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PhD Thesis

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The Mathematics of Charged Particles interacting with Electromagnetic Fields

This thesis has been submitted to the PhD School of The Faculty of Science,
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Contents

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Summary

The mathematical problems treated in this thesis concern the dynamics of charged quantum mechanical particles coupled with their classical self-generated electromagnetic field. This coupling suggests that we not only describe the physical system by the Schrödinger equation of quantum mechanics but also by the Maxwell equations of classical electrodynamics. Through the years experiments have to a very large extent confirmed that each of these fundamental physical laws serves as an accurate description of reality, but the coupling of them raises some interesting mathematical questions – in this thesis we consider two of these questions.

Does the initial value problem associated with the coupled system of equations at all have a mathematical solution? We interpret an affirmative answer as an explanation that macroscopic matter – consisting of several charged particles – can exist and evolve in time (assuming that nature strives to fulfill the Maxwell equations and the Schrödinger equation). In [55], we prove the unique existence of a local in time solution to the many-body Maxwell-Schrödinger initial value problem expressed in Coulomb gauge.

Of course the actual particle motions predicted by the coupled system also ought to correspond with our expectations. For instance we expect a single charged particle to be able to travel in space at a constant velocity so certainly there ought to exist a solution to the coupled system describing this kind of motion. In the joint work [56] with Jan Philip Solovej, we prove the existence of such travelling wave solutions to the one-body Maxwell-Schrödinger system and likewise to the related one-body Maxwell-Pauli system – both of them expressed in Coulomb gauge. Finally, we prove that the effective mass of the particle equals it's bare mass.

Dansk oversættelse

De i denne afhandling behandlede matematiske problemstillinger drejer sig om dynamikken af ladede kvantemekaniske partikler koblet til deres klassiske, selv-genererede elektromagnetiske felt. Denne kobling giver os grund til ikke kun at beskrive det fysiske system ved hjælp af kvantemekanikkens Schrödingerligning, men også ved hjælp af den klassiske elektrodynamiks Maxwell-ligninger. Gennem tiderne har eksperimenter i meget høj grad bekræftet, at hver af disse fundamentale fysiske love fungerer som en akkurat beskrivelse af virkeligheden, men koblingen af dem foranlediger nogle interessante matematiske spørgsmål – i denne afhandling vil vi betragte to af disse spørgsmål.

Har begyndelsesværdiproblemet hørende til det koblede ligningssystem overhovedet en matematisk løsning? Vi fortolker et bekræftende svar som en forklaring på, at makroskopisk stof – bestående af adskillige partikler – kan eksistere og udvikle sig i tiden (under antagelse af, at naturen søger at opfylde Maxwell-ligningerne og Schrödingerligningen). I [55] beviser vi den entydige eksistens af en tidslokal løsning til mange-legeme Maxwell-Schrödinger begyndelsesværdiproblemet udtrykt i Coulomb gauge.

Naturligvis bør de af det koblede system forudsagte partikelbevægelser også stemme overens med vore forventninger. Eksempelvis forventer vi, at en enkelt ladet partikel er i stand til at bevæge sig i rummet med konstant hastighed, så der bør helt bestemt findes en løsning til det koblede ligningssystem, som beskriver denne type bevægelse. I samarbejdet [56] med Jan Philip Solovej beviser vi eksistensen af sådanne solitære bølge-løsninger til en-partikel Maxwell-Schrödinger systemet og tilsvarende til det relaterede en-legeme Maxwell-Pauli system – begge udtrykt i Coulomb gauge. Endelig beviser vi, at partiklens effektive masse er lig med dens bare masse.

Abstract

In this thesis, we study the mathematics used to describe systems of charged quantum mechanical particles coupled with their classical self-generated electromagnetic field. We prove the existence of a unique local in time solution to the many-body Maxwell-Schrödinger initial value problem expressed in Coulomb gauge and we show that the one-body Maxwell-Schrödinger system as well as the related one-body Maxwell-Pauli system both admit travelling wave solutions.

Dansk oversættelse

I denne afhandling studerer vi den matematik, som benyttes til at beskrive systemer af ladede kvantemekaniske partikler koblet til deres klassiske selv-genererede elektromagnetiske felt. Vi beviser den entydige eksistens af en tidslokal løsning til mange-legeme Maxwell-Schrödinger begyndelsesværdi-problemet udtrykt i Coulomb gauge og vi viser, at en-legeme Maxwell-Schrödinger systemet samt det relaterede en-legeme Maxwell-Pauli system har solitære bølge-løsninger.

1 Introduction

Why can matter exist in the form we observe around us every day? Matter is made up of a wealth of charged particles that are all influenced by forces – some of them originating from the Coulomb interactions between the particles and some of them originating from the electromagnetic fields induced by the movement of the particles themselves – so it seems rather remarkable that this enormous system can settle into a stable state. Even the much simpler question concerning stability of the hydrogen atom is quite complex – it can not be answered by means of classical electrodynamics alone and in fact, the explanation of the hydrogen atom's stability is one of the celebrated results of quantum mechanics. In this thesis we will consider a quantum mechanical model for a system of $N \in \mathbb{N}$ nonrelativistic dynamic particles with positive masses $m_1, \ldots, m_N$ and nonzero charges $Q_1, \ldots, Q_N$. In addition, we think of $M \in \mathbb{N}_0$ infinitely heavy nuclei with atomic numbers $Z_1, \ldots, Z_M$ and fixed distinct positions $\boldsymbol{R}_1, \ldots, \boldsymbol{R}_M \in \mathbb{R}^3$ as being present in the system – the configuration of these nuclei will for notational convenience be denoted by $\boldsymbol{R} = (\boldsymbol{R}_1, \ldots, \boldsymbol{R}_M)$. Our main objective will be to investigate the coupling of such a system with some classical electromagnetic field $(\boldsymbol{E}, \nabla \times \boldsymbol{A})$, where the electromagnetic potential is chosen so that the Coulomb gauge condition $\operatorname{div}\boldsymbol{A} = 0$ is satisfied. Suppose that each of the N dynamic particles have $\nu \in \mathbb{N}$ internal (spin-)degrees of freedom so that the possible quantum mechanical states of the system are described by vectors in $\bigotimes^N [L^2(\mathbb{R}^3)]^\nu$, a space that is naturally isomorphic to $[L^2(\mathbb{R}^{3N})]^{\nu^N}$

by the unitary operator U sending a given pure tensor product

$$\psi_1 \otimes \cdots \otimes \psi_N = \begin{pmatrix} \psi_1^1 \\ \vdots \\ \psi_1^\nu \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} \psi_N^1 \\ \vdots \\ \psi_N^\nu \end{pmatrix}$$

into the $[L^2(\mathbb{R}^{3N})]^{\nu^N}$-function with components given by

$$U^{\boldsymbol{s}}(\psi_1 \otimes \cdots \otimes \psi_N)(\boldsymbol{x}) = \psi_1^{s_1}(\boldsymbol{x}_1) \cdots \psi_N^{s_N}(\boldsymbol{x}_N)$$

for $\boldsymbol{s} = (s_1, \ldots, s_N) \in \{1, \ldots, \nu\}^N$ and $\boldsymbol{x} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \in \mathbb{R}^{3N}$. Throughout this thesis we restrict our attention to the cases $\nu = 1$ or $\nu = 2$, corresponding to the situations where either all of the particles are spinless or all of the particles have spin $\frac{1}{2}$.

The total energy of the physical system is in Gaussian units described by the (formal) operator

$$\mathscr{H}\big(\boldsymbol{A}, -\tfrac{P\boldsymbol{E}}{4\pi}\big) = \sum_{j=1}^{N} \mathscr{T}_j[\boldsymbol{A}] + V_C + \mathscr{E}_{\mathrm{EM}}[\boldsymbol{A}, -c^2 P\boldsymbol{E}], \tag{1.1}$$

on the quantum mechanical state space, where $P = 1 - \nabla \mathrm{div} \Delta^{-1}$ is the Helmholtz projection onto divergence free vector fields, $\mathscr{T}_j[\boldsymbol{A}]$ denotes for $j \in \{1, \ldots, N\}$ the kinetic energy of the j'th particle, the electromagnetic field energy $\mathscr{E}_{\mathrm{EM}}[\boldsymbol{A}, -c^2 P\boldsymbol{E}]$ is

$$\mathscr{E}_{\mathrm{EM}}\big[\boldsymbol{A}, -c^2 P\boldsymbol{E}\big] = \frac{1}{8\pi} \int_{\mathbb{R}^3} \big(|\nabla \times \boldsymbol{A}(\boldsymbol{y})|^2 + c^2 |P\boldsymbol{E}(\boldsymbol{y})|^2\big)\, \mathrm{d}\boldsymbol{y}$$

and V_C is the potential energy

$$V_C(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) = \sum_{1 \le j < k \le N} \frac{Q_j Q_k}{|\boldsymbol{x}_j - \boldsymbol{x}_k|} + \sum_{j=1}^{N} \sum_{k=1}^{M} \frac{Q_j Z_k e}{|\boldsymbol{x}_j - \boldsymbol{R}_k|} + \sum_{1 \le j < k \le M} \frac{Z_j Z_k e^2}{|\boldsymbol{R}_j - \boldsymbol{R}_k|}.$$

Here, $c, e > 0$ denote the speed of light and the elementary charge. The three sums in the expression for V_C represent the particle-particle, particle-nucleon and nucleon-nucleon Coulomb interactions. To specify the expression for the kinetic energy operator let $\hbar > 0$ be the reduced Planck constant and let $\boldsymbol{\sigma}$ denote the 3-vector with the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

as components. Then the operator $\mathscr{T}_j[\boldsymbol{A}]$ is often chosen to be either the magnetic Laplacian $\frac{1}{2m_j}\big(i\hbar\nabla_{\boldsymbol{x}_j} + \frac{Q_j}{c}\boldsymbol{A}(\boldsymbol{x}_j)\big)^2$ or – if $\nu = 2$ – the Pauli operator $\frac{1}{2m_j}\big(\boldsymbol{\sigma}\cdot\big(i\hbar\nabla_{\boldsymbol{x}_j} + \frac{Q_j}{c}\boldsymbol{A}(\boldsymbol{x}_j)\big)\big)^2$. The former choice describes the energy originating from the coupling between the magnetic field and the orbital motion of the particles, whereas the latter choice also takes the interactions between the magnetic field and the spin of the particles into account, as can be read off from the Lichnerowicz formula

$$\Big(\boldsymbol{\sigma}\cdot\Big(i\hbar\nabla_{\boldsymbol{x}_j} + \frac{Q_j}{c}\boldsymbol{A}(\boldsymbol{x}_j)\Big)\Big)^2 = \Big(i\hbar\nabla_{\boldsymbol{x}_j} + \frac{Q_j}{c}\boldsymbol{A}(\boldsymbol{x}_j)\Big)^2 - \frac{\hbar Q_j}{c}\boldsymbol{\sigma}\cdot\nabla\times\boldsymbol{A}(\boldsymbol{x}_j). \tag{1.2}$$

The last term on the right hand side of (1.2) is often referred to as the *Zeeman term*. We will always describe all N dynamic particles by the same type of kinetic energy so either we choose all of the operators $\mathscr{T}_1[\boldsymbol{A}],\ldots,\mathscr{T}_N[\boldsymbol{A}]$ as magnetic Laplacians or else we choose them all as Pauli operators.

There are of course several different ways to formulate rigorous criteria expressing stability of the physical system described above. First, we consider an approach where stability is expressed as a condition on the total energy of the system – this notion of stability will therefore be called *energetic stability*. We will give a brief introduction to this concept below, but for an in-depth and very educational discussion of energetic stability and related topics we refer the interested reader to [44].

Energetic Stability of Matter

Whether or not the system is energetically stable – in the sense introduced below – turns out to be significally dependent on the statistics of the dynamic particles involved in the model. Let us consider the physically relevant case where all of the N dynamic particles are (indistinguishable) electrons so that $m_1 = \cdots = m_N = m$ and $Q_1 = \cdots = Q_N = -e$ for some $m > 0$. The fermionic nature of the electrons is reflected in the *Pauli exclusion principle* according to which the wave function ψ is *totally* *antisymmetric*, meaning that

$$\begin{aligned}&\psi^{(s_1,\ldots,s_j,\ldots,s_k,\ldots,s_N)}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_j,\ldots,\boldsymbol{x}_k,\ldots,\boldsymbol{x}_N)\\ &= -\psi^{(s_1,\ldots,s_k,\ldots,s_j,\ldots,s_N)}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_k,\ldots,\boldsymbol{x}_j,\ldots,\boldsymbol{x}_N)\end{aligned} \tag{1.3}$$

for all $j < k$, $(s_1, \ldots, s_N) \in \{1, \ldots, \nu\}^N$ and $(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \in \mathbb{R}^{3N}$. For comparison, boson wave functions have the property (1.3) with the minus sign being absent. The square integrable, totally antisymmetric functions form a closed subspace $\bigwedge^N [L^2(\mathbb{R}^3)]^\nu$ of $\bigotimes^N [L^2(\mathbb{R}^3)]^\nu$ – thereby $\bigwedge^N [L^2(\mathbb{R}^3)]^\nu$ is itself a Hilbert space with the inner product inherited from $\bigotimes^N [L^2(\mathbb{R}^3)]^\nu$.

Stability of the First Kind

The principle of minimum energy says that in a closed system (with constant entropy) the internal energy will always decrease and approach the least possible value. This suggests a natural criterion for stability of a physical system, namely boundedness from below of the total energy. We formalize this idea in the following way. Given any $(\boldsymbol{E}, \boldsymbol{A}) \in L^2(\mathbb{R}^3; \mathbb{R}^3) \times L^4_{\mathrm{loc}}(\mathbb{R}^3; \mathbb{R}^3)$ with $\nabla \times \boldsymbol{A} \in L^2(\mathbb{R}^3; \mathbb{R}^3)$ and $\mathrm{div}\boldsymbol{A} \in L^2_{\mathrm{loc}}(\mathbb{R}^3; \mathbb{R})$ we realize (1.1) as a symmetric unbounded operator $\mathscr{H}\big(\boldsymbol{A}, -\frac{P\boldsymbol{E}}{4\pi}\big)$ in $\bigwedge^N [L^2(\mathbb{R}^3)]^\nu$ with dense domain $[C_0^\infty(\mathbb{R}^{3N})]^{\nu^N} \cap \bigwedge^N [L^2(\mathbb{R}^3)]^\nu$. Suppose that the quantity

$$\begin{aligned}E_N^M(\boldsymbol{R}) &= \inf_{\substack{\psi\in[C_0^\infty(\mathbb{R}^{3N})]^{\nu^N}\cap\bigwedge^N[L^2(\mathbb{R}^3)]^\nu,\|\psi\|_{\bigwedge^N[L^2]^\nu}=1\\ \boldsymbol{A}\in L^4_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{R}^3),\nabla\times\boldsymbol{A}\in L^2(\mathbb{R}^3;\mathbb{R}^3),\mathrm{div}\boldsymbol{A}\in L^2_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{R})\\ \boldsymbol{E}\in L^2(\mathbb{R}^3;\mathbb{R}^3)}} \Big(\psi, \mathscr{H}\big(\boldsymbol{A}, -\tfrac{P\boldsymbol{E}}{4\pi}\big)\psi\Big)_{\bigwedge^N[L^2]^\nu}\\ &= \inf_{\substack{\psi\in[C_0^\infty(\mathbb{R}^{3N})]^{\nu^N}\cap\bigwedge^N[L^2(\mathbb{R}^3)]^\nu,\|\psi\|_{\bigwedge^N[L^2]^\nu}=1\\ \boldsymbol{A}\in L^4_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{R}^3),\nabla\times\boldsymbol{A}\in L^2(\mathbb{R}^3;\mathbb{R}^3),\mathrm{div}\boldsymbol{A}\in L^2_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{R})}} \big(\psi, \mathscr{H}(\boldsymbol{A}, \boldsymbol{0})\psi\big)_{\bigwedge^N[L^2]^\nu}\end{aligned}$$

is finite. This means that all of the operators $\mathscr{H}\big(\boldsymbol{A}, -\frac{P\boldsymbol{E}}{4\pi}\big)$ are bounded from below, uniformly in $\big(\boldsymbol{A}, -\frac{P\boldsymbol{E}}{4\pi}\big)$, whereby their Friedrichs extensions are well defined and have the common lower bound $E_N^M(\boldsymbol{R})$. Being selfadjoint operators these extensions can serve as energy observables and with this understanding we can interpret $E_N^M(\boldsymbol{R})$ as the least possible energy the system can achieve, no matter which electromagnetic field the particles are being exposed to. A system satisfying the condition $E_N^M(\boldsymbol{R}) > -\infty$ for all vectors $\boldsymbol{R} = (\boldsymbol{R}_1, \ldots, \boldsymbol{R}_M) \in \mathbb{R}^{3M}$ with distinct coordinates is thus said to be *stable of the first kind* and $E_N^M(\boldsymbol{R})$ is called the ground state energy associated with the configuration $\boldsymbol{R}$ of the nuclei.

When the kinetic energies of the N particles are measured by means of the magnetic Laplacian one can use the diamagnetic and Sobolev inequalities to dominate the average potential energy $(\psi, V_C\psi)_{\bigotimes^N [L^2]^\nu}$ by the average kinetic energy, thus obtaining stability of the first kind even if one subtracts the nonnegative field energy term from the Hamiltonian. It turns out that things change dramatically when spin-effects are taken into consideration by letting the Pauli operator represent the kinetic energies of the particles. To see this let us consider the simple situation of one particle and one oppositely charged nucleus.

Example (Hydrogenic atom with Pauli kinetic energy). Consider a system of $N = 1$ electron and $M = 1$ infinitely heavy nucleus fixed at the origin $\boldsymbol{R}_1 = \boldsymbol{0}$ with atomic number $Z_1 = Z$ – such a system is called a hydrogenic atom. When the kinetic energy of the electron is measured by means of the Pauli operator the total energy of the electron-nucleus pair is described by the unbounded operator

$$\mathscr{H}\big(\boldsymbol{A}, -\tfrac{P\boldsymbol{E}}{4\pi}\big) = \frac{1}{2m}\Big(\boldsymbol{\sigma}\cdot\Big(i\hbar\nabla - \frac{e}{c}\boldsymbol{A}\Big)\Big)^2 - \frac{Ze^2}{|\boldsymbol{x}|} + \mathscr{E}_{\mathrm{EM}}[\boldsymbol{A}, -c^2P\boldsymbol{E}]$$

in the Hilbert space $[L^2(\mathbb{R}^3)]^2$ with domain $[C_0^\infty(\mathbb{R}^3)]^2$. Showing that this system is stable of the first kind is a matter of controlling $-\frac{Ze^2}{|\boldsymbol{x}|}$ by the two nonnegative terms $\frac{1}{2m}\big(\boldsymbol{\sigma}\cdot\big(i\hbar\nabla - \frac{e}{c}\boldsymbol{A}\big)\big)^2$ and $\mathscr{E}_{\mathrm{EM}}[\boldsymbol{A}, -c^2P\boldsymbol{E}]$. For this purpose the kinetic energy term is not particularly useful because unlike the magnetic Laplacian the Pauli operator has *zero modes*, i.e. nontrivial states with zero kinetic energy. Were it not for the presence of the field energy term in the Hamiltonian this property of the Pauli operator would in fact imply instability of *all* hydrogenic atom-systems with Pauli kinetic energy – as will be apparent below the field energy term is strong enough to stabilize the system in some (but not all) cases.

For an arbitrary normalized $\phi_0 \in \mathbb{C}^2$ introduce $\boldsymbol{w} = \langle\phi_0, \boldsymbol{\sigma}\phi_0\rangle_{\mathbb{C}^2}$ and set

$$\begin{aligned}&(\psi, \boldsymbol{A})(\boldsymbol{x})\\ &= \left(\frac{1}{\pi}\frac{1 + i\boldsymbol{\sigma}\cdot\boldsymbol{x}}{(1+\boldsymbol{x}^2)^{\frac{3}{2}}}\phi_0, \frac{3}{(1+\boldsymbol{x}^2)^2}\big((1-\boldsymbol{x}^2)\boldsymbol{w} + 2(\boldsymbol{w}\cdot\boldsymbol{x})\boldsymbol{x} + 2\boldsymbol{w}\times\boldsymbol{x}\big)\right).\end{aligned} \tag{1.4}$$

The pair $(\psi, \boldsymbol{A}) \in [C^\infty(\mathbb{R}^3)]^2 \times C^\infty(\mathbb{R}^3; \mathbb{R}^3)$ is a zero mode for the Pauli operator in the sense that

$$\boldsymbol{\sigma} \cdot (i\nabla + \boldsymbol{A})\psi = 0$$

and $\psi, \nabla \times \boldsymbol{A}$ are both square integrable – in fact, (1.4) was the first of the Pauli operator's zero modes that was discovered (see Loss and Yau [51]). We now scale and regularize the zero mode, i.e. for any $\ell \in \mathbb{N}$ and sufficiently large $n \in \mathbb{N}$ we set

$$(\psi_{\ell,n}, \boldsymbol{A}_\ell)(\boldsymbol{x}) = \left(\frac{\ell^{\frac{3}{2}} \chi\big(\frac{\boldsymbol{x}}{n}\big)\psi(\ell \boldsymbol{x})}{\big(\int_{\mathbb{R}^3} \big|\chi\big(\frac{\boldsymbol{y}}{\ell n}\big)\big|^2 |\psi(\boldsymbol{y})|^2 \,\mathrm{d}\boldsymbol{y}\big)^{\frac{1}{2}}}, -\frac{\hbar c}{e} \ell \boldsymbol{A}(\ell \boldsymbol{x}) \right),$$

where χ is any cut-off function with

$$\chi(\boldsymbol{x}) \begin{cases} = 1 & \text{if } |\boldsymbol{x}| \leq 1 \\ \in [0,1] & \text{if } 1 < |\boldsymbol{x}| < 2 \\ = 0 & \text{if } |\boldsymbol{x}| \geq 2 \end{cases}.$$

For any ℓ and large enough n the pair $(\psi_{\ell,n}, \boldsymbol{A}_\ell) \in [C_0^\infty(\mathbb{R}^3)]^2 \times C^\infty(\mathbb{R}^3;\mathbb{R}^3)$ then satisfies $\|\psi_{\ell,n}\|_{L^2} = 1$, $\nabla \times \boldsymbol{A}_\ell \in L^2(\mathbb{R}^3;\mathbb{R}^3)$ and

$$\begin{aligned}\big(\psi_{\ell,n}, \mathscr{H}\big(\boldsymbol{A}_\ell, \boldsymbol{0}\big)\psi_{\ell,n}\big)_{L^2} &= \frac{\hbar^2}{2mn^2} \frac{\int_{\mathbb{R}^3} \big|\nabla\chi\big(\frac{\boldsymbol{y}}{\ell n}\big)\big|^2 |\psi(\boldsymbol{y})|^2 \,\mathrm{d}\boldsymbol{y}}{\int_{\mathbb{R}^3} \big|\chi\big(\frac{\boldsymbol{y}}{\ell n}\big)\big|^2 |\psi(\boldsymbol{y})|^2 \,\mathrm{d}\boldsymbol{y}} + \frac{\ell \hbar^2 c^2}{e^2} \mathscr{E}_{\mathrm{EM}}[\boldsymbol{A}, \boldsymbol{0}] \\ &\quad - \ell Z e^2 \frac{\int_{\mathbb{R}^3} \frac{1}{|\boldsymbol{y}|} \big|\chi\big(\frac{\boldsymbol{y}}{\ell n}\big)\big|^2 |\psi(\boldsymbol{y})|^2 \,\mathrm{d}\boldsymbol{y}}{\int_{\mathbb{R}^3} \big|\chi\big(\frac{\boldsymbol{y}}{\ell n}\big)\big|^2 |\psi(\boldsymbol{y})|^2 \,\mathrm{d}\boldsymbol{y}} \\ &\xrightarrow[n\to\infty]{} \ell\Big(-Ze^2\Big(\psi, \frac{1}{|\boldsymbol{x}|}\psi\Big)_{L^2} + \frac{\hbar^2 c^2}{e^2} \mathscr{E}_{\mathrm{EM}}[\boldsymbol{A}, \boldsymbol{0}]\Big),\end{aligned}$$

so under the condition $Z\alpha^2 > \frac{\mathscr{E}_{\mathrm{EM}}[\boldsymbol{A},\boldsymbol{0}]}{(\psi, \frac{1}{|\boldsymbol{x}|}\psi)_{L^2}}$ we can make $\big(\psi_{\ell,n}, \mathscr{H}\big(\boldsymbol{A}_\ell, \boldsymbol{0}\big)\psi_{\ell,n}\big)_{L^2}$ arbitrarily negative by choosing ℓ and n appropriately large[1]. In other words, the hydrogenic atom with Pauli kinetic energy is unstable for large values of $Z\alpha^2$. Fröhlich, Lieb and Loss show in [24] that the critical value of $Z\alpha^2$ is $\Theta = \inf \frac{\mathscr{E}_{\mathrm{EM}}[\boldsymbol{A},\boldsymbol{0}]}{(\psi, \frac{1}{|\boldsymbol{x}|}\psi)_{L^2}}$, where the infimum is taken over all zero modes of

[1] $\alpha = \frac{e^2}{\hbar c}$ denotes the *fine-structure constant*. It is dimensionless and has a numerical value of approximately $\frac{1}{137}$. However, in the literature concerning energetic stability of matter α is often thought of as a parameter that can take any positive value.

the Pauli operator. This means that the system is stable for $Z\alpha^2 < \Theta$ and unstable for $Z\alpha^2 > \Theta$. The authors also provide a positive lower bound on Θ explaining stability of the hydrogenic atom in more than all physically relevant cases, but to determine the precise value of Θ is still an open problem. Lieb and Loss study in [40] the N-electron atom with a single nucleus fixed at the origin and prove that it's ground state energy is finite provided that $Z\alpha^{\frac{12}{7}}$ is sufficiently small.

Stability of the Second Kind

Let us return to the full many-body problem. Suppose that we do manage to show that the system is stable of the first kind, but suppose also that it's ground state energy decreases superlinearly as a function of the total particle number $N + M$. In principle, we would then be able to extract an arbitrarily large amount of energy simply by bringing together two sufficiently large objects – this property is certainly not in agreement with what we observe in our daily life. The system is said to be *stable of the second kind* if the ground state energy decreases at most linearly as a function of the number of particles, i.e.

$$E_N^M(\boldsymbol{R}) \geq -C(N + M) \tag{1.5}$$

for some constant $C > 0$ that is independent of the nuclear configuration $\boldsymbol{R}$ but might depend on the atomic numbers $Z_1, \ldots, Z_M$ of the nuclei. Demanding that C is independent of $\boldsymbol{R}$ corresponds to allowing the nuclei to move around (but still neglecting their kinetic energies). For completeness we mention that the inequality (1.5) can be used as an important step when proving that the free energy per particle in an infinite system at fixed temperature and density has a thermodynamic limit, at least when $\boldsymbol{A} = \boldsymbol{0}$ (see [36, 39]).

Dyson and Lenard [15, 37] were the first to prove stability of the second kind in the case $\boldsymbol{A} = \boldsymbol{0}$ and the result was later rederived – both by Federbush [17] as well as by Lieb and Thirring [47]. In the latter paper, systems without magnetic fields are proven to be stable of the second kind by an argument based on a new type of inequalities – now referred to as Lieb-Thirring inequalities. As noted earlier it is in this context essential that the dynamic particles are fermions – systems of bosonic dynamic particles are unstable of the second kind (even though they are stable of the first kind).

For electrically neutral systems of N bosons and M infinitely heavy nuclei the ground state energy has been shown [15, 5, 38] to behave like $-KN^{\frac{5}{3}}$ for some constant $K > 0$ and if the nuclei have finite positive masses the ground state energy behaves like $-K'N^{\frac{7}{5}}$ for some other constant $K' > 0$, as demonstrated in [14, 10, 46, 58].

Due to the diamagnetic inequality the inclusion of magnetic fields in the model introduces no further complications to the stability question when using the magnetic Laplacian as the kinetic energy observable. When the kinetic energy is measured by the Pauli operator we can of course only expect stability of the second kind to hold for sufficiently small values of $\max\{Z_1, \ldots, Z_M\}\alpha^2$, as is apparent from the example above. However, as shown by Lieb and Loss' treatment [40] of the one-electron molecule a bound on the fine-structure constant itself is also necessary to ensure stability. The full many-body stability problem with Pauli kinetic energy is solved by Fefferman in [19, 20] for $Z_1 = \cdots = Z_M = 1$ and sufficiently small α. Finally, Lieb, Loss and Solovej prove in [43] that as long as $\max\{Z_1, \ldots, Z_M\}\alpha^2$ and α are sufficiently small then any system of N fermions with Pauli kinetic energy interacting with M nuclei will be stable of the second kind. More precisely, they show the existence of a constant $C > 0$, depending only on the atomic numbers of the nuclei, such that the estimate $E_N^M(\boldsymbol{R}) \geq -CN^{\frac{1}{3}}M^{\frac{2}{3}}$ holds true for all $\boldsymbol{R}$, provided that

$$\max\{Z_1, \ldots, Z_M\}\alpha^2 \leq 0.041 \text{ and } \alpha \leq 0.06. \tag{1.6}$$

Here, (1.6) easily covers all physically relevant cases. Interestingly, one also finds the necessity of a bound on the fine-structure constant itself to ensure stability in the analogous theory for relativistic systems. In order to prove that such systems are stable it is necessary and sufficient that $\max\{Z_1, \ldots, Z_M\}\alpha$ and α are small – this holds both in the absence of magnetic fields [11, 9, 18, 48], in the presence of magnetic fields (that are only coupled to the orbital motion of the particles) [22, 42] and in the modified Brown-Ravenhall model [45]. Let us finally mention that models with quantized electromagnetic fields have also been studied – see for instance [7, 21, 6, 41, 32].

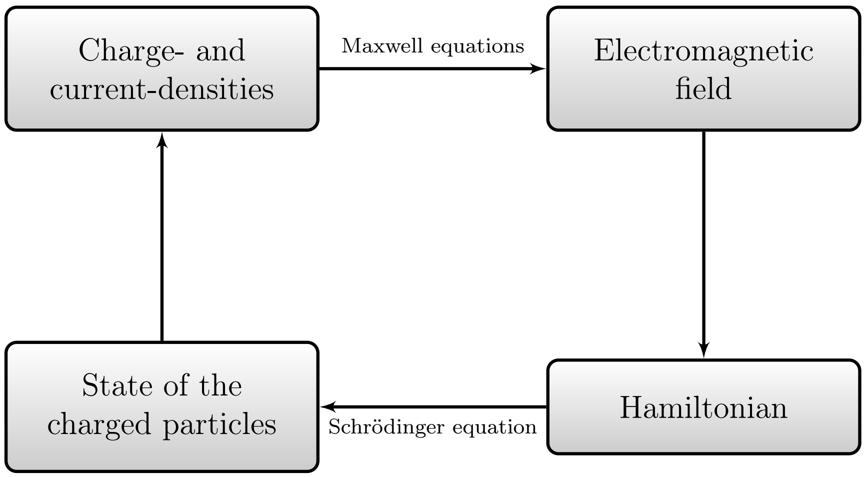

Figure 1: When modelling a system of charged dynamic particles we think of the Schrödinger equation as being coupled to the Maxwell equations.

Dynamic Stability of Matter

We will study a completely different approach to the concept of stability focusing on the dynamics of the particles instead of their total energy. The idea of describing complex systems' time evolution by simple fundamental laws is a cornerstone in science. For instance we describe the evolution of systems in quantum mechanics by the Schrödinger equation and likewise we describe the evolution of systems in electrodynamics by the Maxwell equations – the results of several experiments suggest that each of these equations serve as accurate descriptions of reality. The matter system introduced above is of both quantum mechanical and electrodynamical nature and thus it seems reasonable to model it's dynamics by the Schrödinger equation *and* the Maxwell equations. This leaves us with a nonlinear system of partial differential equations and as shown schematically in Figure 1 these equations are genuinely coupled in the sense that the dependent variable of one equation enters as an independent variable of the other. Considered from a mathematical point of view there is of course no guarantee whatsoever that this coupled system has a solution – we interpret the affirmative case as an explanation that matter can at all exist and evolve in time (under the assumption that nature acts in accordance with the Schrödinger equation and the Maxwell equations).

From now on we consider the case with $M = 0$ infinitely heavy nuclei so in other words the system simply consists of N nonrelativistic dynamic particles with charges $Q_1, \ldots, Q_N$ and masses $m_1, \ldots, m_N$. In fact, setting $M = 0$ is no restriction since the model describing dynamic as well as static particles can be restored in a suitable large mass limit.

The many-body Maxwell-Schrödinger System

We first consider the particles as being spinless and describe their kinetic energies by the magnetic Laplacian. Then for any given $\big(\boldsymbol{A}, -\frac{P\boldsymbol{E}}{4\pi}\big)$ with $\mathrm{div}\boldsymbol{A} = 0$ we can express the quantum mechanical Hamiltonian of the system formally by

$$\begin{aligned}&\mathscr{H}\big(\boldsymbol{A}, -\tfrac{P\boldsymbol{E}}{4\pi}\big)\\ &= \sum_{j=1}^{N} \frac{1}{2m_j}\Big(i\hbar\nabla_{\boldsymbol{x}_j} + \frac{Q_j}{c}\boldsymbol{A}(\boldsymbol{x}_j)\Big)^2 + \sum_{1\le j<k\le N} \frac{Q_j Q_k}{|\boldsymbol{x}_j - \boldsymbol{x}_k|} + \mathscr{E}_{\mathrm{EM}}[\boldsymbol{A}, -c^2 P\boldsymbol{E}].\end{aligned} \tag{1.7}$$

The associated *many-body Maxwell-Schrödinger system* in Coulomb gauge reads

$$\begin{aligned}\Box\boldsymbol{A} &= \frac{4\pi}{c}\sum_{j=1}^{N} P\boldsymbol{J}_j^{\mathrm{S}}[\psi, \boldsymbol{A}],\\ i\hbar\partial_t\psi &= \left(\sum_{j=1}^{N}\frac{\big(i\hbar\nabla_{\boldsymbol{x}_j} + \frac{Q_j}{c}\boldsymbol{A}(\boldsymbol{x}_j)\big)^2}{2m_j} + \sum_{1\le j<k\le N}\frac{Q_j Q_k}{|\boldsymbol{x}_j - \boldsymbol{x}_k|} + \mathscr{E}_{\mathrm{EM}}[\boldsymbol{A}, \partial_t\boldsymbol{A}]\right)\psi,\\ \mathrm{div}\boldsymbol{A} &= 0,\end{aligned} \tag{1.8}$$

where we now think of $\psi(t) : \mathbb{R}^{3N} \to \mathbb{C}$ and $\boldsymbol{A}(t) : \mathbb{R}^3 \to \mathbb{R}^3$ as time-dependent functions. We have also set $\Box = \frac{1}{c^2}\partial_t^2 - \Delta$ and for $j \in \{1, \ldots, N\}$ we have introduced the probability current density $\boldsymbol{J}_j^{\mathrm{S}}[\psi, \boldsymbol{A}](t) : \mathbb{R}^3 \to \mathbb{R}^3$ associated with the j'th particle that is given by

$$\begin{aligned}&\boldsymbol{J}_j^{\mathrm{S}}[\psi, \boldsymbol{A}](t)(\boldsymbol{x}_j)\\ &= -\frac{Q_j}{m_j}\mathrm{Re}\int_{\mathbb{R}^{3(N-1)}} \overline{\psi}(t)(\boldsymbol{x})\Big(i\hbar\nabla_{\boldsymbol{x}_j} + \frac{Q_j}{c}\boldsymbol{A}(t)(\boldsymbol{x}_j)\Big)\psi(t)(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}_j',\end{aligned}$$

where $\boldsymbol{x} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)$ and $\boldsymbol{x}'_j = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{j-1}, \boldsymbol{x}_{j+1}, \ldots, \boldsymbol{x}_N)$. As motivated above we will say that matter is *dynamically stable* if there exists a solution to the Cauchy problem corresponding to (1.8), where the initial conditions are formulated in the form

$$\psi(0) = \psi_0, \boldsymbol{A}(0) = \boldsymbol{A}_0 \text{ and } \partial_t \boldsymbol{A}(0) = \boldsymbol{A}_1 \tag{1.9}$$

for some $(\psi_0, \boldsymbol{A}_0, \boldsymbol{A}_1)$ that is given beforehand. The main objective of the paper [55] is to show this kind of dynamic stability of matter.

Objective 1: Prove that for any sufficiently regular initial state $(\psi_0, \boldsymbol{A}_0, \boldsymbol{A}_1)$ with $\mathrm{div}\boldsymbol{A}_0 = \mathrm{div}\boldsymbol{A}_1 = 0$ there exists a unique local in time solution of appropriate regularity to (1.8)–(1.9).

Of course it would be desirable to prove that the solution exists globally in time and that it depends continuously on the initial data, sometimes expressed by saying that the many-body Maxwell-Schrödinger system is *globally well-posed*. This is still an open problem. However, there are global well-posedness results for other formulations of the Maxwell-Schrödinger system – we elaborate on this in Chapter 2. Before stating the precise result of [55] we give a (formal) motivation for the appearance of (1.8).

Derivation of the many-body Maxwell-Schrödinger System

In quantum mechanics one often derives the equations of motion directly from the analogous classical equations of motion by a procedure called quantization. Let us apply this procedure to the matter system introduced above and thereby derive (1.7)–(1.8). As already mentioned, one can describe systems of charged particles classically by the Maxwell equations and in combination with the Lorentz force law these equations in principle allow us to calculate the time evolution of all forces influencing the system. However, the Maxwell-Lorentz coupling turns out to be ill-posed for point particles, as observed in [59]. In the outset, we will therefore consider the Abraham model of charged particles, where the charge Q_j of the j'th particle is thought of as being smeared out on a small region of positive measure. At time t the charge- and current-densities associated with the j'th particle are thus represented by

$$\rho_{R,j}(t) : \boldsymbol{y} \mapsto Q_j \chi_R(\boldsymbol{x}_j(t) - \boldsymbol{y}) \text{ respectively } \boldsymbol{J}_{R,j}(t) : \boldsymbol{y} \mapsto \frac{\mathrm{d}\boldsymbol{x}_j}{\mathrm{d}t}(t) \rho_{R,j}(t)(\boldsymbol{y}),$$

where $\boldsymbol{x}_j(t)$ denotes the location of the particle and χ_R can be written on the form $\boldsymbol{y} \mapsto \frac{1}{R^3}\chi\big(\frac{\boldsymbol{y}}{R}\big)$ for some $R > 0$ and some positive cut-off function $\chi \in C_0^\infty(\mathbb{R}^3)$ satisfying $\int_{\mathbb{R}^3} \chi(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = 1$. It is instructive to think of the charge Q_j as being distributed throughout the ball of radius R, centered at $\boldsymbol{x}_j(t)$ – our intention of studying genuine point particles will later lead us to take the limit $R \to 0^+$. Applying the Maxwell equations to the Abraham model tells us that the electromagnetic field $(\boldsymbol{E}, \boldsymbol{B})$ induced by the particles satisfies

$$\operatorname{div}\boldsymbol{B}(t) = 0, \tag{1.10}$$

$$\nabla \times \boldsymbol{E}(t) = -\frac{1}{c}\partial_t \boldsymbol{B}(t), \tag{1.11}$$

$$\operatorname{div}\boldsymbol{E}(t) = 4\pi \sum_{j=1}^{N} \rho_{R,j}(t), \tag{1.12}$$

$$\nabla \times \boldsymbol{B}(t) = \frac{1}{c}\Big(\partial_t \boldsymbol{E}(t) + 4\pi \sum_{j=1}^{N} \boldsymbol{J}_{R,j}(t)\Big), \tag{1.13}$$

and the Lorentz force law reads

$$m_j \frac{\mathrm{d}^2 \boldsymbol{x}_j}{\mathrm{d}t^2}(t) = Q_j\Big(\frac{1}{c}\frac{\mathrm{d}\boldsymbol{x}_j}{\mathrm{d}t}(t) \times \boldsymbol{B}(t) + \boldsymbol{E}(t)\Big) * \chi_R(\boldsymbol{x}_j(t)) \text{ for } j \in \{1, \dots, N\}. \tag{1.14}$$

Here, (1.10)–(1.11) ensure that we can find an electromagnetic potential corresponding to $(\boldsymbol{E}, \boldsymbol{B})$, meaning a pair $(V, \boldsymbol{A})$ of mappings $V(t) : \mathbb{R}^3 \to \mathbb{R}$ and $\boldsymbol{A}(t) : \mathbb{R}^3 \to \mathbb{R}^3$ satisfying

$$\boldsymbol{B}(t) = \nabla \times \boldsymbol{A}(t) \text{ and } \boldsymbol{E}(t) = -\frac{1}{c}\partial_t \boldsymbol{A}(t) - \nabla V(t) \tag{1.15}$$

at all times t. The potential is clearly not unique since $\big(V - \frac{1}{c}\partial_t \eta, \boldsymbol{A} + \nabla\eta\big)$ also serves as an electromagnetic potential for any $\eta(t) : \mathbb{R}^3 \to \mathbb{R}$. This leaves us with freedom to choose a potential $(V, \boldsymbol{A})$ fulfilling the Coulomb gauge condition

$$\operatorname{div}\boldsymbol{A}(t) = 0$$

for all t.

We now formulate the classical equations of motion in a more concise form, namely as the Euler-Lagrange equations associated with a certain Lagrangian. For this purpose we choose $\mathscr{Q}^0 = \mathbb{R}^{3N} \times D^1 \times PL^2$ as configuration space, where D^1 is the space of locally integrable functions that vanish at infinity and have first order derivatives in L^2. With $\mathscr{Q}^1$ denoting the manifold domain $\mathbb{R}^{3N} \times D^1 \times PH^1$ of $\mathscr{Q}^0$ we then define the Lagrangian $\mathscr{L}_R$ on $T\mathscr{Q}^0|\mathscr{Q}^1 \cong \mathscr{Q}^1 \times \mathscr{Q}^0$ by setting

$$\mathscr{L}_R\big(\boldsymbol{x}, V, \boldsymbol{A}, \dot{\boldsymbol{x}}, \dot{V}, \dot{\boldsymbol{A}}\big) = \sum_{j=1}^{N} \Big(\frac{1}{2} m_j \dot{\boldsymbol{x}}_j^2 + \frac{Q_j}{c} \dot{\boldsymbol{x}}_j \cdot \boldsymbol{A} * \chi_R(\boldsymbol{x}_j) - Q_j V * \chi_R(\boldsymbol{x}_j)\Big) + \frac{1}{8\pi} \int_{\mathbb{R}^3} \Big(\Big|\frac{1}{c}\dot{\boldsymbol{A}}(\boldsymbol{y}) + \nabla V(\boldsymbol{y})\Big|^2 - |\nabla \times \boldsymbol{A}(\boldsymbol{y})|^2\Big)\, \mathrm{d}\boldsymbol{y},$$

where H^1 is the usual Sobolev space of order 1. The corresponding Euler-Lagrange equations are indeed (1.12)–(1.14), but unfortunately the Lagrangian formalism does not lend itself to the quantization process as effectively as the Hamiltonian formalism does. To pass between these two formalisms we introduce the fiber derivative $\mathbb{F}\mathscr{L}_R$ mapping from velocity phase space $T\mathscr{Q}^0|\mathscr{Q}^1$ to momentum phase space $T^*\mathscr{Q}^0|\mathscr{Q}^1$ by the prescription

$$\mathbb{F}\mathscr{L}_R(v)(w) = \frac{\mathrm{d}}{\mathrm{d}t}\mathscr{L}_R(v + tw)\Big|_{t=0} \quad \text{for } q \in \mathscr{Q}^1 \text{ and } v, w \in T_q\mathscr{Q}^0.$$

In coordinates, $\mathbb{F}\mathscr{L}_R$ can be expressed as

$$\begin{aligned} &\mathbb{F}\mathscr{L}_R(\boldsymbol{x}, V, \boldsymbol{A}, \dot{\boldsymbol{x}}, \dot{V}, \dot{\boldsymbol{A}})(\boldsymbol{x}, V, \boldsymbol{A}, \dot{\boldsymbol{x}}', \dot{V}', \dot{\boldsymbol{A}}') \\ &= \sum_{j=1}^{N} \dot{\boldsymbol{x}}_j' \cdot \Big(m_j \dot{\boldsymbol{x}}_j + \frac{Q_j}{c} \boldsymbol{A} * \chi_R(\boldsymbol{x}_j)\Big) + \int_{\mathbb{R}^3} \dot{\boldsymbol{A}}'(\boldsymbol{y}) \cdot \frac{\dot{\boldsymbol{A}}(\boldsymbol{y})}{4\pi c^2}\, \mathrm{d}\boldsymbol{y}, \end{aligned}$$

whereby the total energy $\mathscr{E}_R : T\mathscr{Q}^0|\mathscr{Q}^1 \ni v \mapsto \big(\mathbb{F}\mathscr{L}_R(v)(v) - \mathscr{L}_R(v)\big) \in \mathbb{R}$ as a function of coordinates and velocities is given by

$$\mathscr{E}_R(\boldsymbol{x}, V, \boldsymbol{A}, \dot{\boldsymbol{x}}, \dot{V}, \dot{\boldsymbol{A}}) = \sum_{j=1}^{N} \Big(\frac{1}{2} m_j \dot{\boldsymbol{x}}_j^2 + Q_j V * \chi_R(\boldsymbol{x}_j)\Big) - \frac{1}{8\pi} \int_{\mathbb{R}^3} |\nabla V(\boldsymbol{y})|^2\, \mathrm{d}\boldsymbol{y} + \frac{1}{8\pi} \int_{\mathbb{R}^3} \Big(\Big|\frac{1}{c}\dot{\boldsymbol{A}}(\boldsymbol{y})\Big|^2 + |\nabla \times \boldsymbol{A}(\boldsymbol{y})|^2\Big).$$

The Hamiltonian, on the other hand, expresses the total energy as a function of coordinates and momenta. Thus, our goal is to find a function $\mathscr{H}_R$ that is defined on (part of) $T^*\mathcal{Q}^0|\mathcal{Q}^1 \cong \mathcal{Q}^1 \times (\mathcal{Q}^0)^*$ and satisfies

$$\mathscr{H}_R \circ \mathbb{F}\mathscr{L}_R = \mathscr{E}_R. \tag{1.16}$$

The identity (1.16) does happen to uniquely determine a function defined on the image $\mathcal{M}_1 = \mathbb{F}\mathscr{L}_R(T\mathcal{Q}^0|\mathcal{Q}^1)$, namely the function

$$\begin{aligned}\big(\boldsymbol{x}, V, \boldsymbol{A}, \boldsymbol{p}, U, -\tfrac{P\boldsymbol{E}}{4\pi}\big) \mapsto \sum_{j=1}^{N} \left(\frac{1}{2m_j}\Big(\boldsymbol{p}_j - \frac{Q_j}{c}\boldsymbol{A} * \chi_R(\boldsymbol{x}_j)\Big)^2 + Q_j V * \chi_R(\boldsymbol{x}_j) \right) \\ - \frac{1}{8\pi}\int_{\mathbb{R}^3} |\nabla V(\boldsymbol{y})|^2 \,\mathrm{d}\boldsymbol{y} + \mathscr{E}_{\mathrm{EM}}[\boldsymbol{A}, -c^2 P\boldsymbol{E}],\end{aligned} \tag{1.17}$$

where we, as usual, identify the spaces $\mathcal{Q}^0$ and $(\mathcal{Q}^0)^*$ by the isometric isomorphism $\mathcal{Q}^0 \to (\mathcal{Q}^0)^*$,

$$\big(\boldsymbol{p}, U, -\tfrac{P\boldsymbol{E}}{4\pi}\big) \mapsto \left((\dot{\boldsymbol{x}}, \dot{V}, \dot{\boldsymbol{A}}) \mapsto \Big(\dot{\boldsymbol{x}} \cdot \boldsymbol{p} + \int_{\mathbb{R}^3} \big(\nabla\dot{V} \cdot \nabla U - \dot{\boldsymbol{A}} \cdot \tfrac{P\boldsymbol{E}}{4\pi}\big)(\boldsymbol{y}) \,\mathrm{d}\boldsymbol{y}\Big) \right).$$

But the formalism requires that the domain of the Hamiltonian is a (weak) symplectic manifold – $\mathcal{M}_1$ does not fit into this scheme when equipped with the pull-back $\omega_1 = j_1^*\Omega$ of the canonical 2-form Ω on $T^*\mathcal{Q}^0$ via the inclusion $\mathcal{M}_1 \overset{j_1}{\hookrightarrow} T^*\mathcal{Q}^0$ since ω_1 is a degenerate 2-form. We therefore aim to restrict (1.17) to some embedded submanifold $\mathcal{M}_2 \overset{j_2}{\hookrightarrow} \mathcal{M}_1$ that when equipped with $\omega_2 = j_2^*\omega_1$ becomes a (weak) symplectic manifold. To find such a submanifold we use the algorithm invented by Gotay, Nester and Hinds [31] as a further development of Anderson, Bergmann and Dirac's constraint theory [1, 12, 13]. The algorithm produces the manifold

$$\mathcal{M}_2 = \left\{ \big(\boldsymbol{x}, V, \boldsymbol{A}, \boldsymbol{p}, 0, -\tfrac{P\boldsymbol{E}}{4\pi}\big) \in \mathcal{Q}^1 \times \mathcal{Q}^0 \,\middle|\, -\Delta V = 4\pi \sum_{j=1}^{N} Q_j \chi_R(\boldsymbol{x}_j - \cdot) \right\} \tag{1.18}$$

and restricting (1.17) to this manifold results in the function sending an element $\left(\boldsymbol{x}, V, \boldsymbol{A}, \boldsymbol{p}, 0, -\frac{P\boldsymbol{E}}{4\pi}\right) \in \mathcal{M}_2$ into the number

$$\sum_{j=1}^{N} \frac{1}{2m_j}\Big(\boldsymbol{p}_j - \frac{Q_j}{c}\boldsymbol{A} * \chi_R(\boldsymbol{x}_j)\Big)^2 + \mathscr{E}_{\mathrm{EM}}[\boldsymbol{A}, -c^2 P\boldsymbol{E}]$$
$$+\sum_{j=1}^{N}\sum_{k=1}^{N} \frac{Q_j Q_k}{2} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\chi_R(\boldsymbol{x}_j - \boldsymbol{y})\chi_R(\boldsymbol{x}_k - \boldsymbol{z})}{|\boldsymbol{y} - \boldsymbol{z}|}\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{z}, \tag{1.19}$$

where we have explicitly used that V is not an independent variable but is given uniquely in terms of $\boldsymbol{x}$ as the function $\boldsymbol{z} \mapsto \sum_{j=1}^{N} Q_j \int_{\mathbb{R}^3} \frac{\chi_R(\boldsymbol{x}_j - \boldsymbol{y})}{|\boldsymbol{y}-\boldsymbol{z}|}\,\mathrm{d}\boldsymbol{y}$. Due to this dependence of V on $\boldsymbol{x}$ we might as well identify $\mathcal{M}_2$ with the space $\mathbb{R}^{3N} \times PH^1 \times \mathbb{R}^{3N} \times PL^2$ by the isomorphism

$$\mathcal{M}_2 \ni \left(\boldsymbol{x}, V, \boldsymbol{A}, \boldsymbol{p}, 0, -\tfrac{P\boldsymbol{E}}{4\pi}\right) \mapsto \left(\boldsymbol{x}, \boldsymbol{A}, \boldsymbol{p}, -\tfrac{P\boldsymbol{E}}{4\pi}\right) \in \mathbb{R}^{3N} \times PH^1 \times \mathbb{R}^{3N} \times PL^2$$

and correspondingly let the Hamiltonian $\mathscr{H}_R : \mathbb{R}^{3N} \times PH^1 \times \mathbb{R}^{3N} \times PL^2 \to \mathbb{R}$ be the function sending a given $\left(\boldsymbol{x}, \boldsymbol{A}, \boldsymbol{p}, -\frac{P\boldsymbol{E}}{4\pi}\right)$ into the expression written in (1.19).

