PhD Thesis

Harmonic Analysis on Triple Spaces

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PhD thesis submitted to the PhD School of Science, Faculty of Science, University of Copenhagen, Denmark in May 2013.

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Day of defense:	June 19, 2013

ISBN 978-87-7078-991-2

Abstract

In this thesis we study examples of triple spaces, both their structure theory, their invariant differential operators as well as analysis on them. The first major results provide us with some examples of triple spaces which are strongly spherical, i.e. satisfy some conditions reminiscent of properties of symmetric spaces. The algebras of invariant differential operators for these spaces are studied and the conclusion is that most of them are non-commutative. Finally, we restrict our attention to a single triple space, giving a specific polar decomposition and corresponding integration formula, and studying the relations between open orbits of parabolic subgroups, multiplicities and distribution vectors.

Resumé

Fokus for denne afhandling er tripelrum, det være sig deres strukturteori, deres invariante differentialoperatorer, såvel som analyse på dem. Det første hovedresultat giver os en række eksempler på triplerum, som er stærkt sfæriske, dvs. opfylder nogle betingelser, der tilsvarer nogle egenskaber ved symmetriske rum. Algebraen af invariante differentialoperatorer for disse rum studeres, og det vises, at de fleste af dem er ikke-kommutative. Til sidst fokuserer vi udelukkende på et enkelt tripelrum, hvor vi giver en specifik polardekomposition og tilhørende integrationsformel, og studerer sammenhængen mellem åbne baner af parabolske undergrupper, multipliciteter og distributionsvektorer.

CONTENTS

PREFACE

The harmonic analysis on semisimple/reductive symmetric spaces is well understood thanks to the effort of Helgason, Flensted-Jensen, Oshima, Matsuki, van den Ban, Schlichtkrull, Delorme and others which again rests on the work of Harish-Chandra. However, the requirement on symmetric spaces - the existence of an involution - seems unnatural, of more like a technical rather than a geometric requirement on the space. Two properties seem to be of great importance for symmetric spaces, namely sphericality and polarity - properties of a more geometric nature than the existence of an involution. Thus, the overall idea is to generalize the Plancherel formula from symmetric spaces to spaces which are only spherical and polar. This, however, seems to be an overwhelming task and by no means the scope of this thesis. Here we will focus on certain examples of spherical and polar spaces, to see which phenomena we can expect or not expect to occur in general.

The thesis is divided into three chapters as follows: The first chapter contains background material on more or less advanced topics from representation theory and harmonic analysis. The results obtained during the studies are described in the following 2 chapters. In Chapter 2, we study triple spaces in as much generality as we can (the number of spherical and polar triple spaces turns out to be quite limited). The main result here is a polar decomposition for certain triple spaces, a result obtained jointly with Schlichtkrull and Krötz. The resulting paper [10] is attached as an appendix. Finally, in Chapter 3, we study a particular example of a triple space. Here we are able to determine the Plancherel measure (up to absolute continuity) and describe a relation between the multiplicities of the different representations and the number of open orbits of the parabolic subgroup from which this representation is induced. Also the algebra of invariant differential operators is determined.

Acknowledgements
I am indebted to many people who have helped, aided or otherwise supported me during the past 4 years of ph.d.-studies. Where to begin?

First, I would like to thank Mogens Flensted-Jensen and Bernhard Krötz for fruitful discussions. I would also like to convey my thanks to "Rejselegat for matematikere" for giving me the wonderful opportunity of traveling one year abroad. In this connection I would like thank heartily Prof. Karl-Hermann Neeb from the University of Erlangen, and Prof. Sigurdur Helgason from MIT for hosting me at my stays at their respective institutes!

A special thank goes to my (now former) fellow ph.d.-students, Adam, Hannes, Henrik, and Tim for many joyous moments. We had so much fun!

And finally, I wish to express my deep gratitude to Henrik Schlichtkrull, my supervisor, for his aid, his patience, and his inspiration!

Copenhagen, June 2013. *Thomas Hjortgaard Danielsen.*

CHAPTER 1

PRELIMINARIES

In this first chapter we recall different notions and concepts from representation theory and harmonic analysis that will be used throughout this thesis. I make no claims to novelty in this chapter.

Most results are stated without proofs (indeed, some of the theorems mentioned are so profound, that in-depth proofs would make up a thesis of its own) but references to relevant literature are given. In a few cases, where I was unable to find a proper reference, I have added my own proofs (most notably the proof of Proposition 1.36 and of the lemmas preceding it), and in a few cases I have added a proof from the literature, if I thought it to be an illustrative application of a previously stated theorem (e.g. the proof of Lemma 1.42).

1.1 Topological Vector Spaces

Definition 1.1. A *topological vector space* V is a vector space over a field $\mathbb{K}$ equipped with a topology such that addition $V \times V \longrightarrow V$ and scalar multiplication $\mathbb{K} \times V \longrightarrow V$ are continuous maps.

Obviously, the definition implies that the translation map $v \longmapsto v+a$ for some fixed vector a is a homeomorphism. Thus, the topology of V is determined solely by the system of neighborhoods of 0, and a linear map between two topological vector spaces is continuous if and only if it is continuous at 0.

A topological vector space is called *locally convex* if there exists a neighborhood basis of 0 consisting of convex sets. An obvious example is a normed space, where the system $(B_r(0))$ of balls of radius $r > 0$ is a convex neighborhood basis. More generally if $\mathcal{P}$ is a system of seminorms on a vector space V, such that for each seminorm $p \in \mathcal{P}$, there exists another seminorm $q \in \mathcal{P}$ and a constant $c > 0$, such that

$$\forall v \in V : \; p(v) \leq cq(v), \tag{1.1}$$

then the system $(B_r^p(0))_{p\in\mathcal{P},r>0}$ of balls is a neighborhood basis of 0 and by translation defines a locally convex topology on V. The converse also holds: if V is a locally convex topological vector space, then there exists a family of seminorms satisfying (1.1), which generates the topology on V (cf. Proposition 7.6 of [30]).

In other words: a locally convex topology on a vector space means a topology generated by seminorms.

If there a exists a countable family $(p_n)_{n\in\mathbb{N}}$ of seminorms generating the topology (equivalently, if there exists a countable neighborhood basis of 0), we can define a metric, by

$$d(v, w) := \sum_{n\in\mathbb{N}} \frac{1}{2^n} \frac{p_n(v-w)}{1 + p_n(v-w)}$$

and this metric, which is obviously translation invariant, defines the topology on the vector space. The space is called a *Fréchet space* if it is complete as a metric space. One example of a Fréchet space is the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. The classical Schwartz space allows several generalizations to Schwartz spaces on a group G. Here we outline the definition of a Schwartz space as given in [5] in Section 2.5: First, pick an inner product on T_eG and extend this to a left-invariant Riemannian metric on G. The metric gives rise to a distance function $d(g_1, g_2)$ (note that a change in the inner product in T_eG just amounts to a scaling, and thus the resulting distance functions will just differ by a factor). Define the *scale function* : $s(g) := \exp(d(g, e))$. Then we define the *Schwartz space* (or rather the *L^1-Schwartz space*)

$$\mathcal{S}(G) := \{f \in C^\infty(G) \mid \forall D \in U(\mathfrak{g}_\mathbb{C}) \, \forall n \in \mathbb{N} : \; s(g)^n Df(g) \in L^1(G)\}. \tag{1.2}$$

The Schwartz space becomes a Fréchet space by equipping it with the topology generated by the seminorms $\|f\|_{n,D} := \|s(g)^n Df(g)\|_{L^1}$ for $n \in \mathbb{N}$ and $D \in U(\mathfrak{g})$. We return to this space later.

For a given topological vector space we denote by V^* the *algebraic dual* of V, i.e. the vector space of linear maps $V \longrightarrow \mathbb{K}$. By V' we denote the *continuous dual* , i.e. the subspace of V^* consisting of continuous linear functionals. The continuous dual can be equipped with several different vector space topologies. The *weak topology*, is the topology given by the seminorms $(p_v)_{v \in V}$ where $p_v(\varphi) := |\varphi(v)|$. This is the weakest vector space topology on V' for which it holds that the dual of V' equals V (as a vector space) in the sense that the canonical map $V \longrightarrow (V')'$ is surjective. The strongest topology on V' for which it holds that $(V')' = V$ is called the *Mackey topology* . A third important topology is the *strong topology* which is the topology of uniform convergence on bounded subsets. In general, the strong topology is stronger than the Mackey topology, i.e. the dual of V' is larger than V. In the case where the strong topology and the Mackey topology coincide, i.e. when $(V')' = V$, we say that V is a *semi-reflexive* space . If the identity holds as topological vector spaces when $(V')'$ is given the strong dual topology, then we say that V is *reflexive* .

A matter which is even more delicate than giving the dual space a topology, is that of topologizing a tensor product of two topological vector spaces. Given two locally convex topological vector spaces V and W we denote by $V \otimes W$ the *algebraic tensor product* of V and W, i.e. the space of finite linear combinations of elements of the form $v \otimes w$. There are two "extreme" topologies on the tensor product: The first is the *projective topology* or *π-topology* we understand the strongest vector space topology on $V \otimes W$ making the canonical bilinear map $V \times W \longrightarrow V \otimes W$ continuous. $V \otimes W$ equipped with this topology will be denoted $V \otimes_\pi W$ and the completion of this space by $V \widehat{\otimes}_\pi W$. By [30] Proposition 43.4 the projective tensor product satisfies the following *topological* universal property: any continuous bilinear form $\psi : V \times W \longrightarrow X$ to some topological vector space X extends uniquely to a continuous linear map $\overline{\psi} : V \otimes_\pi W \longrightarrow X$. The projective topology on $V \otimes W$ is the unique topology with this property.

If $\varphi_1 \in V'$ and $\varphi_2 \in W'$ (the continuous linear duals), then $\varphi_1 \otimes \varphi_2$ given by

$$(\varphi_1 \otimes \varphi_2)(v, w) := \varphi_1(v)\varphi_2(w)$$

is a linear functional on $V \otimes W$, and by the universal property this functional is indeed continuous if the tensor product is equipped with the projective topology, i.e. $\varphi_1 \otimes \varphi_2 \in (V \otimes_\pi W)'$.

By construction, the projective topology is in some sense the strongest topology we could meaningfully put on a tensor product. At the other end of the scale we find the *injective* topology or *ε-topology* which is the weakest vector space topology on $V \otimes W$ such that linear functionals of the form $\varphi_1 \otimes \varphi_2$ are continuous. These were already found to be continuous in the projective topology, hence the injective topology is weaker than the projective topology. By $V \otimes_\varepsilon W$ and $V \widehat{\otimes}_\varepsilon W$ we denote the tensor product equipped with the ε-topology resp. the completion of this. The identity map on the algebraic tensor product induces an injection $V \widehat{\otimes}_\pi W \longrightarrow V \widehat{\otimes}_\varepsilon W$.

A locally convex topological space V is called *nuclear* if for any locally convex vector space W this injection is a homeomorphism. In other words V is nuclear if and only if the injective and projective topology coincide. Examples of nuclear spaces are $\mathscr{D}(M)$, $\mathcal{S}(\mathbb{R}^n)$, $\mathcal{S}(G)$, $C^\infty(M)$ (for M a manifold) as well as their subspaces and their strong duals (cf. [30] p. 530 as well as Proposition 50.1 and Proposition 50.6). Taking the strong continuous dual of a tensor product where at least one of the spaces is nuclear is very easy (cf. *loc. cit.* Proposition 50.7):

$$(V \widehat{\otimes} W)' = V' \widehat{\otimes} W'. \tag{1.3}$$

Here V' and W' are both equipped with the strong dual topology.

If V and W happen to be Hilbert spaces, we have a natural Hilbert space topology on $V \otimes W$, namely that given by the inner product

$$\langle v_1 \otimes w_2, v_2 \otimes w_2 \rangle := \langle v_1, v_2 \rangle \langle w_1, w_2 \rangle.$$

We denote the completion of the tensor product of two Hilbert spaces simply by $V \widehat{\otimes} W$ underlining that this choice of topology is the canonical one. However, in general, this topology is neither equal to the projective nor the injective topology, it is somewhere in between. In fact, a Hilbert space (or even a normed space) is nuclear if and only if it is finite-dimensional. In particular the topological universal property fails to hold in general.

1.2 Basic Representation Theory

Definition 1.2 (**Representation**)**.** A *continuous representation* or just a *representation* of a Lie group G on a topological vector space V_π is a group homomorphism $\pi : G \longrightarrow \mathrm{Aut}(V_\pi)$ (where $\mathrm{Aut}(V_\pi)$ is the group of linear bijections $V_\pi \xrightarrow{\sim} V_\pi$) such that the map $(g, v) \longmapsto \pi(g)v$ is continuous $G \times V_\pi \longrightarrow V_\pi$. We will often refer to the space V_π as a *G-module.*

The definition obviously implies that $\pi(g)$ is a linear homeomorphism of V_π to itself.

Definition 1.3. A representation is called *irreducible* if the only *closed* invariant subspaces are the trivial ones.

A vector $v \in V_\pi$ is called *smooth* , if the map $g \longmapsto \pi(g)v$ is a smooth map $g \longmapsto \pi(g)v$. By V_π^∞ we denote the space of smooth vectors. The space can be embedded into $C^\infty(G, V_\pi)$ (the vector v representing the map $g \longmapsto \pi(g)v$ which was smooth by definition). The latter space is given the topology of uniform convergence of functions and its derivatives over compact subsets of G, and V_π^∞ is given the subspace topology. It is not hard to see that V_π^∞ is a G-invariant subspace of V_π (albeit not a closed one) and that π restricts to a continuous representation of G on V_π^∞.

If $V_\pi = \mathcal{H}_\pi$ is a Hilbert space, π is called *unitary* if $\pi(g)$ is a unitary map for each g. For a Lie group G we denote by $\widehat{G}$ the set of equivalence classes of irreducible unitary representations. We will reserve the notation $\mathcal{H}_\pi$ for infinite-dimensional Hilbert spaces. For a finite-dimensional representation we will use V_π.

Given two groups G_1 and G_2, and two unitary representations π_1 and π_2 of G_1 and G_2 respectively we form the so-called *outer product* $\pi_1 \times \pi_2$ of $G_1 \times G_2$ on $\mathcal{H}_{\pi_1} \widehat{\otimes} \mathcal{H}_{\pi_2}$ given by

$$(\pi_1 \times \pi_2)(g_1, g_2) v_1 \otimes v_2 := (\pi_1(g_1)v_1) \otimes (\pi_2(g_2)v_2)$$

(this is sometimes also denoted $\pi_1 \boxtimes \pi_2$ but we will not use this notation here). If $G_1 = G_2$, the restriction to the diagonal is called the *tensor product* and denoted $\pi_1 \otimes \pi_2$ [1]:

$$(\pi_1 \otimes \pi_2)(g) v_1 \otimes v_2 := (\pi_1(g)v_1) \otimes (\pi_2(g)v_2).$$

Note that the canonical unitary map $\mathcal{H}_{\pi_1} \widehat{\otimes} \mathcal{H}_{\pi_2} \xrightarrow{\sim} \mathcal{H}_{\pi_2} \widehat{\otimes} \mathcal{H}_{\pi_1}$ intertwines $\pi_1 \otimes \pi_2$ and $\pi_2 \otimes \pi_1$ which are thus unitarily equivalent, unlike $\pi_1 \times \pi_2$ and $\pi_2 \times \pi_1$ which are usually *not* necessarily equivalent.

In the unitary case, the topology on $\mathcal{H}_\pi^\infty$ is a Fréchet topology and it is generated by the seminorms

$$\|x\|_U := \|Ux\|_{\mathcal{H}}$$

for $U \in U(\mathfrak{g}_\mathbb{C})$, the universal enveloping algebra of $\mathfrak{g}_\mathbb{C}$. By a theorem of Gaarding, $\mathcal{H}_\pi^\infty$ is dense in $\mathcal{H}_\pi$.

In what follows, our focus will be almost exclusively on unitary representations. The main source of non-unitary representations will be restrictions of

[1] Generally in the literature $\pi_1 \otimes \pi_2$ is used both for the tensor product and the outer product construction. However, at certain places in this thesis we need to distinguish them, and therefore I found it prudent to have two different notations for them.

unitary representations to the space of smooth vectors or extensions to *distribution vectors* : If $(\pi, \mathcal{H}_\pi)$ is a unitary representation, then by $\mathcal{H}_\pi^{-\infty}$ we denote the continuous linear dual of $\overline{\mathcal{H}}_\pi^\infty$, i.e. the space of continuous conjugate linear functionals $\mathcal{H}_\pi^\infty \longrightarrow \mathbb{C}$. The elements in this space are called *distribution vectors*. The space is equipped with the strong dual topology and is turned into a G-representation space by the *dual representation* of $\mathcal{H}_\pi^\infty$, i.e.

$$(\pi(g)\eta)(v) := \eta(\pi(g^{-1})v).$$

Since $\overline{\mathcal{H}}_\pi^\infty \subseteq \overline{\mathcal{H}}_\pi$ we get by dualizing (using the fact that the dual of $\overline{\mathcal{H}}_\pi$ is $\mathcal{H}_\pi$): $\mathcal{H}_\pi \subseteq \mathcal{H}_\pi^{-\infty}$, and hence we obtain the following string of (topological) inclusions

$$\mathcal{H}_\pi^\infty \subseteq \mathcal{H}_\pi \subseteq \mathcal{H}_\pi^{-\infty}.$$

This explains the conjugation in the definition of distribution vectors.

For the remainder of this section we assume G to be a semisimple Lie group with finite center and with maximally compact subgroup K. We turn now to the definition of some *very* important representations, namely the so-called *principal series representations* . Let $P = MAN$ be the Langlands decomposition of a parabolic subgroup (not necessarily a minimal one), let ξ be an irreducible unitary representation of M on some Hilbert space $\mathcal{H}_\xi$ and let $\lambda \in \mathfrak{a}_\mathbb{C}^*$. Then we define a representation (ξ, λ) of P on $\mathcal{H}_\xi$ by [2]

$$(\xi, \lambda)(man) := a^{\lambda+\rho}\xi(m)$$

and we induce from this a representation $\pi_{P,\xi,\lambda}$ of G. To be more specific this representation is constructed as follows: Consider the space of continuous functions $f : G \longrightarrow \mathcal{H}_\xi$ with the following equivariance $f(xman) = a^{-(\lambda+\rho)}\xi(m)^{-1}f(x)$. Equip it with the norm

$$\|f\| := \int_K |f(k)|^2 dk,$$

which is indeed a norm on the given space, since a function in there is determined completely by its behavior on K (because of the decomposition $G = KMAN$ and the equivariance above). Let $\mathcal{H}_{P,\xi,\lambda}$ denote the norm closure and define the continuous representation $\pi_{P,\xi,\lambda}$ on this space by

$$(\pi_{P,\xi,\lambda}(g)f)(x) := f(g^{-1}x).$$

[2]Here $\rho = \frac{1}{2}\sum_{\alpha\in\Sigma^+(\mathfrak{g},\mathfrak{a})}(\dim \mathfrak{g}_\alpha)\alpha$ is half the sum of the positive roots for the root system $\Sigma(\mathfrak{g}, \mathfrak{a})$.

The representation is unitary if λ is imaginary.

