$ and a net (T_α) of normal, completely bounded maps on M such that

(1) $\|T_\alpha\|_{\mathrm{cb}} \leq C$ for every α,
(2) $\langle T_\alpha x, y\rangle_\tau = \langle x, T_\alpha y\rangle_\tau$ for every $x, y \in M$,
(3) each T_α extends to a *compact* operator on $L^2(M, \tau)$,
(4) $T_\alpha x \to x$ ultraweakly for every $x \in M$.

The weak Haagerup constant $\Lambda_{\mathrm{WH}}(M, \tau)$ is defined as the infimum of those C for which such a net (T_α) exists, and if no such net exists we write $\Lambda_{\mathrm{WH}}(M, \tau) = \infty$. It is not hard to see that the infimum is actually a minimum and that $\Lambda_{\mathrm{WH}}(M, \tau) \geq 1$. If τ is implicit from the context (which will always be the case later on), we simply write $\Lambda_{\mathrm{WH}}(M)$ for $\Lambda_{\mathrm{WH}}(M, \tau)$.

Remark 7.3. The weak Haagerup constant of M is actually independent of the choice of faithful normal trace on M, that is, $\Lambda_{\mathrm{WH}}(M, \tau) = \Lambda_{\mathrm{WH}}(M, \tau')$ for any two faithful, normal traces τ and τ' on M (Proposition 8.4). Because of this, we sometimes write $\Lambda_{\mathrm{WH}}(M)$ instead of $\Lambda_{\mathrm{WH}}(M, \tau)$.

Remark 7.4. Note that by Proposition 7.1, condition (2) ensures that each T_α extends to a bounded operator on $L^2(M, \tau)$, and the extension is a self-adjoint operator on $L^2(M, \tau)$ with norm at most $\|T_\alpha\|$.

Remark 7.5. The choice of topology in which the net (T_α) converges to the identity map on M could be one of many without affecting the definition, as we will see now.

Suppose we are given a net (T_α) of normal, completely bounded maps on M such that

(1) $\|T_\alpha\|_{\mathrm{cb}} \leq C$ for every α,
(2) $\langle T_\alpha x, y\rangle_\tau = \langle x, T_\alpha y\rangle_\tau$ for every $x, y \in M$,
(3) each T_α extends to a compact operator on $L^2(M, \tau)$,
(4) $T_\alpha \to 1_M$ in the point-weak operator topology.

Since the closure of any convex set in $B(M,M)$ in the point-weak operator topology coincides with its closure in the point-strong operator topology, there is a net (S_β) such that $S_\beta \in \operatorname{conv}\{T_\alpha\}_\alpha$ and

(1') $\|S_\beta\|_{\mathrm{cb}} \leq C$ for every β,
(2') $\langle S_\beta x, y\rangle_\tau = \langle x, S_\beta y\rangle_\tau$ for every $x,y \in M$,
(3') each S_β extends to a compact operator on $L^2(M,\tau)$,
(4') $S_\beta \to 1_M$ in the point-strong operator topology.

Since the net (S_β) is norm-bounded and the strong operator topology coincides with the trace norm topology on bounded sets of M, condition (4') is equivalent to

(4") $\|S_\beta x - x\|_2 \to 0$ for any $x \in M$.

If we let $\widetilde{S}_\beta$ denote the extension of S_β to an operator on $L^2(M)$, then by Proposition 7.1 $\|\widetilde{S}_\beta\| \leq \|S_\beta\|$, so the net $(\widetilde{S}_\beta)$ is bounded, and hence (4") is equivalent to the condition that

(4"') $\widetilde{S}_\beta \to 1_{L^2(M)}$ strongly.

Using that $\|y^*\|_2 = \|y\|_2$ for any $y \in M$, condition (4") implies that

(4"") $\|(S_\beta x)^* - x^*\|_2 \to 0$ for any $x \in M$

so also, $S_\beta \to 1_M$ in the point-strong* operator topology. Finally, since the net (S_β) is bounded in norm, and since the ultrastrong and strong operator topologies coincide on bounded sets, we also obtain

(4""') $S_\beta \to 1_M$ in the point-ultrastrong* operator topology.

Let us see that the weak Haagerup property is indeed weaker than the (usual) Haagerup property. Let M be a von Neumann algebra with a faithful normal trace τ. We recall (see [1],[37]) that (M,τ) has the Haagerup property if there exists a net $(T_\alpha)_{\alpha\in A}$ of normal completely positive maps from M to itself such that

(1) $\tau \circ T_\alpha \leq \tau$ for every α,
(2) T_α extends to a compact operator on $L^2(M)$,
(3) $\|T_\alpha x - x\|_2 \to 0$ for every $x \in M$.

One can actually assume that $\tau \circ T_\alpha = \tau$ and that T_α is unital (see [37, Proposition 2.2]). Moreover, the Haagerup property does not depend on the choice of τ (see [37, Proposition 2.4]).

Proposition 7.6. *Let M be a von Neumann algebra with a faithful normal trace τ. If (M,τ) has the Haagerup property, then (M,τ) has the weak Haagerup property. In fact, $\Lambda_{\mathrm{WH}}(M,\tau) = 1$.*

Proof. The proof is merely an application of the following result (see [2, Lemma 2.5]). If T is a normal unital completely positive map on M, then $\tau \circ T = \tau$ if and only if there is a normal unital completely positive map $S\colon M \to M$ such that $\langle Tx, y\rangle_\tau = \langle x, Sy\rangle_\tau$ for every $x,y \in M$.

Suppose M has the Haagerup property and let $(T_\alpha)_{\alpha\in A}$ be a net of normal unital completely positive maps from M to itself such that

- $\tau \circ T_\alpha = \tau$ for every α,
- T_α extends to a compact operator on $L^2(M)$,
- $\|T_\alpha x - x\|_2 \to 0$ for every $x \in M$.

Then there are normal unital completely positive maps $S_\alpha\colon M \to M$ such that $\langle T_\alpha x, y\rangle_\tau = \langle x, S_\alpha y\rangle_\tau$ for every $x, y \in M$. Let $R_\alpha = \frac{1}{2}(T_\alpha + S_\alpha)$. Then R_α is normal unital completely positive and

- $\langle R_\alpha x, y\rangle_\tau = \langle x, R_\alpha y\rangle_\tau$ for every α,
- R_α extends to a compact operator on $L^2(M)$,
- $\|R_\alpha x - x\|_2 \to 0$ for every $x \in M$.

Since unital completely positive maps have completely bounded norm 1, this shows that $\Lambda_{\mathrm{WH}}(M,\tau) \leq 1$. This completes the proof. □

It is mentioned in [37] that injective finite von Neumann algebras have the Haagerup property. Indeed, it is a deep, and by now classical, result that injective von Neumann algebras are semidiscrete [11], [12], [13] (see [7, Theorem 9.3.4] for a proof of the finite case based on [58]). It then follows from [52, Proposition 4.6] that injective von Neumann algebras which admit a faithful normal trace have the Haagerup property. In particular, injective von Neumann algebras with a faithful normal trace have the weak Haagerup property.

We now turn to discrete groups and their group von Neumann algebras. For the moment, fix a discrete group Γ. We let λ denote the left regular representation of Γ on $\ell^2(\Gamma)$. The von Neumann algebra generated by $\lambda(\Gamma)$ inside $B(\ell^2(\Gamma))$ is the *group von Neumann algebra* denoted $L(\Gamma)$. It is equipped with the faithful normal trace τ given by $\tau(x) = \langle x\delta_e, \delta_e\rangle$ for $x \in L(\Gamma)$.

Theorem B. *Let Γ be a discrete group. The following conditions are equivalent.*

(1) *The group Γ has the weak Haagerup property.*
(2) *The group von Neumann algebra $L(\Gamma)$ (equipped with its canonical trace) has the weak Haagerup property.*

More precisely, $\Lambda_{\mathrm{WH}}(\Gamma) = \Lambda_{\mathrm{WH}}(L(\Gamma))$.

Proof. Suppose the net (u_α) of maps in $B_2(\Gamma) \cap C_0(\Gamma)$ witnesses the weak Haagerup property of Γ with $\|u_\alpha\|_{B_2} \leq C$ for every α. Upon replacing u_α with $\frac{1}{2}(u_\alpha + \bar{u}_\alpha)$ we may assume that u_α is real. Let $T_\alpha = M_{u_\alpha}$ be the corresponding multiplier on $L(\Gamma)$, that is

$$T_\alpha \lambda(g) = u_\alpha(g)\lambda(g), \quad g \in \Gamma. \tag{7.1}$$

Then T_α is normal and completely bounded on $L(\Gamma)$ with $\|T_\alpha\|_{\mathrm{cb}} = \|u_\alpha\|_{B_2}$. From (7.1) it follows that T_α extends to a diagonal operator $\widetilde{T}_\alpha$ on $L^2(L(\Gamma))$, when $L^2(L(\Gamma))$ has the standard basis $\{\lambda(g)\}_{g\in G}$. Since u_α is real, $\widetilde{T}_\alpha$ is self-adjoint. In particular $\langle T_\alpha x, y\rangle_\tau = \langle x, T_\alpha y\rangle_\tau$ for all $x, y \in L(\Gamma)$. Also, $\widetilde{T}_\alpha$ is compact, because $u_\alpha \in C_0(\Gamma)$. Since $u_\alpha \to 1$ pointwise and $\|u_\alpha\|_\infty \leq C$, it follows that $\widetilde{T}_\alpha \to 1_{L^2}$ strongly on $L^2(L(\Gamma))$. By Remark 7.5, this proves that $L(\Gamma)$ has the weak Haagerup property with $\Lambda_{\mathrm{WH}}(L(\Gamma)) \leq C$.

Conversely, suppose there is a net (T_α) of maps on $L(\Gamma)$ witnessing the weak Haagerup property of $L(\Gamma)$ with $\|T_\alpha\|_{\mathrm{cb}} \leq C$ for every α. Let

$$u_\alpha(g) = \tau(\lambda(g)^* T_\alpha(\lambda(g))).$$

Since $T_\alpha \to \mathrm{id}_{L(\Gamma)}$ point-ultraweakly, and τ is normal, it follows that $u_\alpha \to 1$ pointwise.

Let $V : \ell^2(\Gamma) \to \ell^2(\Gamma) \otimes \ell^2(\Gamma)$ be the isometry given by $V\delta_g = \delta_g \otimes \delta_g$. Observe then that

$$V^*(\lambda(g) \otimes \lambda(h))V = \begin{cases} \lambda(g) & \text{if } g = h, \\ 0 & \text{if } g \neq h, \end{cases}$$

so

$$V^*(\lambda(g) \otimes a)V = \tau(\lambda(g)^* a)\lambda(g).$$

By Fell's absorption principle [7, Theorem 2.5.5] there is a normal *-homomorphism $\sigma : L(\Gamma) \to L(\Gamma) \otimes L(\Gamma)$ such that $\sigma(\lambda(g)) = \lambda(g) \otimes \lambda(g)$. Using Lemma 8.1 we see that the operator $\mathrm{id}_{L(\Gamma)} \otimes T_\alpha$ on $L(\Gamma) \otimes L(\Gamma)$ exists, and it is easily verified that

$$V^*((\mathrm{id}_{L(\Gamma)} \otimes T_\alpha)(\lambda(g) \otimes \lambda(g)))V = u_\alpha(g)\lambda(g),$$

when $g \in \Gamma$, and so

$$V^*((\mathrm{id}_{L(\Gamma)} \otimes T_\alpha)(\sigma(a)))V = M_{u_\alpha}(a) \quad \text{for all } a \in L(\Gamma).$$

It follows that M_{u_α} is completely bounded and $u_\alpha \in B_2(\Gamma)$ with

$$\|u_\alpha\|_{B_2} = \|M_{u_\alpha}\|_{\mathrm{cb}} \leq \|T_\alpha\|_{\mathrm{cb}} \leq C,$$

where the first inequality follows from Proposition D.6 in [7].

It remains to show that $u_\alpha \in C_0(\Gamma)$. We may of course suppose that Γ is infinite. Since T_α extends to a compact operator on $L^2(L(\Gamma))$, it follows that

$$\lim_g \|T_\alpha(\lambda(g))\|_2 = 0,$$

because $(\lambda(g))_{g\in\Gamma}$ is orthonormal in $L^2(\Gamma)$. By the Cauchy-Schwarz inequality

$$|u_\alpha(g)| \leq \|T_\alpha \lambda(g)\|_2 \to 0 \qquad \text{as } g \to \infty.$$

This completes the proof. ☐

8. Hereditary properties II

In this section we prove hereditary results for the weak Haagerup property of von Neumann algebras. As an application we are able to show that the weak Haagerup property of a von Neumann algebra does not depend on the choice of the faithful normal trace.

When M is a finite von Neumann algebra with a faithful normal trace τ, and $p \in M$ is a non-zero projection, we let τ_p denote the faithful normal trace on pMp given as $\tau_p(x) = \tau(p)^{-1}\tau(x)$.

Since we have not yet proved that the weak Haagerup property of a von Neumann algebra does not depend on the choice of faithful normal trace (Proposition 8.4), we state Theorem C in the following more cumbersome way. Once we have shown Proposition 8.4, Theorem C makes sense and is Theorem C'.

Theorem C'. *Let (M,τ), (M_1,τ_1) and (M_2,τ_2) be a finite von Neumann algebras with faithful normal traces.*

(1) *Suppose* (M, τ) *is weakly Haagerup with constant* C*, and* $N \subseteq M$ *is a von Neumann subalgebra. Then* (N, τ) *is weakly Haagerup with constant at most* C*.*
(2) *Suppose* (M, τ) *is weakly Haagerup with constant* C*,* $p \in M$ *is a non-zero projection. Then* (pMp, τ_p) *is weakly Haagerup with constant at most* C*.*
(3) *Suppose* $1 \in N_1 \subseteq N_2 \subseteq \cdots$ *are von Neumann subalgebras of* M *generating all of* M*, and that there is an increasing sequence of non-zero projections* $p_n \in N_n$ *with strong limit* 1*. If each* $(p_n N_n p_n, \tau_{p_n})$ *is weakly Haagerup with constant at most* C*, then* (M, τ) *is weakly Haagerup with constant at most* C*.*
(4) *Suppose* (M_1, τ_1)*,* (M_2, τ_2)*,* $\ldots$ *is a (possibly finite) sequence of von Neumann algebras with faithful normal traces, and that* $\alpha_1, \alpha_2, \ldots$ *are strictly positive numbers with* $\sum_n \alpha_n = 1$*. Then the weak Haagerup constant of*

$$\left(\bigoplus_n M_n, \bigoplus_n \alpha_n \tau_n \right)$$

equals $\sup_n \Lambda_{\mathrm{WH}}(M_n, \tau_n)$*, where* $\bigoplus_n \alpha_n \tau_n$ *denotes the trace defined by*

$$\left(\bigoplus_n \alpha_n \tau_n \right) (x_n) = \sum_n \alpha_n \tau_n(x_n), \quad (x_n)_n \in \bigoplus_n M_n.$$

.
(5) *Suppose* (M_1, τ_1) *and* (M_2, τ_2) *are weakly Haagerup with constant* C_1 *and* C_2*, respectively. Then the tensor product* $(M_1 \bar{\otimes} M_2, \tau_1 \bar{\otimes} \tau_2)$ *is weakly Haagerup with constant at most* $C_1 C_2$*.*

Proof.

(1) Let $E : M \to N$ be the unique trace-preserving conditional expectation. Given a net (T_α) witnessing the weak Haagerup property of M we let $S_\alpha = E \circ T_\alpha|_N$. Clearly, $\|S_\alpha\|_{\mathrm{cb}} \leq \|T_\alpha\|_{\mathrm{cb}}$. Since E is an N-bimodule map, trace-preserving and positive, an easy calculation shows that $\langle S_\alpha x, y \rangle = \langle x, S_\alpha y \rangle$ for every $x, y \in N$.

As is customary, the Hilbert space $L^2(N)$ is naturally identified with the closed subspace of $L^2(M)$ spanned by $N \subseteq M \subseteq L^2(M)$, and the conditional expectation $E \colon M \to N$ extends to a projection $e_N : L^2(M) \to L^2(N)$. Since T_α extends to a compact operator $\widetilde{T}_\alpha$ on $L^2(M)$, it follows that $E \circ T_\alpha$ extends to the compact operator $e_N \widetilde{T}_\alpha$ on $L^2(M)$. Hence S_α extends to the compact operator $e_N \widetilde{T}_\alpha|_{L^2(N)}$ on $L^2(N)$.

Since E is normal, $E|_N = 1_N$, and $T_\alpha \to 1_M$ point-ultraweakly, we obtain $S_\alpha \to 1_N$ point-ultraweakly.

(2) Let $P : M \to pMp$ be the map $P(x) = pxp$, $x \in M$. Then P is unital and completely positive. Given a net (T_α) witnessing the weak Haagerup property of M we let $S_\alpha = P \circ T_\alpha|_{pMp}$. Clearly, $\|S_\alpha\|_{\mathrm{cb}} \leq \|T_\alpha|_{pMp}\|_{\mathrm{cb}} \leq \|T_\alpha\|_{\mathrm{cb}}$. An easy calculation shows that

$$\langle S_\alpha x, y \rangle_{\tau_p} = \langle x, S_\alpha y \rangle_{\tau_p} \quad \text{for all } x, y \in pMp.$$

Let $V : L^2(pMp) \to L^2(M)$ be the map $Vx = \tau(p)^{-1/2} x$. Then V is an isometry, and evidently $V^* x = \tau(p)^{1/2} pxp$ for every $x \in M$. It follows that on pMp we have $S_\alpha = V^* T_\alpha V$. Hence S_α extends to the compact operator

$$\widetilde{S}_\alpha = V^* \widetilde{T}_\alpha V,$$

on $L^2(pMp)$, where $\widetilde{T}_\alpha$ denotes the extension of T_α to a compact operator on $L^2(M)$.

Since P is normal, it follows that $S_\alpha \to 1$ point-ultraweakly.

(3) We denote the trace-preserving conditional expectation $M \to N_n$ by E_n and its extension to a projection $L^2(M) \to L^2(N_n)$ by e_n. Note first that since M is generated by the sequence N_n, for each $x \in M$ we have $E_n(x) \to x$ strongly. Indeed, the union of the increasing sequence of Hilbert spaces $L^2(N_n)$ is a norm dense subspace of the Hilbert space $L^2(M)$, and thus $e_n \nearrow 1_{L^2(M)}$ strongly. In other words, $\|E_n(x) - x\|_\tau \to 0$.

For each $n \in \mathbb{N}$ we define $S_n : M \to p_n N_n p_n$ by $S_n(x) = p_n E_n(x) p_n$. It follows that $S_n(x) \to x$ strongly.

Let $F \subseteq M$ be a finite set, and let $\varepsilon > 0$ be given. Choose n such that

$$\|S_n(x) - x\|_\tau \leq \varepsilon \quad \text{for all } x \in F.$$

By assumption there is a completely bounded map $R : p_n N_n p_n \to p_n N_n p_n$ such that $\|R\|_{\mathrm{cb}} \leq C$, R extends to a self-adjoint compact operator on $L^2(p_n N_n p_n)$, and

$$\|R(S_n(x)) - S_n(x)\|_{\tau_{p_n}} \leq \varepsilon \quad \text{for all } x \in F.$$

Let $T_\alpha = R \circ S_n$, where $\alpha = (F, \varepsilon)$. Clearly,

$$\|T_\alpha x - x\|_\tau \leq 2\varepsilon \quad \text{when } x \in F.$$

It follows that $T_\alpha \to 1$ point-strongly.

Since S_n is unital and completely positive, we get $\|T_\alpha\|_{\mathrm{cb}} \leq \|R\|_{\mathrm{cb}} \leq C$. When $x, y \in M$ we have

$$\begin{aligned} \langle T_\alpha x, y\rangle_\tau &= \langle R(p_n E_n(x) p_n), p_n E_n(y) p_n\rangle_\tau \\ &= \langle p_n E_n(x) p_n, R(p_n E_n(y) p_n)\rangle_\tau = \langle x, T_\alpha y\rangle_\tau \end{aligned}$$

using the properties of E_n and R. Since T_α is the composition

$$M \xrightarrow{E_n} N_n \xrightarrow{P_n} p_n N_n p_n \xrightarrow{R} p_n N_n p_n \xrightarrow{\iota} N_n \xrightarrow{\iota} M,$$

where ι denotes inclusion, it follows that the extension of T_α to $L^2(M)$ is compact, because the extension of R to $L^2(p_n N_n p_n)$ is compact:

$$L^2(M) \xrightarrow{e_n} L^2(N_n) \xrightarrow{\widetilde{P}_n} L^2(p_n N_n p_n) \xrightarrow{\widetilde{R}} L^2(p_n N_n p_n) \xrightarrow{\widetilde{\iota}} L^2(N_n) \xrightarrow{\widetilde{\iota}} L^2(M).$$

The net $(T_\alpha)_{\alpha\in A}$ indexed by $A = \{(F, \varepsilon) \mid F \subseteq M \text{ finite}, \ \varepsilon > 0\}$ shows that the weak Haagerup constant of M is at most C.

(4) It is enough to show that the weak Haagerup constant of $M_1 \oplus M_2$ with respect to the trace $\tau = \lambda\tau_1 \oplus (1-\lambda)\tau_2$ equals

$$\max\{\Lambda_{\mathrm{WH}}(M_1, \tau_1), \Lambda_{\mathrm{WH}}(M_2, \tau_2)\}$$

for any $0 < \lambda < 1$, and then apply induction and (3) to obtain the general case of (4). We only prove

$$\Lambda_{\mathrm{WH}}(M_1 \oplus M_2) \leq \max\{\Lambda_{\mathrm{WH}}(M_1), \Lambda_{\mathrm{WH}}(M_2)\}, \tag{8.1}$$

since the other inequality is clear from (2).

Two points should be made. Firstly, if T_1 and T_2 are normal completely bounded maps on M_1 and M_2 respectively, then $T_1 \oplus T_2$ is a normal completely bounded map on M with completely bounded norm

$$\|T_1 \oplus T_2\|_{\mathrm{cb}} = \max\{\|T_1\|_{\mathrm{cb}}, \|T_2\|_{\mathrm{cb}}\}.$$

Secondly, the map $V(x \oplus y) = \lambda^{1/2}x \oplus (1-\lambda)^{1/2}y$ on $M_1 \oplus M_2$ extends to a unitary operator

$$V\colon L^2(M_1 \oplus M_2, \tau) \to L^2(M_1, \tau_1) \oplus L^2(M_2, \tau_2).$$

Now, let $\varepsilon > 0$ be given and let $(S_\alpha)_{\alpha \in A}$ and $(T_\beta)_{\beta \in B}$ be normal completely bounded maps on M_1 and M_2, respectively such that

- $\|S_\alpha\|_{\mathrm{cb}} \le \Lambda_{\mathrm{WH}}(M_1, \tau_1) + \varepsilon$ for every α,
- $\langle S_\alpha x, y\rangle_{\tau_1} = \langle x, S_\alpha y\rangle_{\tau_1}$ for every $x, y \in M_1$,
- each S_α extends to a *compact* operator on $L^2(M_1, \tau_1)$,
- $S_\alpha x \to x$ ultraweakly for every $x \in M_1$,

and similar properties hold for $(T_\beta)_{\beta \in B}$ and M_2. We may assume that $A = B$. Now, let $R_\alpha = S_\alpha \oplus T_\alpha$. Using the net (R_α) it is easy to show that

$$\Lambda_{\mathrm{WH}}(M_1 \oplus M_2) \le \max\{\Lambda_{\mathrm{WH}}(M_1), \Lambda_{\mathrm{WH}}(M_2)\} + \varepsilon.$$

Letting $\varepsilon \to 0$ we obtain (8.1).

(5) We remark that the product trace $\tau_1 \bar{\otimes} \tau_2$ on the von Neumann algebraic tensor product $M_1 \bar{\otimes} M_2$ is a faithful normal trace (see [56, Corollary IV.5.12]). Suppose we are given nets $(S_\alpha)_{\alpha \in A}$ and $(T_\beta)_{\beta \in B}$ witnessing the weak Haagerup property of M_1 and M_2, respectively. By Remark 7.5 we may assume that

$$\widetilde{S}_\alpha \to 1_{L^2(M_1)} \text{ strongly} \quad \text{and} \quad \widetilde{T}_\beta \to 1_{L^2(M_2)} \text{ strongly}, \tag{8.2}$$

where $\widetilde{S}_\alpha$ and $\widetilde{T}_\beta$ denote the extensions to operators on $L^2(M_1)$ and $L^2(M_2)$, respectively. For each $\gamma = (\alpha, \beta) \in A \times B$, we consider the map $R_\gamma = S_\alpha \bar{\otimes} T_\beta$ given by Lemma 8.1 below. Then R_γ is a normal, completely bounded map on $M \bar{\otimes} N$ with $\|R_\gamma\|_{\mathrm{cb}} \le \|S_\alpha\|_{\mathrm{cb}} \|T_\beta\|_{\mathrm{cb}}$. Let $\tau = \tau_1 \bar{\otimes} \tau_2$ be the product trace. We claim that when $A \times B$ is given the product order, the net $(R_\gamma)_{\gamma \in A \times B}$ witnesses the weak Haagerup property of $M_1 \bar{\otimes} M_2$, i.e. that

(a) $\langle R_\gamma x, y\rangle_\tau = \langle x, R_\gamma y\rangle_\tau$ for every $x, y \in M_1 \bar{\otimes} M_2$.
(b) Each R_γ extends to a *compact* operator $\widetilde{R}_\gamma$ on $L^2(M_1 \bar{\otimes} M_2, \tau)$.
(c) $\widetilde{R}_\gamma \to 1_{L^2(M_1 \bar{\otimes} M_2)}$ strongly.

Condition (a) is easy to check on elementary tensors, and then when x and y are in the algebraic tensor product $M_1 \otimes M_2$. Since the unit ball of the algebraic tensor product $M_1 \otimes M_2$ is dense in the unit ball of $M_1 \bar{\otimes} M_2$ in the strong* operator topology, it follows that (a) holds for arbitrary $x, y \in M_1 \bar{\otimes} M_2$.

If $V : L^2(M_1) \otimes L^2(M_2) \to L^2(M_1 \bar{\otimes} M_2)$ is the unitary which is the identity on $M_1 \otimes M_2$, then

$$R_\gamma = V(S_\alpha \bar{\otimes} T_\beta)V^*$$

Thus, since the tensor product of two compact operators is compact, R_γ extends to a compact operator on $L^2(M_1 \bar{\otimes} M_2)$.

Condition (c) follows easily from (8.2) and the general fact that if two bounded nets (V_α) and (W_β) of operators on Hilbert spaces converge strongly with limits V and W, then the net $V_\alpha \otimes W_\beta$ converges strongly to $V \otimes W$. □

In the course of proving (5) above, we postponed the proof of Lemma 8.1 concerning the existence of the tensor product of two normal, completely bounded map between von

Neumann algebras. A version of the lemma exists for completely contractive maps between operator spaces, when the tensor product under consideration is the operator space projective tensor product (see [22, Proposition 7.1.3]) or the operator space injective tensor product (see [22, Proposition 8.1.5]). The operator space injective tensor product coincides with the minimal C^*-algebraic tensor product, when the operator spaces are von Neumann algebras (see [22, Proposition 8.1.6]). Also, a version of the lemma exists for normal, completely positive maps between von Neumann algebras ([56, Proposition IV.5.13]). See also [17, Lemma 1.5].

Lemma 8.1. *Suppose M_i and N_i ($i = 1, 2$) are von Neumann algebras and $T_i : M_i \to N_i$ are normal, completely contractive maps. Then there is a normal, completely contractive map $T_1 \bar{\otimes} T_2 : M_1 \bar{\otimes} M_2 \to N_1 \bar{\otimes} N_2$ such that*

$$T_1 \bar{\otimes} T_2(x_1 \otimes x_2) = T_1 x_1 \otimes T_2 x_2 \quad \textit{for all } x_i \in M_i \ (i = 1, 2).$$

Proof. It follows from [22, Proposition 8.1.5] and [22, Proposition 8.1.6] that there is a completely contractive map $T_1 \otimes T_2 : M_1 \otimes_{\min} M_2 \to N_1 \otimes_{\min} N_2$ between the minimal tensor products such that

$$T_1 \otimes T_2(x_1 \otimes x_2) = T_1 x_1 \otimes T_2 x_2 \quad \text{for all } x_i \in M_i \ (i = 1, 2).$$

We must show that $T_1 \otimes T_2$ extends continuously to a completely contractive map from the ultraweak closure $M_1 \bar{\otimes} M_2$ of $M_1 \otimes_{\min} M_2$. First we show that $T_1 \otimes T_2$ is ultraweakly continuous. For this, it will suffice to show that $\rho \circ T_1 \otimes T_2$ is ultraweakly continuous on $M_1 \otimes_{\min} M_2$ for each ultraweakly continuous functional $\rho \in (N_1 \bar{\otimes} N_2)_*$.

Suppose first that ρ is of the form $\rho_1 \otimes \rho_2$ for some $\rho_1 \in (N_1)_*$ and $\rho_2 \in (N_2)_*$. Then if we let $\sigma_i = \rho_i \circ T_i$, it is clear that $\sigma_1 \otimes \sigma_2$ is ultraweakly continuous [39, 11.2.7], and $\rho \circ (T_1 \otimes T_2) = \sigma_1 \otimes \sigma_2$. In general, ρ is the norm limit of a sequence of functionals ρ_n where each ρ_n is a finite linear combination of ultraweakly continuous product functionals [39, 11.2.8], and it then follows from [39, 10.1.15] that $\rho \circ T_1 \otimes T_2$ is ultraweakly continuous.

Now, from [39, 10.1.10] it follows that $T_1 \otimes T_2$ extends (uniquely) to an ultraweakly continuous contraction $M_1 \bar{\otimes} M_2 \to N_1 \bar{\otimes} N_2$. The same argument applied to $T_1 \otimes T_2 \otimes \mathrm{id}_n$, where $\mathrm{id}_n \colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is the identity, shows that $T_1 \bar{\otimes} T_2$ is not only contractive, but completely contractive. □

Remark 8.2. Theorem C (1)–(3) may conveniently be expressed as the following inequalities.

$$\begin{aligned} \Lambda_{\mathrm{WH}}(N) &\leq \Lambda_{\mathrm{WH}}(M), \\ \Lambda_{\mathrm{WH}}(pMp) &\leq \Lambda_{\mathrm{WH}}(M), \\ \Lambda_{\mathrm{WH}}(M) &= \sup_{n \in \mathbb{N}} \Lambda_{\mathrm{WH}}(p_n N_n p_n), \end{aligned}$$

when $N \subseteq M$ is a subalgebra, $p \in M$ is a non-zero projection, $(N_n)_{n \geq 1}$ is an increasing sequence of subalgebras generating M with projections $p_n \in N_n$, $p_n \nearrow 1$. Theorem C (5) reads

$$\Lambda_{\mathrm{WH}}(M_1 \bar{\otimes} M_2) \leq \Lambda_{\mathrm{WH}}(M_1) \Lambda_{\mathrm{WH}}(M_2). \tag{8.3}$$

Remark 8.3. We do not know if $\Lambda_{\mathrm{WH}}(M_1 \bar{\otimes} M_2) = \Lambda_{\mathrm{WH}}(M_1)\Lambda_{\mathrm{WH}}(M_2)$ holds for any two finite von Neumann algebras M_1 and M_2. The corresponding result for the weak amenability constant Λ_{WA} is known to be true, [53, Theorem 4.1]. If either $\Lambda_{\mathrm{WH}}(M_1) = 1$ or $\Lambda_{\mathrm{WH}}(M_2) = 1$, then equality holds in (8.3).

We will now show that the weak Haagerup property does not depend on the choice of the faithful normal trace. The basic idea of the proof is to apply the noncommutative Radon-Nikodym theorem. Since the Radon-Nikodym derivative in general may be an unbounded operator, we will need to cut it into pieces that are bounded and then apply Theorem C (4) in the end.

Let M be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$, and let M' denote the commutant of M. A (possibly unbounded) closed operator h is affiliated with M if $hu = uh$ (with agreement of domains) for every unitary $u \in M'$. If h is bounded, then by the bicommutant theorem h is affiliated with M if and only if $h \in M$. In general, if h is affiliated with M, then $f(h)$ lies in M for every bounded Borel function f on $[0, \infty[$. See e.g. [54, Appendix B] for details.

We recall the version of the Radon-Nikodym theorem that we will need. We refer to [48] for more details. We denote the center of M by $Z(M)$. Let τ be a faithful normal trace on M and suppose h is a self-adjoint, positive operator affiliated with $Z(M)$. For $\varepsilon > 0$ put $h_\varepsilon = h(1 + \varepsilon h)^{-1}$. Then $h_\varepsilon \in Z(M)_+$ for every $\varepsilon > 0$. When $x \in M_+$, define the number $\tau(hx)$ by

$$\tau(hx) = \lim_{\varepsilon \to 0} \tau(h_\varepsilon x). \tag{8.4}$$

Then τ' defined by $\tau'(x) = \tau(hx)$ is a normal semifinite weight on M. If moreover $\lim_\varepsilon \tau(h_\varepsilon) = 1$, then (8.4) makes sense for all $x \in M$ and defines a normal trace τ' on M. The Radon-Nikodym theorem [48, Theorem 5.4] gives a converse to this: Given any normal trace τ' on M there is a unique self-adjoint positive operator h affiliated with $Z(M)$ such that $\tau'(x) = \tau(hx)$ for every $x \in M$.

Proposition 8.4. *Let M be a von Neumann algebra with two faithful normal traces τ and τ'. Then M has the weak Haagerup property with respect to τ if and only if M has the weak Haagerup property with respect to τ'. More precisely,*

$$\Lambda_{\mathrm{WH}}(M, \tau) = \Lambda_{\mathrm{WH}}(M, \tau').$$

Proof. Let $\varepsilon > 0$ be arbitrary. We will show that

$$\Lambda_{\mathrm{WH}}(M, \tau') \leq \Lambda_{\mathrm{WH}}(M, \tau)(1 + \varepsilon). \tag{8.5}$$

By symmetry and letting $\varepsilon \to 0$, this will complete the proof. We may of course assume that $\Lambda_{\mathrm{WH}}(M, \tau) < \infty$, since otherwise (8.5) obviously holds.

We let $Z(M)$ denote the center of M. Suppose first that there is a positive, invertible operator $h \in Z(M)$ such that $\tau'(x) = \tau(hx)$ for every $x \in M$ and h has spectrum $\sigma(h)$ contained in the interval $[c(1+\varepsilon)^n, c(1+\varepsilon)^{n+1}]$ for some $c > 0$ and some integer n. Note that then

$$\|h^{1/2}\| \|h^{-1/2}\| \leq (1 + \varepsilon)^{1/2}.$$

Let (T_α) be a net of normal, completely bounded operators on M such that

(1) $\|T_\alpha\|_{\mathrm{cb}} \leq \Lambda_{\mathrm{WH}}(M, \tau)(1 + \varepsilon)^{1/2}$ for every α,
(2) $\langle T_\alpha x, y \rangle_\tau = \langle x, T_\alpha y \rangle_\tau$ for every $x, y \in M$,
(3) each T_α extends to a compact operator on $L^2(M, \tau)$,
(4) $T_\alpha x \to x$ ultraweakly for every $x \in M$.

Since h belongs to $Z(M)_+$, it is easily verified that the map $U\colon M \to M$ defined by $Ux = h^{1/2}x$ extends to an isometry $L^2(M,\tau') \to L^2(M,\tau)$, and since h is invertible, U is actually a unitary. We let S_α be the operator on M defined as $S_\alpha = U^* T_\alpha U$, that is $S_\alpha x = h^{-1/2} T_\alpha(h^{1/2}x)$. Then S_α is normal and completely bounded with

$$\|S_\alpha\|_{\mathrm{cb}} \le \|h^{1/2}\| \|h^{-1/2}\| \|T_\alpha\|_{\mathrm{cb}} \le \Lambda_{\mathrm{WH}}(M,\tau)(1+\varepsilon).$$

Since U is a unitary, it is clear from (2), (3) and (4) that S_α extends to a self-adjoint, compact operator on $L^2(M,\tau')$ and that $S_\alpha x \to x$ ultraweakly for every $x \in M$. This shows that

$$\Lambda_{\mathrm{WH}}(M,\tau') \le \Lambda_{\mathrm{WH}}(M,\tau)(1+\varepsilon).$$

In general, there is a (possibly unbounded) unique self-adjoint positive operator h affiliated with $Z(M)$ such that $\tau'(x) = \tau(hx)$. For each $n \in \mathbb{Z}$ let p_n denote the spectral projection of h defined as

$$p_n = 1_{[(1+\varepsilon)^n,(1+\varepsilon)^{n+1}[}(h),$$

and let $q = 1_{\{0\}}(h)$. Then p_n and q are projections in $Z(M)$. Since (the closure of) hq is zero we see that

$$\tau'(q) = \tau(hq) = \tau(0) = 0,$$

and then we must have $q = 0$, since τ' is faithful. Hence

$$\sum_{n=-\infty}^{\infty} p_n = 1_{]0,\infty[}(h) = 1.$$

Let I be the set of those $n \in \mathbb{Z}$ for which $p_n \neq 0$, and for $n \in I$ let M_n denote the von Neumann algebra $p_n M$ with faithful normal trace $\tau_n = \tau(p_n)^{-1}\tau$. Then from the decomposition

$$M = \bigoplus_{n\in I} M_n$$

we get by Theorem C (4) that

$$\Lambda_{\mathrm{WH}}(M,\tau) = \sup_{n\in I} \Lambda_{\mathrm{WH}}(M_n,\tau_n).$$

Similarly,

$$\Lambda_{\mathrm{WH}}(M,\tau') = \sup_{n\in I} \Lambda_{\mathrm{WH}}(M_n,\tau_n'),$$

where $\tau_n' = \tau'(p_n)^{-1}\tau'$.