Having defined $\mathscr{H}_R$ we will now take the point particle limit $R \to 0^+$. The (j,k)'th term of the double sum in (1.19) represents the potential energy of particle j due to the Coulomb force exerted by particle k. But in reality a given particle does not Coulomb interact with itself so it is reasonable to neglect the diagonal terms $\frac{Q_j^2}{2R}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\chi(\boldsymbol{y})\chi(\boldsymbol{z})}{|\boldsymbol{y}-\boldsymbol{z}|}\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{z}$ from the Hamiltonian. The resulting function converges pointwise to

$$\mathscr{H}_0\left(\boldsymbol{x}, \boldsymbol{A}, \boldsymbol{p}, -\tfrac{P\boldsymbol{E}}{4\pi}\right) = \sum_{j=1}^{N} \frac{1}{2m_j}\Big(\boldsymbol{p}_j - \frac{Q_j}{c}\boldsymbol{A}(\boldsymbol{x}_j)\Big)^2 + \sum_{1 \le j < k \le N} \frac{Q_j Q_k}{|\boldsymbol{x}_j - \boldsymbol{x}_k|} + \mathscr{E}_{\mathrm{EM}}[\boldsymbol{A}, -c^2 P\boldsymbol{E}], \tag{1.20}$$

at least if $\boldsymbol{A}$ is continuous at $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$. We are now in position to quantize the charged particles in the model, which formally takes place by reinterpreting (1.20) as an (electromagnetic potential-dependent) operator on the quantum mechanical state space: For given $\left(\boldsymbol{A}, -\frac{P\boldsymbol{E}}{4\pi}\right) \in PH^1 \times PL^2$ we thus let the quantum mechanical Hamiltonian $\mathscr{H}\left(\boldsymbol{A}, -\frac{P\boldsymbol{E}}{4\pi}\right)$ act as prescribed in (1.20), where we perceive $\boldsymbol{x} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)$ as the independent variable of functions belonging to the state space, we interpret $\boldsymbol{A}(\boldsymbol{x}_j)$, $\frac{1}{|\boldsymbol{x}_j - \boldsymbol{x}_k|}$ and

$\mathscr{E}_{\mathrm{EM}}[\boldsymbol{A}, -c^2 P\boldsymbol{E}]$ as multiplication operators acting on this space and we replace $\boldsymbol{p}_j$ by the differentiation operator $-i\hbar\nabla_{\boldsymbol{x}_j}$. The resulting Hamiltonian is exactly the one described in (1.7), where we under the Coulomb gauge condition $\mathrm{div}\boldsymbol{A} = 0$ interpret the square $\big(i\hbar\nabla_{\boldsymbol{x}_j} + \frac{Q_j}{c}\boldsymbol{A}(\boldsymbol{x}_j)\big)^2$ as

$$\Big(i\hbar\nabla_{\boldsymbol{x}_j} + \frac{Q_j}{c}\boldsymbol{A}(\boldsymbol{x}_j)\Big)^2 = -\hbar^2\Delta_{\boldsymbol{x}_j} + 2i\frac{\hbar Q_j}{c}\boldsymbol{A}(\boldsymbol{x}_j)\cdot\nabla_{\boldsymbol{x}_j} + \frac{Q_j^2}{c^2}\big[\boldsymbol{A}(\boldsymbol{x}_j)\big]^2$$

for $j \in \{1, \ldots, N\}$.

Finally, to derive (1.8) we let $\psi : \mathbb{R}^{3N} \to \mathbb{C}$ be a normalized state and consider the average energy $\big(\boldsymbol{A}, -\frac{P\boldsymbol{E}}{4\pi}\big) \mapsto \big(\psi, \mathscr{H}\big(\boldsymbol{A}, -\frac{P\boldsymbol{E}}{4\pi}\big)\psi\big)_{L^2}$ as a classical Hamiltonian defined on the weak symplectic manifold $PH^1 \times PL^2$ equipped with the 2-form ω given by

$$\omega_m\big(m, \boldsymbol{A}_1, -\tfrac{P\boldsymbol{E}_1}{4\pi}, m, \boldsymbol{A}_2, -\tfrac{P\boldsymbol{E}_2}{4\pi}\big) = \frac{1}{4\pi}\int_{\mathbb{R}^3}\big(P\boldsymbol{E}_1\cdot\boldsymbol{A}_2 - P\boldsymbol{E}_2\cdot\boldsymbol{A}_1\big)(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}$$

for $m, \big(\boldsymbol{A}_1, -\frac{P\boldsymbol{E}_1}{4\pi}\big), \big(\boldsymbol{A}_2, -\frac{P\boldsymbol{E}_2}{4\pi}\big) \in PH^1 \times PL^2$. The associated Hamilton equations read

$$\frac{1}{c^2}\partial_t\boldsymbol{A}(t) = -P\boldsymbol{E}(t) \text{ and } -\partial_t P\boldsymbol{E}(t) = \Delta\boldsymbol{A}(t) + \frac{4\pi}{c}\sum_{j=1}^{N} P\boldsymbol{J}_j^{\mathrm{S}}[\psi, \boldsymbol{A}(t)]. \tag{1.21}$$

Here, the electromagnetic field is regarded as a dynamic quantity, while the wave function ψ is considered to be fixed in time. However, according to the postulates of quantum mechanics the state ψ is also expected to evolve in time, namely as governed by the Schrödinger equation

$$i\hbar\partial_t\psi(t) = \mathscr{H}\big(\boldsymbol{A}(t), -\tfrac{P\boldsymbol{E}(t)}{4\pi}\big)\psi(t). \tag{1.22}$$

The system (1.8) can now be obtained by coupling (1.21) with (1.22) in the sense that we replace the fixed ψ appearing in (1.21) with the time-dependent $\psi(t)$ that is present in (1.22).

The many-body Maxwell-Pauli System

If all of the N particles have spin $\frac{1}{2}$ and we describe their kinetic energies by means of the Pauli operator then the (electromagnetic potential-dependent)

Hamiltonian of the system is given by

$$\mathscr{H}\big(\boldsymbol{A}, -\tfrac{P\boldsymbol{E}}{4\pi}\big) = \sum_{j=1}^{N} \frac{\big(\boldsymbol{\sigma}\cdot\big(i\hbar\nabla_{\boldsymbol{x}_j} + \frac{Q_j}{c}\boldsymbol{A}(\boldsymbol{x}_j)\big)\big)^2}{2m_j} + \sum_{1\leq j<k\leq N} \frac{Q_jQ_k}{|\boldsymbol{x}_j - \boldsymbol{x}_k|} + \mathscr{E}_{\mathrm{EM}}[\boldsymbol{A}, -c^2P\boldsymbol{E}]$$

and the corresponding *many-body Maxwell-Pauli system* expressed in Coulomb gauge reads

$$\begin{aligned}
\Box\boldsymbol{A} &= \frac{4\pi}{c}\sum_{j=1}^{N} P\boldsymbol{J}_j^{\mathrm{P}}[\psi,\boldsymbol{A}],\\
i\hbar\partial_t\psi &= \left(\sum_{j=1}^{N} \frac{\big(\boldsymbol{\sigma}\cdot\big(i\hbar\nabla_{\boldsymbol{x}_j} + \frac{Q_j}{c}\boldsymbol{A}(\boldsymbol{x}_j)\big)\big)^2}{2m_j} + \sum_{1\leq j<k\leq N} \frac{Q_jQ_k}{|\boldsymbol{x}_j - \boldsymbol{x}_k|}\right)\psi \\
&\quad + \mathscr{E}_{\mathrm{EM}}[\boldsymbol{A}, \partial_t\boldsymbol{A}]\psi,\\
\mathrm{div}\boldsymbol{A} &= 0.
\end{aligned} \tag{1.23}$$

Here, the ℓ'th component of the current density $\boldsymbol{J}_j^{\mathrm{P}}[\psi,\boldsymbol{A}](t) : \mathbb{R}^3 \to \mathbb{R}^3$ is for $\ell \in \{1,2,3\}$ given by

$$\begin{aligned}
&\big\{\boldsymbol{J}_j^{\mathrm{P}}[\psi,\boldsymbol{A}](t)\big\}^{\ell}(\boldsymbol{x}_j)\\
&= -\frac{Q_j}{m_j}\sum_{\boldsymbol{s}\in\{1,2\}^N}\sum_{k=1}^{3}\sum_{v=1}^{2}\sum_{w=1}^{2}\mathrm{Re}\int_{\mathbb{R}^{3(N-1)}} \overline{\big\{\psi(t)\big\}^{(s_1,\dots,s_j,\dots,s_N)}}(\boldsymbol{x}_1,\dots,\boldsymbol{x}_j,\dots,\boldsymbol{x}_N)\\
&\cdot\sigma^{\ell}_{s_jw}\sigma^{k}_{wv}\Big(i\hbar\partial_{x_j^k} + \frac{Q_j}{c}A^k(t)(\boldsymbol{x}_j)\Big)\big\{\psi(t)\big\}^{(s_1,\dots,v,\dots,s_N)}(\boldsymbol{x}_1,\dots,\boldsymbol{x}_j,\dots,\boldsymbol{x}_N)\,\mathrm{d}\boldsymbol{x}_j',
\end{aligned}$$

where $\boldsymbol{x}_j = (x_j^1, x_j^2, x_j^3)$, $\boldsymbol{x}_j' = (\boldsymbol{x}_1,\dots,\boldsymbol{x}_{j-1},\boldsymbol{x}_{j+1},\dots,\boldsymbol{x}_N)$, $\sigma^k = \big(\sigma^k_{wv}\big)^2_{w,v=1}$, $\boldsymbol{A} = (A^1, A^2, A^3)$ and $\boldsymbol{s} = (s_1,\dots,s_N)$ for $(j,k) \in \{1,\dots,N\}\times\{1,2,3\}$.

For simplicity let us consider the one-body Maxwell-Pauli system that models a single spin-$\frac{1}{2}$ particle of mass $m > 0$ and charge $Q \in \mathbb{R}\setminus\{0\}$ interacting with it's self-generated electromagnetic field. By using the Lichnerowicz formula (1.2) we can formulate this system in Coulomb gauge as

$$\begin{aligned}
\Box\boldsymbol{A} &= \frac{4\pi}{c}P\boldsymbol{J}^{\mathrm{P}}[\psi,\boldsymbol{A}],\\
i\hbar\partial_t\psi &= \left(\frac{1}{2m}\Big(i\hbar\nabla + \frac{Q}{c}\boldsymbol{A}\Big)^2 - \frac{\hbar Q}{2mc}\boldsymbol{\sigma}\cdot\nabla\times\boldsymbol{A} + \mathscr{E}_{\mathrm{EM}}[\boldsymbol{A},\partial_t\boldsymbol{A}]\right)\psi,\\
\mathrm{div}\boldsymbol{A} &= 0
\end{aligned} \tag{1.24}$$

and the probability current density $\boldsymbol{J}^{\mathrm{P}}[\psi, \boldsymbol{A}](t) : \mathbb{R}^3 \to \mathbb{R}^3$ reduces to

$$\begin{aligned}\boldsymbol{J}^{\mathrm{P}}[\psi, \boldsymbol{A}](t)(\boldsymbol{x}) = -\frac{Q}{m}\mathrm{Re}\Big\langle \psi(t)(\boldsymbol{x}), \Big(i\hbar\nabla + \frac{Q}{c}\boldsymbol{A}\Big)\psi(t)(\boldsymbol{x})\Big\rangle_{\mathbb{C}^2} \\ + \frac{\hbar Q}{2m}\nabla \times \big\langle \psi(t)(\boldsymbol{x}), \boldsymbol{\sigma}\psi(t)(\boldsymbol{x})\big\rangle_{\mathbb{C}^2}.\end{aligned}$$

Note that this system only differs from the $(N = 1)$-case of (1.8) by the presence of $-\frac{\hbar Q}{2mc}\boldsymbol{\sigma} \cdot \nabla \times \boldsymbol{A}\psi$ in (1.24) and the presence of $\frac{\hbar Q}{2m}\nabla \times \big\langle \psi, \boldsymbol{\sigma}\psi\big\rangle_{\mathbb{C}^2}$ in the expression for the probability current density. To our knowledge the well-posedness of the Maxwell-Pauli system has not yet been studied by anyone, even in the simplest case $N = 1$. We have tried several different approaches to proving local in time existence of a solution to the Cauchy problem associated with (1.24), but all of these methods seem to break down due to the presence of the two derivative-terms mentioned above. Instead, we have turned to the problem of proving existence of travelling wave solutions to the one-body Maxwell-Pauli (and Maxwell-Schrödinger) system, as explained in the following section.

Travelling Waves

Until now we have only considered whether the laws of quantum mechanics and electrodynamics admit matter to exist and evolve in time. But in order to give an appropriate description of reality these laws also ought to predict particle motions that correspond with our expectations. For instance, we expect a single charged particle to be able to move in space at a constant velocity $\boldsymbol{v} \in \mathbb{R}^3$, while still obeying the Maxwell equations as well as the Schrödinger equation. Put another way, there should exist a solution $(\psi_{\mathrm{t}}, \boldsymbol{A}_{\mathrm{t}})$ to the $(N = 1)$-case of (1.8) (or (1.24)) in the form

$$\begin{aligned}\psi_{\mathrm{t}}(t)(\boldsymbol{x}) &= \mathrm{e}^{-i\omega t}\psi(\boldsymbol{x} - \boldsymbol{v}t), \\ \boldsymbol{A}_{\mathrm{t}}(t)(\boldsymbol{x}) &= \boldsymbol{A}(\boldsymbol{x} - \boldsymbol{v}t),\end{aligned} \tag{1.25}$$

where $(\psi, \boldsymbol{A})$ is defined on $\mathbb{R}^3$ and ω is a real number. Such a solution $(\psi_{\mathrm{t}}, \boldsymbol{A}_{\mathrm{t}})$ is called a *travelling wave* solution for obvious reasons.

> *Objective 2*: Given any appropriate velocity $\boldsymbol{v} \in \mathbb{R}^3$ prove that we can find an $\omega \in \mathbb{R}$ such that the pair $(\psi_{\mathrm{t}}, \boldsymbol{A}_{\mathrm{t}})$ defined in (1.25) solves the $(N = 1)$-case of (1.8) (or (1.24)).

Under suitable conditions on the velocity $\boldsymbol{v}$ we manage to meet *Objective 2* in [56]. If the particle were uncharged it's total energy would naturally be given by $\frac{1}{2}m|\boldsymbol{v}|^2$ as a function of the speed $|\boldsymbol{v}|$ of motion. But a particle with nonzero charge has to drag it's self-generated electromagnetic field along with it – to what extent does this affect the dependence of the total energy on the speed $|\boldsymbol{v}|$? We provide an answer to this question in [56].

Overview of the Results 2

We will now state the precise results of the two papers contained in this thesis. For a more indepth explanation of the notation we refer the reader to the Introduction.

Existence of a Unique Local Solution to the Many-body Maxwell-Schrödinger Initial Value Problem

The many-body Maxwell-Schrödinger system, describing a system of N charged particles, reads

$$\begin{aligned} \Box \boldsymbol{A} &= \frac{4\pi}{c} \sum_{j=1}^{N} P \boldsymbol{J}_j[\psi, \boldsymbol{A}], \\ i\hbar \partial_t \psi &= \left(\sum_{j=1}^{N} \frac{\left(i\hbar \nabla_{\boldsymbol{x}_j} + \frac{Q_j}{c} \boldsymbol{A}(\boldsymbol{x}_j)\right)^2}{2m_j} + \sum_{1 \le j < k \le N} \frac{Q_j Q_k}{|\boldsymbol{x}_j - \boldsymbol{x}_k|} + \mathscr{E}_{\mathrm{EM}}[\boldsymbol{A}, \partial_t \boldsymbol{A}] \right) \psi, \end{aligned} \tag{2.1}$$

where $\boldsymbol{A}(t) : \mathbb{R}^3 \to \mathbb{R}^3$ is the magnetic vector potential expected to satisfy the Coulomb gauge condition $\operatorname{div} \boldsymbol{A}(t) = 0$ at all times, $\psi(t) : \mathbb{R}^{3N} \to \mathbb{C}$ is the quantum mechanical wave function and $\boldsymbol{J}_j[\psi, \boldsymbol{A}](t) : \mathbb{R}^3 \to \mathbb{R}^3$ is the probability current density

$$\boldsymbol{x}_j \mapsto \left(-\frac{Q_j}{m_j} \mathrm{Re} \int_{\mathbb{R}^{3(N-1)}} \overline{\psi}(t)(\boldsymbol{x}) \Big(i\hbar \nabla_{\boldsymbol{x}_j} + \frac{Q_j}{c} \boldsymbol{A}(t)(\boldsymbol{x}_j) \Big) \psi(t)(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}_j' \right)$$

associated with the j'th particle. It seems that well-posedness questions concerning this system have not yet been considered in the literature. On the other hand, several authors have studied the *d-dimensional Maxwell-Schrödinger system*

$$\begin{aligned} -\Delta_d\varphi - \frac{1}{c}\partial_t \mathrm{div}_d \boldsymbol{A} &= 4\pi Q|\psi|^2, \\ \Box_d \boldsymbol{A} + \nabla_d\Big(\frac{1}{c}\partial_t\varphi + \mathrm{div}_d\boldsymbol{A}\Big) &= -\frac{4\pi}{c}\frac{Q}{m}\mathrm{Re}\Big(\overline{\psi}\Big(i\hbar\nabla_d + \frac{Q}{c}\boldsymbol{A}\Big)\psi\Big), \\ i\hbar\partial_t\psi &= \Big(\frac{1}{2m}\Big(i\hbar\nabla_d + \frac{Q}{c}\boldsymbol{A}\Big)^2 + Q\varphi\Big)\psi, \end{aligned} \tag{2.2}$$

where $(\psi,\varphi,\boldsymbol{A})(t) : \mathbb{R}^d \to \mathbb{C}\times\mathbb{R}\times\mathbb{R}^d$, $\Delta_d = \sum_{k=1}^d \partial_{x^k}^2$, $\Box_d = \frac{1}{c^2}\partial_t^2 - \Delta_d$, $\nabla_d = (\partial_{x^1},\dots,\partial_{x^d})$ and $\mathrm{div}_d\boldsymbol{A} = \sum_{k=1}^d \partial_{x^k}A^k$. However, the known results concerning (2.2) can not be directly applied to (2.1) due to the presence of the Coulomb singularities $\frac{1}{|\boldsymbol{x}_j-\boldsymbol{x}_k|}$ in (2.1). For $d=3$ the system (2.2) can be expressed in Coulomb gauge by

$$\begin{aligned} \Box\boldsymbol{A} &= \frac{4\pi}{c}P\Big(-\frac{Q}{m}\mathrm{Re}\Big(\overline{\psi}\Big(i\hbar\nabla + \frac{Q}{c}\boldsymbol{A}\Big)\psi\Big)\Big) \\ i\hbar\partial_t\psi &= \Big(\frac{1}{2m}\Big(i\hbar\nabla + \frac{Q}{c}\boldsymbol{A}\Big)^2 + Q^2\big(|\boldsymbol{x}|^{-1} * |\psi|^2\big)\Big)\psi, \end{aligned} \tag{2.3}$$

which only deviates from the $(N=1)$-case of (2.1) by the presence of the term $Q^2\big(|\boldsymbol{x}|^{-1} * |\psi|^2\big)\psi$ and the absence of the term $\mathscr{E}_{\mathrm{EM}}[\boldsymbol{A},\partial_t\boldsymbol{A}]$ on the right hand side of (2.3)'s second equation. The term $Q^2\big(|\boldsymbol{x}|^{-1} * |\psi|^2\big)\psi$ should definitely not be included in the equations of motion describing a single charged particle so at least in this simple case (2.1) is evidently a better description of reality than (2.3). Nakamitsu and Tsutsumi treat (2.2) expressed in Lorenz gauge in the paper [52], where they prove local well-posedness of the system for all dimensions d and in the special cases $d \in \{1,2\}$ they even prove global well-posedness. The well-posedness theory regarding (2.2) is developed further in the papers [27, 28, 29, 30, 33, 52, 53, 54, 57, 61] and the research culminates with Bejenaru and Tataru [2] proving global well-posedness in the energy space of the three-dimensional version of (2.2) expressed in Coulomb gauge and with Wada [62] proving the analogous result for the two-dimensional variant of (2.2) expressed in Lorenz gauge. Our main result concerning the many-body Maxwell-Schrödinger system says the following.

Theorem 1. *Let* $(\psi_0, \boldsymbol{A}_0, \boldsymbol{A}_1) \in H^2(\mathbb{R}^{3N}) \times H^{\frac{3}{2}}(\mathbb{R}^3;\mathbb{R}^3) \times H^{\frac{1}{2}}(\mathbb{R}^3;\mathbb{R}^3)$ *satisfy* $\mathrm{div}\boldsymbol{A}_0 = \mathrm{div}\boldsymbol{A}_1 = 0$. *Then there exist a* $T > 0$ *and a unique* $(\psi, \boldsymbol{A})$ *in* $C([0,T]; H^2(\mathbb{R}^{3N})) \times \big(C([0,T]; H^{\frac{3}{2}}(\mathbb{R}^3;\mathbb{R}^3)) \cap C^1([0,T]; H^{\frac{1}{2}}(\mathbb{R}^3;\mathbb{R}^3))\big)$ *that solves* (2.1)*, fulfills the condition* $\mathrm{div}\boldsymbol{A}(t) = 0$ *for all* $t \in [0,T]$*, and takes the values*

$$\psi(0) = \psi_0, \boldsymbol{A}(0) = \boldsymbol{A}_0 \text{ and } \partial_t \boldsymbol{A}(0) = \boldsymbol{A}_1$$

at time $t = 0$.

To prove Theorem 1 we use the same strategy as in [53, 54]: We first introduce a solution mapping $(\psi, \boldsymbol{A}) \mapsto (\xi, \boldsymbol{B})$ associated with the linearization

$$(\Box + 1)\boldsymbol{B} = \frac{4\pi}{c} \sum_{j=1}^{N} P\boldsymbol{J}_j[\psi, \boldsymbol{A}] + \boldsymbol{A} \tag{2.4}$$

$$i\hbar\partial_t \xi = \Big(\sum_{j=1}^{N} \frac{1}{2m_j}\Big(i\hbar\nabla_{\boldsymbol{x}_j} + \frac{Q_j}{c}\boldsymbol{A}(\boldsymbol{x}_j)\Big)^2 + \sum_{1\le j<k\le N} \frac{Q_j Q_k}{|\boldsymbol{x}_j - \boldsymbol{x}_k|}\Big)\xi, \tag{2.5}$$

of (2.1) supplied with the initial conditions

$$\xi(0) = \psi_0, \quad \boldsymbol{B}(0) = \boldsymbol{A}_0 \quad \text{and} \quad \partial_t \boldsymbol{B}(0) = \boldsymbol{A}_1.$$

Then we show that this mapping is a contraction on an appropriate complete normed space – this implies the desired result by the Banach fixed-point theorem. The solution operator corresponding to (2.5) is defined by means of Kato's result [34, 35] concerning Cauchy problems related to quite general linear evolution equations – the required estimate on a certain norm of this operator is obtained by using Gronwall's inequality. To treat the solution to the linear Klein-Gordon equation (2.4) we use a known Strichartz estimate [4, 25, 26, 60].

Existence of Travelling Wave Solutions to the Maxwell-Pauli and Maxwell-Schrödinger Systems

In this paper that is joint work with Jan Philip Solovej, we prove the existence of travelling wave solutions to the Maxwell-Schrödinger system ($j = \mathrm{S}$)

provided that the velocity of the wave is not too large – we also prove the analogous result for the Maxwell-Pauli system ($j = \mathrm{P}$). These systems can be formulated in Coulomb gauge as

$$\begin{aligned} \Box \boldsymbol{A}_{\mathrm{t}} &= \frac{4\pi}{c} P \boldsymbol{J}_j[\psi_{\mathrm{t}}, \boldsymbol{A}_{\mathrm{t}}], \\ i\hbar \partial_t \psi_{\mathrm{t}} &= \Big(\frac{1}{2m} \nabla_{j,\boldsymbol{A}_{\mathrm{t}}}^2 + \mathscr{E}_{\mathrm{EM}}[\boldsymbol{A}_{\mathrm{t}}, \partial_t \boldsymbol{A}_{\mathrm{t}}] \Big) \psi_{\mathrm{t}} \end{aligned} \tag{2.6}$$

for $j \in \{\mathrm{S}, \mathrm{P}\}$, where $\psi_{\mathrm{t}}(t) : \mathbb{R}^3 \to \mathbb{C}^2$ denotes the spinor wave function, the magnetic vector potential $\boldsymbol{A}_{\mathrm{t}}(t) : \mathbb{R}^3 \to \mathbb{R}^3$ should satisfy $\mathrm{div} \boldsymbol{A}_{\mathrm{t}}(t) = 0$ at all times t and we use the abbreviations

$$\nabla_{j,\boldsymbol{A}_{\mathrm{t}}} = \begin{cases} i\hbar \nabla + \dfrac{Q}{c} \boldsymbol{A}_{\mathrm{t}} & \text{for } j = \mathrm{S} \\ \boldsymbol{\sigma} \cdot \Big(i\hbar \nabla + \dfrac{Q}{c} \boldsymbol{A}_{\mathrm{t}} \Big) & \text{for } j = \mathrm{P} \end{cases}$$

as well as

$$\boldsymbol{J}_j[\psi_{\mathrm{t}}, \boldsymbol{A}_{\mathrm{t}}](t)(\boldsymbol{x}) = \begin{cases} -\dfrac{Q}{m} \mathrm{Re} \big\langle \psi_{\mathrm{t}}(t)(\boldsymbol{x}), \nabla_{\mathrm{S},\boldsymbol{A}_{\mathrm{t}}} \psi_{\mathrm{t}}(t)(\boldsymbol{x}) \big\rangle_{\mathbb{C}^2} & \text{for } j = \mathrm{S} \\ -\dfrac{Q}{m} \mathrm{Re} \big\langle \psi_{\mathrm{t}}(t)(\boldsymbol{x}), \boldsymbol{\sigma} \nabla_{\mathrm{P},\boldsymbol{A}_{\mathrm{t}}} \psi_{\mathrm{t}}(t)(\boldsymbol{x}) \big\rangle_{\mathbb{C}^2} & \text{for } j = \mathrm{P} \end{cases}.$$

By a travelling wave – at velocity $\boldsymbol{v} \in \mathbb{R}^3$ – we mean a pair $(\psi_{\mathrm{t}}, \boldsymbol{A}_{\mathrm{t}})$ that can be written in the form

$$\begin{aligned} \psi_{\mathrm{t}}(t)(\boldsymbol{x}) &= \mathrm{e}^{-i\omega t} \psi(\boldsymbol{x} - \boldsymbol{v}t), \\ \boldsymbol{A}_{\mathrm{t}}(t)(\boldsymbol{x}) &= \boldsymbol{A}(\boldsymbol{x} - \boldsymbol{v}t) \end{aligned}$$

for some real number ω and some time independent $(\psi, \boldsymbol{A}) : \mathbb{R}^3 \to \mathbb{C}^2 \times \mathbb{R}^3$. There are not that many known results concerning travelling wave solutions to the equations of motion for a single charged particle. Coclite and Georgiev prove in [8] that the three-dimensional version of (2.2) expressed in Lorenz gauge has no nontrivial solutions $(\psi_{\mathrm{t}}, \boldsymbol{A}_{\mathrm{t}}, \varphi_{\mathrm{t}})$ in the form

$$\begin{aligned} \psi_{\mathrm{t}}(t)(\boldsymbol{x}) &= \mathrm{e}^{-i\omega t} \psi(\boldsymbol{x}), \\ \boldsymbol{A}_{\mathrm{t}}(t)(\boldsymbol{x}) &= \boldsymbol{0}, \\ \varphi_{\mathrm{t}}(t)(\boldsymbol{x}) &= \varphi(\boldsymbol{x}) \end{aligned}$$

and they also find that such solutions do exist when one adds to the Schrödinger equation a potential energy term originating from an attractive Coulomb force. In the paper [3], Benci and Fortunato study the corresponding problem in a bounded space region and in [16], Esteban, Georgiev and Séré prove the existence of stationary solutions to the Maxwell-Dirac and Klein-Gordon-Dirac systems. Finally, we mention that Fröhlich, Jonsson and Lenzmann [23] show the existence of travelling solitary wave solutions to an equation describing the dynamics of pseudo-relativistic boson stars in the mean field limit – again under a smallness assumption on the speed of the wave. As mentioned above the system (2.6) serves as a better description of a single charged particle than (2.2) – the existence of travelling wave solutions to (2.6) has never been studied before. Let us formulate our main theorem concerning the Maxwell-Schrödinger system. We remind the reader that $H^1(\mathbb{R}^3;\mathbb{C}^2)$ denotes the usual Sobolev space of order 1 and $D^1(\mathbb{R}^3;\mathbb{R}^3)$ denotes the space of locally integrable maps $\mathbb{R}^3 \to \mathbb{R}^3$ that vanish at infinity and have square integrable first derivatives.

Theorem 2. *For all* $\lambda > 0$ *and* $\boldsymbol{v} \in \mathbb{R}^3$ *with* $0 < |\boldsymbol{v}| < c$ *there exist* $\omega \in \mathbb{R}$ *and* $(\psi, \boldsymbol{A}) \in H^1(\mathbb{R}^3;\mathbb{C}^2) \times D^1(\mathbb{R}^3;\mathbb{R}^3)$ *satisfying* $\|\psi\|_{L^2}^2 = \lambda$ *and* $\mathrm{div}\boldsymbol{A} = 0$ *such that* $(\psi_{\mathrm{t}}, \boldsymbol{A}_{\mathrm{t}})(t)(\boldsymbol{x}) = \big(\mathrm{e}^{-i\omega t}\psi(\boldsymbol{x} - \boldsymbol{v}t), \boldsymbol{A}(\boldsymbol{x} - \boldsymbol{v}t)\big)$ *solves the* $(j = \mathrm{S})$*-case of* (2.6).

Here, $\lambda = 1$ is the physically relevant case since $\|\psi\|_{L^2}^2$ should be interpreted as the total probability that the charged particle is located somewhere in $\mathbb{R}^3$. We see that the Maxwell-Schrödinger system admits charged particles (and their self-generated electromagnetic fields) to travel with nonzero speeds less than that of the light's – this upper bound on the allowed speeds of motion is in agreement with the special theory of relativity so travelling wave solutions to (2.6) with $|\boldsymbol{v}| \geq c$ do most likely not exist. However, we have not proven this. Our main theorem about the Maxwell-Pauli system says the following – we let K_S be the constant occurring in the Sobolev inequality $\|A\|_{L^6} \leq K_S \|\nabla A\|_{L^2}$.

Theorem 3. *Consider an arbitrary* $\lambda > 0$ *as well as some given velocity* $\boldsymbol{v} \in \mathbb{R}^3$ *satisfying*

$$0 < |\boldsymbol{v}| < -\frac{8\pi K_S^3 Q^2 \lambda}{\hbar} + \sqrt{\frac{(8\pi)^2 K_S^6 Q^4 \lambda^2}{\hbar^2} + c^2}.$$

Then there exist an $\omega \in \mathbb{R}$ *and a pair* $(\psi, \boldsymbol{A}) \in H^1(\mathbb{R}^3;\mathbb{C}^2) \times D^1(\mathbb{R}^3;\mathbb{R}^3)$ *fulfilling the identities* $\|\psi\|_{L^2}^2 = \lambda$ *and* $\mathrm{div}\boldsymbol{A} = 0$ *such that the travelling* *wave* $(\psi_{\rm t}, \boldsymbol{A}_{\rm t})(t)(\boldsymbol{x}) = \big(\mathrm{e}^{-i\omega t}\psi(\boldsymbol{x}-\boldsymbol{v}t), \boldsymbol{A}(\boldsymbol{x}-\boldsymbol{v}t)\big)$ *solves the* $(j = \mathrm{P})$*-case of* (2.6).

As illustrated on Figure 2 the upper bound on the values of $|\boldsymbol{v}|$ presented in Theorem 3 is strictly less than c – this is supposedly an outcome of the technique used to prove Theorem 3 and should probably not be attributed any physical importance. In other words, we expect that there exist travel-

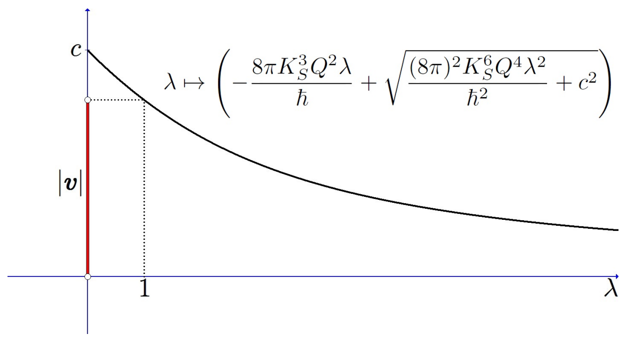

Figure 2: The allowed speeds $|\boldsymbol{v}|$ of motion in the physical case $\lambda = 1$ are highlighted in red.

ling wave solutions to the Maxwell-Pauli system with any velocity $\boldsymbol{v} \in \mathbb{R}^3$ satisfying $0 < |\boldsymbol{v}| < c$.

The Theorems 2 and 3 are proven by minimizing the functional

$$\mathscr{E}_j^{\boldsymbol{v}}(\psi, \boldsymbol{A}) = \frac{1}{2m}\big\|\nabla_{j,\boldsymbol{A}}\psi\big\|_{L^2}^2 + \frac{1}{8\pi}\Big(\|\nabla\otimes\boldsymbol{A}\|_{L^2}^2 - \Big\|\Big(\frac{\boldsymbol{v}}{c}\cdot\nabla\Big)\boldsymbol{A}\Big\|_{L^2}^2\Big) + (\psi, i\hbar\boldsymbol{v}\cdot\nabla\psi)_{L^2}$$

on the set

$$\mathcal{S}_\lambda = \big\{(\psi, \boldsymbol{A}) \in H^1 \times D^1 \,\big|\, \|\psi\|_{L^2}^2 = \lambda, \mathrm{div}\boldsymbol{A} = 0\big\}$$

for $j \in \{\mathrm{S}, \mathrm{P}\}$. Here, the translation invariance of $\mathscr{E}_j^{\boldsymbol{v}}(\psi, \boldsymbol{A})$ complicates the minimization problem considerably, but we resolve this issue by deriving and applying a variant of the concentration-compactness principle by Lions [49, 50]. Unfortunately, this method of proof does not allow us to draw any conclusions regarding uniqueness of the solution.

Instead, we investigate the dependence of the found travelling wave solutions' energies on the speed $|\boldsymbol{v}|$ of the particle for small values of $|\boldsymbol{v}|$. By the energy of a (sufficiently nice) solution $(\psi_{\mathrm{t}}, \boldsymbol{A}_{\mathrm{t}})$ to (2.6) we here mean the average $\big(\psi_{\mathrm{t}}, \mathscr{H}_j\big(\boldsymbol{A}_{\mathrm{t}}, \frac{\partial_t \boldsymbol{A}_{\mathrm{t}}}{4\pi c^2}\big)\psi_{\mathrm{t}}\big)_{L^2}$ of the energy observable given by

$$\mathscr{H}_j\big(\boldsymbol{A}, -\tfrac{P\boldsymbol{E}}{4\pi}\big) = \frac{1}{2m}\nabla_{j,\boldsymbol{A}}^2 + \mathscr{E}_{\mathrm{EM}}[\boldsymbol{A}, -c^2 P\boldsymbol{E}].$$

For travelling wave solutions $(\psi_{\mathrm{t}}, \boldsymbol{A}_{\mathrm{t}})(t)(\boldsymbol{x}) = \big(\mathrm{e}^{-i\omega t}\psi(\boldsymbol{x} - \boldsymbol{v}t), \boldsymbol{A}(\boldsymbol{x} - \boldsymbol{v}t)\big)$ the energy takes the form

$$E_j(\boldsymbol{v}, \psi, \boldsymbol{A}) = \frac{1}{2m}\|\nabla_{j,\boldsymbol{A}}\psi\|_{L^2}^2 + \frac{1}{8\pi}\int_{\mathbb{R}^3}\Big(|\nabla \times \boldsymbol{A}|^2 + \Big|\Big(\frac{\boldsymbol{v}}{c}\cdot\nabla\Big)\boldsymbol{A}\Big|^2\Big)\,\mathrm{d}\boldsymbol{x}\lambda.$$

We have found the following result concerning this energy function.

Theorem 4. *Let $j \in \{\mathrm{S}, \mathrm{P}\}$ and $\lambda > 0$ be arbitrary. Then there exist universal constants $\theta_j, \kappa_j > 0$ such that the inequality*

$$\Big|E_j(\boldsymbol{v}, \psi, \boldsymbol{A}) - \frac{m\boldsymbol{v}^2}{2}\lambda\Big| \le \kappa_j|\boldsymbol{v}|^3$$

holds for all velocities $\boldsymbol{v} \in \mathbb{R}^3$ with $0 < |\boldsymbol{v}| < \theta_j$ and all minimizers $(\psi, \boldsymbol{A})$ of $\mathscr{E}_j^{\boldsymbol{v}}$ on $\mathcal{S}_\lambda$.

Thus, the energy of the charged particle essentially behaves as if the particle were uncharged (for small velocities).

Conclusions and Perspectives 3

Let us now mention some interesting problems that are related to the ones studied in this thesis and would be natural to pursue in further research.

Problem 1: Global well-posedness of the many-body Maxwell-Schrödinger System

In [55], we have shown local existence of a unique solution to the many-body Maxwell-Schrödinger initial value problem expressed in Coulomb gauge. As is apparent from Theorem 1 the triple $(\psi_0, \boldsymbol{A}_0, \boldsymbol{A}_1)$ of initial data is required to belong to $H^2(\mathbb{R}^{3N}) \times H^{\frac{3}{2}}(\mathbb{R}^3; \mathbb{R}^3) \times H^{\frac{1}{2}}(\mathbb{R}^3; \mathbb{R}^3)$ and then as a function of time the corresponding solution $(\psi, \boldsymbol{A}, \partial_t \boldsymbol{A})$ will map continuously into the same space. However, it would be desirable to derive the same result for a space of less regularity, for instance $H^2(\mathbb{R}^{3N}) \times H^1(\mathbb{R}^3; \mathbb{R}^3) \times L^2(\mathbb{R}^3; \mathbb{R}^3)$ or $H^2(\mathbb{R}^{3N}) \times D^1(\mathbb{R}^3; \mathbb{R}^3) \times L^2(\mathbb{R}^3; \mathbb{R}^3)$. On another note, our intuition tells us that changing the initial data a "little bit" should only imply a "small" change in the solution. It would be desirable to confirm this intuition by showing that the solution depends continuously on the initial data. Finally, we would like to prove that the solution exists globally in time.

Problem 2: Well-posedness of the Maxwell-Pauli system

As already mentioned, even proving the existence of a local solution to the one-body Maxwell-Pauli initial value problem is an open problem. Well-posedness of the Maxwell-Pauli system is an intriguing topic, especially when seen in the light of the already known results concerning energetic stability. Remember that systems of charged particles with Pauli kinetic energy are only energetically stable under certain conditions on the fine-structure constant and on the atomic numbers of the nuclei. This raises some interesting questions regarding the influence of these parameters on the well-posedness of the Maxwell-Pauli system. Does global (or even local) well-posedness for instance fail to hold for large values of α? We do not know.

Problem 3: Uniqueness of travelling wave solutions

In [56], we prove the existence of travelling wave solutions to the Maxwell-Schrödinger and Maxwell-Pauli systems, provided that the speed of the wave is not too large – this is done by minimizing some functional $\mathscr{E}_j^{\boldsymbol{v}}$ on a certain set. The solution produced by this method is clearly not unique due to translation invariance of $\mathscr{E}_j^{\boldsymbol{v}}$, but it would be satisfactory to show uniqueness of the solution up to symmetries.

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Existence of a Unique Local Solution to the Many-body Maxwell-Schrödinger Initial Value Problem

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Abstract. We study the many-body problem of charged particles interacting with their self-generated electromagnetic field. We model the dynamics of the particles by the many-body Maxwell-Schrödinger system, where the particles are treated quantum mechanically and the electromagnetic field is a classical quantity. We prove the existence of a unique local in time solution to this nonlinear initial value problem using a contraction mapping argument.