Not all of the principal series representations are irreducible, and they are not all inequivalent. It is a delicate issue, and here we only focus on the case where $P = MAN$ is a minimal parabolic and λ imaginary. Bruhat showed that the dimension of the space of self-intertwiners of the representation $\pi_{P,\xi,\lambda}$ is bounded by $|W_{\xi,\lambda}|$ [3] where

$$W_{\xi,\lambda} = \{w \in W(G,A) \mid w\xi = \xi,\ w\lambda = \lambda\}$$

and $W(G,A) = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ is the Weyl group. Since the Weyl group is generated by root reflections of simple roots, it follows that there exists a non-trivial w mapping λ to itself, if and only if there exists a root α such that $\langle \lambda, \alpha \rangle = 0$. In particular, $\pi_{P,1,\lambda}$ (these are the so-called *spherical principal series representations*) is reducible if and only if there exists a root α such that $\langle \lambda, \alpha \rangle = 0$. The set of such λ's is clearly a set of measure 0 in ß$\mathfrak{a}$. Moreover two representations π_{P,ξ_1,λ_1} and π_{P,ξ_2,λ_2} are equivalent if and only if there exists a $w \in W(G,A)$ such that $\xi_2 = w\xi_1$ and $\lambda_2 = w\lambda_1$. The construction of an actual intertwiner is very involved as it is first constructed for λ in a certain subset of $\mathfrak{a}^*_{\mathbb{C}}$ (which does not necessarily contain $i\mathfrak{a}$) and then meromorphically extended.

We shall return to the principal series representations in a later section, when we discuss the representation theory of $SL(2,\mathbb{R})$.

1.3 K-Finite Vectors

We retain the notation from the previous section and let G be a semisimple Lie group with finite center, and let K be its maximally compact subgroup. Let π be a representation of G on some Hilbert space $\mathcal{H}_\pi$. For $\delta \in \widehat{K}$ we denote by $\mathcal{H}_\pi[\delta]$ the space of *K-finite vectors of δ type*, i.e. vectors v for which the K-module $\operatorname{span}\{\pi(K)v\}$ is equivalent to $\delta^{\oplus n}$ for some finite n. If $\pi|_K$ is unitary we can, by Peter-Weyl, decompose $\pi|_K$ into K-types

$$\mathcal{H}_\pi \cong \widehat{\bigoplus_{\delta \in \widehat{K}}} V_\delta^{\oplus n_\delta} \tag{1.4}$$

where $n_\delta \in \mathbb{N}_0 \cup \{\infty\}$. Under this isomorphism $\mathcal{H}_{\pi,\delta}$ is simply $V_\delta^{\oplus n_\delta}$.

Definition 1.4 (Admissible representation). A representation π on a Hilbert space $\mathcal{H}_\pi$ for which $\pi|_K$ is unitary and which allows a decomposition (1.4) with $n_\delta < \infty$ for all $\delta \in \widehat{K}$, is called an *admissible* representation.

[3] See Theorem 7.2 of [21].

This notion was introduced by Harish-Chandra. Admissible representations constitute a particularly well-behaved class of representations, where a lot of information can be extracted from its corresponding infinitesimal Lie algebra representation (or rather of its corresponding $(\mathfrak{g}, K)$-module - a notion to be defined shortly). Harish-Chandra showed that all irreducible unitary representations of G are indeed admissible ([21] Theorem 8.1). However, irreducible non-unitary representations need not be admissible, as was proved by Soergel [29].

Let

$$\mathcal{H}_{\pi,K} := \bigoplus_{\delta \in \widehat{K}} \mathcal{H}_\pi[\delta]$$

denote the space of K-finite vectors (note that the direct sum here is algebraic!). From (1.4) it follows that $\mathcal{H}_{\pi,K}$ is dense in $\mathcal{H}_\pi$. Moreover ([21] Proposition 8.5):

Theorem 1.5 (Harish-Chandra). *For π an admissible representation, any K-finite vector is smooth, i.e. $\mathcal{H}_{\pi,K} \subseteq \mathcal{H}_\pi^\infty$.*

Since $\pi(k)\pi(X)v = \pi(\mathrm{Ad}(k)X)\pi(k)v$ it follows easily that $\mathcal{H}_{\pi,K}$ is invariant under $\pi(\mathfrak{g})$. Obviously, it is also invariant under the K-action. Motivated by this we therefore define

Definition 1.6. A $(\mathfrak{g}, K)$-*module* is a vector space V (no topology involved) on which we have a representation of the Lie algebra $\mathfrak{g}$ and a representation of the group K satisfying the following requirements

1) $k \cdot X \cdot v = \mathrm{Ad}(k)X \cdot k \cdot v$.

2) For each $v \in V$, $\mathrm{span}\{K \cdot v\}$ is finite-dimensional and the K-action on this space is continuous.

3) For $X \in \mathfrak{k}$ we have $\frac{d}{dt}\big|_{t=0} \exp(tX) \cdot v = X \cdot v$.

Note that 3) makes sense due to 2). For a unitary representation π which is not necessarily admissible, the space of K-finite vectors is in general *not* a $(\mathfrak{g}, K)$-module, as it may not admit an action of $\mathfrak{g}$. On the other hand it is also clear that $\mathcal{H}_\pi^\infty$ is in general not a $(\mathfrak{g}, K)$-module since requirement 2 is not necessarily satisfied for an arbitrary smooth vector. However, the space $\mathcal{H}_{\pi,K} \cap \mathcal{H}_\pi^\infty$ is in fact a $(\mathfrak{g}, K)$-module and we call this the $(\mathfrak{g}, K)$-module *associated with* π. If π happens to be admissible, then $\mathcal{H}_{\pi,K} \cap \mathcal{H}_\pi^\infty = \mathcal{H}_{\pi,K}$ is the associated $(\mathfrak{g}, K)$-module.

Theorem 1.7 (**Harish-Chandra**). *If π is any representation on $\mathcal{H}_\pi$, then $\mathcal{H}_{\pi,K} \cap \mathcal{H}_\pi^\infty$ is dense in $\mathcal{H}_\pi^\infty$. In particular, if π is admissible, then $\mathcal{H}_{\pi,K}$ is dense in $\mathcal{H}_\pi^\infty$.*

For a proof see for instance [34], Theorem 4.4.2.1 on p. 261.

If V and W are two $(\mathfrak{g}, K)$-modules, $\mathrm{Hom}_{\mathfrak{g},K}(V, W)$ is the set of linear maps $V \longrightarrow W$ which intertwine both the $\mathfrak{g}$-action and the K-action and V and W are said to be *equivalent* as $(\mathfrak{g}, K)$-modules if there exists an invertible element in $\mathrm{Hom}_{\mathfrak{g},K}(V, W)$. A $(\mathfrak{g}, K)$-module V is called *irreducible* if the only subspaces invariant under both the $\mathfrak{g}$-action and the K-action are $\{0\}$ and V. By a version of Schur's Lemma : if V and W are irreducible, then $\mathrm{Hom}_{\mathfrak{g},K}(V, W) \cong \mathbb{C}$ if the modules are equivalent, and $\mathrm{Hom}_{\mathfrak{g},K}(V, W) = 0$ if they are non-equivalent.

Just as we have dual representations, we also have *dual* $(\mathfrak{g}, K)$*-modules* . For a module V, this is defined as $\widetilde{V} := (V^*)_K$, as the K-finite vectors in the algebraic dual of V.

Lemma 1.8. *Let $(\pi, \mathcal{H}_\pi)$ be an admissible G-module. The dual of the $(\mathfrak{g}, K)$-module $\mathcal{H}_{\pi,K}$ is the module associated with the dual representation of π on $\mathcal{H}'_\pi = \overline{\mathcal{H}}_\pi$, in other words, the restriction map*

$$(\overline{\mathcal{H}}_\pi)_K = (\mathcal{H}'_\pi)_K \longrightarrow \widetilde{\mathcal{H}_{\pi,K}}$$

is a linear isomorphism.

Proof. Since $\mathcal{H}_{\pi,K}$ is dense in $\mathcal{H}_\pi$, it is clear that the restriction map is injective.

We decompose $\mathcal{H}_\pi$ into irreducible K-modules:

$$\mathcal{H}_\pi = \widehat{\bigoplus_{\delta \in \widehat{K}}} V_\delta^{\oplus n_\delta}$$

and since π is admissible, each n_δ is finite. Thus

$$\mathcal{H}_{\pi,K} \cong \bigoplus_{\delta \in \widehat{K}} V_\delta^{\oplus n_\delta}.$$

This implies that

$$\mathcal{H}^*_{\pi,K} = \prod_{\delta \in \widehat{K}} (V_\delta^{\oplus n_\delta})^*.$$

$\mathcal{H}^*_{\pi,K}$ is a K-module and the map that restricts an element in $\mathcal{H}^*_{\pi,K}$ to $V_\delta^{\oplus n_\delta}$ is a K-intertwiner (the dual map of an intertwiner is an intertwiner, and restriction is the dual map of inclusion map, which is an intertwiner). It means that an element $\xi \in (\mathcal{H}^*_{\pi,K})_K$ when restricted to $V_\delta^{\oplus n_\delta}$ is non-zero only for finitely many δ's. Hence

$$(\mathcal{H}^*_{\pi,K})_K \cong \bigoplus_{\delta \in \widehat{K}} (V_\delta^{\oplus n_\delta})^*.$$

On the other hand, we have

$$\mathcal{H}'_\pi = \widehat{\bigoplus_{\delta \in \widehat{K}}} (V_\delta^{\oplus n_\delta})^*$$

and hence

$$(\mathcal{H}'_\pi)_K = \bigoplus_{\delta \in \widehat{K}} (\mathcal{H}_\delta^{\oplus n_\delta})^*$$

which equals $(\mathcal{H}^*_{\pi,K})_K$. This proves the lemma. □

A unitary representation of G is said to be *infinitesimally irreducible* if its associated $(\mathfrak{g}, K)$-module is irreducible, and two unitary representations are said to be *infinitesimally equivalent* if there associated $(\mathfrak{g}, K)$-modules are equivalent. It is a deep result by Harish-Chandra that a unitary representation of G is irreducible if and only if it is infinitesimally irreducible and that two irreducible unitary representations are unitarily equivalent if and only if they are infinitesimally equivalent. For more on $(\mathfrak{g}, K)$-modules see [32] Ch. 3.

A $(\mathfrak{g}, K)$-module V is called *admissible* if $\mathrm{Hom}_{\mathfrak{g},K}(V_\delta, V)$ is finite-dimensional for all $\delta \in \widehat{K}$ [4]. Obviously, the $(\mathfrak{g}, K)$-module associated with an admissible representation is admissible.

Definition 1.9 (Harish-Chandra module). An admissible $(\mathfrak{g}, K)$-module V is called a *Harish-Chandra module* [5] if it is *finitely generated*, i.e. if there exists a finite set $\{v_1, \ldots, v_n\} \subseteq V$ such that

$$V = \bigoplus_{j=1}^{n} U(\mathfrak{g}_\mathbb{C}) v_j.$$

We see that an irreducible $(\mathfrak{g}, K)$-module is finitely generated, since it is generated by any non-zero vector. Thus if π is an irreducible admissible representation, its associated $(\mathfrak{g}, K)$-module is a Harish-Chandra module. The requirement

[4] In [5] this is called weak admissibility.
[5] In [5] a Harish-Chandra module is also called an admissible module.

of being finitely generated might seem a bit awkward. The reason is that we want this category to contain also the (associated $(\mathfrak{g}, K)$-modules of) non-irreducible principal series representations.

One of the fundamental results in the theory of Harish-Chandra modules is the celebrated *Casselman Subrepresentation Theorem*

Theorem 1.10 (**Casselman**). *Any Harish-Chandra module can be embedded into the* $(\mathfrak{g}, K)$*-module associated to a principal series representation.*

So not only does the category of Harish-Chandra modules contain all the principal series representations - these and their submodules are the *only* Harish-Chandra modules.

1.4 Invariant Differential Operators

Invariant differential operators play a very important role in all of harmonic analysis. It is visible already in the Fourier theory on $\mathbb{R}$: here the invariant differential operators are the constant coefficient ones, and we expand functions in integrals involving exponential maps which are exactly the eigenfunctions of the constant coefficient differential operators.

We begin with some basic definitions. Let $\varphi : M \longrightarrow M$ be a diffeomorphism of a manifold. Then we define an automorphism of $C^\infty(M)$ by $\varphi \longmapsto f^\varphi := f \circ \varphi^{-1}$. Similarly, we define an automorphism on the set of differential operators on M by

$$D^\varphi f := (Df^{\varphi^{-1}})^\varphi = (D(f \circ \varphi)) \circ \varphi^{-1}.$$

We say that D is *invariant* under the mapping φ, if $D^\varphi = D$, i.e. if $D(f \circ \varphi) = (Df) \circ \varphi$.

On a Lie group we have some standard diffeomorphisms: $\ell_g : G \longrightarrow G$, given by $h \longmapsto gh$ as well as r_g mapping $h \longmapsto hg^{-1}$. For a subgroup H the left action descends to diffeomorphisms of the quotient $\ell_g : G/H \longrightarrow G/H$.

Definition 1.11 (**Invariant Differential Operator**). A differential operator on a Lie group G is called G-*invariant* or just *invariant* if it is invariant under all left translations ℓ_g. The space of all such differential operators is denoted $\mathbb{D}(G)$. More generally, a differential operator on a homogenous space G/H is called *invariant* if it is invariant under all left translations ℓ_g of G/H. The space of such operators is denoted $\mathbb{D}(G/H)$.

It is not hard to check that $(D_1 D_2)^\varphi = D_1^\varphi D_2^\varphi$ and hence the spaces $\mathbb{D}(G)$ and $\mathbb{D}(G/H)$ are actually algebras. Note also the inclusion $\mathfrak{g} \subseteq \mathbb{D}(G)$, since elements of $\mathfrak{g}$ are (by definition) first order invariant differential operators [6].

We would like some kind of characterization of the algebra $\mathbb{D}(G)$ or $\mathbb{D}(G/H)$ in terms of the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. The algebra $\mathbb{D}(G)$ is easy to handle. Recall the universal enveloping algebra $U(\mathfrak{g}_\mathbb{C})$ which as a vector space can be identified with the symmetric algebra $S(\mathfrak{g}_\mathbb{C})$ consisting of complex polynomials

$$\sum a_{i_1 \ldots i_n} X_1^{i_1} \cdots X_n^{i_n}$$

for some basis $X_1, \ldots, X_n$ for $\mathfrak{g}$. Since compositions of differential operators are again invariant, the extension of the map $X \longmapsto \widetilde{X}$ given in the footnote above, to $U(\mathfrak{g}_\mathbb{C})$ (which consists of compositions of elements of $\mathfrak{g}_\mathbb{C}$ subject to the Lie-algebra relations) should embed $U(\mathfrak{g}_\mathbb{C})$ into $\mathbb{D}(G)$. They even turn out to be equal:

Theorem 1.12. *The map $\lambda : U(\mathfrak{g}_\mathbb{C}) \longrightarrow \mathbb{D}(G)$ given by*

$$(\lambda(P)f)(g) = P(\partial_1, \ldots, \partial_n) f(g \exp(t_1 X_1 + \cdots + t_n X_n))\big|_{t=0}, \quad f \in C^\infty(G)$$

is an algebra homomorphism.

By $\mathbb{Z}(G)$ we denote the center of $\mathbb{D}(G) \cong U(\mathfrak{g}_\mathbb{C})$. The center turns out to have an interesting interpretation. First note, that since $\mathrm{Ad}(g) : \mathfrak{g} \longrightarrow \mathfrak{g}$ is a Lie algebra homomorphism, it extends uniquely to an algebra homomorphism $\mathrm{Ad}(g) : U(\mathfrak{g}_\mathbb{C}) \longrightarrow U(\mathfrak{g}_\mathbb{C})$. By $I(\mathfrak{g}_\mathbb{C})$ we denote the subset of elements which are invariant under all $\mathrm{Ad}(g)$, i.e. satisfy $\mathrm{Ad}(g)x = x$ for all g. These are called the Ad-invariant polynomials.

For the differential operators we define

$$\mathrm{Ad}(g)D := D^{r_g}.$$

This is again an algebra homomorphism. The motivation for this definition is that $\mathrm{Ad}(g)$ in some sense is a conjugation with g and since D is left-invariant

[6] However, we may also view $\mathfrak{g}$ as the tangent space T_eG. The connection is that we can view a tangent vector X as the vector field or differential operator given by

$$\widetilde{X} f(g) := \frac{d}{dt}\bigg|_{t=0} f(g \exp(tX)).$$

only the right translation is left. Furthermore, a simple calculation reveals that for $X \in \mathfrak{g}$:

$$(\mathrm{Ad}(g)X) = X^{r_g}$$

which shows that $\lambda(\mathrm{Ad}(g)x) = \mathrm{Ad}(g)\lambda(x)$ for $x \in U(\mathfrak{g}_{\mathbb{C}})$.

Corollary 1.13. *Let G be a connected Lie group. The isomorphism λ maps $I(\mathfrak{g}_{\mathbb{C}})$ bijectively onto $\mathbb{Z}(G)$, in other words the center $\mathbb{Z}(G)$ consists of invariant differential operators which are* $\mathrm{Ad}(g)$*-invariant i.e. invariant both from the right and from the left.*

For the proof one needs to note that D commutes with X if and only if D is right $\exp(tX)$-invariant. Since any element in G is a finite product of the form $\exp X_1 \cdots \exp X_n$ the equality of $\mathbb{Z}(G)$ and the set of Ad-invariants follows.

One prominent element which is always in $\mathbb{Z}(G)$ (when G is semisimple) is the *Casimir element*. It is defined as follows: Let $\{X_1, \ldots, X_n\}$ be some basis for $\mathfrak{g}$, and let $\{\widetilde{X}_1, \ldots, \widetilde{X}_n\}$ be the dual basis w.r.t. the Killing form B (which is non-degenerate as G is semisimple). The *Casimir element* is defined as follows:

$$\omega := \sum_{i,j=1}^{n} B(X_i, X_j)\widetilde{X}_i \widetilde{X}_j.$$

The center of the universal enveloping algebra can be used to define what is knows as an infinitesimal character. First we recall the Schur lemma, which says that the only bounded operators commuting with an irreducible unitary representation are scalar operators [7] Dixmier proved an infinitesimal version of this: Let $\mathfrak{g}_{\mathbb{C}}$ be a complex Lie algebra and $U(\mathfrak{g}_{\mathbb{C}})$ its universal enveloping algebra and $Z(\mathfrak{g}_{\mathbb{C}})$ the center of $U(\mathfrak{g}_{\mathbb{C}})$ (not to be confused with the center of $\mathfrak{g}_{\mathbb{C}}$ which we denote $Z_{\mathfrak{g}_{\mathbb{C}}}$). In the following a *unital* $U(\mathfrak{g}_{\mathbb{C}})$-module is a $U(\mathfrak{g}_{\mathbb{C}})$-module where $1 \in U(\mathfrak{g}_{\mathbb{C}})$ (the enveloping algebra, by definition, always contains a unit) acts as the identity.