For $n \in I$, let $f_n\colon \mathbb{R}_+ \to \mathbb{R}_+$ be defined by $f_n(t) = t1_{[(1+\varepsilon)^n,(1+\varepsilon)^{n+1}[}(t)$ and put $h_n = c_n f_n(h)$, where $c_n = \tau(p_n)\tau'(p_n)^{-1}$. Then $h_n \in Z(M_n)_+$ is invertible in M_n with spectrum $\sigma(h_n) \subseteq [c_n(1+\varepsilon)^n, c_n(1+\varepsilon)^{n+1}]$ and

$$\tau_n'(x) = \tau_n(h_n x) \quad \text{for every } x \in M_n.$$

By the first part of the proof applied to M_n we get that $\Lambda_{\mathrm{WH}}(M_n,\tau_n') \le \Lambda_{\mathrm{WH}}(M_n,\tau_n)(1+\varepsilon)$ for every $n \in I$. Putting things together we obtain

$$\Lambda_{\mathrm{WH}}(M,\tau') = \sup_{n\in I} \Lambda_{\mathrm{WH}}(M_n,\tau_n') \le \sup_{n\in I} \Lambda_{\mathrm{WH}}(M_n,\tau_n)(1+\varepsilon) = \Lambda_{\mathrm{WH}}(M,\tau)(1+\varepsilon).$$

This proves (8.5), and the proof is complete.

□

9. AN EXAMPLE

In this section we give an example of two von Neumann algebras, in fact II_1 factors arising from discrete groups, with different weak Haagerup constants. None of the other approximation properties mentioned in the introduction (see Figure 1) are useful as invariants to distinguish precisely these two factors (see Remark 9.1).

It is well-known that if Γ is an infinite discrete group, then $L(\Gamma)$ is a II_1 factor if and only if all conjugacy classes in Γ are infinite except for the conjugacy class of the neutral element. Such groups are called ICC (infinite conjugacy classes).

It is known from [4] that every arithmetic subgroup of $\mathrm{Sp}(1,n)$ is a lattice. Let $\mathbb{H}_{\mathrm{int}}$ be the quaternion integers $\mathbb{Z}+\mathbb{Z}i+\mathbb{Z}j+\mathbb{Z}k$ inside the quaternion division ring $\mathbb{H}$, and let $n\geq 2$ be fixed. Then the group Γ consisting of matrices in $\mathrm{Sp}(1,n)$ with entries in $\mathbb{H}_{\mathrm{int}}$ is an arithmetic subgroup of $\mathrm{Sp}(1,n)$ and hence a lattice. To be explicit, Γ consists of $(n+1)\times(n+1)$ matrices with entries in $\mathbb{H}_{\mathrm{int}}$ that preserve the Hermitian form

$$h(x,y)=x_0\overline{y_0}-\sum_{k=1}^{n}x_m\overline{y_m},\quad x=(x_i),\ y=(y_i)\in\mathbb{H}^{n+1}.$$

Here $\mathbb{H}^{n+1}$ is regarded as a right $\mathbb{H}$-module. If I denotes the identity matrix in $\mathrm{Sp}(1,n)$, then the center of $\mathrm{Sp}(1,n)$ is $\{\pm I\}$, and it is proved in [16, p. 547] that $\Gamma_0=\Gamma/\{\pm I\}$ is an ICC group.

Let $H=\mathbb{Z}/2\wr\mathbb{F}_2$ be the wreath product of $\mathbb{Z}/2$ and $\mathbb{F}_2$ (see Section 1). Then H is ICC (see [51, Corollary 4.2]) and the direct product group $\Gamma_1=\Gamma_0\times H$ is also ICC (see [51, p. 74]).

Let $\Gamma_2=\mathbb{Z}^2\rtimes\mathrm{SL}_2(\mathbb{Z})$. It is well-known that Γ_2 is ICC and a lattice in $\mathbb{R}^2\rtimes\mathrm{SL}_2(\mathbb{R})$. We claim that the II_1 factors $L(\Gamma_1)$ and $L(\Gamma_2)$ are not isomorphic. Indeed, we show below that their weak Haagerup constants differ. Since both von Neumann algebras are II_1 factors, there is a unique trace on each of them, so any isomorphism would necessarily be trace-preserving.

Using Theorem B, Proposition 5.5/Remark 5.6, Proposition 5.4, Corollary 5.17 and Theorem 6.2 we get

$$\begin{aligned}\Lambda_{\mathrm{WH}}(L(\Gamma_1))&=\Lambda_{\mathrm{WH}}(\Gamma_1)\\&=\Lambda_{\mathrm{WH}}(\Gamma_0)\Lambda_{\mathrm{WH}}(H)\\&=\Lambda_{\mathrm{WH}}(\Gamma)\\&=\Lambda_{\mathrm{WH}}(\mathrm{Sp}(1,n))\\&\leq\Lambda_{\mathrm{WA}}(\mathrm{Sp}(1,n))\\&=2n-1<\infty.\end{aligned}$$

In [30, Theorem D] it is proved that $\mathbb{R}^2\rtimes\mathrm{SL}_2(\mathbb{R})$ does not have the weak Haagerup property. Thus, using also Theorem B and Corollary 5.17 we get

$$\Lambda_{\mathrm{WH}}(L(\mathbb{Z}^2\rtimes\mathrm{SL}_2(\mathbb{Z})))=\Lambda_{\mathrm{WH}}(\mathbb{Z}^2\rtimes\mathrm{SL}_2(\mathbb{Z}))=\Lambda_{\mathrm{WH}}(\mathbb{R}^2\rtimes\mathrm{SL}_2(\mathbb{R}))=\infty.$$

In view of Theorem C this shows that $L(\Gamma_2)$ cannot be not embedded into any corner of any subalgebra of $L(\Gamma_1)$. In particular, $L(\Gamma_1)$ and $L(\Gamma_2)$ are not isomorphic.

Remark 9.1. We remark that Γ_1 and Γ_2 do not have the Haagerup property. Also,

$$\Lambda_{\mathrm{WA}}(\Gamma_1) = \Lambda_{\mathrm{WA}}(\Gamma_2) = \infty,$$

and Γ_1 and Γ_2 both have AP. Thus, none of these three approximation properties distinguish $L(\Gamma_1)$ and $L(\Gamma_2)$.

Appendices

The appendices contain a collection of results that are used to show the equivalence of several definitions of the weak Haagerup property and of weak amenability. The results are certainly known to experts, but some of the results below do not appear explicitly or in this generality in the literature.

In all of the following G is a locally compact group equipped with left Haar measure dx. For definitions concerning the Fourier algebra $A(G)$, the Herz-Schur multipliers $B_2(G)$ and its predual $Q(G)$ we refer to Section 3.

Appendix A. Topologies on the unit ball of $B_2(G)$

We are concerned with three different topologies on bounded sets in $B_2(G)$ besides the norm topology: The first topology is the weak*-topology, where we view $B_2(G)$ as the dual space of $Q(G)$. It will be referred to as the $\sigma(B_2, Q)$-topology. The second topology is the locally uniform topology, i.e., the topology determined by uniform convergence on compact subsets of G. The third topology is the point-norm topology, where we think of elements in $B_2(G)$ as operators on $A(G)$. The following lemma reveals the relations between these topologies.

Lemma A.1. *Let (u_α) be a net in $B_2(G)$ and let $u \in B_2(G)$.*

1. *If $\|(u_\alpha - u)w\|_A \to 0$ for every $w \in A(G)$, then $u_\alpha \to u$ uniformly on compacts.*
2. *If the net is bounded and $u_\alpha \to u$ uniformly on compacts, then $u_\alpha \to u$ in the $\sigma(B_2, Q)$-topology.*

Proof. Suppose $\|(u_\alpha - u)w\|_A \to 0$ for every $w \in A(G)$, and let $L \subseteq G$ be a compact subset. By [23, Lemma 3.2] there is a $w \in A(G)$ which takes the value 1 on L. Hence

$$\sup_{x\in L} |u_\alpha(x) - u(x)| \leq \|(u_\alpha - u)w\|_\infty \leq \|(u_\alpha - u)w\|_A \to 0.$$

This proves (1).

Suppose $u_\alpha \to u$ uniformly on compacts. Since the net (u_α) is bounded, and $C_c(G)$ is dense in $Q(G)$, it will suffice to prove $\langle u_\alpha, f\rangle \to \langle u, f\rangle$ for every $f \in C_c(G)$. Let $L = \operatorname{supp} f$. Then since $u_\alpha \to u$ uniformly on L, we obtain

$$\langle f, u_\alpha\rangle = \int_L f(x)u_\alpha(x)\, dx \to \int_L f(x)u(x)\, dx = \langle f, u\rangle.$$

This proves (2). ◻

Remark A.2. In the proof of (2), the assumption of boundedness is essential. In general, there always exist (possibly unbounded) nets (u_α) in $A(G) \subseteq B_2(G)$ converging to 1 uniformly on compacts (use [23, Lemma 3.2]), but for groups without the Approximation Property (AP) such as $\mathrm{SL}_3(\mathbb{Z})$ no such net can converge to 1 in the $\sigma(B_2, Q)$-topology (see [31] and [43, Theorem C]).

Lemma A.3. *The unit ball of $B_2(G)$ is closed in $C(G)$ under locally uniform convergence and even pointwise convergence.*

Proof. This is obvious from the equivalence (1) $\iff$ (2) in Proposition 3.1. □

Lemma A.4. *The unit ball of $B_2(G)$ is closed in $B_2(G)$ in the $\sigma(B_2, Q)$-topology.*

Proof. This is a consequence of Banach-Alaoglu's Theorem. □

APPENDIX B. AVERAGE TRICKS

B.1. **The convolution trick.** In all of the following h is a continuous, non-negative, compactly supported function on G such that $\int h(x)\,dx = 1$. Such functions exist, and if G is a Lie group, one can even take h to be smooth.

The convolution trick consists of replacing a given convergent net (u_α) in $B_2(G)$ with the convoluted net $h * u_\alpha$ to obtain convergence in a stronger topology. Recall that the convolution of h with $u \in L^p(G)$ is defined by

$$(h * u)(x) = \int_G h(y)u(y^{-1}x)\,dy = \int_G h(xy)u(y^{-1})\,dy, \quad x \in G.$$

Lemma B.1 (The convolution trick – Part I). *Let $u \in C(G)$ be given and let h be as above.*

(1) *If $u \in C_c(G)$, then $h * u \in C_c(G)$.*
(2) *If $u \in C_0(G)$, then $h * u \in C_0(G)$.*
(3) *If u is uniformly bounded, then $\|h * u\|_\infty \le \|u\|_\infty$.*
(4) *If $u \in A(G)$, then $h * u \in A(G)$ and $\|h * u\|_A \le \|u\|_A$.*
(5) *If $u \in B_2(G)$, then $h * u \in B_2(G)$ and $\|h * u\|_{B_2(G)} \le \|u\|_{B_2(G)}$.*
(6) *If G is a Lie group and $h \in C_c^\infty(G)$, then $h * u \in C^\infty(G)$.*

Proof.

We leave (1)–(3) as an exercise.

(4) If $u \in A(G)$, then $u = f * \check{g}$ for some $f, g \in L^2(G)$ with $\|u\|_A = \|f\|_2\,\|g\|_2$. Then $h * u = (h * f) * \check{g}$. Since $h * f \in L^2(G)$ with $\|h * f\|_2 \le \|f\|_2$ (see [24, p. 52]), it follows that $h * u \in A(G)$ with $\|h * u\|_A \le \|u\|_A$.

(5) We use the characterization of Herz-Schur multipliers given in Proposition 3.1. Given $y \in G$ we let $y.u$ be defined by $(y.u)(x) = u(y^{-1}x)$ for $x \in G$. Clearly, $y.u \in B_2(G)$ and $\|y.u\|_{B_2} = \|u\|_{B_2}$.

Let $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in G$ in be given and let $m \in M_n(\mathbb{C})$ be the $n \times n$ matrix

$$m = \left(u(x_j^{-1}x_i)\right)_{i,j=1}^n.$$

More generally, for any $y \in G$, let $y.m$ denote the matrix

$$y.m = \big(u(y^{-1}x_j^{-1}x_i)\big)_{i,j=1}^n$$

Clearly, $\|y.m\|_S \le \|y.u\|_{B_2} = \|u\|_{B_2}$ and $y \mapsto y.m$ is continuous from G into $M_n(\mathbb{C})$, when $M_n(\mathbb{C})$ is equipped with the Schur norm. Thus, by usual Banach space integration theory,

$$\big((h*u)(x_j^{-1}x_i)\big)_{i,j=1}^n = \int_G h(y)(y.m)\,dy$$

has Schur norm at most $\|u\|_{B_2}$. By Proposition 3.1 (2) it follows that the Herz-Schur norm of $h*u$ satisfies

$$\|h*u\|_{B_2(G)} \le \|u\|_{B_2(G)}.$$

(6) This is elementary. □

The proof of (1) in the lemma below is taken from [16, p. 510]. Although the authors of [16] assume that $u_\alpha \in A(G)$ and $u = 1$, the proof carries over without changes.

Lemma B.2 (The convolution trick – Part II)**.** *Let (u_α) be a bounded net in $B_2(G)$, let $u \in B_2(G)$ and let h be as above. We set*

$$v_\alpha = h*u_\alpha \quad \textit{and} \quad v = h*u.$$

(1) *If $u_\alpha \to u$ uniformly on compacts then $\|(v_\alpha - v)w\|_A \to 0$ for every $w \in A(G)$.*
(2) *If $u_\alpha \to u$ in the $\sigma(B_2, Q)$-topology then $v_\alpha \to v$ uniformly on compacts.*

Proof.
(1) Assume $u_\alpha \to u$ uniformly on compacts. Since the net (u_α) is bounded in B_2-norm, and since $A(G) \cap C_c(G)$ is dense in $A(G)$, it follows from (3.3) that it will suffice to prove that

$$\|(v_\alpha - v)w\|_A \to 0$$

for every $w \in A(G) \cap C_c(G)$. We let S denote the compact set $\operatorname{supp}(h)^{-1}\operatorname{supp}(w)$ and 1_S its characteristic function. Then if $x \in \operatorname{supp}(w)$

$$(h*u_\alpha)(x) = \int_G h(y)u_\alpha(y^{-1}x)\,dy = \int_G h(y)(1_S u_\alpha)(y^{-1}x)\,dy$$

because if $y^{-1}x \notin S$, then $h(y) = 0$. It follows that

$$(v_\alpha w)(x) = ((h*1_S u_\alpha)w)(x). \tag{B.1}$$

Note that (B.1) actually holds for all $x \in G$, since if $x \notin \operatorname{supp}(w)$, then both sides vanish. Similarly one can show

$$(vw)(x) = ((h*1_S u)w)(x) \quad \text{for all } x \in G.$$

By assumption, $1_S u_\alpha \to 1_S u$ uniformly, and hence

$$\|h*1_S u_\alpha - h*1_S u\|_A \le \|h\|_2\,\|\widetilde{1_S u_\alpha} - \widetilde{1_S u}\|_2 \to 0.$$

Since multiplication in $A(G)$ is continuous we also have

$$\|(v_\alpha - v)w\|_A \to 0.$$

This completes the proof of (1).

(2) For each $x \in G$, let $t(x) = h_x \in C_c(G)$ be the function $h_x(y) = h(xy)$. The map $t : G \to C_c(G)$ is continuous, when $C_c(G)$ is equipped with the L^1-norm (see [24,

Proposition 2.41]). Since the Q-norm is dominated by the L^1-norm, it follows that t is continuous into $Q(G)$.

Assume that $u_\alpha \to u$ in the $\sigma(B_2, Q)$-topology, and let $L \subseteq G$ be compact. Since the net (u_α) is bounded, the convergence is uniform on compact subsets of $Q(G)$. By continuity of t, the set

$$T = \{h_x \in C_c(G) \mid x \in L\}$$

is a compact subset of $Q(G)$. Hence

$$(h * u_\alpha)(x) = \langle h_x, \check{u}_\alpha \rangle \to \langle h_x, \check{u} \rangle = (h * u)(x)$$

uniformly on L. □

Remark B.3. In applications, u will often be the constant function $1 \in B_2(G)$, and in that case $h * u = 1$.

Lemma B.4. *Let (u_α) be a net in $B_2(G)$ and let $u \in B_2(G)$. We set*

$$v_\alpha = h * u_\alpha \quad \textit{and} \quad v = h * u.$$

(1) *If $u_\alpha \to u$ uniformly on compacts then $v_\alpha \to v$ uniformly on compacts.*
(2) *If $u_\alpha \to u$ in the $\sigma(B_2, Q)$-topology then $v_\alpha \to v$ in the $\sigma(B_2, Q)$-topology.*

Proof.

(1) For any subset $L \subseteq G$ we observe that

$$\sup_{x \in L} |v_\alpha(x) - v(x)| \le \sup_{x \in \operatorname{supp}(h)^{-1} L} |u_\alpha(x) - u(x)| \to 0.$$

If L is compact, then $\operatorname{supp}(h)^{-1}L$ is compact as well. This is sufficient to conclude (1).

(2) Let $\Delta : G \to \mathbb{R}_+$ be the modular function. When $f \in L^1(G)$ we let

$$(Rf)(y) = \Delta(y^{-1}) \int_G f(x) h(xy^{-1})\, dx.$$

It is not hard to show that $\|Rf\|_1 \le \|f\|_1$ and in particular $Rf \in L^1(G)$. We observe that if $w \in B_2(G)$ then

$$\begin{aligned} \langle f, h * w \rangle &= \int_G f(x)(h * w)(x)\, dx \\ &= \int_{G \times G} f(x) h(xy^{-1}) w(y) \Delta(y^{-1})\, dy dx \\ &= \langle Rf, w \rangle. \end{aligned}$$

It now follows from Lemma B.1 (5) that R extends uniquely to a linear contraction $R : Q(G) \to Q(G)$, and that the dual operator $R^* : B_2(G) \to B_2(G)$ satisfies $R^* w = h * w$. Since R^* is weak*-continuous we conclude

$$\langle f, v_\alpha \rangle = \langle f, R^* u_\alpha \rangle \to \langle f, R^* u \rangle = \langle f, v_\alpha \rangle$$

for any $f \in Q(G)$ as desired. □

B.2. **The bi-invariance trick.** In all of the following K is a compact subgroup of G equipped with normalized Haar measure dk.

Lemma B.5 (The bi-invariance trick – Part I)**.** *Let $u \in C(G)$ or $u \in L^1(G)$ be given, and define*

$$u^K(x) = \int_{K\times K} u(kxk')\, dkdk', \qquad x \in G. \tag{B.2}$$

Then u^K is a K-bi-invariant function on G. Moreover, the following holds.

(1) *If $u \in C(G)$, then $u^K \in C(G)$.*
(2) *If $u \in C_c(G)$, then $u^K \in C_c(G)$.*
(3) *If $u \in C_0(G)$, then $u^K \in C_0(G)$.*
(4) $\|u^K\|_\infty \leq \|u\|_\infty$.
(5) *If $u \in L^1(G)$, then $u^K \in L^1(G)$ and $\|u^K\|_1 \leq \|u\|_1$.*
(6) *If $u \in A(G)$, then $u^K \in A(G)$ and $\|u^K\|_A \leq \|u\|_A$.*
(7) *If $u \in B_2(G)$, then $u^K \in B_2(G)$ and $\|u^K\|_{B_2(G)} \leq \|u\|_{B_2(G)}$.*
(8) *If G is a Lie group and $u \in C^\infty(G)$, then $u^K \in C^\infty(G)$.*

Proof.

(1) Suppose $u \in C(G)$. To simplify matters, we first show that u_K given by

$$u_K(x) = \int_K u(kx)\, dk, \qquad x \in G$$

is a continuous function on G. A similar argument will then show that u^K is continuous, because

$$u^K(x) = \int_K u_K(xk)\, dk, \qquad x \in G.$$

Let $x \in G$ and $\varepsilon > 0$ be given. We will find a neighborhood V of the identity such that

$$|u_K(x) - u_K(zx)| \leq \varepsilon \quad \text{for all } z \in V.$$

Actually, it will be sufficient to verify that

$$|u(kx) - u(kzx)| \leq \varepsilon \quad \text{for all } z \in V \text{ and } k \in K.$$

For each $k \in K$, the function $x \mapsto u(kx)$ is continuous, so there exists a neighborhood U_k of the identity such that

$$|u(kx) - u(kzx)| \leq \varepsilon/2 \quad \text{for all } z \in U_k.$$

Let V_k be a neighborhood of the identity such that $V_kV_k \subseteq U_k$. Observe that the sets kV_k where $k \in K$ together cover K, so by compactness

$$K \subseteq k_1V_{k_1} \cup \cdots \cup k_nV_{k_n}$$

for some $k_1, \ldots, k_n \in K$. Let $V = \bigcap_{i=1}^n V_{k_i}$. Now, let $k \in K$ and $z \in V$ be arbitrary, and choose $i \in \{1, \ldots, n\}$ such that $k \in k_iV_{k_i}$. Note that then $k_i^{-1}k \in V_{k_i} \subseteq U_{k_i}$ and $k_i^{-1}kz \in V_{k_i}V_{k_i} \subseteq U_{k_i}$. Thus,

$$|u(kx) - u(kzx)| \leq |u(k_i(k_i^{-1}k)x) - u(k_ix)| + |u(k_ix) - u(k_i(k_i^{-1}kz)x)| \leq \varepsilon$$

as desired.

(2)-(4) we leave as an exercise.

(5) Recall (see [24, Section 2.4]) the fundamental relation of the modular function Δ,

$$\Delta(y) \int_G f(xy)\, dx = \int_G f(x)\, dx.$$

Also, $\Delta|_K = 1$, since K is compact. We now compute

$$\begin{aligned} \int_G |u^K(x)|\, dx &\leq \int_{K\times K} \int_G |u(kxk')|\, dxdkdk' \\ &= \int_{K\times K} \Delta(k') \int_G |u(x)|\, dxdkdk' \\ &= \|f\|_1. \end{aligned}$$

This proves (5).

(6) It suffices to note that $A(G)$ is a Banach space, that left and right translation on $A(G)$ is continuous and isometric, and then apply usual Banach space integration theory.

(7) This is mentioned in [16]. An argument similar the proof of Lemma B.1 (5) applies. Alternatively, one can use the proof from [55, Section 3].

(8) This is elementary. □

Lemma B.6 (The bi-invariance trick – Part II). *Let (u_α) be a net in $B_2(G)$ and let $u \in B_2(G)$. We set*

$$u_\alpha^K(x) = \int_{K\times K} u_\alpha(kxk')\, dkdk' \quad \textit{and} \quad u^K(x) = \int_{K\times K} u(kxk')\, dkdk'$$

(1) *If $u_\alpha \to u$ uniformly on compacts then $u_\alpha^K \to u^K$ uniformly on compacts.*
(2) *If $u_\alpha \to u$ in the $\sigma(B_2, Q)$-topology then $u_\alpha^K \to u^K$ in the $\sigma(B_2, Q)$-topology.*

Proof.

(1) Suppose $u_\alpha \to u$ uniformly on compacts. Let $L \subseteq G$ be compact. Then since $u_\alpha \to u$ uniformly on the compact set KLK, we have

$$\begin{aligned} \sup_{x\in L} |u_\alpha^K(x) - u^K(x)| &\leq \sup_{x\in L} \int_{K\times K} |u_\alpha(kxk') - u(kxk')|\, dkdk' \\ &\leq \sup_{y\in KLK} |u_\alpha(y) - u(y)| \to 0. \end{aligned}$$

This shows that $u_\alpha^K \to u^K$ uniformly on L.

(2) This is proved in [28, Lemma 2.5]. We sketch the proof here. Observe that

$$\langle f, v^K\rangle = \langle f^K, v\rangle$$

for any $v \in B_2(G)$ and $f \in L^1(G)$. Thus $\|f^K\|_Q \leq \|f\|_Q$ by Lemma B.5 (7), and the map $f \mapsto f^K$ extends uniquely to a linear contraction $R : Q(G) \to Q(G)$. The dual operator $R^* : B_2(G) \to B_2(G)$ obviously satisfies $R^*v = v^K$ and is weak*-continuous. Hence

$$\langle f, u_\alpha^K\rangle = \langle f, R^*u_\alpha\rangle \to \langle f, R^*u\rangle = \langle f, u^K\rangle$$

for any $f \in Q(G)$ as desired. □

Appendix C. Continuity of Herz-Schur multipliers

U. Haagerup has allowed us to include the following lemma whose proof is taken from Appendix A in the unpublished manuscript [27].

Lemma C.1 ([27]). *Let G be a locally compact group, let $u : G \to \mathbb{C}$ be a function, and suppose there exist a separable Hilbert space $\mathcal{H}$ and two bounded Borel maps $P, Q : G \to \mathcal{H}$ such that*

$$u(y^{-1}x) = \langle P(x), Q(y) \rangle \qquad \textit{for all } x, y \in G.$$

Then u is continuous, $u \in B_2(G)$ and

$$\|u\|_{B_2} \le \|P\|_\infty \|Q\|_\infty.$$

Proof. We construct another Hilbert space $\mathcal{K}$ and two *continuous* bounded maps $\widehat{P}, \widehat{Q} : G \to \mathcal{K}$ such that

$$u(y^{-1}x) = \langle \widehat{P}(x), \widehat{Q}(y) \rangle \qquad \text{for all } x, y \in G$$

and

$$\|\widehat{P}\|_\infty \|\widehat{Q}\|_\infty \le \|P\|_\infty \|Q\|_\infty.$$

This will complete the proof in the light of Proposition 3.1 (4).

Take $h \in C_c(G)$ satisfying $\|h\|_2 = 1$, and define

$$\widehat{P}(x), \widehat{Q}(x) \in L^2(G, \mathcal{H}) \quad \text{for all } x \in G$$

by

$$\widehat{P}(x)(z) = h(z)P(zx), \quad z \in G;$$
$$\widehat{Q}(x)(z) = h(z)Q(zx), \quad z \in G.$$

We find

$$\begin{aligned} \langle \widehat{P}(x), \widehat{Q}(y) \rangle &= \int_G |h(z)|^2 \langle P(zx), Q(zy) \rangle \, dz \\ &= \int_G |h(z)|^2 u(y^{-1}x) \, dz \\ &= u(y^{-1}x). \end{aligned}$$

It is also easy to see that

$$\sup_{x\in G} \|\widehat{P}(x)\|_2 \le \sup_{x\in G} \|P(x)\|, \quad \sup_{x\in G} \|\widehat{Q}(x)\|_2 \le \sup_{x\in G} \|Q(x)\|,$$

so in particular the maps $\widehat{P}, \widehat{Q} : G \to L^2(G, \mathcal{H})$ are bounded. It remains only to check continuity of $x \mapsto \widehat{P}(x)$ and $x \mapsto \widehat{Q}(x)$. Let $\rho : G \to B(L^2(G))$ be the right regular representation

$$\rho_x(f)(z) = \Delta^{1/2}(x) f(zx), \quad f \in L^2(G), \ x, z \in G.$$

We let R be the representation $\rho \otimes 1$ of G on $L^2(G, \mathcal{H}) = L^2(G) \otimes \mathcal{H}$, that is,

$$R_x(f)(z) = \Delta^{1/2}(x) f(zx), \quad f \in L^2(G, \mathcal{H}), \ x, z \in G.$$

It is well-known that ρ is strongly continuous (see Proposition 2.41 in [24]), and hence R is strongly continuous.

Suppose $x_n \to x$ in G. Then $R_{x_n x^{-1}}\widehat{P}(x) \to \widehat{P}(x)$ in $L^2(G, \mathcal{H})$. Also,

$$\begin{aligned}\|R_{x_n x^{-1}}\widehat{P}(x) - \widehat{P}(x_n)\|^2 &= \int_G \|R_{x_n x^{-1}}\widehat{P}(x)(z) - \widehat{P}(x_n)(z)\|^2 \, dz \\ &= \int_G \|\Delta(x_n x^{-1})^{1/2} h(z x_n x^{-1}) P(z x_n) - h(z) P(z x_n)\|^2 \, dz \\ &= \int_G |\Delta(x_n x^{-1})^{1/2} h(z x_n x^{-1}) - h(z)|^2 \|P(z x_n)\|^2 \, dz \\ &\leq \int_G |\Delta(x_n x^{-1})^{1/2} h(z x_n x^{-1}) - h(z)|^2 \|P\|_\infty^2 \, dz \\ &= \|P\|_\infty^2 \|\rho_{x_n x^{-1}} h - h\|^2 \to 0.\end{aligned}$$

Hence $\widehat{P}(x_n) \to \widehat{P}(x)$ as desired. Continuity of $\widehat{Q}$ is verified similarly. □

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ARTICLE C

The weak Haagerup property II: Examples

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THE WEAK HAAGERUP PROPERTY II: EXAMPLES

UFFE HAAGERUP AND SØREN KNUDBY

Abstract. The weak Haagerup property for locally compact groups and the weak Haagerup constant was recently introduced by the second author in [27]. The weak Haagerup property is weaker than both weak amenability introduced by Cowling and the first author in [9] and the Haagerup property introduced by Connes [6] and Choda [5].

In this paper it is shown that a connected simple Lie group G has the weak Haagerup property if and only if the real rank of G is zero or one. Hence for connected simple Lie groups the weak Haagerup property coincides with weak amenability. Moreover, it turns out that for connected simple Lie groups the weak Haagerup constant coincides with the weak amenability constant, although this is not true for locally compact groups in general.

It is also shown that the semidirect product $\mathbb{R}^2 \rtimes \mathrm{SL}(2,\mathbb{R})$ does not have the weak Haagerup property.

1. Introduction

Amenability is a fundamental concept for locally compact groups, see, for example, the books [15], [30]. In the 1980s, two weaker properties for locally compact groups were introduced, first the *Haagerup property* by Connes [6] and Choda [5] and next *weak amenability* by Cowling and the first author [9]. Both properties have been studied extensively (see [3, Chapter 12], [4] and [8] and the references therein). It is well known that amenability of a locally compact group G is equivalent to the existence of a net $(u_\alpha)_{\alpha\in A}$ of continuous, compactly supported, positive definite functions on G such that $(u_\alpha)_{\alpha\in A}$ converges to the constant function 1_G uniformly on compact subsets of G.

Definition 1.1 ([6],[4])**.** A locally compact group G has the *Haagerup property* if there exists a net $(u_\alpha)_{\alpha\in A}$ of continuous positive definite functions on G vanishing at infinity such that $u_\alpha \to 1_G$ uniformly on compact sets.

As usual we let $C_0(G)$ denote the continuous (complex) functions on G vanishing at infinity and let $C_c(G)$ be the subspace of functions with compact support. Also, $B_2(G)$ denotes the space of Herz–Schur multipliers on G with the Herz–Schur norm $\| \ \|_{B_2}$ (see Section 2 for more details).

Definition 1.2 ([9])**.** A locally compact group G is *weakly amenable* if there exist a constant $C > 0$ and a net $(u_\alpha)_{\alpha\in A}$ in $B_2(G)\cap C_c(G)$ such that

$$\|u_\alpha\|_{B_2} \le C \quad \text{for every } \alpha \in A, \tag{1.1}$$

$$u_\alpha \to 1 \text{ uniformly on compacts.} \tag{1.2}$$

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The best possible constant C in (1.1) is called the *weak amenability constant* denoted $\Lambda_{\mathrm{WA}}(G)$. If G is not weakly amenable, then we put $\Lambda_{\mathrm{WA}}(G) = \infty$. The weak amenability constant $\Lambda_{\mathrm{WA}}(G)$ is also called the Cowling–Haagerup constant and denoted $\Lambda_{\mathrm{cb}}(G)$ or Λ_G in the literature.

The definition of weak amenability given here is different from the definition given in [9], but the definitions are equivalent. In one direction, this follows from [9, Proposition 1.1] and the fact that $A(G) \subseteq B_2(G)$, where $A(G)$ denotes the Fourier algebra of G (see Section 2). In the other direction, one can apply the convolution trick (see [27, Appendix B]) together with the standard fact that $C_c(G) * C_c(G) \subseteq A(G)$.

Definition 1.3 ([26],[27])**.** A locally compact group G has the *weak Haagerup property* if there exist a constant $C > 0$ and a net $(u_\alpha)_{\alpha\in A}$ in $B_2(G) \cap C_0(G)$ such that

$$\|u_\alpha\|_{B_2} \leq C \quad \text{for every } \alpha \in A, \tag{1.3}$$

$$u_\alpha \to 1 \text{ uniformly on compacts.} \tag{1.4}$$

The best possible constant C in (1.3) is called the *weak Haagerup constant* denoted $\Lambda_{\mathrm{WH}}(G)$. If G does not have the weak Haagerup property, then we put $\Lambda_{\mathrm{WH}}(G) = \infty$.

Clearly, the weak Haagerup property is weaker than both the Haagerup property and weak amenability, and hence there are many known examples of groups with the weak Haagerup property. Moreover, there exist examples of groups that fail the first two properties but nevertheless have the weak Haagerup property (see [27, Corollary 5.7]).

Our first result is the following theorem.

Theorem A. *The groups* $\mathrm{SL}(3,\mathbb{R})$*,* $\mathrm{Sp}(2,\mathbb{R})$*, and* $\widetilde{\mathrm{Sp}}(2,\mathbb{R})$ *do not have the weak Haagerup property.*

The case of $\mathrm{SL}(3,\mathbb{R})$ can also be found in [28, Theorem 5.1] (take $p = \infty$). Our proof of Theorem A is a fairly simple application of the recent methods and results of de Laat and the first author [17], [18], where it is proved that a connected simple Lie group G of real rank at least two does not have the Approximation Property (AP), that is, there is no net $(u_\alpha)_{\alpha\in A}$ in $B_2(G) \cap C_c(G)$ which converges to the constant function 1_G in the natural weak*-topology on $B_2(G)$. By inspection of their proofs in the case of the three groups mentioned in Theorem A, one gets that for those three groups the net $(u_\alpha)_{\alpha\in A}$ cannot even be chosen as functions in $B_2(G) \cap C_0(G)$, which proves Theorem A.

By standard structure theory of connected simple Lie groups, it now follows that the conclusion of Theorem A holds for all connected simple Lie groups of real rank at least two. Moreover, by [7], [9], [10], [20] every connected simple Lie group of real rank zero or one is weakly amenable. We thus obtain the following theorem.

Theorem B. *Let G be a connected simple Lie group. Then G has the weak Haagerup property if and only if the real rank of G is at most one.*

For connected simple Lie groups G the constants $\Lambda_{\mathrm{WA}}(G)$ are known: if the real rank is zero, then G is compact and $\Lambda_{\mathrm{WA}}(G) = 1$. If the real rank is at least two, then by [16], [12] the group G is not weakly amenable and hence $\Lambda_{\mathrm{WA}}(G) = \infty$. Finally, in the real rank one case, one has by [7], [9], [10], [20] that

$$\Lambda_{\mathrm{WA}}(G) = \begin{cases} 1 & \text{for } G \approx \mathrm{SO}_0(1,n) \\ 1 & \text{for } G \approx \mathrm{SU}(1,n) \\ 2n-1 & \text{for } G \approx \mathrm{Sp}(1,n) \\ 21 & \text{for } G \approx \mathrm{F}_{4(-20)} \end{cases} \tag{1.5}$$

where $G \approx H$ means that G is locally isomorphic to H. We prove the following theorem.

Theorem C. *For every connected simple Lie group G, $\Lambda_{\mathrm{WA}}(G) = \Lambda_{\mathrm{WH}}(G)$.*

It is clear that for every locally compact group G one has $1 \leq \Lambda_{\mathrm{WH}}(G) \leq \Lambda_{\mathrm{WA}}(G)$. Theorem C then amounts to show that $\Lambda_{\mathrm{WH}}(G) = \Lambda_{\mathrm{WA}}(G)$ when G is locally isomorphic to $\mathrm{Sp}(1,n)$ or $\mathrm{F}_{4(-20)}$. Moreover, since the groups $\mathrm{Sp}(1,n)$ and $\mathrm{F}_{4(-20)}$ are simply connected and have finite center, one can actually restrict to the case when G is either $\mathrm{Sp}(1,n)$ or $\mathrm{F}_{4(-20)}$. The proof of Theorem C in these two cases relies heavily on a result from [25], namely that for $\mathrm{Sp}(1,n)$ and $\mathrm{F}_{4(-20)}$ the minimal parabolic subgroup $P = MAN$ of these groups has the property that $A(P) = B(P) \cap C_0(P)$. Here, $A(P)$ and $B(P)$ denote, respectively, the Fourier algebra and the Fourier–Stieltjes algebra of P (see Section 2).

For all the groups mentioned so far, the weak Haagerup property coincides with weak amenability and with the AP. As an example of a group with the AP which fails the weak Haagerup property we have the following theorem.

Theorem D. *The group $\mathbb{R}^2 \rtimes \mathrm{SL}(2,\mathbb{R})$ does not have the weak Haagerup property.*

Combining Theorem D with [27, Theorem A] we observe that the discrete group $\mathbb{Z}^2 \rtimes \mathrm{SL}(2,\mathbb{Z})$, which is a lattice in $\mathbb{R}^2 \rtimes \mathrm{SL}(2,\mathbb{R})$, also does not have the weak Haagerup property.