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1 Introduction

The three-dimensional many-body Maxwell-Schrödinger system in Coulomb gauge is a system of partial differential equations that models the dynamics of several charged point particles interacting via their self-generated electromagnetic fields – in Gaussian units it reads

$$\begin{aligned} \Box \boldsymbol{A} &= \frac{4\pi}{c}\sum_{j=1}^{N} P\boldsymbol{J}_j[\psi,\boldsymbol{A}], \\ i\hbar\partial_t\psi &= \Big(\sum_{j=1}^{N}\frac{1}{2m_j}\nabla_{j,\boldsymbol{A}}^2 + \sum_{1\le j<k\le N}\frac{Q_jQ_k}{|\boldsymbol{x}_j-\boldsymbol{x}_k|} + \mathscr{E}_{\mathrm{EM}}[\boldsymbol{A},\partial_t\boldsymbol{A}]\Big)\psi, \end{aligned} \tag{1}$$

where $\hbar > 0$ is the reduced Planck constant, $c > 0$ is the speed of light, $N \in \mathbb{N}$ is the number of particles, $m_1, \dots, m_N > 0$ are the particles' respective masses, $Q_1, \dots, Q_N \in \mathbb{R}$ are their charges, $\psi(t) : \mathbb{R}^{3N} \to \mathbb{C}$ is the wave function,

$\boldsymbol{A}(t) : \mathbb{R}^3 \to \mathbb{R}^3$ is the vector potential, $\nabla_{j,\boldsymbol{A}} = i\hbar\nabla_{\boldsymbol{x}_j} + \frac{Q_j}{c}\boldsymbol{A}(\boldsymbol{x}_j)$ is the covariant derivative with respect to $\boldsymbol{A}$ acting on the j'th particle, $\Box = \frac{1}{c^2}\partial_t^2 - \Delta$ denotes the d'Alembertian, $P = 1 - \nabla\mathrm{div}\Delta^{-1}$ is the Helmholtz projection, $\mathscr{E}_{\mathrm{EM}}[\boldsymbol{A}, \partial_t\boldsymbol{A}](t)$ is the field energy

$$\mathscr{E}_{\mathrm{EM}}[\boldsymbol{A}, \partial_t\boldsymbol{A}](t) = \frac{1}{8\pi}\int_{\mathbb{R}^3}\Big(|\nabla \times \boldsymbol{A}(t)(\boldsymbol{y})|^2 + \Big|\frac{1}{c}\partial_t\boldsymbol{A}(t)(\boldsymbol{y})\Big|^2\Big)\,\mathrm{d}\boldsymbol{y}$$

and $\boldsymbol{J}_j[\psi, \boldsymbol{A}](t)$ denotes the j'th particle's probability current density

$$\boldsymbol{J}_j[\psi, \boldsymbol{A}](t) : \mathbb{R}^3 \ni \boldsymbol{x}_j \mapsto \Big(-\frac{Q_j}{m_j}\mathrm{Re}\int_{\mathbb{R}^{3(N-1)}}\overline{\psi}(t)(\boldsymbol{x})\nabla_{j,\boldsymbol{A}}\psi(t)(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}'_j\Big) \in \mathbb{R}^3$$

where $\boldsymbol{x} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)$ and $\boldsymbol{x}'_j = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{j-1}, \boldsymbol{x}_{j+1}, \ldots, \boldsymbol{x}_N)$. That we have chosen Coulomb gauge means that the magnetic vector potential $\boldsymbol{A}$ should satisfy

$$\mathrm{div}\boldsymbol{A}(t) = 0. \tag{2}$$

As we will see later we might just as well study the system

$$\begin{aligned} \Box\boldsymbol{A} &= \frac{4\pi}{c}\sum_{j=1}^{N} P\boldsymbol{J}_j[\psi, \boldsymbol{A}], \\ i\hbar\partial_t\psi &= \Big(\sum_{j=1}^{N}\frac{1}{2m_j}\nabla_{j,\boldsymbol{A}}^2 + \sum_{1\le j<k\le N}\frac{Q_jQ_k}{|\boldsymbol{x}_j - \boldsymbol{x}_k|}\Big)\psi, \end{aligned} \tag{3}$$

where the field energy-term $\mathscr{E}_{\mathrm{EM}}[\boldsymbol{A}, \partial_t\boldsymbol{A}]$ is no longer present in the Schrödinger equation. In the literature the d-dimensional Maxwell-Schrödinger system often refers to the coupled equations

$$\begin{aligned} -\Delta_d\varphi - \frac{1}{c}\partial_t\mathrm{div}_d\boldsymbol{A} &= 4\pi Q|\psi|^2, \\ \Box_d\boldsymbol{A} + \nabla_d\Big(\frac{1}{c}\partial_t\varphi + \mathrm{div}_d\boldsymbol{A}\Big) &= -\frac{4\pi}{c}\frac{Q}{m}\mathrm{Re}\Big(\overline{\psi}\Big(i\hbar\nabla_d + \frac{Q}{c}\boldsymbol{A}\Big)\psi\Big), \\ i\hbar\partial_t\psi &= \Big(\frac{1}{2m}\Big(i\hbar\nabla_d + \frac{Q}{c}\boldsymbol{A}\Big)^2 + Q\varphi\Big)\psi, \end{aligned} \tag{4}$$

with unknowns $\psi(t) : \mathbb{R}^d \to \mathbb{C}$, $\boldsymbol{A}(t) : \mathbb{R}^d \to \mathbb{R}^d$, $\varphi(t) : \mathbb{R}^d \to \mathbb{R}$ and a hopefully obvious notation. If $d = 3$ and $\boldsymbol{A}$ satisfies the Coulomb gauge-condition (2) the system (4) reads

$$\begin{aligned} \Box\boldsymbol{A} &= P\Big(-\frac{4\pi}{c}\frac{Q}{m}\mathrm{Re}\Big(\overline{\psi}\Big(i\hbar\nabla + \frac{Q}{c}\boldsymbol{A}\Big)\psi\Big)\Big), \\ i\hbar\partial_t\psi &= \Big(\frac{1}{2m}\Big(i\hbar\nabla + \frac{Q}{c}\boldsymbol{A}\Big)^2 + Q^2\big(|\boldsymbol{x}|^{-1} * |\psi|^2\big)\Big)\psi, \end{aligned} \tag{5}$$

which only differs from the ($N = 1$)-case of (3) by the presence of the nonlinear term $Q^2(|\boldsymbol{x}|^{-1} * |\psi|^2)\psi$ in the Schrödinger equation. This term comes from the particles' Coulomb self-interactions. From a physical point of view it is wrong to include self-interactions in this context. In fact, the system (5) may be considered as a mean field approximation to the many-body description (3). In the ($c \to \infty$)-limit the second equation in (3) reduces to the standard many-body Coulomb problem

$$i\hbar\partial_t\psi = \Big(-\sum_{j=1}^{N}\frac{\hbar^2}{2m_j}\Delta_{\boldsymbol{x}_j} + \sum_{1\le j<k\le N}\frac{Q_jQ_k}{|\boldsymbol{x}_j - \boldsymbol{x}_k|}\Big)\psi$$

which is the basis of almost all work done in quantum chemistry. On the other hand, the system (5) reduces in the ($c \to \infty$)-limit to a mean field approximation, which at best would be good in the large N limit. The system (4) has been studied (up to different choices of units) by several authors, both when expressed in the Coulomb gauge [4, 12, 13, 14, 15, 17, 21, 22, 24, 28], the Lorenz gauge [17, 20, 21, 22, 29] and the temporal gauge [17, 21, 22]. In [20], Nakamitsu and Tsutsumi prove local well-posedness in sufficiently regular Sobolev spaces of the d-dimensional Maxwell-Schrödinger initial value problem – for $d \in \{1, 2\}$ they also show global existence of the solution. Tsutsumi shows in [28] that for $d = 3$ the problem has a global solution for a certain set of final states (i.e. data given at $t = +\infty$) and studies the asymptotic behavior of such a solution. In [17], Guo, Nakamitsu and Strauss prove global solvability of the three-dimensional system in Coulomb gauge (but not uniqueness of the solution) for initial data $(\psi(0), \boldsymbol{A}(0), \partial_t\boldsymbol{A}(0))$ in the space of $H^1 \times H^1 \times L^2$-functions satisfying $\mathrm{div}\boldsymbol{A}(0) = \mathrm{div}\partial_t\boldsymbol{A}(0) = 0$. Using techniques on which the arguments in the present paper are based, Nakamura and Wada [21] prove local well-posedness of the three-dimensional problem in Sobolev spaces of sufficient regularity, expanding significantly on the previously known results – in [22] they even prove global existence of unique solutions. Bejenaru and Tataru [4] prove global well-posedness in the energy-space of the three-dimensional initial value problem and in the recent paper [29], Wada proves unique solvability in the energy space of the two-dimensional analogue. The scattering theory for (5) has also been studied by several authors – see the papers by Tsutsumi [28], Shimomura [24] as well as Ginibre and Velo [12, 13, 14, 15]. It seems that the solvability of the system (1) has not yet been studied and the known results concerning (4) are not directly applicable to this system due to the presence of the Coulomb singularities $\frac{1}{|\boldsymbol{x}_j - \boldsymbol{x}_k|}$ in (1). The aim of this paper is to prove the unique existence of a local solution to (1) as expressed in the following main theorem.

Theorem 1. *For all* $(\psi_0, \boldsymbol{A}_0, \boldsymbol{A}_1) \in H^2(\mathbb{R}^{3N}) \times H^{\frac{3}{2}}(\mathbb{R}^3;\mathbb{R}^3) \times H^{\frac{1}{2}}(\mathbb{R}^3;\mathbb{R}^3)$ *with* $\mathrm{div}\boldsymbol{A}_0 = \mathrm{div}\boldsymbol{A}_1 = 0$ *there exist a number* $T > 0$ *and a unique solution* $(\psi, \boldsymbol{A}) \in C([0,T]; H^2(\mathbb{R}^{3N})) \times \big(C([0,T]; H^{\frac{3}{2}}(\mathbb{R}^3;\mathbb{R}^3)) \cap C^1([0,T]; H^{\frac{1}{2}}(\mathbb{R}^3;\mathbb{R}^3))\big)$

to (1) *such that* $\mathrm{div}\boldsymbol{A}(t) = 0$ *for all* $t \in [0,T]$ *and the initial conditions*

$$\psi(0) = \psi_0, \boldsymbol{A}(0) = \boldsymbol{A}_0 \text{ and } \partial_t \boldsymbol{A}(0) = \boldsymbol{A}_1 \tag{6}$$

are satisfied.

Remark 2. In this paper, we consider all of the charged particles as being spinless. Let us just mention that by thinking of the particles as having spin and by including the interaction between this spin and the electromagnetic field in the kinetic energy operator we are led to another interesting system of partial differential equations: The many-body Maxwell-Pauli system. For now, let us just write up the one-body Maxwell-Pauli system – in Coulomb gauge it reads

$$\begin{aligned} \Box \boldsymbol{A} &= \frac{4\pi}{c} P \boldsymbol{\mathcal{J}}[\psi, \boldsymbol{A}], \\ i\hbar\partial_t \psi &= \left(\frac{1}{2m}\Big(i\hbar\nabla + \frac{Q}{c}\boldsymbol{A}\Big)^2 - \frac{\hbar Q}{2mc}\boldsymbol{\sigma}\cdot\nabla\times\boldsymbol{A} + \mathscr{E}_{\mathrm{EM}}[\boldsymbol{A}, \partial_t\boldsymbol{A}] \right)\psi, \end{aligned} \tag{7}$$

where the probability current density $\boldsymbol{\mathcal{J}}[\psi, \boldsymbol{A}](t) : \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$\begin{aligned} \boldsymbol{\mathcal{J}}[\psi, \boldsymbol{A}](t)(\boldsymbol{x}) = -\frac{Q}{m}\mathrm{Re}\Big\langle \psi(t)(\boldsymbol{x}), \Big(i\hbar\nabla + \frac{Q}{c}\boldsymbol{A}\Big)\psi(t)(\boldsymbol{x})\Big\rangle_{\mathbb{C}^2} \\ + \frac{\hbar Q}{2m}\nabla\times\langle\psi(t)(\boldsymbol{x}), \boldsymbol{\sigma}\psi(t)(\boldsymbol{x})\rangle_{\mathbb{C}^2} \end{aligned}$$

and $\boldsymbol{\sigma}$ is the vector with the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

as components. The techniques used in this paper to treat the many-body Maxwell-Schrödinger system do not seem to be immediately adaptable to the Maxwell-Pauli system and so the existence of a local solution to the initial value problem corresponding to (7) is an open problem.

Remark 3. Suppose that $m_1 = \cdots = m_N$ and $Q_1 = \cdots = Q_N$ so that the N particles are indistinguishable and consider an initial state ψ_0 where either all of the particles are bosonic ($s = 0$) or all of the particles are fermionic ($s = 1$). If $e_{\ell n} : \mathbb{R}^{3N} \to \mathbb{R}^{3N}$ is the coordinate exchange map given by

$$e_{\ell n}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_\ell, \ldots, \boldsymbol{x}_n, \ldots, \boldsymbol{x}_N) = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n, \ldots, \boldsymbol{x}_\ell, \ldots, \boldsymbol{x}_N)$$

this means that $\psi_0 = (-1)^s \psi_0 \circ e_{\ell n}$ for all $\ell, n \in \{1, \ldots, N\}$ with $\ell < n$. With $(\psi, \boldsymbol{A})$ denoting the solution to (1)+(6) whose existence is established in Theorem 1 one can easily verify that $t \mapsto \big((-1)^s\psi(t)\circ e_{\ell n}, \boldsymbol{A}(t)\big)$ solves (1)+(6) too. But then the uniqueness result of Theorem 1 implies that the identity $\psi(t) = (-1)^s\psi(t) \circ e_{\ell n}$ holds at all times t of existence so in other words the particles will continue to obey the same particle statistics as they did in the initial state.

The paper is organized as follows. We will end this introduction by establishing some notation and in Section 2 we (formally) motivate the model (1). In Section 3 we take the first steps towards proving Theorem 1 – the basic strategy for obtaining the existence part of the theorem will be to find a fixed point for the solution mapping associated with a certain linearization of the many-body Maxwell-Schrödinger system. The linear equations constituting this linearization are studied in Sections 4 and 5 – more specifically, the many-body Schrödinger equation is studied in Section 4 by means of a result by Kato [18, 19] and in Section 5 we recall a result developed by Brenner [5], Strichartz [27], Ginibre and Velo [10, 11] concerning the Klein-Gordon equation. Finally, we prove existence of the desired solution in Section 6 and the uniqueness part is proven in Section 7.

As can be seen from the statement of Theorem 1 the values of the time variable will vary in some closed interval $\mathcal{I}_T = [0, T]$ where $T > 0$. For some given reflexive Banach space $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ we will let $C(\mathcal{I}_T; \mathcal{X})$ denote the space of continuous mappings $\mathcal{I}_T \to \mathcal{X}$ and $C^1(\mathcal{I}_T; \mathcal{X})$ will denote the subspace of maps $\psi \in C(\mathcal{I}_T; \mathcal{X})$ whose strong derivative

$$\partial_t \psi(t) = \begin{cases} \lim\limits_{h\to 0^+} \dfrac{\psi(t+h)-\psi(t)}{h} & \text{for } t = 0 \\ \lim\limits_{h\to 0} \dfrac{\psi(t+h)-\psi(t)}{h} & \text{for } t \in (0,T) \\ \lim\limits_{h\to 0^-} \dfrac{\psi(t+h)-\psi(t)}{h} & \text{for } t = T \end{cases}$$

is well defined and continuous everywhere in $\mathcal{I}_T$. For $p \in [1, \infty]$ we let $L^p(\mathcal{I}_T; \mathcal{X})$ denote the space of (equivalence classes of) strongly Lebesgue-measurable functions $\psi : \mathcal{I}_T \to \mathcal{X}$ with the property that

$$\|\psi\|_{L^p_T \mathcal{X}} = \begin{cases} \Big(\displaystyle\int_{\mathcal{I}_T} \|\psi(t)\|^p_{\mathcal{X}} \,\mathrm{d}t\Big)^{\frac{1}{p}} & \text{if } 1 \le p < \infty \\ \operatorname*{ess\,sup}\limits_{t\in\mathcal{I}_T} \|\psi(t)\|_{\mathcal{X}} & \text{if } p = \infty \end{cases}$$

is finite. Equipping $L^p(\mathcal{I}_T; \mathcal{X})$ with the norm $\|\cdot\|_{L^p_T \mathcal{X}}$ results in a Banach space. Just as in the case where $\mathcal{X} = \mathbb{C}$ any given $\psi \in L^p(\mathcal{I}_T; \mathcal{X})$ can be identified with the $\mathcal{X}$-valued distribution that sends $f \in C_0^\infty(\mathcal{I}_T^\circ)$ into the Bochner integral $\int_{\mathcal{I}_T} \psi(t) f(t) \,\mathrm{d}t \in \mathcal{X}$; thus, it makes sense to consider the space $W^{1,p}(\mathcal{I}_T; \mathcal{X})$ of $L^p(\mathcal{I}_T; \mathcal{X})$-functions with distributional derivative $\partial_t \psi$ in $L^p(\mathcal{I}_T; \mathcal{X})$, which is a Banach space when endowed with the norm

$$\|\psi\|_{W^{1,p}_T \mathcal{X}} = \big(\|\psi\|^2_{L^p_T \mathcal{X}} + \|\partial_t \psi\|^2_{L^p_T \mathcal{X}}\big)^{\frac{1}{2}}.$$

For a nice introduction to the spaces $W^{1,p}(\mathcal{I}_T, \mathcal{X})$ we refer to Section 1.4 in [3]. Let us just mention one result that we will often use: For $\psi \in L^p(\mathcal{I}_T; \mathcal{X})$ the condition that $\psi \in W^{1,p}(\mathcal{I}_T;\mathcal{X})$ is equivalent to the existence of an absolutely continuous $\psi_0 : \mathcal{I}_T \to \mathcal{X}$ with strong derivative $\partial_t\psi_0 : t \mapsto \lim_{h\to 0} \frac{\psi_0(t+h)-\psi_0(t)}{h}$ in $L^p(\mathcal{I}_T;\mathcal{X})$ such that $\psi(t) = \psi_0(t)$ for almost all $t \in \mathcal{I}_T$. Moreover, the Sobolev embedding $W^{1,p}(\mathcal{I}_T;\mathcal{X}) \hookrightarrow_{p,T} L^\infty(\mathcal{I}_T;\mathcal{X})$ holds true. If $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ is another Banach space we let $(\mathcal{L}(\mathcal{X},\mathcal{Y}), \|\cdot\|_{\mathcal{L}(\mathcal{X},\mathcal{Y})})$ denote the Banach space of bounded linear operators $\mathcal{X} \to \mathcal{Y}$ and set $\mathcal{L}(\mathcal{X}) = \mathcal{L}(\mathcal{X},\mathcal{X})$. By $A \lesssim B$ we mean that there exists a universal constant $c > 0$ such that $A \leq cB$. Finally, we let $p' = \frac{p}{p-1}$ denote the Hölder conjugate to a given $p \in [1,\infty]$ and set $\langle s\rangle = \sqrt{1+s^2}$ for $s \in \mathbb{R}$.

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2 Motivation for the Model

As our starting point we use the Abraham model of charged particles. So for some arbitrary $R > 0$ and some positive C_0^∞-function χ with $\int_{\mathbb{R}^3} \chi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 1$ we set $\chi_R : \boldsymbol{x} \mapsto \frac{1}{R^3}\chi\big(\frac{\boldsymbol{x}}{R}\big)$ and associate the smeared out charge distribution $\rho_{R,j} : \boldsymbol{x} \mapsto Q_j\chi_R(\boldsymbol{x}_j - \boldsymbol{x})$ to the j'th particle – the corresponding Maxwell equations can be written as

$$\mathrm{div}\boldsymbol{B}(t) = 0, \tag{8}$$

$$\nabla \times \boldsymbol{E}(t) = -\frac{1}{c}\partial_t\boldsymbol{B}(t), \tag{9}$$

$$\mathrm{div}\boldsymbol{E}(t) = 4\pi\sum_{j=1}^{N}\rho_{R,j}(t), \tag{10}$$

$$\nabla \times \boldsymbol{B}(t) = \frac{1}{c}\Big(\partial_t\boldsymbol{E}(t) + 4\pi\sum_{j=1}^{N}\frac{\mathrm{d}\boldsymbol{x}_j}{\mathrm{d}t}(t)\rho_{R,j}(t)\Big), \tag{11}$$

and the Lorentz force law states that

$$m_j\frac{\mathrm{d}^2\boldsymbol{x}_j}{\mathrm{d}t^2}(t) = Q_j\Big(\frac{1}{c}\frac{\mathrm{d}\boldsymbol{x}_j}{\mathrm{d}t}(t) \times \boldsymbol{B}(t) + \boldsymbol{E}(t)\Big) * \chi_R(\boldsymbol{x}_j(t)) \quad \text{for } j \in \{1,\dots,N\}, \tag{12}$$

where we interpret the coordinates of $\boldsymbol{x} = (\boldsymbol{x}_1,\dots,\boldsymbol{x}_N) \in \mathbb{R}^{3N}$ as the positions of the N particles, $\boldsymbol{B}$ is the magnetic field and $\boldsymbol{E}$ denotes the electric field. The reason for smearing out the charges is that the coupled Maxwell-Lorentz system does not make sense in the point particle case as explained in [26].

Now, (8) ensures that $\boldsymbol{B}(t) : \mathbb{R}^3 \to \mathbb{R}^3$ can be written as the curl of some magnetic vector potential $\boldsymbol{A}(t) : \mathbb{R}^3 \to \mathbb{R}^3$, whereby (9) allows us to write

$-\boldsymbol{E}(t) - \frac{1}{c}\partial_t \boldsymbol{A}(t) : \mathbb{R}^3 \to \mathbb{R}^3$ as the gradient of some electric scalar potential $V : \mathbb{R}^3 \to \mathbb{R}$. In other words,

$$\boldsymbol{B}(t) = \nabla \times \boldsymbol{A}(t) \text{ and } \boldsymbol{E}(t) = -\frac{1}{c}\partial_t \boldsymbol{A}(t) - \nabla V(t). \tag{13}$$

The choice of potentials is not unique – if $(V, \boldsymbol{A})$ is an electromagnetic potential corresponding to the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ then for any $\eta(t) : \mathbb{R}^3 \to \mathbb{R}$ the pair $\big(V - \frac{1}{c}\partial_t \eta, \boldsymbol{A} + \nabla\eta\big)$ will also serve as such a potential. This freedom of choice allows us to demand that $\boldsymbol{A}$ satisfies the Coulomb gauge condition (2).

To formulate the problem in the Lagrangian formalism we choose the Hilbert manifold $\mathcal{Q}^0 = \mathbb{R}^{3N} \times D^1 \times PL^2$ as configuration space, where D^1 is the space of locally integrable mappings that vanish at infinity and have square integrable first derivatives. Then the formulas (10)–(12) are the Euler-Lagrange equations associated with the Lagrangian

$$\begin{aligned}\mathscr{L}_R\big(\boldsymbol{x}, V, \boldsymbol{A}, \dot{\boldsymbol{x}}, \dot{V}, \dot{\boldsymbol{A}}\big) = \sum_{j=1}^{N}\Big(\frac{1}{2}m_j\dot{\boldsymbol{x}}_j^2 + \frac{Q_j}{c}\dot{\boldsymbol{x}}_j \cdot \boldsymbol{A} * \chi_R(\boldsymbol{x}_j) - Q_j V * \chi_R(\boldsymbol{x}_j)\Big)\\ + \frac{1}{8\pi}\int_{\mathbb{R}^3}\Big(\Big|\frac{1}{c}\dot{\boldsymbol{A}}(\boldsymbol{y}) + \nabla V(\boldsymbol{y})\Big|^2 - |\nabla \times \boldsymbol{A}(\boldsymbol{y})|^2\Big)\,\mathrm{d}\boldsymbol{y}.\end{aligned}$$

defined on the restricted tangent bundle $T\mathcal{Q}^0|\mathcal{Q}^1 \cong \mathcal{Q}^1 \times \mathcal{Q}^0$, where $\mathcal{Q}^1$ denotes the manifold domain $\mathbb{R}^{3N} \times D^1 \times PH^1$ of $\mathcal{Q}^0$. The associated energy function is $\mathscr{E}_R : T\mathcal{Q}^0|\mathcal{Q}^1 \ni v \mapsto \big(\mathbb{F}\mathscr{L}_R(v)(v) - \mathscr{L}_R(v)\big) \in \mathbb{R}$, where the fiber derivative $\mathbb{F}\mathscr{L}_R : T\mathcal{Q}^0|\mathcal{Q}^1 \to T^*\mathcal{Q}^0|\mathcal{Q}^1$ is given by

$$\mathbb{F}\mathscr{L}_R(v)(w) = \frac{\mathrm{d}}{\mathrm{d}t}\mathscr{L}_R(v + tw)\Big|_{t=0} \text{ for } q \in \mathcal{Q}^1 \text{ and } v, w \in T_q\mathcal{Q}^0.$$

With the intention of later passing to a quantum mechanical description of the charged particles we would like to define a Hamiltonian corresponding to $\mathscr{L}_R$ – such a Hamiltonian expresses the energy in terms of coordinates and momenta, in the sense that the identity

$$\mathscr{H}_R \circ \mathbb{F}\mathscr{L}_R = \mathscr{E}_R \tag{14}$$

holds on some appropriate subset of $T\mathcal{Q}^0|\mathcal{Q}^1$ as we shall explain. The Lagrangian $\mathscr{L}_R$ is degenerate since it does not at all depend on $\dot{V}$ and so $\mathbb{F}\mathscr{L}_R$ is not even locally invertible, but as can easily be verified (14) does define a mapping $\mathscr{H}_R$ on all of the image $\mathcal{M}_1 = \mathbb{F}\mathscr{L}_R(T\mathcal{Q}^0|\mathcal{Q}^1) \subset T^*\mathcal{Q}^0$. The pull-back $\omega_1 = j_1^*\Omega$ to $\mathcal{M}_1$ of the canonical 2-form Ω on $T^*\mathcal{Q}^0$ via the inclusion $\mathcal{M}_1 \xhookrightarrow{j_1} T^*\mathcal{Q}^0$ is degenerate and so $(\mathcal{M}_1, \omega_1)$ is not a symplectic manifold. To remedy this problem we can restrict $\mathbb{F}\mathscr{L}_R$ to the subset of elements $(\boldsymbol{x}, V, \boldsymbol{A}, \dot{\boldsymbol{x}}, \dot{V}, \dot{\boldsymbol{A}}) \in T\mathcal{Q}^0|\mathcal{Q}^1$ satisfying Gauss' law

$$-\Delta V(\boldsymbol{z}) = 4\pi\sum_{j=1}^{N} Q_j \chi_R(\boldsymbol{x}_j - \boldsymbol{z}),$$

meaning that V is the function $\boldsymbol{z} \mapsto \sum_{j=1}^{N} Q_j \int_{\mathbb{R}^3} \frac{\chi_R(\boldsymbol{x}_j - \boldsymbol{y})}{|\boldsymbol{y}-\boldsymbol{z}|}\,\mathrm{d}\boldsymbol{y}$. The image $\mathcal{M}_2$ of this set under the map $\mathbb{F}\mathscr{L}_R$ becomes a weak symplectic manifold in the sense of [1] and this procedure is completely natural in the framework devised by Gotay, Nester and Hinds [16] as a further development of Anderson, Bergmann and Dirac's constraint theory [2, 7, 8] – see also [23]. Identifying $\mathcal{M}_2$ with $\mathbb{R}^{3N} \times PH^1 \times \mathbb{R}^{3N} \times PL^2$ we can write the Hamiltonian $\mathscr{H}_R$ as

$$\begin{aligned}
&\mathscr{H}_R\big(\boldsymbol{x}, \boldsymbol{A}, \boldsymbol{p}, -\tfrac{P\boldsymbol{E}}{4\pi}\big) \\
&= \sum_{j=1}^{N} \frac{1}{2m_j}\Big(\boldsymbol{p}_j - \frac{Q_j}{c}\boldsymbol{A} * \chi_R(\boldsymbol{x}_j)\Big)^2 + \frac{1}{8\pi}\int_{\mathbb{R}^{3N}} \big(c^2|P\boldsymbol{E}(\boldsymbol{y})|^2 + |\nabla \times \boldsymbol{A}(\boldsymbol{y})|^2\big)\,\mathrm{d}\boldsymbol{y} \\
&+ \frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N} Q_j Q_k \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\chi_R(\boldsymbol{x}_j - \boldsymbol{y})\chi_R(\boldsymbol{x}_k - \boldsymbol{z})}{|\boldsymbol{y}-\boldsymbol{z}|}\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{z}. \qquad (15)
\end{aligned}$$

Now take the point particle-limit $R \to 0^+$ in the following (formal) sense: Consider the mapping $\mathscr{H}_R$ acting as prescribed in (15) on the R-independent space $\mathbb{R}^{3N} \times PH^1 \times \mathbb{R}^{3N} \times PL^2$. The first term on the right hand side of (15) represents the kinetic energy of the N particles, the second term is the energy stored in the electromagnetic field and the double sum is the potential energy induced by the Coulomb interactions between the N particles. In particular, the double sum's diagonal term $\frac{Q_j^2}{2R}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\chi(\boldsymbol{y})\chi(\boldsymbol{z})}{|\boldsymbol{y}-\boldsymbol{z}|}\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{z}$ is the energy coming from the j'th particle's interaction with itself. We subtract this self-energy from $\mathscr{H}_R$ and note that as $R \to 0^+$ the result converges pointwise to the mapping

$$\begin{aligned}
\mathscr{H}_0\big(\boldsymbol{x}, \boldsymbol{A}, \boldsymbol{p}, -\tfrac{P\boldsymbol{E}}{4\pi}\big) &= \sum_{j=1}^{N} \frac{1}{2m_j}\Big(\boldsymbol{p}_j - \frac{Q_j}{c}\boldsymbol{A}(\boldsymbol{x}_j)\Big)^2 + \sum_{1\le j<k\le N} \frac{Q_j Q_k}{|\boldsymbol{x}_j - \boldsymbol{x}_k|} \\
&+ \frac{1}{8\pi}\int_{\mathbb{R}^{3N}} \big(c^2|P\boldsymbol{E}(\boldsymbol{y})|^2 + |\nabla \times \boldsymbol{A}(\boldsymbol{y})|^2\big)\,\mathrm{d}\boldsymbol{y},
\end{aligned}$$

provided $\boldsymbol{A}$ is continuous at the points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$. We now quantize the charged particles in our model and obtain the Hamilton operator

$$\begin{aligned}
\mathscr{H}\big(\boldsymbol{A}, -\tfrac{P\boldsymbol{E}}{4\pi}\big) &= \sum_{j=1}^{N} \frac{1}{2m_j}\Big(i\hbar\nabla_{\boldsymbol{x}_j} + \frac{Q_j}{c}\boldsymbol{A}(\boldsymbol{x}_j)\Big)^2 + \sum_{1\le j<k\le N} \frac{Q_j Q_k}{|\boldsymbol{x}_j - \boldsymbol{x}_k|} \\
&+ \frac{1}{8\pi}\int_{\mathbb{R}^{3N}} \big(c^2|P\boldsymbol{E}(\boldsymbol{y})|^2 + |\nabla \times \boldsymbol{A}(\boldsymbol{y})|^2\big)\,\mathrm{d}\boldsymbol{y},
\end{aligned}$$

acting on a certain dense subspace of the Hilbert space $L^2(\mathbb{R}^{3N})$. Instead of also quantizing the fields $\boldsymbol{A}$ and $-\frac{P\boldsymbol{E}}{4\pi}$ we leave them as classical variables. In this spirit we will for a given (normalized) quantum state $\psi : \mathbb{R}^{3N} \to \mathbb{C}$ of the particles regard the average energy $(\boldsymbol{A}, -\frac{P\boldsymbol{E}}{4\pi}) \mapsto \big(\psi, \mathscr{H}\big(\boldsymbol{A}, -\frac{P\boldsymbol{E}}{4\pi}\big)\psi\big)_{L^2}$ as a classical Hamiltonian defined on the weak symplectic manifold $(PH^1\times PL^2,\omega)$ with

$$\omega_m\big(m,\boldsymbol{A}_1,-\tfrac{P\boldsymbol{E}_1}{4\pi},m,\boldsymbol{A}_2,-\tfrac{P\boldsymbol{E}_2}{4\pi}\big)=\frac{1}{4\pi}\int_{\mathbb{R}^3}\big(P\boldsymbol{E}_1\cdot\boldsymbol{A}_2-P\boldsymbol{E}_2\cdot\boldsymbol{A}_1\big)(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}$$

for $m,\big(\boldsymbol{A}_1,-\tfrac{P\boldsymbol{E}_1}{4\pi}\big),\big(\boldsymbol{A}_2,-\tfrac{P\boldsymbol{E}_2}{4\pi}\big)\in PH^1\times PL^2$. The corresponding Hamilton equations express that

$$\frac{1}{c^2}\partial_t\boldsymbol{A}(t)=-P\boldsymbol{E}(t)\text{ and }-\partial_tP\boldsymbol{E}(t)=\Delta\boldsymbol{A}(t)+\frac{4\pi}{c}\sum_{j=1}^{N}P\boldsymbol{J}_j[\psi,\boldsymbol{A}(t)].\quad(16)$$

In reality, we do of course not expect the quantum state of the charged particles to be time independent – the time evolution of ψ is governed by the Schrödinger equation

$$i\hbar\partial_t\psi(t)=\mathscr{H}\big(\boldsymbol{A}(t),-\tfrac{P\boldsymbol{E}(t)}{4\pi}\big)\psi(t).\quad(17)$$

We investigate the situation where the fixed time-independent state ψ appearing in (16) is replaced by the time-dependent state $\psi(t)$ satisfying the Schrödinger equation (17). (1) is precisely obtained by doing this coupling of (16) with (17).

3 Preliminaries

First, we collect some simple estimates that will be useful to us later.

Lemma 4. *For all $1\le j\le N$, $\boldsymbol{A}\in\big[L^4(\mathbb{R}^3)\big]^3$, $B\in L^2(\mathbb{R}^3)$, $\psi\in L^2(\mathbb{R}^{3N})$ with $\Delta_{\boldsymbol{x}_j}\psi\in L^2(\mathbb{R}^{3N})$, $0<\varepsilon<1$ and $0<\delta<\frac12$ we have*

$$\begin{aligned}\big\|\boldsymbol{A}(\boldsymbol{x}_j)\cdot\nabla_{\boldsymbol{x}_j}\psi\big\|_{L^2}&\lesssim\|\boldsymbol{A}\|_{L^4}\big(\varepsilon^{-7}\|\psi\|_{L^2}+\varepsilon\|\Delta_{\boldsymbol{x}_j}\psi\|_{L^2}\big)\\ \|B(\boldsymbol{x}_j)\psi\|_{L^2}&\lesssim\|B\|_{L^2}\big(\varepsilon^{-\frac{3+2\delta}{1-2\delta}}\|\psi\|_{L^2}+\varepsilon\|\Delta_{\boldsymbol{x}_j}\psi\|_{L^2}\big)\\ \left\|\frac{1}{|\boldsymbol{x}_j-\boldsymbol{x}_k|}\psi\right\|_{L^2}&\lesssim\varepsilon^{-\frac{3+2\delta}{1-2\delta}}\|\psi\|_{L^2}+\varepsilon\|\Delta_{\boldsymbol{x}_j}\psi\|_{L^2}\text{ for }1\le j<k\le N\end{aligned}\quad(18)$$

and for all $1\le j\le N$, $\boldsymbol{A}\in\big[L^4(\mathbb{R}^3)\big]^3$, $B\in L^2(\mathbb{R}^3)$ and $\psi\in L^2(\mathbb{R}^{3N})$ the estimates

$$\begin{aligned}\|\mathrm{div}_{\boldsymbol{x}_j}(\boldsymbol{A}(\boldsymbol{x}_j)\psi)\|_{H^{-2}}&\lesssim\|\boldsymbol{A}\|_{L^4}\|\psi\|_{L^2},\\ \|B(\boldsymbol{x}_j)\psi\|_{H^{-2}}&\lesssim\|B\|_{L^2}\|\psi\|_{L^2},\\ \left\|\frac{1}{|\boldsymbol{x}_j-\boldsymbol{x}_k|}\psi\right\|_{H^{-2}}&\lesssim\|\psi\|_{L^2}\text{ for }1\le j<k\le N\end{aligned}\quad(19)$$

hold true. Moreover, we have

$$\begin{aligned}&\big\|\boldsymbol{J}_j[\psi_1,\boldsymbol{A}_1]-\boldsymbol{J}_j[\psi_2,\boldsymbol{A}_2]\big\|_{H^1}\\&\lesssim\sum_{k=1}^{2}\big((1+\|\boldsymbol{A}_k\|_{D^1})\|\psi_k\|_{H^2}\big)\|\psi_1-\psi_2\|_{H^2}+\|\psi_1\|_{H^2}\|\psi_2\|_{H^2}\|\boldsymbol{A}_1-\boldsymbol{A}_2\|_{D^1}\end{aligned}\tag{20}$$

for any $1\leq j\leq N$ *and* $(\psi_1,\boldsymbol{A}_1),(\psi_2,\boldsymbol{A}_2)\in H^2(\mathbb{R}^{3N})\times D^1(\mathbb{R}^3)$*.*

Proof. For instance we can use Tonelli's theorem, Hölder's inequality, the Sobolev embedding $H^{\frac{3}{4}}\hookrightarrow L^4$ as well as the Young inequalities $\boldsymbol{p}_j^2\leq\frac{1}{2\varepsilon^2}+\frac{\varepsilon^2}{2}\boldsymbol{p}_j^4$ and $|\boldsymbol{p}_j|^{\frac{7}{2}}\leq\frac{1}{8\varepsilon^{14}}+\frac{7\varepsilon^2}{8}\boldsymbol{p}_j^4$ to obtain

$$\begin{aligned}\|\boldsymbol{A}(\boldsymbol{x}_j)\cdot\nabla_{\boldsymbol{x}_j}\psi\|_{L^2}^2&\leq\int_{\mathbb{R}^{3(N-1)}}\Big(\int_{\mathbb{R}^3}|\boldsymbol{A}(\boldsymbol{x}_j)|^4\,\mathrm{d}\boldsymbol{x}_j\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{R}^3}|\nabla_{\boldsymbol{x}_j}\psi(\boldsymbol{x})|^4\,\mathrm{d}\boldsymbol{x}_j\Big)^{\frac{1}{2}}\mathrm{d}\boldsymbol{x}_j'\\&\lesssim\|\boldsymbol{A}\|_{L^4}^2\int_{\mathbb{R}^{3(N-1)}}\int_{\mathbb{R}^3}\big|(1-\Delta)^{\frac{3}{8}}\nabla\psi^{j,\boldsymbol{x}_j'}(\boldsymbol{x}_j)\big|^2\,\mathrm{d}\boldsymbol{x}_j\,\mathrm{d}\boldsymbol{x}_j'\\&\lesssim\|\boldsymbol{A}\|_{L^4}^2\big(\varepsilon^{-14}\|\psi\|_{L^2}^2+\varepsilon^2\|\Delta_{\boldsymbol{x}_j}\psi\|_{L^2}^2\big),\end{aligned}$$

where we for $\boldsymbol{x}_j'=(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_{j-1},\boldsymbol{x}_{j+1},\ldots,\boldsymbol{x}_N)\in\mathbb{R}^{3(N-1)}$ introduce the mapping $\psi^{j,\boldsymbol{x}_j'}:\boldsymbol{x}_j\mapsto\psi(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_{j-1},\boldsymbol{x}_j,\boldsymbol{x}_{j+1},\ldots,\boldsymbol{x}_N)$ that for almost all vectors $\boldsymbol{x}_j'$ is contained in the Sobolev space $H^2(\mathbb{R}^3)$ and satisfies the identities $(1-\Delta)^{\frac{s}{2}}\big[\psi^{j,\boldsymbol{x}_j'}\big]=\big[(1-\Delta_{\boldsymbol{x}_j})^{\frac{s}{2}}\psi\big]^{j,\boldsymbol{x}_j'}$ and $\partial^\alpha\big[\psi^{j,\boldsymbol{x}_j'}\big]=[\partial^\alpha_{\boldsymbol{x}_j}\psi]^{j,\boldsymbol{x}_j'}$ for any $s\leq 2$ and any multi-index α with $|\alpha|\leq 2$. The other estimates in (18) follow analogously by using the Sobolev embedding $H^{\frac{3}{2}+\delta}\hookrightarrow L^\infty$ instead of $H^{\frac{3}{4}}\hookrightarrow L^4$.

To prove the first inequality in (19) we first note that for $\xi\in C_0^\infty$,

$$\begin{aligned}\|\mathrm{div}_{\boldsymbol{x}_j}(\boldsymbol{A}(\boldsymbol{x}_j)\xi)\|_{H^{-2}}&\leq(2\pi)^{3N}\sum_{k=1}^{3}\sup_{\|\eta\|_{L^2}=1}\Big|\Big(\overline{A^k}(\boldsymbol{x}_j)\mathscr{F}^{-1}\big[(1+\boldsymbol{p}^2)^{-\frac{1}{2}}\eta\big],\xi\Big)_{L^2}\Big|\\&\lesssim\|\boldsymbol{A}\|_{L^4}\|\xi\|_{L^2},\end{aligned}\tag{21}$$

where we use the Riesz-Fréchet theorem and the Sobolev embedding $H^1\hookrightarrow L^4$. For a given $\psi\in L^2$ we can therefore choose a sequence $(\psi_n)_{n\in\mathbb{N}}$ of C_0^∞-functions converging in L^2 to ψ and use (21) to conclude that $(\mathrm{div}_{\boldsymbol{x}_j}(\boldsymbol{A}(\boldsymbol{x}_j)\psi_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in the Hilbert space H^{-2}, whereby it must converge to some limit in H^{-2}. But this limit has to be $\mathrm{div}_{\boldsymbol{x}_j}(\boldsymbol{A}(\boldsymbol{x}_j)\psi)$ since the convergence $\mathrm{div}_{\boldsymbol{x}_j}(\boldsymbol{A}(\boldsymbol{x}_j)\psi_n)\xrightarrow[n\to\infty]{}\mathrm{div}_{\boldsymbol{x}_j}(\boldsymbol{A}(\boldsymbol{x}_j)\psi)$ holds in the space $\mathscr{D}'$ of distributions. Thus, the first estimate of (19) is true and each of the remaining two inequalities follow by combining the Riesz-Fréchet theorem with the corresponding estimate in (18).

Finally, (20) is easy to derive from the general estimates

$$\Big(\int_{\mathbb{R}^3}\Big|\int_{\mathbb{R}^{3(N-1)}} \Psi_1(\boldsymbol{x})\nabla_{\boldsymbol{x}_j}\Psi_2(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}_j'\Big|^2\,\mathrm{d}\boldsymbol{x}_j\Big)^{\frac{1}{2}}$$
$$\lesssim \min\big\{\|(1-\Delta_{\boldsymbol{x}_j})^{\frac{1}{4}}\Psi_1\|_{L^2}\|\nabla_{\boldsymbol{x}_j}\otimes\nabla_{\boldsymbol{x}_j}\Psi_2\|_{L^2},$$
$$\|(1-\Delta_{\boldsymbol{x}_j})^{\frac{3}{4}+\frac{\delta}{2}}\Psi_1\|_{L^2}\|\nabla_{\boldsymbol{x}_j}\Psi_2\|_{L^2}\big\}$$

and

$$\Big(\int_{\mathbb{R}^3}\Big|\int_{\mathbb{R}^{3(N-1)}} \boldsymbol{A}(\boldsymbol{x}_j)\Psi_1(\boldsymbol{x})\Psi_2(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}_j'\Big|^2\,\mathrm{d}\boldsymbol{x}_j\Big)^{\frac{1}{2}}$$
$$\lesssim \min\Big\{\|\boldsymbol{A}\|_{L^6}\|\nabla_{\boldsymbol{x}_j}\Psi_1\|_{L^2}\|\nabla_{\boldsymbol{x}_j}\Psi_2\|_{L^2}, \|\boldsymbol{A}\|_{L^2}\prod_{k=1}^{2}\|(1-\Delta_{\boldsymbol{x}_j})^{\frac{3}{4}+\frac{\delta}{2}}\Psi_k\|_{L^2}\Big\}$$

on mappings $\Psi_1, \Psi_2 : \mathbb{R}^{3N} \to \mathbb{C}$ and $\boldsymbol{A} : \mathbb{R}^3 \to \mathbb{C}^3$ that follow for $\delta > 0$ from Minkowski's integral inequality, the Sobolev embeddings $D^1 \hookrightarrow L^6$, $H^{\frac{1}{2}} \hookrightarrow L^3$, $H^{\frac{3}{2}+\delta} \hookrightarrow L^\infty$ and Hölder's inequality. By $\nabla_{\boldsymbol{x}_j}\otimes\nabla_{\boldsymbol{x}_j}\Psi_2$ we here mean a 9-vector with the derivatives $\partial_{x_j^k}\partial_{x_j^\ell}\Psi_2$ as components ($k,\ell \in \{1,2,3\}$). $\square$

Remark 5. The lemma above allows us to clarify the exact meaning of a solution to (1). If for some given pair $(\psi, \boldsymbol{A}) \in C(\mathcal{I}_T, H^2) \times C\big(\mathcal{I}_T; H^{\frac{3}{2}}\big)$ the derivative $\partial_t\boldsymbol{A}$ of $\boldsymbol{A} \in \mathscr{D}'\big(\mathcal{I}_T^\circ; H^{\frac{3}{2}}\big)$ is a continuous mapping $\mathcal{I}_T \to H^{\frac{1}{2}}$ then by boundedness of $P : H^1 \to H^1$ and the estimates in Lemma 4 we have

$$c^2\Big(\Delta\boldsymbol{A} + \frac{4\pi}{c}\sum_{j=1}^{N} P\boldsymbol{J}_j[\psi,\boldsymbol{A}]\Big) \in C\big(\mathcal{I}_T; H^{-\frac{1}{2}}\big) \tag{22}$$

and

$$-\frac{i}{\hbar}\Big(\sum_{j=1}^{N}\frac{1}{2m_j}\nabla_{j,\boldsymbol{A}}^2\psi + \sum_{1\le j<k\le N}\frac{Q_jQ_k}{|\boldsymbol{x}_j-\boldsymbol{x}_k|}\psi + \mathscr{E}_{\mathrm{EM}}[\boldsymbol{A},\partial_t\boldsymbol{A}]\psi\Big) \in C(\mathcal{I}_T; L^2). \tag{23}$$

A pair $(\psi, \boldsymbol{A}) \in C(\mathcal{I}_T; H^2) \times \big(C\big(\mathcal{I}_T; H^{\frac{3}{2}}\big) \cap C^1\big(\mathcal{I}_T; H^{\frac{1}{2}}\big)\big)$ is said to solve (1) if the second derivative $\partial_t^2\boldsymbol{A}$ of $\boldsymbol{A} \in \mathscr{D}'\big(\mathcal{I}_T^\circ; H^{\frac{3}{2}}\big)$ equals (22) and the derivative $\partial_t\psi$ of $\psi \in \mathscr{D}'(\mathcal{I}_T^\circ; H^2)$ equals (23).

For any solution $(\psi, \boldsymbol{A}) \in C(\mathcal{I}_T; H^2) \times \big(C\big(\mathcal{I}_T; H^{\frac{3}{2}}\big) \cap C^1\big(\mathcal{I}_T; H^{\frac{1}{2}}\big)\big)$ to (3) the pair $\big(\mathrm{e}^{-\frac{i}{\hbar}\int_0^t \mathscr{E}_{\mathrm{EM}}[\boldsymbol{A},\partial_t\boldsymbol{A}](s)\,\mathrm{d}s}\psi, \boldsymbol{A}\big)$ will solve (1) – here, the field energy $\mathscr{E}_{\mathrm{EM}}[\boldsymbol{A},\partial_t\boldsymbol{A}]$ is absolutely continuous $\mathcal{I}_T \to \mathbb{R}$ because $\partial_t\boldsymbol{A}$ and $\nabla\times\boldsymbol{A}$ are both absolutely continuous $\mathcal{I}_T \to H^{-\frac{1}{2}}$ and continuous $\mathcal{I}_T \to H^{\frac{1}{2}}$. Conversely, any $(\psi, \boldsymbol{A}) \in C(\mathcal{I}_T; H^2) \times \big(C\big(\mathcal{I}_T; H^{\frac{3}{2}}\big) \cap C^1\big(\mathcal{I}_T; H^{\frac{1}{2}}\big)\big)$ solving (1) gives rise to the solution $\left(\mathrm{e}^{\frac{i}{\hbar}\int_0^t \mathscr{E}_{\mathrm{EM}}[\boldsymbol{A},\partial_t\boldsymbol{A}](s)\,\mathrm{d}s}\psi, \boldsymbol{A}\right)$ to (3). Therefore we can concentrate on uniquely solving the simplified initial value problem (3)+(6) instead of (1)+(6).

It is noteworthy that for any solution $(\psi, \boldsymbol{A})$ to the system (3) (or (1) for that matter) the norm $\|\psi\|_{L^2} : \mathcal{I}_T \ni t \mapsto \|\psi(t)\|_{L^2} \in \mathbb{R}$ will be a constant of the motion. The absolute continuity of $\psi : \mathcal{I}_T \to L^2$ implies namely that $\|\psi\|^2_{L^2}$ is absolutely continuous and for almost all $t \in \mathcal{I}_T$

$$\partial_t\|\psi\|^2_{L^2}(t) = \frac{2}{\hbar}\mathrm{Im}\left(\psi(t), \sum_{j=1}^N \frac{1}{2m_j}\nabla^2_{j,\boldsymbol{A}}\psi(t) + \sum_{1\le j<k\le N}\frac{Q_jQ_k}{|\boldsymbol{x}_j - \boldsymbol{x}_k|}\psi(t)\right)_{L^2} = 0. \tag{24}$$

So if the initial condition ψ_0 is a unit vector in L^2 then the wave function ψ will continue to be a unit vector in L^2 at all later times of existence – this is consistent with the quantum mechanical interpretation of $|\psi(t)(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)|^2$ as the probability density at time t for finding particle 1 at $\boldsymbol{x}_1$, particle 2 at $\boldsymbol{x}_2$ etc.

Let us emphasize a final important consequence of Lemma 4 – namely that for any choice of divergence free vector potential $\boldsymbol{A} \in L^4(\mathbb{R}^3; \mathbb{R}^3)$ the formal operator acting on ψ on the right hand side of the second equation in (3) can be realized as a symmetric operator in $L^2(\mathbb{R}^{3N})$ with dense domain $H^2(\mathbb{R}^{3N})$. By the Kato-Rellich theorem the selfadjointness of the nonnegative operator $-\sum_{j=1}^N \frac{\hbar^2}{2m_j}\Delta_{\boldsymbol{x}_j} : H^2(\mathbb{R}^{3N}) \to L^2(\mathbb{R}^{3N})$ and the estimates (18) even imply that $\sum_{j=1}^N \frac{1}{2m_j}\nabla^2_{j,\boldsymbol{A}} + \sum_{1\le j<k\le N}\frac{Q_jQ_k}{|\boldsymbol{x}_j - \boldsymbol{x}_k|}$ is selfadjoint on the domain $H^2(\mathbb{R}^{3N})$ with a lower bound that goes like some power of $\langle\|\boldsymbol{A}\|_{L^4}\rangle$.