Theorem 1.14 (Dixmier). *Let ρ be an irreducible unital left $U(\mathfrak{g}_{\mathbb{C}})$ module. Then the only $U(\mathfrak{g}_{\mathbb{C}})$-linear maps commuting with ρ are scalar operators.*

Now, let π be an irreducible admissible representation of a connected reductive Lie group G. This induces a representation π_* of the Lie algebra $\mathfrak{g}$ on the (dense) space of K-finite vectors. We can complexify this representation and extend to

[7] In fact, by definition *quasisimple representations* are representations satisfying that the only bounded operators commuting with then are scalars. In that sense, Schur's lemma asserts that irreducible unitary representations are quasisimple.

an algebra representation, also denoted π_*, of the universal enveloping algebra $U(\mathfrak{g}_\mathbb{C})$ on V^K. Dixmier's Theorem implies that $\pi_*(X)$ is just a scalar multiple of the identity map id_{V^K} when $X \in Z(\mathfrak{g})$. By the homomorphism property of π_* this implies the existence of a homomorphism $\chi_\pi : Z(\mathfrak{g}_\mathbb{C}) \longrightarrow \mathbb{C}$ such that

$$\pi_*(X) = \chi_\pi(X)\,\mathrm{id}_{V^K}$$

for all $X \in Z(\mathfrak{g}_\mathbb{C})$. Here χ_π is called the *infinitesimal character* of π. Thus we are interested in classifying the characters on the algebra $Z(\mathfrak{g}_\mathbb{C})$. This was one of the first of Harish-Chandra's great achievements.

The strategy is to identify $Z(\mathfrak{g}_\mathbb{C})$ with a more manageable algebra whose characters are easier to classify. This identification is via the *Harish-Chandra homomorphism* which we now define. We fix a Cartan subalgebra $\mathfrak{h}_\mathbb{C} \subseteq \mathfrak{g}_\mathbb{C}$ and let $\Sigma^+ = \Sigma^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ denote the set of positive roots w.r.t. $\mathfrak{h}_\mathbb{C}$ and some choice of positivity. In this case we know that $(\mathfrak{g}_\mathbb{C})_\alpha$ is 1-dimensional so pick a basis E_α for each of the root spaces as well as a basis $\{H_1, \ldots, H_k\}$ for the Cartan subalgebra. Inside $U(\mathfrak{g}_\mathbb{C})$ we consider the two subspaces $U(\mathfrak{h}_\mathbb{C})$ (which is obviously a commutative subalgebra) and

$$\mathscr{P} := \bigoplus_{\alpha \in \Delta^+} U(\mathfrak{g}_\mathbb{C}) E_\alpha. \tag{1.5}$$

Then one can show that $U(\mathfrak{h}_\mathbb{C}) \cap \mathscr{P} = \{0\}$ and that $Z(\mathfrak{g}_\mathbb{C}) \subseteq U(\mathfrak{h}_\mathbb{C}) \oplus \mathscr{P}$. What does this mean? It means the following: By the Poincaré-Birkhoff-Witt-theorem a basis for $U(\mathfrak{g}_\mathbb{C})$ is given by elements of the form

$$E_{-\alpha_1}^{q_1} \cdots E_{-\alpha_l}^{q_l} H_1^{m_1} \cdots H_k^{m_k} E_{\alpha_1}^{p_1} \cdots E_{\alpha_l}^{p_l},$$

and if this belongs to the center, then $p_1 = \cdots = p_l = 0$ implies $q_1 = \cdots = q_l = 0$. In other words, assume v to be a highest weight vector in a $U(\mathfrak{g}_\mathbb{C})$-module of highest weight λ, then a basis element of the form (1.5) would act on v either by 0 (if there exists $p_j \neq 0$) or by the scalar

$$\lambda(H_1)^{m_1} \cdots \lambda(H_k)^{m_k}$$

(if $p_1 = \cdots = p_l = 0$ in which case also $q_1 = \cdots = q_l = 0$).

Since $\mathfrak{h}_\mathbb{C}$ and $\mathbb{C}$ are both abelian, λ extends via the universal property to an algebra homomorphism $\lambda : U(\mathfrak{h}_\mathbb{C}) \longrightarrow \mathbb{C}$. Thus letting $\gamma'_{\Sigma^+} : Z(\mathfrak{g}_\mathbb{C}) \longrightarrow U(\mathfrak{h}_\mathbb{C})$ denote the projection onto the first component we see that

$$X \cdot v = \lambda(\gamma'_{\Sigma^+}(X))v \tag{1.6}$$

for all $X \in Z(\mathfrak{g}_{\mathbb{C}})$.

Remembering that $\rho = \frac{1}{2}\sum_{\alpha\in\Sigma^+}\alpha$ (for the root system $\Sigma(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ all the roots automatically have multiplicity 1) we define a map $\sigma_{\Sigma^+} : \mathfrak{h}_{\mathbb{C}} \longrightarrow U(\mathfrak{h}_{\mathbb{C}})$ by

$$\sigma_{\Delta^+}(H) := H - \rho(H)1,$$

we more or less translate by ρ. Again by the universal property, this map extends to a map $\sigma_{\Sigma^+} : U(\mathfrak{h}_{\mathbb{C}}) \longrightarrow U(\mathfrak{h}_{\mathbb{C}})$.

Definition 1.15 (Harish-Chandra homomorphism). The *Harish-Chandra homomorphism* is the homomorphism $\gamma : Z(\mathfrak{g}_{\mathbb{C}}) \longrightarrow U(\mathfrak{h}_{\mathbb{C}})$ defined by

$$\gamma := \sigma_{\Sigma^+} \circ \gamma'_{\Sigma^+}. \tag{1.7}$$

Then, bearing (1.6) in mind, it should come as no surprise that in the highest weight $U(\mathfrak{g}_{\mathbb{C}})$-module of highest weight λ considered before, we have

$$X \cdot v = (\lambda - \rho)(\gamma(X))v. \tag{1.8}$$

Even though σ_{Σ^+} and γ'_{Σ^+} depend on the choice of positivity, the Harish-Chandra homomorphism does not. Finally, recall that the Weyl group acts on $\mathfrak{h}_{\mathbb{C}}$ (it acts by the adjoint action of certain equivalence classes of elements in K) and this action extends to an algebra action on $U(\mathfrak{h}_{\mathbb{C}})$. Let $U(\mathfrak{h}_{\mathbb{C}})^W$ denote the subalgebra of elements fixed by the Weyl group action.

Theorem 1.16. *The Harish-Chandra homomorphism is an algebra isomorphism*

$$\gamma : Z(\mathfrak{g}_{\mathbb{C}}) \xrightarrow{\sim} U(\mathfrak{h}_{\mathbb{C}})^W. \tag{1.9}$$

As, mentioned, any $\Lambda \in \mathfrak{h}_{\mathbb{C}}$ extends to an algebra homomorphism $\Lambda : U(\mathfrak{h}_{\mathbb{C}}) \longrightarrow \mathbb{C}$. Composing with the Harish-Chandra homomorphism gives us

$$\chi_\Lambda := \Lambda \circ \gamma : Z(\mathfrak{g}_{\mathbb{C}}) \longrightarrow \mathbb{C}.$$

Not all of these maps are different, since $\gamma(X) \in U(\mathfrak{h}_{\mathbb{C}})^W$ we see

$$\chi_{w\cdot\Lambda}(X) = (w \cdot \Lambda)(\gamma(X)) = \Lambda(\mathrm{Ad}(w^{-1})\gamma(X)) = \Lambda(\gamma(X)) = \chi_\Lambda(X)$$

and hence $\chi_{w\cdot\Lambda} = \chi_\Lambda$. Actually the converse statement also holds: if $\chi_\Lambda = \chi_{\Lambda'}$ then $\Lambda' = w \cdot \Lambda$ for some w in the Weyl group. Thus [8]:

[8] This is Proposition 8.20 and 8.21 in [21]. The last claim, that a closed Weyl chamber is a fundamental domain for $\mathfrak{h}_{\mathbb{C}}/W$ is Lemma 10.3.B in [20].

Theorem 1.17. *Every homomorphism $\chi : Z(\mathfrak{g}_{\mathbb{C}}) \longrightarrow \mathbb{C}$ is of the form χ_Λ for some $\Lambda \in \mathfrak{h}'_{\mathbb{C}}$, i.e. the set of all characters on $Z(\mathfrak{g}_{\mathbb{C}})$ is in* 1-1 *correspondence with $(\mathfrak{h}^*_{\mathbb{C}})^W$ which in turn is in 1-1 correspondence with a closed Weyl chamber.*

We end this section with some general remarks on the algebra $\mathscr{D}(G/H)$. Let $\pi : G \longrightarrow G/H$ denote the natural projection. This gives us a map $C^\infty(G/H) \longrightarrow C^\infty(G)$, by

$$f \longmapsto \widetilde{f} := f \circ \pi.$$

It is not hard to check that the image of this map inside $C^\infty(G)$ is $C^\infty(G)_H$ which are the functions which are right H-invariant, i.e. satisfy $f(gh) = f(g)$ for all $h \in H$. We use this map to identify the spaces $C^\infty(G/H)$ and $C^\infty(G)_H$.

In the following, we let $\mathbb{D}_H(G)$ denote the subset of $\mathbb{D}(G)$ consisting of differential operators which in addition to the left invariance are also right H-invariant, i.e. satisfy $D^{r_h} = D$ for $h \in H$ (another way to phrase this is that it should be left G-invariant and $\mathrm{Ad}_G(H)$-invariant). Our first task is to relate $\mathbb{D}_H(G)$ to $\mathbb{D}(G/H)$. Unfortunately, it is too much to hope for that the two spaces can be identified as in the case of smooth functions. If $D \in \mathbb{D}_H(G)$ and $f \in C^\infty(G/H)$, it is not hard to check that $D\widetilde{f} \in C^\infty(G)$ is actually right H-invariant, thus it is of the form $\widetilde{F}$ for some unique $F \in C^\infty(G/H)$. Let $\mu(D)$ denote the operator mapping f to F, i.e. satisfying

$$(\mu(D)f)^\sim = D\widetilde{f}.$$

It is not hard to see that $\mu(D)$ is linear and decreases support, and hence is a differential operator on G/H. Checking that it is left invariant is also straightforward. Thus we get a map $\mu : \mathbb{D}_H(G) \longrightarrow \mathbb{D}(G/H)$

Theorem 1.18. *The map $\mu : \mathbb{D}_H(G) \longrightarrow \mathbb{D}(G/H)$ is a surjective algebra homomorphism whose kernel equals $\mathbb{D}_H(G) \cap \mathbb{D}(G)\mathfrak{h}$. Thus it induces an algebra isomorphism*

$$\mathbb{D}_H(G)/(\mathbb{D}_H(G) \cap \mathbb{D}(G)\mathfrak{h}) \xrightarrow{\sim} \mathbb{D}(G/H). \tag{1.10}$$

A much more explicit isomorphism can be given, in the case where $G/H = G/K$ is a Riemannian symmetric space of non-compact type, but we will not go into that here. We will eventually return to the explicit calculation of two such algebras in the final chapter.

1.5 Direct Integrals

For compact groups we know (Peter-Weyl) that a unitary representation always decomposes in a direct sum of irreducible representations. For representations of a non-compact group, however, it is no longer the case, that any representation decomposes discretely, as it may have "continuous" components also. Hence we need a framework which allows us to deal with non-discrete decompositions of representations. This framework is provided by the *direct integral* construction. Throughout this section, G will denote an arbitrary Lie group.

Let (X, μ) be a measure space and assume we have a family of separable $\mathbb{K}$-Hilbert spaces $\mathcal{H}_x$ parametrized by X. A *measurable family of Hilbert spaces* is a family of Hilbert spaces together with a countable set $(e_i)_{i=1}^{\infty}$ of vector fields satisfying that the linear span of $\{e_i(x)\}_{i=1}^{\infty}$ is dense in $\mathcal{H}_x$ and satisfying that $x \longmapsto \langle e_i(x), e_j(x)\rangle_{\mathcal{H}_x}$ is a measurable function. From this definition it follows (cf. Proposition (7.27) in [15]) that there exists disjoint measurable subsets $X_n \subseteq X$, for $n \in \mathbb{N} \cup \{\infty\}$ such that $X = \bigcup_{i=1}^{\infty}$ and such that

$$\forall x \in X_n : \ \mathcal{H}_x \cong \mathbb{K}^n$$

where $\mathbb{K}^\infty$ stands for the Hilbert space $\ell^2(\mathbb{N}, \mathbb{K})$. A *section* of a measurable family of Hilbert spaces is a map

$$s : X \longrightarrow \coprod_{x \in X} \mathcal{H}_x$$

such that $s(x) \in \mathcal{H}_x$ and such that $x \longmapsto \langle s(x), e_i(x)\rangle_{H_x}$ is a measurable function for all i. Note that e_i is a section. We identify two sections if they agree almost everywhere.

Definition 1.19 (Direct integral). For a measurable family $(\mathcal{H}_x)$ of Hilbert spaces, we define the *direct integral*

$$\int_X^{\oplus} \mathcal{H}_x d\mu(x)$$

to be the set of equivalence classes of measurable sections satisfying

$$\int_X \|s(x)\|^2_{\mathcal{H}_x} d\mu(x) < \infty.$$

Endowed with the inner product

$$\langle s_1, s_2\rangle := \int_X \langle s_1(x), s_2(x)\rangle_{\mathcal{H}_x} d\mu(x)$$

this becomes a Hilbert space.

There are two things to note in this connection: 1) the direct integral depends on the choice of (e_i) (the sections used to define measurability). However, for a different choice of (e_i) the corresponding direct integral Hilbert space will, in a canonical way, be isomorphic to the first one. 2) if ν is another measure on X such that μ and ν are mutually absolutely continuous, then the map $s \longmapsto s\sqrt{d\mu/d\nu}$ is a unitary map $\int_X^\oplus \mathcal{H}_x d\mu(x) \xrightarrow{\sim} \int_X^\oplus \mathcal{H}_x d\nu(x)$ (here $d\mu/d\nu$ is the Radon-Nikodym derivative of μ w.r.t. ν).

We have two obvious examples of direct integrals: 1) a Hilbert (i.e. completed) direct sum $\widehat{\bigoplus}_{i\in I}\mathcal{H}_i$ where the measure space is some discrete countable set I with the counting measure, and 2) the space $L^2(X, \mathbb{K}, \mu)$ where the measure space is (X, μ) and $\mathcal{H}_x = \mathbb{K}$.

Proposition 1.20. *For the direct integral construction the following hold*

1) *If $\mathcal{H}_{x,y}$ is a doubly indexed measurable family of Hilbert spaces over $X \times Y$, with X carrying the measure μ and Y carrying the measure ν, then*

$$\begin{aligned}\int_{X\times Y}^\oplus \mathcal{H}_{x,y} d(\mu\otimes\nu)(x,y) &\cong \int_X^\oplus \int_Y^\oplus \mathcal{H}_{x,y} d\nu(y) d\mu(x)\\ &\cong \int_Y^\oplus \int_X^\oplus \mathcal{H}_{x,y} d\mu(x) d\nu(y).\end{aligned}$$

2) *If $\mathcal{H}_x$ and $\mathcal{H}_y$ are Hilbert spaces indexed over X and Y, then*

$$\Big(\int_X^\oplus \mathcal{H}_x d\mu(x)\Big) \otimes \Big(\int_Y^\oplus \mathcal{H}_y d\nu(y)\Big) \cong \int_{X\times Y}^\oplus \mathcal{H}_x \otimes \mathcal{H}_y d(\mu\otimes\nu)(x,y).$$

The first claim is a simple consequence of the Tonelli Theorem. For the second claim, simply consider two orthonormal bases and check that the natural map maps one orthonormal basis to the other.

An operator A on $\mathcal{H} = \int_X^\oplus \mathcal{H}_x d\mu(x)$ is called *decomposable* if there exists a family of operators A_x on $\mathcal{H}_x$ such that for all $s \in \mathcal{H}$ and almost all $x \in X$:

$$(As)(x) = A_x(s(x)).$$

Should this be the case, we write $A = \int_X^\oplus A_x d\mu(x)$ and call A the *direct integral* of the family of operators (A_x).

This carries over to representations where a direct integral decomposition is a decomposition of each of the operators $\pi(g)$ in a "compatible way":

Definition 1.21. A unitary representation $(\pi, \mathcal{H}_\pi)$ of a topological group on a direct integral Hilbert space $\mathcal{H}_\pi = \int_X^\oplus \mathcal{H}_x d\mu(x)$ is called a *direct integral representation* and is written

$$\pi = \int_X^\oplus \pi_x d\mu(x)$$

if there exist representations π_x of G on $\mathcal{H}_x$ so that each operator $\pi(g)$ is decomposable with $(\pi(g))_x = \pi_x(g)$.

Stated differently, if s is a section of the direct integral over X, then $\pi = \int_X^\oplus \pi_x d\mu(x)$ simply means that

$$(\pi(g)s)(x) = \pi_x(g)(s(x))$$

for almost all x.

For representations we see as a consequence of Proposition 1.20 above, that if (π_x) and (π_y) are families of representations of G_1 resp. G_2, then

$$\Big(\int_X^\oplus \pi_x d\mu(x)\Big) \times \Big(\int_Y^\oplus \pi_y d\nu(y)\Big) = \int_{X\times Y}^\oplus (\pi_x \times \pi_y) d(\mu \otimes \nu)(x, y)$$

as representations of $G_1 \times G_2$. In particular, if $G_1 = G_2$, we get by restricting to the diagonal in $G \times G$, that

$$\Big(\int_X^\oplus \pi_x d\mu(x)\Big) \otimes \Big(\int_Y^\oplus \pi_y d\nu(y)\Big) = \int_{X\times Y}^\oplus (\pi_x \otimes \pi_y) d(\mu \otimes \nu)(x, y)$$

as representations of G.

The following can be seen as a generalization of the Peter-Weyl theorem to non-compact groups (cf. Theorem 14.10.5 in [33] - the unitary dual of G is denoted by $\mathcal{E}(G)$, see p. 311).

Theorem 1.22. *For any unitary representation π of a locally compact group G of type I* [9] *there exists a Radon measure on $\widehat{G}$ (depending on π) and a measurable*

[9] A topological group G is said to be of *type I*, if any unitary representation π for which the space of self-intertwiners is just $\mathbb{C}I$, decomposes into a direct sum of irreducibles. The terminology comes from operator algebra theory: if A_π denotes the closure in the weak operator topology of the sub-algebra of $B(\mathcal{H}_\pi)$ generated by $\pi(g)$ for $g \in G$, then the requirement that there are no self-intertwiners of π except the trivial ones, translates into the fact that the von Neumann algebra A_π has trivial center, i.e. is a so-called *factor* , and the condition of reducibility translates into all A_π being factors of so-called type I. Examples of type I groups are compact groups, abelian groups, as well as semisimple and nilpotent groups (cf. [15] Theorem (7.8)). The result for semisimple groups is due to Harish-Chandra cf. [17], Theorem 7.

function $m : \widehat{G} \longrightarrow \mathbb{N}_0 \cup \{\infty\}$ *such that*

$$\mathcal{H}_\pi = \int_{\widehat{G}}^{\oplus} \mathcal{H}_\lambda^{\oplus m(\lambda)} d\mu(\lambda) \qquad \text{and} \qquad \pi = \int_{\widehat{G}}^{\oplus} \pi_\lambda^{\oplus m(\lambda)} d\mu(\lambda).$$

The measure μ *is unique up to multiplication with an a.e.-bounded measurable function with a.e.-bounded inverse.*

The number $m(\lambda)$ is the *multiplicity* of the irreducible representation π_λ in the decomposition of π.

In the case of G being a Lie group of type I, this decomposition gives rise to corresponding decompositions of $\mathcal{H}_\pi^\infty$ and $\mathcal{H}_\pi^{-\infty}$.