Theorem D generalizes a result from [16] where it is shown that $\mathbb{R}^2 \rtimes \mathrm{SL}(2,\mathbb{R})$ is not weakly amenable. Crucial to our proof of Theorem D are some of the techniques developed in [16]. These techniques are further developed here using a result from [19], namely that $\mathbb{R}^2 \rtimes \mathrm{SL}(2,\mathbb{R})$ satisfies the AP. Also, [25, Theorem 2] is essential in the proof of Theorem D.

Both groups $\mathbb{R}^2$ and $\mathrm{SL}(2,\mathbb{R})$ enjoy the Haagerup property and hence also the weak Haagerup property. Theorem D thus shows that extensions of groups with the (weak) Haagerup property need not have the weak Haagerup property.

2. Preliminaries

Let G be a locally compact group equipped with a left Haar measure. We denote the left regular representation of G on $L^2(G)$ by λ. As usual, $C(G)$ denotes the (complex) continuous functions on G. When G is a Lie group, $C^\infty(G)$ is the space of smooth functions on G.

We first describe the Fourier–Stieltjes algebra and the Fourier algebra of G. These were originally introduced in the seminal paper [14] to which we refer for further details about these algebras. Afterwards we describe the Herz–Schur multiplier algebra.

The *Fourier–Stieltjes algebra* $B(G)$ can be defined as set of matrix coefficients of strongly continuous unitary representations of G, that is, $u \in B(G)$ if and only if there are a strongly continuous unitary representation $\pi\colon G \to U(\mathcal{H})$ of G on a Hilbert space $\mathcal{H}$ and vectors $x, y \in \mathcal{H}$ such that

$$u(g) = \langle \pi(g)x, y \rangle \quad \text{for all } g \in G. \tag{2.1}$$

The norm $\|u\|_B$ of $u \in B(G)$ is defined as the infimum (actually a minimum) of all numbers $\|x\|\|y\|$, where x, y are vectors in some representation $(\pi, \mathcal{H})$ such that (2.1) holds. With this norm $B(G)$ is a unital Banach algebra. The Fourier–Stieltjes algebra coincides with the linear span of the continuous positive definite functions on G. For any $u \in B(G)$ the inequality $\|u\|_\infty \leq \|u\|_B$ holds, where $\| \ \|_\infty$ denotes the uniform norm.

The compactly supported functions in $B(G)$ form an ideal in $B(G)$, and the closure of this ideal is the *Fourier algebra* $A(G)$, which is then also an ideal. The Fourier algebra coincides with the set of matrix coefficients of the left regular representation λ, that is, $u \in A(G)$ if and only if there are vectors $x, y \in L^2(G)$ such that

$$u(g) = \langle \lambda(g)x, y \rangle \quad \text{for all } g \in G. \tag{2.2}$$

The norm of $u \in A(G)$ is the infimum of all numbers $\|x\|\|y\|$, where $x, y \in L^2(G)$ satisfy (2.2). We often write $\|u\|_A$ for the norm $\|u\|_B$ when $u \in A(G)$.

The dual space of $A(G)$ can be identified with the group von Neumann algebra $L(G)$ of G via the duality

$$\langle a, u \rangle = \langle ax, y \rangle = \int_G (ax)(g)\overline{y(g)}\ dg$$

where $a \in L(G)$ and $u \in A(G)$ is of the form (2.2).

When G is a Lie group, it is known that $C_c^\infty(G) \subseteq A(G)$ (see [14, Proposition 3.26]).

Since the uniform norm is bounded by the Fourier–Stieltjes norm, it follows that $A(G) \subseteq B(G) \cap C_0(G)$. For many groups this inclusion is strict (see e.g [25]), but in some cases it is not. We will need the following result when proving Theorem C.

Theorem 2.1 ([25, Theorem 3])**.** *Let G be one of the groups* $\mathrm{SO}(1,n)$, $\mathrm{SU}(1,n)$, $\mathrm{Sp}(1,n)$ *or* $\mathrm{F}_{4(-20)}$, *and let $G = KAN$ be the Iwasawa decomposition. The group N is contained in a closed amenable group P satisfying $A(P) = B(P) \cap C_0(P)$.*

We will need the following lemma in Section 5. For a demonstration, see the proof of Proposition 1.12 in [10].

Lemma 2.2 ([10])**.** *Let G be a locally compact group with a closed subgroup $H \subseteq G$. If $u \in A(G)$, then $u|_H \in A(H)$. Moreover, $\|u|_H\|_{A(H)} \leq \|u\|_{A(G)}$. Conversely, if $u \in A(H)$, then there is $\widetilde{u} \in A(G)$ such that $u = \widetilde{u}|_H$ and $\|u\|_{A(H)} = \inf\{\|\widetilde{u}\|_{A(G)} \mid \widetilde{u} \in A(G),\ \widetilde{u}|_H = u\}$.*

We now recall the definition of the *Herz–Schur multiplier algebra* $B_2(G)$. A function $k\colon G \times G \to \mathbb{C}$ is a Schur multiplier on G if for every bounded operator $A = [a_{xy}]_{x,y \in G} \in B(\ell^2(G))$ the matrix $[k(x,y)a_{xy}]_{x,y\in G}$ represents a bounded operator on $\ell^2(G)$, denoted $m_k(A)$. If this is the case, then by the closed graph theorem m_k defines a *bounded* operator on $B(\ell^2(G))$, and the Schur norm $\|k\|_S$ is defined as the operator norm of m_k.

A continuous function $u\colon G \to \mathbb{C}$ is a Herz–Schur multiplier, if $k(x, y) = u(y^{-1}x)$ is a Schur multiplier on G, and the Herz–Schur norm $\|u\|_{B_2}$ is defined as $\|k\|_S$. We let $B_2(G)$ denote the space of Herz–Schur multipliers, which is a Banach space, in fact a unital Banach algebra, with the Herz–Schur norm $\| \ \|_{B_2}$. The Herz–Schur norm dominates the uniform norm.

It is known that $B(G) \subseteq B_2(G)$, and $\|u\|_{B_2} \leq \|u\|_B$ for every $u \in B(G)$. In [22, Theoreme 1(ii)], it is shown that $B_2(G)$ multiplies the Fourier algebra $A(G)$ into itself and $\|uv\|_A \leq \|u\|_{B_2}\|v\|_A$ for every $u \in B_2(G)$, $v \in A(G)$. In this way, we can view $B_2(G)$ as bounded operators on $A(G)$, and $B_2(G)$ inherits a point-norm (or strong operator) topology and a point-weak (or weak operator) topology.

It is known that the space of Herz–Schur multipliers coincides isometrically with the completely bounded Fourier multipliers, usually denoted $M_0A(G)$ (see [2] or [23]). It is well known that if G is amenable then $B(G) = B_2(G)$ isometrically. The converse is known to hold, when G is discrete (see [1]).

Given $f \in L^1(G)$ and $u \in B_2(G)$ define

$$\langle f, u\rangle = \int_G f(x)u(x)\ dx \tag{2.3}$$

and

$$\|f\|_Q = \sup\{|\langle f, u\rangle| \mid u \in B_2(G),\ \|u\|_{B_2} \leq 1\}.$$

Then $\|\ \|_Q$ is a norm on $L^1(G)$, and the completion of $L^1(G)$ with respect to this norm is a Banach space $Q(G)$ whose dual space is identified with $B_2(G)$ via (2.3) (see [10, Proposition 1.10(b)]). In this way $B_2(G)$ is equipped with a weak*-topology coming from its predual $Q(G)$. The weak*-topology is also denoted $\sigma(B_2, Q)$.

We recall that G has the Approximation Property (AP) if there is a net $(u_\alpha)_{\alpha\in A}$ in $B_2(G)\cap C_c(G)$ which converges to the constant function 1_G in the $\sigma(B_2, Q)$-topology. As with weak amenability, the definition of the AP just given can be seen to be equivalent to the original definition by use of the convolution trick (see [27, Appendix B]). For more on the $\sigma(B_2, Q)$-topology and the AP we refer to the original paper [19].

The following lemma is a variant of [19, Proposition 1.3 (a)]. The statement of [19, Proposition 1.3 (a)] involves an infinite dimensional Hilbert space $\mathscr{H}$, but going through the proof of [19, Proposition 1.3 (a)] one can check that the statement remains true, if $\mathscr{H}$ is just the one-dimensional space $\mathbb{C}$. Hence, we have the following lemma.

Lemma 2.3 ([19])**.** *Let G be a locally compact group. Suppose $a \in L(G)$, $v \in A(G)$ and that $f \in A(G)$ is a compactly supported, positive function with integral 1. Then the functional $\omega_{a,v,f} : B_2(G) \to \mathbb{C}$ defined as*

$$\omega_{a,v,f}(u) = \langle a, (f * u)v\rangle, \quad u \in B_2(G)$$

is bounded, that is, $\omega_{a,v,f} \in Q(G)$.

It is known that weakly amenable groups have the AP [19, Theorem 1.12], and extensions of groups with the AP have the AP [19, Theorem 1.15]. In particular, the group $\mathbb{R}^2 \rtimes \mathrm{SL}(2,\mathbb{R})$ has the AP.

Given a compact subgroup K of G we say that a continuous function $f\colon G \to \mathbb{C}$ is K-bi-invariant, if $f(kx) = f(xk) = f(x)$ for every $k \in K$ and $x \in G$. The space of continuous K-bi-invariant functions on G is denoted $C(K\backslash G/K)$.

The following two lemmas concerning weak amenability and the AP are standard averaging arguments. For the convenience of the reader, we include a proof of the second lemma. A proof of the first can be manufactured in basically the same way. We note that the special cases where K is the trivial subgroup follow from [9, Proposition 1.1] and [19, Theorem 1.11], respectively.

Lemma 2.4. *Let G be a locally compact group with compact subgroup K. If G is weakly amenable, say $\Lambda_{\mathrm{WA}}(G) \leq C$, then there is a net (v_β) in $A(G) \cap C_c(K\backslash G/K)$ such that*

$$\|v_\beta v - v\|_{A(G)} \to 0 \quad \text{for every } v \in A(G)$$

and $\sup_\beta \|v_\beta\|_{B_2} \leq C$. Moreover, if G is a Lie group, we may arrange that each v_β is smooth.

Lemma 2.5. *Let G be a locally compact group with compact subgroup K. If G has the AP, then there is a net (v_β) in $A(G) \cap C_c(K\backslash G/K)$ such that*

$$\|v_\beta v - v\|_{A(G)} \to 0 \quad \textit{for every } v \in A(G).$$

Moreover, if G is a Lie group, we may arrange that each v_β is smooth.

Proof. We suppose G has the AP. Then there is a net (u_α) in $A(G) \cap C_c(G)$ such that $u_\alpha \to 1$ in the $\sigma(B_2, Q)$-topology (see [19, Remark 1.2]). Choose a positive function $f \in A(G)$ with compact support and integral 1. By averaging from left and right over K (see Appendix B in [27]), we may further assume that f and each u_α is K-bi-invariant. Let $w_\alpha = f * u_\alpha$. Then $w_\alpha \in A(G) \cap C_c(K\backslash G/K)$.

Given $a \in L(G)$ and $v \in A(G)$ we have the following equation:

$$\langle a, w_\alpha v\rangle = \omega_{a,v,f}(u_\alpha) \to \omega_{a,v,f}(1) = \langle a, v\rangle.$$

Hence $w_\alpha \to 1$ in the point-weak topology on $B_2(G)$. It follows from [13, Corollary VI.1.5] that there is a net (v_β) where each v_β lies in the convex hull of $\{w_\alpha\}$ such that $v_\beta \to 1$ in the point-norm topology. In other words, there is a net (v_β) in $A(G) \cap C_c(K\backslash G/K)$ such that

$$\|v_\beta v - v\|_{A(G)} \to 0 \quad \text{for every } v \in A(G).$$

If G is a Lie group, we may further assume that $f \in C_c^\infty(G)$, in which case v_β becomes smooth. ☐

3. Simple Lie groups of higher real rank

It is known that a connected simple Lie group of real rank at least two is not weakly amenable [12],[16]. In fact, an even stronger result was proved recently [28], [17],[18]. One could ask if such Lie groups also fail the weak Haagerup property. Using results from [17],[18] we completely settle this question in the affirmative. We thus prove Theorems A and Theorem B.

3.1. **Three groups of real rank two.** We will prove that the three groups $\mathrm{SL}(3,\mathbb{R})$, $\mathrm{Sp}(2,\mathbb{R})$ and the universal covering group $\widetilde{\mathrm{Sp}}(2,\mathbb{R})$ of $\mathrm{Sp}(2,\mathbb{R})$ do not have the weak Haagerup property. The cases of $\mathrm{SL}(3,\mathbb{R})$ and $\mathrm{Sp}(2,\mathbb{R})$ are similar and are treated together. The case of $\widetilde{\mathrm{Sp}}(2,\mathbb{R})$ is more difficult, essentially because $\widetilde{\mathrm{Sp}}(2,\mathbb{R})$ is not a matrix Lie group, and we will go into more details in this case.

When we consider the special linear group $\mathrm{SL}(3,\mathbb{R})$, then $K = \mathrm{SO}(3)$ will be its maximal compact subgroup. We now describe the group $\mathrm{Sp}(2,\mathbb{R})$ and a maximal compact subgroup. Consider the matrix 4×4 matrix

$$J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$$

where I_2 denotes the 2×2 identity matrix. The symplectic group $\mathrm{Sp}(2,\mathbb{R})$ is defined as

$$\mathrm{Sp}(2,\mathbb{R}) = \{g \in \mathrm{GL}(4,\mathbb{R}) \mid g^t J g = J\}.$$

Here g^t denotes the transpose of g. The symplectic group $\mathrm{Sp}(2,\mathbb{R})$ is a connected simple Lie group of real rank two. It has a maximal compact subgroup

$$K = \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in M_4(\mathbb{R}) \mid A + iB \in \mathrm{U}(2) \right\} \tag{3.1}$$

which is isomorphic to $\mathrm{U}(2)$.

The following is immediate from [27, Proposition 4.3, Lemma A.1(2)].

Lemma 3.1. *Let G be locally compact group with a compact subgroup K. If G has the weak Haagerup property, then there is a bounded net (u_α) in $B_2(G) \cap C_0(K\backslash G/K)$ such that $u_\alpha \to 1$ in the weak*-topology.*

We remind the reader that $B_2(G)$ coincides isometrically with the completely bounded Fourier multipliers $M_0A(G)$. The following result is then extracted from [17, p. 937 + 957].

Theorem 3.2 ([17])**.** *If G is one of the groups $\mathrm{SL}(3,\mathbb{R})$ or $\mathrm{Sp}(2,\mathbb{R})$ and K is the corresponding maximal compact subgroup in G, then $B_2(G) \cap C_0(K\backslash G/K)$ is closed in $B_2(G)$ in the weak*-topology.*

Theorem 3.3. *The groups $\mathrm{SL}(3,\mathbb{R})$ and $\mathrm{Sp}(2,\mathbb{R})$ do not have the weak Haagerup property.*

Proof. Let G be one of the groups $\mathrm{SL}(3,\mathbb{R})$ or $\mathrm{Sp}(2,\mathbb{R})$. Obviously, $1 \notin B_2(G) \cap C_0(K\backslash G/K)$. Since $B_2(G) \cap C_0(K\backslash G/K)$ is weak*-closed, there can be no net $u_\alpha \in B_2(G) \cap C_0(K\backslash G/K)$ such that $u_\alpha \to 1$ in the weak*-topology. Using Lemma 3.1, we conclude that G does not have the weak Haagerup property. □

Remark 3.4. An alternative proof of Theorem 3.3 for the group $\mathrm{SL}(3,\mathbb{R})$, avoiding the use of the difficult Theorem 3.2, is to use the fact that $\mathbb{R}^2 \rtimes \mathrm{SL}(2,\mathbb{R})$ is a closed subgroup of $\mathrm{SL}(3,\mathbb{R})$. From Theorem D (to be proved in Section 5), we know that $\mathbb{R}^2 \rtimes \mathrm{SL}(2,\mathbb{R})$ does not have the weak Haagerup property, and this is sufficient to conclude that $\mathrm{SL}(3,\mathbb{R})$ also fails to have the weak Haagerup property (see [27, Theorem A(1)]).

We now turn to the case of $\widetilde{\mathrm{Sp}}(2,\mathbb{R})$. To ease notation a bit, in the rest of this section we let $G = \mathrm{Sp}(2,\mathbb{R})$ and $\widetilde{G} = \widetilde{\mathrm{Sp}}(2,\mathbb{R})$. We now describe the group $\widetilde{G}$. This is based on [31] and [18, Section 3].

By definition, $\widetilde{G}$ is the universal covering group of G. The group G has fundamental group $\pi_1(G) \simeq \pi_1(\mathrm{U}(2))$ which is the group $\mathbb{Z}$ of integers. There is a smooth function $c : G \to \mathbb{T}$, where $\mathbb{T}$ denotes the unit circle in $\mathbb{C}$, such that c induces an isomorphism of the fundamental groups of G and $\mathbb{T}$ (such a c is called a circle function). An explicit description of c can be found in [31] and [18]. The circle function c satisfies

$$c(1) = 1 \quad \text{and} \quad c(g^{-1}) = c(g)^{-1}.$$

There is a unique smooth map $\eta : G \times G \to \mathbb{R}$ such that

$$c(g_1 g_2) = c(g_1)c(g_2)e^{i\eta(g_1,g_2)} \quad \text{and} \quad \eta(1,1) = 0$$

for all $g_1, g_2 \in G$. The map η is also explicitly described in [31] and [18]. The universal cover $\widetilde{G}$ of G can be realized as the smooth manifold

$$\widetilde{G} = \{(g,t) \in G \times \mathbb{R} \mid c(g) = e^{it}\}$$

with multiplication given by

$$(g_1,t_1)(g_2,t_2) = (g_1 g_2, t_1 + t_2 + \eta(g_1,g_2)).$$

The identity in $\widetilde{G}$ is $(1,0)$, where 1 denotes the identity in G, and the inverse is given by $(g,t)^{-1} = (g^{-1}, -t)$. The map $\sigma : \widetilde{G} \to G$ given by $\sigma(g,t) = g$ is the universal covering homomorphism, and the kernel of σ is $\{(1, 2\pi k) \in G \times \mathbb{R} \mid k \in \mathbb{Z}\}$, which is of course isomorphic to $\mathbb{Z}$.

Let K be the maximal compact subgroup of G given in (3.1). Then one can show that

$$\eta(g,h) = 0 \quad \text{for all } g,h \in K. \tag{3.2}$$

Under the obvious identification $K \simeq \mathrm{U}(2)$, we consider $\mathrm{SU}(2) \subseteq \mathrm{U}(2)$ as a subgroup of K. Define a compact subgroup $\widetilde{H}$ of $\widetilde{G}$ by

$$\widetilde{H} = \{(g,0) \in G \times \mathbb{R} \mid g \in \mathrm{SU}(2)\}.$$

By (3.2) $\widetilde{H}$ is indeed a subgroup of $\widetilde{G}$.

When $t \in \mathbb{R}$ let $v_t \in G$ be the element

$$v_t = \begin{pmatrix} \cos t & 0 & -\sin t & 0 \\ 0 & \cos t & 0 & -\sin t \\ \sin t & 0 & \cos t & 0 \\ 0 & \sin t & 0 & \cos t \end{pmatrix},$$

and define $\widetilde{v}_t = (v_t, 2t) \in \widetilde{G}$. Then $\eta(v_t, g) = \eta(g, v_t) = 0$ for any $g \in G$. Obviously, $(\widetilde{v}_t)_{t\in\mathbb{R}}$ is a one-parameter family in $\widetilde{G}$, and it is a simple matter to check that conjugation by $\widetilde{v}_t$ is π-periodic. A simple computation will also show that if $g \in K$, then $gv_t = v_t g$ and hence $h\widetilde{v}_t = \widetilde{v}_t h$ for every $h \in \widetilde{H}$.

Consider the subspace $\mathcal{C}$ of $C(\widetilde{G})$ defined by

$$\mathcal{C} = \{u \in C(\widetilde{G}) \mid u \text{ is } \widetilde{H}\text{-bi-invariant and } u(\widetilde{v}_t g \widetilde{v}_t^{-1}) = u(g) \text{ for all } t \in \mathbb{R}\}.$$

Further, we let $\mathcal{C}_0 = \mathcal{C} \cap C_0(\widetilde{G})$. For any $f \in C(\widetilde{G})$ or $f \in L^1(\widetilde{G})$, let $f^{\mathcal{C}} : \widetilde{G} \to \mathbb{C}$ be defined by

$$f^{\mathcal{C}}(x) = \frac{1}{\pi}\int_0^\pi \int_{\widetilde{H}} \int_{\widetilde{H}} f(h_1 \widetilde{v}_t x \widetilde{v}_t^{-1} h_2)\, dh_1 dh_2 dt, \quad x \in \widetilde{G},$$

where dh_1 and dh_2 both denote the normalized Haar measure on the compact group $\widetilde{H}$.

Lemma 3.5. *With the notation as above the following holds:*

1. *If $u \in C(\widetilde{G})$, then $u^{\mathcal{C}} \in \mathcal{C}$.*
2. *If $f \in L^1(\widetilde{G})$, then $f^{\mathcal{C}} \in L^1(\widetilde{G})$ and $\|f^{\mathcal{C}}\|_Q \le \|f\|_Q$.*
3. *If $u \in B_2(\widetilde{G})$, then $u^{\mathcal{C}} \in B_2(\widetilde{G})$ and $\|u^{\mathcal{C}}\|_{B_2} \le \|u\|_{B_2}$.*
4. *If $u \in C_0(\widetilde{G})$, then $u^{\mathcal{C}} \in C_0(\widetilde{G})$.*

Proof.

(1) This is elementary.

(2) Suppose $f \in L^1(\widetilde{G})$. Connected simple Lie groups are unimodular (see [24, Corollary 8.31]), and hence each left or right translate of f is also in $L^1(\widetilde{G})$ with the same norm. Since left and right translation on $L^1(G)$ is norm continuous, it now follows from usual Banach space integration theory that $f^{\mathcal{C}} \in L^1(\widetilde{G})$.

We complete the proof of (2) after we have proved (3).

(3) This statement is implicit in [18] in the proof of [18, Lemma 3.10]. We have chosen to include a proof.

For each $g \in B_2(\widetilde{G})$ or $g \in L^1(\widetilde{G})$ and $\alpha = (h_1, h_2, t) \in \widetilde{H} \times \widetilde{H} \times \mathbb{R}$ define

$$g_\alpha(x) = g(h_1 \widetilde{v}_t x \widetilde{v}_t^{-1} h_2), \quad x \in \widetilde{G}.$$

If $g \in B_2(\widetilde{G})$, then $g_\alpha \in B_2(\widetilde{G})$ and $\|g_\alpha\|_{B_2} = \|g\|_{B_2}$. Similarly, if $g \in L^1(\widetilde{G})$, then $g_\alpha \in L^1(\widetilde{G})$ and $\|g_\alpha\|_1 = \|g\|_1$. Note also that $\langle g, f_\alpha\rangle = \langle g_{\alpha^{-1}}, f\rangle$ where $\alpha^{-1} = (h_1^{-1}, h_2^{-1}, -t)$. In particular,

$$|\langle g, u_\alpha\rangle - \langle g, u_\beta\rangle| \leq \|g_{\alpha^{-1}} - g_{\beta^{-1}}\|_1 \|u\|_{B_2}$$

for $\alpha, \beta \in \widetilde{H} \times \widetilde{H} \times \mathbb{R}$, and $\alpha \mapsto u_\alpha$ is weak*-continuous.

The set $S = \{u_\alpha \mid \alpha \in \widetilde{H} \times \widetilde{H} \times [0, \pi]\}$ is a norm bounded subset of $B_2(\widetilde{G})$. If $T = \overline{\mathrm{conv}}^{\sigma(B_2,Q)}(S)$ is the weak*-closed convex hull of S, then T is weak*-compact by Banach–Alaoglu's Theorem. By [32, Theorem 3.27] the integral

$$u^{\mathcal{C}} = \frac{1}{\pi} \int_{\widetilde{H} \times \widetilde{H} \times [0,\pi]} u_\alpha \, d\mu(\alpha)$$

exists in $B_2(\widetilde{G})$. Here $d\mu(\alpha) = dh_1 dh_2 dt$. Since the set T is bounded in norm by $\|u\|_{B_2}$, and because it follows from [32, Theorem 3.27] that $u^{\mathcal{C}} \in T$, we obtain the inequality $\|u^{\mathcal{C}}\|_{B_2} \leq \|u\|_{B_2}$.

(2) Continued. Let $u \in B_2(\widetilde{G})$ be arbitrary. Observe that $\langle f^{\mathcal{C}}, u\rangle = \langle f, u^{\mathcal{C}}\rangle$. Hence the norm estimate $\|f^{\mathcal{C}}\|_Q \leq \|f\|_Q$ follows from (3).

(4) This is elementary. □

Proposition 3.6. *If $\widetilde{G}$ had the weak Haagerup property, then there would exist a bounded net (v_α) in $B_2(\widetilde{G}) \cap \mathcal{C}_0$ such that $v_\alpha \to 1$ in the weak*-topology.*

Proof. We suppose $\widetilde{G}$ has the weak Haagerup property. Using [27, Proposition 4.2] we see that there exist a constant $C > 0$ and a net (u_α) in $B_2(\widetilde{G}) \cap C_0(\widetilde{G})$ such that

$$\|u_\alpha\|_{B_2} \leq C \quad \text{for every } \alpha,$$

$$u_\alpha \to 1 \text{ in the } \sigma(B_2, Q)\text{-topology}.$$

Let $u_\alpha^{\mathcal{C}}$ be given by

$$u_\alpha^{\mathcal{C}}(x) = \frac{1}{2\pi} \int_0^{2\pi} \int_{\widetilde{H} \times \widetilde{H}} u_\alpha(h_1 \widetilde{v}_t g \widetilde{v}_t^{-1} h_2) \, dh_1 dh_2 dt, \quad x \in \widetilde{G},$$

where dh_1 and dh_2 both denote the normalized Haar measure on $\widetilde{H}$. By Lemma 3.5 we see that $u_\alpha^{\mathcal{C}} \in B_2(\widetilde{G}) \cap \mathcal{C}_0$ and that $(u_\alpha^{\mathcal{C}})$ is a bounded net. Thus, it suffices to prove that $u_\alpha^{\mathcal{C}} \to 1$ in the weak*-topology.

By Lemma 3.5, the map $L^1(\widetilde{G}) \to L^1(\widetilde{G})$ given by $f \mapsto f^{\mathcal{C}}$ extends uniquely to a linear contraction $R : Q(\widetilde{G}) \to Q(\widetilde{G})$. The dual operator $R^* : B_2(\widetilde{G}) \to B_2(\widetilde{G})$ obviously satisfies $R^* v = v^{\mathcal{C}}$ and is weak*-continuous. Hence

$$\langle f, u_\alpha^{\mathcal{C}}\rangle = \langle f, R^* u_\alpha\rangle \to \langle f, R^* 1\rangle = \langle f, 1\rangle$$

for any $f \in Q(G)$. This proves that $u_\alpha^{\mathcal{C}} \to 1$ in the weak*-topology. □

For $\beta \geq \gamma \geq 0$, we let $D(\beta, \gamma)$ denote the element in G given as

$$D(\beta, \gamma) = \begin{pmatrix} e^\beta & 0 & 0 & 0 \\ 0 & e^\gamma & 0 & 0 \\ 0 & 0 & e^{-\beta} & 0 \\ 0 & 0 & 0 & e^{-\gamma} \end{pmatrix}.$$

We define $\widetilde{D}(\beta,\gamma)$ as the element $(D(\beta,\gamma),0)$ in $\widetilde{G}$. Let $u \in B_2(\widetilde{G}) \cap \mathcal{C}$ be given. If we put

$$\dot{u}(\beta,\gamma,t) = u(\widetilde{v}_{\frac{t}{2}}\widetilde{D}(\beta,\gamma)),$$

then it is shown in [18, Proposition 3.11] that the limit $\lim_{s\to\infty} \dot{u}(2s,s,t)$ exists for any $t \in \mathbb{R}$. If we let

$$\mathcal{T} = \{u \in B_2(\widetilde{G}) \cap \mathcal{C} \mid \lim_{s\to\infty} \dot{u}(2s,s,t) = 0 \text{ for all } t \in \mathbb{R}\},$$

then we can phrase part of the main result of [18] in the following way.

Lemma 3.7 ([18, Lemma 3.12])**.** *The space $\mathcal{T}$ is closed in the weak*-topology.*

Using Lemma 3.7, it is not hard to show that $\widetilde{G}$ does not have the weak Haagerup property. The argument goes as follows.

Obviously, $1 \notin \mathcal{T}$. We claim that $B_2(\widetilde{G}) \cap \mathcal{C}_0 \subseteq \mathcal{T}$. Indeed, if $u \in C_0(\widetilde{G})$ and $t \in \mathbb{R}$, then

$$\dot{u}(2s,s,t) = u(\widetilde{v}_{\frac{t}{2}}\widetilde{D}(2s,s)) \to 0 \quad \text{ as } s \to \infty.$$

Since $\mathcal{T}$ is weak*-closed, we conclude by Proposition 3.6 that $\widetilde{G}$ does not have the weak Haagerup property.

Theorem 3.8. *The group $\widetilde{G} = \widetilde{\mathrm{Sp}}(2,\mathbb{R})$ does not have the weak Haagerup property.*

3.2. **The general case.** Knowing that the three groups $\mathrm{SL}(3,\mathbb{R})$, $\mathrm{Sp}(2,\mathbb{R})$, and $\widetilde{\mathrm{Sp}}(2,\mathbb{R})$ do not have the weak Haagerup property, it is a simple matter to generalize this result to include all connected simple Lie groups of real rank at least two. The idea behind the general case is basically that inside any connected simple Lie group of real rank at least two one can find a subgroup that looks like one of the three mentioned groups. We will make this statement more precise now. The following is certainly well known.

Lemma 3.9. *Let G be a connected simple Lie group of real rank at least two. Then G contains a closed connected subgroup H locally isomorphic to either* $\mathrm{SL}(3,\mathbb{R})$ *or* $\mathrm{Sp}(2,\mathbb{R})$.

Proof. Consider a connected simple Lie group G of real rank at least two. It is well known that the Lie algebra of such a group contains one of the Lie algebras $\mathfrak{sl}(3,\mathbb{R})$ or $\mathfrak{sp}(2,\mathbb{R})$ (see [29, Proposition 1.6.2]). Hence there is a connected Lie subgroup H of G whose Lie algebra is either $\mathfrak{sl}(3,\mathbb{R})$ or $\mathfrak{sp}(2,\mathbb{R})$ (see [21, Theorem II.2.1]). By [21, Theorem II.1.11] we get that H is locally isomorphic to $\mathrm{SL}(3,\mathbb{R})$ or $\mathrm{Sp}(2,\mathbb{R})$. It remains only to see that H is closed. This is [12, Corollary 1]. □

Theorem B. *A connected simple Lie group has the weak Haagerup property if and only if it has real rank zero or one.*

Proof. It is known that connected simple Lie groups of real rank zero and one have the weak Haagerup property. Indeed, connected simple Lie groups of real rank zero are compact, and connected simple Lie groups of real rank one are weakly amenable (see [9],[20]). This is clearly enough to conclude that such groups have the weak Haagerup property. Thus, we must prove that connected simple Lie groups of real rank at least two do not have the weak Haagerup property.

Let G be a connected simple Lie group of real rank at least two. Then G contains a closed connected subgroup H locally isomorphic to $\mathrm{SL}(3,\mathbb{R})$ or $\mathrm{Sp}(2,\mathbb{R})$. Because of [27, Theorem A(1)], it is sufficient to show that H does not have the weak Haagerup property.

Suppose first that H is locally isomorphic to $\mathrm{SL}(3,\mathbb{R})$. The fundamental group of $\mathrm{SL}(3,\mathbb{R})$ has order two, and $\mathrm{SL}(3,\mathbb{R})$ has trivial center. Hence the universal covering group of $\mathrm{SL}(3,\mathbb{R})$ has center of order two, and H must have finite center Z of order one or two. Then $\mathrm{SL}(3,\mathbb{R}) \simeq H/Z$. Since $\mathrm{SL}(3,\mathbb{R})$ does not have the weak Haagerup property, we deduce from [27, Theorem A(2)] that H does not have the weak Haagerup property.

Suppose instead that H is locally isomorphic to $\mathrm{Sp}(2,\mathbb{R})$. Then there is a central subgroup $Z \subseteq \widetilde{\mathrm{Sp}}(2,\mathbb{R})$ such that $\widetilde{\mathrm{Sp}}(2,\mathbb{R})/Z \simeq H$. Since the center of $\widetilde{\mathrm{Sp}}(2,\mathbb{R})$ is isomorphic to $\pi_1(\mathrm{Sp}(2,\mathbb{R})) \simeq \mathbb{Z}$, every nontrivial subgroup of the center of $\widetilde{\mathrm{Sp}}(2,\mathbb{R})$ is infinite and of finite index. Hence, if H has infinite center, then $\widetilde{\mathrm{Sp}}(2,\mathbb{R}) \simeq H$. In that case, H does not have the weak Haagerup property. Otherwise, H has finite center Z, and then $H/Z \simeq \mathrm{Sp}(2,\mathbb{R})/\{\pm 1\}$. Since $\mathrm{Sp}(2,\mathbb{R})$ does not have the weak Haagerup property, we deduce from [27, Theorem A(2)] that H does not have the weak Haagerup property.

□

4. Simple Lie groups of real rank one

In this section we compute the weak Haagerup constant of the groups $\mathrm{Sp}(1,n)$ and $\mathrm{F}_{4(-20)}$. We thus prove Theorem C. Throughout this section G denotes one of the groups $\mathrm{Sp}(1,n)$, $n \geq 2$ or $\mathrm{F}_{4(-20)}$. The symbol $\mathbb{R}_+$ denotes the nonnegative reals, that is, $\mathbb{R}_+ = [0,\infty[$.

4.1. **Preparations.** The group $\mathrm{Sp}(1,n)$ is defined as the group of quaternion matrices of size $n+1$ that preserve the Hermitian form

$$\langle x, y\rangle = \bar{y}_1 x_1 - \sum_{k=2}^{n+1} \bar{y}_k x_k, \quad x = (x_k)_{k=1}^{n+1}, y = (y_k)_{k=1}^{n+1} \in \mathbb{H}^{n+1}.$$

Equivalently,

$$\mathrm{Sp}(1,n) = \{g \in \mathrm{GL}(n+1,\mathbb{H}) \mid g^* I_{1,n} g = I_{1,n}\}$$

where $I_{1,n}$ is the $(n+1)\times(n+1)$ diagonal matrix

$$I_{1,n} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix}.$$

The exceptional Lie group $\mathrm{F}_{4(-20)}$ is described in [33].

For details about general structure theory of semisimple Lie groups we refer to [24, Chapters VI-VII] and [21, Chapter IX]. The proof of Theorem C builds on [9], where (1.5) is proved. We adopt the following from [9].

Recall that throughout this section G denotes one of the connected simple real rank one Lie groups $\mathrm{Sp}(1,n)$, $n \geq 2$ or $\mathrm{F}_{4(-20)}$. Let $\mathfrak{g}$ be the Lie algebra of G. Let θ be a Cartan involution, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the corresponding Cartan decomposition and K the analytic subgroup corresponding to $\mathfrak{k}$. Then K is a maximal compact subgroup of G. Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p}$, and decompose $\mathfrak{g}$ into root spaces,

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \sum_{\beta\in\Sigma} \mathfrak{g}_\beta,$$

where $\mathfrak{m}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$ and Σ is the set of roots. Then $\mathfrak{a}$ is one dimensional and $\Sigma = \{-2\alpha, -\alpha, \alpha, 2\alpha\}$. Let $\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$. We have the Iwasawa decomposition at the Lie algebra level

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$$

and at the group level

$$G = KAN$$

where A and N are the analytic subgroups of G with Lie algebras $\mathfrak{a}$ and $\mathfrak{n}$, respectively. The group A is abelian and simply connected, and N is nilpotent and simply connected.

Let B be the Killing form of $\mathfrak{g}$. Let $\mathfrak{v} = \mathfrak{g}_\alpha$, $\mathfrak{z} = \mathfrak{g}_{2\alpha}$ and equip the Lie algebra $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ with the inner product

$$\langle v + z, v' + z' \rangle = \frac{-1}{2p + 4q} B\left(\frac{v}{2} + \frac{z}{4}, \theta\left(\frac{v'}{2} + \frac{z'}{4}\right)\right)$$

$v, v' \in \mathfrak{v}$, $z, z' \in \mathfrak{z}$ where, as in [9],

$$2p = \dim \mathfrak{v}, \quad q = \dim \mathfrak{z}.$$

The inner product on $\mathfrak{n}$ of course gives rise to a norm $|\ \ |$ on $\mathfrak{n}$ defined by $|n| = \sqrt{\langle n, n \rangle}$, $n \in \mathfrak{n}$.

The following convenient notation is taken from [9]. Let

$$(v, z) = \exp(v + z/4), \quad v \in \mathfrak{v},\ z \in \mathfrak{z}. \tag{4.1}$$

Then $(v, z) \in N$. Since N is connected, nilpotent and simply connected, the exponential mapping is a diffeomorphism of $\mathfrak{n}$ onto N ([24, Theorem 1.127]), and hence every element of N can in a unique way be written in the form (4.1).