4 The Many-Body Schrödinger Equation

We will eventually solve (3) by applying the Banach fixed-point theorem to the solution operator of a certain linearization of (3). In this section we approach the many-body Schrödinger equation

$$i\hbar\partial_t\xi = \Big(\sum_{j=1}^N \frac{1}{2m_j}\nabla^2_{j,\boldsymbol{A}} + \sum_{1\le j<k\le N}\frac{Q_jQ_k}{|\boldsymbol{x}_j - \boldsymbol{x}_k|}\Big)\xi \tag{25}$$

by considering $\boldsymbol{A}$ as a fixed (time-dependent) vector potential. We supply (25) with the initial condition

$$\xi(\tau) = \psi_0, \tag{26}$$

where $\tau \in \mathcal{I}_T$ and ψ_0 are also fixed and thought of as given beforehand. We will show that this initial value problem is well-posed by applying the following fundamental result by Kato concerning general linear evolution equations of the type

$$\begin{aligned}\partial_t \xi + \mathbb{A}(t)\xi &= \mathbb{F}(t),\\ \xi(\tau) &= \psi_0\end{aligned}$$

in a Banach space $\mathcal{X}$.

Theorem 6. [19, Theorem I] *Suppose that*

(i') *For all $t \in \mathcal{I}_T$ the operator $-\mathbb{A}(t)$ generates a strongly continuous one-parameter semigroup $[0,\infty) \ni s \mapsto \exp\big(-s\mathbb{A}(t)\big) \in \mathcal{L}(\mathcal{X})$ and the family $\{\mathbb{A}(t) \,|\, t \in \mathcal{I}_T\}$ is quasi-stable with stability index (M,β), in the sense that*

$$\Big\|\prod_{j=1}^{k} \exp(-s_j \mathbb{A}(t_j))\Big\|_{\mathcal{L}(\mathcal{X})} \le M \exp\Big(\sum_{j=1}^{k} s_j \beta(t_j)\Big)$$

for all $k \in \mathbb{N}$, $0 \le t_1 \le \cdots \le t_k \le T$ and $s_1, \ldots, s_k \in [0,\infty)$, where M is a constant, $\beta : \mathcal{I}_T \to \mathbb{R}$ is upper Lebesgue integrable and the product on the left hand side is time-ordered so that a factor with larger t_j stands to the left of ones with smaller t_j.

(ii''') *There exists a Banach space $\mathcal{Y}$, continuously and densely embedded in $\mathcal{X}$, and a family $\{\mathbb{S}(t) \,|\, t \in \mathcal{I}_T\}$ of isomorphisms $\mathcal{Y} \to \mathcal{X}$, such that*

$$\mathbb{S}(t)\mathbb{A}(t)\mathbb{S}(t)^{-1} = \mathbb{A}(t) + \mathbb{B}(t) \text{ for almost all } t \in \mathcal{I}_T,$$

where $\mathbb{B}$ maps into $\mathcal{L}(\mathcal{X})$, $\mathbb{B}(\cdot)x$ is strongly measurable (as an $\mathcal{X}$-valued mapping) for all $x \in \mathcal{X}$ and $\|\mathbb{B}(\cdot)\|_{\mathcal{L}(\mathcal{X})}$ is upper Lebesgue integrable. Furthermore, there exists a function $\dot{\mathbb{S}}$ defined almost everywhere on $\mathcal{I}_T$ and mapping into $\mathcal{L}(\mathcal{Y},\mathcal{X})$ such that $\dot{\mathbb{S}}(\cdot)y$ is strongly measurable for all $y \in \mathcal{Y}$, $\big\|\dot{\mathbb{S}}(\cdot)\big\|_{\mathcal{L}(\mathcal{Y},\mathcal{X})}$ is upper Lebesgue integrable and $\mathbb{S}$ is a strong indefinite integral of $\dot{\mathbb{S}}$.

(iii) *For all $t \in \mathcal{I}_T$ the domain of the operator $\mathbb{A}(t)$ in $\mathcal{X}$ contains $\mathcal{Y}$ and $\mathbb{A} : \mathcal{I}_T \to \mathcal{L}(\mathcal{Y},\mathcal{X})$ is norm-continuous.*

Then there exists a unique $\mathscr{U}$ defined on the triangle $\mathcal{T}_T = \{(t,\tau) \in \mathcal{I}_T^2 \mid t \ge \tau\}$ with the following properties.

(a) *$\mathscr{U}$ is strongly continuous $\mathcal{T}_T \to \mathcal{L}(\mathcal{X})$ with $\mathscr{U}(t,t) = 1$ for all $t \in \mathcal{I}_T$,*

(b) *$\mathscr{U}(t,\tau)\mathscr{U}(\tau,s) = \mathscr{U}(t,s)$ for all (t,τ,s) satisfying $0 \le s \le \tau \le t \le T$,*

(c) *For all $(t,\tau) \in \mathcal{T}_T$ the inclusion $\mathscr{U}(t,\tau)\mathcal{Y} \subset \mathcal{Y}$ holds and $\mathscr{U}$ is strongly continuous $\mathcal{T}_T \to \mathcal{L}(\mathcal{Y})$,*

(d) *The strong partial derivatives* $\partial_t \mathscr{U}(t,\tau)y = -\mathbb{A}(t)\mathscr{U}(t,\tau)y$ *as well as* $\partial_\tau \mathscr{U}(t,\tau)y = \mathscr{U}(t,\tau)\mathbb{A}(\tau)y$ *exist in* $\mathcal{X}$ *for all* $(t,\tau,y) \in \mathcal{T}_T \times \mathcal{Y}$ *and* $\partial_t \mathscr{U}, \partial_\tau \mathscr{U} : \mathcal{T}_T \to \mathcal{L}(\mathcal{Y},\mathcal{X})$ *are both strongly continuous.*

Remark 7. If $\mathbb{A}$ satisfies the points (i'), (ii''') and (iii) then $\mathbb{A}' = -\mathbb{A} \circ \mathfrak{R}$ with $\mathfrak{R} : \mathcal{I}_T \ni t \mapsto (T-t) \in \mathcal{I}_T$ will automatically fulfill (ii''') and (iii). This can easily be checked by choosing $(\mathbb{S}', \mathbb{B}', \dot{\mathbb{S}}') = (\mathbb{S}, -\mathbb{B}, -\dot{\mathbb{S}}) \circ \mathfrak{R}$ (with a hopefully obvious notation) and using that for any Banach space $\mathcal{Z}$ the function $f \mapsto (-f \circ \mathfrak{R})$ not only conserves the property of strong measurability $\mathcal{I}_T \to \mathcal{Z}$, but it also maps $L^1(\mathcal{I}_T; \mathcal{Z})$ isometrically onto itself. If $\mathbb{A}'$ also happens to satisfy (i') in the sense that $-\mathbb{A}'(t)$ generates a C_0-semigroup for all $t \in \mathcal{I}_T$ and the family $\{\mathbb{A}'(t) \mid t \in \mathcal{I}_T\}$ is quasi-stable with stability index $(M, \beta \circ \mathfrak{R})$, then we can combine the evolution operators $\mathscr{U}_\mathbb{A}$ and $\mathscr{U}_{\mathbb{A}'}$ – whose existence are ensured by Theorem 6 – into a single evolution operator $\mathscr{U}$ defined in *all* points $(t,\tau) \in \mathcal{I}_T^2$ by setting

$$\mathscr{U}(t,\tau) = \begin{cases} \mathscr{U}_\mathbb{A}(t,\tau) & \text{for } t \geq \tau \\ \mathscr{U}_{\mathbb{A}'}(T-t, T-\tau) & \text{for } t < \tau \end{cases}.$$

This operator satisfies

(a') $\mathscr{U}$ is strongly continuous $\mathcal{I}_T^2 \to \mathcal{L}(\mathcal{X})$ with $\mathscr{U}(t,t) = 1$ for all $t \in \mathcal{I}_T$,

(b') $\mathscr{U}(t,\tau)\mathscr{U}(\tau,s) = \mathscr{U}(t,s)$ for all $(t,\tau,s) \in \mathcal{I}_T^3$,

(c') For all $(t,\tau) \in \mathcal{I}_T^2$ the inclusion $\mathscr{U}(t,\tau)\mathcal{Y} \subset \mathcal{Y}$ holds and $\mathscr{U}$ is strongly continuous $\mathcal{I}_T^2 \to \mathcal{L}(\mathcal{Y})$,

(d') The strong partial derivatives $\partial_t \mathscr{U}(t,\tau)y = -\mathbb{A}(t)\mathscr{U}(t,\tau)y$ as well as $\partial_\tau \mathscr{U}(t,\tau)y = \mathscr{U}(t,\tau)\mathbb{A}(\tau)y$ exist in $\mathcal{X}$ for all $(t,\tau,y) \in \mathcal{I}_T^2 \times \mathcal{Y}$ and $\partial_t \mathscr{U}, \partial_\tau \mathscr{U} : \mathcal{I}_T^2 \to \mathcal{L}(\mathcal{Y},\mathcal{X})$ are both strongly continuous.

Here, (b') is the only point that does not follow immediately from the properties listed in Theorem 6 of the individual operators $\mathscr{U}_\mathbb{A}$ and $\mathscr{U}_{\mathbb{A}'}$ – however, it suffices to prove the identities

$$\mathscr{U}_\mathbb{A}(t_0,\tau_0)\mathscr{U}_{\mathbb{A}'}(T-\tau_0, T-t_0) = \mathscr{U}_{\mathbb{A}'}(T-\tau_0, T-t_0)\mathscr{U}_\mathbb{A}(t_0,\tau_0) = 1 \tag{27}$$

for all $(t_0,\tau_0) \in \mathcal{T}_T$. To prove (27) note first that by [18, Proposition 4.4] the operator $\widetilde{\mathbb{A}}(t)$ (resp. $\widetilde{\mathbb{A}'}(t)$) in $\mathcal{Y}$ acting like $\mathbb{A}(t)$ (resp. $\mathbb{A}'(t)$) on the domain $\{y \in \mathcal{Y} \mid \mathbb{A}(t)y \in \mathcal{Y}\}$ (resp. $\{y \in \mathcal{Y} \mid \mathbb{A}'(t)y \in \mathcal{Y}\}$) is quasi-stable and the second coordinate of it's stability index can be chosen to be $\widetilde{\beta} = \beta + M\|\mathbb{B}(\cdot)\|_{\mathcal{L}(\mathcal{X})}$ (resp. $\widetilde{\beta} \circ \mathfrak{R}$). Without loss of generality we can here assume that β and $\widetilde{\beta}$ are integrable $\mathcal{I}_T \to [0,\infty)$ (otherwise replace them by integrable majorants). With the help of [19, Lemma A1] and the remark after [19, Lemma

A2] consider now a sequence $\big(\{\mathcal{I}_T^{n1},\dots,\mathcal{I}_T^{nm_n}\}\big)_{n\in\mathbb{N}}$ of partitions of the interval $\mathcal{I}_T$ into subintervals with $\sup_j|\mathcal{I}_T^{nj}| \xrightarrow[n\to\infty]{} 0$ and a corresponding sequence $\big(\{t^{n1},\dots,t^{nm_n}\}\big)_{n\in\mathbb{N}}$ with $t^{nj}\in\mathcal{I}_T^{nj}$ for $n\in\mathbb{N}$ and $j\in\{1,\dots,m_n\}$ such that the Riemann step functions $\sum_{j=1}^{m_n}\beta\big(t^{nj}\big)1_{\mathcal{I}_T^{nj}}$ and $\sum_{j=1}^{m_n}\widetilde{\beta}\big(t^{nj}\big)1_{\mathcal{I}_T^{nj}}$ approximate β respectively $\widetilde{\beta}$, in $L^1(\mathcal{I}_T)$ as well as pointwise almost everywhere. Then by the proof of [19, Theorem I] the operator $\mathscr{U}_{\mathbb{A}}(t,\tau)$ is the strong limit in $\mathcal{L}(L^2)$ (uniformly in $(t,\tau)\in\mathcal{T}_T$) of a sequence $\big(\mathscr{U}_{\mathbb{A}}^n(t,\tau)\big)_{n\in\mathbb{N}}$ of operators satisfying

- $\mathscr{U}_{\mathbb{A}}^n(t,\tau)=\mathrm{e}^{-(t-\tau)\mathbb{A}(t^{nj})}$ for $t,\tau\in\overline{\mathcal{I}_T^{nj}}$ with $t\geq\tau$,
- $\mathscr{U}_{\mathbb{A}}^n(t,\tau)=\mathscr{U}_{\mathbb{A}}^n(t,s)\mathscr{U}_{\mathbb{A}}^n(s,\tau)$ for $t\geq s\geq\tau$.

But here the sequence $\big(\{T-\mathcal{I}_T^{n1},\dots,T-\mathcal{I}_T^{nm_n}\}\big)_{n\in\mathbb{N}}$ of partitions of $\mathcal{I}_T$ satisfies $\sup_j|T-\mathcal{I}_T^{nj}| \xrightarrow[n\to\infty]{} 0$ and the corresponding Riemann step functions $\sum_{j=1}^{m_n}(\beta\circ\mathfrak{R})\big(T-t^{nj}\big)1_{T-\mathcal{I}_T^{nj}}$ and $\sum_{j=1}^{m_n}\big(\widetilde{\beta}\circ\mathfrak{R}\big)\big(T-t^{nj}\big)1_{T-\mathcal{I}_T^{nj}}$ approximate $\beta\circ\mathfrak{R}$ respectively $\widetilde{\beta}\circ\mathfrak{R}$, in $L^1(\mathcal{I}_T)$ as well as pointwise almost everywhere. Consequently, $\mathscr{U}_{\mathbb{A}'}(T-\tau,T-t)$ is also the strong limit in $\mathcal{L}(L^2)$ (uniformly in $(t,\tau)\in\mathcal{T}_T$) of a sequence $\big(\mathscr{U}_{\mathbb{A}'}^n(T-\tau,T-t)\big)_{n\in\mathbb{N}}$ satisfying

- $\mathscr{U}_{\mathbb{A}'}^n(T-\tau,T-t)=\mathrm{e}^{(t-\tau)\mathbb{A}(t^{nj})}$ for $t,\tau\in\overline{\mathcal{I}_T^{nj}}$ with $t\geq\tau$,
- $\mathscr{U}_{\mathbb{A}'}^n(T-\tau,T-t)=\mathscr{U}_{\mathbb{A}'}^n(T-\tau,T-s)\mathscr{U}_{\mathbb{A}'}^n(T-s,T-t)$ for $t\geq s\geq\tau$.

Now, (27) follows immediately from the four properties of $\mathscr{U}_{\mathbb{A}}^n$ and $\mathscr{U}_{\mathbb{A}'}^n$ listed above.

We now apply Theorem 6 to the problem (25)–(26).

Corollary 8. *For all $T>0$ and all $\boldsymbol{A}\in W^{1,1}\big(\mathcal{I}_T;L^4(\mathbb{R}^3;\mathbb{R}^3)\big)$ whose continuous representative is divergence free at all times there exists a unique evolution operator $\mathscr{U}_{\boldsymbol{A}}$ defined on $\mathcal{I}_T^2$ such that*

(A) *$\mathscr{U}_{\boldsymbol{A}}$ is strongly continuous $\mathcal{I}_T^2\to\mathcal{L}(L^2)$ with $\mathscr{U}_{\boldsymbol{A}}(t,t)=1$ for $t\in\mathcal{I}_T$,*

(B) *$\mathscr{U}_{\boldsymbol{A}}(t,\tau)\mathscr{U}_{\boldsymbol{A}}(\tau,s)=\mathscr{U}_{\boldsymbol{A}}(t,s)$ for all $(t,\tau,s)\in\mathcal{I}_T^3$,*

(C) *$\mathscr{U}_{\boldsymbol{A}}(t,\tau)H^2\subset H^2$ for $(t,\tau)\in\mathcal{I}_T^2$ and $\mathscr{U}_{\boldsymbol{A}}:\mathcal{I}_T^2\to\mathcal{L}(H^2)$ is strongly continuous,*

(D) *The strong partial derivatives $\partial_t\mathscr{U}_{\boldsymbol{A}}(t,\tau)\psi_0$ and $\partial_\tau\mathscr{U}_{\boldsymbol{A}}(t,\tau)\psi_0$ exist in L^2 for all $(t,\tau)\in\mathcal{I}_T^2$ and $\psi_0\in H^2$ and are given by*

$$i\hbar\partial_t\mathscr{U}_{\boldsymbol{A}}(t,\tau)\psi_0=\Big(\sum_{j=1}^{N}\frac{1}{2m_j}\nabla_{j,\boldsymbol{A}(t)}^2+\sum_{1\leq j<k\leq N}\frac{Q_jQ_k}{|\boldsymbol{x}_j-\boldsymbol{x}_k|}\Big)\mathscr{U}_{\boldsymbol{A}}(t,\tau)\psi_0$$

respectively

$$\hbar\partial_\tau \mathscr{U}_{\boldsymbol{A}}(t,\tau)\psi_0 = i\mathscr{U}_{\boldsymbol{A}}(t,\tau)\Big(\sum_{j=1}^N \frac{1}{2m_j}\nabla^2_{j,\boldsymbol{A}(\tau)} + \sum_{1\le j<k\le N} \frac{Q_jQ_k}{|\boldsymbol{x}_j-\boldsymbol{x}_k|}\Big)\psi_0.$$

Moreover, $\partial_t\mathscr{U}_{\boldsymbol{A}}, \partial_\tau\mathscr{U}_{\boldsymbol{A}} : \mathcal{I}_T^2 \to \mathcal{L}(H^2, L^2)$ *are strongly continuous.*

Proof. Let $\boldsymbol{A} : \mathcal{I}_T \to L^4$ denote (the absolutely continuous representative of) a magnetic vector potential satisfying the hypotheses of the corollary and consider it's strong derivative $\partial_t\boldsymbol{A}$ that is defined almost everywhere on $\mathcal{I}_T$ and contained in $L^1(\mathcal{I}_T; L^4)$. Our goal will be to apply Theorem 6 and Remark 7 to the family of operators

$$\mathbb{A}(t) = \frac{i}{\hbar}\Big(\sum_{j=1}^N \frac{1}{2m_j}\nabla^2_{j,\boldsymbol{A}(t)} + \sum_{1\le j<k\le N} \frac{Q_jQ_k}{|\boldsymbol{x}_j-\boldsymbol{x}_k|}\Big)$$

in $\mathcal{X} = L^2(\mathbb{R}^{3N})$ with domain $\mathcal{Y} = H^2(\mathbb{R}^{3N})$. By Stone's theorem the selfadjointness of $i\mathbb{A}(t)$ implies that $-\mathbb{A}(t)$ generates a strongly continuous one-parameter group $\mathbb{R} \ni s \mapsto \exp(-s\mathbb{A}(t)) \in \mathcal{L}(L^2)$ of unitary operators for each $t \in \mathcal{I}_T$. Thereby $[0,\infty) \ni s \mapsto \exp(-s\mathbb{A}(t))$ and $[0,\infty) \ni s \mapsto \exp(s\mathbb{A}(T-t))$ are strongly continuous one-parameter semigroups generated by $-\mathbb{A}(t)$ respectively $\mathbb{A}(T-t)$. Moreover, the unitarity of the operators $\exp(-s\mathbb{A}(t))$ for $t \in \mathcal{I}_T$ and $s \in \mathbb{R}$ ensures that both of the families $\{\mathbb{A}(t) \mid t \in \mathcal{I}_T\}$ and $\{-\mathbb{A}(T-t) \mid t \in \mathcal{I}_T\}$ are (quasi-)stable with the common stability index $(1,0)$. Thus, $\mathbb{A}$ and $-\mathbb{A}\circ\mathfrak{R}$ both satisfy the point (i') from Theorem 6.

The operator $-i\mathbb{A}(t)$ in L^2 is selfadjoint and bounded from below, uniformly in t, by some constant $-M$ so by setting

$$\mathbb{S}(t) = M + 1 - i\mathbb{A}(t) \text{ for } t \in \mathcal{I}_T,$$

we obtain a family of selfadjoint operators in L^2 that all have lower bounds ≥ 1 and thereby map their common domain H^2 bijectively onto L^2. Lemma 4 even gives that $\mathbb{S}(t)$ is bounded, when considered as an operator from the Hilbert space H^2 to the Hilbert space L^2, whereby it's inverse must also be bounded according to the bounded inverse theorem. Consequently, $\mathbb{S}(t)$ is an isomorphism $H^2 \to L^2$ and the identity $\mathbb{S}(t)\mathbb{A}(t)\mathbb{S}(t)^{-1} = \mathbb{A}(t)$ holds by construction for all $t \in \mathcal{I}_T$. To show the final part of (ii''') we define

$$\dot{\mathbb{S}}(t) = \sum_{j=1}^N \frac{Q_j}{\hbar m_j c}\partial_t\boldsymbol{A}(t)(\boldsymbol{x}_j)\cdot\nabla_{j,\boldsymbol{A}(t)}$$

as an $\mathcal{L}(H^2, L^2)$-element for almost all points $t \in \mathcal{I}_T$ – namely the points where $\partial_t\boldsymbol{A}$ is well-defined. Lemma 4 and the strong measurability $\mathcal{I}_T \to L^4$ of $\boldsymbol{A}$ and

$\partial_t \boldsymbol{A}$ allow us to conclude that $\mathbb{S}$ and $\dot{\mathbb{S}}$ are strongly measurable $\mathcal{I}_T \to \mathcal{L}(H^2, L^2)$ with the estimates

$$\|\mathbb{S}(t)\|_{\mathcal{L}(H^2,L^2)} \lesssim 1 + \|\boldsymbol{A}(t)\|_{L^4}^2, \left\|\dot{\mathbb{S}}(t)\right\|_{\mathcal{L}(H^2,L^2)} \lesssim \|\partial_t \boldsymbol{A}(t)\|_{L^4}(1 + \|\boldsymbol{A}(t)\|_{L^4})$$

holding true for almost all $t \in \mathcal{I}_T$. Consequently, $\mathbb{S}$ and $\dot{\mathbb{S}}$ are both Bochner integrable $\mathcal{I}_T \to \mathcal{L}(H^2, L^2)$ – in fact, it follows from (30) that $\mathbb{S}$ is continuous. Given an arbitrary $C_0^\infty(\mathcal{I}_T^\circ)$-function g we now get

$$\begin{aligned}
&\int_0^T \dot{\mathbb{S}}(t) g(t)\,\mathrm{d}t \\
&= \sum_{j=1}^N \frac{Q_j}{m_j c\hbar}\Big(i\hbar \int_0^T \partial_t \boldsymbol{A}(t) g(t)\,\mathrm{d}t(\boldsymbol{x}_j) \cdot \nabla_{\boldsymbol{x}_j} + \frac{Q_j}{c}\int_0^T \big(\partial_t \boldsymbol{A} \cdot \boldsymbol{A}\big)(t) g(t)\,\mathrm{d}t(\boldsymbol{x}_j)\Big) \\
&= -\int_0^T \mathbb{S}(t) g'(t)\,\mathrm{d}t. \qquad (28)
\end{aligned}$$

where we use that $\boldsymbol{A}^2 \in W^{1,1}\big(\mathcal{I}_T; L^2(\mathbb{R}^3)\big)$ with

$$\partial_t \boldsymbol{A}^2(t) = 2\partial_t \boldsymbol{A}(t) \cdot \boldsymbol{A}(t) \text{ for almost all } t \in \mathcal{I}_T, \qquad (29)$$

which follows from approximating $\boldsymbol{A}$ in $W^{1,1}\big(\mathcal{I}_T; L^4(\mathbb{R}^3;\mathbb{R}^3)\big)$ by functions in the form $\boldsymbol{A}^n : t \mapsto \sum_{m=1}^{M^n} \boldsymbol{a}_m^n f_m^n(t)$ with $M^n \in \mathbb{N}$, $\boldsymbol{a}_1^n, \ldots, \boldsymbol{a}_{M^n}^n \in L^4(\mathbb{R}^3;\mathbb{R}^3)$ and $f_1^n, \ldots, f_{M^n}^n \in C^\infty(\mathcal{I}_T)$ for $n \in \mathbb{N}$. We conclude from (28) that the function $\mathbb{S} \in W^{1,1}\big(\mathcal{I}_T, \mathcal{L}(H^2, L^2)\big)$ has $\dot{\mathbb{S}}$ as it's derivative, whereby (ii''') from Theorem 6 has been verified.

Finally, we obtain from Lemma 4 that for all $t, t' \in \mathcal{I}_T$

$$\|\mathbb{A}(t) - \mathbb{A}(t')\|_{\mathcal{L}(H^2,L^2)} \lesssim \big(1 + \|\boldsymbol{A}(t) + \boldsymbol{A}(t')\|_{L^4}\big)\|\boldsymbol{A}(t) - \boldsymbol{A}(t')\|_{L^4} \qquad (30)$$

so the continuity of $\boldsymbol{A} : \mathcal{I}_T \to L^4$ implies that $\mathbb{A} : \mathcal{I}_T \to \mathcal{L}(H^2, L^2)$ is norm-continuous. Thus, also the point (iii) of Theorem 6 is satisfied. ☐

Remark 9. Let $\psi_0 \in H^2$ and $\tau \in \mathcal{I}_T$ be given and set $\xi(t) = \mathscr{U}_{\boldsymbol{A}}(t, \tau)\psi_0$ for $t \in \mathcal{I}_T$. Being strongly differentiable $\mathcal{I}_T \to L^2$ with continuous derivative the function ξ can be expressed as

$$\xi(t) = \xi(0) + \int_0^t \partial_t \xi(s)\,\mathrm{d}s \text{ for all } t \in \mathcal{I}_T,$$

since the right hand side as a function of t is strongly differentiable in L^2 with $\partial_t \xi$ as it's derivative by the mean value theorem. Thus, ξ is absolutely continuous $\mathcal{I}_T \to L^2$, which in turn means that $\xi \in W^{1,1}(\mathcal{I}_T; L^2)$ and that it's distributional derivative agrees with it's strong derivative.

Remark 10. By the same argument as in (24) the mapping $\xi(t) = \mathscr{U}_{\boldsymbol{A}}(t,\tau)\psi_0$ has a conserved L^2-norm for any $\psi_0 \in H^2$ and $\tau \in \mathcal{I}_T$. This together with the continuity of $\mathscr{U}_{\boldsymbol{A}}(t,\tau) : L^2 \to L^2$ implies that the L^2-norm of $\xi(t)$ is in fact a constant of the motion for all $\psi_0 \in L^2$ and $\tau \in \mathcal{I}_T$.

Given a potential $\boldsymbol{A} \in W^{1,1}\big(\mathcal{I}_T; L^4(\mathbb{R}^3;\mathbb{R}^3)\big)$ whose continuous representative is divergence free at all times we can according to Corollary 8 apply $\mathscr{U}_{\boldsymbol{A}}(t,\tau)$ to any L^2-function ψ_0 and thereby obtain another L^2-function, even though we are only guaranteed that the result $\mathscr{U}_{\boldsymbol{A}}(t,\tau)\psi_0$ actually solves (25) if $\psi_0 \in H^2$. However, by the estimates (19) the right hand side of (25) is in fact meaningful (as an H^{-2}-element) when $\xi(t)$ is merely an L^2-function, provided that we interpret $\nabla^2_{j,\boldsymbol{A}(t)}\xi(t)$ as the sum

$$-\hbar^2\Delta_{\boldsymbol{x}_j}\xi(t) + 2i\frac{\hbar Q_j}{c}\mathrm{div}_{\boldsymbol{x}_j}\big(\boldsymbol{A}(t)(\boldsymbol{x}_j)\xi(t)\big) + \frac{Q_j^2}{c^2}\big[\boldsymbol{A}(t)(\boldsymbol{x}_j)\big]^2\xi(t). \tag{31}$$

A special case of the result below shows that for $\psi_0 \in L^2$ there can not be any other $C(\mathcal{I}_T;L^2) \cap W^{1,1}(\mathcal{I}_T;H^{-2})$-solutions to the initial value problem (25)–(26) than $\mathscr{U}_{\boldsymbol{A}}(t,\tau)\psi_0$. In order to formulate this result we introduce for $(t,\tau) \in \mathcal{I}_T^2$ the linear operator $H^{-2} \to H^{-2}$ (that we will again call $\mathscr{U}_{\boldsymbol{A}}(t,\tau)$) by setting

$$\langle \mathscr{U}_{\boldsymbol{A}}(t,\tau)\xi,\zeta\rangle_{H^{-2},H^2} = \langle \xi, \mathscr{U}_{\boldsymbol{A}}(\tau,t)\zeta\rangle_{H^{-2},H^2}$$

for $\xi \in H^{-2}$ and $\zeta \in H^2$, where we remember that H^{-s} is isometrically anti-isomorphic to the dual space $(H^s)^*$ of H^s by the mapping

$$H^{-s} \ni \xi \mapsto \Big(\langle \xi,\cdot\rangle_{H^{-s},H^s} : \zeta \mapsto \frac{1}{(2\pi)^3}\big(\langle \boldsymbol{p}\rangle^{-s}\widehat{\xi}, \langle \boldsymbol{p}\rangle^{s}\widehat{\zeta}\big)_{L^2}\Big) \in \big(H^s\big)^*.$$

Then $\mathscr{U}_{\boldsymbol{A}}(t,\tau)$ is bounded with

$$\|\mathscr{U}_{\boldsymbol{A}}(t,\tau)\|_{\mathcal{L}(H^{-2})} \le \|\mathscr{U}_{\boldsymbol{A}}(\tau,t)\|_{\mathcal{L}(H^2)} \le \sup_{(t',\tau')\in\mathcal{I}_T^2}\big\|\mathscr{U}_{\boldsymbol{A}}(t',\tau')\big\|_{\mathcal{L}(H^2)}, \tag{32}$$

for $(t,\tau) \in \mathcal{I}_T^2$, where the right hand side is finite by the uniform boundedness principle. Moreover, $\mathscr{U}_{\boldsymbol{A}}(t,\tau) : H^{-2} \to H^{-2}$ is an extension of the unitary operator $\mathscr{U}_{\boldsymbol{A}}(t,\tau) : L^2 \to L^2$ in the sense that they agree on L^2-functions.

Lemma 11. *Let the continuous representative of $\boldsymbol{A} \in W^{1,1}\big(\mathcal{I}_T; L^4(\mathbb{R}^3;\mathbb{R}^3)\big)$ be divergence free at all times and consider some arbitrary $f \in L^1\big(\mathcal{I}_T; H^{-2}\big)$. Then if $\xi \in C(\mathcal{I}_T;L^2) \cap W^{1,1}\big(\mathcal{I}_T;H^{-2}\big)$ satisfies the inhomogeneous many-body Schrödinger equation*

$$i\hbar\partial_t\xi = \Big(\sum_{j=1}^{N}\frac{1}{2m_j}\nabla^2_{j,\boldsymbol{A}} + \sum_{1\le j<k\le N}\frac{Q_jQ_k}{|\boldsymbol{x}_j - \boldsymbol{x}_k|}\Big)\xi + f$$

then

$$\xi(t) = \mathscr{U}_{\boldsymbol{A}}(t,\tau)\xi(\tau) - \frac{i}{\hbar}\int_\tau^t \mathscr{U}_{\boldsymbol{A}}(t,s)f(s)\,\mathrm{d}s, \tag{33}$$

for all $(t,\tau) \in \mathcal{I}_T^2$.

Proof. Given some $t \in \mathcal{I}_T$ and $\zeta \in H^2$ the map $\langle \mathscr{U}_{\boldsymbol{A}}(t,\cdot)\xi(\cdot),\zeta\rangle_{H^{-2},H^2}$ is absolutely continuous since $\xi : \mathcal{I}_T \to H^{-2}$, $\mathscr{U}_{\boldsymbol{A}}(\cdot,t)\zeta : \mathcal{I}_T \to L^2$ are absolutely continuous (see Remark 9) and $\xi : \mathcal{I}_T \to L^2$, $\mathscr{U}_{\boldsymbol{A}}(\cdot,t)\zeta : \mathcal{I}_T \to H^2$ are continuous. It's derivative is well defined almost everywhere in $\mathcal{I}_T$ and for almost all $s \in \mathcal{I}_T$

$$\begin{aligned}\partial_s\big\langle \mathscr{U}_{\boldsymbol{A}}(t,s)\xi(s),\zeta\big\rangle_{H^{-2},H^2} &= \big\langle \partial_s\xi(s),\mathscr{U}_{\boldsymbol{A}}(s,t)\zeta\big\rangle_{H^{-2},H^2} + \big(\xi(s),\partial_s\mathscr{U}_{\boldsymbol{A}}(s,t)\zeta\big)_{L^2}\\ &= \frac{i}{\hbar}\big\langle \mathscr{U}_{\boldsymbol{A}}(t,s)f(s),\zeta\big\rangle_{H^{-2},H^2},\end{aligned} \tag{34}$$

where $\big\langle\big(\sum_{j=1}^N \frac{1}{2m_j}\nabla^2_{j,\boldsymbol{A}(s)} + \sum_{j<k}\frac{Q_jQ_k}{|\boldsymbol{x}_j-\boldsymbol{x}_k|}\big)\xi(s),\mathscr{U}_{\boldsymbol{A}}(s,t)\zeta\big\rangle_{H^{-2},H^2}$ is seen to be equal to $\big(\xi(s),\big(\sum_{j=1}^N \frac{1}{2m_j}\nabla^2_{j,\boldsymbol{A}(s)} + \sum_{j<k}\frac{Q_jQ_k}{|\boldsymbol{x}_j-\boldsymbol{x}_k|}\big)\mathscr{U}_{\boldsymbol{A}}(s,t)\zeta\big)_{L^2}$ by approximating $\xi(s)$ and $\mathscr{U}_{\boldsymbol{A}}(s,t)\zeta$ in L^2 respectively H^2 by sequences of C_0^∞-functions and using the estimates (18) and (19). Thus,

$$\langle\xi(t),\zeta\rangle_{H^{-2},H^2} = \langle\mathscr{U}_{\boldsymbol{A}}(t,\tau)\xi(\tau),\zeta\rangle_{H^{-2},H^2} + \frac{i}{\hbar}\int_\tau^t\langle\mathscr{U}_{\boldsymbol{A}}(t,s)f(s),\zeta\rangle_{H^{-2},H^2}\,\mathrm{d}s$$

for all $\tau \in \mathcal{I}_T$. Here, (32) and the assumption that $f \in L^1(\mathcal{I}_T, H^{-2})$ give that $\mathscr{U}_{\boldsymbol{A}}(t,\cdot)f(\cdot)$ is Bochner integrable $\mathcal{I}_T \to H^{-2}$, whereby we can use [30, Corollary V.5.2] to commute the integral with the bounded anti-linear operator $\langle\cdot,\zeta\rangle_{H^{-2},H^2} : H^{-2} \to \mathbb{C}$ and obtain

$$\Big\langle\xi(t) - \mathscr{U}_{\boldsymbol{A}}(t,\tau)\xi(\tau) + \frac{i}{\hbar}\int_\tau^t\mathscr{U}_{\boldsymbol{A}}(t,s)f(s)\,\mathrm{d}s,\zeta\Big\rangle_{H^{-2},H^2} = 0$$

for all $\tau \in \mathcal{I}$, whereby the identity (33) follows. □

As already mentioned in (32) the norms $\|\mathscr{U}_{\boldsymbol{A}}(t,\tau)\|_{\mathcal{L}(H^2)}$ are uniformly bounded in $(t,\tau) \in \mathcal{I}_T^2$. We will now find an explicit upper bound.

Lemma 12. *Consider a vector potential* $\boldsymbol{A} \in W^{1,1}\big(\mathcal{I}_T; L^4(\mathbb{R}^3;\mathbb{R}^3)\big)$ *whose continuous representative is divergence free at all times. Then for all* $0 < \delta < \frac{1}{2}$ *there exists a constant* $C > 0$ *(depending on c,* $\hbar$*,* δ*,* N*,* $m_1,\dots,m_N$ *and* $Q_1,\dots,Q_N$*) such that*

$$\|\mathscr{U}_{\boldsymbol{A}}(t,\tau)\|_{\mathcal{L}(H^2)} \le C\langle\|\boldsymbol{A}\|_{L_T^\infty L^4}\rangle^{\frac{8}{1-2\delta}}\exp\Big(C\int_\tau^t\langle\|\boldsymbol{A}(s)\|_{L^4}\rangle\|\partial_t\boldsymbol{A}(s)\|_{L^4}\,\mathrm{d}s\Big) \tag{35}$$

for all $(t,\tau) \in \mathcal{T}_T$.

Proof. Given $\psi_0 \in H^2$ and $\tau \in \mathcal{I}_T$ we set $\xi(\cdot) = \mathscr{U}_{\boldsymbol{A}}(\cdot,\tau)\psi_0 \in C(\mathcal{I}_T; H^2)$ and note that the time derivative

$$\partial_t \xi = -\frac{i}{\hbar}\Big(\sum_{j=1}^{N} \frac{1}{2m_j}\nabla^2_{j,\boldsymbol{A}} + \sum_{1\le j<k\le N} \frac{Q_j Q_k}{|\boldsymbol{x}_j - \boldsymbol{x}_k|}\Big)\xi$$

has the distributional derivative given by

$$\partial_t^2 \xi = -\frac{i}{\hbar}\Big(\sum_{j=1}^{N} \frac{1}{2m_j}\nabla^2_{j,\boldsymbol{A}} + \sum_{1\le j<k\le N} \frac{Q_j Q_k}{|\boldsymbol{x}_j - \boldsymbol{x}_k|}\Big)\partial_t\xi - \frac{i}{\hbar} f \tag{36}$$

where $\nabla^2_{j,\boldsymbol{A}}$ is interpreted as in (31) and we introduce the $L^1(\mathcal{I}_T; L^2)$-map

$$f(t) = i\sum_{j=1}^{N} \frac{\hbar Q_j}{c m_j}\mathrm{div}_{\boldsymbol{x}_j}\big(\partial_t \boldsymbol{A}(t)(\boldsymbol{x}_j)\xi(t)\big) + \sum_{j=1}^{N} \frac{Q_j^2}{c^2 m_j}\boldsymbol{A}(t)(\boldsymbol{x}_j)\cdot\partial_t\boldsymbol{A}(t)(\boldsymbol{x}_j)\xi(t).$$

This can be shown by approximating ξ in $W^{1,1}\big(\mathcal{I}_T; L^2(\mathbb{R}^{3N})\big)$ by a sequence of maps $\xi^n : t \mapsto \sum_{m=1}^{M^n} \xi^n_m f^n_m(t)$ with $M^n \in \mathbb{N}$, $\xi^n_1, \dots, \xi^n_{M^n} \in L^2\big(\mathbb{R}^{3N}\big)$ and $f^n_1, \dots, f^n_{M^n} \in C^\infty(\mathcal{I}_T)$ for $n \in \mathbb{N}$. From (19) and (29) it follows for example that

$$\int_0^T \mathrm{div}_{\boldsymbol{x}_j}\big(\boldsymbol{A}(t)(\boldsymbol{x}_j)\xi(t)\big)g'(t)\,\mathrm{d}t = \lim_{n\to\infty}\sum_{m=1}^{M^n}\mathrm{div}_{\boldsymbol{x}_j}\Big(\int_0^T \boldsymbol{A}(t)(f^n_m g')(t)\,\mathrm{d}t(\boldsymbol{x}_j)\xi^n_m\Big)$$
$$= -\int_0^T \mathrm{div}_{\boldsymbol{x}_j}\big(\partial_t\boldsymbol{A}(t)(\boldsymbol{x}_j)\xi(t) + \boldsymbol{A}(t)(\boldsymbol{x}_j)\partial_t\xi(t)\big)g(t)\,\mathrm{d}t$$

and

$$\int_0^T \big[\boldsymbol{A}(t)(\boldsymbol{x}_j)\big]^2\xi(t)g'(t)\,\mathrm{d}t = \lim_{n\to\infty}\sum_{m=1}^{M^n}\int_0^T \big[\boldsymbol{A}(t)\big]^2(f^n_m g')(t)\,\mathrm{d}t(\boldsymbol{x}_j)\xi^n_m$$
$$= -\int_0^T \big[\boldsymbol{A}(t)(\boldsymbol{x}_j)\big]^2\partial_t\xi(t)g(t)\,\mathrm{d}t - 2\int_0^T \boldsymbol{A}(t)(\boldsymbol{x}_j)\cdot\partial_t\boldsymbol{A}(t)(\boldsymbol{x}_j)\xi(t)g(t)\,\mathrm{d}t,$$

for all $j \in \{1,\dots,N\}$ and $g \in C_0^\infty(\mathcal{I}_T^\circ)$, where the limits are taken in H^{-2}. From (18), (19), (36), Corollary 8 and Lemma 11 we get for all $t \in \mathcal{I}_T$ that

$$\partial_t\xi(t) = \mathscr{U}_{\boldsymbol{A}}(t,\tau)\partial_t\xi(\tau) - \frac{i}{\hbar}\int_\tau^t \mathscr{U}_{\boldsymbol{A}}(t,s)f(s)\,\mathrm{d}s.$$

By using (18) and Remark 10 we therefore get the existence of a constant $K > 0$ such that

$$\|\xi(t)\|_{H^2}$$
$$\le K\Big(\langle\|\boldsymbol{A}\|_{L^\infty_T L^4}\rangle^{\frac{8}{1-2\delta}}\|\xi(\tau)\|_{H^2} + \int_\tau^t \|\partial_t\boldsymbol{A}(s)\|_{L^4}\langle\|\boldsymbol{A}(s)\|_{L^4}\rangle\|\xi(s)\|_{H^2}\,\mathrm{d}s\Big)$$

for all $t \in [\tau, T]$ so (35) holds by Gronwall's inequality. □

Given $\sigma \in \mathbb{R}$, $(\boldsymbol{A}_0, \boldsymbol{A}_1) \in H^\sigma \times H^{\sigma-1}$ and $\boldsymbol{F} \in L^1(\mathcal{I}_T; H^{\sigma-1})$ define the continuous function $\mathscr{V}_{\boldsymbol{F}}(\cdot,0)[\boldsymbol{A}_0, \boldsymbol{A}_1] : \mathcal{I}_T \to H^\sigma$ by

$$\mathscr{V}_{\boldsymbol{F}}(t,0)[\boldsymbol{A}_0, \boldsymbol{A}_1] = \dot{\mathfrak{s}}(t)\boldsymbol{A}_0 + \mathfrak{s}(t)\boldsymbol{A}_1 + c^2 \int_0^t \mathfrak{s}(t-\tau)\boldsymbol{F}(\tau)\,\mathrm{d}\tau, \tag{37}$$

where the two linear operators $\dot{\mathfrak{s}}(t) = \cos\big(c(1-\Delta)^{1/2}t\big) : H^\sigma \to H^\sigma$ and $\mathfrak{s}(t) = \frac{\sin(c(1-\Delta)^{1/2}t)}{c(1-\Delta)^{1/2}} : H^{\sigma-1} \to H^\sigma$ are defined as Fourier multipliers for $t \in \mathcal{I}_T$. Then $\mathscr{V}_{\boldsymbol{F}}(\cdot,0)[\boldsymbol{A}_0, \boldsymbol{A}_1]$ has the $C(\mathcal{I}_T; H^{\sigma-1})$-mapping

$$\partial_t \mathscr{V}_{\boldsymbol{F}}(t,0)[\boldsymbol{A}_0, \boldsymbol{A}_1] = c^2(\Delta-1)\mathfrak{s}(t)\boldsymbol{A}_0 + \dot{\mathfrak{s}}(t)\boldsymbol{A}_1 + c^2 \int_0^t \dot{\mathfrak{s}}(t-\tau)\boldsymbol{F}(\tau)\,\mathrm{d}\tau$$

as distributional first derivative and the $L^1(\mathcal{I}_T; H^{\sigma-2})$-function

$$\partial_t^2 \mathscr{V}_{\boldsymbol{F}}(t,0)[\boldsymbol{A}_0, \boldsymbol{A}_1] = c^2(\Delta-1)\mathscr{V}_{\boldsymbol{F}}(t,0)[\boldsymbol{A}_0, \boldsymbol{A}_1] + c^2\boldsymbol{F}(t).$$

as distributional second derivative. In other words, $\mathscr{V}_{\boldsymbol{F}}(\cdot,0)[\boldsymbol{A}_0, \boldsymbol{A}_1]$ solves the Klein-Gordon equation

$$(\Box + 1)\boldsymbol{B} = \boldsymbol{F} \tag{38}$$

with initial conditions

$$\boldsymbol{B}(0) = \boldsymbol{A}_0 \text{ and } \partial_t \boldsymbol{B}(0) = \boldsymbol{A}_1. \tag{39}$$

As expressed below in Lemma 13 the function (37) can be shown to be a $C(\mathcal{I}_T; H^\sigma) \cap C^1(\mathcal{I}_T; H^{\sigma-1})$-solution to the initial value problem (38)–(39) for even more general choices of inhomogeneity $\boldsymbol{F}$. We will need the accompanying Strichartz estimate. The result is due to Brenner [5], Strichartz [27], Ginibre and Velo [10, 11], but is formulated on the basis of [21, Lemma 4.1].

Lemma 13. [21, Lemma 4.1] *Let $0 \le \frac{2}{q_k} = 1 - \frac{2}{r_k} < 1$ for $k \in \{0,1\}$. Then for $\sigma \in \mathbb{R}$, $(\boldsymbol{A}_0, \boldsymbol{A}_1) \in H^\sigma \times H^{\sigma-1}$ and $\boldsymbol{F} \in L^{q_1'}\big(\mathcal{I}_T; W^{\sigma-1+\frac{2}{q_1}, r_1'}\big)$ the function $\boldsymbol{B}(t) = \mathscr{V}_{\boldsymbol{F}}(\cdot,0)[\boldsymbol{A}_0, \boldsymbol{A}_1]$ in (37) is contained in $C(\mathcal{I}_T; H^\sigma) \cap C^1(\mathcal{I}_T; H^{\sigma-1})$ and the Strichartz estimate*

$$\max_{k\in\{0,1\}} \|\partial_t^k \boldsymbol{B}\|_{L_T^{q_0} W^{\sigma-k-\frac{2}{q_0}, r_0}} \lesssim \|(\boldsymbol{A}_0, \boldsymbol{A}_1)\|_{H^\sigma \times H^{\sigma-1}} + \|\boldsymbol{F}\|_{L_T^{q_1'} W^{\sigma-1+\frac{2}{q_1}, r_1'}}$$

holds true.

6 The Contraction Argument

Let $(\psi_0, \boldsymbol{A}_0, \boldsymbol{A}_1) \in H^2(\mathbb{R}^{3N}) \times H^{\frac{3}{2}}(\mathbb{R}^3; \mathbb{R}^3) \times H^{\frac{1}{2}}(\mathbb{R}^3; \mathbb{R}^3)$ satisfy the identities $\mathrm{div}\boldsymbol{A}_0 = \mathrm{div}\boldsymbol{A}_1 = 0$ and consider for $T, R_1, R_2 \in (0,\infty)$ the mapping Φ sending a pair $(\psi, \boldsymbol{A})$ from the (T, R_1, R_2)-dependent space

$$\mathscr{Z}_T = \big\{(\psi, \boldsymbol{A}) \in L^\infty(\mathcal{I}_T; H^2) \times \big(L^\infty\big(\mathcal{I}_T; H^1(\mathbb{R}^3;\mathbb{R}^3)\big) \cap W^{1,4}\big(\mathcal{I}_T; L^4(\mathbb{R}^3;\mathbb{R}^3)\big)\big) \,\big|\, \text{the continuous representative } \mathcal{I}_T \to L^4(\mathbb{R}^3;\mathbb{R}^3) \text{ of } \boldsymbol{A} \text{ is divergence free at all times}, \|\psi\|_{L^\infty_T H^2} \le R_1, \max\{\|\boldsymbol{A}\|_{L^\infty_T H^1}, \|\boldsymbol{A}\|_{W^{1,4}_T L^4}\} \le R_2\big\}$$

into the solution $\Phi(\psi, \boldsymbol{A}) = \big(\mathscr{U}_{\boldsymbol{A}}(\cdot, 0)\psi_0, \mathscr{V}_{\frac{4\pi}{c}\sum_{j=1}^N P\boldsymbol{J}_j[\psi,\boldsymbol{A}]+\boldsymbol{A}}(\cdot, 0)[\boldsymbol{A}_0, \boldsymbol{A}_1]\big)$ to the linearized system

$$i\hbar\partial_t\xi = \Big(\sum_{j=1}^N \frac{1}{2m_j}\nabla^2_{j,\boldsymbol{A}} + \sum_{1\le j<k\le N} \frac{Q_jQ_k}{|\boldsymbol{x}_j - \boldsymbol{x}_k|}\Big)\xi, \tag{40}$$

$$(\Box + 1)\boldsymbol{B} = \frac{4\pi}{c}\sum_{j=1}^N P\boldsymbol{J}_j[\psi, \boldsymbol{A}] + \boldsymbol{A} \tag{41}$$

with initial data

$$\xi(0) = \psi_0, \quad \boldsymbol{B}(0) = \boldsymbol{A}_0 \quad \text{and} \quad \partial_t\boldsymbol{B}(0) = \boldsymbol{A}_1,$$

where we observe that $W^{1,4}\big(\mathcal{I}_T; L^4(\mathbb{R}^3;\mathbb{R}^3)\big) \hookrightarrow W^{1,1}\big(\mathcal{I}_T; L^4(\mathbb{R}^3;\mathbb{R}^3)\big)$ and $P\boldsymbol{J}_j[\psi, \boldsymbol{A}] \in L^\infty(\mathcal{I}_T; H^1(\mathbb{R}^3;\mathbb{R}^3))$ for $j \in \{1, \dots, N\}$ by (20) and the boundedness of the Helmholtz projection $H^1 \to H^1$. Combining Corollary 8 with Lemma 13 gives that $\Phi(\psi, \boldsymbol{A}) \in C(\mathcal{I}_T; H^2) \times \big(C(\mathcal{I}_T; H^{\frac{3}{2}}) \cap C^1(\mathcal{I}_T; H^{\frac{1}{2}})\big)$ and we observe directly from (37) that the second coordinate of $\Phi(\psi, \boldsymbol{A})$ must be divergence free at all times, whereby a fixed point of Φ will have the desired properties. Our strategy will therefore be to invoke the Banach fixed-point theorem and for this we equip $\mathscr{Z}_T$ with the metric d given by

$$d\big((\psi, \boldsymbol{A}), (\psi', \boldsymbol{A}')\big) = \max\big\{\|\psi - \psi'\|_{L^\infty_T L^2}, \|\boldsymbol{A} - \boldsymbol{A}'\|_{L^\infty_T H^{\frac{1}{2}}}, \|\boldsymbol{A} - \boldsymbol{A}'\|_{L^4_T L^4}\big\}$$

for $(\psi, \boldsymbol{A}), (\psi', \boldsymbol{A}') \in \mathscr{Z}_T$.