Theorem 1.23. *Let* $\pi = \int_X^{\oplus} \pi_x d\mu(x)$ *be a direct integral representation. Then the following hold*:

1) *If* $v \in \mathcal{H}_\pi^\infty$ *and* $v = (v_\lambda)$ *as an element in* $\int_X^{\oplus} \mathcal{H}_\lambda d\mu(\lambda)$, *then* $v_\lambda \in \mathcal{H}_\lambda^\infty$ *for a.e.* $\lambda \in X$ *and* $\pi(X_0)v = (\pi_\lambda(X_0)v_\lambda)$ *for* $X_0 \in \mathfrak{g}$. *Conversely, if* $v = (v_\lambda)$ *and* $v_\lambda \in \mathcal{H}_\lambda^\infty$ *for a.e.* $\lambda \in X$, *and if* $(\pi_\lambda(X_0)v_\lambda) \in \mathcal{H}_\pi$ *for* $X_0 \in \mathfrak{g}$, *then* $v \in \mathcal{H}_\pi^\infty$.

2) *If* $\eta \in \mathcal{H}_\pi^{-\infty}$, *then for a.e.* $\lambda \in X$ *there exists* $\eta_\lambda \in \mathcal{H}_\lambda^{-\infty}$, *so*

$$\eta(v) = \int_X \eta_\lambda(v_\lambda) d\mu(\lambda)$$

for all $v = (v_\lambda) \in \mathcal{H}_\pi^\infty$. *The integral is absolutely convergent and the* η_λ *are a.e. unique. Conversely, if* $\eta_\lambda \in \mathcal{H}_\lambda^{-\infty}$ *is a collection of distribution vectors, such that* $\lambda \longmapsto \eta_\lambda(v_\lambda)$ *is integrable for all* $v \in \mathcal{H}_\pi^\infty$, *then*

$$v \longmapsto \int_X \eta_\lambda(v_\lambda) d\mu(\lambda)$$

defines an element of $\mathcal{H}_\pi^{-\infty}$.

For a proof of the second claim and for a reference to a proof of the first, consult [26], Theorem C and Corollary C.I.

With this theorem at hand we can justify a notation like

$$\mathcal{H}_\pi^\infty = \int_X^{\oplus} \mathcal{H}_\lambda^\infty d\mu(\lambda) \qquad \text{and} \qquad \mathcal{H}_\pi^{-\infty} = \int_X^{\oplus} \mathcal{H}_\lambda^{-\infty} d\mu(\lambda)$$

even though it is not a direct integral of Hilbert spaces.

A distribution vector η is called *cyclic* if $v \in \mathcal{H}_\pi^\infty$ and $\eta(\pi(g)v) = 0$ for all g implies that $v = 0$. In particular, a cyclic distribution vector is non-zero.

Corollary 1.24. *If* $\eta = (\eta_\lambda) \in \mathcal{H}_\pi^{-\infty} = \int_{\widehat{G}}^{\oplus} \mathcal{H}_\lambda^{-\infty} d\mu(\lambda)$ *is a cyclic distribution vector, then* η_λ *is cyclic for almost all* λ.

This is Theorem (II.5) of [26].

1.6 Fourier Transforms

In the previous section we recorded a theorem guaranteeing the existence of a decomposition of any unitary representation into a direct integral of irreducibles. In harmonic analysis on a group, one is usually interested in one particular representation (or rather 3 related representations) namely the regular representations: the *left-regular representation* L, the *right-regular representation* R and the outer product of these two - the *bi-regular representation* T. The first 2 are representations of G on $L^2(G)$ given by

$$L_{g_0} f(g) := f(g_0^{-1} g) \qquad \text{and} \qquad R_{g_0} f(g) := f(g g_0)$$

and the bi-regular representation is a representation of $G \times G$ on $L^2(G)$ given by

$$T_{(g_1, g_2)} f(g) = f(g_1^{-1} g g_2)$$

The left- and right-regular representations are the dual of the actions ℓ and r of G on itself defined in Section 1.4.

For a homogenous space G/H, we only have a left-regular representation - a representation of G on $L^2(G/H)$. The goal is the decomposition of that into irreducibles. Note that if we view a group G as the homogenous space $(G \times G)/G$ (this is an example of a symmetric space, more on this later), then the regular representation of this space is the same as the bi-regular representation of G.

A Fourier transform is a specific unitary intertwiner between the left-regular representation and its direct integral decomposition. In the following we only deal with Fourier transforms on groups. Fourier transforms on homogenous spaces (to the extend that they can even be defined) are much more complicated, one reason being that higher multiplicities may occur.

For the Abelian group $\mathbb{R}^n$ we know all the irreducible unitary representations, they are maps $\mathbb{R}^n \longrightarrow \mathbb{C}$ of the form $x \longmapsto e^{i\xi \cdot x}$ where ξ runs through $\mathbb{R}^n$. The classical Fourier transformed of an L^1-function, defined as

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-i\xi \cdot x} dx$$

can thus be viewed as a function on $\widehat{\mathbb{R}^n}$ where we integrate f against the irreducible unitary representation of $\mathbb{R}^n$ corresponding to ξ. This is the idea that we generalize to a general group:

Definition 1.25 (Fourier transform). Let G be a topological group and $\widehat{G}$ its unitary dual. For $f \in L^1(G)$ we define the Fourier transformed $\widehat{f}$ of f to be the operator-valued function on $\widehat{G}$ given by

$$\widehat{f}(\pi) = \int_G f(g)\pi(g^{-1})dg \in U(\mathcal{H}_\pi). \tag{1.11}$$

The map $f \longmapsto \widehat{f}$ is denoted $\mathcal{F}$.

First we note, some basic properties of the Fourier transform:

Proposition 1.26. *For the Fourier transform the following hold*:

1) *The map $\mathcal{F}$ is linear, in the sense that for each $\pi \in \widehat{G}$:*

$$(af_1 + bf_2)^\wedge(\pi) = a\widehat{f_1}(\pi) + b\widehat{f_2}(\pi).$$

2) $(\widehat{f_1 * f_2})(\pi) = \widehat{f_2}(\pi)\widehat{f_1}(\pi)$.

3) *If $f^*(g) = \overline{f(g^{-1})}$, then $\widehat{(f^*)}(\pi) = \widehat{f}(\pi)^*$.*

4) *When L_g and R_g are the left and right regular representations respectively, then*

$$\widehat{(L_g f)}(\pi) = \widehat{f}(\pi)\pi(g^{-1}) \qquad \textit{and} \qquad \widehat{(R_g f)}(\pi) = \pi(g)\widehat{f}(\pi).$$

One of the high points of Fourier theory, is the existence of a measure on $\widehat{G}$ w.r.t. which one can define an inverse Fourier transform. Before we can state the inversion formula, we briefly recall the definition of Hilbert-Schmidt and trace class operators. Let $\mathcal{H}$ be a Hilbert space and consider the algebraic tensor product $\mathcal{H} \otimes \overline{\mathcal{H}}$. This tensor product is identifiable with the space $F(\mathcal{H})$ of finite-rank operators $\mathcal{H} \longrightarrow \mathcal{H}$. We can equip this space with (at least) 3 norms, 1) the operator norm, $\|\cdot\|$, 2) the so-called *trace norm* : $\|A\|_1 := \sum s_k(|A|)$ where $s_k(|A|)$ are the eigenvalues of the operator $|A| := (A^*A)^{\frac{1}{2}}$, and 3) with the so-called *Hilbert-Schmidt norm*

$$\|A\|_2 := \mathrm{Tr}(A^*A)$$

(note that the trace and the sum of eigenvalues are well-defined, as the operators have finite rank). The Hilbert-Schmidt norm comes from the inner product on $(A \mid B) = \operatorname{Tr}(A^*B)$ on $F(\mathcal{H})$. Thus the completion of $F(\mathcal{H}) = \mathcal{H} \otimes \overline{\mathcal{H}}$ in this norm equals the Hilbert tensor product $\mathcal{H}\widehat{\otimes}\overline{\mathcal{H}}$.

The three norms are related as follows

$$\|A\| \leq \|A\|_2 \leq \|A\|_1.$$

The completion of $F(\mathcal{H})$ in the operator norm is nothing but the space of compact operators $K(\mathcal{H})$, the completion in the trace norm, $S_1(\mathcal{H})$ is the space of so-called *trace class operators* and the completion $S_2(\mathcal{H}) = \mathcal{H}\widehat{\otimes}\overline{\mathcal{H}}$ in the Hilbert-Schmidt norm consists of the so-called *Hilbert-Schmidt operators* . From the relations among the norms we have the following inclusions of spaces

$$S_1(\mathcal{H}) \subseteq S_2(\mathcal{H}) \subseteq K(\mathcal{H}).$$

The trace class operators are characterized by the fact that for any orthonormal basis (e_n) for $\mathcal{H}$, the series

$$\sum_i \langle Ae_i, e_i \rangle$$

is absolutely convergent, and inspired by the finite-dimensional case, we define the *trace* $\operatorname{Tr} A$ of A to be the above sum (which can be shown to be independent of the choice of orthonormal basis). From the definition of Hilbert-Schmidt operators we see that $S_2(\mathcal{H})S_2(\mathcal{H}) \subseteq S_1(\mathcal{H})$, i.e. that a product of two Hilbert-Schmidt operators has a trace.

Theorem 1.22 guaranteed, for each representation, the existence of a certain equivalence class of measures, such that the representation decomposes in a direct integral w.r.t. this measure. For the bi-regular representation of $G \times G$ we can be more specific. The following general version of the Plancherel theorem says that the measure is supported on the diagonal of $(G \times G)^\wedge = \widehat{G} \times \widehat{G}$, that the multiplicity is at most 1, and that the Fourier transform is an intertwiner

Theorem 1.27 (Plancherel). [10] *Let G be a second countable, unimodular topological group of type I with a fixed Haar measure dg. There exists a unique measure μ on $\widehat{G}$ such that $\pi \longmapsto \mathcal{H}_\pi\widehat{\otimes}\overline{\mathcal{H}}_\pi$ is a measurable field of Hilbert spaces and such that $\mathcal{F}$ as defined in* (1.11) *maps $L^1(G) \cap L^2(G)$ into $\int_{\widehat{G}}^{\oplus} \mathcal{H}_\pi\widehat{\otimes}\overline{\mathcal{H}}_\pi d\mu(\pi)$ and extends to a unitary map*

$$\mathcal{F} : L^2(G) \xrightarrow{\sim} \int_{\widehat{G}}^{\oplus} \mathcal{H}_\pi\widehat{\otimes}\overline{\mathcal{H}}_\pi d\mu(\pi)$$

[10] Cf. [15] Theorem (7.44).

which intertwines the bi-regular representation T and $\int_{\widehat{G}}^{\oplus} \pi \times \pi^* d\mu(\pi)$.

For this measure it holds for $f_1, f_2 \in L^1(G) \cap L^2(G)$ that

$$\int_G f_1(g)\overline{f_2(g)}dg = \int_{\widehat{G}} \mathrm{Tr}\big(\widehat{f_1}(\pi)\widehat{f_2}(\pi)^*\big)d\mu(\pi) \tag{1.12}$$

and in particular

$$\|f\|^2_{L^2(G)} = \int_{\widehat{G}} \|\widehat{f}(\pi)\|_2^2 d\mu(\pi). \tag{1.13}$$

For $f \in L^2(G)*L^2(G)$ (i.e. f being a linear combination of convolution products) we have the inversion formula

$$f(g) = \int_{\widehat{G}} \mathrm{Tr}\big(\widehat{f}(\pi)\pi(g)^*\big)d\mu(\pi). \tag{1.14}$$

Note that the first part of the theorem states that $\widehat{f}(\pi)$ is a Hilbert-Schmidt operator for almost all π, and hence the trace in the Plancherel formula (1.12) is well-defined. Similarly, for $f \in L^2(G) * L^2(G)$, $\widehat{f}(\pi)$ will be a linear combination of products of the form $\widehat{f_1}(\pi)\widehat{f_2}(\pi)$ which are all trace class operators (for almost all π) and hence the trace in the inversion formula (1.14) well-defined.

Definition 1.28 (**Plancherel measure**)**.** The measure on $\widehat{G}$ whose existence is guaranteed in the theorem above is called the *Plancherel measure* of G. The support $\widehat{G}_r := \operatorname{supp} \mu$ of the Plancherel measure is called the *reduced dual* of G.

Summing it all up briefly, relative to the Plancherel measure we have the following unitary equivalences (2 and 3 following from restriction to the first resp. second factor)

$$\begin{aligned} T &\sim \int_{\widehat{G}}^{\oplus} (\pi \times \pi^*) d\mu(\pi), \\ L &\sim \int_{\widehat{G}}^{\oplus} (\pi \otimes \mathbb{I}) d\mu(\pi), \\ R &\sim \int_{\widehat{G}}^{\oplus} (\mathbb{I} \otimes \pi^*) d\mu(\pi). \end{aligned}$$

where the Fourier transform acts as the unitary intertwiner. The first equivalence is a decomposition into irreducibles, the two last are not.

The explicit determination of the Plancherel measure is one of the really difficult tasks in harmonic analysis. In the 60s and 70s Harish-Chandra succeeded in determining the reduced dual and the Plancherel measure for general non-compact semisimple Lie groups. However, the full dual $\widehat{G}$ is still unknown for all but a few of these groups.

The Plancherel formula (1.13) actually follows from the inversion formula at $e \in G$ for $f \in C_c^\infty(G)$:

$$f(e) = \int_{\widehat{G}} \mathrm{Tr}\big(\widehat{f}(\pi)\big) d\mu(\pi)$$

simply replace f by $f^* * f$ where $f^*(g) := \overline{f(g^{-1})}$ and perform a limit argument to go from $C_c^\infty(G)$ to $L^2(G)$. Hence Fourier decomposition smooth compactly supported functions at the identity element is sometimes called a Plancherel formula .

For Abelian and compact groups, we can write down explicitly the unitary dual and the Plancherel measure. For an Abelian group A, we know that the dual $\widehat{A}$ is again a topological group. The Plancherel measure turns out to be nothing but the Haar measure (properly normalized) and the direct integral $\int_{\widehat{A}}^{\oplus} \mathcal{H}_\pi \otimes \overline{\mathcal{H}}_\pi d\mu(\pi)$ is

$$\int_{\widehat{A}}^{\oplus} \mathbb{C}\, d\mu(\pi) = L^2(\widehat{A}, \mu).$$

Thus the Fourier transform is a unitary map $L^2(A) \xrightarrow{\sim} L^2(\widehat{A})$.

To determine the Plancherel measure for a compact group K, we observe the following identity

$$\mathrm{Tr}\big(\widehat{f}(\delta)\pi(k)\big) = f * \chi_\delta(k)$$

for $\delta \in \widehat{K}$ and $\chi_\delta(k) := \mathrm{Tr}(\delta(k))$. Hence the Peter-Weyl decomposition of $f \in L^2(K)$ can be written as

$$f(k) = \sum_{\delta \in \widehat{K}} d(\delta)\, \mathrm{Tr}\big(\widehat{f}(\delta)\pi(k)\big)$$

where $d(\delta)$ is the dimension of V_δ. This is a Fourier inversion formula for compact groups. Thus, Fourier theory of a compact group reduces to Peter-Weyl theory, and that the Plancherel measure on $\widehat{K}$ is nothing but

$$\mu(E) = \sum_{\delta \in E} d(\delta),$$

i.e. the counting measure weighted by the function d.

More generally, instead of considering decompositions of $L^2(G)$ for a given group G, we can consider decomposing $L^2(G/H)$ for a homogenous space G/H. The space has a natural left G-action $g_0(gH) := (g_0g)H$ which induces a *left-regular representation* of G on $L^2(G/H)$ by

$$L(g_0)f(gH) = f(g_0^{-1}gH).$$

As an example: If we view a group G' as the homogenous space $G' = (G' \times G')/G'$ (where G' is embedded in $G' \times G'$ as the diagonal), we see that the left-regular representation of $G' \times G'$ on G' (viewed as a homogenous space) equals the bi-regular representation. We refer to this example as the *group case* . Another example is a G/K where G is a semisimple group and K is a maximally compact subgroup. Both are examples of *symmetric spaces.* The latter is referred to as a *Riemannian symmetric space of non-compact type* . We return to symmetric spaces later in this chapter.

Unlike the decomposition of a group, it is no longer to be expected, that $L^2(G/H)$ will have a multiplicity free decomposition. In principle, an irreducible unitary representation could occur with infinite multiplicity. This makes it very difficult to write down a general Fourier transform for homogenous spaces.

1.7 More on Smooth Vectors

After the previous two quite general sections, we return, to the setting of G being a semisimple Lie group with finite center, and K its maximally compact subgroup. We denote by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the Cartan decomposition of the Lie algebra of G.

Definition 1.29 (Moderate Growth). A representation π of a group G with scale function s on a locally convex topological vector space V is said to be of *moderate growth* if for each seminorm p on V there exists a seminorm q on V and an integer $N > 0$ such that

$$\forall g \in G\, \forall v \in V: \; p(\pi(g)v) \leq s(g)^N q(v).$$

On p. 272 of [9] it is shown that any continuous representation of a group on a Banach space, as well as its submodule of smooth vectors are always of moderate growth.

Definition 1.30 (Smooth Representation). A continuous representation on a topological vector space V is called *smooth* if all vectors are smooth, i.e. if $V = V^\infty$.

A unitary representation π on $\mathcal{H}_\pi$ restricts to a continuous representation on the Fréchet space $\mathcal{H}_\pi^\infty$, and the smooth vectors of this representation turn out to be the same space, i.e.

$$(\mathcal{H}_\pi^\infty)^\infty = \mathcal{H}_\pi^\infty,$$

so $\mathcal{H}_\pi^\infty$ is a smooth G-representation. More contraintuitively, also $\mathcal{H}_\pi^{-\infty}$ is a smooth G-representation, i.e. $(\mathcal{H}_\pi^{-\infty})^\infty$ [11].

So smooth distribution vectors aren't necessarily vectors. But K-finite distribution vectors are:

Proposition 1.31. *If π is admissible, then $(\mathcal{H}_\pi^{-\infty})_K = \mathcal{H}_{\pi,K}$.*

PROOF. Since $\mathcal{H}_\pi \subseteq \mathcal{H}_\pi^{-\infty}$ the inclusion "$\supseteq$" is obvious. On the other hand, we have

$$(\mathcal{H}_\pi^{-\infty})_K \subseteq ((\overline{\mathcal{H}_{\pi,K}})^*)_K = (\overline{\mathcal{H}_{\pi,K}})^\sim = \mathcal{H}_{\pi,K}$$

where the last identity follows from Lemma 1.8. $\square$

Another way of transforming distribution vectors into actual vectors is to apply $\pi(f)$ for $f \in C_c^\infty(G)$ where

$$\pi(f)\eta := \int_G f(g)\pi(g)\eta dg$$

which is to be understood as an integral in $\mathcal{H}_\pi^{-\infty}$. If π is the left-regular representation, then $\pi(f)\eta$ is actually nothing but a convolution with f. One of the most important applications of a convolution in analysis is to turn "nasty" functions into nice ones. This is indeed also the case for this generalized convolution $\pi(f)$:

Proposition 1.32. *If π is unitary and $f \in C_c^\infty(G)$ and $\eta \in \mathcal{H}_\pi^{-\infty}$, then $\pi(f)\eta \in \mathcal{H}_\pi^\infty$. If η is in $\mathcal{H}_\pi^{-\infty,H}$, then the conclusion holds with a function $f \in C^\infty(G/H)$.*

PROOF. A priori we only know that $\pi(f)\eta$ is in $\mathcal{H}_\pi^{-\infty}$, i.e. there exists a $U \in U(\mathfrak{g})$ such that $|\eta(v)| \leq C\|v\|_U$ for all $v \in \mathcal{H}_\pi^\infty$. The first goal is to show that η is continuous w.r.t. the Hilbert space norm, i.e. that we can pick $U = 1$.