We let $a = p/2$. It is well known that the values of p, q, and a are as follows:

Group	p	q	a
$\mathrm{Sp}(1, n)$	$2n - 2$	3	$n - 1$
$\mathrm{F}_{4(-20)}$	4	7	2

(4.2)

As $\mathfrak{a}$ is one-dimensional there is a unique element H in $\mathfrak{a}$ such that $\mathrm{ad}(H)|_{\mathfrak{g}_\alpha} = \mathrm{id}_{\mathfrak{g}_\alpha}$. Let

$$a_t = \exp(tH) \in A, \quad t \in \mathbb{R},$$

and $\overline{A^+} = \overline{\{\exp tH \mid t > 0\}} = \{a_t \in A \mid t \geq 0\}$. Then we have the KAK decomposition of G (see [21, Theorem IX.1.1])

$$G = K\overline{A^+}K. \tag{4.3}$$

More precisely, for each $g \in G$ there is a unique $t \geq 0$ such that $g \in Ka_tK$. Concerning the KAK decomposition of elements of N we can be even more specific. The following lemma is completely analogous to part of [9, Proposition 2.1], and thus we leave out the proof.

Lemma 4.1. *For every $v \in \mathfrak{v}$ and $z \in \mathfrak{z}$ exists a unique $t \in \mathbb{R}_+$ such that*

$$(v, z) \in Ka_tK.$$

Moreover, t satisfies

$$4 \sinh^2 t = 4|v|^2 + |v|^4 + |z|^2.$$

The following fact proved by Whitney [34, Theorem 1] identifies the smooth even functions on $\mathbb{R}$ with smooth functions on $\mathbb{R}_+ = [0, \infty[$.

Lemma 4.2 ([34])**.** *An even function g on $\mathbb{R}$ is smooth if and only if it has the form $g(x) = f(x^2)$ for some (necessarily unique) $f \in C^\infty(\mathbb{R}_+)$.*

The following proposition is inspired by Theorem 2.5(b) in [9].

Proposition 4.3. *Suppose $u \in C(N)$. Then u is the restriction to N of a K-bi-invariant function on G if and only if u is of the form*

$$(v, z) \mapsto f(4|v|^2 + |v|^4 + |z|^2) \tag{4.4}$$

for some $f \in C(\mathbb{R}_+)$. In that case, the function f is uniquely determined by u.

The function f is in $C^\infty(\mathbb{R}_+)$, $C_c(\mathbb{R}_+)$, or $C_0(\mathbb{R}_+)$ if and only if u is in $C^\infty(N)$, $C_c(N)$, or $C_0(N)$, respectively.

Proof. Assume $u \in C(N)$ is the restriction to N of a K-bi-invariant function on G. Then by Lemma 4.1, $u(v, z)$ only depends on $4|v|^2 + |v|^4 + |z|^2$ when $v \in \mathfrak{v}$, $z \in \mathfrak{z}$. Hence there is a unique function f on $\mathbb{R}_+$ such that

$$u(v, z) = f(4|v|^2 + |v|^4 + |z|^2), \quad v \in \mathfrak{v},\ z \in \mathfrak{z}.$$

If we fix a unit vector $z_0 \in \mathfrak{z}$ then $t \mapsto u(0, \sqrt{t} z_0) = f(t)$ is continuous on $\mathbb{R}_+$, since u is continuous. In other words, $f \in C(\mathbb{R}_+)$.

Assume conversely that u is of the form (4.4) for some (necessarily unique) $f \in C(\mathbb{R}_+)$. We define a function $\widetilde{u}$ on G using the $K\overline{A^+}K$ decomposition as follows. For an element ka_tk' in G where $k, k' \in K$ and $t \in \mathbb{R}_+$ we let

$$\widetilde{u}(ka_tk') = f(4\sinh^2 t).$$

By the uniqueness of t in the $K\overline{A^+}K$ decomposition, this is well-defined. Clearly, $\widetilde{u}$ is a K-bi-invariant function on G. When $(v, z) \in N$ we find by Lemma 4.1 that

$$\widetilde{u}(v, z) = f(4|v|^2 + |v|^4 + |z|^2) = u(v, z)$$

so that $\widetilde{u}$ restricts to u on the subgroup N.

It is easy to see that u has compact support if and only if f has compact support, and similarly that u vanishes at infinity if and only if f vanishes at infinity. It is also clear that smoothness of f implies smoothness of u.

Finally, assume that u is smooth. If again $z_0 \in \mathfrak{z}$ is a unit vector, then $t \mapsto u(0, tz_0) = f(t^2)$ is a smooth even function on $\mathbb{R}$. By Lemma 4.2 we obtain $f \in C^\infty(\mathbb{R}_+)$. □

We remark that $\|u\|_\infty = \|f\|_\infty$.

Lemma 4.4. *Let (u_k) be a sequence, where $u_k \in C(N)$ is the restrictions to N of a K-bi-invariant function in $C(G)$, and let $f_k \in C(\mathbb{R}_+)$ be as in Proposition 4.3. If $u_k \to 1$ pointwise, then $f_k \to 1$ pointwise.*

Proof. This is obvious, since the map $(v, z) \mapsto 4|v|^2 + |v|^4 + |z|^2$ is a surjection of N onto $\mathbb{R}_+$. □

4.2. **Proof of Theorem C.** With almost all the notational preparations in place, we are now ready to aim for the proof of Theorem C. The starting point is the inequality in Proposition 4.5 which is taken almost directly from [9]. To ease notation a bit, let

$$C = \frac{2^{p+1}\Gamma\left(\frac{p+q}{2}\right)}{\Gamma(p)\Gamma\left(\frac{q}{2}\right)}. \tag{4.5}$$

We remark that with this definition C and (1.5) and (4.2) in mind, then

$$\frac{C}{4}\Gamma(a) = \Lambda_{\mathrm{WA}}(G). \tag{4.6}$$

Combining Theorem 2.5(b), Proposition 5.1, and Proposition 5.2 in [9] one obtains the following proposition.

Proposition 4.5 ([9]). *If $u \in C_c^\infty(N)$ is the restriction of a K-bi-invariant function on G, then u is of the form*

$$u(z,v) = f(4|v|^2 + |v|^4 + |z|^2)$$

for some $f \in C_c^\infty(\mathbb{R})$, and

$$\left| C \int_{\mathbb{R}_+} f^{(a)}(4t^2 + t^4) t^{2p-1}\ dt \right| \leq \|u\|_{A(N)}.$$

We now aim to prove a variation of the above proposition where we no longer require the function u to be compactly supported.

Following [9], we let $h\colon]0,\infty[\to \mathbb{R}$ be defined by $h(s) = (s^{1/2} - 2)^{p-1} s^{-1/2}$, and let g be the $(a-1)$'th derivative of h. It is known (see [9, p. 544]) that

$$\int_4^\infty |g'(s)|\ ds < \infty \quad \text{and} \quad \int_4^\infty g'(s)\ ds = \Gamma(a). \tag{4.7}$$

Proposition 4.6. *If $u \in A(N) \cap C^\infty(N)$ is the restriction of a K-bi-invariant function on G, then u is of the form*

$$u(z,v) = f(4|v|^2 + |v|^4 + |z|^2)$$

for some $f \in C_0^\infty(\mathbb{R}_+)$, and

$$\left| \frac{C}{4} \int_4^\infty f(s-4) g'(s)\ ds \right| \leq \|u\|_{A(N)}.$$

Proof. We use the fact that G is weakly amenable [9]. We will then approximate u by functions in $C_c^\infty(N)$ and apply Proposition 4.5 to those functions.

Choose a sequence $v_k \in C_c^\infty(G)$ of K-bi-invariant functions such that $\|v_k\|_{B_2} \leq \Lambda_{\mathrm{WA}}(G)$ and

$$\|v_k v - v\|_{A(G)} \to 0 \quad \text{for every } v \in A(G)$$

(see Lemma 2.4). Put $w_k = v_k|_N$. Then by Lemma 2.2 we have

$$\|w_k v - v\|_{A(N)} \to 0 \quad \text{for every } v \in A(N).$$

If we put $u_k = w_k u$, then we get $u_k \to u$ uniformly. Note that $u_k \in C_c^\infty(N)$. Let $f \in C_0^\infty(\mathbb{R}_+)$ and $f_k \in C_c^\infty(\mathbb{R}_+)$ be chosen according to Proposition 4.3 such that

$$u(v,z) = f(4|v|^2 + |v|^4 + |z|^2), \qquad u_k(v,z) = f_k(4|v|^2 + |v|^4 + |z|^2).$$

Using the substitution $s = 4 + 4t^2 + t^4$ and then partial integration we get

$$\begin{aligned}\|u\|_{A(N)} &= \lim_k \|u_k\|_{A(N)} \\ &\geq \lim_k \left| C \int_{\mathbb{R}_+} f_k^{(a)}(4t^2 + t^4) t^{2p-1}\, dt \right| \\ &= \lim_k \left| \frac{C}{4} \int_4^\infty f_k^{(a)}(s-4) h(s)\, ds \right| \\ &= \lim_k \left| \frac{C}{4} \int_4^\infty f_k(s-4) g'(s)\, ds \right|.\end{aligned}$$

There are no boundary terms, since f_k has compact support, and because the first $p-2$ derivatives of h vanish at $s = 4$. We observe that

$$\|f_k\|_\infty = \|u_k\|_\infty \leq \|u_k\|_{B_2} \leq \|w_k\|_{B_2} \|u\|_{B_2} \leq \|v_k\|_{B_2} \|u\|_{B_2} \leq \Lambda_{\mathrm{WA}}(G)\ \|u\|_{B_2},$$

so in particular, $\sup_k \|f_k\|_\infty < \infty$. Finally, since $g'(s)$ is integrable (see (4.7)), we can apply Lebesgue's Dominated Convergence Theorem and get

$$\|u\|_{A(N)} \geq \left| \frac{C}{4} \int_4^\infty f(s-4) g'(s)\, ds \right|.$$

□

Proposition 4.7. *Suppose $u_k \in A(N) \cap C^\infty(N)$ is the restriction of a K-bi-invariant function on G, and suppose further that $u_k \to 1$ pointwise as $k \to \infty$. Then*

$$\sup_k \|u_k\|_{A(N)} \geq \Lambda_{\mathrm{WA}}(G).$$

Proof. If $\sup_k \|u_k\|_{A(N)} = \infty$, there is nothing to prove. So we assume that $\sup_k \|u_k\|_{A(N)} < \infty$.

Let $f_k \in C_0^\infty(\mathbb{R}_+)$ be chosen according to Proposition 4.3 such that

$$u_k(v, z) = f_k(4|v|^2 + |v|^4 + |z|^2).$$

Observe that $f_k \to 1$ pointwise, and $\sup_k \|f_k\|_\infty < \infty$. By Lebesgue's Dominated Convergence Theorem we have

$$\sup_k \|u_k\|_{A(N)} \geq \lim_k \left| \frac{C}{4} \int_4^\infty f_k(s-4) g'(s)\, ds \right| = \frac{C}{4} \int_4^\infty g'(s)\, ds = \frac{C}{4}\ \Gamma(a).$$

As mentioned in (4.6), $\Lambda_{\mathrm{WA}}(G) = C\Gamma(a)/4$. □

Theorem C is an immediate consequence of the following, since we already know the value $\Lambda_{\mathrm{WA}}(G)$ and that $\Lambda_{\mathrm{WH}}(G) \leq \Lambda_{\mathrm{WA}}(G)$.

Proposition 4.8. *If G is either* $\mathrm{Sp}(1, n)$*, $n \geq 2$, or* $\mathrm{F}_{4(-20)}$*, then $\Lambda_{\mathrm{WH}}(G) = \Lambda_{\mathrm{WA}}(G)$.*

Proof. We only prove $\Lambda_{\mathrm{WH}}(G) \geq \Lambda_{\mathrm{WA}}(G)$, since the other inequality holds trivially. Using Proposition 4.3 in [27], it is enough to prove that if a sequence $v_k \in B_2(G) \cap C_0^\infty(G)$ consisting of K-bi-invariant functions satisfies

$$\|v_k\|_{B_2} \leq L \quad \text{for all } k,$$

$$v_k \to 1 \quad \text{uniformly on compacts as } k \to \infty,$$

then $L \geq \Lambda_{\mathrm{WA}}(G)$. So suppose such a sequence is given. Consider the subgroup P from Theorem 2.1. Since P is amenable, $B_2(P) = B(P)$ isometrically. Then

$$v_k|_P \in B_2(P) \cap C_0(P) = B(P) \cap C_0(P) = A(P)$$

by Theorem 2.1, and so $v_k|_N \in A(N)$. To ease notation, we let $u_k = v_k|_N$. Then (using amenability of N)

$$\|u_k\|_{A(N)} = \|u_k\|_{B(N)} = \|u_k\|_{B_2(N)} \leq \|v_k\|_{B_2(G)}.$$

Hence by Proposition 4.7 and the above inequalities we conclude

$$\Lambda_{\mathrm{WA}}(G) \leq \sup_k \|u_k\|_{A(N)} \leq L.$$

This shows that $\Lambda_{\mathrm{WA}}(G) \leq \Lambda_{\mathrm{WH}}(G)$, and the proof is complete. □

Proof of Theorem C. Suppose G is a connected simple Lie group. If the real rank of G is zero, then G is compact and $\Lambda_{\mathrm{WA}}(G) = \Lambda_{\mathrm{WH}}(G) = 1$. If the real rank of G is at least two, then $\Lambda_{\mathrm{WA}}(G) = \infty$ by [16], [12]. By Theorem B also $\Lambda_{\mathrm{WH}}(G) = \infty$.

Only the case when the real rank of G equals one remains. Then G is locally isomorphic to either $\mathrm{SO}_0(1,n)$, $\mathrm{SU}(1,n)$, $\mathrm{Sp}(1,n)$ where $n \geq 2$ or locally isomorphic to $\mathrm{F}_{4(-20)}$ (see, for example, the list [24, p. 426] and [21, Theorem II.1.11]). If G is locally isomorphic to $\mathrm{SO}_0(1,n)$ or $\mathrm{SU}(1,n)$ then by (1.5) we conclude that $\Lambda_{\mathrm{WA}}(G) = \Lambda_{\mathrm{WH}}(G) = 1$.

Finally, let $\widetilde{G}$ be either $\mathrm{Sp}(1,n)$ or $\mathrm{F}_{4(-20)}$ and suppose G is locally isomorphic to $\widetilde{G}$. If KAN is the Iwasawa decomposition of $\widetilde{G}$ then K is $\mathrm{Sp}(n) \times \mathrm{Sp}(1)$ or $\mathrm{Spin}(9)$, respectively (see Section 4, Proposition 1 and Section 5, Theorem 1 in [33] for the latter). Here $\mathrm{Spin}(9)$ is the two-fold simply connected cover of $\mathrm{SO}(9)$. In any case, K is simply connected and compact, so it follows that $\widetilde{G}$ is simply connected with finite center ([24, Theorem 6.31]).

From Proposition 4.8, we get that $\Lambda_{\mathrm{WH}}(G) = \Lambda_{\mathrm{WA}}(G)$ if $G = \widetilde{G}$. Otherwise G is a quotient of $\widetilde{G}$ by a finite central subgroup, and then it follows from (1.5), Proposition 4.8 and [27, Proposition 5.4] that $\Lambda_{\mathrm{WH}}(G) = \Lambda_{\mathrm{WH}}(\widetilde{G}) = \Lambda_{\mathrm{WA}}(G)$. □

5. Another group without the weak Haagerup property

Throughout this section, we let G be the group $G = \mathbb{R}^2 \rtimes \mathrm{SL}(2,\mathbb{R})$. We show here that this group does not have the weak Haagerup property. In short, we prove $\Lambda_{\mathrm{WH}}(G) = \infty$. This generalizes a result from [11] and [16], where it is proved that $\Lambda_{\mathrm{WA}}(G) = \infty$.

We shall think of G as a subgroup of $\mathrm{SL}(3,\mathbb{R})$ in the following way:

$$G = \left(\begin{array}{c|c} \mathrm{SL}(2,\mathbb{R}) & \mathbb{R}^2 \\ \hline 0 & 1 \end{array}\right).$$

We consider the compact group $K = \mathrm{SO}_2(\mathbb{R})$ as a subgroup of G using the inclusions

$$\mathrm{SO}_2(\mathbb{R}) \subseteq \mathrm{SL}(2,\mathbb{R}) \subseteq \mathbb{R}^2 \rtimes \mathrm{SL}(2,\mathbb{R}).$$

We will make use of the following closed subgroups of G.

$$N = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| \, x,y,z \in \mathbb{R} \right\}, \qquad P = \left\{ \begin{pmatrix} \lambda & x & z \\ 0 & \lambda^{-1} & y \\ 0 & 0 & 1 \end{pmatrix} \middle| \, x,y,z \in \mathbb{R},\ \lambda > 0 \right\}. \tag{5.1}$$

The group N is the Heisenberg group. The following is proved in [11, Section 10] (see also [16, Lemma A and Lemma E]).

Proposition 5.1 ([11]). *Consider the Heisenberg group N. If $u \in C_c^\infty(N)$ is the restriction of a K-bi-invariant function in $C^\infty(G)$, then*

$$\left| \int_{-\infty}^{\infty} \frac{u(x,0,0)}{\sqrt{1+x^2/4}}\, dx \right| \leq 12\pi \|u\|_{A(N)}.$$

We now prove a variation of the above lemma where we no longer require the function in question to be compactly supported.

Proposition 5.2. *Suppose $u \in A(N) \cap C^\infty(N)$ is the restriction of a K-bi-invariant function in $C^\infty(G)$. Then*

$$\int_{-\infty}^{\infty} \frac{|u(x,0,0)|^2}{\sqrt{1+x^2/4}}\, dx \leq 12\pi \|u\|_{A(N)}^2.$$

Proof. The idea is to use the fact (see [19, p. 670]) that $G = \mathbb{R}^2 \rtimes \mathrm{SL}(2,\mathbb{R})$ has the AP. We will approximate u with compactly supported, smooth functions on N and then apply Proposition 5.1.

By Lemma 2.2, there is an extension $\widetilde{u} \in A(G)$ of u. It follows from Lemma 2.5 that there is a *sequence* (v_k) in $C_c^\infty(K\backslash G/K)$ such that

$$\|v_k \widetilde{u} - \widetilde{u}\|_{A(G)} \to 0.$$

We let $w_k = v_k|_N$. Since restriction does not increase the norm, we have

$$\|w_k u - u\|_{A(N)} = \|(v_k \widetilde{u} - \widetilde{u})|_N\|_{A(N)} \leq \|v_k \widetilde{u} - \widetilde{u}\|_{A(G)} \to 0.$$

Since $w_k u \to u$ pointwise we have by Fatou's Lemma and Proposition 5.1 applied to $|w_k u|^2$

$$\begin{aligned} \|u\|_{A(N)}^2 \geq \liminf_{k\to\infty} \||w_k u|^2\|_{A(N)} &\geq \frac{1}{12\pi} \int_{-\infty}^{\infty} \liminf_{k\to\infty} \frac{|w_k u(x,0,0)|^2}{\sqrt{1+x^2/4}}\, dx \\ &= \frac{1}{12\pi} \int_{-\infty}^{\infty} \frac{|u(x,0,0)|^2}{\sqrt{1+x^2/4}}\, dx. \end{aligned}$$

In the first inequality we have used that for every $v \in A(N)$ we have $|v|^2 = v\overline{v} \in A(N)$ and $\||v|^2\|_A \leq \|v\|_A \|\overline{v}\|_A = \|v\|_A^2$. ☐

Having done all the necessary preparations, we are now ready for

Theorem D. *The group $\mathbb{R}^2 \rtimes \mathrm{SL}(2,\mathbb{R})$ does not have the weak Haagerup property.*

Proof. Suppose there is a net (u_n) of Herz–Schur multipliers on $G = \mathbb{R}^2 \rtimes \mathrm{SL}(2,\mathbb{R})$ vanishing at infinity and converging uniformly to the constant function 1_G on compacts. We will show that $\sup_n \|u_n\|_{B_2} = \infty$. By Proposition 4.3 in [27], we may assume that $u_n \in C_0^\infty(K\backslash G/K)$, and since G is second countable, we may assume that the net is a sequence.

Consider the group P defined in (5.1). Since P is amenable, even solvable, we know that $B_2(P) = B(P)$ isometrically. We also know that $A(P) = B(P) \cap C_0(P)$ (see [25, Theorem 2]). Then

$$u_n|_P \in B_2(P) \cap C_0(P) = B(P) \cap C_0(P) = A(P),$$

and so $u_n|_N \in A(N)$. To ease notation, we let $w_n = u_n|_N$. Then, using amenability of N,

$$\|w_n\|_{A(N)} = \|w_n\|_{B(N)} = \|w_n\|_{B_2(N)} \leq \|u_n\|_{B_2(G)}.$$

Hence it will suffice to show that $\sup_n \|w_n\|_{A(N)} = \infty$. By Proposition 5.2, we have

$$\int_{-\infty}^{\infty} \frac{|w_n(x,0,0)|^2}{\sqrt{1+x^2/4}}\, dx \le 12\pi \|w_n\|_{A(N)}^2.$$

Since $u_n \to 1_G$ uniformly on compacts, we have in particular $w_n(x,0,0) \to 1$ as $n \to \infty$. It follows that

$$\liminf_{n\to\infty} \|w_n\|_{A(N)}^2 \ge \frac{1}{12\pi} \int_{-\infty}^{\infty} \frac{1}{\sqrt{1+x^2/4}}\, dx = \infty.$$

This completes the proof. □

Remark 5.3. It was proved by the first author [16] that the group $\mathbb{R}^2 \rtimes \mathrm{SL}(2,\mathbb{R})$ is not weakly amenable. This result was later generalized by Dorofaeff [11] to include the groups $\mathbb{R}^n \rtimes \mathrm{SL}(2,\mathbb{R})$ where $n \ge 2$. Here the action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{R}^n$ is by the unique irreducible representation of dimension n.

In view of Theorem D, and especially since our proof of Theorem D uses the same techniques as [16] and [11], it is natural to ask if the groups $\mathbb{R}^n \rtimes \mathrm{SL}(2,\mathbb{R})$ also fail to have the weak Haagerup property when $n \ge 3$.

We note that an affirmative answer in the case $n = 3$ would give a different proof of Theorem A. This is because $\mathrm{SL}(3,\mathbb{R})$ contains $\mathbb{R}^2 \rtimes \mathrm{SL}(2,\mathbb{R})$ as a closed subgroup, and both groups $\mathrm{Sp}(2,\mathbb{R})$ and $\widetilde{\mathrm{Sp}}(2,\mathbb{R})$ contain $\mathbb{R}^3 \rtimes \mathrm{SL}(2,\mathbb{R})$ as a closed subgroup (see [12]).

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ARTICLE D

Fourier algebras of parabolic subgroups

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FOURIER ALGEBRAS OF PARABOLIC SUBGROUPS

SØREN KNUDBY

Abstract. We study the following question: Given a locally compact group when does its Fourier algebra coincide with the subalgebra of the Fourier-Stieltjes algebra consisting of functions vanishing at infinity? We provide sufficient conditions for this to be the case.

As an application we show that when P is the minimal parabolic subgroup in one of the classical simple Lie groups of real rank one or the exceptional such group, then the Fourier algebra of P coincides with the subalgebra of the Fourier-Stieltjes algebra of P consisting of functions vanishing at infinity. In particular, the regular representation of P decomposes as a direct sum of irreducible representations although P is not compact.

We also show that P contains a non-compact closed normal subgroup with the relative Howe-Moore property.

1. Introduction

If G is a locally compact abelian group with dual group $\widehat{G}$, then the Fourier transform on $\widehat{G}$ maps the group algebra $L^1(\widehat{G})$ injectively onto a subset $A(G)$ of the continuous functions on G. Also, the Fourier-Stieltjes transform on $\widehat{G}$ maps the measure algebra $M(\widehat{G})$ injectively onto a subset $B(G)$ of the continuous functions on G. Using the usual identification $L^1(\widehat{G}) \subseteq M(\widehat{G})$ we see that $A(G) \subseteq B(G)$. Every function in $B(G)$ is bounded, and every function in $A(G)$ vanishes at infinity. In the very special case when $\widehat{G} = \mathbb{R}^n$, the fact that functions in $A(G)$ vanish at infinity is the Riemann-Lebesgue lemma.

In the paper [11], Eymard introduced the algebras $A(G)$ and $B(G)$ in the setting where G is no longer assumed to be abelian. Let G be a locally compact group. The *Fourier-Stieltjes algebra* $B(G)$ is defined as the linear span of the continuous positive definite functions on G. There is a natural identification of $B(G)$ with the Banach space dual of the full group C^*-algebra $C^*(G)$, and under this identification $B(G)$ inherits a norm with which it is a Banach space. The *Fourier algebra* $A(G)$ is the closed subspace in $B(G)$ generated by the compactly supported functions in $B(G)$. Other descriptions of $A(G)$ and $B(G)$ are available (see Section 2). The Fourier and Fourier-Stieltjes algebras play an important role in non-commutative harmonic analysis.

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For general locally compact groups it is still true that $A(G) \subseteq C_0(G)$ just as in the abelian case, and it is natural to ask whether every function in $B(G)$ which vanishes at infinity belongs to $A(G)$.

Question 1. *Let G be a locally compact group. Does the equality*

$$A(G) = B(G) \cap C_0(G) \tag{1.1}$$

hold?

Of course, if G is compact then $B(G) = A(G)$, and (1.1) obviously holds. But for non-compact groups the question is more delicate.

In 1916, Menchoff [26] proved the existence of a singular probability measure μ on the circle such that its Fourier-Stieltjes transform $\widehat{\mu}$ satisfies $\widehat{\mu}(n) \to 0$ as $|n| \to \infty$. In other words, $\widehat{\mu} \in B(\mathbb{Z}) \cap C_0(\mathbb{Z})$, but $\widehat{\mu} \notin A(\mathbb{Z})$, and thus the answer to Question 1 is negative when G is the group $\mathbb{Z}$ of integers. In 1966, Hewitt and Zuckerman [15] proved that for any abelian locally compact group G the answer to Question 1 is always negative, unless G is compact. In 1983 it was shown that for any countable, discrete group G one has $A(G) \neq B(G) \cap C_0(G)$, unless G is finite (see [30, p. 190] and [5]).

The first non-compact example of a group satisfying (1.1) was given by Khalil in [19] and is the (non-unimodular) $ax + b$ group consisting of affine transformations $x \mapsto ax+b$ of the real line, where $a > 0$ and $b \in \mathbb{R}$. We remark that the $ax+b$ group is isomorphic to the minimal parabolic subgroup in the simple Lie group $\mathrm{PSL}_2(\mathbb{R})$ of real rank one.

It is proved in [12],[5] that if (1.1) holds for some second countable, locally compact group G, then the regular representation of G is completely reducible, i.e., a direct sum of irreducible representations. For a while, this was thought to be a characterization of groups satisfying (1.1), but this was shown not to be the case (see [4] or [25]). However, it follows from the fact that second countable, locally compact groups satisfying (1.1) have completely reducible regular representations combined with [30] that (1.1) fails for second countable, locally compact IN-groups, unless they are compact. Recall that an IN-group is a group which has a compact neighborhood of the identity which is invariant under all inner automorphisms. In particular, abelian, discrete and compact groups are all IN-groups.

It follows from Baggett's work [3] that if G is a locally compact, second countable group which is also connected, unimodular and has a completely reducible regular representation, then G is compact (see [31, Theorem 3]). In particular, Question 1 has a negative answer for locally compact second countable connected unimodular groups which are non-compact. This gives an abundance of examples of groups where Question 1 has a negative answer. An example given in [25] and [32] (independently) of a unimodular group satisfying (1.1) shows that the assumption about connectedness cannot be removed from the previous statement, and of course the assumption about unimodularity cannot be removed as the $ax + b$ group shows.

It should be apparent from the above that there are plenty of examples of groups for which Question 1 has a negative answer. In this paper we provide new examples of groups answering Question 1 in the affirmative. Our main source of examples is formed by the minimal parabolic subgroups in connected simple Lie groups of real rank one. But first we give a more straightforward example which is a subgroup of $\mathrm{SL}_3(\mathbb{R})$. The method of proof in this example can be seen as an easy version of what follows after. We prove the following.

Theorem 2. *For the group*

$$P = \left\{ \begin{pmatrix} \lambda & a & c \\ 0 & \lambda^{-1} & b \\ 0 & 0 & 1 \end{pmatrix} \middle| a, b, c \in \mathbb{R},\ \lambda > 0 \right\} \tag{1.2}$$

we have $A(P) = B(P) \cap C_0(P)$.

If we think of $\mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2$ as a subgroup of $\mathrm{SL}_3(\mathbb{R})$ in the following way

$$\left(\begin{array}{c|c} \mathrm{SL}_2(\mathbb{R}) & \mathbb{R}^2 \\ \hline 0 & 1 \end{array} \right),$$

then we can think of P as a subgroup of $\mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2$. This viewpoint will be relevant in a forthcoming paper [14] by the author and U. Haagerup.

Apart from the group in (1.2), our examples of groups satisfying (1.1) arise in the following way. Let $n \geq 2$, let G be one of the classical simple Lie groups $\mathrm{SO}_0(n,1)$, $\mathrm{SU}(n,1)$, $\mathrm{Sp}(n,1)$ or the exceptional group $\mathrm{F}_{4(-20)}$, and let $G = KAN$ be the Iwasawa decomposition. If M is the centralizer of A in K, then $P = MAN$ is the minimal parabolic subgroup of G. We refer to Section 6 for explicit descriptions of the groups G, K, A, N and M. We prove the following theorem concerning the Fourier algebra of the minimal parabolic subgroup.

Theorem 3. *Let P be the minimal parabolic subgroup in one of the simple Lie groups* $\mathrm{SO}_0(n,1)$, $\mathrm{SU}(n,1)$, $\mathrm{Sp}(n,1)$ *or* $\mathrm{F}_{4(-20)}$. *Then* $A(P) = B(P) \cap C_0(P)$.

In order to prove Theorem 2 and Theorem 3 we develop a general strategy for providing examples of groups that answer Question 1 affirmatively. The strategy is based on (1) determining all irreducible representations of the group, (2) determining the irreducible subrepresentations of the regular representation and (3) disintegration theory. An often useful tool for (1) is the Mackey Machine (see [13, Chapter 6] and [17]).

Our strategy for proving Theorem 2 and Theorem 3 is contained in the following theorem.

Theorem 4. *Let G be a second countable, locally compact group satisfying the following two conditions.*

(1) *G is type I.*
(2) *There is a non-compact, closed subgroup H of G such that every irreducible unitary representation of G is either trivial on H or is a subrepresentation of the left regular representation λ_G.*

Then

$$A(G) = B(G) \cap C_0(G).$$

In particular, the left regular representation λ_G *is completely reducible.*

It was pointed out to the author by T. de Laat that with the assumptions of Theorem 4 one can deduce that (G, H) has the *relative Howe-Moore property* (defined in [6]). In fact, condition (1) can be dropped, and condition (2) still implies that (G, H) has the relative Howe-Moore property as is immediately seen from [6, Proposition 2.3] and the well-known fact that the regular representation is a C_0-representation.

Since we prove Theorem 2 and Theorem 3 by verifying the conditions in Theorem 4 for the groups in question, we obtain the following corollary.

Corollary 5. *Let* P *be the group in* (1.2) *or the minimal parabolic subgroup in one of the simple Lie groups* $\mathrm{SO}_0(n,1)$*,* $\mathrm{SU}(n,1)$*,* $\mathrm{Sp}(n,1)$ *or* $\mathrm{F}_{4(-20)}$*. Then there is a normal, non-compact closed subgroup* H *in* P *such that* (P, H) *has the relative Howe-Moore property.*

The non-compact subgroup H can be described explicitly. For a more precise statement see Corollary 28 below. We refer to [6] for a treatment of the relative Howe-Moore property.

In order to verify the two conditions in Theorem 4 for the minimal parabolic subgroups P, we rely primarily on earlier work of J.A. Wolf. In [33] the irreducible representations of some parabolic subgroups are determined by employing the Mackey Machine, and the approach of [33] carries over to our situation almost without changes. Using [20] we can easily determine the irreducible subrepresentations of the regular representation of P.

The paper is organized as follows. In Section 2 we describe the basic properties of the Fourier and Fourier-Stieltjes algebra, and Section 3 contains the proof of Theorem 4. Section 4 contains a few results to be used later when we verify condition (2) of Theorem 4 for the groups in question. In Section 5 we prove Theorem 2. This includes determining all irreducible unitary representations of the group (1.2), determining the Plancherel measure for the group and finally verifying conditions (1) and (2) of Theorem 4 for the group.

In Section 6 we turn to the minimal parabolic subgroups P in the simple Lie groups of real rank one that we will be working with. We give an explicit description of the groups as matrix groups (at least in the classical cases). In Section 7 we describe the irreducible representations of the minimal parabolic subgroups, and then, in Section 8, we verify the two conditions in Theorem 4 for the minimal parabolic subgroups. Theorem 3 then follows immediately.

Section 9 contains the proof of Corollary 5 concerning the Howe-Moore property, and Section 10 contains some concluding remarks.

2. The Fourier and Fourier-Stieltjes algebra

This section contains a brief description of the Fourier and Fourier-Stieltjes algebra of a locally compact group introduced by Eymard in [11]. We refer to the original paper [11] for more details. Let G be a locally compact group equipped with a left Haar measure. By a representation of G we always mean a continuous unitary representation of G on some Hilbert space (except for the vector and spin representations in Section 6.2). If π is a unitary representation of G on a Hilbert space $\mathcal{H}$, and $x, y \in \mathcal{H}$, then the continuous complex function

$$\varphi(g) = \langle \pi(g)x, y\rangle, \qquad (g \in G)$$

is a *matrix coefficient* of π. The Fourier-Stieltjes algebra of G is denoted $B(G)$ and consists of the complex linear span of continuous positive definite functions on G. It coincides with the set of all matrix coefficients of representations of G,

$$B(G) = \{\langle \pi(\cdot)x, y\rangle \mid (\pi, \mathcal{H}) \text{ is a representation of } G \text{ and } x, y \in \mathcal{H}\}.$$

Since the pointwise product of two positive definite functions is again positive definite, $B(G)$ is an algebra under pointwise multiplication. Given $\varphi \in B(G)$, the map

$$f \mapsto \langle f, \varphi\rangle = \int_G f(x)\varphi(x)\, dx$$

is a linear functional on $L^1(G)$ which is bounded, when $L^1(G)$ is equipped with the universal C^*-norm. Hence φ defines a functional on $C^*(G)$, the full group C^*-algebra of G, and this gives the identification of $B(G)$ with $C^*(G)^*$ as vector spaces. The Fourier-Stieltjes algebra inherits the norm

$$\|\varphi\| = \sup\{|\langle f, \varphi\rangle| \mid f \in L^1(G),\ \|f\|_{C^*(G)} \leq 1\}$$

of $C^*(G)^*$ from this identification. With this norm $B(G)$ is a unital Banach algebra.

Given $\varphi \in B(G)$, a representation $(\pi, \mathcal{H})$ and $x, y \in \mathcal{H}$ such that $\varphi(g) = \langle \pi(g)x, y\rangle$ we have

$$\|\varphi\| \leq \|x\|\|y\|,$$

and conversely, it is always possible to find $(\pi, \mathcal{H})$ and $x, y \in \mathcal{H}$ such that $\varphi(g) = \langle \pi(g)x, y\rangle$ and $\|\varphi\| = \|x\|\|y\|$.

The Fourier algebra of G is denoted $A(G)$ and is the closure of the set of compactly supported functions in $B(G)$, and $A(G)$ is in fact an ideal. The Fourier algebra coincides with the set of all matrix coefficients of the left regular representation of G,

$$A(G) = \{\langle \lambda(\cdot)x, y\rangle \mid x, y \in L^2(G)\},$$

and given any $\varphi \in A(G)$, there are $x, y \in L^2(G)$ such that $\varphi(g) = \langle \lambda(g)x, y\rangle$ and $\|\varphi\| = \|x\|\|y\|$. This can be rephrased as follows. Given $\varphi \in A(G)$, there are $f, h \in L^2(G)$ such that $\varphi = f * \check{h}$ and $\|\varphi\| = \|f\|\|h\|$, where $\check{h}(g) = h(g^{-1})$. This is often written as

$$A(G) = L^2(G) * L^2(G).$$

It is known that $\|\varphi\|_\infty \leq \|\varphi\|$ for any $\varphi \in B(G)$, and hence $A(G) \subseteq C_0(G)$.

Although we will not study von Neumann algebras in this paper, we note that $A(G)$ may be identified with the predual of the group von Neumann algebra $L(G)$

of G. When G is abelian, the Fourier transform provides an isometric isomorphism between $L^1(\widehat{G})$ and $A(G)$, and in this way $A(G)$ is identified isometrically with the predual of group von Neumann algebra $L(G) \simeq L^\infty(\widehat{G})$. In the non-abelian case it is still true that $A(G)$ identifies isometrically with the predual of the group von Neumann algebra via the duality

$$\langle T, \varphi \rangle = \langle Tf, h \rangle,$$

where $T \in L(G)$ and $\varphi = \bar{h} * \check{f}$ for some $f, h \in L^2(G)$.

3. Proof of Theorem 4

In this section we prove Theorem 4, which is the basis of proving Theorems 2 and 3. We first prove that the conditions in Theorem 4 ensure that the regular representation is completely reducible.

Lemma 6. *Let G be a locally compact group. Any unitary representation of G on a separable Hilbert space has at most countably many inequivalent (with respect to unitary equivalence) irreducible subrepresentations.*

Proof. Let π be a unitary representation of G. The subrepresentations of π are in correspondence with the projections in the commutant $\pi(G)'$, equivalent subrepresentations correspond to projections that are equivalent in $\pi(G)'$ (in the sense of Murray-von Neumann), and the irreducible subrepresentations correspond to minimal projections in $\pi(G)'$. It is therefore enough to show that a von Neumann algebra on a separable Hilbert space has at most countably many inequivalent minimal projections. Let M be such a von Neumann algebra.