Lemma 14. *For all choices of positive numbers T, R_1 and R_2 the metric space $(\mathscr{Z}_T, d)$ is complete.*

Proof. Let $\big((\psi_n, \boldsymbol{A}_n)\big)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(\mathscr{Z}_T, d)$. Then $(\psi_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $L^\infty(\mathcal{I}_T; L^2)$ and $(\psi_n)_{n\in\mathbb{N}}$ is furthermore known to be bounded by the constant R_1 in the space $L^\infty(\mathcal{I}_T; H^2)$ – a space that can be identified with the dual of the separable space $L^1(\mathcal{I}_T; H^{-2})$ by the isometric anti-isomorphism

$$L^\infty(\mathcal{I}_T; H^2) \ni F \mapsto \Big(G \mapsto \int_0^T \langle F(t), G(t)\rangle_{H^2,H^{-2}}\,\mathrm{d}t\Big) \in \big(L^1(\mathcal{I}_T; H^{-2})\big)^*$$

as expressed in [9, Theorem 8.18.3]. Therefore we can use the Banach-Alaoglu theorem to conclude that there exist $\psi \in L^\infty(\mathcal{I}_T; L^2)$ and $\psi^* \in L^\infty(\mathcal{I}_T; H^2)$ such that

$$\psi_n \xrightarrow[n\to\infty]{} \psi \text{ in } L^\infty(\mathcal{I}_T; L^2) \text{ and } \psi_{n_k} \xrightarrow[k\to\infty]{w*} \psi^* \text{ in } L^\infty(\mathcal{I}_T; H^2). \tag{42}$$

For $\varphi \in L^2(\mathcal{I}_T; L^2)$ the sequence $\big((\psi_{n_k}, \varphi)_{L^2L^2}\big)_{k\in\mathbb{N}}$ then converges to both of the numbers $(\psi, \varphi)_{L^2L^2}$ and $(\psi^*, \varphi)_{L^2L^2}$ so the functions ψ and ψ^* must be identical. Likewise, $(\boldsymbol{A}_n)_{n\in\mathbb{N}}$ is bounded by the constant R_2 in the dual $L^\infty(\mathcal{I}_T; H^1)$ of the separable space $L^1(\mathcal{I}_T; H^{-1})$ and in addition $(\boldsymbol{A}_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in each of the two Banach spaces $L^\infty\big(\mathcal{I}_T; H^{\frac{1}{2}}\big)$ and $L^4(\mathcal{I}_T; L^4)$. Consequently, there exists an $\boldsymbol{A} \in L^\infty(\mathcal{I}_T, H^1) \cap L^4(\mathcal{I}_T, L^4)$ such that

$$\boldsymbol{A}_n \xrightarrow[n\to\infty]{} \boldsymbol{A} \text{ in } L^\infty\big(\mathcal{I}_T; H^{\frac{1}{2}}\big) \text{ and } L^4(\mathcal{I}_T; L^4), \boldsymbol{A}_{n'_k} \xrightarrow[k\to\infty]{w*} \boldsymbol{A} \text{ in } L^\infty(\mathcal{I}_T; H^1). \tag{43}$$

Moreover, the boundedness of the sequence $(\partial_t \boldsymbol{A}_n)_{n\in\mathbb{N}}$ in the reflexive space $L^4(\mathcal{I}_T; L^4)$ gives the existence of an $\dot{\boldsymbol{A}} \in L^4(\mathcal{I}_T; L^4)$ such that the weak convergence

$$\partial_t \boldsymbol{A}_{n''_k} \xrightarrow[k\to\infty]{} \dot{\boldsymbol{A}} \text{ in } L^4(\mathcal{I}_T; L^4) \tag{44}$$

holds. But for any $k \in \mathbb{N}$, $\eta \in L^{\frac{4}{3}}(\mathbb{R}^3)$ and $\varphi \in C_0^\infty(\mathcal{I}_T^\circ)$ we then have

$$\int_0^T \int_{\mathbb{R}^3} \boldsymbol{A}_{n''_k}(t)(\boldsymbol{x})\eta(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\varphi'(t)\,\mathrm{d}t = -\int_0^T \int_{\mathbb{R}^3} \partial_t \boldsymbol{A}_{n''_k}(t)(\boldsymbol{x})\eta(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\varphi(t)\,\mathrm{d}t$$

whereby letting $k \to \infty$ and using (43)–(44) gives that $\dot{\boldsymbol{A}}$ is the distributional time derivative of $\boldsymbol{A}$. Concerning the divergence of $\boldsymbol{A}$ we observe that

$$\int_0^T \|\mathrm{div}\boldsymbol{A}(t)\|_{W^{-1,4}}^4\,\mathrm{d}t \leq \|\boldsymbol{A} - \boldsymbol{A}_n\|_{L_T^4 L^4}^4 \xrightarrow[n\to\infty]{} 0$$

so $\mathrm{div}\boldsymbol{A}(t) = \mathrm{div}\partial_t\boldsymbol{A}(t) = 0$ for almost all $t \in \mathcal{I}_T$. For any $t \in (0, T]$ the continuous representative $\mathcal{I}_T \to L^4(\mathbb{R}^3; \mathbb{R}^3)$ of $\boldsymbol{A}$ therefore satisfies

$$\mathrm{div}\boldsymbol{A}(t) = \mathrm{div}\boldsymbol{A}(t') + \int_{t'}^t \mathrm{div}\partial_t\boldsymbol{A}(s)\,\mathrm{d}s = 0,$$

where we have chosen some time $t' \in [0, t]$ in which $\boldsymbol{A}$ takes a divergence free value – the identity $\mathrm{div}\boldsymbol{A}(0) = 0$ then follows by using the continuity of $\mathcal{I}_T \ni t \mapsto \mathrm{div}\boldsymbol{A}(t) \in W^{-1,4}(\mathbb{R}^3)$. Finally, [6, Propositions 3.5 and 3.13] concerning boundedness of weakly (respectively weak-$*$) convergent sequences combined with (42)–(44) give

$$\|\psi\|_{L_T^\infty H^2} \leq R_1 \text{ and } \max\big\{\|\boldsymbol{A}\|_{L_T^\infty H^1}, \|\boldsymbol{A}\|_{W_T^{1,4} L^4}\big\} \leq R_2,$$

whereby we are in position to conclude that $(\psi, \boldsymbol{A})$ is contained in $\mathcal{Z}_T$ and that $d\big((\psi, \boldsymbol{A}), (\psi_n, \boldsymbol{A}_n)\big) \xrightarrow[n\to\infty]{} 0$. □

Next, we investigate the properties of the mapping Φ.

Lemma 15. *Given any* $(\psi_0, \boldsymbol{A}_0, \boldsymbol{A}_1) \in H^2(\mathbb{R}^{3N}) \times H^{\frac{3}{2}}(\mathbb{R}^3;\mathbb{R}^3) \times H^{\frac{1}{2}}(\mathbb{R}^3;\mathbb{R}^3)$ *satisfying* $\mathrm{div}\boldsymbol{A}_0 = \mathrm{div}\boldsymbol{A}_1 = 0$ *and any* $R > 0$ *there exist* $R_1, R_2 \in (R,\infty)$ *and* $T_\dagger > 0$ *such that for all* $T \in (0, T_\dagger]$ *the function* Φ *maps* $\mathcal{Z}_T$ *into itself.*

Proof. Let $(\psi_0, \boldsymbol{A}_0, \boldsymbol{A}_1) \in H^2(\mathbb{R}^{3N}) \times H^{\frac{3}{2}}(\mathbb{R}^3;\mathbb{R}^3) \times H^{\frac{1}{2}}(\mathbb{R}^3;\mathbb{R}^3)$ satisfy the identities $\mathrm{div}\boldsymbol{A}_0 = \mathrm{div}\boldsymbol{A}_1 = 0$ and consider arbitrary positive constants T, R_1 and R_2. For any fixed pair $(\psi, \boldsymbol{A}) \in \mathcal{Z}_T$ we get from Lemma 12, the Sobolev embedding $H^{\frac{3}{4}} \hookrightarrow L^4$, Lemma 13 and (20) that not only is $\Phi(\psi,\boldsymbol{A}) = (\xi, \boldsymbol{B})$ contained in $C(\mathcal{I}_T; H^2) \times \big(C(\mathcal{I}_T; H^{\frac{3}{2}}) \cap C^1(\mathcal{I}_T; H^{\frac{1}{2}})\big)$ as noted above, but we also have $\boldsymbol{B} \in W^{1,4}(\mathcal{I}_T, L^4)$ with the two estimates

$$\|\xi\|_{L^\infty_T H^2} \leq C\langle R_2\rangle^{\frac{8}{1-2\delta}} \exp\big(CT^{\frac{3}{4}}\langle R_2\rangle R_2\big)\|\psi_0\|_{H^2}$$

and

$$\begin{aligned}&\max\big\{\|\boldsymbol{B}\|_{L^\infty_T H^{\frac{3}{2}}}, \|\boldsymbol{B}\|_{L^4_T L^4}, \|\partial_t \boldsymbol{B}\|_{L^4_T L^4}\big\}\\ &\leq C\big(\|(\boldsymbol{A}_0, \boldsymbol{A}_1)\|_{H^{\frac{3}{2}} \times H^{\frac{1}{2}}} + T(1+R_2)R_1^2 + TR_2\big)\end{aligned}$$

holding true for some constant $C > 0$ (depending on c, $\hbar$, N, $m_1, \ldots, m_N$ and $Q_1, \ldots, Q_N$). Given some $(\psi_0, \boldsymbol{A}_0, \boldsymbol{A}_1) \in H^2(\mathbb{R}^{3N}) \times H^{\frac{3}{2}}(\mathbb{R}^3;\mathbb{R}^3) \times H^{\frac{1}{2}}(\mathbb{R}^3;\mathbb{R}^3)$ with $\mathrm{div}\boldsymbol{A}_0 = \mathrm{div}\boldsymbol{A}_1 = 0$ and some positive number R we can therefore choose $R_2 > \max\big\{2\sqrt{2}C\|(\boldsymbol{A}_0, \boldsymbol{A}_1)\|_{H^{\frac{3}{2}} \times H^{\frac{1}{2}}}, R\big\}$, $R_1 > \max\big\{2C\langle R_2\rangle^{\frac{8}{1-2\delta}}\|\psi_0\|_{H^2}, R\big\}$ and $T_\dagger = \min\big\{\frac{R_2}{2\sqrt{2}C((1+R_2)R_1^2+R_2)}, \big(\frac{\log 2}{CR_2\langle R_2\rangle}\big)^{\frac{4}{3}}\big\}$ to make sure that Φ maps $\mathcal{Z}_T$ into itself for any $T \in (0, T_\dagger]$. $\square$

Finally, we show that by choosing T sufficiently small we can make Φ a contraction on $(\mathcal{Z}_T, d)$, which by the Banach fixed-point theorem guarantees the existence of a unique fixed point for Φ.

Lemma 16. *For any* $(\psi_0, \boldsymbol{A}_0, \boldsymbol{A}_1) \in H^2(\mathbb{R}^{3N}) \times H^{\frac{3}{2}}(\mathbb{R}^3;\mathbb{R}^3) \times H^{\frac{1}{2}}(\mathbb{R}^3;\mathbb{R}^3)$ *with* $\mathrm{div}\boldsymbol{A}_0 = \mathrm{div}\boldsymbol{A}_1 = 0$ *and any* $R \geq 0$ *there exist* $R_1, R_2 \in (R, \infty)$ *and* $T_* > 0$ *such that* Φ *is a contraction on* $\big(\mathcal{Z}_T, d\big)$ *for all* $T \in (0, T_*]$.

Proof. Given $R \geq 0$ and $(\psi_0, \boldsymbol{A}_0, \boldsymbol{A}_1) \in H^2(\mathbb{R}^{3N}) \times H^{\frac{3}{2}}(\mathbb{R}^3;\mathbb{R}^3) \times H^{\frac{1}{2}}(\mathbb{R}^3;\mathbb{R}^3)$ satisfying $\mathrm{div}\boldsymbol{A}_0 = \mathrm{div}\boldsymbol{A}_1 = 0$ we use Lemma 15 to choose $R_1, R_2 \in (R,\infty)$ and $T_\dagger > 0$ such that Φ maps $\mathcal{Z}_T$ into itself for any time span $T \in (0, T_\dagger]$. Given an arbitrary such $T \in (0, T_\dagger]$ we consider $(\psi, \boldsymbol{A}), (\psi', \boldsymbol{A}') \in \mathcal{Z}_T$ and write $\Phi(\psi, \boldsymbol{A}) = (\xi, \boldsymbol{B})$ as well as $\Phi(\psi', \boldsymbol{A}') = (\xi', \boldsymbol{B}')$. After introducing $f \in C(\mathcal{I}_T; L^2)$ by setting

$$f(t) = \sum_{j=1}^{N} \frac{1}{2m_j}\big(\nabla^2_{j,\boldsymbol{A}(t)} - \nabla^2_{j,\boldsymbol{A}'(t)}\big)\xi'(t) \text{ for } t \in \mathcal{I}_T$$

we observe that $\xi - \xi'$ solves the initial value problem

$$\begin{aligned} i\hbar\partial_t(\xi - \xi') &= \Big(\sum_{j=1}^{N} \frac{1}{2m_j}\nabla^2_{j,\boldsymbol{A}} + \sum_{1\le j<k\le N} \frac{Q_jQ_k}{|\boldsymbol{x}_j - \boldsymbol{x}_k|}\Big)(\xi - \xi') + f \\ (\xi - \xi')(0) &= 0. \end{aligned}$$

Combining this with Lemma 11 gives that $(\xi - \xi')(t) = -\frac{i}{\hbar}\int_0^t \mathscr{U}_{\boldsymbol{A}}(t,s)f(s)\,\mathrm{d}s$ for all $t \in \mathcal{I}_T$, whereby Remark 10, Lemma 4 and Hölder's inequality help us obtain the estimate

$$\begin{aligned} \|\xi - \xi'\|_{L^\infty_T L^2} &\lesssim \int_0^T (1 + \|\boldsymbol{A}(s) + \boldsymbol{A}'(s)\|_{L^4})\|\boldsymbol{A}(s) - \boldsymbol{A}'(s)\|_{L^4}\|\xi'(s)\|_{H^2}\,\mathrm{d}s \\ &\le R_1\big(T^{\frac{3}{4}} + 2R_2T^{\frac{1}{2}}\big)\|\boldsymbol{A} - \boldsymbol{A}'\|_{L^4_T L^4}. \end{aligned} \tag{45}$$

The map $\boldsymbol{B} - \boldsymbol{B}' = \mathscr{V}_{\frac{4\pi}{c}\sum_{j=1}^N P(\boldsymbol{J}_j[\psi,\boldsymbol{A}] - \boldsymbol{J}_j[\psi',\boldsymbol{A}'])}(\cdot,0)[\boldsymbol{0},\boldsymbol{0}] + \mathscr{V}_{\boldsymbol{A}-\boldsymbol{A}'}(\cdot,0)[\boldsymbol{0},\boldsymbol{0}]$ satisfies

$$\begin{aligned} &\max\big\{\|\boldsymbol{B} - \boldsymbol{B}'\|_{L^\infty_T H^{\frac{1}{2}}}, \|\boldsymbol{B} - \boldsymbol{B}'\|_{L^4_T L^4}\big\} \\ &\lesssim \sum_{j=1}^{N}\big\|P\big(\boldsymbol{J}_j[\psi,\boldsymbol{A}] - \boldsymbol{J}_j[\psi',\boldsymbol{A}']\big)\big\|_{L^{\frac{4}{3}}_T L^{\frac{4}{3}}} + \|\boldsymbol{A} - \boldsymbol{A}'\|_{L^1_T H^{-\frac{1}{2}}} \end{aligned} \tag{46}$$

by Lemma 13. To estimate the first term on the right hand side of (46) we write $\big(\boldsymbol{J}_j[\psi,\boldsymbol{A}] - \boldsymbol{J}_j[\psi',\boldsymbol{A}']\big)(t)$ for almost all $t \in \mathcal{I}_T$ as a sum of the three $L^{\frac{4}{3}}$-functions

$$\begin{aligned} g^1_j(t) &: \boldsymbol{x}_j \mapsto \frac{Q_j}{m_j}\mathrm{Re}\int_{\mathbb{R}^{3(N-1)}} \overline{(\psi' - \psi)}(t)(\boldsymbol{x})\nabla_{j,\boldsymbol{A}(t)}\psi(t)(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}'_j, \\ g^2_j(t) &: \boldsymbol{x}_j \mapsto \frac{Q_j^2}{m_jc}(\boldsymbol{A}' - \boldsymbol{A})(t)(\boldsymbol{x}_j)\mathrm{Re}\int_{\mathbb{R}^{3(N-1)}} \big(\overline{\psi}'\psi\big)(t)(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}'_j \end{aligned}$$

and

$$\begin{aligned} g^3_j(t) : \boldsymbol{x}_j \mapsto \Big\{&-\frac{Q_j}{m_j}\mathrm{Re}\int_{\mathbb{R}^{3(N-1)}} \nabla_{j,-\boldsymbol{A}'(t)}\overline{\psi}'(t)(\boldsymbol{x})(\psi' - \psi)(t)(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}'_j \\ &- \frac{Q_j\hbar}{m_j}\nabla_{\boldsymbol{x}_j}\mathrm{Im}\int_{\mathbb{R}^{3(N-1)}} \big[\overline{\psi}'(\psi' - \psi)\big](t)(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}'_j\Big\}. \end{aligned} \tag{47}$$

where the expression for the third function can also be written more compactly as $\boldsymbol{x}_j \mapsto \frac{Q_j}{m_j}\mathrm{Re}\int_{\mathbb{R}^{3(N-1)}} \overline{\psi}'(t)(\boldsymbol{x})\nabla_{j,\boldsymbol{A}'(t)}(\psi' - \psi)(t)(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}'_j$. However, in the present context we prefer to express g^3_j in the form (47) since applying the Helmholtz projection kills the last term in (47) and leaves us with a term with no derivatives applied to the difference $(\psi' - \psi)(t)$. As in the proof of Lemma 4 we can therefore use Minkowski's integral inequality, the Sobolev embeddings

$H^{\frac{3}{4}} \hookrightarrow L^4$, $H^{\frac{3}{2}+\delta} \hookrightarrow L^\infty$, boundedness of the Helmholtz projection $L^{\frac{4}{3}} \to L^{\frac{4}{3}}$ and Hölder's inequality to obtain that for almost all $t \in \mathcal{I}_T$,

$$\begin{aligned}
&\left\|P\big(\boldsymbol{J}_j[\psi, \boldsymbol{A}] - \boldsymbol{J}_j[\psi', \boldsymbol{A}']\big)(t)\right\|_{L^{\frac{4}{3}}} \\
&\lesssim \|g_j^1(t)\|_{L^{\frac{4}{3}}} + \|g_j^2(t)\|_{L^{\frac{4}{3}}} + \left\|\boldsymbol{x}_j \mapsto \int \nabla_{j,-\boldsymbol{A}'(t)}\overline{\psi}'(t)(\boldsymbol{x})(\psi' - \psi)(t)(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}_j'\right\|_{L^{\frac{4}{3}}} \\
&\lesssim \big\{(1 + \|\boldsymbol{A}(t)\|_{L^4})\|\psi(t)\|_{H^2} + (1 + \|\boldsymbol{A}'(t)\|_{L^4})\|\psi'(t)\|_{H^2}\big\}\|(\psi' - \psi)(t)\|_{L^2} \\
&\qquad\qquad + \|\psi'(t)\|_{L^2}\|\psi(t)\|_{H^2}\|(\boldsymbol{A}' - \boldsymbol{A})(t)\|_{L^4}.
\end{aligned}$$

and so

$$\begin{aligned}
&\left\|P\big(\boldsymbol{J}_j[\psi, \boldsymbol{A}] - \boldsymbol{J}_j[\psi', \boldsymbol{A}']\big)\right\|_{L^{\frac{4}{3}} L^{\frac{4}{3}}} \\
&\lesssim R_1\big(T^{\frac{3}{4}} + R_2 T^{\frac{1}{2}}\big)\|\psi' - \psi\|_{L_T^\infty L^2} + R_1^2 T^{\frac{1}{2}}\|\boldsymbol{A}' - \boldsymbol{A}\|_{L_T^4 L^4}.
\end{aligned} \tag{48}$$

From (45), (46) and (48) we realize that there exists a constant $C > 0$ such that

$$d\big(\Phi(\psi, \boldsymbol{A}), \Phi(\psi', \boldsymbol{A}')\big) \le C\big(R_1\big(T^{\frac{3}{4}} + R_2 T^{\frac{1}{2}}\big) + R_1^2 T^{\frac{1}{2}} + T\big) d\big((\psi, \boldsymbol{A}), (\psi', \boldsymbol{A}')\big)$$

so for small enough T the mapping Φ will be a contraction on $(\mathcal{Z}_T, d)$. ☐

The existence part of Theorem 1 has now been proven.

7 Uniqueness

We now turn our attention to the uniqueness question.

Lemma 17. *Let* $(\psi_0, \boldsymbol{A}_0, \boldsymbol{A}_1) \in H^2(\mathbb{R}^{3N}) \times H^{\frac{3}{2}}(\mathbb{R}^3; \mathbb{R}^3) \times H^{\frac{1}{2}}(\mathbb{R}^3; \mathbb{R}^3)$ *with* $\mathrm{div}\boldsymbol{A}_0 = \mathrm{div}\boldsymbol{A}_1 = 0$ *and* $T > 0$ *be given. Then if the pairs* $(\psi^1, \boldsymbol{A}^1)$ *and* $(\psi^2, \boldsymbol{A}^2)$ *belong to* $C(\mathcal{I}_T; H^2) \times \big(C\big(\mathcal{I}_T; H^{\frac{3}{2}}\big) \cap C^1\big(\mathcal{I}_T; H^{\frac{1}{2}}\big)\big)$*, solve* (3)+(6) *and both of the vector fields* $\boldsymbol{A}^1$*,* $\boldsymbol{A}^2$ *are divergence free at all times in* $[0, T]$ *then there exists a* $T_* \in (0, T]$ *such that* $(\psi^1, \boldsymbol{A}^1)$ *and* $(\psi^2, \boldsymbol{A}^2)$ *agree on the time interval* $[0, T_*]$.

Proof. For $\ell \in \{1, 2\}$ let $(\psi^\ell, \boldsymbol{A}^\ell)$ satisfy the hypotheses of the lemma and choose with the help of Lemma 16 some radii

$$R_1, R_2 > \max\big\{\|\psi^\ell\|_{L_T^\infty H^2}, \|\boldsymbol{A}^\ell\|_{L_T^\infty H^1}, \|\boldsymbol{A}^\ell\|_{W_T^{1,4} L^4}\big|\ell \in \{1, 2\}\big\}$$

and a time $T_* \in (0, T]$ such that Φ is a contraction on $\mathcal{Z}_{T_*}$. Then the vector field $\boldsymbol{B} = \boldsymbol{A}^\ell|_{\mathcal{I}_{T_*}} \in C\big(\mathcal{I}_{T_*}; H^{\frac{3}{2}}\big) \cap C^1\big(\mathcal{I}_{T_*}; H^{\frac{1}{2}}\big)$ solves the initial value problem (41)+(39) on $\mathcal{I}_{T_*}$ with $(\psi, \boldsymbol{A}) = \big(\psi^\ell|_{\mathcal{I}_{T_*}}, \boldsymbol{A}^\ell|_{\mathcal{I}_{T_*}}\big)$ so by uniqueness of solutions to the Klein-Gordon initial value problem [25, Theorem 3.2] we have

$$\boldsymbol{A}^\ell|_{\mathcal{I}_{T_*}}(t) = \mathscr{V}_{\frac{4\pi}{c}\sum_{j=1}^N P\boldsymbol{J}_j[\psi^\ell|_{\mathcal{I}_{T_*}}, \boldsymbol{A}^\ell|_{\mathcal{I}_{T_*}}] + \boldsymbol{A}^\ell|_{\mathcal{I}_{T_*}}}(t, 0)[\boldsymbol{A}_0, \boldsymbol{A}_1]$$

for $t \in \mathcal{I}_{T_*}$. We conclude that $\boldsymbol{A}^1|_{\mathcal{I}_{T_*}}, \boldsymbol{A}^2|_{\mathcal{I}_{T_*}} \in W^{1,4}\big(\mathcal{I}_{T_*}; L^4(\mathbb{R}^3;\mathbb{R}^3)\big)$ by Lemma 13. Likewise, for $\ell \in \{1,2\}$ the map $\xi = \psi^\ell|_{\mathcal{I}_{T_*}} \in C(\mathcal{I}_{T_*}; H^2)$ solves the initial value problem (25)+(26) on $\mathcal{I}_{T_*}$ with $\boldsymbol{A} = \boldsymbol{A}^\ell|_{\mathcal{I}_{T_*}}$ so Lemma 11 gives that

$$\psi^\ell|_{\mathcal{I}_{T_*}}(t) = \mathscr{U}_{\boldsymbol{A}^\ell|_{\mathcal{I}_{T_*}}}(t,0)\psi_0$$

for $t \in \mathcal{I}_{T_*}$. Consequently, $\big(\psi^1|_{\mathcal{I}_{T_*}}, \boldsymbol{A}^1|_{\mathcal{I}_{T_*}}\big)$ and $\big(\psi^2|_{\mathcal{I}_{T_*}}, \boldsymbol{A}^2|_{\mathcal{I}_{T_*}}\big)$ are both fixed points for the contraction $\Phi : \mathcal{Z}_{T_*} \to \mathcal{Z}_{T_*}$, whereby we must have $\big(\psi^1|_{\mathcal{I}_{T_*}}, \boldsymbol{A}^1|_{\mathcal{I}_{T_*}}\big) = \big(\psi^2|_{\mathcal{I}_{T_*}}, \boldsymbol{A}^2|_{\mathcal{I}_{T_*}}\big)$. □

In fact, Lemma 17 holds true with $T_* = T$.

Lemma 18. *Given $(\psi_0, \boldsymbol{A}_0, \boldsymbol{A}_1) \in H^2(\mathbb{R}^{3N}) \times H^{\frac{3}{2}}(\mathbb{R}^3;\mathbb{R}^3) \times H^{\frac{1}{2}}(\mathbb{R}^3;\mathbb{R}^3)$ with* $\mathrm{div}\boldsymbol{A}_0 = \mathrm{div}\boldsymbol{A}_1 = 0$ *and $T > 0$ there exists at most one pair $(\psi, \boldsymbol{A})$ in the space $C(\mathcal{I}_T; H^2) \times \big(C\big(\mathcal{I}_T; H^{\frac{3}{2}}\big) \cap C^1\big(\mathcal{I}_T; H^{\frac{1}{2}}\big)\big)$ that solves (3)+(6) and satisfies* $\mathrm{div}\boldsymbol{A}(t) = 0$ *for all $t \in \mathcal{I}_T$.*

Proof. For $T > 0$ and $(\psi_0, \boldsymbol{A}_0, \boldsymbol{A}_1) \in H^2(\mathbb{R}^{3N}) \times H^{\frac{3}{2}}(\mathbb{R}^3;\mathbb{R}^3) \times H^{\frac{1}{2}}(\mathbb{R}^3;\mathbb{R}^3)$ with $\mathrm{div}\boldsymbol{A}_0 = \mathrm{div}\boldsymbol{A}_1 = 0$ consider two solutions $(\psi^1, \boldsymbol{A}^1)$ and $(\psi^2, \boldsymbol{A}^2)$ to (3)+(6) that belong to $C(\mathcal{I}_T; H^2) \times \big(C\big(\mathcal{I}_T; H^{\frac{3}{2}}\big) \cap C^1\big(\mathcal{I}_T; H^{\frac{1}{2}}\big)\big)$ and satisfy $\mathrm{div}\boldsymbol{A}^1(t) = \mathrm{div}\boldsymbol{A}^2(t) = 0$ for all $t \in \mathcal{I}_T$. Then the continuity of the mappings $\psi^1, \psi^2 : \mathcal{I}_T \to H^2$, $\boldsymbol{A}^1, \boldsymbol{A}^2 : \mathcal{I}_T \to H^{\frac{3}{2}}$ and $\partial_t\boldsymbol{A}^1, \partial_t\boldsymbol{A}^2 : \mathcal{I}_T \to H^{\frac{1}{2}}$ gives that the number

$$t_0 = \sup\big\{t \in [0,T] \,\big|\, (\psi^1, \boldsymbol{A}^1) = (\psi^2, \boldsymbol{A}^2) \text{ on } [0,t]\big\}$$

satisfies

$$(\psi^1(t_0), \boldsymbol{A}^1(t_0), \partial_t\boldsymbol{A}^1(t_0)) = (\psi^2(t_0), \boldsymbol{A}^2(t_0), \partial_t\boldsymbol{A}^2(t_0)).$$

With the intention of reaching a contradiction we assume that $t_0 < T$. Then for $\ell \in \{1,2\}$ the pair $\big(\widetilde{\psi}^\ell, \widetilde{\boldsymbol{A}}^\ell\big) \in C(\mathcal{I}_{T-t_0}; H^2) \times \big(C\big(\mathcal{I}_{T-t_0}; H^{\frac{3}{2}}\big) \cap C^1\big(\mathcal{I}_{T-t_0}; H^{\frac{1}{2}}\big)\big)$ given by

$$\big(\widetilde{\psi}^\ell(t), \widetilde{\boldsymbol{A}}^\ell(t)\big) = \big(\psi^\ell(t+t_0), \boldsymbol{A}^\ell(t+t_0)\big) \text{ for } t \in \mathcal{I}_{T-t_0}$$

takes the initial values

$$\widetilde{\psi}^\ell(0) = \psi^1(t_0), \widetilde{\boldsymbol{A}}^\ell(0) = \boldsymbol{A}^1(t_0) \text{ and } \partial_t\widetilde{\boldsymbol{A}}^\ell(0) = \partial_t\boldsymbol{A}^1(t_0)$$

and satisfies (3) on $\mathcal{I}_{T-t_0}$. Thus, Lemma 17 gives the existence of some time $T_* \in (0, T-t_0]$ such that $\big(\widetilde{\psi}^1, \widetilde{\boldsymbol{A}}^1\big)$ and $\big(\widetilde{\psi}^2, \widetilde{\boldsymbol{A}}^2\big)$ agree on $[0, T_*]$, whereby the pairs $(\psi^1, \boldsymbol{A}^1)$ and $(\psi^2, \boldsymbol{A}^2)$ agree on $[t_0, t_0 + T_*]$. This contradicts the definition of t_0, whereby we can conclude that $(\psi^1, \boldsymbol{A}^1)$ and $(\psi^2, \boldsymbol{A}^2)$ agree on all of the interval $\mathcal{I}_T$. □

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Existence of Travelling Wave Solutions to the Maxwell-Pauli and Maxwell-Schrödinger Systems

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Abstract. We study two mathematical descriptions of a charged particle interacting with it's self-generated electromagnetic field. The first model is the one-body Maxwell-Schrödinger system where the interaction of the spin with the magnetic field is neglected and the second model is the related one-body Maxwell-Pauli system where the spin-field interaction is included. We prove that there exist travelling wave solutions to both of these systems provided that the speed $|\boldsymbol{v}|$ of the wave is not too large. Moreover, we observe that the energies of these solutions behave like $\frac{m\boldsymbol{v}^2}{2}$ for small velocities of the particle, which may be interpreted as saying that the effective mass of the particle is the same as it's bare mass.

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1 Introduction

Consider a single spin-$\frac{1}{2}$ particle of mass $m > 0$ and charge $Q \in \mathbb{R} \setminus \{0\}$ interacting with it's self-generated electromagnetic field – the Maxwell-Schrödinger system in Coulomb gauge says[1] that

$$\begin{aligned}
\Box \boldsymbol{A}_{\mathrm{t}} &= \frac{4\pi}{c} P \boldsymbol{J}_{\mathrm{S}}[\psi_{\mathrm{t}}, \boldsymbol{A}_{\mathrm{t}}], \\
i\hbar \partial_t \psi_{\mathrm{t}} &= \Big(\frac{1}{2m} \nabla^2_{\mathrm{S}, \boldsymbol{A}_{\mathrm{t}}} + \mathscr{E}_{\mathrm{EM}}[\boldsymbol{A}_{\mathrm{t}}, \partial_t \boldsymbol{A}_{\mathrm{t}}] \Big) \psi_{\mathrm{t}}, \qquad (1) \\
\mathrm{div} \boldsymbol{A}_{\mathrm{t}} &= 0.
\end{aligned}$$

[1] We use Gaussian units.

where $\hbar > 0$ is the reduced Planck constant, $\psi_{\mathrm{t}}(t) : \mathbb{R}^3 \to \mathbb{C}^2$ is the quantum mechanical wave function describing the particle, $\boldsymbol{A}_{\mathrm{t}}(t) : \mathbb{R}^3 \to \mathbb{R}^3$ is the classical magnetic vector potential induced by the particle, with $c > 0$ denoting the speed of light we let $\nabla_{\mathrm{S},\boldsymbol{A}_{\mathrm{t}}} = i\hbar\nabla + \frac{Q}{c}\boldsymbol{A}_{\mathrm{t}}$ denote the covariant derivative with respect to $\boldsymbol{A}_{\mathrm{t}}$, $\Box = \frac{1}{c^2}\partial_t^2 - \Delta$ is the d'Alembertian, $P = 1 - \nabla\mathrm{div}\Delta^{-1}$ is the Helmholtz projection onto the solenoidal subspace of divergence free vector fields, $\mathscr{E}_{\mathrm{EM}}[\boldsymbol{A}_{\mathrm{t}}, \partial_t\boldsymbol{A}_{\mathrm{t}}](t)$ is the energy

$$\mathscr{E}_{\mathrm{EM}}[\boldsymbol{A}_{\mathrm{t}}, \partial_t\boldsymbol{A}_{\mathrm{t}}](t) = \frac{1}{8\pi}\int_{\mathbb{R}^3}\Big(|\nabla \times \boldsymbol{A}_{\mathrm{t}}(t)(\boldsymbol{y})|^2 + \Big|\frac{1}{c}\partial_t\boldsymbol{A}_{\mathrm{t}}(t)(\boldsymbol{y})\Big|^2\Big)\,\mathrm{d}\boldsymbol{y}$$

associated with the electromagnetic field and $\boldsymbol{J}_{\mathrm{S}}[\psi_{\mathrm{t}}, \boldsymbol{A}_{\mathrm{t}}](t) : \mathbb{R}^3 \to \mathbb{R}^3$ denotes the probability current density given by

$$\boldsymbol{J}_{\mathrm{S}}[\psi_{\mathrm{t}}, \boldsymbol{A}_{\mathrm{t}}](t)(\boldsymbol{x}) = -\frac{Q}{m}\mathrm{Re}\big\langle\psi_{\mathrm{t}}(t)(\boldsymbol{x}), \nabla_{\mathrm{S},\boldsymbol{A}_{\mathrm{t}}}\psi_{\mathrm{t}}(t)(\boldsymbol{x})\big\rangle.$$

Here, $\langle\cdot,\cdot\rangle$ is the usual inner product in $\mathbb{C}^2$ and $|\cdot|$ denotes the norm induced by this inner product. We will also study an alternative (more accurate) description of the physical system's time evolution that takes the interactions between the magnetic field and the quantum mechanical spin of the particle into account. By letting $\boldsymbol{\sigma}$ denote the 3-vector with the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

as components we can write the Maxwell-Pauli system in Coulomb gauge as

$$\begin{aligned} \Box\boldsymbol{A}_{\mathrm{t}} &= \frac{4\pi}{c}P\boldsymbol{J}_{\mathrm{P}}[\psi_{\mathrm{t}}, \boldsymbol{A}_{\mathrm{t}}], \\ i\hbar\partial_t\psi_{\mathrm{t}} &= \Big(\frac{1}{2m}\nabla_{\mathrm{P},\boldsymbol{A}_{\mathrm{t}}}^2 + \mathscr{E}_{\mathrm{EM}}[\boldsymbol{A}_{\mathrm{t}}, \partial_t\boldsymbol{A}_{\mathrm{t}}]\Big)\psi_{\mathrm{t}}, \\ \mathrm{div}\boldsymbol{A}_{\mathrm{t}} &= 0. \end{aligned} \tag{2}$$

Here, $\nabla_{\mathrm{P},\boldsymbol{A}_{\mathrm{t}}}$ is short for $\boldsymbol{\sigma}\cdot\nabla_{\mathrm{S},\boldsymbol{A}_{\mathrm{t}}}$ whose square by definition is the Pauli operator – the Lichnerowicz formula says that this operator can alternatively be expressed as

$$\nabla_{\mathrm{P},\boldsymbol{A}_{\mathrm{t}}}^2 = \nabla_{\mathrm{S},\boldsymbol{A}_{\mathrm{t}}}^2 - \frac{\hbar Q}{c}\boldsymbol{\sigma}\cdot\nabla\times\boldsymbol{A}_{\mathrm{t}}. \tag{3}$$

The probability current density $\boldsymbol{J}_{\mathrm{P}}[\psi_{\mathrm{t}}, \boldsymbol{A}_{\mathrm{t}}](t) : \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$\boldsymbol{J}_{\mathrm{P}}[\psi_{\mathrm{t}}, \boldsymbol{A}_{\mathrm{t}}](t)(\boldsymbol{x}) = -\frac{Q}{m}\mathrm{Re}\big\langle\psi_{\mathrm{t}}(t)(\boldsymbol{x}), \boldsymbol{\sigma}\nabla_{\mathrm{P},\boldsymbol{A}_{\mathrm{t}}}\psi_{\mathrm{t}}(t)(\boldsymbol{x})\big\rangle.$$

In the literature, the Maxwell-Schrödinger system often refers to the following equations in $\psi_{\mathrm{t}}(t):\mathbb{R}^3\to\mathbb{C}^2$, $\boldsymbol{A}_{\mathrm{t}}(t):\mathbb{R}^3\to\mathbb{R}^3$ and $\varphi_{\mathrm{t}}(t):\mathbb{R}^3\to\mathbb{R}$

$$\begin{aligned} -\Delta\varphi_{\mathrm{t}} - \frac{1}{c}\partial_t \mathrm{div}\boldsymbol{A}_{\mathrm{t}} &= 4\pi Q|\psi_{\mathrm{t}}|^2, \\ \Box\boldsymbol{A}_{\mathrm{t}} + \nabla\Big(\frac{1}{c}\partial_t\varphi_{\mathrm{t}} + \mathrm{div}\boldsymbol{A}_{\mathrm{t}}\Big) &= \frac{4\pi}{c}\boldsymbol{J}_{\mathrm{S}}[\psi_{\mathrm{t}},\boldsymbol{A}_{\mathrm{t}}], \\ i\hbar\partial_t\psi_{\mathrm{t}} &= \Big(\frac{1}{2m}\nabla^2_{\mathrm{S},\boldsymbol{A}_{\mathrm{t}}} + Q\varphi_{\mathrm{t}}\Big)\psi_{\mathrm{t}}. \end{aligned} \tag{4}$$

This system approximates the quantum field equations for an electrodynamical nonrelativistic many-body system. When expressed in Coulomb gauge it reads

$$\begin{aligned} \Box\boldsymbol{A}_{\mathrm{t}} &= \frac{4\pi}{c}P\boldsymbol{J}_{\mathrm{S}}[\psi_{\mathrm{t}},\boldsymbol{A}_{\mathrm{t}}], \\ i\hbar\partial_t\psi_{\mathrm{t}} &= \Big(\frac{1}{2m}\nabla^2_{\mathrm{S},\boldsymbol{A}_{\mathrm{t}}} + Q^2\big(|\boldsymbol{x}|^{-1} * |\psi_{\mathrm{t}}|^2\big)\Big)\psi_{\mathrm{t}}, \\ \mathrm{div}\boldsymbol{A}_{\mathrm{t}} &= 0, \end{aligned} \tag{5}$$

which only deviates from (1) by the absence of the term $\mathscr{E}_{\mathrm{EM}}[\boldsymbol{A}_{\mathrm{t}},\partial_t\boldsymbol{A}_{\mathrm{t}}]\psi_{\mathrm{t}}$ (making no difference for the existence question studied in this paper) and by the presence of $Q^2\big(|\boldsymbol{x}|^{-1} * |\psi_{\mathrm{t}}|^2\big)\psi_{\mathrm{t}}$ on the right hand side of the Schrödinger equation. The nonlinear term $Q^2\big(|\boldsymbol{x}|^{-1} * |\psi_{\mathrm{t}}|^2\big)\psi_{\mathrm{t}}$ should be perceived as a mean field originating from the Coulomb interactions between the particles present in the many-body system – when the system only consists of a single particle there simply are no other particles to interact with and so (1) is a better decription than (5) of the one-body system. In [6], Coclite and Georgiev observe that there do not exist any nontrivial solutions in the form $(\psi_{\mathrm{t}},\boldsymbol{A}_{\mathrm{t}},\varphi_{\mathrm{t}})(t)(\boldsymbol{x}) = (\mathrm{e}^{-i\omega t}\psi(\boldsymbol{x}),\boldsymbol{0},\varphi(\boldsymbol{x}))$ to the system (4) expressed in Lorenz gauge – they also prove that such solutions do exist when one adds an attractive potential of Coulomb type to the Schrödinger equation. The analogous problem in a bounded space region has been studied by Benci and Fortunato [3]. Several authors have studied the existence of solitary solutions to other systems than (1) and (2). For example Esteban, Georgiev and Séré [7] prove the existence of stationary solutions to the Maxwell-Dirac system in Lorenz gauge – in the same paper they also treat the Klein-Gordon-Dirac system. The existence of travelling wave solutions to a certain nonlinear equation describing the dynamics of pseudo-relativistic boson stars in the mean field limit has been proven by Fröhlich, Jonsson and Lenzmann [8] and also the existence of solitary water waves has been studied extensively – let us mention the recent paper by Buffoni, Groves, Sun and Wahlén [5]. Finally, we mention that the well-posedness of the initial value problem associated with (4) expressed in different gauges has been subject to a lot of research – see [2, 9, 15, 16] and references therein. In [18], the unique existence of a local solution to the many-body Maxwell-Schrödinger initial value problem expressed in Coulomb gauge is proven. For $j \in \{\mathrm{S},\mathrm{P}\}$ the aim of the present paper is to show that

$$\begin{aligned}\Box \boldsymbol{A}_{\mathrm{t}} &= \frac{4\pi}{c} P \boldsymbol{J}_j[\psi_{\mathrm{t}}, \boldsymbol{A}_{\mathrm{t}}],\\ i\hbar\partial_t\psi_{\mathrm{t}} &= \Big(\frac{1}{2m}\nabla^2_{j,\boldsymbol{A}_{\mathrm{t}}} + \mathscr{E}_{\mathrm{EM}}[\boldsymbol{A}_{\mathrm{t}}, \partial_t \boldsymbol{A}_{\mathrm{t}}]\Big)\psi_{\mathrm{t}},\\ \mathrm{div}\boldsymbol{A}_{\mathrm{t}} &= 0\end{aligned} \tag{6}$$

admits solutions $(\psi_{\mathrm{t}}, \boldsymbol{A}_{\mathrm{t}})$ in the form

$$\begin{aligned}\psi_{\mathrm{t}}(t)(\boldsymbol{x}) &= \mathrm{e}^{-i\omega t}\psi(\boldsymbol{x}-\boldsymbol{v}t),\\ \boldsymbol{A}_{\mathrm{t}}(t)(\boldsymbol{x}) &= \boldsymbol{A}(\boldsymbol{x}-\boldsymbol{v}t),\end{aligned} \tag{7}$$

with $\omega \in \mathbb{R}$, $\boldsymbol{v} \in \mathbb{R}^3$ and both of the functions ψ and $\boldsymbol{A}$ defined on $\mathbb{R}^3$. As time evolves the shapes of these functions do not change – the initial states ψ and $\boldsymbol{A}$ are simply translated in space with constant velocity $\boldsymbol{v}$ (and in case $\omega \neq 0$ the phase of the wave function oscillates too). For this reason solutions in the form (7) are often called *travelling waves.* To formulate our main theorem ensuring

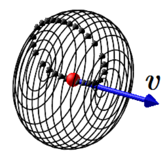

Figure 1: A travelling wave solution models the situation where a particle travels in space at a constant velocity $\boldsymbol{v}$ and it's self-generated electromagnetic field travels along with it.

the existence of travelling wave solutions to (6) we let H^1 denote the usual Sobolev space of order 1 and introduce the space D^1 of locally integrable functions A on $\mathbb{R}^3$ that have distributional first order derivatives in L^2 and vanish at infinity, in the sense that the (Lebesgue-)measure of the set

$$\{\boldsymbol{x} \in \mathbb{R}^3 \mid |A(\boldsymbol{x})| > t\}$$

is finite for all $t > 0$. The elements in the space D^1 satisfy the Sobolev inequality $\|A\|_{L^6} \leq K_S\|\nabla A\|_{L^2}$ and by equipping D^1 with the inner product $(A, B) \mapsto (\nabla A, \nabla B)_{L^2}$ we obtain a Hilbert space in which C_0^∞ is a dense subspace. Also, for $\lambda > 0$ we define the quantities

$$\Theta^\lambda_{j,\pm} = \begin{cases} \pm c & \text{if } j = \mathrm{S},\\ -\frac{8\pi K_S^3 Q^2 \lambda}{\hbar} \pm \sqrt{\frac{(8\pi)^2 K_S^6 Q^4 \lambda^2}{\hbar^2} + c^2} & \text{if } j = \mathrm{P}\end{cases}.$$

Our main theorem then asserts the following.