First we note that

$$\pi(g_0)\pi(f)v = \pi(L_{g_0}f)v$$

[11] See [5], Remark 2.12.

when v is a smooth vector. From that we get

$$\begin{aligned}
\pi(X)\pi(f)v &= \frac{d}{dt}\Big|_{t=0} \pi(\exp tX)\pi(f)v = \frac{d}{dt}\Big|_{t=0} \pi(L_{\exp tX} f)v \\
&= \frac{d}{dt}\Big|_{t=0} \int_G (L_{\exp(tX)} f)(g)\pi(g)v dg \\
&= \int_G \Big(\frac{d}{dt}\Big|_{t=0} L_{\exp(tX)} f\Big)(g)\pi(g)v dg \\
&= \int_G (Xf)(g)\pi(g)v dg = \pi(Xf)v
\end{aligned}$$

when $X \in \mathfrak{g}$. Consequently, $\pi(U)\pi(f)v = \pi(Uf)v$ for $U \in U(\mathfrak{g}_{\mathbb{C}})$.

A quick calculation shows that for $v \in \mathcal{H}_\pi^\infty$ we have

$$(\pi(f)\eta)(v) = \eta(\pi(\check{f})v)$$

where $\check{f}(g) := f(g^{-1})$. Then:

$$\begin{aligned}
(\pi(f)\eta)(v) &= |\eta(\pi(\check{f})v)| \leq C\|\pi(\check{f})v\|_U \\
&= C\|\pi(U)\pi(\check{f})v\| = C\|\pi(U\check{f})v\| \\
&\leq C'\|v\|
\end{aligned}$$

where continuity of $\pi(U\check{f})$ follows from the fact that $U\check{f}$ is compactly supported. Thus $\pi(f)\eta \in \mathcal{H}_\pi$.

Now we need to show that it is actually a smooth element in $\mathcal{H}_\pi$. By the same calculations as above, one shows that the X-derivative of $\pi(f)\eta$ exists and equals $\pi(Xf)\eta$ which is an element of $\mathcal{H}_\pi$. Since f is smooth, we can differentiate arbitrarily often, hence $\pi(f)\eta$ is smooth.

If $\eta \in \mathcal{H}_\pi^{-\infty,H}$ and $f \in C_c^\infty(G/H)$ it makes sense to define

$$\pi(f)\eta := \int_{G/H} f(x)\pi(x)\eta dx$$

as an integral in $\mathcal{H}_\pi^{-\infty,H}$. We can find a function $F \in C_c^\infty(G)$ such that

$$f(gH) = \int_H F(gh)dh$$

hence $\pi(f)\eta = \pi(F)\eta$ which is then actually in $\mathcal{H}_\pi^\infty$ by the first part of the proof. $\square$

If $v = \eta$ in the proposition above is actually in $\mathcal{H}_\pi$, then $\pi(f)v$ (and linear combinations of such) is called a *Gaarding vector* and the space of all Gaarding vectors is denoted $C_c^\infty(G)*\mathcal{H}_\pi$ (the star is there to indicate convolution). This is a subspace of the smooth vectors, and Gaarding proved that this space is dense. A famous theorem by Dixmier and Malliavin sharpens this result, as they proved that $\mathcal{S}(G) * \mathcal{H}_\pi^\infty$ (where $\mathcal{S}(G)$ denotes the space of Schwartz functions defined in (1.2)) actually *equals* the space of smooth vectors. Casselman and Wallach were able to push this even further for admissible representations

Theorem 1.33 (Casselman-Wallach). *For any admissible representation π on a Hilbert space it holds that*

$$\mathcal{H}_\pi^\infty = \mathcal{S}(G) * \mathcal{H}_{\pi,K}.$$

The result can be found in [9], Théorème 3.1.2. It is crucial that the G-module $\mathcal{H}_\pi^\infty$ is of (at most) moderate growth, otherwise it would grow too fast to be balanced by the rapidly decreasing functions in $\mathcal{S}(G)$ and the convolution wouldn't make any sense.

We will be using this result shortly when we prove that the operation of taking smooth vectors behaves nicely under tensor products. However, this involves the intricacies of topological tensor products of Fréchet spaces. To cope with these difficulties, the following result (which can be found in [5]) is extremely useful [12].

Theorem 1.34. *If $(\pi, \mathcal{H}_\pi)$ is a representation such that its associated $(\mathfrak{g}, K)$-module is a Harish-Chandra module, then the space of smooth vectors $\mathcal{H}_\pi^\infty$ with the standard Fréchet topology, is a nuclear space. In particular $\mathcal{H}_\pi^\infty$ is nuclear for $\pi \in \widehat{G}$.*

Intuitively the theorem follows from the Casselman-Wallach theorem since $\mathcal{S}(G)$ is nuclear and $\mathcal{H}_{\pi,K}$ is finitely generated.

In order to study tensor products of smooth vectors, it is fruitful to first examine how the K-finite vectors behave under tensor products of representations: Consider the outer product representation $\pi_1 \times \pi_2$ of $G_1 \times G_2$. The maximally compact subgroup is $K_1 \times K_2$ and the K-finite vectors of type $\delta_1 \times \delta_2 \in \widehat{K}_1 \times \widehat{K}_2$ are given by

$$(\mathcal{H}_{\pi_1 \times \pi_2})[\delta_1 \times \delta_2] = \mathcal{H}_{\pi_1}[\delta_1] \otimes \mathcal{H}_{\pi_2}[\delta_2]. \tag{1.15}$$

[12] The result was known to Harish-Chandra in the case where the representation was irreducible unitary. The theorem here generalizes the statement to encompass e.g. non-irreducible or non-unitary principal series representations.

The inclusion "$\supseteq$" follows trivially and the inclusion "$\subseteq$" follows from the fact that any $v \otimes w \in (\mathcal{H}_{\pi_1 \times \pi_2})_{\delta_1 \times \delta_2}$ is in the image of $(\pi_1 \times \pi_2)(\chi_{\delta_1}\chi_{\delta_2})$ (note that $\chi_{\delta_1 \times \delta_2}(k_1, k_2) = \chi_{\delta_1}(k_1)\chi_{\delta_2}(k_2)$), i.e.

$$\begin{aligned} v \otimes w &= (\pi_1 \times \pi_2)(\chi_{\delta_1 \times \delta_2})v' \otimes w' \\ &= \int_{K_1 \times K_2} (\chi_{\delta_1}(k_1)\pi_1(k_1)v') \otimes (\chi_{\delta_2}(k_2)\pi_2(k_2)w')\, dk_1 dk_2 \\ &= \Big(\int_{K_1} \chi_{\delta_1}(k_1)\pi_1(k_1)v'\, dk_1\Big) \otimes \Big(\int_{K_2} \chi_{\delta_2}(k_2)\pi_2(k_2)w' dk_2\Big) \end{aligned}$$

which is clearly in $\mathcal{H}_{\pi_1}[\delta_1] \otimes \mathcal{H}_{\pi_2}[\delta_2]$.

Consequently, by taking a direct sum over $K_1 \times K_2$-types, we arrive at

Lemma 1.35. *For any two representations π_1 and π_2 of G_1 and G_2, it holds that*

$$(\mathcal{H}_{\pi_1 \times \pi_2})_{K_1 \times K_2} = \mathcal{H}_{\pi_1, K_1} \otimes \mathcal{H}_{\pi_2, K_2} \tag{1.16}$$

where the tensor product to the right is the algebraic tensor product.

The next step, is then to ask if this holds also on the level of smooth vectors, and indeed, this is the case:

Proposition 1.36. *Let π_1 and π_2 be admissible representations of G_1 and G_2 respectively and assume that either $\mathcal{H}_{\pi_1}^\infty$ or $\mathcal{H}_{\pi_2}^\infty$ is nuclear, then it holds that $\mathcal{H}_{\pi_1 \times \pi_2}^\infty = \mathcal{H}_{\pi_1}^\infty \widehat{\otimes} \mathcal{H}_{\pi_2}^\infty$ as Fréchet spaces.*

Proof. Consider the map

$$\mathcal{H}_{\pi_1}^\infty \times \mathcal{H}_{\pi_2}^\infty \longrightarrow \mathcal{H}_{\pi_1 \times \pi_2}^\infty, \qquad (v, w) \longmapsto v \otimes w. \tag{1.17}$$

This map is continuous, for if $(v_n, w_n) \to (0, 0)$, then $\|v_n\|_{U_1} \to 0$ and $\|w_n\|_{U_2} \to 0$ for all $U_1 \in U(\mathfrak{g}_1)$ and $U_2 \in U(\mathfrak{g}_2)$. From this we get

$$\begin{aligned} \|v_n \otimes w_n\|_{U_1 \otimes U_2} &= \|\pi_1(U_1)v_n \otimes \pi_2(U_2)w_n\| \\ &= \|\pi_1(U_1)v_n\|\|\pi_2(U_2)w_n\| = \|v_n\|_{U_1}\|w_n\|_{U_2} \to 0 \end{aligned}$$

and hence that the map (1.17) is continuous. By the universal property of the projective topology (recall, the projective and injective topologies on the tensor product are identical) the map extends to a continuous injection

$$\mathcal{H}_{\pi_1}^\infty \widehat{\otimes} \mathcal{H}_{\pi_2}^\infty \hookrightarrow \mathcal{H}_{\pi_1 \times \pi_2}^\infty.$$

But by the Casselman-Wallach Theorem combined with the lemma above, we get

$$\begin{aligned}\mathcal{H}^\infty_{\pi_1\times\pi_2} &= \mathcal{S}(G_1\times G_2) * \mathcal{H}_{\pi_1\times\pi_2, K_1\times K_2} \\ &= (\mathcal{S}(G_1)\widehat{\otimes}\mathcal{S}(G_2)) * (\mathcal{H}_{\pi_1,K_1}\otimes\mathcal{H}_{\pi_2,K_2}) \\ &= \bigoplus_{(\delta_1,\delta_2)\in\widehat{K}_1\times\widehat{K}_2} (\mathcal{S}(G_1)\widehat{\otimes}\mathcal{S}(G_2)) * (\mathcal{H}_{\pi_1}[\delta_1]\otimes\mathcal{H}_{\pi_2}[\delta_2])\end{aligned}$$

and as $(\mathcal{S}(G_1)\widehat{\otimes}\mathcal{S}(G_2)) * (\mathcal{H}_{\pi_1,\delta_1}\otimes\mathcal{H}_{\pi_2,\delta_2}) = (\mathcal{S}(G_1) * \mathcal{H}_{\pi_1,\delta_1})\widehat{\otimes}(\mathcal{S}(G_2) * \mathcal{H}_{\pi_2,\delta_2})$, we conclude that $\mathcal{H}^\infty_{\pi_1\times\pi_2} = \mathcal{H}^\infty_{\pi_1}\widehat{\otimes}\mathcal{H}^\infty_{\pi_2}$ as vector spaces, i.e. that the map (1.17) is bijective. By the Banach isomorphism theorem for Fréchet spaces, the map is a homeomorphism. $\square$

1.8 Discrete Series Representations

Definition 1.37 (**Matrix coefficient**). If π is a representation of G on $\mathcal{H}_\pi$ and $v, w \in \mathcal{H}_\pi$, the continuous function on G by

$$g \longmapsto \langle v, \pi(g)w\rangle =: M_{v,w} \tag{1.18}$$

is called a *matrix coefficient* of π.

Matrix coefficients play a very important role in the Peter-Weyl theory of compact groups, as they provide a concrete realization of an irreducible subrepresentation in $L^2(K)$.

For non-compact groups, the situation is more complicated, as matrix coefficients need not be square integrable over G. For instance, the matrix coefficients for the principal series representations defined earlier are never square integrable. The ones for which the matrix coefficients are indeed square integrable, have a special name

Definition 1.38 (**Discrete series representations**). If $\pi \in \widehat{G}$ is such that all matrix coefficients are square integrable functions on G, then π is called a *discrete series representation*

For the record, we list some equivalent conditions

Proposition 1.39. *For a representation $\pi \in \widehat{G}$, the following are equivalent*

1) *π is a discrete series representation.*

2) *There exists a matrix coefficient which is square integrable.*

3) *The space* $\mathrm{Hom}_G(\mathcal{H}_\pi, L^2(G))$ *is non-trivial.*

4) *The singleton set* $\{\pi\} \subseteq \widehat{G}$ *has strictly positive Plancherel measure.*

Matrix coefficients are important, since they give an explicit realization of a discrete series representation inside $L^2(G)$. More precisely, from the abstract Plancherel theorem, we know that $L^2(G)$ decomposes as

$$L^2(G) \cong \Big(\bigoplus_{\pi \in \widehat{G}_d} \mathcal{H}_\pi \widehat{\otimes} \overline{\mathcal{H}}_\pi \Big) \oplus \int_{\widehat{G}_c}^{\oplus} \mathcal{H}_\pi \widehat{\otimes} \overline{\mathcal{H}}_\pi d\mu(\pi)$$

and the embedding of $\mathcal{H}_\pi \widehat{\otimes} \overline{\mathcal{H}}_\pi$ into $L^2(G)$ for π a discrete series representation, is by the mapping $v \otimes w \longmapsto M_{v,w}$ (thanks to the conjugation on the second factor, this is indeed a linear map).

The question of which semisimple Lie groups (with finite center) admit discrete series representations, was completely solved by Harish-Chandra in the 60's. We give here a brief description of the result as well as the classification of discrete series representations which he was also able to obtain.

Theorem 1.40 (**Harish-Chandra**). *A semisimple Lie group* G *admits discrete series representations if and only if the rank of* G *equals the rank of its maximally compact subgroup* K [13].

The classical real simple groups: First $SL(n, \mathbb{R})$, this has rank $n-1$, whereas the maximal compact subgroup $SO(n)$ has rank $\lfloor \frac{n}{2} \rfloor$ and these are identical if and only if $n = 2$, thus among the special linear groups only $SL(2, \mathbb{R})$ has a discrete series. Second $SU(p,q)$ whose maximally compact subgroup is $S(U(p) \times U(q))$ and both of these have rank $p+q-1$ and thus these groups always have a discrete series. For $SO(p,g)$ which has rank $\lfloor \frac{p+q}{2} \rfloor$ we have a maximal compact subgroup $S(O(p) \times O(q))$ which has rank $\lfloor \frac{p}{2} \rfloor + \lfloor \frac{q}{2} \rfloor$ and these are equal if and only if pq is even. Finally, $Sp(n, \mathbb{R})$ has the maximal compact subgroup $U(n)$ and both have rank n, and thus the real symplectic groups always have discrete series.

[13] Recall that the rank of a semisimple Lie group is the dimension of a Cartan subgroup. For a compact semisimple group, any maximal torus is automatically a Cartan subgroup, and hence the rank just equals the dimension of a maximal torus. In other words, the condition for the existence of discrete series representations is that a maximal torus in K is also a Cartan subgroup of G.

The theorem also accounts for a phenomenon observed by Gelfand and others in the 40s, that complex semisimple Lie groups never admit discrete series representations. This is for the following reason: If G is a complex semisimple Lie group, then it has a compact real form K, i.e. a compact subgroup whose complexification is all of G. The Cartan decomposition of $\mathfrak{g}$ is $\mathfrak{k} \oplus i\mathfrak{k}$. If $\mathfrak{t} \subseteq \mathfrak{k}$ is a maximal torus, then $\mathfrak{t} \oplus i\mathfrak{t}$ is a Cartan subalgebra in $\mathfrak{g}$ and hence

$$\frac{\operatorname{rank} G}{\operatorname{rank} K} = 2.$$

This excludes the possibility of having $\operatorname{rank} K = \operatorname{rank} G$. This also implies that groups like $SL(n, \mathbb{C})$, $SO(n, \mathbb{C})$ and $Sp(n, \mathbb{C})$ have no discrete series representations.

Now for the classification result: Recall that $\mathfrak{g}$ has the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ and assume $\operatorname{rank} G = \operatorname{rank} K$. Then a Cartan subalgebra $\mathfrak{h}_{\mathbb{C}} \subseteq \mathfrak{k}_{\mathbb{C}}$ is also a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$, consequently we can consider the two root systems

$$\Sigma_G := \Sigma(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \qquad \text{and} \qquad \Sigma_K := \Sigma(\mathfrak{k}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}).$$

Since the Cartan subalgebra $\mathfrak{h}_{\mathbb{C}}$ is contained in $\mathfrak{k}_{\mathbb{C}}$ it is seen that the roots in Σ_G are preserved under the Cartan involution, i.e. $\theta(\alpha) = \alpha$, thus the root spaces are θ-invariant and thus lie either inside $\mathfrak{k}_{\mathbb{C}}$ or inside $\mathfrak{s}_{\mathbb{C}}$. A root $\alpha \in \Sigma_G$ is called *compact* resp. *non-compact* if $(\mathfrak{g}_{\mathbb{C}})_\alpha \subseteq \mathfrak{k}_{\mathbb{C}}$ resp. $(\mathfrak{g}_{\mathbb{C}})_\alpha \subseteq \mathfrak{s}_{\mathbb{C}}$. It is easy to see that Σ_K is precisely the set of compact roots. Let W_G and W_K denote the Weyl groups of the two root systems. If we have picked a notion of positivity on Σ_G, it induces a positivity on Σ_K by $\Sigma_K^+ = \Sigma_K \cap \Sigma_G^+$. With respect to these choices of positivity let ρ_G and ρ_K denote half the sum of the positive roots of Σ_G resp. Σ_K. The following theorem and its proof can be found in [21] (Theorem 9.20).

Theorem 1.41 (**Classification of the Discrete Series**). *Under the assumption that G is semisimple with finite center and that* $\operatorname{rank} G = \operatorname{rank} K$ *and $\mathfrak{h}$ is the common Cartan subalgebra, let $\lambda \in (i\mathfrak{h})'$ be a Σ_G-regular element* (*meaning that $\langle \lambda, \alpha \rangle \neq 0$ for all $\alpha \in \Sigma_G$*) *such that $\lambda + \rho_G$ is an analytically integral element. Then there exists a discrete series representation π_λ with the following properties*:

1) *π_λ has infinitesimal character χ_λ.*

2) *$\pi_\lambda|_K$ contains with multiplicity* 1 *the K-type with highest weight $\Lambda = \lambda + \rho_G - 2\rho_K$.*

3) *If Λ' is a highest weight for a K-type appearing as a summand in $\pi_\lambda|_K$, then Λ' is of the form $\Lambda' = \Lambda + \sum_{\alpha\in\Sigma^+(\lambda)} n_\alpha \alpha$ for $n_\alpha \in \mathbb{Z}_{\geq 0}$ where $\Sigma^+(\lambda)$ consists of all the roots α satisfying $\langle \lambda, \alpha\rangle > 0$.*

Two such representations π_λ and $\pi_{\lambda'}$ are equivalent if and only if $\lambda' = w \cdot \lambda$ for $w \in W_K$, and each discrete series representation is of the form π_λ for some λ.

This result is due to Harish-Chandra in 1966, and the parameter λ is called the *Harish-Chandra parameter* . If we would rather parametrize it by its highest/lowest weight, we can parametrize it by $\lambda + \rho_G - 2\rho_K$ which is called the *Blattner parameter* . The problem of finding a global realization of the discrete series wasn't solved until much later by Schmid and others. We return to the discrete series representations in the concrete case $G = SL(2, \mathbb{R})$ in the next section.