Recall that two minimal projections are inequivalent if and only if their central supports are orthogonal (see [16, Proposition 6.1.8]). Let $(p_i)_{i \in I}$ be a family of inequivalent minimal projections, and let c_i be the central support of p_i. Then $(c_i)_{i \in I}$ is a family of orthogonal projections. By separability of the Hilbert space, I must be countable. Hence there are at most countably many inequivalent minimal projections in M. □

Corollary 7. *Let G be a locally compact, second countable group. Then the left regular representation of G has at most countably many inequivalent irreducible subrepresentations.*

Proof. The left regular representation represents G on the Hilbert space $L^2(G)$, which is separable, since G is second countable. The statement now follows. □

We recall that a unitary representation is of type I, if the image of the representation generates a type I von Neumann algebra. A locally compact group is said to be of type I, if all its unitary representations are of type I (see [10, Chapter 13]). Disintegration theory works especially well in the setting of type I groups. We refer to [13, Chapter 7] for more on type I groups and disintegration theory. Several equivalent characterizations of type I groups can also be found in [10, Chapter 9], but let us just mention one characterization here. The unitary equivalence classes of irreducible representations form a set $\widehat{G}$ called the unitary dual of G. The dual

$\widehat{G}$ is equipped with the Mackey Borel structure, and G is of type I if and only if $\widehat{G}$ is a standard Borel space. When G is abelian, the unitary dual coincides with the usual dual group.

Proposition 8. *Let G be a second countable, locally compact group satisfying the following two conditions.*

1. *G is type I.*
2. *There is a non-compact, closed subgroup H of G such that every irreducible unitary representation of G is either trivial on H or is a subrepresentation of the left regular representation λ_G.*

Then the left regular representation λ_G is completely reducible.

Proof. For each $p \in \widehat{G}$, we let π_p denote a representative of the class p, and we assume that the choice of representative is made in a measurable way ([13, Lemma 7.39]). We write the left regular representation as a direct integral of irreducibles,

$$\lambda_G = \int_{\widehat{G}}^{\oplus} n_p \pi_p \, d\mu(p),$$

where μ is a Borel measure on $\widehat{G}$ and $n_p \in \{0, 1, 2, \ldots, \infty\}$ (see [13, Theorem 7.40]). Let $A = \{p \in \widehat{G} \mid \pi_p(h) = 1 \text{ for all } h \in H\}$ and let $B = \widehat{G} \setminus A$. It is not hard to check that $A \subseteq \widehat{G}$ is a Borel set.

We note that if $\pi_p \in B$, then by assumption π_p is a subrepresentation of λ_G. By the previous corollary, B is countable. Since λ_G has no subrepresentation which is trivial on a non-compact subgroup, we must have $\mu(A) = 0$. Then

$$\lambda_G = \int_{B}^{\oplus} n_p \pi_p \, d\mu(p),$$

and since B is countable, λ_G is a direct sum of irreducibles. □

When π is a representation and α is a cardinal number we denote by π^α the direct sum of α copies of π.

Lemma 9. *Let G be a locally compact, second countable group with left regular representation λ and a closed subgroup H such that*

1. *G is type I;*
2. *Every irreducible unitary representation of G is either trivial on H or is a subrepresentation of λ;*
3. *λ is completely reducible.*

Then every unitary representation π of G is a sum $\sigma_1 \oplus \sigma_2$, where σ_1 is trivial on H and σ_2 is a subrepresentation of a multiple of λ. By a multiple of λ we mean a direct sum of copies of λ.

Proof. As in the previous proof, the basic idea is to use disintegration theory. However, this idea only applies if π is a representation on a separable Hilbert space. There is a standard way of getting around the issue of separability.

By a standard application of Zorn's lemma we may write π is a direct sum $\bigoplus_i \pi_i$ of cyclic representations π_i. Since G is second countable, each π_i represents G on a separable Hilbert space.

For each $p \in \widehat{G}$, we let π_p denote a representative of the class p, and we assume that the choice of representative is made in a measurable way ([13, Lemma 7.39]).

For the moment we fix an i. We may write π_i as a direct integral of irreducibles,

$$\pi_i = \int_{\widehat{G}}^{\oplus} n_p \pi_p \, d\mu(p),$$

where μ is a Borel measure on $\widehat{G}$ and $n_p \in \{0, 1, 2, \ldots, \infty\}$ (see [13, Theorem 7.40]). Let $A = \{p \in \widehat{G} \mid \pi_p(h) = 1 \text{ for all } h \in H\}$ and let $B = \widehat{G} \setminus A$. Then $A \subseteq \widehat{G}$ is a Borel set.

By assumption, there is a decomposition

$$\lambda = \bigoplus_{p \in C} m_p \pi_p$$

for some countable $C \subseteq \widehat{G}$ and suitable multiplicities $m_p \in \{1, 2, \ldots, \infty\}$. Also, it follows from our assumptions that $B \subseteq C$. If

$$\sigma_1^i = \int_A^{\oplus} n_p \pi_p \, d\mu(p), \qquad \sigma_2^i = \int_B^{\oplus} n_p \pi_p \, d\mu(p),$$

then we see that

$$\pi_i = \sigma_1^i \oplus \sigma_2^i,$$

where σ_1^i is trivial on H. As B is countable, the integral defining σ_2^i is actually a direct sum, so that σ_2^i is a subrepresentation of

$$\bigoplus_{p \in B} n_p \pi_p$$

which in turn is a subrepresentation of $\lambda \oplus \lambda \oplus \cdots$. Hence σ_2^i is a subrepresentation of a multiple of λ. Finally, let

$$\sigma_1 = \bigoplus_i \sigma_1^i \quad \text{and} \quad \sigma_2 = \bigoplus_i \sigma_2^i.$$

Then $\pi = \sigma_1 \oplus \sigma_2$, where σ_1 is trivial on H and σ_2 is a subrepresentation of a multiple of λ. □

Lemma 10. *Let G be a locally compact group with left regular representation λ and a closed, non-compact subgroup H. Suppose every unitary representation π of G is a sum $\sigma_1 \oplus \sigma_2$, where σ_1 is trivial on H and σ_2 is a subrepresentation of a multiple of λ. Then $A(G) = B(G) \cap C_0(G)$.*

Proof. The inclusion $A(G) \subseteq B(G) \cap C_0(G)$ holds for any locally compact group G. Suppose $\varphi \in B(G) \cap C_0(G)$. Then there is a continuous, unitary representation π of G on some Hilbert space $\mathcal{H}$ and vectors $x, y \in \mathcal{H}$ such that

$$\varphi(g) = \langle \pi(g)x, y \rangle \qquad \text{for all } g \in G.$$

By assumption we may split $\pi = \sigma_1 \oplus \sigma_2$. Accordingly, we split $\varphi = \varphi_1 + \varphi_2$, where φ_1 is a coefficient of σ_1 etc. We will show that $\varphi_1 = 0$ and $\varphi_2 \in A(G)$, which will complete the proof.

Since σ_2 is a subrepresentation of a multiple of λ, we see that φ_2 is of the form

$$\varphi_2(g) = \sum_{i=1}^{\infty} \langle \lambda(g) x_i, y_i \rangle$$

for some $x_i, y_i \in L^2(G)$ with $\sum_i \|x_i\|^2 < \infty$ and $\sum_i \|y_i\|^2 < \infty$. Each of the maps

$$g \mapsto \langle \lambda(g) x_i, y_i \rangle$$

is in $A(G)$ with norm at most $\|x_i\|\|y_i\|$. Since $A(G)$ is a Banach space and $\sum_i \|x_i\|\|y_i\| < \infty$, we deduce that $\varphi_2 \in A(G)$, and in particular $\varphi_2 \in C_0(G)$. It then follows that $\varphi_1 \in C_0(G)$. Since σ_1 is trivial on H, we see that φ_1 is constant on H cosets. Since H is non-compact, we deduce that $\varphi_1 = 0$. Then $\varphi = \varphi_2 \in A(G)$. This proves $B(G) \cap C_0(G) = A(G)$. □

Theorem 4 is an easy consequence of the previous statements.

Proof of Theorem 4. We assume that G is a locally compact, second countable group satisfying the two conditions in the statement of the theorem. It follows from Proposition 8 that λ_G is completely reducible. So by Lemma 9, every unitary representation π of G is a sum $\sigma_1 \oplus \sigma_2$, where σ_1 is trivial on H and $\sigma_2 \leq \lambda^\infty$. From Lemma 10 we conclude that $A(G) = B(G) \cap C_0(G)$. □

4. Invariant measures on homogeneous spaces

To describe the irreducible representations of the groups P in question, we rely on a general method known to the common man as the Mackey Machine. Essential in the Mackey Machine is the notion of induced representations. For a general introduction to the theory of induced representations we refer to [13, Chapter 6] which also contains a description of (a simple version of) the Mackey Machine. The general results about the Mackey Machine can be found in the original paper [22]. See also the book [17].

The construction of an induced representation from a closed subgroup H to a group G is more easily described when the homogeneous space G/H admits an invariant measure for the G-action given by left translation. Regarding homogeneous spaces and invariant measures we record the following easy (and well-known) facts.

Proposition 11. *Consider topological groups G, N, H, K, A, B and topological spaces X and Y.*

(1) *Suppose G is the semi-direct product $G = N \rtimes H$, where N is normal in G. If $K \leq H$ is a closed subgroup of H, then there is a canonical isomorphism*

$$NH/NK \simeq H/K$$

as G-spaces. Here the G-action on H/K is the H-action, and N acts trivially on H/K.

(2) *Suppose $G = N \times H$, and $A \leq N$, $B \leq H$ are closed subgroups. Then there is a canonical isomorphism*

$$(N \times H)/(A \times B) \simeq N/A \times H/B$$

as G-spaces, where the G-action on $N/A \times H/B$ is the product action of $N \times H$.

(3) *Suppose $G \curvearrowright X$ and $H \curvearrowright Y$ have invariant, σ-finite Borel measures. Then the product $G \times H \curvearrowright X \times Y$ has an invariant, σ-finite Borel measure.*

(4) *Suppose G is compact (or just amenable) and X is compact. Then any action $G \curvearrowright X$ has an invariant probability measure.*

Proof.

(1) The map $[nh]_{NK} \mapsto [h]_K$ is a well-defined, equivariant homeomorphism.
(2) The map $[(n,h)]_{A \times B} \mapsto ([n]_A, [h]_B)$ is a well-defined, equivariant homeomorphism.
(3) Take the product measure on $X \times Y$ of the invariant measures on X and Y.
(4) This is Proposition 5.4 in [28].

□

The following lemma will be relevant in Section 5 and Section 8 when we verify condition (2) of Theorem 4 for the minimal parabolic groups P.

Lemma 12. *Let G be a locally compact group with closed subgroups $N \subseteq H \subseteq G$, and suppose $N \triangleleft G$. If σ is a unitary representation of H which is trivial on N, and if G/H admits a G-invariant measure, then the induced representation $\operatorname{Ind}_H^G \sigma$ is also trivial on N.*

Proof. Let $\mathcal{H}$ denote the Hilbert space of σ, and let $q : G \to G/H$ be the quotient map. The induced representation $\pi = \operatorname{Ind}_H^G \sigma$ acts on a completion of the space

$$\mathcal{F}_0 = \left\{ f \in C(G, \mathcal{H}) \middle| \begin{array}{l} f(gh) = \sigma(h^{-1})f(g) \text{ for } g \in G, h \in H \\ \text{and } q(\operatorname{supp} f) \text{ is compact} \end{array} \right\}.$$

Since G/H admits an invariant measure, the action on $\mathcal{F}_0$ is simply given by left translation, $(\pi(x)f)(g) = f(x^{-1}g)$. With $f \in \mathcal{F}_0$, $g \in G$ and $n \in N$, we compute $(\pi(n)f)(g)$. We get

$$(\pi(n)f)(g) = f(n^{-1}g) = f(g(g^{-1}n^{-1}g)) = \sigma(g^{-1}ng)f(g) = f(g),$$

since $g^{-1}ng \in N$. It follows that $\pi(n) = 1$. □

5. The first example

In this section we prove Theorem 2. Let P be the group defined in (1.2). In the following proposition we describe the unitary dual of P, i.e. the equivalence classes of the irreducible representations of P. To do so we apply the Mackey Machine, which works particularly well in our case, where P decomposes as a semidirect product $N_0 \rtimes P_0$ with N_0 abelian. For an account on the Mackey Machine we refer to [13, Chapter 6] and [17].

Consider the following closed subgroups of P.

$$P_0 = \left\{ \begin{pmatrix} \lambda & a & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \; a \in \mathbb{R},\ \lambda > 0 \right\} \tag{5.1}$$

$$P_1 = \left\{ \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \; a \in \mathbb{R} \right\} \tag{5.2}$$

$$N_0 = \left\{ \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \middle| \; b, c \in \mathbb{R} \right\} \tag{5.3}$$

$$N_1 = \left\{ \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \; c \in \mathbb{R} \right\} \tag{5.4}$$

Observe that $P = N_0 \rtimes P_0$. We note that P_0 is isomorphic to the $ax+b$ group, i.e. the group of affine transformations $x \mapsto ax + b$ of the real line, where $a > 0$ and $b \in \mathbb{R}$. The dual of the $ax+b$ group is well-known (see for instance [13, Section 6.7]). The dual of $N_0 \simeq \mathbb{R}^2$ is $\widehat{N}_0 \simeq \widehat{\mathbb{R}}^2$ which we as usual identify with $\mathbb{R}^2$.

Proposition 13. *Let π be an irreducible representation of P. Then π is equivalent to one of the following representations (and these are all inequivalent).*

(1) $\pi_1 = \mathrm{Ind}_{N_0}^P(\nu)$, *where $\nu \in \widehat{N}_0$ is $\nu = (1, 0)$.*
(2) $\pi_2 = \mathrm{Ind}_{N_0}^P(\nu)$, *where $\nu \in \widehat{N}_0$ is $\nu = (-1, 0)$.*
(3) $\pi_{3,\rho} = \mathrm{Ind}_{N_0 P_1}^P(\nu\rho)$, *where $\nu \in \widehat{N}_0$ is $\nu = (0, 1)$ and ρ is a character in $\widehat{P_1} \simeq \mathbb{R}$.*
(4) $\pi_{4,\rho} = \mathrm{Ind}_{N_0 P_1}^P(\nu\rho)$, *where $\nu \in \widehat{N}_0$ is $\nu = (0, -1)$ and ρ is a character in $\widehat{P_1} \simeq \mathbb{R}$.*
(5) $\pi_{5,\sigma} = \sigma \circ q$, *where $\sigma \in \widehat{P}_0$ and $q : P \to P_0$ is the quotient map.*

Proof. We follow the strategy of the Mackey Machine as described in Theorem 6.42 in [13], which gives a complete description of the unitary dual of P. We think of $P = N_0 \rtimes P_0$ as a subgroup of $\mathbb{R}^2 \rtimes \mathrm{SL}_2(\mathbb{R})$, where $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{R}^2$ by matrix multiplication. The action $P_0 \curvearrowright N_0$ is then simply matrix multiplication, and the dual action $P_0 \curvearrowright \widehat{N}_0$ is given by $(p.\nu)(n) = \nu(p^{-1}.n)$ for $p \in P_0$, $\nu \in \widehat{N}_0$ and $n \in N_0$. Under the usual identification $\widehat{N}_0 \simeq \mathbb{R}^2$ we see that $p \in P_0$ acts on $\mathbb{R}^2$ by matrix multiplication by the transpose of the inverse of p. Thus, if p has the form in (5.1), then the action of p on $\mathbb{R}^2$ is

$$\begin{pmatrix} s \\ t \end{pmatrix} \mapsto \begin{pmatrix} \lambda^{-1} & 0 \\ -a & \lambda \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix}.$$

There are five orbits in $\widehat{N}_0$ under this action, which give the five alternatives in the proposition. The orbits are

$$\begin{aligned}\mathcal{O}_1 &= \{(s,t) \mid s > 0\},\\ \mathcal{O}_2 &= \{(s,t) \mid s < 0\},\\ \mathcal{O}_3 &= \{(0,t) \mid t > 0\},\\ \mathcal{O}_4 &= \{(0,t) \mid t < 0\},\\ \mathcal{O}_5 &= \{(0,0)\}.\end{aligned}$$

Since there are only finitely many orbits, the action of P_0 on $\widehat{N}_0$ is regular. As representatives of the orbits we choose the points

$$(1,0) \in \mathcal{O}_1,\ (-1,0) \in \mathcal{O}_2,\ (0,1) \in \mathcal{O}_3,\ (0,-1) \in \mathcal{O}_4,\ (0,0) \in \mathcal{O}_5.$$

Case 1: $\nu = (1,0)$. In this case the stabilizer subgroup of ν inside P_0 is trivial, and hence we obtain the representation $\pi = \mathrm{Ind}_{N_0}^{P}(\nu)$.

Case 2: $\nu = (-1,0)$. This is similar to case 1.

Case 3: $\nu = (0,1)$. The stabilizer subgroup of ν inside P_0 is P_1, and hence we obtain $\pi = \mathrm{Ind}_{N_0P_1}^{P}(\nu\rho)$, where $\rho \in \widehat{P_1}$. Here the representation $\nu\rho$ on N_0P_1 is given by

$$(\nu\rho)(nh) = \nu(n)\rho(h), \qquad \text{for all } n \in N_0,\ h \in P_1.$$

Case 4: $\nu = (0,-1)$. This is similar to case 3.

Case 5: $\nu = (0,0)$. In this case the stabilizer subgroup of ν inside P_0 is everything. It follows that π is a representation which satisfies $\pi(n) = \langle n,\nu\rangle$ for every $n \in N_0$. In other words, π is trivial on N_0 and factors to an irreducible representation σ of P_0. That is, $\pi = \sigma \circ q$. □

The Plancherel measure of a group describes how the left regular representation decomposes as a direct integral of irreducible representations. For example, the Plancherel measure of a locally compact abelian group is simply the Haar measure on the dual group. This is seen using the Fourier transform. The following proposition determines the Plancherel measure of P and shows in particular that the measure is purely atomic. Hence the left regular representation of P is completely reducible.

Proposition 14. *The left regular representation λ_P of P is (equivalent to) the countably infinite direct sum of $\pi_1 \oplus \pi_2$, where π_1 and π_2 are as in Proposition 13.*

Proof. Again it is useful to view P as the semidirect product $P = N_0 \rtimes P_0$. We follow the approach described in [4, Section 1]. Their results are stated for the right regular representation, but everything works mutatis mutandis for the left regular. As before, the dual group $\widehat{N}_0$ is identified with $\mathbb{R}^2$, and the Plancherel measure on $\widehat{N}_0$ is simply Lebesgue measure. The orbits under the dual action $P_0 \curvearrowright \widehat{N}_0$ which have positive Lebesgue measure are $\mathcal{O}_1$ and $\mathcal{O}_2$, and their complement in $\widehat{N}_0$ is a null set (the y-axis). The stabilizer subgroups inside P_0 of the points $(1,0)$

and $(-1,0)$ are trivial, so in particular these stabilizer subgroups have completely reducible regular representations. Thus criteria (a) and (b) of [4] are satisfied, and it follows from their calculation on page 595 that

$$\lambda_P = \bigoplus_{n=1}^{\infty} (\pi_1 \oplus \pi_2).$$

□

Lemma 15. *Consider the groups $P = N_0 \rtimes P_0$ and $N_1 \subseteq P$. If π is an irreducible unitary representation of P, then one (and only one) of the following holds.*

(1) $\pi(g) = 1$ *for every g in the subgroup N_1,*
(2) π *is a subrepresentation of λ_P.*

Proof. We divide the proof into the cases according to the description in Proposition 13.

If $\pi = \pi_1$ or $\pi = \pi_2$, then it follows from Proposition 14 that π is a subrepresentation of λ_P.

Suppose now $\pi = \pi_{3,\rho}$, where $\rho \in \widehat{P}_1$. If we let $\nu = (0,1) \in \widehat{N}_0$, then we see that $N_1 = \ker \nu$. Hence the representation $\nu\rho$ of $N_0 P_1$ is trivial on N_1 which is normal in P. Since $N_0 P_1$ is a normal subgroup of P, the homogeneous space $P/(N_0 P_1)$ has a P-invariant measure, Haar measure. It follows from Lemma 12 that $\pi = \mathrm{Ind}_{N_0 P_1}^{P}(\nu\rho)$ is trivial on N_1.

The case $\pi = \pi_{4,\rho}$ is similar to the previous case. We simply note that $\ker \bar{\nu} = N_1$, where $\bar{\nu} = (0, -1)$.

In the case $\pi = \pi_{5,\sigma}$, it is clear that $\pi(g) = 1$ for every $g \in N_0$, and hence in particular for every $g \in N_1$. □

Lemma 16. *The group P is of type I.*

Proof. The group P is a connected, real algebraic group, and such groups are of type I according to [8, Theorem 1].

□

We collect the previous results of this section in the following proposition, which together with Theorem 4 immediately implies Theorem 2.

Proposition 17. *Let P and N_1 be the groups in* (1.2) *and* (5.4). *The following holds.*

(1) *P is of type I.*
(2) *Every irreducible unitary representation of P is either trivial on the non-compact closed subgroup N_1 or is a subrepresentation of λ_P.*

6. Simple Lie groups of real rank one

Let G be a connected simple Lie group with finite center and of real rank one. Let $G = KAN$ be an Iwasawa decomposition of G. Then K is a maximal compact subgroup, A is abelian of dimension 1, and N is nilpotent. Let M be the centralizer of A in K, and let $P = MAN$ be the minimal parabolic subgroup of G.

It is known that G is locally isomorphic to one of the classical groups $\mathrm{SO}_0(n,1)$, $\mathrm{SU}(n,1)$, $\mathrm{Sp}(n,1)$ or the exceptional group $\mathrm{F}_{4(-20)}$ (see for instance the list on p. 426 in [21]), and we now describe these groups in more detail, including explicit descriptions of the Iwasawa subgroups and the minimal parabolic subgroup P.

6.1. **The classical cases.** Let $\mathbb{F}$ be one of the three finite-dimensional division algebras over the reals, the real field $\mathbb{R}$, the complex field $\mathbb{C}$ or the quaternion division ring $\mathbb{H}$. In the exceptional case treated later, $\mathbb{F}$ will be the non-associative real algebra $\mathbb{O}$ of octonions, also known as the Cayley algebra. We let $\operatorname{Re}\mathbb{F}$ and $\operatorname{Im}\mathbb{F}$ denote the real and imaginary part of $\mathbb{F}$, so that $\mathbb{F} = \operatorname{Re}\mathbb{F} + \operatorname{Im}\mathbb{F}$, and in the standard notation

$$\operatorname{Im}\mathbb{R} = 0, \quad \operatorname{Im}\mathbb{C} = \mathbb{R}i, \quad \operatorname{Im}\mathbb{H} = \mathbb{R}i + \mathbb{R}j + \mathbb{R}k.$$

We use the notation $\mathbb{F}'$ to denote the unit sphere in $\mathbb{F}$,

$$\mathbb{F}' = \{x \in \mathbb{F} \mid \|x\| = 1\},$$

and $\mathbb{F}^* = \mathbb{F} \setminus \{0\}$.

Let $\mathbb{F}^{p,q}$ denote the real vector space $\mathbb{F}^{p+q}$ equipped with the hermitian form

$$\langle x, y\rangle = \sum_{i=1}^{p} x_i \bar{y}_i - \sum_{i=p+1}^{p+q} x_i \bar{y}_i.$$

We also think of $\mathbb{F}^{p,q}$ as a right $\mathbb{F}$-module. Of course, $\mathbb{F}^n = \mathbb{F}^{n,0}$. We write w^t for the row vector which is the transpose of a column vector $w \in \mathbb{F}^n$. Also, $w^* = \bar{w}^t$ and $|w|^2 = w^* w = \langle w, w\rangle$, when $w \in \mathbb{F}^n$.

Let $\mathrm{U}(p,q,\mathbb{F})$ denote the unitary group of $\mathbb{F}^{p,q}$, i.e. the square matrices over $\mathbb{F}$ of size $p+q$ that preserve the hermitian form. If $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$, we will be concerned with the unitaries of determinant 1. We write $\mathrm{SU}(p,q,\mathbb{F})$ for this group. It is customary to write

$$\mathrm{U}(p,q,\mathbb{R}) = \mathrm{O}(p,q), \quad \mathrm{SU}(p,q,\mathbb{R}) = \mathrm{SO}(p,q),$$
$$\mathrm{SU}(p,q,\mathbb{C}) = \mathrm{SU}(p,q), \quad \mathrm{U}(p,q,\mathbb{H}) = \mathrm{Sp}(p,q),$$

The groups $\mathrm{SU}(p,q)$ and $\mathrm{Sp}(p,q)$ are connected, and $\mathrm{Sp}(p,q)$ is even simply connected (see [21, Section I.17]).

We let $\mathrm{U}_0(p,q,\mathbb{F})$ denote the connected component of $\mathrm{U}(p,q,\mathbb{F})$. Note that

$$\mathrm{U}_0(n,\mathbb{R}) = \mathrm{SO}(n), \quad \mathrm{U}_0(n,\mathbb{C}) = \mathrm{U}(n), \quad \mathrm{U}_0(n,\mathbb{H}) = \mathrm{Sp}(n),$$

and in particular $\mathrm{U}_0(1,\mathbb{R}) = \{1\}$. We remark that $\mathrm{U}_0(n,\mathbb{F})$ acts transitively on the unit sphere in $\mathbb{F}^n$ except for the case $n = 1$ and $\mathbb{F} = \mathbb{R}$.

The following is taken from [24]. Let G be one of $\mathrm{SO}_0(n,1)$, $\mathrm{SU}(n,1)$, $\mathrm{Sp}(n,1)$. Then the subgroups related to the Iwasawa decomposition of G are the following.

$$K = \begin{pmatrix} k & 0 \\ 0 & \beta \end{pmatrix}, \qquad \begin{array}{l} k \in \mathrm{U}_0(n,\mathbb{F}), \quad \beta \in \mathrm{U}_0(1,\mathbb{F}), \\ \beta \det k = 1 \text{ if } \mathbb{F} \neq \mathbb{H} \end{array} \tag{6.1}$$

$$M = \begin{pmatrix} \beta & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & \beta \end{pmatrix}, \qquad \begin{array}{l} u \in \mathrm{U}_0(n-1,\mathbb{F}), \quad \beta \in \mathrm{U}_0(1,\mathbb{F}), \\ \beta^2 \det u = 1 \text{ if } \mathbb{F} \neq \mathbb{H} \end{array} \tag{6.2}$$

$$A = \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & I & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix}, \quad t \in \mathbb{R}$$

$$N = \begin{pmatrix} 1 + z - \frac{1}{2}|w|^2 & w^* & -z + \frac{1}{2}|w|^2 \\ -w & I & w \\ z - \frac{1}{2}|w|^2 & w^* & 1 - z + \frac{1}{2}|w|^2 \end{pmatrix}, \quad w \in \mathbb{F}^{n-1}, \quad z \in \operatorname{Im} \mathbb{F} \tag{6.3}$$

The subgroups M and A of P commute. The group N is normal in P, and P is the semi-direct product of MA and N. To describe the action of M and A on N, it will be easier to work with a group isomorphic to P (but no longer a subgroup of G) obtained by conjugating P by the orthogonal matrix

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & I & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Then A and N become, with $\alpha = e^t$,

$$A = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \alpha^{-1} \end{pmatrix}, \quad \alpha > 0 \tag{6.4}$$

$$N = \begin{pmatrix} 1 & w^t & z + \frac{1}{2}|w|^2 \\ 0 & I & \bar{w} \\ 0 & 0 & 1 \end{pmatrix}, \quad w \in \mathbb{F}^{n-1}, \quad z \in \operatorname{Im} \mathbb{F} \tag{6.5}$$

while M remains the same. We have chosen to rescale the parameter z in (6.5) by a factor of two compared with (6.3) and replace w by its conjugate $\bar{w}$, so that the group law in N matches the one from [33]. We think of the group N as $\mathbb{F}^{n-1} \times \operatorname{Im} \mathbb{F}$ with group structure

$$(w_1, z_1)(w_2, z_2) = (w_1 + w_2, z_1 + z_2 + \operatorname{Im}\langle w_1, w_2 \rangle)$$

and write (w, z) for the matrix in (6.5). The action $MA \curvearrowright N$ is given by

$$\alpha.(w, z) = (\alpha w, \alpha^2 z) \tag{6.6}$$

and

$$\operatorname{diag}(\beta, u, \beta).(w, z) = (uw\beta^{-1}, \beta z\beta^{-1}). \tag{6.7}$$

Note that the three subsets $\{0\} \times \operatorname{Im} \mathbb{F}$, $\{0\} \times \operatorname{Im} \mathbb{F}^*$ and $\mathbb{F}^{n-1} \times \{0\}$ are invariant under the action MA. If $\mathbb{F} = \mathbb{R}$, then N is abelian, and otherwise the center of N is

$$Z(N) = \{(0, z) \mid z \in \operatorname{Im} \mathbb{F}\}.$$

6.2. **The exceptional case.** The exceptional group $\mathrm{F}_{4(-20)}$ has a realization as automorphisms of a Jordan algebra. A detailed treatment of the group $\mathrm{F}_{4(-20)}$ can be found in [29] including a description of the Iwasawa decomposition $\mathrm{F}_{4(-20)} = KAN$ (see [29, §5 Theoreme 1]). Here we only describe the components M, A and N of the minimal parabolic subgroup $P = MAN$ and not the group $\mathrm{F}_{4(-20)}$ itself. The group P is best described using the octonion non-associative division algebra $\mathbb{O}$. For a detailed description of the octonions we refer to [29, §1]. Another reference is [1, 2].

We recall that $\mathbb{O}$ is an 8-dimensional real vector space, and thus we usually identify $\mathbb{O}$ with $\mathbb{R}^8$. We use the notation $\bar{y}$ for the conjugate of $y \in \mathbb{O}$, and we let $\langle x, y\rangle = x\bar{y}$. The real bilinear form $(x|y) = \operatorname{Re}\langle x, y\rangle$ corresponds to the usual inner product on $\mathbb{R}^8$. The imaginary octonions $\operatorname{Im} \mathbb{O}$ form a subspace identified with $\mathbb{R}^7$.

The group N is $\mathbb{O} \times \operatorname{Im} \mathbb{O}$ with group product

$$(w_1, z_1)(w_2, z_2) = (w_1 + w_2, z_1 + z_2 + \operatorname{Im}\langle w_1, w_2\rangle).$$

The center of N is $Z(N) = \{(0, z) \mid z \in \operatorname{Im} \mathbb{O}\}$, and the quotient $N/Z(N)$ is then isomorphic to $(\mathbb{O}, +)$. The group N is connected and nilpotent.

The group A is $\mathbb{R}_+$, and the action $A \curvearrowright N$ is given by

$$\alpha.(w, z) = (\alpha w, \alpha^2 z), \quad \alpha \in \mathbb{R}_+.$$

The group M is the spin group $\mathrm{Spin}(7)$, which is the (2-sheeted) universal cover of $\mathrm{SO}(7)$. In order to describe the action $M \curvearrowright N$, we need to consider two orthogonal representations of $\mathrm{Spin}(7)$, the spin representation $\sigma : M \to \mathrm{SO}(8)$ and the vector representation $\nu : M \to \mathrm{SO}(7)$. Then the action $M \curvearrowright N$ is then given as

$$u.(w, z) = (\sigma(u)w, \nu(u)z), \quad u \in \mathrm{Spin}(7).$$

If we identify $\operatorname{Im} \mathbb{O}$ with $\mathbb{R}^7$ in the usual way, then $\mathrm{SO}(7)$ acts on $\operatorname{Im} \mathbb{O}$ by matrix multiplication. The vector representation ν is simply the covering homomorphism $\nu : \mathrm{Spin}(7) \to \mathrm{SO}(7)$. Under the identification of $\operatorname{Im} \mathbb{O}$ with $\mathbb{R}^7$, the purely imaginary unit octonions are identified with the unit sphere S^6. Since $\mathrm{SO}(7)$ acts transitively on S^6, it follows that MA acts transitively on $\operatorname{Im} \mathbb{O}^*$.

The spin representation $\sigma : \mathrm{Spin}(7) \to \mathrm{SO}(8)$ gives a transitive action of $\mathrm{Spin}(7)$ on S^7 (see [29, §4 Lemme 1]).

The actions of M and A on N commute and thus give an action $M \times A \curvearrowright N$. The group P is the semidirect product $P = MA \ltimes N$.

Note that the three subsets $\{0\}\times\mathrm{Im}\,\mathbb{O}$, $\{0\}\times\mathrm{Im}\,\mathbb{O}^*$ and $\mathbb{O}\times\{0\}$ of N are invariant under the action MA.

7. The irreducible representations of parabolic subgroups

In this section we describe the unitary dual of the minimal parabolic subgroups P from the previous section. The result is contained in Theorem 18 and Theorem 20. We also prove that P and N are of type I.

7.1. **The classical cases.** Let G be one of the classical groups $\mathrm{SO}_0(n,1)$, $\mathrm{SU}(n,1)$, $\mathrm{Sp}(n,1)$, and let $P = MAN$ be the minimal parabolic subgroup of G. To describe the irreducible representations of P we rely on the work of [33], in which groups very similar to our P are considered as well as many other groups. In fact, they consider the group $\widetilde{M}AN$, where $\widetilde{M}$ is

$$\widetilde{M} = \begin{pmatrix} \beta & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & \beta \end{pmatrix}, \quad u \in \mathrm{U}(n-1,\mathbb{F}),\ \beta \in \mathrm{U}(1,\mathbb{F})$$

Their conclusion about the irreducible representations is contained in [33, Proposition 7.8]. Actually, if $\mathbb{F} = \mathbb{H}$, which is the case we are most interested in because of future applications [14], then $\widetilde{M} = M$, and Theorem 18 is a special case of [33, Proposition 7.8].

The discussion below is based on Section 4 and 7 from [33] to which we refer for proofs and more details. The arguments carry over without any challenges to our situation. The representations of P fall into three series.

(1) The subgroup N is normal in P, and $P/N \simeq MA$. We let $q : P \to P/N$ denote the quotient map. Of course, any irreducible representation σ of P/N gives rise to the irreducible representation $\sigma \circ q$ of P, and these are precisely the irreducibles of P that annihilate N.

(2) Next we describe the irreducibles of P arising from characters on N. Let $v \in \mathbb{F}^{n-1}$ be non-zero, and define the character χ_v on N by

$$\chi_v(w,z) = \exp(i\mathrm{Re}\langle w,v\rangle).$$

The group MA acts on N by conjugation, and this induces a dual action of MA on $\widehat{N}$. Let L_v be the stabilizer of χ_v in MA under this action. Then χ_v extends to a character of $N \rtimes L_v$ by the formula

$$\chi_v(w,z,g) = \chi_v(w,z) = \exp(i\mathrm{Re}\langle w,v\rangle), \qquad (w,z,g) \in \mathbb{F}^{n-1}\times \mathrm{Im}\,\mathbb{F}\times L_v.$$

Let γ be an irreducible representation of L_v. Extend γ to be the irreducible representation of $N \rtimes L_v$ defined by letting γ be trivial on N. Form the tensor product representation $\chi_v \otimes \gamma$ and induce this representation from $N \rtimes L_v$ to P to get a representation $\pi_{2,v,\gamma}$ of P,

$$\pi_{2,v,\gamma} = \mathrm{Ind}_{NL_v}^{P}(\chi_v \otimes \gamma).$$

Before we move on to the last series in $\widehat{P}$, we describe the action of MA on the non-trivial characters on N in more detail. The action is given by (6.6) and (6.7),

$$(u,\beta,\alpha).\chi_v = \chi_{v'} \qquad \text{where } v' = u^{-1}\alpha^{-1}v\beta. \tag{7.1}$$

We see that unless $G = \mathrm{SO}_0(2,1)$, the action of MA on $\mathbb{F}^{n-1}\setminus\{0\}$ is transitive, and if $G = \mathrm{SO}_0(2,1)$, the action of MA on $\mathbb{R}^*$ has two orbits $\mathbb{R}_+$ and $\mathbb{R}_-$. A set of representatives for the orbits $MA \curvearrowright \mathbb{F}^{n-1}\setminus\{0\}$ is then

$$S_2 = \{-1,1\} \quad \text{if} \quad G = \mathrm{SO}_0(2,1) \qquad \text{and} \qquad S_2 = \{1\} \quad \text{if} \quad G \neq \mathrm{SO}_0(2,1)$$

The stabilizer L_v is $L_v = \{(u,\beta)\in M \mid uv = v\beta\}$, and we note that $L_v \subseteq M$.

(3) Finally, we consider representations that do not come from characters on N. This happens only when $\mathbb{F} \neq \mathbb{R}$. Let $m \in \mathrm{Im}\ \mathbb{F}^*$, and define $\lambda : \mathrm{Im}\ \mathbb{F} \to \mathbb{R}$ by $\lambda(z) = -\mathrm{Re}(m\bar{z})$. Then λ is a non-trivial $\mathbb{R}$-linear map. It is known that there exists an infinite dimensional irreducible representation η_m of N, uniquely determined be the property

$$\eta_m(w,z) = e^{i\lambda(z)}\eta_m(w,0), \qquad (w,z)\in \mathbb{F}^{n-1}\times \mathrm{Im}\ \mathbb{F}.$$

Moreover, η_m is uniquely determined within unitary equivalence by the central character λ (see [33, Lemma 4.4]). The group MA acts on the classes of representations η_m. Let L_m denote the stabilizer in MA of the class $[\eta_m]$,

$$L_m = \{g \in MA \mid g.\eta_m \simeq \eta_m\}.$$

Then η_m extends to a representation of $N \rtimes L_m$ as discussed in [33, Section 7], and the extension is of course still irreducible.