Theorem 1. *For all $j \in \{\mathrm{S},\mathrm{P}\}$, $\lambda > 0$ and $\boldsymbol{v} \in \mathbb{R}^3$ with $0 < |\boldsymbol{v}| < \Theta_{j,+}^{\lambda}$ there exist $\omega \in \mathbb{R}$ and functions $(\psi, \boldsymbol{A}) \in H^1 \times D^1$ satisfying $\|\psi\|_{L^2}^2 = \lambda$ such that $(\psi_{\mathrm{t}}, \boldsymbol{A}_{\mathrm{t}})(t)(\boldsymbol{x}) = \big(\mathrm{e}^{-i\omega t}\psi(\boldsymbol{x}-\boldsymbol{v}t), \boldsymbol{A}(\boldsymbol{x}-\boldsymbol{v}t)\big)$ solves (6).*

Remark 2. In quantum mechanics the quantity $\|\psi\|_{L^2}^2$ is interpreted as the total probability of the particle being located somewhere in space. Therefore $\lambda = 1$ is the physically interesting case.

We do not prove any uniqueness results concerning the travelling wave solutions, but in Theorem 18 we show that the energies of the solutions produced by the proof of Theorem 1 behave like $\frac{m\boldsymbol{v}^2}{2}\lambda$ for small $|\boldsymbol{v}|$, meaning that the effective mass of the particle equals it's bare mass. Here, the energy of a (sufficiently nice) solution $(\psi_{\mathrm{t}}, \boldsymbol{A}_{\mathrm{t}})$ to (6) refers to the inner product $\big(\psi_{\mathrm{t}}, \mathscr{H}_j\big(\boldsymbol{A}_{\mathrm{t}}, \frac{\partial_t \boldsymbol{A}_{\mathrm{t}}}{4\pi c^2}\big)\psi_{\mathrm{t}}\big)_{L^2}$, where

$$\mathscr{H}_j\big(\boldsymbol{A}, -\tfrac{P\boldsymbol{E}}{4\pi}\big) = \frac{1}{2m}\nabla_{j,\boldsymbol{A}}^2 + \mathscr{E}_{\mathrm{EM}}[\boldsymbol{A}, -c^2 P\boldsymbol{E}]. \tag{8}$$

is the quantum mechanical (electromagnetic potential-dependent) Hamiltonian of the system. In [18, 17], we have motivated the expression for (8) in the case $j = \mathrm{S}$. For any given normalized state ψ the Hamilton equations associated with the classical Hamiltonian $\big(\boldsymbol{A}, -\frac{P\boldsymbol{E}}{4\pi}\big) \mapsto \big(\psi, \mathscr{H}_j\big(\boldsymbol{A}, -\frac{P\boldsymbol{E}}{4\pi}\big)\psi\big)_{L^2}$ defined on the symplectic manifold $PH^1 \times PL^2$ say that

$$\frac{1}{c^2}\partial_t \boldsymbol{A}_{\mathrm{t}}(t) = -P\boldsymbol{E}_{\mathrm{t}}(t) \text{ and } -\partial_t P\boldsymbol{E}_{\mathrm{t}}(t) = \Delta \boldsymbol{A}_{\mathrm{t}}(t) + \frac{4\pi}{c}P\boldsymbol{J}_j[\psi, \boldsymbol{A}_{\mathrm{t}}(t)]. \tag{9}$$

In light of (9)'s first equation it is natural to represent the energy of a given solution $(\psi_{\mathrm{t}}, \boldsymbol{A}_{\mathrm{t}})$ by the average of $\mathscr{H}_j$ evaluated at the point $\big(\boldsymbol{A}_{\mathrm{t}}, \frac{\partial_t \boldsymbol{A}_{\mathrm{t}}}{4\pi c^2}\big)$. Observe also that the operator acting on the right hand side of (6)'s second equation is exactly $\mathscr{H}_j\big(\boldsymbol{A}_{\mathrm{t}}, \frac{\partial_t \boldsymbol{A}_{\mathrm{t}}}{4\pi c^2}\big)$ and that replacing ψ in (9) by the time-dependent wave function ψ_{t} produces the first equation in (6). Note that the energy of any solution $(\psi_{\mathrm{t}}, \boldsymbol{A}_{\mathrm{t}})$ to (6) with $\|\psi_{\mathrm{t}}\|_{L^2} = 1$ is a conserved quantity – in particular, the energy of a travelling wave solution as in theorem 1 is given by

$$E_j(\boldsymbol{v}, \psi, \boldsymbol{A}) = \frac{1}{2m}\|\nabla_{j,\boldsymbol{A}}\psi\|_{L^2}^2 + \frac{1}{8\pi}\int_{\mathbb{R}^3}\Big(|\nabla \times \boldsymbol{A}|^2 + \Big|\Big(\frac{\boldsymbol{v}}{c}\cdot\nabla\Big)\boldsymbol{A}\Big|^2\Big)\,\mathrm{d}\boldsymbol{x}\lambda. \tag{10}$$

The paper is organized as follows: In Section 2 we show that Theorem 1 can be proven by minimizing a certain functional. This functional is shown to be bounded from below under suitable conditions in Section 3, whereby it is meaningful to consider the functional's infimum under those conditions. In Section 4 we investigate the properties of the infimum and in Section 5 the infimum is shown to be attained by proving a variant of the concentration-compactness principle of Lions [12, 13]. Finally, in Section 6 we consider the behavior of the physical system's energy for small velocities of the particle.

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2 Formulation as a Variational Problem

As a natural step towards proving Theorem 1 we plug the travelling wave expressions (7) into (6), resulting in the system of equations

$$\begin{aligned}\Big(\frac{1}{c^2}(\boldsymbol{v}\cdot\nabla)^2-\Delta\Big)\boldsymbol{A}&=\frac{4\pi}{c}P\boldsymbol{J}_j[\psi,\boldsymbol{A}],\\ -\hbar(\theta+i\boldsymbol{v}\cdot\nabla)\psi&=\frac{1}{2m}\nabla_{j,\boldsymbol{A}}^2\psi,\\ \operatorname{div}\boldsymbol{A}&=0\end{aligned}\tag{11}$$

on $\mathbb{R}^3$, where we have set $\theta=\frac{1}{\hbar}\mathscr{E}_{\mathrm{EM}}[\boldsymbol{A},(\boldsymbol{v}\cdot\nabla)\boldsymbol{A}]-\omega$. The existence of a solution to (11) can be proven by finding a minimum point – or any other type of stationary point for that matter – of the functional

$$\begin{aligned}\mathscr{E}_j^{\boldsymbol{v}}(\psi,\boldsymbol{A})=\frac{1}{2m}\big\|\nabla_{j,\boldsymbol{A}}\psi\big\|_{L^2}^2+\frac{1}{8\pi}\Big(\|\nabla\otimes\boldsymbol{A}\|_{L^2}^2-\Big\|\Big(\frac{\boldsymbol{v}}{c}\cdot\nabla\Big)\boldsymbol{A}\Big\|_{L^2}^2\Big)\\ +\big(\psi,i\hbar\boldsymbol{v}\cdot\nabla\psi\big)_{L^2},\end{aligned}\tag{12}$$

on the set

$$\mathcal{S}_\lambda=\big\{(\psi,\boldsymbol{A})\in H^1\times D^1\,\big|\,\|\psi\|_{L^2}^2=\lambda,\operatorname{div}\boldsymbol{A}=0\big\},$$

where $\nabla\otimes\boldsymbol{A}$ denotes a 9-vector with the first derivatives $\partial_{x^j}A^k$ as components ($j,k\in\{1,2,3\}$). To prove this we will use the boundedness of P as an operator on L^p for all $p\in(1,\infty)$, which follows from the Mikhlin multiplier theorem [14] since any function $\boldsymbol{p}\mapsto\frac{\boldsymbol{p}^\beta}{\boldsymbol{p}^2}$ with $|\beta|=2$ is contained in $C^\infty(\mathbb{R}^3\setminus\{\boldsymbol{0}\})$ with

$$\Big|\partial^\alpha\Big(\frac{\boldsymbol{p}^\beta}{\boldsymbol{p}^2}\Big)\Big|\le\frac{C_{\alpha,\beta}}{|\boldsymbol{p}|^{|\alpha|}}$$

for any multi index α and all $\boldsymbol{p}\in\mathbb{R}^3\setminus\{\boldsymbol{0}\}$.

Lemma 3. *Let $\boldsymbol{v}\in\mathbb{R}^3$, $\lambda>0$ and $j\in\{\mathrm{S},\mathrm{P}\}$ be given. Then any minimizer $(\psi,\boldsymbol{A})$ of $\mathscr{E}_j^{\boldsymbol{v}}$ on $\mathcal{S}_\lambda$ solves (11) for some $\theta\in\mathbb{R}$.*

Proof. Suppose that $\mathscr{E}_j^{\boldsymbol{v}}$ has a minimum point $(\psi,\boldsymbol{A})$ on $\mathcal{S}_\lambda$. Consider also some function $\Psi\in C_0^\infty$ as well as an arbitrary real valued C_0^∞-vector field $\boldsymbol{a}$. Then $P\boldsymbol{a}$ is divergence free and contained in D^1 (in fact, in all positive exponent Sobolev spaces), so the functions f_Ψ and $g_{\boldsymbol{a}}$ given on an open interval containing 0 by

$$f_\Psi:\varepsilon\mapsto\mathscr{E}_j^{\boldsymbol{v}}\left(\frac{(\psi+\varepsilon\overline{\Psi})\sqrt{\lambda}}{\|\psi+\varepsilon\overline{\Psi}\|_{L^2}},\boldsymbol{A}\right)\quad\text{respectively}\quad g_{\boldsymbol{a}}:\varepsilon\mapsto\mathscr{E}_j^{\boldsymbol{v}}(\psi,\boldsymbol{A}+\varepsilon P\boldsymbol{a})$$

have local minima at $\varepsilon = 0$. Now, set $\theta = -\frac{\|\nabla_{j,\boldsymbol{A}}\psi\|_{L^2}^2 + 2m(\psi, i\hbar\boldsymbol{v}\cdot\nabla\psi)_{L^2}}{2m\hbar\lambda}$ and observe that the mappings f_Ψ and $g_{\boldsymbol{a}}$ are both differentiable at 0 with derivatives

$$\frac{\mathrm{d}f_\Psi}{\mathrm{d}\varepsilon}(0) = 2\mathrm{Re}\Big\langle \frac{1}{2m}\nabla_{j,\boldsymbol{A}}^2\psi + \hbar\theta\psi + i\hbar\boldsymbol{v}\cdot\nabla\psi, \Psi\Big\rangle_{\mathscr{D}'} \tag{13}$$

and

$$\begin{aligned}
&\frac{\mathrm{d}g_{\boldsymbol{a}}}{\mathrm{d}\varepsilon}(0)\\
&= \int_{\mathbb{R}^3}\Big(-\frac{1}{c}P\boldsymbol{a}\cdot\boldsymbol{J}_j[\psi,\boldsymbol{A}] + \frac{1}{4\pi}\sum_{k=1}^{3}\partial_k P\boldsymbol{a}\cdot\partial_k\boldsymbol{A} - \frac{1}{4\pi c^2}(\boldsymbol{v}\cdot\nabla)P\boldsymbol{a}\cdot(\boldsymbol{v}\cdot\nabla)\boldsymbol{A}\Big)\mathrm{d}\boldsymbol{x}\\
&= \Big\langle -\frac{1}{c}P\boldsymbol{J}_j[\psi,\boldsymbol{A}] - \frac{1}{4\pi}\Delta\boldsymbol{A} + \frac{1}{4\pi c^2}(\boldsymbol{v}\cdot\nabla)^2\boldsymbol{A}, \boldsymbol{a}\Big\rangle_{\mathscr{D}'}.
\end{aligned} \tag{14}$$

To obtain the expression for $\frac{\mathrm{d}g_j^{\boldsymbol{a}}}{\mathrm{d}\varepsilon}(0)$ we have here used the fact that

$$\int_{\mathbb{R}^3}(1-P)\boldsymbol{b}\cdot P\boldsymbol{K}\,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^3}P\boldsymbol{b}\cdot(1-P)\boldsymbol{K}\,\mathrm{d}\boldsymbol{x} = 0$$

for any choice of fields $\boldsymbol{b}\in C_0^\infty$ and $\boldsymbol{K}\in L^p$ with $p\in(1,2]$. Let us argue that the second of these identities holds true. For this purpose choose a sequence $(\boldsymbol{K}_n)_{n\in\mathbb{N}}$ of C_0^∞-fields converging in L^p to $(1-P)\boldsymbol{K}$. Then by the Hausdorff-Young inequality the sequence $\big(\widehat{\boldsymbol{K}_n}\big)_{n\in\mathbb{N}}$ will converge in $L^{\frac{p}{p-1}}$ to the Fourier transform $\mathscr{F}\big[(1-P)\boldsymbol{K}\big]$ of $(1-P)\boldsymbol{K}$ and so

$$\begin{aligned}
&(2\pi)^3\int_{\mathbb{R}^3}P\boldsymbol{b}(\boldsymbol{x})\cdot(1-P)\boldsymbol{K}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\\
&= \lim_{n\to\infty}\left(\int_{\mathbb{R}^3}\widehat{\boldsymbol{b}}(-\boldsymbol{p})\cdot\widehat{\boldsymbol{K}}_n(\boldsymbol{p})\,\mathrm{d}\boldsymbol{p} - \int_{\mathbb{R}^3}\frac{\big(\boldsymbol{p}\cdot\widehat{\boldsymbol{b}}(-\boldsymbol{p})\big)\big(\boldsymbol{p}\cdot\widehat{\boldsymbol{K}}_n(\boldsymbol{p})\big)}{\boldsymbol{p}^2}\,\mathrm{d}\boldsymbol{p}\right)\\
&= \int_{\mathbb{R}^3}\widehat{\boldsymbol{b}}(-\boldsymbol{p})\cdot\mathscr{F}\big[(1-P)\boldsymbol{K}\big](\boldsymbol{p})\,\mathrm{d}\boldsymbol{p} - \int_{\mathbb{R}^3}\frac{\big(\boldsymbol{p}\cdot\widehat{\boldsymbol{b}}(-\boldsymbol{p})\big)\big(\boldsymbol{p}\cdot\mathscr{F}\big[(1-P)\boldsymbol{K}\big](\boldsymbol{p})\big)}{\boldsymbol{p}^2}\,\mathrm{d}\boldsymbol{p}\\
&= 0,
\end{aligned}$$

where we use the Parseval-Plancherel formula and that $\widehat{\boldsymbol{b}}\in\mathscr{S}$. Thus, we have established the identities (13) and (14) – since the functions f_Ψ, $f_{i\Psi}$ and $g_{\boldsymbol{a}}$ have local minima at $\varepsilon = 0$ we are in position to conclude that $(\psi,\boldsymbol{A})$ solves (11). □

We will finish this section by making two important observations concerning the functional $\mathscr{E}_j^{\boldsymbol{v}}$. First of all, it is sometimes useful to rewrite the expression (12) by using the Hermiticity of the Pauli matrices and the general matrix identity

$$(\boldsymbol{\sigma}\cdot\boldsymbol{F})(\boldsymbol{\sigma}\cdot\boldsymbol{G}) = (\boldsymbol{F}\cdot\boldsymbol{G})\mathrm{id}_{2\times 2} + i\boldsymbol{\sigma}\cdot(\boldsymbol{F}\times\boldsymbol{G}) \tag{15}$$

to obtain

$$\mathscr{E}_j^{\boldsymbol{v}}(\psi,\boldsymbol{A}) = \frac{1}{2m}\big\|\nabla_{j,\boldsymbol{A}+\frac{mc}{Q}\boldsymbol{v}}\psi\big\|_{L^2}^2 - \frac{Q}{c}\big(\psi,\boldsymbol{v}\cdot\boldsymbol{A}\psi\big)_{L^2} - \frac{m\boldsymbol{v}^2}{2}\lambda + \frac{1}{8\pi}\Big(\|\nabla\otimes\boldsymbol{A}\|_{L^2}^2 - \Big\|\Big(\frac{\boldsymbol{v}}{c}\cdot\nabla\Big)\boldsymbol{A}\Big\|_{L^2}^2\Big) \tag{16}$$

for all $(\psi,\boldsymbol{A})\in\mathcal{S}_\lambda$. Secondly, any element O in the rotation group $\mathcal{SO}(3)$ gives rise to the identity

$$\mathscr{E}_j^{\boldsymbol{v}}(\psi,\boldsymbol{A}) = \mathscr{E}_j^{O\boldsymbol{v}}\big(U_O\circ\psi\circ O^{-1}, O\circ\boldsymbol{A}\circ O^{-1}\big) \text{ for } (\psi,\boldsymbol{A})\in\mathcal{S}_\lambda,$$

where U_O is one of the two elements in the preimage of $\{O\}$ under the double cover $\mathcal{SU}(2)\to\mathcal{SO}(3)$ defined by mapping $U\in\mathcal{SU}(2)$ to the matrix representation with respect to the basis $(\sigma^1,\sigma^2,\sigma^3)$ of the endomorphism $M\mapsto UMU^*$ on the space of Hermitean, traceless matrices. Hence, we can without loss of generality think of $\boldsymbol{v}$ as pointing, say, in the x^1-direction.

3 Boundedness from below

At this point we have defined our main goal, namely to minimize the functional $\mathscr{E}_j^{\boldsymbol{v}}$ on the set $\mathcal{S}_\lambda$. In order for this task to even make sense $\mathscr{E}_j^{\boldsymbol{v}}$ of course has to be bounded from below on $\mathcal{S}_\lambda$. In special cases – e.g. for $\boldsymbol{v}=\boldsymbol{0}$ – the question about boundedness from below is trivially answered affirmatively, but it turns out that $\mathscr{E}_j^{\boldsymbol{v}}$ is *not* in general bounded from below on $\mathcal{S}_\lambda$.

Proposition 4. *For all $j\in\{\mathrm{S},\mathrm{P}\}$, $\lambda>0$ and $\boldsymbol{v}\in\mathbb{R}^3$ with sufficiently large length the functional $\mathscr{E}_j^{\boldsymbol{v}}$ is unbounded from below on $\mathcal{S}_\lambda$. On the other hand for any $j\in\{\mathrm{S},\mathrm{P}\}$, $\lambda>0$ and $\boldsymbol{v}\in\mathbb{R}^3$ with $0<|\boldsymbol{v}|<\Theta_{j,+}^\lambda$ the functional $\mathscr{E}_j^{\boldsymbol{v}}$ is bounded from below on $\mathcal{S}_\lambda$.*

Proof. Let $j\in\{\mathrm{S},\mathrm{P}\}$, $\lambda>0$ and $\boldsymbol{v}\in\mathbb{R}^3\setminus\{\boldsymbol{0}\}$ be given. Choose arbitrary real functions $(\psi_0,\boldsymbol{A}_0)\in\mathcal{S}_\lambda$ satisfying $\|(\boldsymbol{v}\cdot\nabla)\boldsymbol{A}_0\|_{L^2}>0$ and $(\psi_0,\boldsymbol{v}\cdot\boldsymbol{A}_0\psi_0)_{L^2}>0$; if we think of $\boldsymbol{v}$ as pointing in the x^1-direction we can set $\boldsymbol{A}_0=(\partial_2\Xi,-\partial_1\Xi,0)$ for some standard cut-off function $\Xi\in C_0^\infty$ and let the components of ψ_0 be some other cut-off function with appropriate L^2-norm which is supported on $\big\{\boldsymbol{x}\in\mathbb{R}^3\,\big|\,\partial_2\Xi(\boldsymbol{x})>0\big\}$. We will show that if $|\boldsymbol{v}|$ is so large that the quantity $c^2\|\nabla\otimes\boldsymbol{A}_0\|_{L^2}^2-\boldsymbol{v}^2\big\|\big(\frac{\boldsymbol{v}}{|\boldsymbol{v}|}\cdot\nabla\big)\boldsymbol{A}_0\big\|_{L^2}^2$ is negative then $\mathscr{E}_j^{\boldsymbol{v}}$ can not be bounded from below on $\mathcal{S}_\lambda$. For this purpose define

$$\psi_R^{\boldsymbol{v}}(\boldsymbol{x}) = R^{-\frac{3}{2}}\mathrm{e}^{i\frac{m\boldsymbol{v}}{\hbar}\cdot\boldsymbol{x}}\psi_0\Big(\frac{\boldsymbol{x}}{R}\Big) \text{ and } \boldsymbol{A}_R^a(\boldsymbol{x}) = \frac{ac}{Q}\boldsymbol{A}_0\Big(\frac{\boldsymbol{x}}{R}\Big) \tag{17}$$

for $a, R > 0$. Then $(\psi_R^{\boldsymbol{v}}, \boldsymbol{A}_R^a) \in \mathcal{S}_\lambda$ and by simply calculating each of the terms on the right hand side of (16) we get

$$\begin{aligned}\mathscr{E}_j^{\boldsymbol{v}}(\psi_R^{\boldsymbol{v}}, \boldsymbol{A}_R^a) = \frac{\hbar^2}{2mR^2}\|\nabla\psi_0\|_{L^2}^2 + \frac{a^2}{2m}\|\boldsymbol{A}_0\psi_0\|_{L^2}^2 - a\big(\psi_0, \boldsymbol{v}\cdot\boldsymbol{A}_0\psi_0\big)_{L^2} - \frac{m\boldsymbol{v}^2}{2}\lambda \\ + \frac{Ra^2}{8\pi Q^2}\Big(c^2\|\nabla\otimes\boldsymbol{A}_0\|_{L^2}^2 - \boldsymbol{v}^2\Big\|\Big(\frac{\boldsymbol{v}}{|\boldsymbol{v}|}\cdot\nabla\Big)\boldsymbol{A}_0\Big\|_{L^2}^2\Big) \\ - \frac{\hbar a 1_{\{\mathrm{P}\}}(j)}{2mR}\int_{\mathbb{R}^3}\langle\psi_0(\boldsymbol{x}), \boldsymbol{\sigma}\cdot\nabla\times\boldsymbol{A}_0(\boldsymbol{x})\psi_0(\boldsymbol{x})\rangle\,\mathrm{d}\boldsymbol{x};\end{aligned} \tag{18}$$

Here, we explicitly use that ψ_0 and $\boldsymbol{A}_0$ are chosen to be real. From (18) we clearly see that when $|\boldsymbol{v}|$ is as described above then for any $a > 0$ we have

$$\lim_{R\to\infty}\mathscr{E}_j^{\boldsymbol{v}}(\psi_R^{\boldsymbol{v}}, \boldsymbol{A}_R^a) = -\infty$$

and consequently $\mathscr{E}_j^{\boldsymbol{v}}$ is not bounded from below on $\mathcal{S}_\lambda$ in this case.

We now let $j \in \{\mathrm{S}, \mathrm{P}\}$, $\lambda > 0$ as well as $\boldsymbol{v} \in \mathbb{R}^3$ with $0 < |\boldsymbol{v}| < \Theta_{j,+}^\lambda$ be arbitrary and consider as a first step the case where some given $(\psi, \boldsymbol{A}) \in \mathcal{S}_\lambda$ satisfies

$$\|\nabla\otimes\boldsymbol{A}\|_{L^2} < 16\pi K_S c|Q|\lambda^{\frac{3}{4}}\frac{|\boldsymbol{v}|}{c^2 - \boldsymbol{v}^2}\|\psi\|_{L^6}^{\frac{1}{2}}. \tag{19}$$

The Lichnerowicz formula (3) and approximation of ψ in H^1 by C_0^∞-functions make it possible to write the quantity $\|\nabla_{j,\boldsymbol{A}+\frac{mc}{Q}\boldsymbol{v}}\psi\|_{L^2}^2$ appearing on the right hand side of (16) as $\|\nabla_{\mathrm{S},\boldsymbol{A}+\frac{mc}{Q}\boldsymbol{v}}\psi\|_{L^2}^2 - 1_{\{\mathrm{P}\}}(j)\frac{\hbar Q}{c}\int_{\mathbb{R}^3}\big\langle\psi, \boldsymbol{\sigma}\cdot\nabla\times\boldsymbol{A}\psi\big\rangle\,\mathrm{d}\boldsymbol{x}$. By using the diamagnetic inequality, the Hölder inequality, the Sobolev inequality and (19) we therefore get

$$\begin{aligned}\|\nabla_{j,\boldsymbol{A}+\frac{mc}{Q}\boldsymbol{v}}\psi\|_{L^2}^2 &\geq \hbar^2\|\nabla|\psi|\|_{L^2}^2 - 1_{\{\mathrm{P}\}}(j)\frac{\hbar|Q|\lambda^{\frac{1}{4}}}{c}\|\nabla\otimes\boldsymbol{A}\|_{L^2}\|\psi\|_{L^6}^{\frac{3}{2}} \\ &\geq \frac{\hbar^2}{K_S^2}\frac{(\Theta_{j,+}^\lambda - |\boldsymbol{v}|)(|\boldsymbol{v}| - \Theta_{j,-}^\lambda)}{c^2 - \boldsymbol{v}^2}\|\psi\|_{L^6}^2\end{aligned}$$

In addition, we apply Young's inequality for products $\big(ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}$ where $\frac{1}{p} + \frac{1}{p'} = 1\big)$ and Sobolev's inequality to the term $-\frac{Q}{c}(\psi, \boldsymbol{v}\cdot\boldsymbol{A}\psi)_{L^2}$ and obtain

$$\begin{aligned}\mathscr{E}_j^{\boldsymbol{v}}(\psi, \boldsymbol{A}) \geq \frac{\hbar^2}{4mK_S^2}\frac{(\Theta_{j,+}^\lambda - |\boldsymbol{v}|)(|\boldsymbol{v}| - \Theta_{j,-}^\lambda)}{c^2 - \boldsymbol{v}^2}\|\psi\|_{L^6}^2 + \frac{1}{8\pi}\Big(1 - \frac{\boldsymbol{v}^2}{c^2}\Big)\|\nabla\otimes\boldsymbol{A}\|_{L^2}^2 \\ - \frac{3K_S^2|Q\boldsymbol{v}|^{\frac{4}{3}}m^{\frac{1}{3}}\lambda}{4\hbar^{\frac{2}{3}}c^{\frac{4}{3}}}\left(\frac{c^2 - \boldsymbol{v}^2}{(\Theta_{j,+}^\lambda - |\boldsymbol{v}|)(|\boldsymbol{v}| - \Theta_{j,-}^\lambda)}\right)^{\frac{1}{3}}\|\nabla\otimes\boldsymbol{A}\|_{L^2}^{\frac{4}{3}} - \frac{m\boldsymbol{v}^2}{2}\lambda.\end{aligned} \tag{20}$$

Another application of Young's inequality for products reveals that

$$\mathscr{E}_j^{\boldsymbol{v}}(\psi, \boldsymbol{A}) \geq \frac{\hbar^2}{4mK_S^2}\frac{(\Theta_{j,+}^\lambda - |\boldsymbol{v}|)(|\boldsymbol{v}| - \Theta_{j,-}^\lambda)}{c^2 - \boldsymbol{v}^2}\|\psi\|_{L^6}^2 + \frac{1}{16\pi}\Big(1 - \frac{\boldsymbol{v}^2}{c^2}\Big)\|\nabla \otimes \boldsymbol{A}\|_{L^2}^2 - \frac{(4\pi)^2 K_S^6 Q^4 m\lambda^3}{\hbar^2}\frac{\boldsymbol{v}^4}{(c^2 - \boldsymbol{v}^2)(\Theta_{j,+}^\lambda - |\boldsymbol{v}|)(|\boldsymbol{v}| - \Theta_{j,-}^\lambda)} - \frac{m\boldsymbol{v}^2}{2}\lambda \tag{21}$$

so for pairs $(\psi, \boldsymbol{A}) \in \mathcal{S}_\lambda$ satisfying (19) there is indeed a lower bound on the possible values of $\mathscr{E}_j^{\boldsymbol{v}}(\psi, \boldsymbol{A})$. Consider now the scenario where the given pair $(\psi, \boldsymbol{A}) \in \mathcal{S}_\lambda$ satisfies the inequality

$$\|\nabla \otimes \boldsymbol{A}\|_{L^2} \geq 16\pi K_S c|Q|\lambda^{\frac{3}{4}}\frac{|\boldsymbol{v}|}{c^2 - \boldsymbol{v}^2}\|\psi\|_{L^6}^{\frac{1}{2}}. \tag{22}$$

In this case we simply use the nonnegativity of the kinetic energy term in (16) to get

$$\begin{aligned}\mathscr{E}_j^{\boldsymbol{v}}(\psi, \boldsymbol{A}) &\geq -\frac{K_S|Q||\boldsymbol{v}|\lambda^{\frac{3}{4}}}{c}\|\nabla \otimes \boldsymbol{A}\|_{L^2}\|\psi\|_{L^6}^{\frac{1}{2}} - \frac{m\boldsymbol{v}^2}{2}\lambda + \frac{1}{8\pi}\Big(1 - \frac{\boldsymbol{v}^2}{c^2}\Big)\|\nabla \otimes \boldsymbol{A}\|_{L^2}^2 \\ &\geq \frac{1}{16\pi}\Big(1 - \frac{\boldsymbol{v}^2}{c^2}\Big)\|\nabla \otimes \boldsymbol{A}\|_{L^2}^2 - \frac{m\boldsymbol{v}^2}{2}\lambda\end{aligned} \tag{23}$$

where the assumption (22) is applied at the final step. Consequently, the values of $\mathscr{E}_j^{\boldsymbol{v}}(\psi, \boldsymbol{A})$ are bounded below by $-\frac{m\boldsymbol{v}^2}{2}$ for pairs $(\psi, \boldsymbol{A}) \in \mathcal{S}_\lambda$ satisfying (22). □

Remark 5. For $j \in \{\mathrm{S}, \mathrm{P}\}$, $\lambda > 0$, $\boldsymbol{v} \in \mathbb{R}^3$ with $0 < |\boldsymbol{v}| < \Theta_{j,+}^\lambda$ as well as $(\psi, \boldsymbol{A}) \in \mathcal{S}_\lambda$ we can bound the quantities $\|\psi\|_{L^6}$ and $\|\nabla \otimes \boldsymbol{A}\|_{L^2}$ from above in terms of $\mathscr{E}_j^{\boldsymbol{v}}(\psi, \boldsymbol{A})$. More precisely, (21) and (23) give that

$$\|\nabla \otimes \boldsymbol{A}\|_{L^2}^2 \leq \frac{2^8\pi^3 K_S^6 c^2 Q^4 m\lambda^3}{\hbar^2}\frac{\boldsymbol{v}^4}{(c^2 - \boldsymbol{v}^2)^2(\Theta_{j,+}^\lambda - |\boldsymbol{v}|)(|\boldsymbol{v}| - \Theta_{j,-}^\lambda)} + \frac{16\pi c^2}{c^2 - \boldsymbol{v}^2}\Big(\mathscr{E}_j^{\boldsymbol{v}}(\psi, \boldsymbol{A}) + \frac{m\boldsymbol{v}^2}{2}\lambda\Big). \tag{24}$$

Moreover, if $(\psi, \boldsymbol{A})$ satisfies (19) we obtain from (21) that

$$\|\psi\|_{L^6}^2 \leq \frac{4mK_S^2}{\hbar^2}\frac{c^2 - \boldsymbol{v}^2}{(\Theta_{j,+}^\lambda - |\boldsymbol{v}|)(|\boldsymbol{v}| - \Theta_{j,-}^\lambda)}\Big(\mathscr{E}_j^{\boldsymbol{v}}(\psi, \boldsymbol{A}) + \frac{m\boldsymbol{v}^2}{2}\lambda\Big) + \frac{2^6\pi^2 K_S^8 Q^4 m^2\lambda^3}{\hbar^4}\frac{\boldsymbol{v}^4}{(\Theta_{j,+}^\lambda - |\boldsymbol{v}|)^2(|\boldsymbol{v}| - \Theta_{j,-}^\lambda)^2} \tag{25}$$

and if $(\psi, \boldsymbol{A})$ on the other hand satisfies (22) then by (23) we have

$$\|\psi\|_{L^6} \leq \frac{c^2 - \boldsymbol{v}^2}{16\pi K_S^2 Q^2 \lambda^{\frac{3}{2}}\boldsymbol{v}^2}\Big(\mathscr{E}_j^{\boldsymbol{v}}(\psi, \boldsymbol{A}) + \frac{m\boldsymbol{v}^2}{2}\lambda\Big). \tag{26}$$

By Proposition 4 it is impossible for the functional $\mathscr{E}_j^{\boldsymbol{v}}$ to attain a minimum on $\mathcal{S}_\lambda$ for sufficiently large values of $|\boldsymbol{v}|$. Of course this does not rule out the existence of solutions to (11), but the nonexistence of such solutions for large $|\boldsymbol{v}|$ would in fact be perfectly compatible with our understanding from the theory of special relativity that a particle with rest mass can not travel faster than light. We therefore guess that the value $\Theta_{\mathrm{S},+}^\lambda$ is optimal in the sense that $\mathscr{E}_\mathrm{S}^{\boldsymbol{v}}$ can not be shown to be bounded from below on $\mathcal{S}_\lambda$ for $|\boldsymbol{v}| > c$. On the other hand, the value of $\Theta_{\mathrm{P},+}^\lambda$ is not optimal.

4 Properties of the infimum

For any given $\boldsymbol{v} \in \mathbb{R}^3$ consider the set

$$\Lambda_j^{\boldsymbol{v}} = \{\lambda > 0 \mid |\boldsymbol{v}| < \Theta_{j,+}^\lambda\}.$$

We have just seen that given $\boldsymbol{v} \in \mathbb{R}^3$ with $0 < |\boldsymbol{v}| < c$ and $\lambda \in \Lambda_j^{\boldsymbol{v}}$ it makes

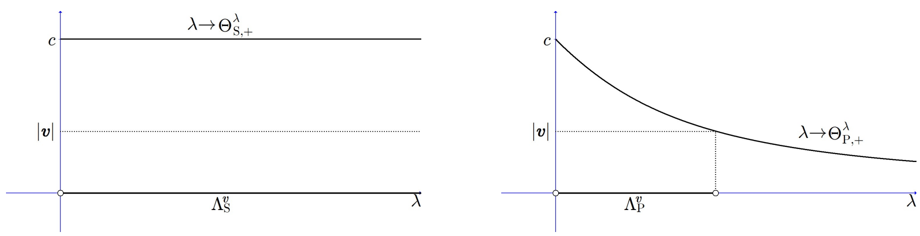


Figure 2: The set $\Lambda_j^{\boldsymbol{v}}$ is an open interval since $\lambda \mapsto \Theta_{j,+}^\lambda$ is decreasing and continuous. In fact, $\Lambda_\mathrm{S}^{\boldsymbol{v}} = (0,\infty)$ regardless of the choice of $\boldsymbol{v}$ with $|\boldsymbol{v}| < c$.

sense to define

$$I_j^\lambda = \inf\{\mathscr{E}_j^{\boldsymbol{v}}(\psi, \boldsymbol{A}) \,|\, (\psi, \boldsymbol{A}) \in \mathcal{S}_\lambda\}$$

and we aim to show that this infimum is attained. Imagine that the functional $\mathscr{E}_j^{\boldsymbol{v}}$ indeed does take the value I_j^λ in some point. It follows from the following simple observation that for such a minimizing point $(\psi, \boldsymbol{A}) \in \mathcal{S}_\lambda$ neither the wave function ψ nor the magnetic vector potential $\boldsymbol{A}$ can be identically equal to zero.

Lemma 6. *Let $j \in \{\mathrm{S}, \mathrm{P}\}$ and $\boldsymbol{v} \in \mathbb{R}^3$ with $0 < |\boldsymbol{v}| < c$ be given. Then*

$$I_j^\lambda < -\frac{m\boldsymbol{v}^2}{2}\lambda$$

for any $\lambda \in \Lambda_j^{\boldsymbol{v}}$.

Proof. Choose the pair $(\psi_0, \boldsymbol{A}_0) \in \mathcal{S}_\lambda$ as in the beginning of the proof of Proposition 4 and define $(\psi_R^{\boldsymbol{v}}, \boldsymbol{A}_R^a) \in \mathcal{S}_\lambda$ for $R, a > 0$ as prescribed in (17). According to (18) we can let R take the specific value

$$R_a = \left(\frac{2\hbar Q}{a}\sqrt{\frac{\pi}{m}}\|\nabla\psi_0\|_{L^2}\right)^{\frac{2}{3}} \left(c^2\|\nabla \otimes \boldsymbol{A}_0\|_{L^2}^2 - \|(\boldsymbol{v}\cdot\nabla)\boldsymbol{A}_0\|_{L^2}^2\right)^{-\frac{1}{3}},$$

and get

$$\begin{aligned}\mathscr{E}_j^{\boldsymbol{v}}(\psi_{R_a}^{\boldsymbol{v}}, \boldsymbol{A}_{R_a}^a) = \left(\tfrac{\hbar^2}{16\pi^2 Q^4 m}\right)^{\frac{1}{3}} \left(c^2\|\nabla \otimes \boldsymbol{A}_0\|_{L^2}^2 - \|(\boldsymbol{v}\cdot\nabla)\boldsymbol{A}_0\|_{L^2}^2\right)^{\frac{2}{3}} \|\nabla\psi_0\|_{L^2}^{\frac{2}{3}} a^{\frac{4}{3}} \\ - \left(\tfrac{1_{\{\mathrm{P}\}}(j)\hbar(c^2\|\nabla\otimes\boldsymbol{A}_0\|_{L^2}^2 - \|(\boldsymbol{v}\cdot\nabla)\boldsymbol{A}_0\|_{L^2}^2)}{2^5 m^2 Q^2 \pi \|\nabla\psi_0\|_{L^2}^2}\right)^{\frac{1}{3}} \int_{\mathbb{R}^3} \langle\psi_0, \boldsymbol{\sigma}\cdot\nabla\times\boldsymbol{A}_0\psi_0\rangle \mathrm{d}\boldsymbol{x}\, a^{\frac{5}{3}} \\ + \tfrac{1}{2m}\|\boldsymbol{A}_0\psi_0\|_{L^2}^2 a^2 - (\psi_0, \boldsymbol{v}\cdot\boldsymbol{A}_0\psi_0)_{L^2} a - \tfrac{m\boldsymbol{v}^2}{2}\lambda.\end{aligned}$$

Thus, $\mathscr{E}_j^{\boldsymbol{v}}(\psi_{R_a}^{\boldsymbol{v}}, \boldsymbol{A}_{R_a}^a)$ can be extended to a continuously differentiable function of a on the entire real line – moreover, the extension takes the value $-\frac{m\boldsymbol{v}^2}{2}\lambda$ and has a negative derivative at $a = 0$. For sufficiently small $a > 0$ the values of $\mathscr{E}_j^{\boldsymbol{v}}(\psi_{R_a}^{\boldsymbol{v}}, \boldsymbol{A}_{R_a}^a)$ must therefore be strictly less than $-\frac{m\boldsymbol{v}^2}{2}\lambda$. ☐

In the following proposition we investigate I_j^λ's dependence on λ.

Lemma 7. *Let $j \in \{\mathrm{S}, \mathrm{P}\}$ and $\boldsymbol{v} \in \mathbb{R}^3$ with $0 < |\boldsymbol{v}| < c$ be given. Then*

$$I_j^{s\nu} < sI_j^\nu \tag{27}$$

for all $\nu \in \Lambda_j^{\boldsymbol{v}}$ and $s > 1$ with $s\nu \in \Lambda_j^{\boldsymbol{v}}$. Moreover,

$$I_j^\lambda < I_j^\mu + I_j^{\lambda-\mu} \tag{28}$$

for $\mu, \lambda \in \Lambda_j^{\boldsymbol{v}}$ with $\mu < \lambda$.

Proof. Let $\nu \in \Lambda_j^{\boldsymbol{v}}$ and $s > 1$ with $s\nu \in \Lambda_j^{\boldsymbol{v}}$ be given and choose (by means of Lemma 6) some constant $k \in \left(0, -\frac{m\boldsymbol{v}^2}{2}\nu - I_j^\nu\right)$. Given a positive ε satisfying

$$\varepsilon < \min\left\{-\frac{m\boldsymbol{v}^2}{2}\nu - I_j^\nu - k, \frac{s-1}{s}\frac{k^{\frac{3}{2}}\hbar\sqrt{\left(c^2 - \boldsymbol{v}^2\right)\left(\Theta_{j,+}^\nu - |\boldsymbol{v}|\right)\left(|\boldsymbol{v}| - \Theta_{j,-}^\nu\right)}}{3^{\frac{3}{2}}\pi K_S^3 Q^2 \sqrt{m}\nu^{\frac{3}{2}}\boldsymbol{v}^2}\right\}$$

we can then choose a pair $(\psi, \boldsymbol{A}) \in \mathcal{S}_\nu$ such that

$$\mathscr{E}_j^{\boldsymbol{v}}(\psi, \boldsymbol{A}) \le I_j^\nu + \varepsilon, \tag{29}$$

which together with (20), (23) and the assumption $\varepsilon < -\frac{m\boldsymbol{v}^2}{2}\nu - I_j^\nu - k$ gives

$$\|\nabla \otimes \boldsymbol{A}\|_{L^2}^2 > \frac{8k^{\frac{3}{2}}\hbar c^2}{3^{\frac{3}{2}} K_S^3 Q^2 \sqrt{m}\nu^{\frac{3}{2}}\boldsymbol{v}^2}\left(\frac{\left(\Theta_{j,+}^\nu - |\boldsymbol{v}|\right)\left(|\boldsymbol{v}| - \Theta_{j,-}^\nu\right)}{c^2 - \boldsymbol{v}^2}\right)^{\frac{1}{2}}. \tag{30}$$

Then (29) and (30) imply that

$$\begin{aligned} I_j^{s\nu} &\le \mathscr{E}_j^{\boldsymbol{v}}(\sqrt{s}\psi, \boldsymbol{A}) \\ &= s\mathscr{E}_j^{\boldsymbol{v}}(\psi, \boldsymbol{A}) + \frac{1-s}{8\pi}\Big(\|\nabla\otimes\boldsymbol{A}\|_{L^2}^2 - \Big\|\Big(\frac{\boldsymbol{v}}{c}\cdot\nabla\Big)\boldsymbol{A}\Big\|_{L^2}^2\Big) \\ &< sI_j^{\nu} + s\varepsilon + (1-s)\frac{k^{\frac{3}{2}}\hbar\sqrt{\big(c^2-\boldsymbol{v}^2\big)\big(\Theta_{j,+}^{\nu}-|\boldsymbol{v}|\big)\big(|\boldsymbol{v}|-\Theta_{j,-}^{\nu}\big)}}{3^{\frac{3}{2}}\pi K_S^3 Q^2\sqrt{m}\nu^{\frac{3}{2}}\boldsymbol{v}^2} \\ &< sI_j^{\nu}, \end{aligned} \tag{31}$$

proving that (27) indeed does hold true.

This enables us to prove (28), so let $\mu, \lambda \in \Lambda_j^{\boldsymbol{v}}$ with $\mu < \lambda$ be given. If $\mu > \lambda - \mu$ is satisfied we can use (27) twice $\big($with $(s,\nu) = \big(\frac{\lambda}{\mu}, \mu\big)$ respectively $(s,\nu) = \big(\frac{\mu}{\lambda-\mu}, \lambda-\mu\big)\big)$ and obtain

$$I_j^{\lambda} = I_j^{\frac{\lambda}{\mu}\mu} < \frac{\lambda}{\mu} I_j^{\mu} = I_j^{\mu} + \frac{\lambda-\mu}{\mu} I_j^{\frac{\mu}{\lambda-\mu}(\lambda-\mu)} < I_j^{\mu} + I_j^{\lambda-\mu}$$

and if on the other hand $\mu \le \lambda - \mu$ we can likewise apply (27) to get

$$I_j^{\lambda} = I_j^{\frac{\lambda}{\lambda-\mu}(\lambda-\mu)} < \frac{\lambda}{\lambda-\mu} I_j^{\lambda-\mu} = \frac{\mu}{\lambda-\mu} I_j^{\frac{\lambda-\mu}{\mu}\mu} + I_j^{\lambda-\mu} \le I_j^{\mu} + I_j^{\lambda-\mu},$$

so (28) also holds true. □

Remark 8. The strict subadditivity expressed in (28) implies that $\nu \mapsto I_j^{\nu}$ is strictly decreasing on $\Lambda_j^{\boldsymbol{v}}$ since the term $I_j^{\lambda-\mu}$ is negative by Lemma 6.

As a consequence of Remark 8 the function $\nu \mapsto I_j^{\nu}$ has limits from the left as well as from the right in all points of $\Lambda_j^{\boldsymbol{v}}$. In fact, we can show the following result.

Lemma 9. *Given* $j \in \{\mathrm{S}, \mathrm{P}\}$ *and* $\boldsymbol{v} \in \mathbb{R}^3$ *with* $0 < |\boldsymbol{v}| < c$ *the mapping* $\nu \mapsto I_j^{\nu}$ *is continuous on* $\Lambda_j^{\boldsymbol{v}}$.

Proof. Let us begin by proving that $\nu \mapsto I_j^{\nu}$ is left continuous: Given $\nu \in \Lambda_j^{\boldsymbol{v}}$, $\varepsilon > 0$ and $0 < s < 1$ we choose $(\psi, \boldsymbol{A}) \in \mathcal{S}_{\nu}$ such that $\mathscr{E}_j^{\boldsymbol{v}}(\psi, \boldsymbol{A}) \le I_j^{\nu} + \varepsilon$ and proceed just as in (31) to obtain

$$I_j^{s\nu} \le sI_j^{\nu} + s\varepsilon + \frac{1-s}{8\pi}\Big(\|\nabla\otimes\boldsymbol{A}\|_{L^2}^2 - \Big\|\Big(\frac{\boldsymbol{v}}{c}\cdot\nabla\Big)\boldsymbol{A}\Big\|_{L^2}^2\Big).$$

Letting $s \to 1^-$ therefore gives $\lim_{s\to1^-} I_j^{s\nu} \le I_j^{\nu} + \varepsilon$ and the fact that this holds true for any $\varepsilon > 0$ implies that $\lim_{s\to1^-} I_j^{s\nu} \le I_j^{\nu}$. Since $\nu \mapsto I_j^{\nu}$ is decreasing the opposite inequality also holds true, whereby

$$\lim_{s\to1^-} I_j^{s\nu} = I_j^{\nu}.$$

To prove right continuity of $\nu \mapsto I_j^\nu$ we let $\nu \in \Lambda_j^{\boldsymbol{v}}$, $\varepsilon > 0$ as well as $s > 1$ with $s\nu \in \Lambda_j^{\boldsymbol{v}}$ be arbitrary and choose a pair $(\psi, \boldsymbol{A}) \in \mathcal{S}_{s\nu}$ such that $\mathscr{E}_j^{\boldsymbol{v}}(\psi, \boldsymbol{A}) \leq I_j^{s\nu} + \varepsilon$. Then (24) and Lemma 6 give that

$$
\begin{aligned}
I_j^{s\nu} + \varepsilon &\geq s\mathscr{E}_j^{\boldsymbol{v}}\big(\tfrac{\psi}{\sqrt{s}}, \boldsymbol{A}\big) + \frac{1-s}{8\pi}\Big(\|\nabla \otimes \boldsymbol{A}\|_{L^2}^2 - \Big\|\Big(\frac{\boldsymbol{v}}{c}\cdot\nabla\Big)\boldsymbol{A}\Big\|_{L^2}^2\Big) \\
&\geq sI_j^\nu + \frac{2c^2(1-s)}{c^2-\boldsymbol{v}^2}\left(\varepsilon + \frac{(4\pi)^2 K_S^6 Q^4 m(s\nu)^3 \boldsymbol{v}^4}{\hbar^2(c^2-\boldsymbol{v}^2)(\Theta_{j,+}^{s\nu} - |\boldsymbol{v}|)(|\boldsymbol{v}| - \Theta_{j,-}^{s\nu})}\right),
\end{aligned}
$$

whereby $\lim_{s\to 1^+} I_j^{s\nu} + \varepsilon \geq I_j^\nu$. By letting $\varepsilon \to 0^+$ we thus obtain the inequality $\lim_{s\to 1^+} I_j^{s\nu} \geq I_j^\nu$ and the opposite inequality follows immediately from (27), which leaves us in position to conclude that the identity

$$\lim_{s\to 1^+} I_j^{s\nu} = I_j^\nu$$

holds true. □

5 Existence of a Minimizer

We will now consider a strategy that is frequently used for approaching minimization problems such as ours – it is often called the direct method in the calculus of variations and was introduced by Zaremba and Hilbert around the year 1900. Here, one first considers a minimizing sequence for the functional at hand.