Now, let's generalize to a homogenous space G/H. Here we can still talk about a discrete series representation, namely a representation $\pi \in \widehat{G}$ for which it holds that $\mathrm{Hom}_G(\mathcal{H}_\pi, L^2(G/H)) \neq 0$. But here we can't form matrix coefficients as we did above, simply because $g \longmapsto \langle v, \pi(g)w\rangle$ need not descend to a function on G/H. It would do so, if w was fixed by the H-action, but such H-fixed vectors need not always exist. Instead we consider for $\eta \in \overline{\mathcal{H}_\pi}^{-\infty}$ (recall that $\mathcal{H}_\pi^{-\infty}$ is the continuous dual of $\overline{\mathcal{H}}_\pi$, hence $\overline{\mathcal{H}_\pi}^{-\infty}$ consists of continuous linear functionals on $\mathcal{H}_\pi^\infty$), and $v \in \mathcal{H}_\pi^\infty$ the *generalized matrix coefficient*

$$M_{\eta,v}(g) := \eta(\pi(g^{-1})v).$$

For this function to be right H-invariant we need η to be an H-invariant distribution vector, and the space of these is denoted $\overline{\mathcal{H}_\pi}^{-\infty,H}$. The set of irreducible unitary representations of G which have non-trivial H-invariant distribution vectors is denoted $\widehat{G}_H$. This is a much weaker requirement on a representation than having an H-fixed *vector* (see also Section 3.3).

The following lemma and its proof has been taken from [18] p. 136:

Lemma 1.42. *The support of the Plancherel measure of G/H is contained in $\widehat{G}_H$, in other words, almost all of the representations occurring in the Plancherel decomposition of $L^2(G/H)$ have a non-trivial H-invariant distribution vector. In particular all discrete series representations for G/H have non-trivial H-invariant distribution vectors.*

Proof. By δ_0 we denote the Dirac distribution in eH. By a Sobolev-embedding argument, we have a continuous injection $L^2(G/H)^\infty \subseteq C^\infty(G/H)$ and hence point-evaluations on $L^2(G/H)^\infty$ are continuous. Clearly, δ_0 is a cyclic distribution vector and by Theorem 1.23 we can decompose δ_0:

$$\delta_0 = \int_{\widehat{G}}^{\oplus} \delta_0^\pi \, d\mu(\pi)$$

where μ is the Plancherel measure for the left-regular representation on G/H and where δ_0^π is cyclic for almost all π. But since δ_0 is H-invariant, we get

$$\int_{\widehat{G}}^{\oplus} \delta_0^\pi \, d\mu(\pi) = \delta_0 = L(h)\delta_0 = \int_{\widehat{G}}^{\oplus} \pi(h)\delta_0^\pi \, d\mu(\pi)$$

and thus by uniqueness of the decomposition, it follows that $\pi(h)\delta_0^\pi = \delta_0^\pi$ for almost all π. Thus, except for π in a set of measure 0, δ_0^π is cyclic (in particular non-zero) and H-invariant. Hence the support of the Plancherel measure is contained in $\widehat{G}_H$. $\square$

By $(\overline{\mathcal{H}_\pi^{-\infty}})_{\mathrm{ds}}^H$ we denote the space of H-fixed distribution vectors η for which the matrix coefficients $M_{\eta,v}$ are in $L^2(G/H)$ for all $v \in \mathcal{H}_\pi^\infty$. For a fixed η, the map $\mathcal{H}_\pi^\infty \ni v \longmapsto M_{\eta,v}$ extends to a linear map $\mathcal{H}_\pi \longrightarrow L^2(G/H)$, and this is in fact (can easily be checked) a G-intertwiner. In other words

$$(\overline{\mathcal{H}_\pi^{-\infty}})_{\mathrm{ds}}^H \cong \mathrm{Hom}_G(\mathcal{H}_\pi, L^2(G/H))$$

and therefore $(\overline{\mathcal{H}_\pi^{-\infty}})_{\mathrm{ds}}^H$ is nontrivial if and only if π is a discrete series representation. The map (embedding) $\mathcal{H}_\pi \longrightarrow L^2(G/H)$ is a realization of $\mathcal{H}_\pi$ inside $L^2(G/H)$, and the multiplicity of $\mathcal{H}_\pi$ inside $L^2(G/H)$ equals $\dim(\overline{\mathcal{H}_\pi^{-\infty}})_{\mathrm{ds}}^H$ which is always finite.

We consider in the following two special cases: the group case and the Riemannian case. If we view a group G' as a homogenous space by identifying it with $(G' \times G')/G'$, a matrix coefficient of some representation $\pi \times \pi^*$ (we know by the abstract Plancherel theory, that any representation appearing in the decomposition is of this form) of $G' \times G'$ is

$$[g_1, g_2] \longmapsto \eta(\pi \times \pi^*(g_1^{-1}, g_2^{-1})v_1 \otimes v_2)$$

where η is a linear form on $\mathcal{H}_{\pi\times\pi^*}^\infty = \mathcal{H}_\pi^\infty \widehat{\otimes} \mathcal{H}_{\pi^*}^\infty$. A natural guess for such a form would be the (restriction of the) inner product on $\mathcal{H}_\pi$:

$$v \otimes w \longmapsto \langle v, w \rangle.$$

This is obviously linear and it is continuous for the following reason: The inner product is continuous as a bilinear map $\mathcal{H}_\pi \times \overline{\mathcal{H}}_\pi \longrightarrow \mathbb{C}$, and so is its restriction to $\mathcal{H}_\pi^\infty \times \overline{\mathcal{H}}_\pi^\infty$ since the topology here is stronger. But these spaces are nuclear, hence a continuous bilinear map extends to a continuous linear map on the tensor product, by the universal property (any tensor product topology equals the projective topology). η is invariant under the diagonal action since

$$\eta(\pi \times \pi^*(g^{-1}, g^{-1})v \otimes w) = \langle \pi(g^{-1})v, \pi(g^{-1})w\rangle = \langle v, w\rangle = \eta(v \otimes w)$$

by unitarity of π. Upon identifying $G' \cong (G' \times G')/G'$ by $g \sim [g, e]$, we see that the matrix coefficient as a function on G' is given by

$$g \longmapsto \eta(\pi \times \pi^*(g^{-1}, e)v \otimes w) = \langle \pi(g^{-1})v, w\rangle = \langle v, \pi(g)w\rangle$$

which is precisely the expression for the matrix coefficient $M_{v,w}$ from (1.18).

Now we consider the Riemannian case where $H = K$ is compact. But then we have

$$\overline{\mathcal{H}}_\pi^{-\infty,K} \subseteq (\overline{\mathcal{H}}_\pi^{-\infty})_K = \overline{\mathcal{H}}_{\pi,K}.$$

Thus in this case η is just the inner product with some K-finite vector, and thus the matrix coefficient is again of the form (1.18).

1.9 Some Representation Theory of $SL(2, \mathbb{R})$

In this section we describe, in more concrete terms, the representation theory of $SL(2, \mathbb{R})$, notably we give concrete realizations of the principal and discrete series representations.

We begin by the discrete series representations, which we know exist from the discussion in the previous section. Here $K = SO(2)$ which is abelian of dimension 1, i.e. its rank is 1 which equals the rang of $SL(2, \mathbb{R})$. Thus in the complexification $\mathfrak{sl}(2, \mathbb{C})$ we can take as common Cartan subalgebra $\mathfrak{so}(2, \mathbb{C})$, i.e.

$$\mathfrak{h}_\mathbb{C} = \left\{ \begin{pmatrix} z & 0 \\ 0 & -z \end{pmatrix} \;\middle|\; z \in \mathbb{C} \right\}.$$

Just to fix notation, the functional $\lambda \in \mathfrak{h}_\mathbb{C}'$ given by

$$\lambda \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = a$$

will just be denoted a.

The root system Σ_K, in this case, is empty, whereas Σ_G consists of two roots $\pm\alpha$ where $\alpha = 2$ and with root spaces

$$(\mathfrak{g}_{\mathbb{C}})_\alpha = \mathbb{C}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad (\mathfrak{g}_{\mathbb{C}})_{-\alpha} = \mathbb{C}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

It is an easy calculation to see that the algebraically integral elements of $\mathfrak{h}'_{\mathbb{C}}$ are exactly the functionals $n \in \mathbb{Z}$ and that this is also equal to the space of analytically integral elements (recall, analytically integral elements are only defined relative to the system $(\mathfrak{k}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$). Finally, we see that $\rho_K = 0$ and that $\rho_G = \pm 1$ - the sign depending on the choice of positivity (recall, that the positivity is determined by the element λ, in this case, if $\lambda = \pm n$ for $n > 0$, then $\rho_G = \pm 1$). Thus the set of discrete series representations of $SL(2,\mathbb{R})$ is thus parametrized by $\mathbb{Z} \setminus \{0\}$. However, in the following, instead of parametrizing the representations according to the Harish-Chandra parameter λ, we will rather parametrize them by their highest or lowest weight, i.e. by their Blattner parameter. The Blattner parameter corresponding to the Harish-Chandra parameter $\pm n$ (for $n > 0$) is $\pm(n+1)$, thus we have discrete series representations $T_{\pm n}$ for $n = 2, 3, 4, \ldots$. The two classes T_n and T_{-n} for $n \geq 2$ are called the *holomorphic discrete series* and *anti-holomorphic discrete series* respectively.

Now, we will describe a concrete model for these representations. Let $\mathbb{C}^+$ denote the set of complex numbers with strictly positive imaginary part (i.e. the upper half plane). The Hilbert space for T_n is

$$\mathcal{H}_n := \Big\{ f : \mathbb{C}^+ \longrightarrow \mathbb{C} \text{ holomorphic} \;\Big|\; \int_{\mathbb{C}^+} |f(x+iy)|^2 y^{n-2} dx dy < \infty \Big\}$$

and the action is

$$T_n \begin{pmatrix} a & b \\ c & d \end{pmatrix} f(z) := (-bz + d)^{-n} f\Big(\frac{az - c}{-bz + d} \Big).$$

The space Hilbert space $\mathcal{H}_{-n}$ for T_{-n} is

$$\mathcal{H}_{-n} := \Big\{ f : \mathbb{C}^+ \longrightarrow \mathbb{C} \text{ anti-holomorphic} \;\Big|\; \int_{\mathbb{C}^+} |f(x+iy)|^2 y^{n-2} dx dy < \infty \Big\}$$

and the action of T_{-n} is given by

$$T_{-n} \begin{pmatrix} a & b \\ c & d \end{pmatrix} f(z) := (-b\overline{z} + d)^{-n} f\Big(\frac{az - c}{-bz + d} \Big).$$

It is easy to see from these definitions that $T_n^* = T_{-n}$. Remembering that anti-holomorphic functions are exactly of the form $\overline{f}$ for f holomorphic, we have

$$T_{-n}(g)\overline{f} = \overline{T_n(g)f}. \tag{1.19}$$

Furthermore, we can realize $\mathcal{H}_{-n}$ as the dual of $\mathcal{H}_n$ in that we let $\overline{f} \in \mathcal{H}_{-n}$ act on $h \in \mathcal{H}_n$ by

$$\overline{f}(h) = \langle h, f\rangle. \tag{1.20}$$

The Hilbert spaces $\mathcal{H}_n$ are the so-called *weighted Bergman spaces* over $\mathbb{C}^+$, in particular they are reproducing kernel Hilbert spaces:

Lemma 1.43. *The spaces $\mathcal{H}_{\pm n}$ are reproducing kernel Hilbert spaces, i.e. point evaluations are continuous.*

PROOF. We show it only for $\mathcal{H}_n$, from this the similar property for $\mathcal{H}_{-n}$ follows easily. Fix $z_0 = x_0 + iy_0 \in \mathbb{C}^+$ and an $R > 0$ such that the closed ball $\overline{B_R(z_0)}$ is contained in $\mathbb{C}^+$. Then for any $0 \leq r \leq R$ we have by the Cauchy formula

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})d\theta$$

and from this

$$\begin{aligned} f(z_0) &= \frac{1}{\pi R^2}\int_0^R 2\pi r f(z_0)dr = \frac{1}{\pi R^2}\int_0^R r\int_0^{2\pi} f(z_0 + re^{i\theta})d\theta dr \\ &= \frac{1}{\pi R^2}\int_{B_R(z_0)} f(z)dz. \end{aligned}$$

If $z = x + iy \in B_R(z_0)$ we have $0 < y_0 - R \leq y \leq y_0 + R$ and hence

$$\begin{aligned} |f(z_0)| &\leq \frac{1}{\pi R^2}\int_{B_R(z_0)} |f(z)|dz \\ &\leq \frac{1}{\pi R^2}\frac{1}{(y_0 - R)^{(n-2)/2}}\int\int_{B_R(z_0)} |f(x+iy)|y^{(n-2)/2}dxdy. \end{aligned}$$

From this we get by Cauchy-Schwartz

$$\begin{aligned} |f(z_0)| &\leq \frac{1}{\pi R^2}\frac{1}{(y_0 - R)^{(n-2)/2}}\Big(\int\int_{B_R(z_0)} |f(x+iy)|^2 y^{n-2}dxdy\Big)^{\frac{1}{2}}\sqrt{\pi}R \\ &= \frac{1}{\sqrt{\pi}R}\frac{1}{(y_0 - R)^{(n-2)/2}}\|f\| \end{aligned}$$

which shows that evaluation at z_0 is continuous. $\square$

In particular this simplifies the process of calculating the derived Lie algebra representation: For some $X \in \mathfrak{sl}(2,\mathbb{R})$ and $f \in \mathcal{H}_n^\infty$ the derived $T_n(X)f$ is again an element in $\mathcal{H}_n^\infty \subseteq \mathcal{H}_n$ in particular a holomorphic function on $\mathbb{C}^+$. By continuity of point evaluations

$$(T_n(X)f)(z_0) = \mathrm{ev}_{z_0}\Big(\frac{d}{dt}\Big|_{t=0} T_n(\exp tX)f\Big) = \frac{d}{dt}\Big|_{t=0}[T_n(\exp tX)f(z_0)]$$

we reduce from differentiation in a Hilbert space to differentiation of scalar-valued functions.

For the basis

$$H_0 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad X_0 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Y_0 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

the following formulas are trivially checked:

$$\begin{aligned} T_n(H_0)f(z) &= nf(z) + 2zf'(z) \\ T_n(X_0)f(z) &= nzf(z) + z^2 f'(z) \\ T_n(Y_0)f(z) &= -f'(z). \end{aligned}$$

This $\mathfrak{sl}_2$ triple has one disadvantage, however, namely that none of its elements lies in the compact Lie subalgebra $\mathfrak{k}$. Therefore we will also have to consider the following triple

$$T := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad X_+ := \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \qquad X_- := \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} \tag{1.21}$$

which satisfies

$$[T, X_+] = 2iX_+, \qquad [T, X_-] = -2iX_-, \qquad [X_+, X_-] = iT$$

i.e. (iT, X_+, X_-) is an $\mathfrak{sl}_2$-triple. Note that we now have complex matrices! As we are using these as building blocks for invariant differential operators which are elements in the enveloping algebra of the *complexification* of $\mathfrak{g}$, this is no problem.

For later use we record the following transformation formulas

$$H_0 = X_+ + X_-, \qquad X_0 = \tfrac{1}{2}T - \tfrac{i}{2}X_+ + \tfrac{i}{2}X_-, \qquad Y_0 = -\tfrac{1}{2}T - \tfrac{i}{2}X_+ + \tfrac{i}{2}X_-.$$

In particular we get

$$\omega = -\tfrac{1}{2}T^2 + X_+X_- + X_-X_+.$$

Extending T_n by complexification to a representation of $\mathfrak{sl}(2,\mathbb{C})$ we get by use of the above formulas

$$\begin{aligned} T_n(T)f(z) &= nzf(z) + (z-i)(z+i)f'(z) \\ T_n(X_+)f(z) &= \tfrac{in}{2}(z-i)f(z) + \tfrac{i}{2}(z-i)^2 f'(z) \\ T_n(X_-)f(z) &= -\tfrac{in}{2}(z+i)f(z) - \tfrac{i}{2}(z+i)^2 f'(z). \end{aligned}$$

Complexifying T_{-n} we get by (1.19)

$$T_{-n}(X)\overline{f} = \overline{T_n(\overline{X})f}. \tag{1.22}$$

The discrete series representations have a particularly nice weight structure. Namely, consider the representation T_n and define for $k \geq 0$

$$\psi_n^k(z) := \frac{(z-i)^k}{(z+i)^{k+n}}.$$

It can be checked that $\psi_n^k \in \mathcal{H}_n$. Defining $k_\theta := \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \exp(\theta T)$ we calculate that

$$T_n(k_\theta)\psi_n^k = e^{i(n+2k)\theta}\psi_n^k \qquad \text{and} \qquad T_n(T)\psi_n^k = i(n+2k)\psi_n^k.$$

Similarly, we define $\psi_{-n}^{-k} := \overline{\psi_n^k} \in \mathcal{H}_{-n}$ (i.e. the complex conjugate) and from (1.19) we get

$$T_{-n}(k_\theta)\psi_{-n}^{-k} = e^{-i(n+2k)\theta}\psi_{-n}^{-k} \qquad \text{and} \qquad T_{-n}(T)\psi_{-n}^{-k} = -i(n+2k)\psi_{-n}^{-k}$$

We phrase this by saying that ψ_n^k is a K-weight vector for T_n with weight $n+2k$ and that ψ_{-n}^{-k} is a K-weight vector for T_{-n} with weight $-n-2k$. In particular we see that T_n has a lowest weight, namely n and that T_{-n} has a highest weight, namely $-n$.

From the formulas above for $T_n(X_+)$ and $T_n(X_-)$ as well as the identity $\psi_{n-1}^k - 2i\psi_n^k = \psi_{n-1}^{k+1}$ we get

$$T_n(X_+)\psi_n^k = -(k+n)\psi_n^{k+1} \qquad \text{and} \qquad T_n(X_-)\psi_n^k = k\psi_n^{k-1}.$$

In particular we see that $T_n(X_+)$ raises the K-weight by $2i$ whereas $T_n(X_-)$ lowers the weight by $2i$ which was to be expected, given the commutation relations above. Also we see $T_n(X_-)\psi_n^0 = 0$.

For the anti-holomorphic discrete series we get by (1.22)

$$T_{-n}(X_+)\psi_{-n}^{-k} = k\psi_{-n}^{-k+1} \qquad \text{and} \qquad T_{-n}(X_-)\psi_{-n}^{-k} = -(k+n)\psi_{-n}^{-k-1}.$$

So much for the discrete series representation. We end this section by describing a concrete model for the principal series representations. In $SL(2,\mathbb{R})$ we have, up to conjugation, only 1 proper parabolic subgroup, namely the subgroup of upper triangular matrices

$$P' = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \;\Big|\; a \neq 0,\ b \in \mathbb{R} \right\}.$$

The Langlands decomposition is as follows:

$$\begin{aligned} M &= \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ A &= \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \;\Big|\; a > 0 \right\} \\ N &= \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \;\Big|\; b \in \mathbb{R} \right\} \end{aligned}$$

Hence $\widehat{M} \cong \mathbb{Z}_2$ in the sense that an irreducible unitary representation of M is either the trivial one, mapping everything to the identity, or the defining representation. We also note that A is 1-dimensional. Hence principal series representations are parametrized by the parameters $\xi = \pm 1$ and $\lambda \in \mathbb{C}$. A concrete model for the representation $\pi_{\xi,\lambda}$ is provided by the Hilbert space

$$\mathcal{H}_{\xi,\lambda} \cong L^2(\mathbb{R}, (1+x^2)^{\operatorname{Re}\lambda} dx)$$

with the following $SL(2,\mathbb{R})$-action:

$$\begin{aligned} \pi_{+,\lambda} \begin{pmatrix} a & b \\ c & d \end{pmatrix} f(x) &= |-bx+d|^{-1-\lambda} f(\frac{ax-c}{-bx+d}) \\ \pi_{-,\lambda} \begin{pmatrix} a & b \\ c & d \end{pmatrix} f(x) &= \frac{\operatorname{sign}(-bx+d)}{|-bx+d|^{-1-\lambda}} f(\frac{ax-c}{-bx+d}). \end{aligned}$$

The representations are unitary if and only if λ is purely imaginary. Moreover, they are all irreducible, unless $(\xi,\lambda) = ((-1)^{n+1}, n)$ for some $n \in \mathbb{Z}$, and if λ is imaginary, it holds that $\pi_{\xi,\lambda}$ is equivalent to $\pi_{\xi,-\lambda}$ (this follows from the general theory of principal series, since $w\xi = \xi$ for w being one of the two elements in $W(G,A)$).