Let γ be an irreducible representation of L_m. Extend γ to be the irreducible representation of $N \rtimes L_m$ defined by letting γ be trivial on N. Form the tensor product representation $\eta_m \otimes \gamma$ and induce this representation to get a representation $\pi_{3,m,\gamma}$ of P,

$$\pi_{3,m,\gamma} = \mathrm{Ind}_{NL_m}^{P}(\eta_m\otimes\gamma).$$

We now describe the action of MA on the infinite dimensional representations of N in more detail. Since η_m is uniquely determined within unitary equivalence by λ (or equivalently by m), the action is best described by the action $MA \curvearrowright \mathrm{Im}\ \mathbb{F}^*$ given by (6.6) and (6.7),

$$(u,\beta,\alpha).\eta_m = \eta_{m'} \qquad \text{where } m' = \beta\alpha^{-2}m\beta^{-1}. \tag{7.2}$$

If $\mathbb{F} = \mathbb{C}$, there are two orbits under this action, $i\mathbb{R}_+$ and $i\mathbb{R}_-$, and if $\mathbb{F} = \mathbb{H}$, there is only one orbit $\mathrm{Im}\ \mathbb{F}^*$. As a set of representatives for the orbits we choose

$$S_3 = \{-i,i\} \quad \text{if} \quad \mathbb{F} = \mathbb{C} \qquad \text{and} \qquad S_3 = \{i\} \quad \text{if} \quad \mathbb{F} = \mathbb{H}.$$

The stabilizer of $m \in \{-i,i\}$ is

$$L_m = \{(u,\beta)\in M \mid \beta \in \mathbb{R} + \mathbb{R}i\}$$

We note that the stabilizer $L_m \subseteq M$.

The three constructions given above exhaust the unitary dual of P.

Theorem 18. *Let G be one of the classical groups* $\mathrm{SO}_0(n,1)$*,* $\mathrm{SU}(n,1)$*,* $\mathrm{Sp}(n,1)$*, let $\mathbb{F}$ be the corresponding division algebra ($\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$), and let $P = MAN$ be the minimal parabolic subgroup of G. The irreducible representations of P fall into three series as follows.*

(1) *The series*

$$\pi_{1,\sigma} = \sigma \circ q,$$

where $q : P \to P/N$ is the quotient map, and $\sigma \in \widehat{P/N} = \widehat{MA}$. The classes are parametrized by $\sigma \in \widehat{P/N}$.

(2) *The series*

$$\pi_{2,v,\gamma} = \mathrm{Ind}_{NL_v}^{P}(\chi_v \otimes \gamma),$$

where $v \in \mathbb{F}^{n-1}$ is non-zero and $\gamma \in \widehat{L}_v$. The classes are parametrized by $v \in S_2$ and $\gamma \in \widehat{L}_v$.

(3) *The series (only when $\mathbb{F} \neq \mathbb{R}$)*

$$\pi_{3,m,\gamma} = \mathrm{Ind}_{NL_m}^{P}(\eta_m \otimes \gamma)$$

where $m \in \mathrm{Im}\,\mathbb{F}^*$ and $\gamma \in \widehat{L}_m$. The classes are parametrized by $m \in S_3$ and $\gamma \in \widehat{L}_m$.

Lemma 19. *Consider the minimal parabolic subgroup $P = MAN$ in one of the classical groups* $\mathrm{SO}_0(n,1)$*,* $\mathrm{SU}(n,1)$*,* $\mathrm{Sp}(n,1)$*. Then P and N are of type I.*

Proof. It is known that connected nilpotent Lie groups are of type I (see [9, Corollaire 4]), and it follows that N is of type I.

Theorem 9.3 in [22] provides a way of establishing that P is type I. First of all, $\widehat{N}$ is a standard Borel space, because N is of type I. The action $MA \curvearrowright \widehat{N}$ has only finitely many orbits (the exact number depends on $\mathbb{F}$ and n), so in particular there is a Borel set in $\widehat{N}$ which meets each orbit exactly once. By [22, Theorem 9.2] the action $MA \curvearrowright \widehat{N}$ is *regular*, that is, N is *regularly embedded* in P.

We now verify that when $\pi \in \widehat{N}$, the stabilizer $L_\pi = \{g \in MA \mid g.\pi \simeq \pi\}$ is of type I. Indeed, if π is the trivial character on N, then $L_\pi = MA$ which is a direct product of the compact group M and the abelian group A. Hence the stabilizer MA is of type I. If π is not the trivial character, then $\pi = \eta_m$ or $\pi = \chi_v$, where $m \in \mathbb{F}^*$ or $v \in \mathrm{Im}\,\mathbb{F}^*$, and we already saw that L_m and L_v are closed subgroups of M and hence compact. In particular the stabilizers are of type I.

According to [22, Theorem 9.3] we may now conclude that P is of type I. □

7.2. **The exceptional case.** Let $P = MAN$ be the minimal parabolic subgroup of $\mathrm{F}_{4(-20)}$. We will now describe the irreducible representations of P. Again, this is based on [33]. They consider the group $\widetilde{M}AN$, where $\widetilde{M} = \mathrm{Spin}(7) \times \{\pm 1\}$. The complete description of the unitary dual of $\widetilde{M}AN$ can be found in (8.12) and (8.15) in [33]. The discussion below is based on Section 8 in [33] to which we refer for proofs and more details. The representations fall into two series.

(1) Irreducible representations of N that annihilate the center $Z(N) = \mathrm{Im}\ \mathbb{O}$ are characters of the form χ_v for some $v \in \mathbb{O}$, where χ_v is given by

$$\chi_v(w,z) = e^{i\mathrm{Re}\langle w,v\rangle} = e^{i(w|v)}$$

The group MA acts on N, and this induces a dual action of MA on $\widehat{N}$.

Let L_v be the stabilizer of χ_v in MA. Then χ_v extends to a character of $N \rtimes L_v$ by the formula

$$\chi_v(w,z,g) = \chi_v(w,z) = e^{i(w|v)}, \qquad (w,z,g) \in \mathbb{O} \times \mathrm{Im}\ \mathbb{O} \times L_v.$$

Let γ be an irreducible representation of L_v. Extend γ to be the irreducible representation of $N \rtimes L_v$ defined by letting γ be trivial on N. Form the tensor product representation $\chi_v \otimes \gamma$ and induce this representation from $N \rtimes L_v$ to P to get a representation $\pi_{1,v,\gamma}$ of P,

$$\pi_{1,v,\gamma} = \mathrm{Ind}_{NL_v}^P(\chi_v \otimes \gamma).$$

This representation $\pi_{1,v,\gamma}$ is a representation in the first series.

From the definition of the action $MA \curvearrowright N$ we see that

$$(u,\alpha).\chi_v(w,z) = e^{i((\alpha\sigma(u))^{-1}w|v)} = e^{i(w|\alpha\sigma(u)v)} = \chi_{\alpha\sigma(u)v}(w,z)$$

Since M acts transitively on $S^7 \subseteq \mathbb{O}$, we see that MA acts transitively $\mathbb{O}^*$ and thus on the characters $\{\chi_v\}_{v\in\mathbb{O}^*}$.

If $v = 0$, the stabilizer L_v is of course all of MA. Otherwise the stabilizer L_v is

$$L_v = \{(u,\alpha) \in MA \mid \sigma(u)\alpha v = v\}.$$

Since $\sigma(u)$ preserves the norm of elements in $\mathbb{O}$, we see that if $(u,\alpha) \in L_v$, then $\alpha = 1$. Hence $L_v \subseteq M$.

(2) Let $m \in \mathrm{Im}\ \mathbb{O}^*$ be non-zero, and define $\lambda : \mathrm{Im}\ \mathbb{O} \to \mathbb{R}$ by $\lambda(z) = -\mathrm{Re}(m\bar{z})$. Then λ is a non-trivial $\mathbb{R}$-linear map which is uniquely determined by m. Irreducible representations of N that do not annihilate the center are infinite dimensional and of the form η_m for some $m \in \mathrm{Im}\ \mathbb{O}^*$, where η_m is uniquely determined by the property

$$\eta_m(w,z) = e^{i\lambda(z)}\eta_m(w,0), \quad (w,z) \in N.$$

Moreover, the equivalence class of η_m is uniquely determined by the central character λ and hence by m. Since the action of MA on $\mathrm{Im}\ \mathbb{O}^*$ is transitive, MA acts transitively on the set $\{\eta_m\}_{m\in\mathrm{Im}\ \mathbb{O}^*}$.

Let L_m denote the stabilizer in MA of the class of η_m. Then

$$L_m = \{u \in M \mid \nu(u)m = m\}.$$

It follows from [33, Lemma 8.14] that η_m extends to a representation of $N \rtimes L_m$. Let γ be an irreducible representation of L_m, and extend γ to $N \rtimes L_m$ by letting γ be trivial on N. Form the tensor product representation $\eta_m \otimes \gamma$ and induce this representation to get a representation $\pi_{2,m,\gamma}$ of P,

$$\pi_{2,m,\gamma} = \mathrm{Ind}_{NL_m}^P(\eta_m \otimes \gamma).$$

Theorem 20. *Let $P = MAN$ be the minimal parabolic subgroup of* $\mathrm{F}_{4(-20)}$ *and let π be an irreducible representation of P. Then π is unitarily equivalent to one of the following.*

(1) $\pi_{1,v,\gamma} = \mathrm{Ind}_{NL_v}^{P}(\chi_v \otimes \gamma)$ *for some* $v \in \mathbb{O}$ *and* $\gamma \in \widehat{L}_v$.
(2) $\pi_{2,m,\gamma} = \mathrm{Ind}_{NL_m}^{P}(\eta_m \otimes \gamma)$ *for some* $m \in \mathrm{Im}\ \mathbb{O}^*$ *and* $\gamma \in \widehat{L}_m$.

Lemma 21. *Consider the minimal parabolic subgroup $P = MAN$ of* $\mathrm{F}_{4(-20)}$*. Then P and N are of type I.*

Proof. Since N is a connected nilpotent Lie group, N is of type I (see [9, Corollaire 4]).

Theorem 9.3 in [22] provides a way of establishing that P is type I. First of all, $\widehat{N}$ is a standard Borel space, because N is of type I. The action $MA \curvearrowright \widehat{N}$ has only three orbits,

$$\mathcal{O}_1 = \{\chi_0\}, \quad \mathcal{O}_2 = \{\chi_v\}_{v \in \mathbb{O}^*}, \quad \mathcal{O}_3 = \{\eta_m\}_{m \in \mathrm{Im}\ \mathbb{O}^*},$$

where χ_0 is the trivial representation. Then, clearly, there is a Borel set in $\widehat{N}$ which meets each orbit exactly once. By [22, Theorem 9.2] the action $MA \curvearrowright \widehat{N}$ is regular.

We now verify that when $\pi \in \widehat{N}$, the stabilizer $L_\pi = \{g \in MA \mid g.\pi \simeq \pi\}$ is of type I. Indeed, if $\pi = \eta_m$ or $\pi = \chi_v$, where $m \in \mathbb{O}^*$ or $v \in \mathrm{Im}\ \mathbb{O}^*$, then we already saw that L_m and L_v are compact and in particular of type I. If $\pi = \chi_0$, then the stabilizer is MA which is a direct product of the compact group M and the abelian group A. Hence the stabilizer MA is of type I.

According to [22, Theorem 9.3] we may now conclude that P is of type I. □

8. The Fourier algebra of P

In this section we verify the last condition in Theorem 4 for the minimal parabolic subgroups P. The result is contained in Proposition 27.

Recall that part of the Peter-Weyl Theorem asserts that the left regular representation of a compact group is completely reducible, and every irreducible representation of the compact group occurs as a direct summand (see [13, Theorem 5.12]).

We now set out to determine which irreducible representations of P that occur as subrepresentations of the left regular representation. For this we will rely on Corollary 11.1 in [20]. In order to apply the corollary we first need to verify the assumptions I-IV from [20]. For this it will suffice to observe that N and P are of type I (see Lemma 19 and Lemma 21), and all stabilizers L_v and L_m are closed and contained in M, so they are compact and in particular of type I.

In the case of the representation $\pi_{1,v,\gamma}$, Corollary 11.1 in [20] then applies to show that $\pi_{1,v,\gamma}$ is a subrepresentation of the left regular representation of P if and only if γ is a subrepresentation of the left regular representation of the stabilizer group L_v

and the orbit of χ_v inside $\widehat{N}$ has positive Plancherel measure. Similar conclusions hold in the other cases.

8.1. **The classical cases.** We first consider the case where $\mathbb{F} = \mathbb{R}$.

Lemma 22. *Let G be the group* $\mathrm{SO}_0(n,1)$*, and let $P = MAN$ be the minimal parabolic subgroup of G. Any irreducible unitary representation π of P is either trivial on the non-compact subgroup N or is a subrepresentation of λ_P.*

Proof. We divide the proof into the cases according to the description in Theorem 18.

In the case $\pi = \pi_{1,\sigma}$, it is clear that $\pi(g) = 1$ for every $g \in N$.

Consider now a representation $\pi = \pi_{2,v,\gamma}$ where v is non-zero. Since L_v is compact, $\gamma \in \widehat{L}_v$ is a subrepresentation of the regular representation of L_v. If $n \neq 2$, then the action of MA on the non-zero characters of N is transitive. In particular, the orbit has positive Plancherel measure in $\widehat{N}$. If $n = 2$, then the orbit of χ_v is either $\mathbb{R}_+$ or $\mathbb{R}_-$ inside $\widehat{N} \simeq \mathbb{R}$, and both of these sets have positive measure. By Corollary 11.1 in [20] we conclude that π is a subrepresentation of λ_P.

The case $\pi = \pi_{3,m,\gamma}$ does not occur, when $\mathbb{F} = \mathbb{R}$. □

From Proposition 8 we can now conclude that the left regular representation of P is completely reducible. From the proof of Lemma 22 we then obtain the following.

Corollary 23. *Let G be the group* $\mathrm{SO}_0(n,1)$*, and let $P = MAN$ be the minimal parabolic subgroup of G. The left regular representation of P is completely reducible with the representations $\pi_{2,v,\gamma}$ as its subrepresentations. Here $v \in S_2$ and $\gamma \in \widehat{L}_v$.*

When $\mathbb{F}$ equals $\mathbb{C}$ or $\mathbb{H}$ we have the following.

Lemma 24. *Let G be one of the groups* $\mathrm{SU}(n,1)$, $\mathrm{Sp}(n,1)$*, and let $P = MAN$ be the minimal parabolic subgroup of G. Any irreducible unitary representation π of P is either trivial on the non-compact subgroup $Z(N)$ or is a subrepresentation of λ_P.*

Proof. We divide the proof into the cases according to the description in Theorem 18.

In the case $\pi = \pi_{1,\sigma}$, it is clear that $\pi(g) = 1$ for every $g \in N$, and hence in particular for every $g \in Z(N)$.

Suppose now $\pi = \pi_{2,v,\gamma}$. Since χ_v is trivial on $Z(N)$ and γ is trivial on N, it follows that $\chi_v \otimes \gamma$ is trivial on $Z(N)$. Since $Z(N) \triangleleft P$, it now follows from Lemma 12 that π is trivial on $Z(N)$, once we show that the homogeneous space P/NL_v admits a P-invariant measure. Using Proposition 11 we find

$$P/NL_v \simeq MA/L_v \simeq M/L_v \times A.$$

The left translation action $A \curvearrowright A$ has the Haar measure as an invariant measure. Since M is compact, the action $M \curvearrowright M/L_v$ has an invariant measure. It follows that $P \curvearrowright NL_v$ has an invariant measure, and then by Lemma 12 the representation π is trivial on $Z(N)$.

Consider now a representation $\pi = \pi_{3,m,\gamma}$. Since L_m is compact, $\gamma \in \widehat{L}_m$ is a subrepresentation of the regular representation of L_m. It remains to show that the orbit of η_m in $\widehat{N}$ has positive Plancherel measure.

If $\mathbb{F} = \mathbb{H}$, then the third series of Theorem 18 forms a single orbit, which must then have positive Plancherel measure, because all other irreducible representations of N are trivial on $Z(N)$ and hence must form a null set for the Plancherel measure.

If $\mathbb{F} = \mathbb{C}$, then the action of MA on the representations $\{\eta_m \in \widehat{N} \mid m \in \mathrm{Im}\,\mathbb{F}^*\}$ has two orbits, so the simple argument for $\mathbb{H}$ does not apply. Luckily, the Plancherel measure of N is well-known. In fact, N is the Heisenberg group of dimension $2n-1$, and the Plancherel measure for the Heisenberg group can be found on p. 241 in [13]. We see that the measure of the orbit of η_i is

$$\mu_N(P.\eta_i) = \int_0^\infty |m|^{n-1}\, dm.$$

Hence the orbit of η_i has positive, in fact infinite, measure. Similarly, the orbit of η_{-i} has positive measure. By Corollary 11.1 in [20] we conclude that π is a subrepresentation of λ_P. □

From Proposition 8 we can now conclude that the left regular representation of P is completely reducible. From the proof of Lemma 24 we then obtain the following.

Corollary 25. *Let G be one of the classical groups $\mathrm{SU}(n,1)$, $\mathrm{Sp}(n,1)$, and let $P = MAN$ be the minimal parabolic subgroup of G. The left regular representation of P is completely reducible with the representations $\pi_{3,m,\gamma}$ as its subrepresentations. Here $m \in S_3$ and $\gamma \in \widehat{L}_m$.*

8.2. **The exceptional case.**

Lemma 26. *Let $P = MAN$ be the minimal parabolic subgroup of $\mathrm{F}_{4(-20)}$. Any irreducible unitary representation π of P is either trivial on the non-compact subgroup $Z(N) \simeq \mathrm{Im}\mathbb{O}$ or is a subrepresentation of λ_P.*

Proof. Recall that any irreducible representation of P is given as in Theorem 20. We will show that representations $\pi_{1,v,\gamma}$ are trivial on $Z(N)$ and that representations $\pi_{2,m,\gamma}$ are subrepresentations of λ_P.

Suppose first $\pi = \pi_{1,v,\gamma}$. The proof from Lemma 24 carries over verbatim and shows that π is trivial on $Z(N)$.

Consider now a representation $\pi = \pi_{2,m,\gamma}$. Since L_m is compact, $\gamma \in \widehat{L}_m$ is a subrepresentation of the regular representation of L_m. It remains to show that the orbit of η_m in $\widehat{N}$ has positive Plancherel measure.

Clearly, the characters $\{\chi_v\}_{v\in\mathbb{O}}$ form a null set for the Plancherel measure on $\widehat{N}$, because they are all trivial on the non-compact subgroup $Z(N)$. As mentioned in the proof of Lemma 21, the complement of $\{\chi_v\}_{v\in\mathbb{O}}$ forms a single orbit in $\widehat{N}$ under the action of MA. Thus, the orbit of η_m must have positive Plancherel measure. By Corollary 11.1 in [20] we conclude that π is a subrepresentation of λ_P. □

8.3. **Conclusion.** The following proposition sums up the necessary results from this and last section so that we may apply Theorem 4.

Proposition 27. *Let $n \geq 2$, and let G be one of the simple Lie groups $\mathrm{SO}_0(n,1)$, $\mathrm{SU}(n,1)$, $\mathrm{Sp}(n,1)$ or $\mathrm{F}_{4(-20)}$. Let $P = MAN$ be the minimal parabolic subgroup in G. The following holds.*

1. *P is type I.*
2. *There is a non-compact, closed subgroup H of P such that every irreducible unitary representation of P is either trivial on H or is a subrepresentation of the regular representation λ_P.*

In fact, if $G = \mathrm{SO}_0(n,1)$, then one can take $H = N$, and otherwise one can take $H = Z(N)$.

From Proposition 27 and Theorem 4 we immediately obtain Theorem 3.

9. The relative Howe-Moore property

In this section we prove Corollary 5 concerning the relative Howe-Moore property. We recall from [6] that if H is a closed subgroup of a locally compact group G, then the pair (G,H) has the relative Howe-Moore property, if every representation π of G either has H-invariant vectors, or the restriction $\pi|_H$ is a C_0-representation, i.e. all coefficients of $\pi|_H$ vanish at infinity. Using a direct integral argument, it is proved in [6, Proposition 2.3] that it is sufficient to consider only irreducible representations of G.

From the results in the previous sections we easily obtain the following, which obviously implies Corollary 5.

Corollary 28. *If P is the group in (1.2) and N_1 is the group in (5.4), then N_1 is a normal, non-compact closed subgroup of P, and (P, N_1) has the relative Howe-Moore property.*

Let $n \geq 2$. If $P = MAN$ the minimal parabolic subgroup in the simple Lie group $\mathrm{SO}_0(n,1)$, then N is a normal, non-compact closed subgroup of P, and (P,N) has the relative Howe-Moore property.

If $P = MAN$ the minimal parabolic subgroup in one of the simple Lie groups $\mathrm{SU}(n,1)$, $\mathrm{Sp}(n,1)$ or $\mathrm{F}_{4(-20)}$, then $Z(N)$ is a normal, non-compact closed subgroup of P, and $(P, Z(N))$ has the relative Howe-Moore property.

Proof. Apply Proposition 17 or Proposition 27, respectively. Since any subrepresentation of the left regular representation λ_P is a C_0-representation, we immediately obtain the result. □

10. Concluding remarks

Theorem 3 shows that Question 1 has a positive answer for the minimal parabolic subgroups $P = MAN$ in the groups $\mathrm{SO}_0(n,1)$, $\mathrm{SU}(n,1)$, $\mathrm{Sp}(n,1)$ and $\mathrm{F}_{4(-20)}$. One could ask if the same is true for the smaller groups MN, AN or N. We will now discuss these cases. Recall from the introduction that a non-compact second countable connected unimodular groups never satisfy (1.1).

Let G be one of the groups classical groups $\mathrm{SO}_0(n,1)$, $\mathrm{SU}(n,1)$, $\mathrm{Sp}(n,1)$ with $n \geq 2$ or the exceptional group $\mathrm{F}_{4(-20)}$. Let $\mathbb{F}$ be the corresponding division algebra, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$. We start by discussing the groups N. Since N is nilpotent, N is unimodular. Indeed, a locally compact group G is unimodular if and only if G/Z is unimodular, where Z is the center of G (see [27, p. 92]). Induction on the length of an upper central series then shows that all locally compact nilpotent groups are unimodular. Since N is also connected and second countable, it follows that

$$A(N) \neq B(N) \cap C_0(N).$$

Next we discuss the groups MN. Since MN is a semi-direct product of the unimodular group N by the compact group M, we will argue that MN itself is unimodular. Indeed, this follows directly from [27, Proposition 23] but we also include another argument here. If we use Δ_G to denote the modular function of a locally compact group G, then since N is normal in MN, we see that the quotient space MN/N has an invariant measure, Haar measure on M, and using [13, Theorem 2.49] we see that $\Delta_{MN}|N = \Delta_N = 1$. Also, since M is compact, $\Delta_{MN}|M = 1$ by [13, Proposition 2.27]. Since M and N generate MN, it follows that $\Delta_{MN} = 1$. So MN is connected and unimodular, and hence

$$A(MN) \neq B(MN) \cap C_0(MN).$$

Alternatively, one could show that all orbits in $\widehat{N}$ under the action of M have zero Plancherel measure. This type of argument will be used below for the groups AN.

For the groups $\mathrm{SO}_0(n,1)$, $\mathrm{Sp}(n,1)$ and $\mathrm{F}_{4(-20)}$ it will usually also be the case that Question 1 has a negative answer for the groups AN as well. However, there is one exception. If $G = \mathrm{SO}_0(2,1)$, then M is trivial and P coincides with AN. Hence it follows from Theorem 3 that Question 1 has an affirmative answer for the group AN. In this special case let us remark that in fact AN is isomorphic to the $ax+b$ group, and the result that $A(AN) = B(AN) \cap C_0(AN)$ is actually the original result of Khalil from [19].

The unimodularity argument used for the groups N and MN cannot be replicated for AN, since these groups are not unimodular (see [18, (1.14)]). As mentioned in the introduction, a group satisfying (1.1) has a completely reducible left regular representation, and in particular the left regular representation has irreducible subrepresentations. Then by [20, Corollary 11.1] at least one of the orbits of the action $A \curvearrowright \widehat{N}$ must have positive Plancherel measure. To show that $A(AN) \neq B(AN) \cap C_0(AN)$ it therefore suffices to show that any orbit of $A \curvearrowright \widehat{N}$ has zero Plancherel measure.

It this point we split the argument in cases. Consider first the case when $\mathbb{F} = \mathbb{R}$ and $n \geq 3$. Then $N \simeq \mathbb{R}^{n-1}$, and the Plancherel measure on $\widehat{N} \simeq \mathbb{R}^{n-1}$ is the Lebesgue measure. Since A acts on $\widehat{N}$ by dilation, every orbit except $\{0\}$ is a a half-line. Since $n \geq 3$ every half-line in $\mathbb{R}^{n-1}$ has vanishing Lebesgue measure, and hence every orbit in $\widehat{N}$ has vanishing Plancherel measure.

In the other cases $\mathbb{F}$ is $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$. For convenience, when $\mathbb{F} = \mathbb{O}$, we set $n = 2$. As mentioned earlier, the dual $\widehat{N}$ then consists of the characters $\{\chi_v\}_{v \in \mathbb{F}^{n-1}}$ and the infinite dimensional representations $\widehat{N}_r = \{\eta_m\}_{m \in \operatorname{Im} \mathbb{F}^*}$.

Fortunately, the Plancherel measure for N is known. It is described in [7, Section 3]. Since the characters are trivial on the center $\operatorname{Im} \mathbb{F}$ of N which is non-compact, the characters form a null set for the Plancherel measure. Let k be the dimension of $\operatorname{Im} \mathbb{F}$ as a real vector space so that k is either 1, 3 or 7. If we identify $\widehat{N}_r$ with $\operatorname{Im} \mathbb{F}^*$ which in turn is identified with the punctured Euclidean space $\mathbb{R}^k \setminus \{0\}$, then it follows from [7, p. 524] that the Plancherel measure on $\widehat{N}_r$ is absolutely continuous (has density) with respect to the Lebesgue measure.

Since A acts on $\widehat{N}_r$ by dilation, every orbit in $\widehat{N}_r$ is a a half-line. Every half-line has vanishing Lebesgue measure, unless $k = 1$, and hence every orbit in $\widehat{N}_r$ has vanishing Plancherel measure, except when $\mathbb{F} = \mathbb{C}$. Combined with the fact that the characters have vanishing Plancherel measure, we conclude that every orbit in $\widehat{N}$ has vanishing Plancherel measure. As pointed out, the argument breaks down when $\mathbb{F} = \mathbb{C}$.

We collect the discussion above in the following proposition.

Proposition 29. *Let G be one of the simple Lie groups* $\mathrm{SO}_0(n,1)$ *($n \geq 3$),* $\mathrm{Sp}(n,1)$ *($n \geq 2$) or* $\mathrm{F}_{4(-20)}$*. Let $G = KAN$ be the Iwasawa decomposition of G. Then if H is either N, MN or AN, then*

$$A(H) \neq B(H) \cap C_0(H).$$

Finally, we consider the group AN in $G = \mathrm{SU}(n,1)$.

Proposition 30. *Let G be the simple Lie group* $\mathrm{SU}(n,1)$ *($n \geq 2$) with Iwasawa decomposition $G = KAN$. Then*

$$A(AN) = B(AN) \cap C_0(AN).$$

Proof. We will verify the conditions of Theorem 4 for the group AN.

First we verify that AN is a group of type I. We mimic the proof of Lemma 19. Recall that N is of type I, and hence $\widehat{N}$ is a standard Borel space. Using the notation from Section 7, we identify $\widehat{N}$ with the union of the characters $\{\chi_v\}_{v \in \mathbb{C}^{n-1}}$ and the infinite dimensional representations $\{\eta_m\}_{m \in i\mathbb{R}^*}$. The action $A \curvearrowright \widehat{N}$ is described by (7.1) and (7.2), and it is easy to read off the orbits of the action.

The characters in $\widehat{N}$, which we think of simply as $\mathbb{C}^{n-1}$, form an invariant subset whose orbits consist of the origin $\{0\}$ and half-lines originating at the origin. The infinite dimensional representations in $\widehat{N}$, which we think of simply as $i\mathbb{R}^*$ also form an invariant subset which has two orbits, $i\mathbb{R}_+$ and $i\mathbb{R}_-$.

If S denotes the unit sphere in $\mathbb{C}^{n-1} \simeq \mathbb{R}^{2n-2}$, then $R = \{0\} \cup S \cup \{i, -i\}$ is a set of representatives for the orbits of $A \curvearrowright \widehat{N}$. We claim that R is a Borel subset of $\widehat{N}$. To see this, it suffices to prove that S is a Borel subset, since points are always Borel subsets in a standard Borel space.

The group N is the Heisenberg group of dimension $2n - 1$, and the Fell topology on $\widehat{N}$ is well-known (see e.g. [13, Chapter 7]). The characters $\{\chi_v\}_{v \in \mathbb{C}^{n-1}}$ form a closed subset in $\widehat{N}$, and on the set of characters the Fell topology coincides with the Euclidean topology (on $\mathbb{C}^{n-1}$). In particular S is closed in the Fell topology. By [13, Theorem 7.6], the Mackey Borel structure on $\widehat{N}$ is induced by the Fell topology, since N is of type I. It follows that S is a Borel set.

We may now conclude from [22, Theorem 9.2] that the action $A \curvearrowright \widehat{N}$ is *regular*, that is, N is *regularly embedded* in AN.

Next we verify that if $\pi \in \widehat{N}$, then the stabilizer $L_\pi = \{\alpha \in A \mid \alpha.\pi \simeq \pi\}$ is of type I. Indeed, if π is the trivial character on N, then $L_\pi = A$ which is abelian group. Hence the stabilizer A is of type I. If π is not the trivial character, then the stabilizer L_π is trivial. So all stabilizers are of type I. According to [22, Theorem 9.3] we may now conclude that AN is of type I.

The unitary dual of AN is described in [33, Proposition 7.6]. The irreducible representations of AN fall into three series as follows (retaining earlier notation).

(1) The series
$$\pi_{1,\sigma} = \sigma \circ q,$$
where $q : AN \to A$ is the quotient map, and $\sigma \in \widehat{A}$.

(2) The series
$$\pi_{2,v} = \mathrm{Ind}_N^{AN}(\chi_v),$$
where $v \in \mathbb{C}^{n-1}$ is non-zero. The classes are parametrized by the orbits of $A \curvearrowright \mathbb{C}^{n-1} \setminus \{0\}$.

(3) The series
$$\pi_{3,m} = \mathrm{Ind}_N^{AN}(\eta_m)$$
where $m \in i\mathbb{R}^*$. The classes are parametrized by $m \in \{i, -i\}$.

We claim that $\pi_{1,\sigma}$ and $\pi_{2,v}$ are trivial on the center $Z(N)$ of N, and that $\pi_{3,m}$ is a subrepresentation of the regular representation.

Clearly, $\pi_{1,\sigma}$ annihilates N and in particular $Z(N)$. Consider now a representation $\pi_{2,v} = \mathrm{Ind}_N^{AN}(\chi_v)$, where $v \in \mathbb{C}^{n-1}$ is non-zero. The character $\chi_v \in \widehat{N}$ is trivial on $Z(N)$. Both A and N normalize $Z(N)$, so $Z(N)$ is normal in AN. The representation $\pi_{2,v}$ is induced from N to AN, and when the quotient space AN/N is identified with A in the natural way, it is obvious that AN/N carries an invariant measure for the AN-action, namely the Haar measure on A. From Lemma 12 it now follows that $\pi_{2,v}$ is trivial on $Z(N)$.

Finally, consider a representation $\pi_{3,m} = \mathrm{Ind}_N^{AN}(\eta_m)$ where $m \in i\mathbb{R}^*$. We will show that the orbit of η_m in $\widehat{N}$ has positive Plancherel measure, and then it follows from [20, Corollary 11.1] that $\mathrm{Ind}_N^{AN}(\eta_m)$ is a subrepresentation of the left regular representation of AN.

As mentioned before, the action of A on the representations $\{\eta_m \in \widehat{N} \mid m \in i\mathbb{R}^*\}$ has two orbits, $i\mathbb{R}_+$ and $i\mathbb{R}_-$. The Plancherel measure of N is known and can be found on p. 241 in [13]. We see that the measure of the orbit of η_i is

$$\mu_N(A.\eta_i) = \int_0^\infty |m|^{n-1}\, dm.$$

Hence the orbit of η_i has positive, in fact infinite, measure. Similarly, the orbit of η_{-i} has positive measure. By [20, Corollary 11.1] we conclude that $\pi_{3,m}$ is a subrepresentation of the left regular representation of AN.

The conditions of Theorem 4 have now been verified for the group AN, and our proof is complete. □

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ARTICLE E

Approximation properties of simple Lie groups made discrete

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APPROXIMATION PROPERTIES OF SIMPLE LIE GROUPS MADE DISCRETE

SØREN KNUDBY AND KANG LI

ABSTRACT. In this paper we consider the class of connected simple Lie groups equipped with the discrete topology. We show that within this class of groups the following approximation properties are equivalent: (1) the Haagerup property; (2) weak amenability; (3) the weak Haagerup property (Theorem 1.10). In order to obtain the above result we prove that the discrete group $\mathrm{GL}(2, K)$ is weakly amenable with constant 1 for any field K (Theorem 1.11).

In the final part of the paper we give a contractive Schur multiplier characterization of locally compact groups coarsely embeddable into Hilbert spaces (Theorem 1.12). Consequently, all locally compact groups whose weak Haagerup constant is 1 embed coarsely into Hilbert spaces and hence the Baum-Connes assembly map with coefficients is split-injective for such groups.

1. Introduction

Amenability for groups was first introduced by von Neumann in order to study the Banach-Tarski paradox. It is remarkable that this notion has numerous characterizations and one of them, in terms of an approximation property by positive definite functions, is the following: a locally compact (Hausdorff) group G is amenable if there exists a net of continuous compactly supported, positive definite functions on G tending to the constant function 1 uniformly on compact subsets of G. Later, three weak forms of amenability were introduced: the Haagerup property, weak amenability and the weak Haagerup property. In this paper we will study these approximation properties of groups within the framework of Lie theory and coarse geometry.

Definition 1.1 (Haagerup property [10])**.** A locally compact group G has the *Haagerup property* if there exists a net of positive definite C_0-functions on G, converging uniformly to 1 on compact sets.

Definition 1.2 (Weak amenability [17])**.** A locally compact group G is *weakly amenable* if there exists a net $(\varphi_i)_{i\in I}$ of continuous, compactly supported Herz-Schur multipliers on G, converging uniformly to 1 on compact sets, and such that $\sup_i \|\varphi_i\|_{B_2} < \infty$.

The *weak amenability constant* $\Lambda_{\mathrm{WA}}(G)$ is defined as the best (lowest) possible constant Λ such that $\sup_i \|\varphi_i\|_{B_2} \le \Lambda$, where $(\varphi_i)_{i\in I}$ is as just described.

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Definition 1.3 (The weak Haagerup property [38])**.** A locally compact group G has the *weak Haagerup property* if there exists a net $(\varphi_i)_{i\in I}$ of C_0 Herz-Schur multipliers on G, converging uniformly to 1 on compact sets, and such that $\sup_i \|\varphi_i\|_{B_2} < \infty$.

The *weak Haagerup constant* $\Lambda_{\mathrm{WH}}(G)$ is defined as the best (lowest) possible constant Λ such that $\sup_i \|\varphi_i\|_{B_2} \leq \Lambda$, where $(\varphi_i)_{i\in I}$ is as just described.

Clearly, amenable groups have the Haagerup property. It is also easy to see that amenable groups are weakly amenable with $\Lambda_{\mathrm{WA}}(G) = 1$ and that groups with the Haagerup property have the weak Haagerup property with $\Lambda_{\mathrm{WH}}(G) = 1$. Also, $1 \leq \Lambda_{\mathrm{WH}}(G) \leq \Lambda_{\mathrm{WA}}(G)$ for any locally compact group G, so weakly amenable groups have the weak Haagerup property.

It is natural to ask about the relation between the Haagerup property and weak amenability. The two notions agree in many cases, like generalized Baumslag-Solitar groups (see [14, Theorem 1.6]) and connected simple Lie groups with the discrete topology (see Theorem 1.10). In general, weak amenability does not imply the Haagerup property and vice versa. In one direction, the group $\mathbb{Z}/2 \wr \mathbb{F}_2$ has the Haagerup property [19], but is not weakly amenable [44]. In the other direction, the simple Lie groups $\mathrm{Sp}(1,n)$, $n \geq 2$, are weakly amenable [17], but since these non-compact groups also have Property (T) [5, Section 3.3], they cannot have the Haagerup property. However, since the weak amenability constant of $\mathrm{Sp}(1,n)$ is $2n-1$, it is still reasonable to ask whether $\Lambda_{\mathrm{WA}}(G) = 1$ implies that G has the Haagerup property. In order to study this, the weak Haagerup property was introduced in [37, 38], and the following questions were considered.

Question 1.4. *For which locally compact groups G do we have $\Lambda_{\mathrm{WA}}(G) = \Lambda_{\mathrm{WH}}(G)$?*

Question 1.5. *Is $\Lambda_{\mathrm{WH}}(G) = 1$ if and only if G has the Haagerup property?*

It is clear that if the weak amenability constant of a group G is 1, then so is the weak Haagerup constant, and Question 1.4 has a positive answer. In general, the constants differ by the example $\mathbb{Z}/2 \wr \mathbb{F}_2$ mentioned before. There is an another class of groups for which the two constants are known to be the same.