Definition 10. Let $j \in \{\mathrm{S}, \mathrm{P}\}$, $\boldsymbol{v} \in \mathbb{R}^3$ with $0 < |\boldsymbol{v}| < c$ and $\lambda \in \Lambda_j^{\boldsymbol{v}}$ be given. By a *minimizing sequence* for $\mathscr{E}_j^{\boldsymbol{v}}$ we mean a sequence of points $(\psi_n, \boldsymbol{A}_n) \in \mathcal{S}_\lambda$ such that $\big(\mathscr{E}_j^{\boldsymbol{v}}(\psi_n, \boldsymbol{A}_n)\big)_{n\in\mathbb{N}}$ converges to I_j^λ in $\mathbb{R}$.

The philosophy of the direct method in the calculus of variations is to first argue that a given minimizing sequence must have a subsequence converging weakly to some point $(\psi, \boldsymbol{A})$ and then as a second step one hopes to show lower semicontinuity properties of $\mathscr{E}_j^{\boldsymbol{v}}$ ensuring that the identity $\mathscr{E}_j^{\boldsymbol{v}}(\psi, \boldsymbol{A}) = I_j^\lambda$ holds true. However, our specific functional $\mathscr{E}_j^{\boldsymbol{v}}$ is translation invariant – meaning that any translation $\tau_{\boldsymbol{y}} : \boldsymbol{x} \mapsto (\boldsymbol{x} + \boldsymbol{y})$ in space gives rise to the identity $\mathscr{E}_j^{\boldsymbol{v}}(\psi\circ\tau_{\boldsymbol{y}}, \boldsymbol{A}\circ\tau_{\boldsymbol{y}}) = \mathscr{E}_j^{\boldsymbol{v}}(\psi, \boldsymbol{A})$. Thus, even if $\mathscr{E}_j^{\boldsymbol{v}}$ indeed does have a minimizer, there will exist lots of minimizing sequences whose ψ-part converges weakly in L^2 to the zero function and the possible limit of (any subsequence of) such a minimizing sequence can clearly not serve as a minimizer for $\mathscr{E}_j^{\boldsymbol{v}}$. In other words, we have to break the translation invariance in some way and to do this we will prove a variant of the concentration-compactness principle by Pierre-Louis Lions (see [12, 13]). We can not just apply the result of Lions to our problem since this result concerns a sequence of H^1- (or L^1-)functions ψ_n whereas we are dealing with a sequence of $\mathcal{S}_\lambda$-pairs $(\psi_n, \boldsymbol{A}_n)$. Let us begin by proving the following simple – but important – lemma that provides us with some control over any given minimizing sequence for $\mathscr{E}_j^{\boldsymbol{v}}$.

Lemma 11. *Let $j \in \{\mathrm{S}, \mathrm{P}\}$, $\boldsymbol{v} \in \mathbb{R}^3$ with $0 < |\boldsymbol{v}| < c$ as well as $\lambda \in \Lambda_j^{\boldsymbol{v}}$ be given and consider a minimizing sequence $\big((\psi_n, \boldsymbol{A}_n)\big)_{n\in\mathbb{N}} \subset \mathcal{S}_\lambda$ for $\mathscr{E}_j^{\boldsymbol{v}}$. Then $(\psi_n)_{n\in\mathbb{N}}$ is bounded in H^1 and $(\boldsymbol{A}_n)_{n\in\mathbb{N}}$ is bounded in D^1.*

Proof. The sequence $\big(\mathscr{E}_j^{\boldsymbol{v}}(\psi_n, \boldsymbol{A}_n)\big)_{n\in\mathbb{N}}$ is bounded (because it is convergent) and therefore it follows from the estimates (24), (25) and (26) that $(\|\psi_n\|_{L^6})_{n\in\mathbb{N}}$ and $(\|\nabla\otimes\boldsymbol{A}_n\|_{L^2})_{n\in\mathbb{N}}$ are also bounded. Moreover, the sequence $(\|\psi_n\|_{L^2})_{n\in\mathbb{N}}$ is constant so all that remains to be shown is the boundedness of $(\|\nabla\psi_n\|_{L^2})_{n\in\mathbb{N}}$. For this we expand the kinetic energy in the expression (12) for $\mathscr{E}_j^{\boldsymbol{v}}(\psi_n, \boldsymbol{A}_n)$, use the nonnegativity of $\frac{Q^2}{2mc^2}\|\boldsymbol{A}_n\psi_n\|_{L^2}^2 + \frac{1}{8\pi}\big(\|\nabla\otimes\boldsymbol{A}_n\|_{L^2}^2 - \big\|\big(\frac{\boldsymbol{v}}{c}\cdot\nabla\big)\boldsymbol{A}_n\big\|^2\big)$ and apply Hölder's as well as Sobolev's inequalities to get

$$\frac{\hbar^2}{2m}\|\nabla\psi_n\|_{L^2}^2 \leq |\mathscr{E}_j^{\boldsymbol{v}}(\psi_n, \boldsymbol{A}_n)| + \frac{K_S\hbar|Q|\lambda^{\frac{1}{4}}}{mc}\|\nabla\psi_n\|_{L^2}\|\nabla\otimes\boldsymbol{A}_n\|_{L^2}\|\psi_n\|_{L^6}^{\frac{1}{2}}$$
$$+\hbar\lambda^{\frac{1}{2}}|\boldsymbol{v}|\|\nabla\psi_n\|_{L^2} + 1_{\{\mathrm{P}\}}(j)\frac{\hbar|Q|\lambda^{\frac{1}{4}}}{2mc}\|\nabla\otimes\boldsymbol{A}_n\|_{L^2}\|\psi_n\|_{L^6}^{\frac{3}{2}}. \quad (32)$$

Here, we can use Young's inequality for products to absorb the $\|\nabla\psi_n\|_{L^2}$'s on the right hand side of (32) into the left hand side of (32) and obtain an upper bound on $\|\nabla\psi_n\|_{L^2}^2$. □

Remark 12. From now on we will consider some fixed minimizing sequence $\big((\psi_n, \boldsymbol{A}_n)\big)_{n\in\mathbb{N}}$ for $\mathscr{E}_j^{\boldsymbol{v}}$. It follows from Lemma 11 that all of the terms appearing in the expressions (12) and (16) for $\mathscr{E}_j^{\boldsymbol{v}}(\psi_n, \boldsymbol{A}_n)$ define bounded sequences in $\mathbb{R}$. We will let C denote a constant that majorizes each of the sequences $(\|\psi_n\|_{H^1})_{n\in\mathbb{N}}$, $(\|\nabla\otimes\boldsymbol{A}_n\|_{L^2})_{n\in\mathbb{N}}$ and $(\|\nabla_{j,\boldsymbol{A}_n+\frac{mc}{Q}\boldsymbol{v}}\psi_n\|_{L^2})_{n\in\mathbb{N}}$.

5.1 Breaking the Translation Invariance

We hope to find a sequence $(\boldsymbol{y}_n)_{n\in\mathbb{N}}$ of points in $\mathbb{R}^3$ such that the direct method in the calculus of variations can be applied to the translated minimizing sequence $\big((\psi_n\circ\tau_{\boldsymbol{y}_n}, \boldsymbol{A}_n\circ\tau_{\boldsymbol{y}_n})\big)_{n\in\mathbb{N}}$ for $\mathscr{E}_j^{\boldsymbol{v}}$. As an essential tool in our search for such a sequence we introduce for each $n \in \mathbb{N}$ the nondecreasing *concentration function* $\mathscr{C}_n : (0,\infty) \to (0,\lambda]$ given by

$$\mathscr{C}_n(r) = \sup_{\boldsymbol{y}\in\mathbb{R}^3}\int_{\mathcal{B}(\boldsymbol{y},r)}|\psi_n(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x} \quad \text{for} \quad r > 0. \quad (33)$$

Remember that we think of the ψ-variable as being a quantum particle's wave function and so the physical interpretation of a large value of $\mathscr{C}_n(r)$ (compared to λ) is that the quantum particle is likely to be localized in some ball $\subset \mathbb{R}^3$ with radius r. In this sense $\mathscr{C}_n$ expresses how concentrated the wave function is (see Figure 3). We summarize the most important properties of the functions $\mathscr{C}_n$ in the following lemma.

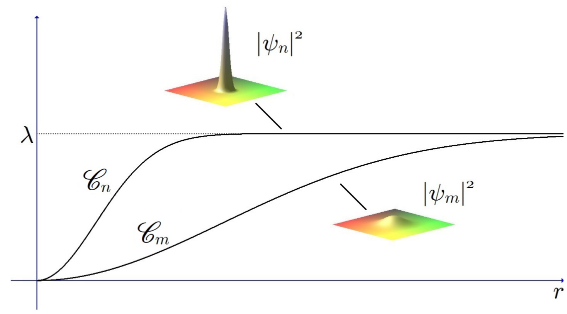


Figure 3: If $\mathscr{C}_n$ increases quickly to the value λ then it means that the corresponding wave function ψ_n is very concentrated around some point $\boldsymbol{y}$ in space, i.e. the quantum particle is with high probability positioned in close vicinity of $\boldsymbol{y}$.

Lemma 13. *Given $\lambda > 0$ let $\psi_n \in H^1$ satisfy $\|\psi_n\|_{L^2}^2 = \lambda$ and $\|\nabla\psi_n\|_{L^2} \leq C$ for all $n \in \mathbb{N}$ and define the function $\mathscr{C}_n : (0,\infty) \to (0,\lambda]$ by (33). Then $\mathscr{C}_n$ is nondecreasing with the limits $\lim_{r\to 0^+} \mathscr{C}_n(r) = 0$ as well as $\lim_{r\to\infty} \mathscr{C}_n(r) = \lambda$ holding true and by passing to a subsequence $(\mathscr{C}_n)_{n\in\mathbb{N}}$ converges pointwise to some nondecreasing mapping $\mathscr{C} : (0,\infty) \to [0,\lambda]$ with $\lim_{r\to 0^+} \mathscr{C}(r) = 0$.*

Proof. For an arbitrary $n \in \mathbb{N}$ the mapping $\mathscr{C}_n : (0,\infty) \to (0,\lambda]$ is obviously nondecreasing and the identity $\lim_{r\to 0^+} \mathscr{C}_n(r) = 0$ holds true since

$$\int_{\mathcal{B}(\boldsymbol{y},r)} |\psi_n(\boldsymbol{x})|^2 \, \mathrm{d}\boldsymbol{x} \leq \left(\frac{4}{3}\pi r^3 K_S^3 C^3\right)^{\frac{2}{3}} \tag{34}$$

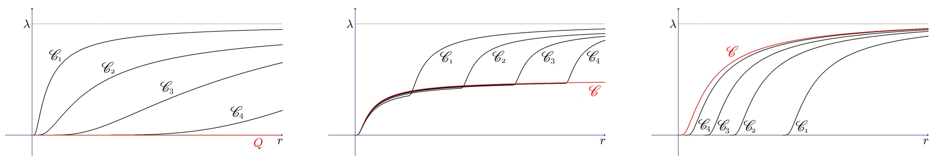
for all $(r,\boldsymbol{y}) \in (0,\infty)\times\mathbb{R}^3$ by Hölder's and Sobolev's inequalities. Moreover, we have $\lim_{r\to\infty} \mathscr{C}_n(r) = \lambda$ since Lebesgue's theorem on dominated convergence gives that the difference

$$\lambda - \mathscr{C}_n(r) \leq \lambda - \int_{\mathcal{B}(\boldsymbol{0},r)} |\psi_n(\boldsymbol{x})|^2 \, \mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^3\setminus\mathcal{B}(\boldsymbol{0},r)} |\psi_n(\boldsymbol{x})|^2 \, \mathrm{d}\boldsymbol{x}$$

can be made arbitrarily small by choosing r sufficiently large. Helly's selection principle [10, Theorem 10.5] ensures the existence of a subsequence of $(\mathscr{C}_n)_{n\in\mathbb{N}}$ converging pointwise to some function $\mathscr{C}$. The limit function $\mathscr{C}$ inherits the nondecreasingness from the $\mathscr{C}_n$'s and (34) gives that $\lim_{r\to 0^+} \mathscr{C}(r) = 0$. □

To simplify notation we will also denote the subsequence described in Lemma 13 by $(\mathscr{C}_n)_{n\in\mathbb{N}}$. It is apparent that $\mathscr{C}$ and the $\mathscr{C}_n$-functions have almost identical properties. But even though the lemma depicts $\lim_{r\to\infty} \mathscr{C}_n(r)$ as being equal to λ it does not at all mention the value of the limit

$$\mu := \lim_{r\to\infty} \mathscr{C}(r), \tag{35}$$



(a) $\mu = 0$ (b) $0 < \mu < \lambda$ (c) $\mu = \lambda$

Figure 4: Examples of possible situations where $\mu = 0$, $0 < \mu < \lambda$ respectively $\mu = \lambda$. Let us consider the behaviour of $|\psi_n|^2$ in each of these cases as n increases: In the (a)-case the wave function spreads out and diminishes, in the case (b) it splits up into lumps that move further and further away from each other whereas $|\psi_n|^2$ approaches a specific probability distribution in a way that conserves the total probability mass of the particle in the (c)-situation.

which is obviously well defined and contained in the interval $[0, \lambda]$. To determine the value of μ we first turn to our physical intuition: Remember that the points $(\psi_n, \boldsymbol{A}_n)$ form a minimizing sequence and we hope to show weak convergence (in some sense) of these points to a pair $(\psi, \boldsymbol{A})$ minimizing $\mathscr{E}_j^{\boldsymbol{v}}$. For a moment let us focus on the ψ-variable: It can be fruitful to think of our quantum particle as being prepared in some initial state and as time evolves we receive snapshots (corresponding to the sequence elements $\psi_1, \psi_2, \psi_3, \ldots$) of the system's intermediate states that steadily approach the limiting state ψ, which has the least possible energy. Of the three scenarios illustrated on Figure 4 the possibility $\mu = \lambda$ seems to be the most reasonable from a physical point of view and as we will see later the identity $\mu = \lambda$ does indeed hold true. We will basically prove this by ruling out the two other alternatives shown on Figure 4.

We begin by proving that it is impossible for μ to be equal to 0. This will be done by first establishing the following lower bound on $\mathscr{E}_j^{\boldsymbol{v}}(\psi_n, \boldsymbol{A}_n)$.

$$\mathscr{E}_j^{\boldsymbol{v}}(\psi_n, \boldsymbol{A}_n) \geq -Q^2 \frac{\boldsymbol{v}^2}{c^2 - \boldsymbol{v}^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|\psi_n(\boldsymbol{x})|^2 |\psi_n(\boldsymbol{y})|^2}{|\boldsymbol{x} - \boldsymbol{y}|} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}\boldsymbol{y} - \frac{m\boldsymbol{v}^2}{2} \lambda. \quad (36)$$

This means that we can control $\mathscr{E}_j^{\boldsymbol{v}}(\psi_n, \boldsymbol{A}_n)$ by information on the wave functions ψ_n alone – we remember from Figure 4(a) that the case $\mu = 0$ would morally correspond to the eventual disappearance of these wave functions. So in that case we expect the first term on the right hand side of (36) to disappear in the large n limit. Our strategy will therefore be to show that the identity $\mu = 0$ would violate the inequality in Lemma 6 stating that I_j^{λ} is *strictly* less than $-\frac{m\boldsymbol{v}^2}{2}$.

Lemma 14. *Let $j \in \{\mathrm{S}, \mathrm{P}\}$, $\boldsymbol{v} \in \mathbb{R}^3$ with $0 < |\boldsymbol{v}| < c$ as well as $\lambda \in \Lambda_j^{\boldsymbol{v}}$ be given and consider a minimizing sequence $\big((\psi_n, \boldsymbol{A}_n)\big)_{n \in \mathbb{N}} \subset \mathcal{S}_\lambda$ for $\mathscr{E}_j^{\boldsymbol{v}}$. Define for $n \in \mathbb{N}$ the concentration function $\mathscr{C}_n$ by (33) and let $\mathscr{C}$ be the pointwise* *limit of (a subsequence of)* $(\mathscr{C}_n)_{n\in\mathbb{N}}$*. Then* $\mu = \lim_{r\to\infty}\mathscr{C}(r)$ *is different from* 0.

Proof. The estimate (36) is actually quite rough because in the first step towards obtaining it we simply dispense with the kinetic energy term on the right hand side of (16), resulting in

$$\mathscr{E}_j^{\boldsymbol{v}}(\psi_n, \boldsymbol{A}_n) \geq \mathscr{G}_n(\boldsymbol{A}_n) - \frac{m\boldsymbol{v}^2}{2}\lambda, \tag{37}$$

where $\mathscr{G}_n : D^1 \to \mathbb{R}$ is defined by

$$\mathscr{G}_n(\boldsymbol{D}) = \frac{1}{8\pi}\Big(1 - \frac{\boldsymbol{v}^2}{c^2}\Big)\|\nabla\otimes\boldsymbol{D}\|_{L^2}^2 - Q\frac{\boldsymbol{v}}{c}\cdot\int_{\mathbb{R}^3}\boldsymbol{D}(\boldsymbol{x})|\psi_n(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x}.$$

This functional is bounded from below since applying the Sobolev and Hölder inequalities as well as optimizing in each of the variables $\left\|D^1\right\|_{L^6}$, $\left\|D^2\right\|_{L^6}$ and $\left\|D^3\right\|_{L^6}$ gives that for any $\boldsymbol{D} \in D^1$

$$\mathscr{G}_n(\boldsymbol{D}) \geq \frac{1}{16\pi}\Big(1 - \frac{\boldsymbol{v}^2}{c^2}\Big)\|\nabla\otimes\boldsymbol{D}\|_{L^2}^2 - 4\pi K_S^3 Q^2 C\lambda^{\frac{3}{2}}\frac{\boldsymbol{v}^2}{c^2 - \boldsymbol{v}^2} \tag{38}$$

so it seems straightforward to meet our intention of obtaining a lower bound on $\mathscr{E}_j^{\boldsymbol{v}}(\psi_n, \boldsymbol{A}_n)$ only depending on ψ_n – we can simply estimate the term $\mathscr{G}_n(\boldsymbol{A}_n)$ appearing on the right hand side of (37) by $\inf_{\boldsymbol{D}\in D^1}\mathscr{G}_n(\boldsymbol{D})$. Therefore it will be worthwhile for us to spend some time studying the properties of this infimum.

We first show the existence of a minimizer $\boldsymbol{D}_n$ for $\mathscr{G}_n$. This will be done by the direct method in the calculus of variations so consider a minimizing sequence for $\mathscr{G}_n$, i.e. a sequence $\big(\boldsymbol{D}_n^k\big)_{k\in\mathbb{N}}$ of D^1-functions such that $\big(\mathscr{G}_n\big(\boldsymbol{D}_n^k\big)\big)_{k\in\mathbb{N}}$ converges to $\inf_{\boldsymbol{D}\in D^1}\mathscr{G}_n(\boldsymbol{D})$. Then (38) together with the Sobolev inequality gives that the sequence $\big(\boldsymbol{D}_n^k\big)_{k\in\mathbb{N}}$ is bounded in the reflexive Banach space L^6 and in addition that $\big(\nabla\otimes\boldsymbol{D}_n^k\big)_{k\in\mathbb{N}}$ is bounded in the Hilbert space L^2. Thereby the Banach-Alaoglu theorem guarantees the existence of a subsequence of $\big(\boldsymbol{D}_n^k\big)_{k\in\mathbb{N}}$ converging weakly in L^6 to some $\boldsymbol{D}_n$ and by passing to yet another subsequence, $\big(\nabla\otimes\boldsymbol{D}_n^k\big)_{k\in\mathbb{N}}$ converges weakly in L^2 to some $\boldsymbol{D}_n'$. But then we have $\boldsymbol{D}_n^k \xrightarrow[k\to\infty]{} \boldsymbol{D}_n$ and $\nabla\otimes\boldsymbol{D}_n^k \xrightarrow[k\to\infty]{} \boldsymbol{D}_n'$ in the distribution sense, whereby we must have $\nabla\otimes\boldsymbol{D}_n = \boldsymbol{D}_n'$. In other words, we have (after passing to a subsequence)

$$\boldsymbol{D}_n^k \underset{k\to\infty}{\rightharpoonup} \boldsymbol{D}_n \text{ in } L^6 \quad\text{and}\quad \nabla\otimes\boldsymbol{D}_n^k \underset{k\to\infty}{\rightharpoonup} \nabla\otimes\boldsymbol{D}_n \text{ in } L^2. \tag{39}$$

That $|\psi_n|^2 \in L^{\frac{6}{5}}$ implies together with the first convergence in (39) that $\lim_{k\to\infty}\boldsymbol{v}\cdot\int_{\mathbb{R}^3}\boldsymbol{D}_n^k(\boldsymbol{x})|\psi_n(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x}$ is equal to $\boldsymbol{v}\cdot\int_{\mathbb{R}^3}\boldsymbol{D}_n(\boldsymbol{x})|\psi_n(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x}$ and the second convergence in (39) gives together with the weak lower semicontinuity

[11, Theorem 2.11] of $\|\cdot\|_{L^2}$ that the quantity $\liminf_{k\to\infty}\big\|\nabla\otimes \boldsymbol{D}_n^k\big\|_{L^2}^2$ is at least $\big\|\nabla\otimes \boldsymbol{D}_n\big\|_{L^2}^2$. Thereby

$$\begin{aligned}\inf_{\boldsymbol{D}\in D^1}\mathscr{G}_n(\boldsymbol{D}) &= \frac{1}{8\pi}\Big(1-\frac{\boldsymbol{v}^2}{c^2}\Big)\liminf_{k\to\infty}\big\|\nabla\otimes \boldsymbol{D}_n^k\big\|_{L^2}^2 - Q\frac{\boldsymbol{v}}{c}\cdot\int_{\mathbb{R}^3}\boldsymbol{D}_n(\boldsymbol{x})|\psi_n(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x}\\ &\geq \mathscr{G}_n(\boldsymbol{D}_n)\end{aligned}$$

and so we must have $\inf_{\boldsymbol{D}\in D^1}\mathscr{G}_n(\boldsymbol{D}) = \mathscr{G}_n(\boldsymbol{D}_n)$. Then the functional derivative $\frac{\delta\mathscr{G}_n}{\delta\boldsymbol{D}}$ must take the value 0 in the point $\boldsymbol{D}_n$, which implies that $\boldsymbol{D}_n$ satisfies the Poisson equation

$$-\Delta\boldsymbol{D}_n = 4\pi Q\frac{c\boldsymbol{v}}{c^2-\boldsymbol{v}^2}|\psi_n|^2 \tag{40}$$

in the distribution sense. The function on the right hand side is contained in $L^1\cap L^3$ and has gradient in $L^1\cap L^{\frac{5}{4}}$ so according to Lemma 19 and Remark 20 we must have

$$\boldsymbol{D}_n(\boldsymbol{x}) = Q\frac{c\boldsymbol{v}}{c^2-\boldsymbol{v}^2}\int_{\mathbb{R}^3}\frac{|\psi_n(\boldsymbol{y})|^2}{|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{d}\boldsymbol{y}$$

for almost every $\boldsymbol{x}\in\mathbb{R}^3$. Consequently, we can continue the estimate (37) and get (36).

We now realize that for almost all $\boldsymbol{y}\in\mathbb{R}^3$ and all choices of positive numbers r and R satisfying $r<R$ we have

$$\begin{aligned}\int_{\mathbb{R}^3}\frac{|\psi_n(\boldsymbol{x})|^2}{|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{d}\boldsymbol{x} &\leq \|\psi_n\|_{L^6}^2\left\|1_{\mathcal{B}(\boldsymbol{0},r)}\frac{1}{|\cdot|}\right\|_{L^{\frac{3}{2}}} + \frac{1}{r}\|1_{\mathcal{B}(\boldsymbol{y},R)}\psi_n\|_{L^2}^2 + \frac{1}{R}\|\psi_n\|_{L^2}^2\\ &\leq \Big(\frac{8\pi}{3}\Big)^{\frac{2}{3}}K_S^2C^2r + \frac{1}{r}\mathscr{C}_n(R) + \frac{1}{R}\lambda;\end{aligned}$$

this is seen by splitting the integral on the left hand side into contributions from $\mathcal{B}(\boldsymbol{y},r)$, $\mathcal{B}(\boldsymbol{y},R)\setminus\mathcal{B}(\boldsymbol{y},r)$ and $\mathbb{R}^3\setminus\mathcal{B}(\boldsymbol{y},R)$. Combining this with (36) gives

$$\mathscr{E}_j^{\boldsymbol{v}}(\psi_n,\boldsymbol{A}_n) \geq -Q^2\frac{\boldsymbol{v}^2}{c^2-\boldsymbol{v}^2}\lambda\left(\Big(\frac{8\pi}{3}\Big)^{\frac{2}{3}}K_S^2C^2r + \frac{1}{r}\mathscr{C}_n(R) + \frac{1}{R}\lambda\right) - \frac{m\boldsymbol{v}^2}{2}\lambda.$$

so sending n to infinity results in

$$I_j^\lambda \geq -Q^2\frac{\boldsymbol{v}^2}{c^2-\boldsymbol{v}^2}\lambda\left(\Big(\frac{8\pi}{3}\Big)^{\frac{2}{3}}K_S^2C^2r + \frac{1}{r}\mathscr{C}(R) + \frac{1}{R}\lambda\right) - \frac{m\boldsymbol{v}^2}{2}\lambda.$$

Under the assumption that $\mu=0$ we can therefore let $R\to\infty$ and get

$$I_j^\lambda \geq -Q^2\frac{\boldsymbol{v}^2}{c^2-\boldsymbol{v}^2}\lambda\Big(\frac{8\pi}{3}\Big)^{\frac{2}{3}}K_S^2C^2r - \frac{m\boldsymbol{v}^2}{2}\lambda,$$

which sets us in position to take the limit $r\to 0^+$ and obtain the inequality $I_j^\lambda \geq -\frac{m\boldsymbol{v}^2}{2}\lambda$, contradicting Lemma 6. □

We now turn to proving that $\mu \notin (0, \lambda)$, which will again be done using the method of proof by contradiction. Remember from Figure 4(b) that if $\mu \in (0, \lambda)$ we expect the wave function to split up into lumps that move further and further away from each other as n increases. It seems reasonable that these lumps will eventually be so far apart that the interaction between them is negligible, whereby we can practically consider them as independent systems. Given a term $(\psi_n, \boldsymbol{A}_n)$ of the minimizing sequence our strategy will therefore be to construct a pair $(\psi_n^{\mathrm{i}}, \boldsymbol{A}_n^{\mathrm{i}})$ which is 'almost' an element of $\mathcal{S}_\mu$ and a pair $(\psi_n^{\mathrm{o}}, \boldsymbol{A}_n^{\mathrm{o}})$ 'almost' belonging to $\mathcal{S}_{\lambda-\mu}$ such that $\mathscr{E}_j^{\boldsymbol{v}}(\psi_n^{\mathrm{i}}, \boldsymbol{A}_n^{\mathrm{i}}) + \mathscr{E}_j^{\boldsymbol{v}}(\psi_n^{\mathrm{o}}, \boldsymbol{A}_n^{\mathrm{o}})$ is at most $\mathscr{E}_j^{\boldsymbol{v}}(\psi_n, \boldsymbol{A}_n)$ (up to a small error). A limiting argument will then give a conclusion contradicting (28). The splitting will of course be done by using cut-off functions, so let us first introduce some mappings χ^{i} and χ^{o} ('i' for 'inner' and 'o' for 'outer') with the following properties: The supports of $\chi^{\mathrm{i}} \in C_0^\infty(\mathbb{R}^3)$ and $\chi^{\mathrm{o}} \in C^\infty(\mathbb{R}^3)$ are disjoint and

$$\chi^{\mathrm{i}}(\boldsymbol{x}) \begin{cases} = 1 & \text{for } |\boldsymbol{x}| \leq 1, \\ \in [0,1] & \text{for } 1 < |\boldsymbol{x}| < 2, \\ = 0 & \text{for } |\boldsymbol{x}| \geq 2, \end{cases} \quad \text{and} \quad \chi^{\mathrm{o}}(\boldsymbol{x}) \begin{cases} = 0 & \text{for } |\boldsymbol{x}| \leq 1, \\ \in [0,1] & \text{for } 1 < |\boldsymbol{x}| < 2, \\ = 1 & \text{for } |\boldsymbol{x}| \geq 2. \end{cases}$$

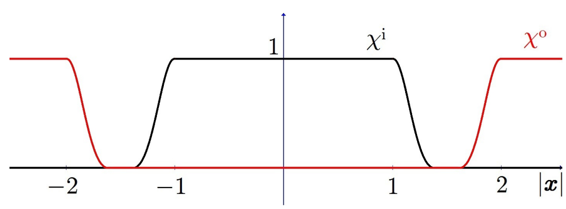


Figure 5: Possible choices for χ^{i} and χ^{o}.

Lemma 15. *Consider $j \in \{\mathrm{S}, \mathrm{P}\}$, $\boldsymbol{v} \in \mathbb{R}^3$ with $0 < |\boldsymbol{v}| < c$ and $\lambda \in \Lambda_j^{\boldsymbol{v}}$. Let also $\big((\psi_n, \boldsymbol{A}_n)\big)_{n\in\mathbb{N}} \subset \mathcal{S}_\lambda$ be a minimizing sequence for $\mathscr{E}_j^{\boldsymbol{v}}$, define $\mathscr{C}_n$ by (33) for $n \in \mathbb{N}$ and consider the pointwise limit $\mathscr{C}$ of (a subsequence of) $(\mathscr{C}_n)_{n\in\mathbb{N}}$. Then $\mu = \lim_{r\to\infty} \mathscr{C}(r)$ is not contained in $(0, \lambda)$.*

Proof. Suppose that $\mu \in (0, \lambda)$. On the basis of ψ_n we want to construct a function ψ_n^{i} whose L^2-norm squared is close to μ, so to which region of space should we localize ψ_n? The answer is of course encoded in the concentration function of ψ_n, so more precisely: Given

$$0 < \varepsilon < \min\left\{ \frac{2^5 3^4 C^6 K_S^6 Q^4}{m^3 c^4 \boldsymbol{v}^2}, 4m\boldsymbol{v}^2\mu, 4m\boldsymbol{v}^2(\lambda-\mu), \frac{2^{\frac{11}{4}} 3 C^{\frac{3}{2}} K_S^{\frac{3}{2}} |Q\boldsymbol{v}|}{c} (\lambda-\mu)^{\frac{3}{4}} \right\} \tag{41}$$

we choose (by the definition of μ) a number

$$R > \max\left\{2^{\frac{3}{2}}\hbar\sqrt{\frac{\lambda}{\varepsilon m}}\|\nabla\chi^{\ell}\|_{L^{\infty}}, \frac{16C\hbar\lambda^{\frac{1}{2}}}{\varepsilon m}\|\nabla\chi^{\ell}\|_{L^{\infty}} \,\middle|\, \ell \in \{\mathrm{i},\mathrm{o}\}\right\}$$
$$\vee \;\frac{2^2 3^4 K_S^2 C^2 Q^2 \lambda^2 \boldsymbol{v}^2}{\varepsilon^2\sqrt{\pi}c^2}\big(\max\{\|\nabla\chi^{\mathrm{i}}\|_{L^{12}}, \|\nabla\chi^{\mathrm{o}}\|_{L^3}\}\big)^2 \tag{42}$$

such that $\mu - \frac{\varepsilon^{4/3}c^{4/3}}{2^{11/3}3^{4/3}C^2K_S^2|Q\boldsymbol{v}|^{4/3}} < \mathscr{C}(R)$. As a first step we will consider n's so large that

$$\mu - \frac{\varepsilon^{\frac{4}{3}}c^{\frac{4}{3}}}{2^{\frac{11}{3}}3^{\frac{4}{3}}C^2K_S^2|Q\boldsymbol{v}|^{\frac{4}{3}}} < \mathscr{C}_n(R) < \mu + \frac{\varepsilon^{\frac{4}{3}}c^{\frac{4}{3}}}{2^{\frac{11}{3}}3^{\frac{4}{3}}C^2K_S^2|Q\boldsymbol{v}|^{\frac{4}{3}}}. \tag{43}$$

Here, the upper bound on $\mathscr{C}_n(R)$ is strictly speaking redundant, since we will later obtain a better upper bound by considering even larger values of n – but already at this point it is advantageous to think of $\mathscr{C}_n(R)$ as being close to μ. We should not just perceive $\mathscr{C}_n(R)$ as being an abstract supremum – it is in fact the probability mass of the particle in the vicinity of some point in space. Because $\psi_n \in L^2$ the continuous function $\boldsymbol{y} \mapsto \int_{\mathcal{B}(\boldsymbol{y},R)} |\psi_n(\boldsymbol{x})|^2 \,\mathrm{d}\boldsymbol{x}$ will namely approach zero as $|\boldsymbol{y}| \to \infty$, whereby we can choose a point $\boldsymbol{y}_n \in \mathbb{R}^3$ such that

$$\mathscr{C}_n(R) = \int_{\mathcal{B}(\boldsymbol{y}_n,R)} |\psi_n(\boldsymbol{x})|^2 \,\mathrm{d}\boldsymbol{x}. \tag{44}$$

So in the ball $\mathcal{B}(\boldsymbol{y}_n, R)$ we have found a ψ_n-lump whose probability mass is essentially μ. The other lumps are expected to move away as n increases, so for large n there should be a large area around $\mathcal{B}(\boldsymbol{y}_n, R)$ where ψ_n has essentially no probability mass. As a consequence we can construct the function ψ_n^{o} by cutting away the values of ψ_n on a ball centered at $\boldsymbol{y}_n$ with quite a large radius. It turns out that we can in fact choose this radius on the form $2^{k_n}R$, where the sequence $(k_n)_{n\in\mathbb{N}}$ of integers satisfies

(I) $k_n \to \infty$ for $n \to \infty$,

(II) $\mathscr{C}_n(2^{k_n}R) \leq \mu + \frac{\varepsilon^{4/3}c^{4/3}}{2^{11/3}3^{4/3}C^2K_S^2|Q\boldsymbol{v}|^{4/3}}$ for all $n \in \mathbb{N}$.

One can namely easily verify that the sequence of numbers

$$k_n = \left\lfloor \log_2 \frac{\sup \mathscr{C}_n^{-1}\left(\left(0, \mu + \frac{\varepsilon^{4/3}c^{4/3}}{2^{11/3}3^{4/3}C^2K_S^2|Q\boldsymbol{v}|^{4/3}}\right]\right)}{2R} \right\rfloor$$

has the desired properties, where $\lfloor\cdot\rfloor$ denotes the floor function and $\log_2$ denotes the binary logarithm[2]. Thus, we will construct ψ_n^{i} and ψ_n^{o} by multiplication

[2]The floor function is $x \mapsto \max\{m \in \mathbb{Z} \mid m \leq x\}$ and the binary logarithm is $x \mapsto \frac{\log(x)}{\log(2)}$, where log denotes the natural logarithm.

with the cut-off functions given by

$$\chi_n^{\mathrm{i},\psi}(\boldsymbol{x}) = \chi^{\mathrm{i}}\Big(\frac{\boldsymbol{x}-\boldsymbol{y}_n}{R}\Big) \quad \text{respectively} \quad \chi_n^{\mathrm{o},\psi}(\boldsymbol{x}) = \chi^{\mathrm{o}}\Big(\frac{\boldsymbol{x}-\boldsymbol{y}_n}{2^{k_n-1}R}\Big)$$

for $\boldsymbol{x} \in \mathbb{R}^3$. Let us emphasize that we use the superscript ψ because these functions will be used to cut the wave function ψ into the two pieces ψ_n^{i} and ψ_n^{o} – later we will define corresponding cut-off functions $\chi_n^{\mathrm{i},\boldsymbol{A}}$ and $\chi_n^{\mathrm{o},\boldsymbol{A}}$ to cut $\boldsymbol{A}$ into two pieces $\boldsymbol{A}_n^{\mathrm{i}}$ and $\boldsymbol{A}_n^{\mathrm{o}}$.

Let us now do this splitting of the $\boldsymbol{A}_n$-field into $\boldsymbol{A}_n^{\mathrm{i}}$- and $\boldsymbol{A}_n^{\mathrm{o}}$-fields. We will aim to make the cuts in the big gap between $\mathcal{B}(\boldsymbol{y}_n, 2R)$ and $\mathbb{R}^3 \setminus \mathcal{B}(\boldsymbol{y}_n, 2^{k_n-1}R)$, where the functions ψ_n^{i} and ψ_n^{o} are guaranteed to vanish. So we decompose space into the disjoint union $\mathbb{R}^3 = \mathcal{B}(\boldsymbol{y}_n, R) \cup \big(\bigcup_{m=1}^{\infty} \mathcal{A}_n^m\big)$, where

$$\mathcal{A}_n^m = \big\{\boldsymbol{x} \in \mathbb{R}^3 \,\big|\, 2^{m-1}R \leq |\boldsymbol{x}-\boldsymbol{y}_n| < 2^m R\big\}$$

for $m \in \mathbb{N}$ (see Figure 6).

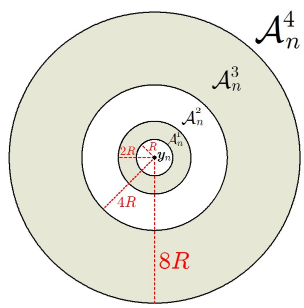


Figure 6: The two-dimensional analogues of the spherical shells $\mathcal{A}_n^0, \mathcal{A}_n^1, \ldots$.

By point (I) from above we have $k_n \geq 4$ for n sufficiently large and for such n's there must exist a number m_n in the set $\{2, \ldots, k_n - 2\}$ such that the inequality $\big\|1_{\mathcal{A}_n^{m_n}} \boldsymbol{A}_n\big\|_{L^6}^6 \leq (k_n-3)^{-1}\big\|1_{\mathcal{A}_n^2 \cup \cdots \cup \mathcal{A}_n^{k_n-2}} \boldsymbol{A}_n\big\|_{L^6}^6$ holds true, whereby we have for n sufficiently large that

$$\begin{aligned}&\big\|1_{\mathcal{A}_n^{m_n}} \boldsymbol{A}_n\big\|_{L^6} \\ &\quad < \min\Big\{\frac{1}{4\|\nabla\chi^\ell\|_{L^3}}\Big(\sqrt{C^2 + \frac{\varepsilon\pi c^2}{c^2+\boldsymbol{v}^2}} - C\Big), \frac{1}{2\|\nabla\chi^\ell\|_{L^3}}\sqrt{\frac{\varepsilon\pi c^2}{c^2+\boldsymbol{v}^2}}\,\Big|\, \ell \in \{\mathrm{i},\mathrm{o}\}\Big\}.\end{aligned} \tag{45}$$

In this way we can control $\boldsymbol{A}_n$ on $\mathcal{A}_n^{m_n}$, so we will define $\boldsymbol{A}_n^{\mathrm{i}}$ and $\boldsymbol{A}_n^{\mathrm{o}}$ using the cut-off functions

$$\chi_n^{\mathrm{i},\boldsymbol{A}}(\boldsymbol{x}) = \chi^{\mathrm{i}}\Big(\frac{\boldsymbol{x}-\boldsymbol{y}_n}{2^{m_n-1}R}\Big) \quad \text{and} \quad \chi_n^{\mathrm{o},\boldsymbol{A}}(\boldsymbol{x}) = \chi^{\mathrm{o}}\Big(\frac{\boldsymbol{x}-\boldsymbol{y}_n}{2^{m_n-1}R}\Big).$$

More precisely, we will for $\ell \in \{\mathrm{i},\mathrm{o}\}$ introduce the mapping $u_n^\ell : \mathbb{R}^3 \to \mathbb{R}$

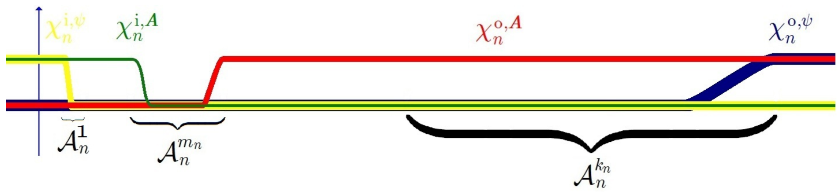


Figure 7: The distance to $\boldsymbol{y}_n$ is measured along the first axis.

given by

$$u_n^\ell(\boldsymbol{x}) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\operatorname{div}\big(\chi_n^{\ell,\boldsymbol{A}}\boldsymbol{A}_n\big)(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} \text{ for almost every } \boldsymbol{x}\in\mathbb{R}^3$$

and define ψ_n^{i}, ψ_n^{o}, $\boldsymbol{A}_n^{\mathrm{i}}$ and $\boldsymbol{A}_n^{\mathrm{o}}$ by

$$\psi_n^\ell = \mathrm{e}^{\frac{iQ}{\hbar c}u_n^\ell}\chi_n^{\ell,\psi}\psi_n \quad \text{and} \quad \boldsymbol{A}_n^\ell = \chi_n^{\ell,\boldsymbol{A}}\boldsymbol{A}_n + \nabla u_n^\ell.$$

We observe that $\operatorname{div}\big(\chi_n^{\ell,\boldsymbol{A}}\boldsymbol{A}_n\big) = \nabla\chi_n^{\ell,\boldsymbol{A}}\cdot\boldsymbol{A}_n$ is contained in H^1 and has compact support so from Lemma 19 we obtain that $\psi_n^\ell\in H^1$, $\boldsymbol{A}_n^\ell\in D^1$ with $\operatorname{div}\boldsymbol{A}_n^\ell = 0$ and

$$\nabla u_n^\ell(\boldsymbol{x}) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\boldsymbol{x}-\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|^3}\operatorname{div}\big(\chi_n^{\ell,\boldsymbol{A}}\boldsymbol{A}_n\big)(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} \text{ for almost every } \boldsymbol{x}\in\mathbb{R}^3. \tag{46}$$

Moreover, ψ_n^{i} and ψ_n^{o} satisfy

$$\max\left\{\left|\mu - \int_{\mathbb{R}^3}|\psi_n^{\mathrm{i}}(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x}\right|, \left|\lambda-\mu-\int_{\mathbb{R}^3}|\psi_n^{\mathrm{o}}(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x}\right|\right\} < \frac{\varepsilon^{\frac{4}{3}}c^{\frac{4}{3}}}{2^{\frac{11}{3}}3^{\frac{4}{3}}C^2K_S^2|Q\boldsymbol{v}|^{\frac{4}{3}}}. \tag{47}$$

which follows from (43), (II) as well as the estimates

$$\mathscr{C}_n(R) \le \int_{\mathbb{R}^3}|\psi_n^{\mathrm{i}}(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x} \le \int_{\mathcal{B}(\boldsymbol{y}_n,2R)}|\psi_n(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x} \le \mathscr{C}_n(2R) \le \mathscr{C}_n(2^{k_n}R)$$

and

$$\mathscr{C}_n(R) \le \lambda - \int_{\mathbb{R}^3}|\psi_n^{\mathrm{o}}(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x} \le \int_{\mathcal{B}(\boldsymbol{y}_n,2^{k_n}R)}|\psi_n(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x} \le \mathscr{C}_n(2^{k_n}R).$$

In the motivational remarks made above Lemma 15 we mentioned the desire to construct ψ_n^ℓ and $\boldsymbol{A}_n^\ell$ in such a way that they 'almost' satisfy $\big(\psi_n^{\mathrm{i}},\boldsymbol{A}_n^{\mathrm{i}}\big)\in\mathcal{S}_\mu$ and $\big(\psi_n^{\mathrm{o}},\boldsymbol{A}_n^{\mathrm{o}}\big)\in\mathcal{S}_{\lambda-\mu}$. The precise meaning of this informal statement is that the pairs $\big(\psi_n^\ell,\boldsymbol{A}_n^\ell\big)\in H^1\times D^1$ have the properties $\operatorname{div}\boldsymbol{A}_n^\ell = 0$ and (47).