It is conspicuous that the principal series are reducible for the same n's for which we have discrete series representations [14]. As a matter of fact, the discrete series representation T_n actually sits inside $\pi_{(-1)^{n+1},n}$ (thus making explicit the Casselman subrepresentation theorem in this case). More precisely, for $n \geq 2$ and $\xi = (-1)^{n+1}$ the following sequences of $(\mathfrak{g}, K)$-modules are exact

$$0 \longrightarrow (\mathcal{H}_n \oplus \mathcal{H}_{-n})_K \longrightarrow (\mathcal{H}_{\xi,n})_K \longrightarrow F_n \longrightarrow 0$$

$$0 \longrightarrow F_n \longrightarrow (\mathcal{H}_{\xi,-n})_K \longrightarrow (\mathcal{H}_n \oplus \mathcal{H}_{-n})_K \longrightarrow 0$$

where F_n is the unique irreducible representation of $SL(2,\mathbb{R})$ of dimension n. By [33] Theorem 11.6.7 (all the $(\mathfrak{g}, K)$-modules here are Harish-Chandra modules) these $(\mathfrak{g}, K)$-maps lift to continuous G-homomorphisms and hence we get exact sequences

$$0 \longrightarrow \mathcal{H}_n^\infty \oplus \mathcal{H}_{-n}^\infty \longrightarrow \mathcal{H}_{\xi,n}^\infty \longrightarrow F_n \longrightarrow 0 \tag{1.24}$$

$$0 \longrightarrow F_n \longrightarrow \mathcal{H}_{\xi,-n}^\infty \longrightarrow \mathcal{H}_n^\infty \oplus \mathcal{H}_{-n}^\infty \longrightarrow 0 \tag{1.25}$$

1.10 Generalities on Symmetric Spaces

In the following 2 chapters we set out to study triple spaces. The overall philosophy is to mimic, as closely as possibly, the theory for symmetric spaces. In this section we give an ultra-short introduction to the topic.

Definition 1.44 (**Symmetric space**)**.** Let σ be an involution on a Lie group G. A *symmetric space* is a homogenous space G/H where H is an open subgroup of the fixed-point set G^σ.

In this section we will focus solely on *semisimple* symmetric spaces, i.e. symmetric spaces where G is semisimple. The structure theory of symmetric spaces is due largely to Rossmann (see [28]) whereas the analysis and representation theory was developed by Flensted-Jensen, Oshima, van den Ban, Schlichtkrull and Delorme.

Let G be a semisimple group with finite center and with Cartan involution θ. The induced involution on the Lie algebra is also denoted θ. Let σ be a second

[14] There actually also exist certain representations of $SL(2,\mathbb{R})$ for $n = \pm 1$ - these are called *Mock discrete series representations* or *limit discrete series representations*, however we will not treat them here.

involution and let G/H be a symmetric space w.r.t. this. We can always assume σ and θ to commute, so this we will do henceforth. The Cartan involution gives a decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ and σ likewise gives a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ (we use the non-standard notation $\mathfrak{s}$ for the negative eigenspace of the Cartan involution to avoid confusion with parabolic subalgebras). As σ and θ commute, it follows that $\mathfrak{k}$ and $\mathfrak{s}$ are σ-invariant and likewise that $\mathfrak{h}$ and $\mathfrak{q}$ are θ-invariant and thus we get a decomposition

$$\mathfrak{g} = (\mathfrak{k} \cap \mathfrak{h}) \oplus (\mathfrak{k} \cap \mathfrak{q}) \oplus (\mathfrak{s} \cap \mathfrak{h}) \oplus (\mathfrak{s} \cap \mathfrak{q}).$$

Putting $\mathfrak{g}^+ := (\mathfrak{k} \cap \mathfrak{h}) \oplus (\mathfrak{s} \cap \mathfrak{q})$ and $\mathfrak{g}^- := (\mathfrak{k} \cap \mathfrak{q}) \oplus (\mathfrak{s} \cap \mathfrak{h})$ it is clear that $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$ is the decomposition of $\mathfrak{g}$ induced from the involution $\theta \circ \sigma = \sigma \circ \theta$. Moreover the decomposition $\mathfrak{g}^+ = (\mathfrak{k} \cap \mathfrak{h}) \oplus (\mathfrak{s} \cap \mathfrak{q})$ is the Cartan decomposition of $\mathfrak{g}^+$.

An abelian subalgebra $\mathfrak{a} \subseteq \mathfrak{g}$ is said to be *split* in $\mathfrak{g}$, if $\mathfrak{g}$ decomposes into a direct sum of root spaces of $\mathfrak{a}$. It turns out that an abelian subalgebra of $\mathfrak{g}$ is split if and only if it is in $\mathfrak{s}$. In particular $\mathfrak{a}$ is θ-invariant. We will consider split algebras which are also σ-invariant, hence we let $\mathfrak{a}_q \subseteq \mathfrak{s} \cap \mathfrak{q}$ be a maximally abelian subspace. The dimension of $\mathfrak{a}_q$ is called the *split rank* of the symmetric space. $\mathfrak{a}_q$ is clearly split in both $\mathfrak{g}$ and in $\mathfrak{g}^+$ so we have two root systems $\Sigma(\mathfrak{g}, \mathfrak{a}_q)$ and $\Sigma(\mathfrak{g}^+, \mathfrak{a}_q)$ and two corresponding Weyl groups $W := W(\mathfrak{g}, \mathfrak{a}_q)$ and $W_{K\cap H} := W(\mathfrak{g}^+, \mathfrak{a}_q)$. As in the Riemannian case we have $W = N_K(\mathfrak{a}_q)/Z_K(\mathfrak{a}_q)$ and $W_{K\cap H}$ is the image of $N_{K\cap H}(\mathfrak{a}_q)$ in W, hence the notation. We denote by $\mathcal{W} \subseteq N_K(\mathfrak{a}_q)$ a set of representatives for the quotient $W/W_{K\cap H}$.

One should carefully distinguish the two root systems, since they are both relevant in different situations. Let $\mathfrak{a}_{q0}^+$ denote a choice of open positive Weyl chamber in $\mathfrak{a}_q$ w.r.t. the big root system $\Sigma(\mathfrak{g}, \mathfrak{a}_q)$, i.e. a connected component of

$$\mathfrak{a}_q \setminus \bigcup_{\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a}_q)} \ker \alpha.$$

These Weyl chambers are of course smaller than the Weyl chambers for the smaller root system $\Sigma(\mathfrak{g}_+, \mathfrak{a}_q)$, and in fact

$$\bigcup_{v \in \mathcal{W}} \mathrm{Ad}(v) \overline{\mathfrak{a}_{q0}^+}$$

is a closed Weyl chamber w.r.t. $\Sigma(\mathfrak{g}_+, \mathfrak{a}_q)$. Let $\mathfrak{a}_q^+$ denote the corresponding open Weyl chamber. For the polar decomposition, it is actually the bigger Weyl chambers that we need. Let

$$\mathfrak{a}_q^{\mathrm{reg}} := \mathfrak{a}_q \setminus \bigcup_{\alpha \in \Sigma(\mathfrak{g}^+, \mathfrak{a}_q)} \ker \alpha$$

then $\mathfrak{a}_q^+$ is a connected component of $\mathfrak{a}_q^{\mathrm{reg}}$. This component determines a positive system $\Sigma^+(\mathfrak{g}^+, \mathfrak{a}_q)$ (all the roots that are positive on $\mathfrak{a}_q^+$). Put $A_q := \exp \mathfrak{a}_q$, $A_q^{\mathrm{reg}} := \exp \mathfrak{a}_q^{\mathrm{reg}}$ and $A_q^+ := \exp \mathfrak{a}_q^+$. The following result (the *polar decomposition*) is due to Flensted-Jensen ([13]):

Theorem 1.45. *Each element $g \in G$ has a decomposition $g = kah$, for $k \in K$, $a \in A_q$ and $h \in H$, i.e. $G = KA_qH$ and if $g \in KaH$, then a is uniquely determined modulo $W_{K\cap H}$. The map $\Phi : K/Z_{K\cap H}(\mathfrak{a}_q) \times \overline{A_q^+} \longrightarrow G/H$, given by*

$$(kZ_{K\cap H}, a) \longmapsto kaH$$

is surjective, and it maps $K/Z_{K\cap H}(\mathfrak{a}_q) \times A_q^+$ diffeomorphically onto an open dense subset of G/H.

The space G/H has a unique (up to a scalar) invariant measure μ. We can push it forward along Φ^{-1} from the theorem above to a measure on $K/Z_{K\cap H}(\mathfrak{a}_q) \times A_q^+$ which is absolutely continuous w.r.t. the canonical invariant measures on $K/Z_{K\cap H}(\mathfrak{a}_q)$ and A_q^+, the first one which has been normalized so that $\int_K f(k) = \int_{K/Z_{K\cap H}(\mathfrak{a}_q)} \Big(\int_{Z_{K\cap H}(\mathfrak{a}_q)} f(kz)dz \Big) d(kZ)$. In particular, for $Z_{K\cap H}(\mathfrak{a}_q)$-invariant functions on K, integration over $K/Z_{K\cap H}(\mathfrak{a}_q)$ is the same as integration over K. Thus we get $\Phi_*^{-1}\mu = J(k,a)dadk$ where $J(k,a)$ is the determinant of the Jacobian of the diffeomorphism. This Jacobian has been explicitly computed by Flensted-Jensen ([14]). To state the expression, we need to introduce some notation (and here we actually need the root system $\Sigma(\mathfrak{g}, \mathfrak{a}_q)$). Let $\Sigma^+(\mathfrak{g}, \mathfrak{a}_q)$ denote a fixed choice of positive system which contains $\Sigma^+(\mathfrak{g}^+, \mathfrak{a}_q)$. Put for $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a}_q)$

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} \mid \forall H \in \mathfrak{a}_q : \ [H, X] = \alpha(H)X\}.$$

Since $\mathfrak{a}_q$ is $\theta \circ \sigma$-invariant, it follows that also $\mathfrak{g}_\alpha$ is $\theta \circ \sigma$-invariant, and if we put $\mathfrak{g}_\alpha^\pm := \mathfrak{g}_\alpha \cap \mathfrak{g}^\pm$, we have a decomposition $\mathfrak{g}_\alpha = \mathfrak{g}_\alpha^+ \oplus \mathfrak{g}_\alpha^-$. Let $m_\alpha^\pm := \dim \mathfrak{g}_\alpha^\pm$, then

$$J(k,a) = \prod_{\alpha \in \Sigma^+} (a^\alpha - a^{-\alpha})^{m_\alpha^+} (a^\alpha + a^{-\alpha})^{m_\alpha^-},$$

(note, in particular, that the map is independent of k). Thus we get the integration formula

Corollary 1.46. *If dx is a choice of invariant measure on G/H and if dk is the normalized Haar measure on K, there exists a unique Haar measure da on A_q such that for any $f \in L^1(G/H)$:*

$$\int_{G/H} f(x)dx = \int_K \int_{A_q^+} f(kaH)J(a)dadk.$$

Example 1.47 (**The Group Case**)**.** Let G' be a semisimple group and put $G := G' \times G'$. Let $H = G'$ viewed as the diagonal in $G' \times G'$. H is the fixed-point group of the involution σ on G given by $\sigma(g_1, g_2) = (g_2, g_1)$, hence $G/H \cong G'$ is a symmetric space. This particular space was studied by Harish-Chandra who developed the full Plancherel theory for it in the 50s, 60s and early 70s. The notion of a triple space, which will be introduced in the next chapter, is a natural generalization of the group case symmetric space.

On the Lie algebra level, we have $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{g}'$ and if $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{s}'$ is the Cartan decomposition of $\mathfrak{g}'$, then $(\mathfrak{k}' \oplus \mathfrak{k}') \oplus (\mathfrak{s}' \oplus \mathfrak{s}') =: \mathfrak{k} \oplus \mathfrak{s}$ is the Cartan decomposition of $\mathfrak{g}$. The induced involution σ gives rise to a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ where $\mathfrak{h} = \{(X, X) \,|\, X \in \mathfrak{g}'\}$ and $\mathfrak{q} = \{(X, -X) \,|\, X \in \mathfrak{g}'\}$. In particular we see that $\mathfrak{s} \cap \mathfrak{q} = \{(X, -X) \mid X \in \mathfrak{s}\}$. If $\mathfrak{a}' \subseteq \mathfrak{s}'$ is maximally abelian, then $\mathfrak{a}_q := \{(X, -X) \mid X \in \mathfrak{a}\}$ is maximally abelian in $\mathfrak{s} \cap \mathfrak{q}$ (and it extends to the maximally abelian subspace $\mathfrak{a} \oplus \mathfrak{a}$ in $\mathfrak{s}$). There is a 1-1 correspondence between the two root systems $\Sigma(\mathfrak{g}', \mathfrak{a}')$ and $\Sigma(\mathfrak{g}, \mathfrak{a}_q)$, namely $\alpha' \in \Sigma(\mathfrak{g}', \mathfrak{a}')$ is mapped to $(\alpha', \alpha') \in \Sigma(\mathfrak{g}, \mathfrak{a}_q)$ and similarly there is a bijection $W(\mathfrak{g}', \mathfrak{a}') \xrightarrow{\sim} W(\mathfrak{g}, \mathfrak{a}_q)$ by mapping $w' \longmapsto (w', w')$. In this way the well-known structure theory for $\mathfrak{g}'$ is carried over to $\mathfrak{g}$.

Example 1.48 (**Hyperbolic Spaces**)**.** Consider $\mathbb{R}^{p+q}$ (for $p, q \geq 1$) equipped with the pseudo inner product

$$\langle x, y \rangle := x_1 y_1 + \cdots + x_p y_p - x_{p+1} y_{p+1} - \cdots - x_{p+q} y_{p+q}$$

and let $X_{p,q} := \{x \in \mathbb{R}^{p+q} \mid \langle x, x \rangle = 1\}$ if $p \geq 2$. If $p = 1$ the set $\{x \in \mathbb{R}^{p+q} \,|\, \langle x, x \rangle = 1\}$ has two connected components, one with $x_1 > 0$ and one with $x_1 < 0$, and we define $X_{1,q}$ to be the connected component with $x_1 > 0$. Note that the group $SO_e(p, q)$ acts transitively on $X_{p,q}$ and that the stabilizer of the point $e_1 = (1, 0, \ldots, 0)$ is $SO_e(p-1, q)$ (viewed as sitting in the lower right corner of $SO_e(p, q)$). Hence $X_{p,q} = SO_e(p, q)/SO_e(p-1, q)$ as a homogenous space. Put $I := \mathrm{Diag}(1, -1, \ldots, -1)$ and $\sigma(g) := IgI$. This is an involution on $SO_e(p, q)$ and it is clear that $SO_e(p-1, q)$ equals the fixed-point set of it. Thus $X_{p,q}$ is a symmetric space.

The hyperbolic spaces turn out to be split rank 1 spaces, for instance one could take as $\mathfrak{a}_q$ the span of $Y := E_{p+q,1} + E_{1,p+q}$ (the matrix with 1 in the upper right and lower left entry and zeros elsewhere). Defining $\alpha \in \mathfrak{a}_q^*$ by $\alpha(Y) = 1$, we get that $\Sigma(\mathfrak{g}, \mathfrak{a}_q) = \{\pm\alpha\}$. In particular $W(\mathfrak{g}, \mathfrak{a}_q) = \{\pm 1\}$. Moreover it holds that the other root space $\Sigma(\mathfrak{g}^+, \mathfrak{a}_q)$ is trivial when $q = 1$ and equal to $\Sigma(\mathfrak{g}, \mathfrak{a}_q)$ when $q > 1$. In particular $W_{K \cap H}$ is trivial in the first case and equal to $W(\mathfrak{g}, \mathfrak{a}_q)$ in the second.

We now briefly discuss parabolic subgroups. Consider the root system $\Sigma(\mathfrak{g}, \mathfrak{a}_q)$ and pick a positive system $\Sigma^+(\mathfrak{g}, \mathfrak{a}_q)$ or, equivalently, a positive Weyl chamber $\mathfrak{a}_q^+$. We let $\Delta \subseteq \Sigma^+(\mathfrak{g}, \mathfrak{a}_q)$ be a set of simple roots. For a subset $F \subseteq \Delta$, we define $\Gamma_F := \Sigma^+(\mathfrak{g}, \mathfrak{a}_q) \cup (\Sigma(\mathfrak{g}, \mathfrak{a}_q) \cap \operatorname{span} F)$. Put

$$\mathfrak{p}_F := Z_{\mathfrak{g}}(\mathfrak{a}_q) \oplus \bigoplus_{\alpha \in \Gamma_F} \mathfrak{g}_\alpha.$$

This is a parabolic subalgebra of $\mathfrak{g}$, meaning that the normalizer of $\mathfrak{p}_F$ in $\mathfrak{g}$ equals $\mathfrak{p}_F$ itself. Moreover, $\mathfrak{p}_F$ is $\sigma\theta$-invariant. Put $P_F := N_G(\mathfrak{p}_F)$, then P_F is a $\sigma\theta$-invariant parabolic subgroup of G, and the map $F \longmapsto P_F$ is a bijection between the set of subsets of Δ and the set of all $\sigma\theta$-invariant subgroup of G containing $\exp \mathfrak{a}_q$. If $F = \emptyset$, then P_F is a minimal $\sigma\theta$-invariant parabolic subgroup.

We get a Langlands decomposition of the parabolic subalgebra $\mathfrak{p} = \mathfrak{p}_F = \mathfrak{m}_{1P} \oplus \mathfrak{n}_P$ where

$$\begin{aligned} \mathfrak{m}_{1P} &= Z_{\mathfrak{g}}(\mathfrak{a}_q) \oplus \bigoplus_{\alpha \in \Gamma_F \cap (-\Gamma_F)} \mathfrak{g}_\alpha \\ \mathfrak{n}_P &= \bigoplus_{\alpha \in \Gamma_F \setminus (-\Gamma_F)} \mathfrak{g}_\alpha \end{aligned}$$

Since $\mathfrak{m}_{1P}$ is reductive, we can split it as $\mathfrak{m}_{1P} = Z(\mathfrak{m}_{1P}) \oplus [\mathfrak{m}_{1P}, \mathfrak{m}_{1P}]$, and hence we can split $\mathfrak{m}_{1P} = \mathfrak{m}_P \oplus \mathfrak{a}_P$ where

$$\begin{aligned} \mathfrak{m}_P &= (\mathfrak{m}_{1P} \cap \mathfrak{k}) \oplus ([\mathfrak{m}_{1P}, \mathfrak{m}_{1P}] \cap \mathfrak{s}) \\ \mathfrak{a}_P &= Z(\mathfrak{m}_{1P}) \cap \mathfrak{s} \end{aligned}$$

and this gives us the Langlands decomposition

$$\mathfrak{p} = \mathfrak{m}_P \oplus \mathfrak{a}_P \oplus \mathfrak{n}_P$$

which gives rise to a Langlands decomposition $P = M_P A_P N_P$ on the group level.