Theorem 1.6 ([31])**.** *Let G be a connected simple Lie group. Then G is weakly amenable if and only if G has the weak Haagerup property. Moreover, $\Lambda_{\mathrm{WA}}(G) = \Lambda_{\mathrm{WH}}(G)$.*

By the work of many authors [16, 17, 18, 24, 30, 33], it is known that a connected simple Lie group G is weakly amenable if and only if the real rank of G is zero or one. Also, the weak amenability constants of these groups are known. Recently, a similar result was proved about the weak Haagerup property [31, Theorem B]. Combining the results on weak amenability and the weak Haagerup property with the classification of connected Lie groups with the Haagerup property [10, Theorem 4.0.1] one obtains the following theorem, which gives a partial answer to both Question 1.4 and Question 1.5.

Theorem 1.7. *Let G be a connected simple Lie group. The following are equivalent.*

(1) *G is compact or locally isomorphic to $\mathrm{SO}(n,1)$ or $\mathrm{SU}(n,1)$ for some $n \geq 2$.*
(2) *G has the Haagerup property.*

(3) *G is weakly amenable with constant 1.*
(4) *G has the weak Haagerup property with constant 1.*

The purpose of this paper is to consider the same class of groups as in theorem above, but made discrete. When G is a locally compact group, we let G_{d} denote the same group equipped with the discrete topology. The idea of considering Lie groups without their topology (or with the discrete topology, depending on the point of view) is not a new one. For instance, a conjecture of Friedlander and Milnor is concerned with computing the (co)homology of the classifying space of G_{d}, when G is a Lie group (see [40] and the survey [45]).

Other papers discussing the relation between G and G_{d} include [13], [4] and [6]. Since our focus is approximation properties, will we be concerned with the following question.

Question 1.8. *Does the Haagerup property/weak amenability/the weak Haagerup property of G_{d} imply the Haagerup property/weak amenability/the weak Haagerup property of G?*

It is not reasonable to expect an implication in the other direction. For instance, many compact groups such as $\mathrm{SO}(n)$, $n \geq 3$, are non-amenable as discrete groups. It follows from Theorem 1.10 below (see also Corollary 4.3) that when $n \geq 5$, then $\mathrm{SO}(n)$ as a discrete group does not even have the weak Haagerup property. It is easy to see that Question 1.8 has a positive answer for second countable, locally compact groups G that admit a lattice Γ. Indeed, G has the Haagerup property if and only if Γ has the Haagerup property. Moreover, $\Lambda_{\mathrm{WA}}(\Gamma) = \Lambda_{\mathrm{WA}}(G)$ and $\Lambda_{\mathrm{WH}}(\Gamma) = \Lambda_{\mathrm{WH}}(G)$.

Remark 1.9. A similar question can of course be asked for amenability. This case is already settled: if G_{d} is amenable, then G is amenable [46, Proposition 4.21], and the converse is not true in general by the counterexamples mentioned above. A sufficient and necessary condition of the converse implication can be found in [4].

Recall that $\mathrm{SL}(2,\mathbb{R})$ is locally isomorphic to $\mathrm{SO}(2,1)$ and that $\mathrm{SL}(2,\mathbb{C})$ is locally isomorphic to $\mathrm{SO}(3,1)$. Thus, Theorem 1.7 and the main theorem below together show in particular that Question 1.8 has a positive answer for connected simple Lie groups. This could however also be deduced (more easily) from the fact that connected simple Lie groups admit lattices [49, Theorem 14.1].

Theorem 1.10 (Main Theorem)**.** *Let G be a connected simple Lie group, and let G_{d} denote the group G equipped with the discrete topology. The following are equivalent.*

(1) *G is locally isomorphic to* $\mathrm{SO}(3)$*,* $\mathrm{SL}(2,\mathbb{R})$*, or* $\mathrm{SL}(2,\mathbb{C})$*.*
(2) *G_{d} has the Haagerup property.*
(3) *G_{d} is weakly amenable with constant 1.*
(4) *G_{d} is weakly amenable.*
(5) *G_{d} has the weak Haagerup property with constant 1.*
(6) *G_{d} has the weak Haagerup property.*

The equivalence of (1) and (2) in Theorem 1.10 was already done by de Cornulier [13, Theorem 1.14] and in greater generality. His methods are the inspiration for our proof of Theorem 1.10. That (1) implies (2) basically follows from a theorem of Guentner, Higson and Weinberger [26, Theorem 5.4], namely that the discrete group $\mathrm{GL}(2, K)$ has the Haagerup property for any field K. Here we prove a similar statement about weak amenability.

Theorem 1.11. *Let K be any field. The discrete group* $\mathrm{GL}(2, K)$ *is weakly amenable with constant 1.*

Theorem 1.11 is certainly known to experts. The result was already mentioned in [48, p. 7] and in [43] with a reference to [26], and indeed our proof of Theorem 1.11 is merely an adaption of the methods developed in [26]. However, since no published proof is available, we felt the need to include a proof.

To obtain Theorem 1.10 we use the classification of simple Lie groups and then combine Theorem 1.11 with the following results proved in Section 4: If G is one of the four groups $\mathrm{SO}(5)$, $\mathrm{SO}_0(1, 4)$, $\mathrm{SU}(3)$ or $\mathrm{SU}(1, 2)$, then G_{d} does not have the weak Haagerup property. Also, if G is the universal covering group of $\mathrm{SU}(1, n)$ where $n \geq 2$, then G_{d} does not have the weak Haagerup property.

In the final part of the paper we study coarse embeddability of locally compact groups into Hilbert spaces. An important application of this concept in [55], [50] and [21] is that the Baum-Connes assembly map with coefficients is split-injective for all locally compact groups that embed coarsely into a Hilbert space (see [3] for more information about the Baum-Connes assembly map). Here, we give a contractive Schur multiplier characterization of locally compact groups coarsely embeddable into Hilbert spaces (see also [22, Theorem 5.3] for the discrete case), and this characterization can be regarded as an answer to the non-equivariant version of Question 1.5. As a result, any locally compact group with weak Haagerup constant 1 embeds coarsely into a Hilbert space and hence the Baum-Connes assembly map with coefficients is split-injective for all these groups.

Theorem 1.12. *Let G be a σ-compact, locally compact group. The following are equivalent.*

(1) *G embeds coarsely into a Hilbert space.*
(2) *There exists a sequence of Schur multipliers $\varphi_n \colon G \times G \to \mathbb{C}$ such that*
   - $\|\varphi_n\|_S \leq 1$ *for every natural number n;*
   - *each φ_n tends to zero off tubes (Definition 6.1);*
   - $\varphi_n \to 1$ *uniformly on tubes.*

If any of these conditions holds, one can moreover arrange that the coarse embedding is continuous and that each φ_n is continuous.

From Theorem 1.12 together with [21, Theorem 3.5] we immediately obtain the following.

Corollary 1.13. *If G is a σ-compact, locally compact group with $\Lambda_{\mathrm{WH}}(G) = 1$, then G embeds coarsely into a Hilbert space. In particular, the Baum-Connes assembly map with coefficients is split-injective for all second countable, locally compact groups G with $\Lambda_{\mathrm{WH}}(G) = 1$.*

2. Preliminaries

Throughout, G will denote a locally compact group. A kernel $\varphi\colon G \times G \to \mathbb{C}$ is a *Schur multiplier* if there exist bounded maps $\xi, \eta\colon G \to \mathcal{H}$ into a Hilbert space $\mathcal{H}$ such that $\varphi(g,h) = \langle \xi(g), \eta(h) \rangle$ for every $g, h \in G$. The Schur norm of φ is defined as

$$\|\varphi\|_S = \inf\{\|\xi\|_\infty \|\eta\|_\infty\}$$

where the infimum is taken over all $\xi, \eta\colon G \to \mathcal{H}$ as above. See [47, Theorem 5.1] for different characterizations of Schur multipliers. Clearly, $\|\varphi \cdot \psi\|_S \leq \|\varphi\|_S \cdot \|\psi\|_S$ and $\|\check{\varphi}\|_S = \|\varphi\|_S$ when φ and ψ are Schur multipliers and $\check{\varphi}(x,y) = \varphi(y,x)$. Also, any positive definite kernel φ on G which is normalized, i.e., $\varphi(x,x) = 1$ for every $x \in G$, is a Schur multiplier of norm 1. The unit ball of Schur multipliers is closed under pointwise limits.

A continuous function $\varphi\colon G \to \mathbb{C}$ is a *Herz-Schur multiplier* if the associated kernel $\widehat{\varphi}(g,h) = \varphi(g^{-1}h)$ is a Schur multiplier. The Herz-Schur norm of φ is defined as $\|\varphi\|_{B_2} = \|\widehat{\varphi}\|_S$. When φ is a Herz-Schur multiplier, the two bounded maps $\xi, \eta\colon G \to \mathcal{H}$ can be chosen to be continuous. The set $B_2(G)$ of Herz-Schur multipliers on G is a unital Banach algebra under pointwise multiplication and $\|\cdot\|_\infty \leq \|\cdot\|_{B_2}$. Any continuous, positive definite function φ on G is a Herz-Schur multiplier with $\|\varphi\|_{B_2} = \varphi(1)$.

Below we list a number of permanence results concerning weak amenability and the weak Haagerup property, which will be useful later on. General references containing almost all of the results are [1], [17], [30] and [38]. Additionally we refer to [15, Theorem III.9] and [9, Corollary 12.3.12].

Suppose Γ_1 is a co-amenable subgroup of a discrete group Γ_2, that is, there exists a left Γ_2-invariant mean on $l^\infty(\Gamma_2/\Gamma_1)$. Then

$$\Lambda_{\mathrm{WA}}(\Gamma_1) = \Lambda_{\mathrm{WA}}(\Gamma_2). \tag{2.1}$$

If $(G_i)_{i \in I}$ is a directed family of open subgroups in a locally compact group G whose union is G, then

$$\Lambda_{\mathrm{WA}}(G) = \sup \Lambda_{\mathrm{WA}}(G_i). \tag{2.2}$$

For any two locally compact groups G and H

$$\Lambda_{\mathrm{WA}}(G \times H) = \Lambda_{\mathrm{WA}}(G)\Lambda_{\mathrm{WA}}(H). \tag{2.3}$$

When H is a closed subgroup of G

$$\Lambda_{\mathrm{WA}}(H) \leq \Lambda_{\mathrm{WA}}(G) \quad \text{and} \quad \Lambda_{\mathrm{WH}}(H) \leq \Lambda_{\mathrm{WH}}(G). \tag{2.4}$$

When K is a compact normal subgroup of G then

$$\Lambda_{\mathrm{WA}}(G/K) = \Lambda_{\mathrm{WA}}(G) \quad \text{and} \quad \Lambda_{\mathrm{WH}}(G/K) = \Lambda_{\mathrm{WH}}(G). \tag{2.5}$$

When Z is a central subgroup of a discrete group G then

$$\Lambda_{\mathrm{WA}}(G) \leq \Lambda_{\mathrm{WA}}(G/Z). \tag{2.6}$$

Recall that a *lattice* in a locally compact group G is a discrete subgroup Γ such that the quotient G/Γ admits a non-trivial finite G-invariant Radon measure. When Γ

is a lattice in a second countable, locally compact G then

$$\Lambda_{\mathrm{WA}}(\Gamma) = \Lambda_{\mathrm{WA}}(G) \quad \text{and} \quad \Lambda_{\mathrm{WH}}(\Gamma) = \Lambda_{\mathrm{WH}}(G). \tag{2.7}$$

When H is a finite index, closed subgroup in a group G then

$$\Lambda_{\mathrm{WH}}(H) = \Lambda_{\mathrm{WH}}(G). \tag{2.8}$$

3. Weak amenability of $\mathrm{GL}(2, K)$

This section is devoted to the proof of Theorem 1.11 (see Theorem 3.7 below). The general idea of our proof follows the idea of [26, Section 5], where it is shown that for any field K the discrete group $\mathrm{GL}(2, K)$ has the Haagerup property. Our proof of Theorem 1.11 also follows the same strategy as used in [28].

Recall that a *pseudo-length function* on a group G is a function $\ell \colon G \to [0, \infty)$ such that

- $\ell(e) = 0$,
- $\ell(g) = \ell(g^{-1})$,
- $\ell(g_1 g_2) \leq \ell(g_1) + \ell(g_2)$.

Moreover, ℓ is a length function on G if, in addition, $\ell(g) = 0 \implies g = e$.

Definition 3.1. We say that the pseudo-length group (G, ℓ) is *weakly amenable* if there exist a sequence (φ_n) of Herz-Schur multipliers on G and a sequence (R_n) of positive numbers such that

- $\sup_n \|\varphi_n\|_{B_2} < \infty$;
- $\operatorname{supp} \varphi_n \subseteq \{g \in G \mid \ell(g) \leq R_n\}$;
- $\varphi_n \to 1$ uniformly on $\{g \in G \mid \ell(g) \leq S\}$ for every $S > 0$.

The *weak amenability constant* $\Lambda_{\mathrm{WA}}(G, \ell)$ is defined as the best possible constant Λ such that $\sup_n \|\varphi_n\|_{B_2} \leq \Lambda$, where (φ_n) is as just described.

Notice that if the group G is discrete and the pseudo-length function l on G is proper (in particular, G is countable), then the weak amenability of (G, l) is equivalent to the weak amenability of G with same weak amenability constant. On other hand, every countable discrete group admits a proper length function, which is unique up to coarse equivalence ([53, Lemma 2.1]). If the group is finitely generated discrete, one can simply take the word-length function associated to any finite set of generators.

The next proposition is a variant of a well-known theorem, which follows from two classical results:

- The graph distance dist on a tree T is a conditionally negative definite kernel [29].
- The Schur multiplier associated with the characteristic function χ_n of the subset $\{(x, y) \in T^2 \mid \operatorname{dist}(x, y) = n\}$ has Schur norm at most $2n$ for every $n \in \mathbb{N}$ [8, Proposition 2.1].

The proof below is similar to the proof of [9, Corollary 12.3.5].

Proposition 3.2. *Suppose a group G acts isometrically on a tree T and that ℓ is a pseudo-length function on G. Suppose moreover* $\mathrm{dist}(g.v, v) \to \infty$ *if and only if* $\ell(g) \to \infty$ *for some (and hence every) vertex $v \in T$. Then* $\Lambda_{\mathrm{WA}}(G, \ell) = 1$.

Proof. Fix a vertex $v \in T$ as in the assumptions. For every $n \in \mathbb{N}$ we consider the functions $\psi_n(g) = \exp(-\frac{1}{n}\mathrm{dist}(g.v, v))$ and $\dot\chi_n(g) = \chi_n(g.v, v)$ defined for $g \in G$. Then

$$\dot\chi_m(g)\psi_n(g) = \exp(-m/n)\dot\chi_m(g)$$

holds for all $g \in G$ and every $n, m \in \mathbb{N}$. As G acts isometrically on T, each ψ_n is a unital positive definite function on G by Schoenberg's theorem and $\|\dot\chi_n\|_{B_2} \leq 2n$ for every $n \in \mathbb{N}$. It follows that $\|\psi_n\|_{B_2} = 1$ and $\|\dot\chi_m\psi_n\|_{B_2} \leq 2m \cdot \exp(-m/n)$ for every $n, m \in \mathbb{N}$. Therefore, for any $M \in \mathbb{N}$, we have

$$\left\| \sum_{m=0}^{M} \dot\chi_m\psi_n \right\|_{B_2} \leq \|\psi_n\|_{B_2} + \left\| \sum_{m>M} \dot\chi_m\psi_n \right\|_{B_2} \leq 1 + \sum_{m>M} 2m \cdot \exp(-m/n).$$

Hence, if we choose M_n suitably for all $n \in \mathbb{N}$, then the functions $\varphi_n = \sum_{m=0}^{M_n} \dot\chi_m\psi_n$ satisfy that $\|\varphi_n\|_{B_2} \leq 1 + \frac{1}{n}$ and $\operatorname{supp}\varphi_n \subseteq \{g \in G \mid \mathrm{dist}(g.v, v) \leq M_n\}$. The assumption

$$\mathrm{dist}(g.v, v) \to \infty \iff \ell(g) \to \infty$$

then insures that $\operatorname{supp}\varphi_n \subseteq \{g \in G \mid \ell(g) \leq R_n\}$ for some suitable R_n and that $\varphi_n \to 1$ uniformly on $\{g \in G \mid \ell(g) \leq S\}$ for every $S > 0$, as desired. □

Remark 3.3. The two classical results listed above have a generalization:

- The combinatorial distance dist on the 1-skeleton of a CAT(0) cube complex X is a conditionally negative definite kernel on the vertex set of X [42].
- The Schur multiplier associated with the characteristic function of the subset $\{(x, y) \in X^2 \mid \mathrm{dist}(x, y) = n\}$ has Schur norm at most $p(n)$ for every $n \in \mathbb{N}$, where p is a polynomial and X is (the vertex set of) a finite-dimensional CAT(0) cube complex [41, Theorem 2].

To see that these results are in fact generalizations, we only have to notice that a tree is exactly a one-dimensional CAT(0) cube complex, and in this case the combinatorial distance is just the graph distance. Because of these generalizations and the fact that the exponential function increases faster than any polynomial, it follows with the same proof as the proof of Proposition 3.2 that the following generalization is true (see also [41, Theorem 3]): suppose a group G acts cellularly (and hence isometrically) on a finite-dimensional CAT(0) cube complex X and that ℓ is a pseudo-length function on G. Suppose moreover $\mathrm{dist}(g.v, v) \to \infty$ if and only if $\ell(g) \to \infty$ for some (and hence every) vertex $v \in X$. Then $\Lambda_{\mathrm{WA}}(G, \ell) = 1$.

In our context, a *norm* on a field K is a map $d\colon K \to [0, \infty)$ satisfying, for all $x, y \in K$

(i) $d(x) = 0$ implies $x = 0$,
(ii) $d(xy) = d(x)d(y)$,
(iii) $d(x + y) \leq d(x) + d(y)$.

A norm obtained as the restriction of the usual absolute value on $\mathbb{C}$ via a field embedding $K \hookrightarrow \mathbb{C}$ is *archimedean.* A norm is *discrete* if the triangle inequality (iii) can be replaced by the stronger ultrametric inequality

(iii') $d(x+y) \leq \max\{d(x), d(y)\}$

and the range of d on $K^\times$ is a discrete subgroup of the multiplicative group $(0, \infty)$.

Theorem 3.4 ([26, Theorem 2.1])**.** *Every finitely generated field K is* discretely embeddable: *For every finitely generated subring A of K there exists a sequence of norms d_n on K, each either archimedean or discrete, such that for every sequence $R_n > 0$, the subset*

$$\{a \in A \mid d_n(a) \leq R_n \text{ for all } n \in \mathbb{N}\}$$

is finite.

Let d be a norm on a field K. Following Guentner, Higson and Weinberger [26] define a pseudo-length function ℓ_d on $\mathrm{GL}(n, K)$ as follows: if d is discrete

$$\ell_d(g) = \log \max_{i,j}\{d(g_{ij}), d(g^{ij})\},$$

where g_{ij} and g^{ij} are the matrix coefficients of g and g^{-1}, respectively; if d is archimedean, coming from an embedding of K into $\mathbb{C}$ then

$$\ell_d(g) = \log \max\{\|g\|, \|g^{-1}\|\},$$

where $\|\cdot\|$ is the operator norm of a matrix in $\mathrm{GL}(n, \mathbb{C})$.

Proposition 3.5. *Let d be an archimedean or a discrete norm on a field K. Then the pseudo-length group $(\mathrm{SL}(2, K), \ell_d)$ is weakly amenable with constant* 1*.*

Proof. The archimedean case: it is clear that the pseudo-length function on $\mathrm{SL}(2, K)$ is the restriction of that on $\mathrm{SL}(2, \mathbb{C})$, so clearly we only have to show $(\mathrm{SL}(2, \mathbb{C}), \ell_d)$ is weakly amenable with constant 1. Since ℓ_d is continuous and proper, this follows from the fact that $\mathrm{SL}(2, \mathbb{C})$ is weakly amenable with constant 1 as a locally compact group ([18, Remark 3.8]).

The discrete case: this is a direct application of [26, Lemma 5.9] and Proposition 3.2. Indeed, [26, Lemma 5.9] states that there exist a tree T and a vertex $v_0 \in T$ such that $\mathrm{SL}(2, K)$ acts isometrically on T and

$$\mathrm{dist}(g.v_0, v_0) = 2 \max_{i,j} -\frac{\log d(g_{ij})}{\log d(\pi)},$$

for all $g = [g_{ij}] \in \mathrm{SL}(2, K)$. Here dist is the graph distance on T and π is certain element of $\{x \in K \mid d(x) < 1\}$. Since the action is isometric, $\mathrm{dist}(g.v_0, v_0) \to \infty$ if and only if $\ell_d(g) \to \infty$. Hence, we are done by Proposition 3.2. ☐

Corollary 3.6. *Let K be a field and G a finitely generated subgroup of* $\mathrm{SL}(2, K)$*. Then there exists a sequence of pseudo-length functions ℓ_n on G such that $\Lambda_{\mathrm{WA}}(G, \ell_n) =$ 1 for every n, and such that for any sequence $R_n > 0$, the set $\bigcap_n \{g \in G \mid \ell_n(g) \leq R_n\}$ is finite.*

Proof. As G is finitely generated, we may assume that K is finitely generated as well. Now, let A be the finitely generated subring of K generated by the matrix coefficients of a finite generating set for G. Clearly, $G \subseteq \mathrm{SL}(2, A) \subseteq \mathrm{SL}(2, K)$.

Since K is discretely embeddable, we may choose a sequence of norms d_n on K according to Theorem 3.4. It follows from Proposition 3.5 that $\Lambda_{\mathrm{WA}}(G, \ell_{d_n}) = 1$. We complete the proof by observing that for any sequence $R_n > 0$,

$$\bigcap_n \{g \in G \mid \ell_{d_n}(g) \leq R_n\} \subseteq \mathrm{SL}(2, F),$$

where F is the finite set $\{a \in A \mid d_n(a) \leq \exp(R_n) \text{ for all } n \in \mathbb{N}\}$. □

Theorem 3.7. *Let K be a field. Every subgroup Γ of $\mathrm{GL}(2, K)$ is weakly amenable with constant 1 (as a discrete group).*

Proof. By the permanence results listed in Section 2 we can reduce our proof to the case where Γ is a finitely generated subgroup of $\mathrm{SL}(2, K)$. It then follows from the previous corollary that there exists a sequence ℓ_n of pseudo-length functions on Γ such that $\Lambda_{\mathrm{WA}}(\Gamma, \ell_n) = 1$ and for any sequence $R_n > 0$, the set $\bigcap_n \{g \in \Gamma \mid \ell_n(g) \leq R_n\}$ is finite.

For each fixed $n \in \mathbb{N}$ there is a sequence $(\varphi_{n,k})_k$ of Herz-Schur multipliers on Γ and a sequence of positive numbers $(R_{n,k})_k$ such that

(1) $\|\varphi_{n,k}\|_{B_2} \leq 1$ for all $k \in \mathbb{N}$;
(2) $\operatorname{supp} \varphi_{n,k} \subseteq \{g \in \Gamma \mid \ell_n(g) \leq R_{n,k}\}$;
(3) $\varphi_{n,k} \to 1$ uniformly on $\{g \in \Gamma \mid \ell_n(g) \leq S\}$ for every $S > 0$ as $k \to \infty$.

Upon replacing $\varphi_{n,k}$ by $|\varphi_{n,k}|^2$ we may further assume that $0 \leq \varphi_{n,k} \leq 1$ for all $n, k \in \mathbb{N}$.

Given any $\varepsilon > 0$ and any finite subset $F \subseteq \Gamma$, we choose a sequence $0 < \varepsilon_n < 1$ such that $\prod_n (1 - \varepsilon_n) > 1 - \varepsilon$. It follows from (3) that for each $n \in \mathbb{N}$ there exists $k_n \in \mathbb{N}$ such that $1 - \varepsilon_n < \varphi_{n,k_n}(g)$ for all $g \in F$. Consider the function $\varphi = \prod_n \varphi_{n,k_n}$. It is not hard to see that φ is well-defined, since $0 \leq \varphi_{n,k_n} \leq 1$. Additionally, since $\|\varphi_{n,k_n}\|_{B_2} \leq 1$ for all $n \in \mathbb{N}$ we also have $\|\varphi\|_{B_2} \leq 1$. Moreover, $\operatorname{supp} \varphi \subseteq \bigcap_n \{g \in \Gamma \mid \ell_n(g) \leq R_{n,k_n}\}$ and

$$\varphi(g) = \prod_n \varphi_{n,k_n}(g) > \prod_n (1 - \varepsilon_n) > 1 - \varepsilon$$

for all $g \in F$. This completes the proof. □

The remaining part of this section follows de Cornulier's idea from [12]. In [12] he proved the same results for Haagerup property, and the same argument actually works for weak amenability with constant 1.

Corollary 3.8. *Let R be a unital commutative ring without nilpotent elements. Then every subgroup Γ of $\mathrm{GL}(2, R)$ is weakly amenable with constant 1 (as a discrete group).*

Proof. Again by the permanence results in Section 2, we may assume that Γ is a finitely generated subgroup of $\mathrm{SL}(2, R)$, and hence that R is also finitely generated. It is well-known that every finitely generated ring is Noetherian and in such a ring there are only finitely many minimal prime ideals. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be the minimal prime ideals in R. The intersection of all minimal prime ideals is the set of nilpotent elements in R, which is trivial by our assumption. So R embeds into the finite product $\prod_{i=1}^n R/\mathfrak{p}_i$. If K_i denotes the fraction field of the integral domain $R/\mathfrak{p}_i$, then Γ embeds into $\mathrm{SL}(2, \prod_{i=1}^n K_i) = \prod_{i=1}^n \mathrm{SL}(2, K_i)$. Now, the result is a direct consequence of Theorem 3.7, (2.3) and (2.4). □

Remark 3.9. In the previous corollary and also in Theorem 3.7, the assumption about commutativity cannot be dropped. Indeed, the group $\mathrm{SL}(2, \mathbb{H})$ with the discrete topology is not weakly amenable, where $\mathbb{H}$ is the skew-field of quaternions. This can be seen from Theorem 1.10. Moreover, $\mathrm{SL}(2, \mathbb{H})_{\mathrm{d}}$ does not even have the weak Haagerup property by the same argument.

Remark 3.10. In the previous corollary, the assumption about the triviality of the nilradical cannot be dropped. Indeed, we show now that the group $\mathrm{SL}(2, \mathbb{Z}[x]/x^2)$ is not weakly amenable. The essential part of the argument is Dorofaeff's result that the locally compact group $\mathbb{R}^3 \rtimes \mathrm{SL}(2, \mathbb{R})$ is not weakly amenable [23]. Here the action $\mathrm{SL}(2, \mathbb{R}) \curvearrowright \mathbb{R}^3$ is the unique irreducible 3-dimensional representation of $\mathrm{SL}(2, \mathbb{R})$.

Consider the ring $R = \mathbb{R}[x]/x^2$. We write elements of R as polynomials $ax+b$ where $a, b \in \mathbb{R}$ and $x^2 = 0$. Consider the unital ring homomorphism $\varphi\colon R \to \mathbb{R}$ given by setting $x = 0$, that is, $\varphi(ax + b) = b$. Then φ induces a group homomorphism $\widetilde{\varphi}\colon \mathrm{SL}(2, R) \to \mathrm{SL}(2, \mathbb{R})$. Embedding $\mathbb{R} \subseteq R$ as constant polynomials, we obtain an embedding $\mathrm{SL}(2, \mathbb{R}) \subseteq \mathrm{SL}(2, R)$ showing that $\widetilde{\varphi}$ splits. The kernel of $\widetilde{\varphi}$ is easily identified as

$$\ker \widetilde{\varphi} = \left\{ \begin{pmatrix} a_{11}x + 1 & a_{12}x \\ a_{21}x & a_{22}x + 1 \end{pmatrix} \middle| a_{ij} \in \mathbb{R},\ a_{11} + a_{22} = 0 \right\} \simeq \mathfrak{sl}(2, \mathbb{R})$$

We deduce that $\mathrm{SL}(2, R)$ is the semidirect product $\mathfrak{sl}(2, \mathbb{R}) \rtimes \mathrm{SL}(2, \mathbb{R})$. A simple computation shows that the action $\mathrm{SL}(2, \mathbb{R}) \curvearrowright \mathfrak{sl}(2, \mathbb{R})$ is the adjoint action. Since $\mathfrak{sl}(2, \mathbb{R})$ is a simple Lie algebra, the adjoint action is irreducible. By uniqueness of the 3-dimensional irreducible representation of $\mathrm{SL}(2, \mathbb{R})$ (see [39, p. 107]) and from [23] we deduce that $\mathfrak{sl}(2, \mathbb{R}) \rtimes \mathrm{SL}(2, \mathbb{R}) \simeq \mathbb{R}^3 \rtimes \mathrm{SL}(2, \mathbb{R})$ is not weakly amenable.

It is easy to see that $\mathrm{SL}(2, \mathbb{Z}[x]/x^2)$ is identified with $\mathfrak{sl}(2, \mathbb{Z}) \rtimes \mathrm{SL}(2, \mathbb{Z})$ under the isomorphism $\mathrm{SL}(2, R) \simeq \mathfrak{sl}(2, \mathbb{R}) \rtimes \mathrm{SL}(2, \mathbb{R})$. Since $\mathfrak{sl}(2, \mathbb{Z}) \rtimes \mathrm{SL}(2, \mathbb{Z})$ is a lattice in $\mathfrak{sl}(2, \mathbb{R}) \rtimes \mathrm{SL}(2, \mathbb{R})$, we conclude from (2.7) that $\mathfrak{sl}(2, \mathbb{Z}) \rtimes \mathrm{SL}(2, \mathbb{Z})$ and hence $\mathrm{SL}(2, \mathbb{Z}[x]/x^2)$ is not weakly amenable.

Remark 3.11. We do not know if $\mathrm{SL}(2, \mathbb{Z}[x]/x^2)$ also fails to have the weak Haagerup property. As $\mathrm{SL}(2, \mathbb{Z}[x]/x^2)$ may be identified with a lattice in $\mathbb{R}^3 \rtimes \mathrm{SL}(2, \mathbb{R})$, by (2.7) the question is equivalent to the question [31, Remark 5.3] raised by Haagerup and the first author concerning the weak Haagerup property of the group $\mathbb{R}^3 \rtimes \mathrm{SL}(2, \mathbb{R})$.

Recall that a group Γ is residually free if for every $g \neq 1$ in Γ, there is a homomorphism f from Γ to a free group F such that $f(g) \neq 1$ in F. Equivalently, Γ embeds into a product of free groups of rank two. A group Γ is residually finite if for every $g \neq 1$ in Γ, there is a homomorphism f from Γ to a finite group F such that $f(g) \neq 1$ in F. Equivalently, Γ embeds into a product of finite groups. Since free groups are residually finite, it is clear that residually free groups are residually finite. On the other hand, residually finite groups need not be residually free as is easily seen by considering e.g. groups with torsion.

Corollary 3.12. *Every residually free group is weakly amenable with constant* 1.

Proof. Since the free group of rank two can be embedded in $\mathrm{SL}(2,\mathbb{Z})$, a residually free group embeds in $\prod_{i\in I}\mathrm{SL}(2,\mathbb{Z}) = \mathrm{SL}(2,\prod_{i\in I}\mathbb{Z})$ for a suitably large set I. We complete the proof by the previous corollary. □

4. Failure of the weak Haagerup property

In this section we will prove the following result, which is the combination of Corollaries 4.3, 4.5 and 4.6.

Proposition 4.1. *If G is one of the four groups $\mathrm{SO}(5)$, $\mathrm{SO}_0(1,4)$, $\mathrm{SU}(3)$ or $\mathrm{SU}(1,2)$, then G_{d} does not have the weak Haagerup property.*

Also, if G is the universal covering group of $\mathrm{SU}(1,n)$ where $n \geq 2$, then G_{d} does not have the weak Haagerup property.

When $p,q \geq 0$ are integers, not both zero, and $n = p+q$, we let $I_{p,q}$ denote the diagonal $n\times n$ matrix with 1 in the first p diagonal entries and -1 in the last q diagonal entries. When g is a complex matrix, g^t denotes the transpose of g, and g^* denotes the adjoint (conjugate transpose) of g. We recall that

$$\begin{aligned} \mathrm{SO}(p,q) &= \{g\in \mathrm{SL}(p+q,\mathbb{R}) \mid g^t I_{p,q} g = I_{p,q}\}\\ \mathrm{SO}(p,q,\mathbb{C}) &= \{g\in \mathrm{SL}(p+q,\mathbb{C}) \mid g^t I_{p,q} g = I_{p,q}\}\\ \mathrm{SU}(p,q) &= \{g\in \mathrm{SL}(p+q,\mathbb{C}) \mid g^* I_{p,q} g = I_{p,q}\}. \end{aligned}$$

When $p,q>0$, the group $\mathrm{SO}(p,q)$ has two connected components, and $\mathrm{SO}_0(p,q)$ denotes the identity component.

To prove Proposition 4.1, we follow a strategy that we have learned from de Cornulier [13], where the same techniques are applied in connection with the Haagerup property. The idea of the proof is the following. We consider the groups as real algebraic groups $G(\mathbb{R})$. Let K be a number field of degree three over $\mathbb{Q}$, not totally real, and let $\mathcal{O}$ be its ring of integers. Then $G(\mathcal{O})$ embeds diagonally as a lattice in $G(\mathbb{R})\times G(\mathbb{C})$. The group $G(\mathbb{C})$ will have real rank two, and we deduce that the group $G(\mathcal{O})$ does not have the weak Haagerup property by combining [31, Theorem B] with (2.7). As $G(\mathcal{O})$ is a subgroup of $G(\mathbb{R})$, (2.4) implies that $G(\mathbb{R})$ also does not have the weak Haagerup property, and we are done. We will now make this argument more precise.

Let K denote the field $\mathbb{Q}(\sqrt[3]{2})$ and $\mathcal{O}$ its ring of integers $\mathbb{Z}[\sqrt[3]{2}]$. Let $\omega = e^{2\pi i/3}$ be a third root of unity and let $\sigma\colon K\to\mathbb{C}$ be the field monomorphism uniquely defined by $\sigma(\sqrt[3]{2}) = \omega\sqrt[3]{2}$. If we denote the image of σ by K^σ, then σ induces a ring isomorphism, also denoted σ, of matrix algebras

$$\sigma\colon M_n(K)\to M_n(K^\sigma) \tag{4.1}$$

by applying σ entry-wise.

The field K is an algebra over $\mathbb{Q}$ with basis 1, $2^{1/3}$, $2^{2/3}$. With respect to this basis, multiplication is given by

$$\begin{pmatrix} a_1\\ b_1\\ c_1\end{pmatrix} \circ \begin{pmatrix} a_2\\ b_2\\ c_2\end{pmatrix} = \begin{pmatrix} a_1a_2+2b_1c_2+2c_1b_2\\ a_1b_2+b_1a_2+2c_1c_2\\ a_1c_2+b_1b_2+c_1a_2\end{pmatrix} \tag{4.2}$$

where $a_i, b_i, c_i \in \mathbb{Q}$ and $i = 1, 2$. Multiplication by an element $x = a+2^{1/3}b+2^{2/3}c \in K$ where $a, b, c \in \mathbb{Q}$ defines an endomorphism of K, and it is clear from (4.2) that the matrix representation of x is

$$\begin{pmatrix} a & 2c & 2b \\ b & a & 2c \\ c & b & a \end{pmatrix}. \tag{4.3}$$

If $\pi(x)$ denotes the matrix in (4.3) then $\pi\colon K \to M_3(\mathbb{Q})$ is an algebra homomorphism.

4.1. **The real case.** Let A be the $\mathbb{R}$-algebra $\mathbb{R}^3$ with multiplication $\circ$ given by (4.2) where $a_i, b_i, c_i \in \mathbb{R}$ and $i = 1, 2$. The unit of A is $(1, 0, 0)$. Let $\xi_1\colon A \to \mathbb{R}$ and $\xi_2\colon A \to \mathbb{C}$ be the algebra homomorphisms defined by

$$\xi_1(a, b, c) = a + 2^{1/3}b + 2^{2/3}c, \quad \text{and} \quad \xi_2(a, b, c) = a + \omega 2^{1/3}b + \varpi 2^{2/3}c, \tag{4.4}$$

where $a, b, c \in \mathbb{R}$. It is easily verified that $\xi = (\xi_1, \xi_2)$ is an algebra isomorphism of A onto $\mathbb{R} \oplus \mathbb{C}$.

More generally, we define $\xi_1^n\colon M_n(A) \to M_n(\mathbb{R})$ and $\xi_2^n\colon M_n(A) \to M_n(\mathbb{C})$ by

$$\xi_i^n([x_{jk}]) = [\xi_i(x_{jk})] \quad \text{when } [x_{jk}] \in M_n(A)$$

for $i = 1, 2$, and we let $\xi^n = (\xi_1^n, \xi_2^n)$. It follows that ξ^n is an $\mathbb{R}$-algebra isomorphism of $M_n(A)$ onto $M_n(\mathbb{R}) \oplus M_n(\mathbb{C})$. We also denote the multiplication in $M_n(A)$ by $\circ$. We note that ξ^n preserves transposition and the determinant in the sense that for every $x \in M_n(A)$

$$\xi^n(x^t) = \xi^n(x)^t \quad \text{and} \quad \det_{\mathbb{R}\oplus\mathbb{C}} \xi^n(x) = \xi(\det_A x).$$

Proposition 4.2. *Let $p, q \geq 0$ be integers with $p+q \geq 3$. If σ is the homomorphism in (4.1), then the homomorphism $1\times\sigma$ embeds the group $\mathrm{SO}(p, q, \mathbb{Z}[\sqrt[3]{2}])$ as a lattice in $\mathrm{SO}(p, q) \times \mathrm{SO}(p, q, \mathbb{C})$.*

Proof. We use the notation introduced before Proposition 4.2. We will show that

$$\Lambda = \{(l, \sigma(l)) \in \mathrm{SO}(p, q) \times \mathrm{SO}(p, q, \mathbb{C}) \mid l \in \mathrm{SO}(p, q, \mathcal{O})\}$$

is a lattice in $\mathrm{SO}(p, q) \times \mathrm{SO}(p, q, \mathbb{C})$.