The next step in our argument is to show that

$$\mathscr{E}_j^{\boldsymbol{v}}(\psi_n, \boldsymbol{A}_n) \geq \mathscr{E}_j^{\boldsymbol{v}}(\psi_n^{\mathrm{i}}, \boldsymbol{A}_n^{\mathrm{i}}) + \mathscr{E}_j^{\boldsymbol{v}}(\psi_n^{\mathrm{o}}, \boldsymbol{A}_n^{\mathrm{o}}) - \varepsilon. \tag{48}$$

We begin by estimating the $\frac{1}{2m}\|\nabla_{j,\boldsymbol{A}_n+\frac{mc}{Q}\boldsymbol{v}}\psi_n\|_{L^2}^2$-term on the right hand side of (16). For this we observe that $\chi_n^{\ell,\boldsymbol{A}}\chi_n^{\ell,\psi} = \chi_n^{\ell,\psi}$, whereby we can rewrite

$$\mathrm{e}^{-\frac{iQ}{\hbar c}u_n^\ell}\nabla_{j,\boldsymbol{A}_n^\ell+\frac{mc}{Q}\boldsymbol{v}}\psi_n^\ell = \nabla_{j,\boldsymbol{0}}\chi_n^{\ell,\psi}\psi_n + \chi_n^{\ell,\psi}\nabla_{j,\boldsymbol{A}_n+\frac{mc}{Q}\boldsymbol{v}}\psi_n,$$

which allows us to apply (15), (42) and Remark 12 to obtain

$$\begin{aligned}
&\|\nabla_{j,\boldsymbol{A}_n^\ell+\frac{mc}{Q}\boldsymbol{v}}\psi_n^\ell\|_{L^2}^2 \\
&\leq \frac{\hbar^2\lambda}{R^2}\|\nabla\chi^\ell\|_{L^\infty}^2 + \|\chi_n^{\ell,\psi}\nabla_{j,\boldsymbol{A}_n+\frac{mc}{Q}\boldsymbol{v}}\psi_n\|_{L^2}^2 + \frac{2C\hbar\lambda^{\frac{1}{2}}}{R}\|\nabla\chi^\ell\|_{L^\infty} \\
&\leq \|\chi_n^{\ell,\psi}\nabla_{j,\boldsymbol{A}_n+\frac{mc}{Q}\boldsymbol{v}}\psi_n\|_{L^2}^2 + \frac{\varepsilon m}{4}.
\end{aligned}$$

and consequently

$$\frac{1}{2m}\|\nabla_{j,\boldsymbol{A}_n+\frac{mc}{Q}\boldsymbol{v}}\psi_n\|_{L^2}^2 \geq \frac{1}{2m}\sum_{\ell\in\{\mathrm{i},\mathrm{o}\}}\|\nabla_{j,\boldsymbol{A}_n^\ell+\frac{mc}{Q}\boldsymbol{v}}\psi_n^\ell\|_{L^2}^2 - \frac{\varepsilon}{4}. \tag{49}$$

To treat the term $-\frac{Q}{c}(\psi_n, \boldsymbol{v}\cdot\boldsymbol{A}_n\psi_n)_{L^2}$ appearing on the right hand side of (16) we establish two auxiliary estimates: The first estimate

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^3}\boldsymbol{v}\cdot\boldsymbol{A}_n\big(|\psi_n|^2 - |\psi_n^{\mathrm{i}}|^2 - |\psi_n^{\mathrm{o}}|^2\big)\,\mathrm{d}\boldsymbol{x}\Big| \\
&\leq |\boldsymbol{v}|\|\boldsymbol{A}_n\|_{L^6}\left\|\sqrt{1-(\chi_n^{\mathrm{i},\psi})^2-(\chi_n^{\mathrm{o},\psi})^2}\psi_n\right\|_{L^6}^{\frac{1}{2}}\Big(\int_{\mathbb{R}^3}\big(|\psi_n|^2 - |\psi_n^{\mathrm{i}}|^2 - |\psi_n^{\mathrm{o}}|^2\big)\,\mathrm{d}\boldsymbol{x}\Big)^{\frac{3}{4}} \\
&\leq \frac{\varepsilon c}{12|Q|}
\end{aligned}$$

follows from (47), Hölder's and Sobolev's inequalities. By choosing n large enough we previously made sure that $k_n \geq 4$ and $2 \leq m_n \leq k_n - 2$ whereby (46), the Hölder inequality and (42) yield the second auxiliary estimate

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^3}\boldsymbol{v}\cdot\nabla u_n^\ell|\psi_n^\ell|^2\,\mathrm{d}\boldsymbol{x}\Big| \\
&\leq \frac{|\boldsymbol{v}|}{4\pi}\int_{\mathbb{R}^3}\int_{\mathcal{A}_n^{m_n}}\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|^2}|\nabla\chi_n^{\ell,\boldsymbol{A}}(\boldsymbol{y})||\boldsymbol{A}_n(\boldsymbol{y})|\,\mathrm{d}\boldsymbol{y}\,|\psi_n^\ell(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x} \\
&\leq \begin{cases}\frac{|\boldsymbol{v}|}{4\pi}\left\|1_{\mathcal{B}(\boldsymbol{0},(2+2^{m_n})R)}\frac{1}{|\cdot|}\right\|_{L^{\frac{8}{3}}}^2\|\nabla\chi_n^{\mathrm{i},\boldsymbol{A}}\|_{L^{12}}\|\boldsymbol{A}_n\|_{L^6}\|\psi_n^{\mathrm{i}}\|_{L^2}^2 & \text{for } \ell = \mathrm{i} \\ \frac{|\boldsymbol{v}|}{4\pi}\left\|1_{\mathbb{R}^3\setminus\mathcal{B}(\boldsymbol{0},(2^{k_n-1}-2^{m_n})R)}\frac{1}{|\cdot|}\right\|_{L^4}^2\|\nabla\chi_n^{\mathrm{o},\boldsymbol{A}}\|_{L^3}\|\boldsymbol{A}_n\|_{L^6}\|\psi_n^{\mathrm{o}}\|_{L^2}^2 & \text{for } \ell = \mathrm{o}\end{cases} \\
&\leq \frac{3|\boldsymbol{v}|K_S C\lambda}{2\pi^{\frac{1}{4}}R^{\frac{1}{2}}}\max\big\{\|\nabla\chi^{\mathrm{i}}\|_{L^{12}}, \|\nabla\chi^{\mathrm{o}}\|_{L^3}\big\} \\
&< \frac{\varepsilon c}{12|Q|}.
\end{aligned}$$

By combining the two previous estimates with the identity $\chi_n^{\ell,\boldsymbol{A}}\chi_n^{\ell,\psi} = \chi_n^{\ell,\psi}$ we obtain

$$\begin{aligned}
&|(\psi_n, \boldsymbol{v}\cdot\boldsymbol{A}_n\psi_n)_{L^2} - (\psi_n^{\mathrm{i}}, \boldsymbol{v}\cdot\boldsymbol{A}_n^{\mathrm{i}}\psi_n^{\mathrm{i}})_{L^2} - (\psi_n^{\mathrm{o}}, \boldsymbol{v}\cdot\boldsymbol{A}_n^{\mathrm{o}}\psi_n^{\mathrm{o}})_{L^2}| \\
&= \Big|\int_{\mathbb{R}^3} \boldsymbol{v}\cdot\boldsymbol{A}_n\big(|\psi_n|^2 - |\psi_n^{\mathrm{i}}|^2 - |\psi_n^{\mathrm{o}}|^2\big)\,\mathrm{d}\boldsymbol{x} - \sum_{\ell\in\{\mathrm{i},\mathrm{o}\}}\int_{\mathbb{R}^3} \boldsymbol{v}\cdot\nabla u_n^\ell|\psi_n^\ell|^2\,\mathrm{d}\boldsymbol{x}\Big| \\
&< \frac{\varepsilon c}{4|Q|}.
\end{aligned} \tag{50}$$

Finally, we estimate the $\frac{1}{8\pi}\big(\|\nabla\otimes\boldsymbol{A}_n\|_{L^2}^2 - \big\|\big(\frac{\boldsymbol{v}}{c}\cdot\nabla\big)\boldsymbol{A}_n\big\|_{L^2}^2\big)$-term on the right hand side of (16) by noting that

$$\begin{aligned}
&\|\nabla\otimes\boldsymbol{A}_n\|_{L^2}^2 - \Big\|\Big(\frac{\boldsymbol{v}}{c}\cdot\nabla\Big)\boldsymbol{A}_n\Big\|_{L^2}^2 \\
&\geq \sum_{\ell\in\{\mathrm{i},\mathrm{o}\}}\Big(\|\chi_n^{\ell,\boldsymbol{A}}\nabla\otimes\boldsymbol{A}_n\|_{L^2}^2 - \Big\|\chi_n^{\ell,\boldsymbol{A}}\Big(\frac{\boldsymbol{v}}{c}\cdot\nabla\Big)\boldsymbol{A}_n\Big\|_{L^2}^2\Big) \\
&\geq \sum_{\ell\in\{\mathrm{i},\mathrm{o}\}}\Big(\|\nabla\otimes\boldsymbol{A}_n^\ell\|_{L^2}^2 - \Big\|\Big(\frac{\boldsymbol{v}}{c}\cdot\nabla\Big)\boldsymbol{A}_n^\ell\Big\|_{L^2}^2 \\
&\qquad - 2\Big(1+\frac{\boldsymbol{v}^2}{c^2}\Big)\|\nabla\otimes\boldsymbol{A}_n^\ell\|_{L^2}\big(\|\nabla\chi_n^{\ell,\boldsymbol{A}}\otimes\boldsymbol{A}_n\|_{L^2} + \|\nabla\otimes\nabla u_n^\ell\|_{L^2}\big) \\
&\qquad\qquad - 2\Big(1+\frac{\boldsymbol{v}^2}{c^2}\Big)\|\nabla\chi_n^{\ell,\boldsymbol{A}}\otimes\boldsymbol{A}_n\|_{L^2}\|\nabla\otimes\nabla u_n^\ell\|_{L^2}\Big),
\end{aligned} \tag{51}$$

where we at the second step use the identities

$$\begin{aligned}
\chi_n^{\ell,\boldsymbol{A}}\nabla\otimes\boldsymbol{A}_n &= \nabla\otimes\boldsymbol{A}_n^\ell - (\nabla\chi_n^{\ell,\boldsymbol{A}}\otimes\boldsymbol{A}_n + \nabla\otimes\nabla u_n^\ell), \\
\chi_n^{\ell,\boldsymbol{A}}\Big(\frac{\boldsymbol{v}}{c}\cdot\nabla\Big)\boldsymbol{A}_n &= \Big(\frac{\boldsymbol{v}}{c}\cdot\nabla\Big)\boldsymbol{A}_n^\ell - \Big(\Big(\frac{\boldsymbol{v}}{c}\cdot\nabla\Big)\chi_n^{\ell,\boldsymbol{A}}\boldsymbol{A}_n + \Big(\frac{\boldsymbol{v}}{c}\cdot\nabla\Big)\nabla u_n^\ell\Big)
\end{aligned} \tag{52}$$

and the nonnegativity of $\big(1-\frac{\boldsymbol{v}^2}{c^2}\big)(\|\nabla\chi_n^{\ell,\boldsymbol{A}}\otimes\boldsymbol{A}_n\|^2 + \|\nabla\otimes\nabla u_n^\ell\|_{L^2}^2)$. As can be seen by approximating ∇u_n^ℓ in D^1 by C_0^∞-functions, applying the Plancherel theorem and using the general vector identity $|\boldsymbol{F}|^2|\boldsymbol{G}|^2 = |\boldsymbol{F}\cdot\boldsymbol{G}|^2 + |\boldsymbol{F}\times\boldsymbol{G}|^2$ we have $\|\nabla\otimes\nabla u_n^\ell\|_{L^2}^2 = \|\mathrm{div}\nabla u_n^\ell\|_{L^2}^2 + \|\nabla\times\nabla u_n^\ell\|_{L^2}^2 = \|\nabla\chi_n^{\ell,\boldsymbol{A}}\cdot\boldsymbol{A}_n\|_{L^2}^2$. Consequently, $\|\nabla\otimes\nabla u_n^\ell\|_{L^2}$ and $\|\nabla\chi_n^{\ell,\boldsymbol{A}}\cdot\boldsymbol{A}_n\|_{L^2}$ have the common upper bound $\|\nabla\chi^\ell\|_{L^3}\big\|1_{\mathcal{A}_n^{m_n}}\boldsymbol{A}_n\big\|_{L^6}$ that is small by (45). Moreover, $\|\nabla\otimes\boldsymbol{A}_n^\ell\|_{L^2}$ is according to (52) bounded from above by $C + 2\|\nabla\chi^\ell\|_{L^3}\big\|1_{\mathcal{A}_n^{m_n}}\boldsymbol{A}_n\big\|_{L^6}$ and so we can use (45) to continue (51) and get

$$\begin{aligned}
&\frac{1}{8\pi}\Big(\|\nabla\otimes\boldsymbol{A}_n\|_{L^2}^2 - \Big\|\Big(\frac{\boldsymbol{v}}{c}\cdot\nabla\Big)\boldsymbol{A}_n\Big\|_{L^2}^2\Big) \\
&\geq \sum_{\ell\in\{\mathrm{i},\mathrm{o}\}}\frac{1}{8\pi}\Big(\|\nabla\otimes\boldsymbol{A}_n^\ell\|_{L^2}^2 - \Big\|\Big(\frac{\boldsymbol{v}}{c}\cdot\nabla\Big)\boldsymbol{A}_n^\ell\Big\|_{L^2}^2\Big) - \frac{\varepsilon}{4}.
\end{aligned} \tag{53}$$

Now, (48) is an immediate consequence of (49), (50), (53) and the inequality

$$\max\left\{\left|\mu - \int_{\mathbb{R}^3} |\psi_n^{\mathrm{i}}(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x}\right|, \left|\lambda - \mu - \int_{\mathbb{R}^3} |\psi_n^{\mathrm{o}}(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x}\right|\right\} < \frac{\varepsilon}{4m\boldsymbol{v}^2} \tag{54}$$

that follows from (41) and (47).

Finally, Remark 8 and (54) give

$$\mathscr{E}_j^{\boldsymbol{v}}(\psi_n, \boldsymbol{A}_n) \geq I_j^{\|\psi_n^{\mathrm{i}}\|_{L^2}^2} + I_j^{\|\psi_n^{\mathrm{o}}\|_{L^2}^2} - \varepsilon \geq I_j^{\mu + \frac{\varepsilon}{4m\boldsymbol{v}^2}} + I_j^{\lambda - \mu + \frac{\varepsilon}{4m\boldsymbol{v}^2}} - \varepsilon,$$

so letting n diverge to infinity produces the estimate

$$I_j^\lambda \geq I_j^{\mu + \frac{\varepsilon}{4m\boldsymbol{v}^2}} + I_j^{\lambda - \mu + \frac{\varepsilon}{4m\boldsymbol{v}^2}} - \varepsilon.$$

By Lemma 9 we can therefore take the limit $\varepsilon \to 0^+$ and obtain the inequality $I_j^\lambda \geq I_j^\mu + I_j^{\lambda-\mu}$ contradicting (28). □

Combining the Lemmas 14 and 15 allows us to reach the conclusion that $\lim_{r\to\infty} \mathscr{C}(r)$ is equal to λ. This is exactly what we need to break the translation invariance of our problem.

Proposition 16. *Given $j \in \{\mathrm{S}, \mathrm{P}\}$, $\boldsymbol{v} \in \mathbb{R}^3$ with $0 < |\boldsymbol{v}| < c$ and $\lambda \in \Lambda_j^{\boldsymbol{v}}$ consider a minimizing sequence $\big((\psi_n, \boldsymbol{A}_n)\big)_{n\in\mathbb{N}} \subset \mathcal{S}_\lambda$ for $\mathscr{E}_j^{\boldsymbol{v}}$. Then there exists a sequence $(\boldsymbol{y}_n)_{n\in\mathbb{N}}$ of points in $\mathbb{R}^3$ with the following property: For every $\varepsilon > 0$ there exists an $R > 0$ such that for all $n \in \mathbb{N}$*

$$\left\|1_{\mathcal{B}(\boldsymbol{y}_n, R)}\psi_n\right\|_{L^2}^2 \geq \lambda - \varepsilon.$$

This property is sometimes expressed by saying that the maps $\boldsymbol{x} \mapsto |\psi_n(\boldsymbol{x}+\boldsymbol{y}_n)|^2$ are tight.

Proof. Given $\nu > 0$ we will first argue that it is possible to find an $r^\nu > 0$ such that

$$\mathscr{C}_n(r^\nu) > \lambda - \nu \quad \text{for all } n \in \mathbb{N}. \tag{55}$$

By the identity $\lim_{r\to\infty} \mathscr{C}(r) = \lambda$ we can namely consider a $\rho > 0$ such that $\mathscr{C}(\rho) > \lambda - \nu$ and then we can choose an $N \in \mathbb{N}$ such that $\mathscr{C}_n(\rho) > \lambda - \nu$ for $n > N$, simply because $\lim_{n\to\infty} \mathscr{C}_n(\rho) = \mathscr{C}(\rho)$. Finally, we can use the fact that $\lim_{r\to\infty} \mathscr{C}_n(r) = \lambda$ for each of the finitely many n's in the set $\{1, \ldots, N\}$ to find a $\rho_n > 0$ satisfying $\mathscr{C}_n(\rho_n) > \lambda - \nu$. Then because $\mathscr{C}_n$ is a nondecreasing function the inequality (55) holds true with $r^\nu = \max\{\rho, \rho_1, \ldots, \rho_N\}$.

Now, the definition of $\mathscr{C}_n(r^\nu)$ guarantees the existence of a sequence $(\boldsymbol{y}_n^\nu)_{n\in\mathbb{N}}$ of points in $\mathbb{R}^3$ satisfying

$$\left\|1_{\mathcal{B}(\boldsymbol{y}_n^\nu, r^\nu)}\psi_n\right\|_{L^2}^2 > \lambda - \nu \quad \text{for all } n \in \mathbb{N}.$$

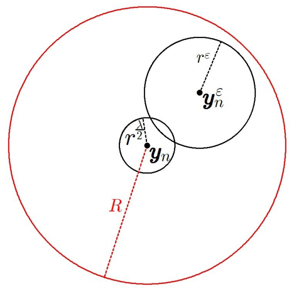


Figure 8: By setting $R = r^{\frac{\lambda}{2}} + 2r^{\varepsilon}$ we get that $\mathcal{B}(\boldsymbol{y}_n^{\varepsilon}, r^{\varepsilon}) \subset \mathcal{B}(\boldsymbol{y}_n, R)$ because the balls $\mathcal{B}(\boldsymbol{y}_n, r^{\frac{\lambda}{2}})$ and $\mathcal{B}(\boldsymbol{y}_n^{\varepsilon}, r^{\varepsilon})$ are guaranteed not to be disjoint.

Our aim will be to prove that by setting $\boldsymbol{y}_n = \boldsymbol{y}_n^{\frac{\lambda}{2}}$ for $n \in \mathbb{N}$ we obtain a sequence $(\boldsymbol{y}_n)_{n\in\mathbb{N}}$ with properties as stated in the proposition. To achieve this goal consider an arbitrary ε in the interval $\left(0, \frac{\lambda}{2}\right)$. Then for every $n \in \mathbb{N}$ both of the integrals $\int_{\mathcal{B}(\boldsymbol{y}_n, r^{\lambda/2})} |\psi_n(\boldsymbol{x})|^2 \,\mathrm{d}\boldsymbol{x}$ and $\int_{\mathcal{B}(\boldsymbol{y}_n^{\varepsilon}, r^{\varepsilon})} |\psi_n(\boldsymbol{x})|^2 \,\mathrm{d}\boldsymbol{x}$ must be strictly larger than $\frac{\lambda}{2}$, which together with the fact that $\int_{\mathbb{R}^3} |\psi_n(\boldsymbol{x})|^2 \,\mathrm{d}\boldsymbol{x} = \lambda$ gives that the balls $\mathcal{B}(\boldsymbol{y}_n, r^{\frac{\lambda}{2}})$ and $\mathcal{B}(\boldsymbol{y}_n^{\varepsilon}, r^{\varepsilon})$ have a nonempty intersection. This enables us to define $R = r^{\frac{\lambda}{2}} + 2r^{\varepsilon}$ and thereby obtain

$$\left\|1_{\mathcal{B}(\boldsymbol{y}_n, R)} \psi_n\right\|_{L^2}^2 \geq \left\|1_{\mathcal{B}(\boldsymbol{y}_n^{\varepsilon}, r^{\varepsilon})} \psi_n\right\|_{L^2}^2 > \lambda - \varepsilon,$$

for any $n \in \mathbb{N}$, which is the desired result. □

5.2 The Lower Semicontinuity Argument

We began by considering an arbitrary minimizing sequence $\big((\psi_n, \boldsymbol{A}_n)\big)_{n\in\mathbb{N}}$ for $\mathscr{E}_j^{\boldsymbol{v}}$. As we will see below our efforts in the previous section enable us to apply the direct method in the calculus of variations to the sequence of translated pairs

$$(\psi_n', \boldsymbol{A}_n') = (\psi_n \circ \tau_{\boldsymbol{y}_n}, \boldsymbol{A}_n \circ \tau_{\boldsymbol{y}_n}),$$

where $(\boldsymbol{y}_n)_{n\in\mathbb{N}}$ denotes the sequence whose existence is guaranteed in Proposition 16. Due to $\mathscr{E}_j^{\boldsymbol{v}}$'s translation invariance $\big((\psi_n', \boldsymbol{A}_n')\big)_{n\in\mathbb{N}}$ will namely be a minimizing sequence for $\mathscr{E}_j^{\boldsymbol{v}}$ and by Proposition 16 we have

$$\forall \varepsilon > 0 \,\exists R > 0 \forall n \in \mathbb{N} : \left\|1_{\mathcal{B}(\boldsymbol{0}, R)} \psi_n'\right\|_{L^2}^2 = \left\|1_{\mathcal{B}(\boldsymbol{y}_n, R)} \psi_n\right\|_{L^2}^2 \geq \lambda - \varepsilon. \qquad (56)$$

This enables us to show the existence of a minimizer for $\mathscr{E}_j^{\boldsymbol{v}}$ on $\mathcal{S}_\lambda$.

Theorem 17. *For every choice of $j \in \{\mathrm{S}, \mathrm{P}\}$, $\boldsymbol{v} \in \mathbb{R}^3$ with $0 < |\boldsymbol{v}| < c$ and $\lambda \in \Lambda_j^{\boldsymbol{v}}$ there exists a pair $(\psi, \boldsymbol{A}) \in \mathcal{S}_\lambda$ such that $\mathscr{E}_j^{\boldsymbol{v}}(\psi, \boldsymbol{A}) = I_j^{\lambda}$.*

Proof. According to Lemma 11 (together with the Sobolev inequality) the sequences $(\psi'_n)_{n\in\mathbb{N}}$, $(\boldsymbol{A}'_n)_{n\in\mathbb{N}}$ and $(\nabla\otimes\boldsymbol{A}'_n)_{n\in\mathbb{N}}$ are bounded in the reflexive Banach spaces H^1, L^6 respectively L^2. Thus, the Banach-Alaoglu theorem gives the existence of functions $\psi\in H^1$ and $\boldsymbol{A}\in L^6$ with square integrable derivatives such that (passing to subsequences)

$$\psi'_n \xrightharpoonup[n\to\infty]{} \psi \text{ in } H^1, \quad \boldsymbol{A}'_n \xrightharpoonup[n\to\infty]{} \boldsymbol{A} \text{ in } L^6 \quad \text{and} \quad \partial_\ell\boldsymbol{A}'_n \xrightharpoonup[n\to\infty]{} \partial_\ell\boldsymbol{A} \text{ in } L^2 \tag{57}$$

for $\ell\in\{1,2,3\}$. Observe that we can (after passing to yet another subsequence) assume that

$$\psi'_n \xrightarrow[n\to\infty]{} \psi \text{ and } \boldsymbol{A}'_n \xrightarrow[n\to\infty]{} \boldsymbol{A} \text{ pointwise almost everywhere in } \mathbb{R}^3 \tag{58}$$

as a consequence of (57) and the result [11, Corollary 8.7] about weak convergence implying a.e. convergence of a subsequence.

The pair $(\psi,\boldsymbol{A})$ is our candidate for a minimizer, so we begin by showing that $(\psi,\boldsymbol{A})\in\mathcal{S}_\lambda$. In this context the only nontrivial condition to check is that the identity $\|\psi\|_{L^2}^2=\lambda$ holds true. The inequality $\|\psi\|_{L^2}^2\le\lambda$ follows immediately from (57) and weak lower semicontinuity of the norm $\|\cdot\|_{L^2}$ (as expressed in [11, Theorem 2.11]). To prove the opposite inequality we let $\varepsilon>0$ be given and use (56) to choose an $R>0$ such that

$$\left\|1_{\mathcal{B}(\mathbf{0},R)}\psi'_n\right\|_{L^2}^2 \ge \lambda-\varepsilon \quad \text{for all } n\in\mathbb{N}. \tag{59}$$

By (57) and the Rellich-Kondrashov theorem [11, Theorem 8.6] the left hand side of (59) converges to $\left\|1_{\mathcal{B}(\mathbf{0},R)}\psi\right\|_{L^2}^2$ as n tends to infinity. Hence we have $\|\psi\|_{L^2}^2\ge\lambda-\varepsilon$ for any $\varepsilon>0$ and consequently $\|\psi\|_{L^2}^2\ge\lambda$. Besides giving the desired conclusion that $(\psi,\boldsymbol{A})\in\mathcal{S}_\lambda$ this also enables us to deduce that

$$\psi'_n \xrightarrow[n\to\infty]{} \psi \text{ in } L^2, \tag{60}$$

simply because the recently gained knowledge that $\|\psi\|_{L^2}^2=\|\psi'_n\|_{L^2}^2=\lambda$ gives together with (57) that

$$\|\psi-\psi'_n\|_{L^2}^2=\|\psi\|_{L^2}^2+\|\psi'_n\|_{L^2}^2-2\mathrm{Re}\big(\psi,\psi'_n\big)_{L^2} \xrightarrow[n\to\infty]{} 0.$$

Finally, we observe that by (58) the sequence $((A'_n)^\ell\psi'_n)_{n\in\mathbb{N}}$ converges pointwise almost everywhere to $A^\ell\psi$ and by the Hölder inequality it is bounded by $K_S^{\frac{3}{2}}C^{\frac{3}{2}}\lambda^{\frac{1}{4}}$ in L^2 for each $\ell\in\{1,2,3\}$. We now use that a bounded sequence of functions converging pointwise a.e. to some L^2-limit also converges weakly in L^2 to the same limit – to prove the weak convergence it suffices namely to test against C_0^∞-functions by [19, Theorem V.1.3] and thus the result follows from Egorov's theorem [19, Section 0.3]. This yields

$$(A'_n)^\ell\psi'_n \xrightharpoonup[n\to\infty]{} A^\ell\psi \text{ in } L^2,$$

which together with (57) implies that

$$\nabla_{j,\boldsymbol{A}'_n}\psi'_n \xrightharpoonup[n\to\infty]{} \nabla_{j,\boldsymbol{A}}\psi \text{ in } L^2. \tag{61}$$

The remaining task to overcome is proving that $I_j^\lambda = \mathscr{E}_j^{\boldsymbol{v}}(\psi, \boldsymbol{A})$ – or rather that $I_j^\lambda \geq \mathscr{E}_j^{\boldsymbol{v}}(\psi, \boldsymbol{A})$ since the opposite inequality is trivially true. By super-additivity of lim inf and (12) we get

$$\begin{aligned} I_j^\lambda \geq{}& \frac{1}{2m}\liminf_{n\to\infty}\|\nabla_{j,\boldsymbol{A}'_n}\psi'_n\|_{L^2}^2 + \hbar\liminf_{n\to\infty}(\psi'_n, i\boldsymbol{v}\cdot\nabla\psi'_n)_{L^2} \\ &+ \frac{1}{8\pi}\liminf_{n\to\infty}\Big(\|\nabla\otimes\boldsymbol{A}'_n\|_{L^2}^2 - \Big\|\Big(\frac{\boldsymbol{v}}{c}\cdot\nabla\Big)\boldsymbol{A}'_n\Big\|_{L^2}^2\Big), \end{aligned} \tag{62}$$

so we will have to estimate each of the terms on the right hand side of (62). That

$$\liminf_{n\to\infty}\|\nabla_{j,\boldsymbol{A}'_n}\psi'_n\|_{L^2}^2 \geq \|\nabla_{j,\boldsymbol{A}}\psi\|_{L^2}^2 \tag{63}$$

follows immediately from (61) and the weak lower semicontinuity of $\|\cdot\|_{L^2}$. Using (60), (57) and Lemma 11 gives

$$\begin{aligned} &\big|(\psi'_n, i\boldsymbol{v}\cdot\nabla\psi'_n)_{L^2} - (\psi, i\boldsymbol{v}\cdot\nabla\psi)_{L^2}\big| \\ &\leq |v|C\|\psi'_n - \psi\|_{L^2} + \sum_{\ell=1}^{3}|v^\ell||(\psi, \partial_\ell\psi'_n - \partial_\ell\psi)_{L^2}| \\ &\xrightarrow[n\to\infty]{} 0 \end{aligned}$$

and therefore

$$\liminf_{n\to\infty}(\psi'_n, i\boldsymbol{v}\cdot\nabla\psi'_n)_{L^2} = (\psi, i\boldsymbol{v}\cdot\nabla\psi)_{L^2}. \tag{64}$$

To treat the last term on the right hand side of (62) we imagine that $\boldsymbol{v}$ points in the direction of the first axis, whereby the functional

$$\mathscr{H} : D^1 \ni \boldsymbol{B} \mapsto \sqrt{\|\nabla\otimes\boldsymbol{B}\|_{L^2}^2 - \Big\|\Big(\frac{\boldsymbol{v}}{c}\cdot\nabla\Big)\boldsymbol{B}\Big\|_{L^2}^2} \in \mathbb{R}$$

simply reduces to

$$\mathscr{H}(\boldsymbol{B}) = \left\|\begin{pmatrix}\sqrt{1-\frac{\boldsymbol{v}^2}{c^2}}\partial_1 \\ \partial_2 \\ \partial_3\end{pmatrix}\otimes\boldsymbol{B}\right\|_{L^2}.$$

It is immediately apparent that $\mathscr{H}$ is a convex functional (by the triangle inequality) and $\mathscr{H}$ is continuous $D^1 \to \mathbb{R}$ since $\mathscr{H}(\boldsymbol{B}) \leq \|\nabla\otimes\boldsymbol{B}\|_{L^2}$ for all

$\boldsymbol{B} \in D^1$. Therefore we get from Mazur's theorem [4, Corollary 3.9] that $\mathscr{H}$ is weakly lower semicontinuous, which together with (57) implies that

$$\liminf_{n\to\infty}\Big(\|\nabla\otimes\boldsymbol{A}'_n\|_{L^2}^2-\Big\|\Big(\frac{\boldsymbol{v}}{c}\cdot\nabla\Big)\boldsymbol{A}'_n\Big\|_{L^2}^2\Big)\geq\|\nabla\otimes\boldsymbol{A}\|_{L^2}^2-\Big\|\Big(\frac{\boldsymbol{v}}{c}\cdot\nabla\Big)\boldsymbol{A}\Big\|_{L^2}^2, \tag{65}$$

since weak convergence of a sequence $(f_n)_{n\in\mathbb{N}}$ to some function f in the Hilbert space D^1 by the Riesz representation theorem is characterized by the limit $\big(\nabla f_n,\nabla g\big)_{L^2}\xrightarrow[n\to\infty]{}\big(\nabla f,\nabla g\big)_{L^2}$ holding true for every choice $g\in D^1$. Finally, combining (62), (63), (64) and (65) results in the inequality $I_j^\lambda\geq\mathscr{E}_j^{\boldsymbol{v}}(\psi,\boldsymbol{A})$.□

Combining Theorem 17 with Lemma 3 now gives the main result.

6 Behavior of the Energy for small velocities of the Particle

In this final section we estimate the energy of our travelling wave solutions. The next Theorem 18 shows that to leading order for small $|\boldsymbol{v}|$ the energy behaves like $\frac{m\boldsymbol{v}^2}{2}\lambda$. We interpret this as saying that there is no change in effective mass due to the electromagnetic field.

Theorem 18. *Let $j\in\{\mathrm{S},\mathrm{P}\}$ and $\lambda>0$ be given. Then there exist $\theta_j,\kappa_j>0$ (only depending on $j,\lambda,\hbar,c,Q$ and m) such that*

$$\Big|E_j(\boldsymbol{v},\psi,\boldsymbol{A})-\frac{m\boldsymbol{v}^2}{2}\lambda\Big|\leq\kappa_j|\boldsymbol{v}|^3$$

for any $\boldsymbol{v}\in\mathbb{R}^3$ with $0<|\boldsymbol{v}|<\theta_j$ and any minimizer $(\psi,\boldsymbol{A})$ of $\mathscr{E}_j^{\boldsymbol{v}}$ on $\mathcal{S}_\lambda$.

Proof. Let $j\in\{\mathrm{S},\mathrm{P}\}$, $\lambda>0$ as well as $\boldsymbol{v}\in\mathbb{R}^3$ with $0<|\boldsymbol{v}|<\Theta_{j,+}^\lambda$ be given and consider an arbitrary minimizer $(\psi,\boldsymbol{A})$ of $\mathscr{E}_j^{\boldsymbol{v}}$ on $\mathcal{S}_\lambda$. Then according to Lemma 6 and (26) the pair $(\psi,\boldsymbol{A})$ must satisfy (19). Thereby (24), (25) and Lemma 6 give that

$$\|\psi\|_{L^6}^2\leq\frac{2^6\pi^2K_S^8Q^4m^2\lambda^3}{\hbar^4}\frac{\boldsymbol{v}^4}{(\Theta_{j,+}^\lambda-|\boldsymbol{v}|)^2(|\boldsymbol{v}|-\Theta_{j,-}^\lambda)^2}$$

and

$$\|\nabla\otimes\boldsymbol{A}\|_{L^2}^2\leq\frac{2^8\pi^3K_S^6c^2Q^4m\lambda^3}{\hbar^2}\frac{\boldsymbol{v}^4}{(c^2-\boldsymbol{v}^2)^2(\Theta_{j,+}^\lambda-|\boldsymbol{v}|)(|\boldsymbol{v}|-\Theta_{j,-}^\lambda)}.$$

Using these estimates together with (16), Lemma 6, Hölder's and Sobolev's inequalities results in the inequality

$$\|\nabla_{j,\boldsymbol{A}+\frac{mc}{Q}\boldsymbol{v}}\psi\|_{L^2}^2\leq\frac{2^6\pi^2K_S^6Q^4m^2\lambda^3}{\hbar^2}\frac{\boldsymbol{v}^4\big(c^2\big(1+\sqrt{2}\big)+\boldsymbol{v}^2\big(1-\sqrt{2}\big)\big)}{(c^2-\boldsymbol{v}^2)^2(\Theta_{j,+}^\lambda-|\boldsymbol{v}|)(|\boldsymbol{v}|-\Theta_{j,-}^\lambda)}$$

and so the desired result follows immediately from the identities

$$E_{\mathrm{S}}(\boldsymbol{v},\psi,\boldsymbol{A}) = \frac{1}{2m}\|\nabla_{\mathrm{S},\boldsymbol{A}+\frac{mc}{Q}\boldsymbol{v}}\psi\|_{L^2}^2 + \frac{m\boldsymbol{v}^2}{2}\lambda - (\psi, \boldsymbol{v}\cdot\nabla_{\mathrm{S},\boldsymbol{A}+\frac{mc}{Q}\boldsymbol{v}}\psi)_{L^2} + \frac{1}{8\pi}\int_{\mathbb{R}^3}\left(\left|\left(\frac{v}{c}\cdot\nabla\right)\boldsymbol{A}\right|^2 + |\nabla\times\boldsymbol{A}|^2\right)\mathrm{d}\boldsymbol{x}\lambda$$

and

$$E_{\mathrm{P}}(\boldsymbol{v},\psi,\boldsymbol{A}) = \frac{1}{2m}\|\nabla_{\mathrm{P},\boldsymbol{A}+\frac{mc}{Q}\boldsymbol{v}}\psi\|_{L^2}^2 + \frac{m\boldsymbol{v}^2}{2}\lambda - \mathrm{Re}(\boldsymbol{\sigma}\cdot\boldsymbol{v}\psi, \nabla_{\mathrm{P},\boldsymbol{A}+\frac{mc}{Q}\boldsymbol{v}}\psi)_{L^2} + \frac{1}{8\pi}\int_{\mathbb{R}^3}\left(\left|\left(\frac{v}{c}\cdot\nabla\right)\boldsymbol{A}\right|^2 + |\nabla\times\boldsymbol{A}|^2\right)\mathrm{d}\boldsymbol{x}\lambda. \square$$

A The Poisson Equation

Given some function f the corresponding Poisson equation reads

$$-\Delta u = f. \tag{66}$$

Let us briefly recall the contents of [11, Theorem 6.21] and [11, Remark 6.21(2)]: If

$$f \in L^1_{\mathrm{loc}}(\mathbb{R}^3) \quad \text{and} \quad \int_{\mathbb{R}^3}\frac{|f(\boldsymbol{y})|}{1+|\boldsymbol{y}|}\,\mathrm{d}\boldsymbol{y} < \infty \tag{67}$$

then defining $u : \mathbb{R}^3 \to \mathbb{C}$ by

$$u(\boldsymbol{x}) = \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}f(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} \tag{68}$$

for almost every $\boldsymbol{x} \in \mathbb{R}^3$ results in a locally integrable solution of (66). Moreover, the distributional gradient ∇u can be identified with the function given by

$$\nabla u(\boldsymbol{x}) = -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{\boldsymbol{x}-\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|^3}f(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} \tag{69}$$

for almost every $\boldsymbol{x} \in \mathbb{R}^3$. We will need the following result.

Lemma 19. *If $f \in L^1 \cap L^3$ and $\nabla f \in L^1 \cap L^{\frac{5}{4}}$ then u defined by* (68) *is a D^1-function solving* (66) *in the distribution sense. Likewise, if $f \in H^1$ has compact support then u solves* (66) *and $\nabla u \in D^1$.*

Proof. Verifying the condition (67) in each of the two scenarios outlined in the statement of the lemma is an easy task, which is left for the reader – in this context it is useful to note that $\boldsymbol{y} \mapsto \frac{1}{1+|\boldsymbol{y}|}$ is e.g. an L^6-function. Thus, the function defined almost everywhere by (68) is indeed a solution of the Poisson equation in those two cases.

Suppose that $f \in L^1 \cap L^3$ and $\nabla f \in L^1 \cap L^{\frac{5}{4}}$. Then we first show that u vanishes at infinity: For this let $\mathcal{N}$ denote the null set on which the identities (68) and (69) do not hold true. Then for each sequence $(\boldsymbol{x}_k)_{k\in\mathbb{N}}$ of elements in $\mathbb{R}^3 \setminus \mathcal{N}$ with $|\boldsymbol{x}_k| \xrightarrow[k\to\infty]{} \infty$ we have

$$|u(\boldsymbol{x}_k)| \leq \frac{1}{4\pi}\left\|1_{\mathcal{B}(\boldsymbol{x}_k,1)}f\right\|_{L^3}\left\|\frac{1_{\mathcal{B}(\boldsymbol{0},1)}}{|\cdot|}\right\|_{L^{\frac{3}{2}}} + \frac{1}{4\pi}\left\|\frac{(1_{\mathbb{R}^3\setminus\mathcal{B}(\boldsymbol{x}_k,1)}f)(\cdot)}{|\boldsymbol{x}_k - \cdot|}\right\|_{L^1} \xrightarrow[k\to\infty]{} 0,$$

where we split the integral involved in the expression for $u(\boldsymbol{x}_k)$ into a contribution from $\mathcal{B}(\boldsymbol{x}_k, 1)$ as well as a contribution from $\mathbb{R}^3 \setminus \mathcal{B}(\boldsymbol{x}_k, 1)$ and treat these by means of the Hölder inequality and Lebesgue's dominated convergence theorem. In order to prove that ∇u is square integrable we use the Hölder inequality and Tonelli's theorem to get

$$\begin{aligned}
&\int_{\mathbb{R}^3} |\nabla u(\boldsymbol{x})|^2 \,\mathrm{d}\boldsymbol{x}\\
&\leq \frac{1}{8\pi^2}\int_{\mathbb{R}^3}\Big(\int_{\mathcal{B}(\boldsymbol{x},1)}\frac{|f(\boldsymbol{y})|}{|\boldsymbol{x}-\boldsymbol{y}|^2}\,\mathrm{d}\boldsymbol{y}\Big)^2\,\mathrm{d}\boldsymbol{x} + \frac{1}{8\pi^2}\int_{\mathbb{R}^3}\Big(\int_{\mathbb{R}^3\setminus\mathcal{B}(\boldsymbol{x},1)}\frac{|f(\boldsymbol{y})|}{|\boldsymbol{x}-\boldsymbol{y}|^2}\,\mathrm{d}\boldsymbol{y}\Big)^2\,\mathrm{d}\boldsymbol{x}\\
&\leq \frac{1}{8\pi^2}\int_{\mathbb{R}^3}\int_{\mathcal{B}(\boldsymbol{x},1)}\frac{|f(\boldsymbol{y})|}{|\boldsymbol{x}-\boldsymbol{y}|^{\frac{5}{2}}}\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x}\left\|\frac{1_{\mathcal{B}(\boldsymbol{0},1)}}{|\cdot|}\right\|_{L^{\frac{5}{2}}}^{\frac{3}{2}}\|f\|_{L^1}^{\frac{1}{10}}\|f\|_{L^3}^{\frac{9}{10}}\\
&+\frac{1}{8\pi^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3\setminus\mathcal{B}(\boldsymbol{x},1)}\frac{|f(\boldsymbol{y})|}{|\boldsymbol{x}-\boldsymbol{y}|^{\frac{7}{2}}}\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x}\left\|\frac{1_{\mathbb{R}^3\setminus\mathcal{B}(\boldsymbol{0},1)}}{|\cdot|}\right\|_{L^{\frac{7}{2}}}^{\frac{1}{2}}\|f\|_{L^3}^{\frac{3}{14}}\|f\|_{L^1}^{\frac{11}{14}}\\
&\leq \frac{1}{8\pi^2}\left\|\frac{1_{\mathcal{B}(\boldsymbol{0},1)}}{|\cdot|}\right\|_{L^{\frac{5}{2}}}^4\|f\|_{L^1}^{\frac{11}{10}}\|f\|_{L^3}^{\frac{9}{10}} + \frac{1}{8\pi^2}\left\|\frac{1_{\mathbb{R}^3\setminus\mathcal{B}(\boldsymbol{0},1)}}{|\cdot|}\right\|_{L^{\frac{7}{2}}}^4\|f\|_{L^3}^{\frac{3}{14}}\|f\|_{L^1}^{\frac{25}{14}}
\end{aligned}$$

and so we conclude that $u \in D^1$.

Assume now that $f \in H^1$ has support in some ball $\mathcal{B}(\boldsymbol{0}, r_f)$. Then for all $\boldsymbol{x} \in \mathbb{R}^3 \setminus \mathcal{N}$ with $|\boldsymbol{x}| > r_f$ we have

$$|\nabla u(\boldsymbol{x})| \leq \frac{1}{4\pi(|\boldsymbol{x}| - r_f)^2}\Big(\frac{4}{3}\pi r_f^3\Big)^{\frac{1}{2}}\|f\|_{L^2}$$

whereby we deduce that ∇u vanishes at infinity. Combining the change of variables $\boldsymbol{z} = \boldsymbol{x} - \boldsymbol{y}$ with a naive differentiation under the integral sign in (69) suggests that

$$\partial_j\partial_k u(\boldsymbol{x}) = -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{x_k - y_k}{|\boldsymbol{x}-\boldsymbol{y}|^3}\partial_j f(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} \tag{70}$$

for $j, k \in \{1, 2, 3\}$ and almost every $\boldsymbol{x} \in \mathbb{R}^3$. Under the assumption that f has square integrable first derivatives and compact support, the right hand side of

(a) $|\boldsymbol{x}-\boldsymbol{y}|<3r_f$ (b) $|\boldsymbol{x}-\boldsymbol{y}|>r_f$

Figure 9: Estimates on the distance from $\boldsymbol{x}$ to any $\boldsymbol{y} \in \operatorname{supp} f \subset \mathcal{B}(\mathbf{0}, r_f)$ in the cases $\boldsymbol{x} \in \mathcal{B}(\mathbf{0}, 2r_f)$ respectively $\boldsymbol{x} \in \mathbb{R}^3 \setminus \mathcal{B}(\mathbf{0}, 2r_f)$. Both estimates follow from the triangle inequality in $\mathbb{R}^3$.

(70) is indeed well defined almost everywhere in $\mathbb{R}^3$: The function $\boldsymbol{y} \mapsto \frac{|\partial_j f(\boldsymbol{y})|}{|\boldsymbol{x}-\boldsymbol{y}|^2}$ (and thereby also $\boldsymbol{y} \mapsto \frac{x_k - y_k}{|\boldsymbol{x}-\boldsymbol{y}|^3}\partial_j f(\boldsymbol{y})$) must namely be integrable for almost all $\boldsymbol{x} \in \mathcal{B}(\mathbf{0}, 2r_f)$, since Tonelli's theorem, the Cauchy-Schwarz inequality and the basic observation stated on Figure 9(a) give that

$$\int_{\mathcal{B}(\mathbf{0},2r_f)} \int_{\mathcal{B}(\mathbf{0},r_f)} \frac{|\partial_j f(\boldsymbol{y})|}{|\boldsymbol{x}-\boldsymbol{y}|^2} \,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x} \leq \left(\frac{4}{3}\pi r_f^3\right)^{\frac{1}{2}} \left\| 1_{\mathcal{B}(\mathbf{0},3r_f)} \frac{1}{|\cdot|} \right\|_{L^2}^2 \|\partial_j f\|_{L^2}. \quad (71)$$

On the other hand $\boldsymbol{y} \mapsto \frac{|\partial_j f(\boldsymbol{y})|}{|\boldsymbol{x}-\boldsymbol{y}|^2}$ is majorized by the integrable function $\frac{|\partial_j f|}{r_f^2}$ for all $\boldsymbol{x} \in \mathbb{R}^3 \setminus \mathcal{B}(\mathbf{0}, 2r_f)$, as shown on Figure 9(b). Thus, it makes sense to consider the function given (a.e.) by the expression on the right hand side of (70) – the identification of this function with the distribution $\partial_j\partial_k u$ then follows from a standard argument utilizing Fubini's theorem (which is outlined in the proof of [11, Theorem 6.21]). It just remains to be proven that the functions $\partial_j\partial_k u$ are square integrable – we first verify this square integrability on the ball $\mathcal{B}(\mathbf{0}, 2r_f)$. This is done by using the Hölder inequality, the Jensen inequality and the Hardy-Littlewood-Sobolev inequality:

$$\begin{aligned}
&\int_{\mathcal{B}(\mathbf{0},2r_f)} |\partial_j\partial_k u(\boldsymbol{x})|^2 \,\mathrm{d}\boldsymbol{x} \\
&\leq \frac{1}{16\pi^2}\int_{\mathcal{B}(\mathbf{0},2r_f)} \Big(\int_{\mathcal{B}(\mathbf{0},r_f)} \frac{|\partial_j f(\boldsymbol{y})|^{\frac{6}{5}}}{|\boldsymbol{x}-\boldsymbol{y}|^{\frac{9}{4}}} \mathrm{d}\boldsymbol{y}\Big)^{\frac{10}{9}} \Big(\int \frac{1_{\mathcal{B}(\mathbf{0},3r_f)}(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^{\frac{27}{10}}} \mathrm{d}\boldsymbol{y}\Big)^{\frac{5}{9}} \|\partial_j f\|_{L^2}^{\frac{2}{3}} \mathrm{d}\boldsymbol{x} \\
&\leq \frac{\left(\frac{4}{3}\pi r_f^3\right)^{\frac{1}{9}}}{16\pi^2} \int_{\mathcal{B}(\mathbf{0},2r_f)} \int_{\mathcal{B}(\mathbf{0},r_f)} \frac{|\partial_j f(\boldsymbol{y})|^{\frac{4}{3}}}{|\boldsymbol{x}-\boldsymbol{y}|^{\frac{5}{2}}} \,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x} \left\| 1_{\mathcal{B}(\mathbf{0},3r_f)} \frac{1}{|\cdot|} \right\|_{L^{\frac{27}{10}}}^{\frac{3}{2}} \|\partial_j f\|_{L^2}^{\frac{2}{3}} \\
&\leq \frac{\left(\frac{4}{3}\pi r_f^3\right)^{\frac{1}{9}}}{8\pi^2} \left(\frac{4}{3}\pi\right)^{\frac{5}{6}} \left(\left(\frac{5}{3}\right)^{\frac{5}{6}} + \left(\frac{5}{2}\right)^{\frac{5}{6}}\right) \left(\frac{4}{3}\pi(2r_f)^3\right)^{\frac{1}{2}} \left\| 1_{\mathcal{B}(\mathbf{0},3r_f)} \frac{1}{|\cdot|} \right\|_{L^{\frac{27}{10}}}^{\frac{3}{2}} \|\partial_j f\|_{L^2}^2.
\end{aligned}$$

Finally, the Cauchy-Schwarz inequality, Tonelli's theorem and the observation on Figure 9(b) give

$$\int_{\mathbb{R}^3\setminus\mathcal{B}(\mathbf{0},2r_f)} |\partial_j\partial_k u(\boldsymbol{x})|^2 \,\mathrm{d}\boldsymbol{x} \le \frac{1}{16\pi^2}\Big(\frac{4}{3}\pi r_f^3\Big)\left\|1_{\mathbb{R}^3\setminus\mathcal{B}(\mathbf{0},r_f)}\frac{1}{|\cdot|}\right\|_{L^4}^4 \|\partial_j f\|_{L^2}^2,$$

whereby $\partial_j\partial_k u$ is also square integrable on $\mathbb{R}^3\setminus\mathcal{B}(\mathbf{0},2r_f)$. Consequently, ∇u is a D^1-function. □

Remark 20. Consider a locally integrable, harmonic function u with square integrable first derivatives. The harmonicity of ∇u ensures the existence of vector fields $\boldsymbol{p}_m$ on $\mathbb{R}^3$ with homogeneous harmonic polynomials of degree m as coordinates such that

$$\nabla u(\boldsymbol{x}) = \sum_{m=0}^{\infty} \boldsymbol{p}_m(\boldsymbol{x})$$

for all $\boldsymbol{x}\in\mathbb{R}^3$ (see [1, Corollary 5.34 and Proposition 1.30]). The series even converges absolutely and uniformly on compact subsets of $\mathbb{R}^3$ so for an arbitrary given $R>0$ we have the series representation $\nabla u = \sum_{m=0}^{\infty}\boldsymbol{p}_m$ in $\big[L^2\big(\overline{\mathcal{B}}(\mathbf{0},R)\big)\big]^3$. Integrating in polar coordinates and using the homogeneity of the functions $\boldsymbol{p}_m$ as well as the spherical harmonic decomposition [1, Theorem 5.12] of $L^2(\partial\mathcal{B}(\mathbf{0},1))$ now gives

$$\begin{aligned}\big\|1_{\overline{\mathcal{B}}(\mathbf{0},R)}\nabla u\big\|_{L^2}^2 &= \sum_{m=0}^{\infty}\sum_{\ell=0}^{\infty}\int_0^R r^{m+\ell+2}\,\mathrm{d}r\big(\boldsymbol{p}_m,\boldsymbol{p}_\ell\big)_{L^2(\partial\mathcal{B}(\mathbf{0},1))}\\ &= \sum_{m=0}^{\infty}\frac{R^{2m+3}}{2m+3}\|\boldsymbol{p}_m\|_{L^2(\partial\mathcal{B}(\mathbf{0},1))}^2. \end{aligned} \tag{72}$$

By Lebesgue's dominated convergence theorem the left hand side of (72) converges to $\|\nabla u\|_{L^2}^2$ as $R\to\infty$ so the same must be true for the right hand side. But the right hand side simply can not converge as $R\to\infty$ unless $\boldsymbol{p}_m\equiv\mathbf{0}$ for all $m\in\mathbb{N}_0$ – so we conclude that $\nabla u\equiv\mathbf{0}$. Therefore u is a constant function – consequently, the Poisson equation can at most have one solution in the space D^1.

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