We put $n = p+q$. Let H be the group consisting of matrices $(a, b, c) \in M_n(A)$ such that

$$(a^t, b^t, c^t) \circ (a, b, c) = (I_{p,q}, 0, 0) \quad \text{and} \quad \det_A[(a, b, c)] = (1, 0, 0). \tag{4.5}$$

Observe that

$$\xi^n(I_{p,q}, 0, 0) = (I_{p,q}, I_{p,q}).$$

Then $(a, b, c) \in H$ if and only if

$$\xi^n(a, b, c)^t \xi^n(a, b, c) = (I_{p,q}, I_{p,q}) \quad \text{and} \quad \det_{\mathbb{R}\oplus\mathbb{C}} \xi^n(a, b, c) = (1, 1),$$

that is, if and only if $\xi^n(a, b, c)$ belongs to $\mathrm{SO}(p, q) \times \mathrm{SO}(p, q, \mathbb{C})$. Thus, ξ^n is a group isomorphism of H onto $\mathrm{SO}(p, q) \times \mathrm{SO}(p, q, \mathbb{C})$.

The next idea is to identify H with an algebraic subgroup of $M_{3n}(\mathbb{R})$ by adopting the matrix representation (4.3) of K. Let $\pi\colon M_n(A) \to M_{3n}(\mathbb{R})$ be the map sending $(a,b,c) \in M_n(A)$ to

$$\begin{pmatrix} a & 2c & 2b \\ b & a & 2c \\ c & b & a \end{pmatrix} \tag{4.6}$$

where $a,b,c \in M_n(\mathbb{R})$. It is not hard to see that π is an injective ring homomorphism.

We let $G = \pi(H)$. Then G is the subgroup of $\mathrm{SL}(3n,\mathbb{R})$ consisting of matrices of the form (4.6), where $a,b,c \in M_n(\mathbb{R})$ satisfies the relations (4.5). The crucial point is that the definition (4.2) of the multiplication $\circ$ in A is given by integral polynomials in the entries, and hence the relations (4.5) are polynomial equations in the entries of a,b,c with integral coefficients. This shows that G is an algebraic subgroup of $\mathrm{SL}(3n,\mathbb{R})$ defined over $\mathbb{Q}$. Moreover, $\rho = \xi^n \circ \pi^{-1}$ is a group isomorphism of G onto $\mathrm{SO}(p,q) \times \mathrm{SO}(p,q,\mathbb{C})$, which is also a diffeomorphism. Since $\mathrm{SO}(p,q) \times \mathrm{SO}(p,q,\mathbb{C})$ is semisimple (here we use $p+q \geq 3$), we deduce that G is semisimple.

By the Borel Harish-Chandra Theorem [7, Theorem 7.8], the subgroup $G_{\mathbb{Z}} = \mathrm{SL}(3n,\mathbb{Z}) \cap G$ is a lattice in G, and hence $\rho(G_{\mathbb{Z}})$ is a lattice in $\mathrm{SO}(p,q) \times \mathrm{SO}(p,q,\mathbb{C})$. It remains to show that $\rho(G_{\mathbb{Z}}) = \Lambda$.

Suppose first that $g \in G_{\mathbb{Z}}$ is of the form (4.6) and put $l = \xi_1^n \circ \pi^{-1}(g) = \xi_1^n(a,b,c)$. Then $l \in \mathrm{SO}(p,q,\mathcal{O})$ and $\xi_2^n(a,b,c) = \sigma(l)$. This shows that $\rho(g) = (l,\sigma(l)) \in \Lambda$.

Conversely, given $(l,\sigma(l)) \in \Lambda$ where $l \in \mathrm{SO}(p,q,\mathcal{O})$ we can in a unique way write $l = a + 2^{1/3}b + 2^{2/3}c = \xi_1^n(a,b,c)$ where $a,b,c \in M_n(\mathbb{Z})$. Then $\sigma(l) = \xi_2^n(a,b,c)$ and if we define g by (4.6) then $g \in G_{\mathbb{Z}}$ and $\rho(g) = l$.

This proves that $\Lambda = \rho(G_{\mathbb{Z}})$, and the proof is complete. □

Corollary 4.3. *If G is* $\mathrm{SO}(5)$ *or* $\mathrm{SO}_0(1,4)$*, then G_{d} does not have the weak Haagerup property.*

Proof. The Lie group $\mathrm{SO}(5,\mathbb{C})$ has real rank two (see Table IV of [34, Ch.X §6]). It is thus a consequence of [31, Theorem B] that $\mathrm{SO}(5,\mathbb{C})$ does not have the weak Haagerup property.

Suppose $(p,q) = (5,0)$ or $(p,q) = (1,4)$ and let $\Gamma = \mathrm{SO}(p,q,\mathbb{Z}[\sqrt[3]{2}])$. As $\mathrm{SO}(p,q,\mathbb{C}) \simeq \mathrm{SO}(p+q,\mathbb{C})$, we see that $\mathrm{SO}(p,q) \times \mathrm{SO}(p,q,\mathbb{C})$ does not have the weak Haagerup property. Since Γ is embedded via $1 \times \sigma$ as a lattice in $\mathrm{SO}(p,q) \times \mathrm{SO}(p,q,\mathbb{C})$, it follows from (2.7) that Γ does not have the weak Haagerup property. Since Γ is a subgroup of $\mathrm{SO}(p,q)$, we conclude that $\mathrm{SO}(p,q)_{\mathrm{d}}$ does not have the weak Haagerup property.

We have now shown that $\mathrm{SO}(5)_{\mathrm{d}}$ and $\mathrm{SO}(1,4)_{\mathrm{d}}$ do not have the weak Haagerup property. To finish the proof, recall that the group $\mathrm{SO}_0(1,4)$ has index two in $\mathrm{SO}(1,4)$, so that by (2.8) we conclude that $\mathrm{SO}_0(1,4)_{\mathrm{d}}$ also does not have the weak Haagerup property. □

4.2. **The complex case.** To prove that $\mathrm{SU}(3)$ and $\mathrm{SU}(2,1)$ do not have the weak Haagerup property we use the same technique as before, but in a complex version.

Let $K = \mathbb{Q}(\sqrt[3]{2}, i)$ and $\mathcal{O} = \mathbb{Z}[\sqrt[3]{2}, i]$, and let $\sigma\colon K \to \mathbb{C}$ be the field homomorphism defined by

$$\sigma(\sqrt[3]{2}) = \omega\sqrt[3]{2}, \quad \sigma(i) = i.$$

We also use σ to denote the ring homomorphism

$$\sigma\colon M_n(K) \to M_n(\mathbb{C}) \tag{4.7}$$

obtained by applying σ entry-wise.

Let A be the $\mathbb{C}$-algebra $\mathbb{C}^3$ with multiplication $\circ$ given by (4.2) where $a_i, b_i, c_i \in \mathbb{C}$ and $i = 1, 2$. Let $\xi_1, \xi_2, \xi_3\colon A \to \mathbb{C}$ be the algebra homomorphisms defined by

$$\begin{aligned} \xi_1(a, b, c) &= a + 2^{1/3}b + 2^{2/3}c, \\ \xi_2(a, b, c) &= a + \omega 2^{1/3}b + \overline{\omega}2^{2/3}c, \\ \xi_3(a, b, c) &= a + \overline{\omega}2^{1/3}b + \omega 2^{2/3}c. \end{aligned} \tag{4.8}$$

Then it is easily verified that $\xi = (\xi_1, \xi_2, \xi_3)$ is an isomorphism of A onto $\mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}$. More generally, for $i = 1, 2, 3$ we define $\xi_i^n\colon M_n(A) \to M_n(\mathbb{C})$ by

$$\xi_i^n([x_{jk}]) = [\xi_i(x_{jk})] \quad \text{when } [x_{jk}] \in M_n(A)$$

and let $\xi^n = (\xi_1^n, \xi_2^n, \xi_3^n)$. It follows that ξ^n is a $\mathbb{C}$-algebra isomorphism of $M_n(A)$ onto $M_n(\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C})$. Multiplication in $M_n(A)$ is also denoted by $\circ$. Elements of $M_n(A)$ are thought of as triples (a, b, c), where $a, b, c \in M_n(\mathbb{C})$. We note that for every (a, b, c) in $M_n(A)$

$$(\xi_1^n, \xi_2^n, \xi_3^n)(a^*, b^*, c^*) = ((\xi_1^n, \xi_3^n, \xi_2^n)(a, b, c))^* \tag{4.9}$$

and

$$\det_{\mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}} \xi^n(a, b, c) = \xi(\det_A(a, b, c)). \tag{4.10}$$

Warning: did you notice the index switch in (4.9)?

Proposition 4.4. *Let $p, q \geq 0$ be integers with $p + q \geq 2$. If σ is as in (4.7), then the homomorphism $1 \times \sigma$ embeds the group $\mathrm{SU}(p, q, \mathbb{Z}[\sqrt[3]{2}, i])$ as a lattice in $\mathrm{SU}(p, q) \times \mathrm{SL}(p + q, \mathbb{C})$.*

Proof. Put $n = p + q$. Let H be the group consisting of matrices $(a, b, c) \in M_n(A)$ such that

$$(a^*, b^*, c^*) \circ (a, b, c) = (I_{p,q}, 0, 0) \quad \text{and} \quad \det_A[(a, b, c)] = (1, 0, 0). \tag{4.11}$$

Observe that

$$\xi^n(I_{p,q}, 0, 0) = (I_{p,q}, I_{p,q}, I_{p,q}).$$

Using (4.9)-(4.10) we see that $(a, b, c) \in H$ if and only if

$$((\xi_1^n, \xi_3^n, \xi_2^n)(a, b, c))^*(\xi_1^n, \xi_2^n, \xi_3^n)(a, b, c) = (I_{p,q}, I_{p,q}, I_{p,q})$$

and

$$\det_{\mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}} \xi^n(a, b, c) = (1, 1, 1).$$

Thus, ξ^n is a group isomorphism of H onto the group L consisting of matrices $(s, z, w) \in M_n(\mathbb{C})^3$ such that

$$s^*s = I_{p,q}, \quad z^*w = I_{p,q}, \quad w^*z = I_{p,q}, \quad \det_{\mathbb{C}} s = \det_{\mathbb{C}} z = \det_{\mathbb{C}} w = 1.$$

It is easily seen that L is in fact

$$L = \{(s, z, (z^*)^{-1}I_{p,q}) \in M_n(\mathbb{C})^3 \mid s \in \mathrm{SU}(p, q),\ z \in \mathrm{SL}(n, \mathbb{C})\}.$$

Let $\eta\colon L \to \mathrm{SU}(p,q) \times \mathrm{SL}(n,\mathbb{C})$ be the isomorphism of L onto $\mathrm{SU}(p,q) \times \mathrm{SL}(n,\mathbb{C})$ given by $\eta(s,z,(z^*)^{-1}) = (s,z)$. Let $\pi\colon M_n(A) \to M_{3n}(\mathbb{C})$ be the map sending $(a,b,c) \in M_n(A)$ to

$$\begin{pmatrix} a & 2c & 2b \\ b & a & 2c \\ c & b & a \end{pmatrix} \tag{4.12}$$

where $a,b,c \in M_n(\mathbb{C})$. It is not hard to see that π is an injective ring homomorphism. We let $G = \pi(H)$. Then G is the subgroup of $\mathrm{SL}(3n,\mathbb{C})$ consisting of matrices of the form (4.12), where $a,b,c \in M_n(\mathbb{C})$ satisfies the relations (4.11). The crucial point is that the definition (4.2) of the multiplication $\circ$ in A is given by integral polynomials in the entries, and hence the relations (4.11) are polynomial equations in the real and imaginary parts of the entries of a,b,c with integral coefficients. Using the standard embedding of $\mathrm{SL}(3n,\mathbb{C})$ into $\mathrm{SL}(6n,\mathbb{R})$ (see e.g. [36, p. 60]) we have then realized G as an algebraic subgroup of $\mathrm{SL}(6n,\mathbb{R})$ defined over $\mathbb{Q}$. Moreover, $\rho = \eta\circ\xi^n\circ\pi^{-1}$ is a group isomorphism of G onto $\mathrm{SU}(p,q)\times\mathrm{SL}(n,\mathbb{C})$, which is also a diffeomorphism. Since $\mathrm{SU}(p,q) \times \mathrm{SL}(n,\mathbb{C})$ is semisimple (here we use $p+q \geq 2$), we deduce that G is semisimple.

By the Borel Harish-Chandra Theorem, the subgroup $G_{\mathbb{Z}+i\mathbb{Z}} = \mathrm{SL}(3n,\mathbb{Z}[i]) \cap G$ is a lattice in G, and hence $\rho(G_{\mathbb{Z}+i\mathbb{Z}})$ is a lattice in $\mathrm{SU}(p,q) \times \mathrm{SL}(n,\mathbb{C})$.

We will finish the proof by showing that

$$\Lambda = \{(l,\sigma(l)) \in \mathrm{SU}(p,q) \times \mathrm{SL}(n,\mathbb{C}) \mid l \in \mathrm{SU}(p,q,\mathcal{O})\}$$

coincides with $\rho(G_{\mathbb{Z}+i\mathbb{Z}})$.

Suppose first that $g \in G_{\mathbb{Z}+i\mathbb{Z}}$ is of the form (4.12) and put $l = \xi_1^n \circ \pi^{-1}(g) = \xi_1^n(a,b,c)$. Then $l \in \mathrm{SU}(p,q,\mathcal{O})$ and $\xi_2^n(a,b,c) = \sigma(l)$. This shows that $\rho(g) = (l,\sigma(l)) \in \Lambda$.

Conversely, given $(l,\sigma(l)) \in \Lambda$ where $l \in \mathrm{SU}(p,q,\mathcal{O})$ we can in a unique way write $l = a+2^{1/3}b+2^{2/3}c = \xi_1^n(a,b,c)$ where $a,b,c \in M_n(\mathbb{Z}+i\mathbb{Z})$. Then $\sigma(l) = \xi_2^n(a,b,c)$ and if we define g by (4.12) then $g \in G_{\mathbb{Z}+i\mathbb{Z}}$ and $\rho(g) = l$.

This proves that $\Lambda = \rho(G_{\mathbb{Z}+i\mathbb{Z}})$, and the proof is complete. □

Corollary 4.5. *If G is $\mathrm{SU}(3)$ or $\mathrm{SU}(1,2)$, then G_{d} does not have the weak Haagerup property.*

Proof. The Lie group $\mathrm{SL}(3,\mathbb{C})$ has real rank two (see Table IV of [34, Ch.X §6]). It is thus a consequence of [31, Theorem B] that $\mathrm{SL}(3,\mathbb{C})$ does not have the weak Haagerup property.

Suppose $(p,q) = (3,0)$ or $(p,q) = (1,2)$ and let $\Gamma = \mathrm{SU}(p,q,\mathbb{Z}[\sqrt[3]{2}])$. Since Γ is embedded via $1 \times \sigma$ as a lattice in $\mathrm{SU}(p,q) \times \mathrm{SL}(3,\mathbb{C})$, it follows from (2.7) that Γ does not have the weak Haagerup property. Since Γ is a subgroup of $\mathrm{SU}(p,q)$, we conclude that $\mathrm{SU}(p,q)_{\mathrm{d}}$ does not have the weak Haagerup property. This completes the proof. □

Corollary 4.6. *Let $\widetilde{G}$ be the universal covering group $\widetilde{\mathrm{SU}}(1,n)$ of $\mathrm{SU}(1,n)$ where $n \geq 2$. Then $\widetilde{G}_{\mathrm{d}}$ does not have the weak Haagerup property.*

Proof. Let $G = \mathrm{SU}(1, n)$, and let $q\colon \widetilde{G} \to G$ be the covering homomorphism. If Γ denotes the image of $\mathrm{SU}(p, q, \mathbb{Z}[\sqrt[3]{2}])$ under $1 \times \sigma$, then Γ is a lattice in $G \times \mathrm{SL}(n+1, \mathbb{C})$. Let $\widetilde{\Gamma}$ be the lift of Γ to $\widetilde{G} \times \mathrm{SL}(n+1, \mathbb{C})$, that is, $\widetilde{\Gamma} = (q \times 1)^{-1}(\Gamma)$. Since $q \times 1$ is a covering homomorphism $\widetilde{G} \times \mathrm{SL}(n+1, \mathbb{C}) \to G \times \mathrm{SL}(n+1, \mathbb{C})$, it is then easy to check that $\widetilde{\Gamma}$ is a lattice in $\widetilde{G} \times \mathrm{SL}(n+1, \mathbb{C})$. The rest of the proof is now similar to the previous proof.

The Lie group $\mathrm{SL}(n+1, \mathbb{C})$ has real rank n (see Table IV of [34, Ch.X §6]). It is thus a consequence of [31, Theorem B] that $\mathrm{SL}(n+1, \mathbb{C})$ does not have the weak Haagerup property. It follows from (2.7) that $\widetilde{\Gamma}$ does not have the weak Haagerup property. The projection $\widetilde{G} \times \mathrm{SL}(n+1, \mathbb{C}) \to \widetilde{G}$ is injective on $\widetilde{\Gamma}$, and hence $\widetilde{\Gamma}$ embeds as a subgroup of $\widetilde{G}$. We conclude that $\widetilde{G}_{\mathrm{d}}$ does not have the weak Haagerup property.

□

5. Proof of the Main Theorem

In this section we prove Theorem 1.10. The theorem is basically a consequence of Theorem 1.11 and Proposition 4.1 together with the permanence results listed in Section 2 and general structure theory of simple Lie groups.

We recall that two Lie groups G and H are *locally isomorphic* if there exist open neighborhoods U and V around the identity elements of G and H, respectively, and an analytic diffeomorphism $f\colon U \to V$ such that

- if $x, y, xy \in U$ then $f(xy) = f(x)f(y)$;
- if $x, y, xy \in V$ then $f^{-1}(xy) = f^{-1}(x)f^{-1}(y)$.

When two Lie groups G and H are locally isomorphic we write $G \approx H$. An important fact about Lie groups and local isomorphims is the following [34, Theorem II.1.11]: Two Lie groups are locally isomorphic if and only if their Lie algebras are isomorphic.

The following is extracted from [11, Chapter II] and [36, Section I.11] to which we refer for details. If G is a connected Lie group, there exists a connected, simply connected Lie group $\widetilde{G}$ and a covering homomorphism $\widetilde{G} \to G$. The kernel of the covering homomorphism is a discrete, central subgroup of $\widetilde{G}$, and it is isomorphic to the fundamental group of G. The group $\widetilde{G}$ is called the *universal covering group* of G. Clearly, $\widetilde{G}$ and G are locally isomorphic. Conversely, any connected Lie group locally isomorphic to G is the quotient of $\widetilde{G}$ by a discrete, central subgroup. If N is a discrete subgroup of the center $Z(\widetilde{G})$ of $\widetilde{G}$, then the center of $\widetilde{G}/N$ is $Z(\widetilde{G})/N$.

Let G_1 and G_2 be locally compact groups. We say that G_1 and G_2 are *strongly locally isomorphic*, if there exist a locally compact group G and finite normal subgroups N_1 and N_2 of G such that $G_1 \simeq G/N_1$ and $G_2 \simeq G/N_2$. In this case we write $G_1 \sim G_2$. It follows from (2.5) that if $G \sim H$, then $\Lambda_{\mathrm{WH}}(G_{\mathrm{d}}) = \Lambda_{\mathrm{WH}}(H_{\mathrm{d}})$.

A theorem due to Weyl states that a connected, simple, compact Lie group has a compact universal cover with finite center [35, Theorem 12.1.17], [34, Theorem II.6.9]. Thus, for connected, simple, compact Lie groups G and H, $G \approx H$ implies $G \sim H$.

Proof of Theorem 1.10. Let G be a connected simple Lie group. As mentioned, the equivalence (1) $\iff$ (2) was already done by de Cornulier [13, Theorem 1.14] in a much more general setting, so we leave out the proof of this part. We only prove the two implications (1) $\implies$ (3) and (6) $\implies$ (1), since the remaining implications then follow trivially.

Suppose (1) holds, that is, G is locally isomorphic to $\mathrm{SO}(3)$, $\mathrm{SL}(2,\mathbb{R})$ or $\mathrm{SL}(2,\mathbb{C})$. If Z denotes the center of G, then by assumption G/Z is isomorphic to $\mathrm{SO}(3)$, $\mathrm{PSL}(2,\mathbb{R})$ or $\mathrm{PSL}(2,\mathbb{C})$. It follows from Theorem 1.11 and (2.5) that the groups $\mathrm{SO}(3)$, $\mathrm{PSL}(2,\mathbb{R})$ and $\mathrm{PSL}(2,\mathbb{C})$ equipped with the discrete topology are weakly amenable with constant 1 (recall that $\mathrm{SO}(3)$ is a subgroup of $\mathrm{PSL}(2,\mathbb{C})$). From (2.6) we deduce that G_{d} is weakly amenable with constant 1. This proves (3).

Suppose (1) does not hold. We prove that (6) fails, that is, G_{d} does not have the weak Haagerup property. We divide the proof into several cases depending on the real rank of G. We recall that with the Iwasawa decomposition $G = KAN$, the real rank of G is the dimension of the abelian group A.

If the real rank of G is at least two, then G does not have the weak Haagerup property [31, Theorem B]. By a theorem of Borel, G contains a lattice (see [49, Theorem 14.1]), and by (2.7) the lattice also does not have the weak Haagerup property. We conclude that G_{d} does not have the weak Haagerup property.

If the real rank of G equals one, then the Lie algebra of G is isomorphic to a Lie algebra in the list [36, (6.109)]. See also [34, Ch.X §6]. In other words, G is locally isomorphic to one of the classical groups $\mathrm{SO}_0(1,n)$, $\mathrm{SU}(1,n)$, $\mathrm{Sp}(1,n)$ for some $n \geq 2$ or locally isomorphic to the exceptional group $\mathrm{F}_{4(-20)}$. Here $\mathrm{SO}_0(1,n)$ denotes the identity component of the group $\mathrm{SO}(1,n)$.

We claim that the universal covering groups of $\mathrm{SO}_0(1,n)$, $\mathrm{Sp}(1,n)$ and $\mathrm{F}_{4(-20)}$ have finite center except for the group $\mathrm{SO}_0(1,2)$. Indeed, $\mathrm{Sp}(1,n)$ and $\mathrm{F}_{4(-20)}$ are already simply connected with finite center. The K-group from the Iwasawa decomposition of $\mathrm{SO}_0(1,n)$ is $\mathrm{SO}(n)$ which has fundamental group of order two, except when $n = 2$, and hence $\mathrm{SO}_0(1,n)$ has fundamental group of order two as well. As the center of the universal cover is an extension of the center of $\mathrm{SO}_0(1,n)$ by the fundamental group of $\mathrm{SO}_0(1,n)$, the claim follows.

The universal covering group $\widetilde{\mathrm{SU}}(1,n)$ of $\mathrm{SU}(1,n)$ has infinite center isomorphic to the group of integers.

We have assumed that G is not locally isomorphic to $\mathrm{SL}(2,\mathbb{R}) \sim \mathrm{SO}_0(1,2)$ or $\mathrm{SL}(2,\mathbb{C}) \sim \mathrm{SO}_0(1,3)$. If G has finite center, it follows that G is strongly locally isomorphic to one of the groups

$$\begin{array}{ll} \mathrm{SO}_0(1,n), & n \geq 4, \\ \mathrm{SU}(1,n), & n \geq 2, \\ \mathrm{Sp}(1,n), & n \geq 2, \\ \mathrm{F}_{4(-20)}, & \end{array}$$

and if G has infinite center, then G is isomorphic to $\widetilde{\mathrm{SU}}(1,n)$. Clearly, there are inclusions

$$\mathrm{SO}_0(1,4) \subseteq \mathrm{SO}_0(1,n),\ n \geq 4,$$
$$\mathrm{SU}(1,2) \subseteq \mathrm{SU}(1,n),\ n \geq 2,$$
$$\mathrm{SU}(1,2) \subseteq \mathrm{Sp}(1,n),\ n \geq 2.$$

The cases where G is strongly locally isomorphic to $\mathrm{SO}_0(1,n)$, $\mathrm{SU}(1,n)$ or $\mathrm{Sp}(1,n)$ are then covered by Proposition 4.1. Since $\mathrm{SO}(5) \subseteq \mathrm{SO}(9) \sim \mathrm{Spin}(9) \subseteq \mathrm{F}_{4(-20)}$ ([52, §.4.Proposition 1]), the case where $G \sim \mathrm{F}_{4(-20)}$ is also covered by Proposition 4.1. Finally, if $G \simeq \widetilde{\mathrm{SU}}(1,n)$, then Proposition 4.1 shows that G_{d} does not have weak Haagerup property.

If the real rank of G is zero, then it is a fairly easy consequence of [35, Theorem 12.1.17] that G is compact. Moreover, the universal covering group of G is compact and with finite center.

By the classification of compact simple Lie groups as in Table IV of [34, Ch.X §6] we know that G is strongly locally isomorphic to one of the groups $\mathrm{SU}(n+1)$ $(n \geq 1)$, $\mathrm{SO}(2n+1)$ $(n \geq 2)$, $\mathrm{Sp}(n)$ $(n \geq 3)$, $\mathrm{SO}(2n)$ $(n \geq 4)$ or one of the five exceptional groups

$$E_6,\ E_7,\ E_8,\ F_4,\ G_2.$$

By assumption G is not strongly locally isomorphic to $\mathrm{SU}(2) \sim \mathrm{SO}(3)$. Using (2.5) it then suffices to show that if G equals any other group in the list, then G_{d} does not have the weak Haagerup property. Clearly, there are inclusions

$$\mathrm{SO}(5) \subseteq \mathrm{SO}(n),\ n \geq 5,$$
$$\mathrm{SU}(3) \subseteq \mathrm{SU}(n),\ n \geq 3,$$
$$\mathrm{SU}(3) \subseteq \mathrm{Sp}(n),\ n \geq 3.$$

Since we also have the following inclusions among Lie algebras (Table V of [34, Ch.X §6])

$$\mathfrak{so}(5) \subseteq \mathfrak{so}(9) \subseteq \mathfrak{f}_4 \subseteq \mathfrak{e}_6 \subseteq \mathfrak{e}_7 \subseteq \mathfrak{e}_8$$

and the inclusion ([54])

$$\mathrm{SU}(3) \subseteq G_2,$$

it is enough to consider the cases where $G = \mathrm{SO}(5)$ or $G = \mathrm{SU}(3)$. These two cases are covered by Proposition 4.1. Hence we have argued that also in the real rank zero case G_{d} does not have the weak Haagerup property. □

6. A Schur multiplier characterization of coarse embeddability

In this section we give a characterization of coarse embeddability into Hilbert spaces in terms of contractive Schur multipliers. It is well-known that the notion of coarse embeddability into Hilbert spaces can be characterized by positive definite kernels (see [27, Theorem 2.3] for the discrete case and [20, Theorem 1.5] for the locally compact case).

If G is a locally compact group, a *(left) tube* in $G \times G$ is a subset of $G \times G$ contained in a set of the form

$$\mathrm{Tube}(K) = \{(x,y) \in G \times G \mid x^{-1}y \in K\}$$

where K is a compact subset of G.

Definition 6.1. A kernel $\varphi\colon G \times G \to \mathbb{C}$ *tends to zero off tubes*, if for any $\varepsilon > 0$ there is a tube $T \subseteq G \times G$ such that $|\varphi(x,y)| < \varepsilon$ whenever $(x,y) \notin T$.

Note that if $\varphi\colon G \to \mathbb{C}$ is a function, then φ vanishes at infinity, if and only if the associated kernel $\widehat{\varphi}\colon G \times G$ defined by $\widehat{\varphi}(x,y) = \varphi(x^{-1}y)$ tends to zero off tubes.

Definition 6.2 ([2, Definition 3.6])**.** Let G be a σ-compact, locally compact group. A map u from G into a Hilbert space H is said to be a *coarse embedding* if u satisfies the following two conditions:

- for every compact subset K of G there exists $R > 0$ such that
$$(s,t) \in \mathrm{Tube}(K) \implies \|u(s) - u(t)\| \leq R;$$
- for every $R > 0$ there exists a compact subset K of G such that
$$\|u(s) - u(t)\| \leq R \implies (s,t) \in \mathrm{Tube}(K).$$

We say that a group G *embeds coarsely into a Hilbert space* or *admits a coarse embedding into a Hilbert space* if there exist a Hilbert space H and a coarse embedding $u : G \to H$.

Every second countable, locally compact group G admits a proper left-invariant metric d, which is unique up to coarse equivalence (see [51] and [32]). So the preceding definition is equivalent to Gromov's notion of coarse embeddability of the metric space (G, d) into Hilbert spaces. We refer to [21, Section 3] for more on coarse embeddability into Hilbert spaces for locally compact groups).

It is not hard to see that the countability assumption in [37, Proposition 4.3] is superfluous. We thus record the following (slightly more general) version of [37, Proposition 4.3].

Lemma 6.3. *Let G be a group with a symmetric kernel $k\colon G \times G \to [0,\infty)$. The following are equivalent.*

1. *For every $t > 0$ one has $\|e^{-tk}\|_S \leq 1$.*
2. *There exist a real Hilbert space $\mathcal{H}$ and maps $R, S\colon G \to \mathcal{H}$ such that*
$$k(x,y) = \|R(x) - R(y)\|^2 + \|S(x) + S(y)\|^2 \quad \text{for every } x, y \in G.$$

Recall that a kernel $k\colon G\times G \to \mathbb{R}$ is *conditionally negative definite* if k is symmetric ($k(x,y) = k(y,x)$), vanishes on the diagonal ($k(x,x) = 0$) and
$$\sum_{i,j=1}^{n} c_i c_j k(x_i, x_j) \leq 0$$
for any finite sequences $x_1, \dots, x_n \in G$ and $c_1, \dots, c_n \in \mathbb{R}$ such that $\sum_{i=1}^{n} c_i = 0$. It is well-known that k is conditionally negative definite if and only if there is a function u from G to a real Hilbert space such that $k(x,y) = \|u(x) - u(y)\|^2$.

If G is a locally compact group we say that a kernel $k\colon G \times G \to \mathbb{C}$ is *proper*, if the set $\{(x,y) \in G \times G \mid |k(x,y)| \leq R\}$ is a tube for every $R > 0$.

Theorem 1.12 is contained in the following theorem, which extends both [22, Theorem 5.3] and [20, Theorem 1.5] in different directions.

Theorem 6.4. *Let G be a σ-compact, locally compact group. The following are equivalent.*

(1) *The group G embeds coarsely into a Hilbert space.*
(2) *There exists a sequence of (not necessarily continuous) Schur multipliers $\varphi_n\colon G \times G \to \mathbb{C}$ such that*
   - $\|\varphi_n\|_S \leq 1$ *for every natural number n;*
   - *each φ_n tends to zero off tubes;*
   - $\varphi_n \to 1$ *uniformly on tubes.*

(3) *There exists a (not necessarily continuous) symmetric kernel $k\colon G \times G \to [0, \infty)$ which is proper, bounded on tubes and satisfies $\|e^{-tk}\|_S \leq 1$ for all $t > 0$.*
(4) *There exists a (not necessarily continuous) conditionally negative definite kernel $h\colon G \times G \to \mathbb{R}$ which is proper and bounded on tubes.*

Moreover, if any of these conditions holds, one can arrange that the coarse embedding in (1)*, each Schur multiplier φ_n in* (2)*, the symmetric kernel k in* (3) *and the conditionally negative definite kernel h in* (4) *are continuous.*

Proof. We show (1) $\iff$ (4) $\iff$ (3) $\iff$ (2).

That (1) implies (4) with h continuous follows directly from [21, Theorem 3.4].

That (4) implies (3) follows from Schoenberg's Theorem and the fact that normalized positive definite kernels are Schur multipliers of norm 1.

Suppose (3) holds. We show that (4) holds. From Lemma 6.3 we see that there are a real Hilbert space $\mathcal{H}$ and maps $R, S\colon G \to \mathcal{H}$ such that

$$k(x, y) = \|R(x) - R(y)\|^2 + \|S(x) + S(y)\|^2 \quad \text{for every } x, y \in G.$$

As k is bounded on tubes, the map S is bounded. If we let

$$h(x, y) = \|R(x) - R(y)\|^2,$$

then it is easily checked that h is proper and bounded on tubes, since k has these properties and S is bounded. It is also clear that h is conditionally negative definite. Thus (4) holds.

Suppose now that (4) holds. By the GNS construction there are a real Hilbert space $\mathcal{H}$ and a map $u\colon G \to \mathcal{H}$ such that

$$h(x, y) = \|u(x) - u(y)\|^2.$$

It is easy to check that the assumptions on h imply that u is a coarse embedding. Thus (1) holds.

If (3) holds, we set $\varphi_n = e^{-k/n}$ when $n \in \mathbb{N}$. It is easy to check that the sequence φ_n has the desired properties so that (2) holds.

Conversely, suppose (2) holds. We verify (3). Essentially, we use the same standard argument as in the proof of [38, Proposition 4.4] and [10, Theorem 2.1.1].

Since G is locally compact and σ-compact, it is the union of an increasing sequence $(U_n)_{n=1}^{\infty}$ of open sets such that the closure K_n of U_n is compact and contained in U_{n+1} (see [25, Proposition 4.39]). Fix an increasing, unbounded sequence (α_n) of positive real numbers and a decreasing sequence (ε_n) tending to zero such that

$\sum_n \alpha_n \varepsilon_n$ converges. By assumption, for every n we can find a Schur multiplier φ_n tending to zero off tubes and such that $\|\varphi_n\|_S \leq 1$ and

$$\sup_{(x,y)\in \mathrm{Tube}(K_n)} |\varphi_n(x,y) - 1| \leq \varepsilon_n/2.$$

Upon replacing φ_n by $|\varphi_n|^2$ one can arrange that $0 \leq \varphi_n \leq 1$ and

$$\sup_{(x,y)\in \mathrm{Tube}(K_n)} |\varphi_n(x,y) - 1| \leq \varepsilon_n.$$

Define kernels $\psi_i : G \times G \to [0,\infty[$ and $\psi : G \times G \to [0,\infty[$ by

$$\psi_i(x,y) = \sum_{n=1}^{i} \alpha_n(1 - \varphi_n(x,y)), \qquad \psi(x,y) = \sum_{n=1}^{\infty} \alpha_n(1 - \varphi_n(x,y)).$$

It is easy to see that ψ is well-defined, bounded on tubes and $\psi_i \to \psi$ pointwise (even uniformly on tubes, but we do not need that).

To see that ψ is proper, let $R > 0$ be given. Choose n large enough such that $\alpha_n \geq 2R$. As φ_n tends to zero off tubes, there is a compact set $K \subseteq G$ such that $|\varphi_n(x,y)| < 1/2$ whenever $(x,y) \notin \mathrm{Tube}(K)$. Now if $\psi(x,y) \leq R$, then $\psi(x,y) \leq \alpha_n/2$, and in particular $\alpha_n(1 - \varphi_n(x,y)) \leq \alpha_n/2$, which implies that $1 - \varphi_n(x,y) \leq 1/2$. We have thus shown that

$$\{(x,y) \in G\times G \mid \psi(x,y) \leq R\} \subseteq \{(x,y) \in G\times G \mid 1-\varphi_n(x,y) \leq 1/2\} \subseteq \mathrm{Tube}(K),$$

and ψ is proper.

We now show that $\|e^{-t\psi}\|_S \leq 1$ for every $t > 0$. Since ψ_i converges pointwise to ψ, it will suffice to prove that $\|e^{-t\psi_i}\|_S \leq 1$, because the set of Schur multipliers of norm at most 1 is closed under pointwise limits. Since

$$e^{-t\psi_i} = \prod_{n=1}^{i} e^{-t\alpha_n(1-\varphi_n)},$$

it is enough to show that $e^{-t\alpha_n(1-\varphi_n)}$ has Schur norm at most 1 for each n. And this is clear:

$$\|e^{-t\alpha_n(1-\varphi_n)}\|_S = e^{-t\alpha_n}\|e^{t\alpha_n\varphi_n}\|_S \leq e^{-t\alpha_n}e^{t\alpha_n\|\varphi_n\|_S} \leq 1.$$

The only thing missing is that ψ need not be symmetric. Put $k = \psi + \check{\psi}$ where $\check{\psi}(x,y) = \psi(y,x)$. Clearly, k is symmetric, bounded on tubes and proper. Finally, for every $t > 0$

$$\|e^{-tk}\|_S \leq \|e^{-t\psi}\|_S\|e^{-t\check{\psi}}\|_S \leq 1,$$

since $\|\check{\varphi}\|_S = \|\varphi\|_S$ for every Schur multiplier φ.

Finally, the statements about continuity follow from [21, Theorem 3.4] and the explicit constructions used in our proof of (1) $\Longrightarrow$ (4) $\Longrightarrow$ (3) $\Longrightarrow$ (2). □

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