Inside the Weyl group $W(\mathfrak{g}, \mathfrak{a}_q)$ we have a subgroup W_P (depending on P) which is the subgroup generated by reflections in the roots in $\Gamma_F \cap (-\Gamma_F)$. Denote by $\mathcal{W}^P$ a set of representatives for the double quotient $W_P \backslash W(\mathfrak{g}, \mathfrak{a}_q) / W_{K \cap H}(\mathfrak{g}, \mathfrak{a}_q)$.

A parabolic subgroup P acts on G/H in a natural way from the left. As proved by Matsuki and Rossmann, the set of orbits in G/H of this action is finite, and the map $w \longmapsto PwH$ gives a bijection between $\mathcal{W}^P$ and the set of *open* orbits.

From these parabolic subgroup we can induce principal series representations, in the same manner as we did above, with some slight modifications. Given a parabolic subgroup P (corresponding to the subset $F \subseteq \Delta$), we define

$$\rho_P := \sum_{\alpha \in \Gamma_F \cap (-\Gamma_F)} (\dim \mathfrak{g}_\alpha)\alpha.$$

Then from $\xi \in \widehat{M_P}$ and $\lambda \in i\mathfrak{a}_q^*$, we can define a principal series representation $\pi_{P,\xi,\lambda}$, in the following way: let $C(P,\xi,\lambda)$ be the space of continuous functions $f : G \longrightarrow \mathcal{H}_\xi$ which satisfy $f(mang) = a^{\lambda+\rho_P}\xi(m)f(g)$. This space is equipped with the inner product

$$\langle f_1, f_2 \rangle := \int_K f_1(k)\overline{f_2(k)}dk$$

and the completion of $C(P,\xi,\lambda)$ w.r.t. this inner product is denoted $\mathcal{H}_{P,\xi,\lambda}$. This is the representation space for the unitary G-representation

$$\pi_{P,\xi,\lambda}(g_0)f(g) = f(g_0^{-1}g)$$

which is induced from the P-representation $(\xi,\lambda)(man) := \xi(m)a^{\lambda+\rho_P}$.

We are interested in the H-invariant distribution vectors for these representations. These can be viewed as $\mathcal{H}_\xi^{-\infty}$-valued distributions on G/H satisfying the equivariance $L_{man}\eta = a^{\lambda+\rho_P}\xi(m)\eta$. The equivariance implies that this distribution is actually a smooth function on the open P-orbits in G/H [15]. In particular it makes sense to talk about $\eta(w)$ for $w \in \mathcal{W}^P$ (which parametrise the open orbits), and it turns out that $\eta(w) \in \mathcal{H}_\xi^{-\infty}$ is actually fixed by $(\xi,\lambda)|_{P\cap wHw^{-1}}$ which again implies that $\eta(w) \in \mathcal{H}_\xi^{-\infty, M_P\cap wHw^{-1}}$ and that $\lambda|_{\mathfrak{a}_P\cap\mathfrak{h}} = 0$. Since only representation with non-trivial H-fixed distribution vectors can occur in the Plancherel decomposition, we infer from the above that these can only be induced from $\xi \in \bigcup_{w\in\mathcal{W}^P} \widehat{M}_P^{M_P\cap wHw^{-1}}$ and $\lambda \in i\mathfrak{a}_{Pq}^*$. So already this puts some restrictions on ξ and λ. But it turns out we have to shrink the possible ξ's even further: we only need the ξ's which are discrete series representations. More formally put

$$V(P,\xi,w) := (\mathcal{H}_\xi^{-\infty})_{\mathrm{ds}}^{M_P\cap wHw^{-1}}$$

[15] For some more details on this see also Section 3.8.

(which is 0, if ξ is not a discrete series representation) and

$$V(P,\xi) := \bigoplus_{w\in\mathcal{W}^P} V(P,\xi,w).$$

This is a finite-dimensional space (a non-trivial fact)!

Theorem 1.49. *The evaluation map $\eta \longmapsto (\eta(w))_{w\in\mathcal{W}^P}$ is linear $\mathcal{H}^{-\infty,H}_{P,\xi,\lambda} \longrightarrow V(P,\xi)$ which is injective for almost all $\lambda \in \mathfrak{a}^*_{Pq\mathbb{C}}$.*

The last claim follows by an application of Bruhat theory.

We can invert this map (at least for almost all λ) as follows: Assume that $\lambda \in \mathfrak{a}^*_{Pq\mathbb{C}}$ satisfies that $\operatorname{Re}\lambda + \rho_P$ is strictly P-dominant, meaning that $\langle \operatorname{Re}\lambda + \rho_P, \alpha\rangle > 0$ for all $\alpha \in \Gamma_F \cap (-\Gamma_F)$, then for $\eta \in V(P,\xi)$ we define a function $j(P,\xi,\lambda,\eta)$ on G/H by

$$j(P,\xi,\lambda,\eta)(manwH) := a^{\lambda+\rho_P}\xi(m)\eta_w$$

on the open orbit given by $w \in \mathcal{W}^P$, and zero outside the union of open orbits. This function turns out to be locally integrable, in particular a distribution, and hence $j(P,\xi,\lambda,\eta) \in \mathcal{H}^{-\infty,H}_{P,\xi,\lambda}$. The map $\eta \longmapsto j(P,\xi,\lambda,\eta)$ is obviously a right inverse of the evaluation map above (for the λ's where the above map was injective). There is one major problem, however, and that is that elements in $i\mathfrak{a}_q$ are *not* strictly P-dominant. The way to circumvent this problem is to meromorphically extend $\lambda \longmapsto j(P,\xi,\lambda,\eta)$ to all of $\mathfrak{a}^*_{Pq\mathbb{C}}$: The first, one has to realise is that one can identify $\mathcal{H}^{-\infty,H}_{P,\xi,\lambda}$ with the dual of $C^\infty(K,\xi)$ which is the space of smooth functions $f : K \longrightarrow \mathcal{H}^\infty_\xi$ satisfying $f(mk) = \xi(m)f(k)$ for $k \in K$ and $m \in K \cap M_P$. Thus we may view $\lambda \longmapsto j(P,\xi,\lambda,\eta)$ as a map from a cone in $\mathfrak{a}^*_{Pq\mathbb{C}}$ into the λ-*independent* space $C^\infty(K,\xi)'$. It is to this map, we perform a meromorphic extension:

Theorem 1.50. *The map $\lambda \longmapsto j(P,\xi,\lambda,\eta)$ extends to a meromorphic map $\mathfrak{a}^*_{Pq\mathbb{C}} \longrightarrow C^\infty(K,\xi)'$. If λ is not a pole of this map, then $j(P,\xi,\lambda,\eta)$ maps $V(P,\xi)$ injectively to $\mathcal{H}^{-\infty,H}_{P,\xi,\lambda}$ and for λ outside the complement of a countable union of complex hypersurfaces in $\mathfrak{a}^*_{Pq\mathbb{C}}$, the map $j(P,\xi,\lambda) : V(P,\xi) \longrightarrow \mathcal{H}^{-\infty,H}_{P,\xi,\lambda}$ is a bijection, the inverse being the evaluation map from above.*

CHAPTER 2

TRIPLE SPACES

2.1 Examples and Elementary Properties

We now embark on our study of so-called *triple spaces* which are a straightforward generalizations of the group case for symmetric spaces:

Definition 2.1 (Triple Space). A *triple space* is a homogenous space G/H where $G = G' \times G' \times G'$ for some Lie group G' and where $H = G' = \{(g, g, g) \in G \mid g \in G'\}$ is the diagonal.

In the following we will only consider triple spaces for G' semisimple.

A triple space G/H is a *reductive homogenous space* , meaning that $\mathfrak{h}$ has a vector space complement $\mathfrak{q}$ which is $\mathfrak{h}$-invariant in the sense that it satisfies $[\mathfrak{h}, \mathfrak{q}] \subseteq \mathfrak{q}$. Unlike in the symmetric space case, where one picks the -1 eigenspace of the involution σ, the is no canonical way of picking a complementary subspace in this case. One example is $\mathfrak{q} := \{(X_1, X_2, X_3) \in \mathfrak{g}' \oplus \mathfrak{g}' \oplus \mathfrak{g}' \mid X_1 + X_2 + X_3 = 0\}$. This is the complement to $\mathfrak{h}$ which is orthogonal to $\mathfrak{h}$ w.r.t. the Killing form, and in this sense we may call this choice of complement "canonical". More generally, if $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$ satisfies that $\lambda_1 + \lambda_2 + \lambda_3 \neq 0$, then

$$\mathfrak{q}_\lambda := \{(X_1, X_2, X_3) \in \mathfrak{g}' \oplus \mathfrak{g}' \oplus \mathfrak{g}' \mid \lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3 = 0\}$$

is an $\mathfrak{h}$-invariant complement to $\mathfrak{h}$. In fact any $\mathfrak{h}$-invariant complement is of this form:

Proposition 2.2. *Let G' be simple, then $\mathfrak{q}_\lambda$ as defined above is an $\mathfrak{h}$-invariant complement to $\mathfrak{g}'$ and any $\mathfrak{h}$-invariant complement is of this form.*

Proof. We have already seen that $\mathfrak{q}_\lambda$ is an $\mathfrak{h}$-invariant complement. Assume $\mathfrak{q}$ to be an $\mathfrak{h}$-invariant complement, and let $\operatorname{proj}_{\mathfrak{h}} : \mathfrak{g} \longrightarrow \mathfrak{h}$ be the projection onto $\mathfrak{h}$ along $\mathfrak{q}$. Note that both $\mathfrak{g}'$ and $\mathfrak{h}$ are $\mathfrak{g}'$-modules in obvious ways (they are even equivalent) and as $\mathfrak{g}'$ is simple, they are irreducible. The two maps $\mathfrak{g}' \longrightarrow \mathfrak{h}$ given by $X \longmapsto \operatorname{proj}_{\mathfrak{h}}(X,0,0)$ and $X \longmapsto (X,X,X)$ respectively, are $\mathfrak{g}'$-homomorphisms. By Schur's lemma there must exist a constant $\lambda_1 \in \mathbb{R}$ such that

$$\operatorname{proj}_{\mathfrak{h}}(X,0,0) = \lambda_1(X,X,X).$$

Similarly there exist constants λ_2 and λ_3 such that

$$\begin{aligned}\operatorname{proj}_{\mathfrak{h}}(0,X,0) &= \lambda_2(X,X,X),\\ \operatorname{proj}_{\mathfrak{h}}(0,0,X) &= \lambda_3(X,X,X).\end{aligned}$$

This means that

$$\begin{aligned}&\operatorname{proj}_{\mathfrak{h}}(X_1,X_2,X_3) =\\ &\qquad(\lambda_1X_1+\lambda_2X_2+\lambda_3X_3, \lambda_1X_1+\lambda_2X_2+\lambda_3X_3, \lambda_1X_1+\lambda_2X_2+\lambda_3X_3)\end{aligned}$$

and hence

$$\mathfrak{q} = \ker \operatorname{proj}_{\mathfrak{h}} = \mathfrak{q}_\lambda.$$

This proves the proposition. □

However, one big difference to the symmetric case is that we can no longer find an $\mathfrak{h}$-invariant $\mathfrak{h}$-complement containing a maximally abelian subalgebra $\mathfrak{a}$: If G' has rank 1 (and that is our sole concern in the following), a maximally abelian subalgebra of $\mathfrak{g}$ is of the form $\mathfrak{a} = \mathbb{R}(X_1,0,0) \oplus \mathbb{R}(0,X_2,0) \oplus \mathbb{R}(0,0,X_3)$ for $X_i \in \mathfrak{g}'$. It is quite clear that there exists no λ for which $\mathfrak{a}$ is contained in $\mathfrak{q}_\lambda$.

For the proof of the Plancherel formula for symmetric spaces, it seems crucial that symmetric pairs exhibit the following 2 features: (1) the polar decomposition and (2) the existence of open orbits in G/H of the action of a minimal parabolic subgroup P. This motivates the following definition which was introduced in [24] (here we restrict to semisimple groups):

Definition 2.3 (**Strongly spherical pair**). Let G be a semisimple Lie group, and let H be a closed subgroup. Let $\mathfrak{a} \subseteq \mathfrak{s}$ be a maximally abelian subalgebra of the negative Cartan eigenspace of G. The pair (G, H) is called *strongly spherical* if there exist minimal parabolic subgroups $P_1, \ldots, P_n$ all containing $\exp \mathfrak{a}$ such that the following are satisfied

1) $P_i H$ is open for all $i = 1, \ldots, n$.

2) $G = \bigcup_{i=1}^{n} K\overline{A_i^+}H$.

In 2) A_i^+ is the open Weyl chamber corresponding to the set of positive roots determined by the parabolic subgroup P_i.

In [24] it is showed that a symmetric pair is in particular also strongly spherical.

In the following we show that the triple spaces for $G' = SL(2, R)$ or $G' = SO_e(n, 1)$ are strongly spherical. The first will be our prime example, and the entire next chapter is devoted to this space alone.

Theorem 2.4. *The triple space with $G' = SL(2, \mathbb{R})$ is strongly spherical.*

PROOF. To see that this is spherical, it suffices to find a minimal parabolic subgroup of the triple product whose Lie algebra $\mathfrak{p}$ satisfies $\mathfrak{p} + \mathfrak{sl}(2, \mathbb{R}) = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$ (the copy of $\mathfrak{sl}(2, \mathbb{R})$ on the left hand side is to be viewed as the diagonal). Since $\mathfrak{p}$ is just a sum of minimal parabolic subalgebras in the three copies of $\mathfrak{sl}(2, \mathbb{R})$, it is 6-dimensional, and thus, for dimension reasons, it suffices to show that $\mathfrak{p} \cap \mathfrak{sl}(2, \mathbb{R}) = \{0\}$ (again viewed as the diagonal in the triple product) is 0. This is equivalent to showing that $\mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{p}_3 = \{0\}$.

But actually, we prove a little more than that. We prove that if $\mathfrak{p}_i = \mathfrak{a}_i \oplus \mathfrak{n}_i$ are parabolic subalgebras [1], where $\mathfrak{a}_i \cap \mathfrak{a}_j = \{0\}$ when $i \neq j$, then $\mathfrak{p}_1 \times \mathfrak{p}_2 \times \mathfrak{p}_3 + \mathfrak{h} = \mathfrak{g}$. W.l.o.g. we may assume that $\mathfrak{a}_1 = \mathbb{R}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Let $\theta_2, \theta_3 \in [0, 2\pi[$ be so that $\mathrm{Ad}(k_{\theta_i})\mathfrak{a}_1 = \mathfrak{a}_i$ where $k_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in SO(2)$ is the standard rotation by the angle θ. It then follows that $\mathfrak{p}_i = \mathrm{Ad}(k_{\theta_i})\mathfrak{p}_1$. By assumption on the $\mathfrak{a}_i$'s, we have $\sin\theta_i \neq 0$. A calculation shows that a general element of $\mathfrak{p}_i$ has the form

$$\begin{pmatrix} a\cos^2\theta_i - b\cos\theta_i\sin\theta_i - a\sin^2\theta_i & 2a\cos\theta_i\sin\theta_i + b\cos^2\theta_i \\ 2a\cos\theta_i\sin\theta_i - b\sin^2\theta_i & a\sin^2\theta_i - a\cos^2\theta_i + b\cos\theta_i\sin\theta_i \end{pmatrix}$$

[1]Note that $SL(2, \mathbb{R})$ is a so-called *split group* , meaning that the $\mathfrak{m}$-part of a minimal parabolic is always 0.

and from this (putting $b = \frac{2a\cos\theta_i}{\sin\theta_i}$) we read off that

$$\mathfrak{p}_1 \cap \mathfrak{p}_i = \mathbb{R}\begin{pmatrix} 1 & -2\cot\theta_i \\ 0 & -1 \end{pmatrix}$$

and since cot is injective and $\theta_2 \neq \theta_3$ we get

$$\mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{p}_3 = (\mathfrak{p}_1 \cap \mathfrak{p}_2) \cap (\mathfrak{p}_1 \cap \mathfrak{p}_3) = \{0\}.$$

Now we want to show that the space is also of polar type. This is a bit more difficult. Here we sketch the idea of the proof. For the details consult [10] (which is added as an appendix to this thesis). Let $A_i := \exp \mathfrak{a}_i$ for $i = 1, 2, 3$ be abelian subgroups of $SL(2,\mathbb{R})$ such that the span of $\mathfrak{a}_1$, $\mathfrak{a}_2$ and $\mathfrak{a}_3$ in $\mathfrak{s}$ is two-dimensional (this condition on the $\mathfrak{a}_i$'s is weaker than above for the sphericality). For the abelian subgroup $A = A_1 \times A_2 \times A_3$, the existence of the KAH-decomposition or equivalently the HAK-decomposition is proved using a geometric argument: Mod'ing out K to the right, we should prove that $G/K = HA\cdot o$, in other words, given a triple of elements $(z_1, z_2, z_3) \in G/K = G'/K' \times G'/K' \times G'/K'$ there should exist an element $g \in H = G'$ and $(a_1, a_2, a_3) \in A$ such that $z_i = ga_i \cdot o$. First we consider the similar situation in $\mathbb{R}^2$, so we have 3 lines ℓ_1, ℓ_2 and ℓ_3 passing through 0, and we have 3 distinct points x_1, x_2 and x_3 and we want an isometry that brings x_i to lie on line ℓ_i. First, we may assume that $x_1 = 0$ and that $x_2 \in \ell_2$, this is by two-point homogeneity of $\mathbb{R}^2$. Now we slide the triangle given by x_1, x_2 and x_3 by isometries in such a way that x_1 stays on line ℓ_1 and x_2 stays on ℓ_2, until x_2 reaches $-x_2$. But then x_3 will have changed to $-x_3$ and it will have done so following a continuous curve. Thus at some point, x_3 will have passed the line ℓ_3. The argument in $SL(2,\mathbb{R})/SO(2)$ is completely analogous, with straight lines through 0 replaced by geodesic curves through eK, and using that $SL(2,\mathbb{R})/SO(2)$ is 2-point homogenous (since it is of rank 1).

So now we have showed that if $\mathfrak{a}_i \cap \mathfrak{a}_j = \{0\}$ (thus automatically spanning a two-dimensional subspace of $\mathfrak{s}$), then we have a KAH-decomposition and for any minimal parabolic subgroup P_i containing $A_i = \exp \mathfrak{a}_i$, we have $(P_1 \times P_2 \times P_3)H$ open. If we split up

$$A = A_1 \times A_2 \times A_3 = \bigcup_{w \in W(G,A)} w^{-1} A^+ w$$

where A^+ is some fixed Weyl chamber in A^+, then we get

$$G = KAH = \bigcup_{w \in W(G,A)} K\overline{A^+} wH.$$

$W(G,A) = W' \times W' \times W'$ where $W' \cong \mathbb{Z}_2$ is the Weyl group of $SL(2,\mathbb{R})$. If we put $W_{K\cap H} = \{\pm(1,1,1)\}$, then we see $K\overline{A^+}wH = K\overline{A^+}H$ if $w \in W_{K\cap H}$, thus we get

$$G = \bigcup_{w\in W(G,A)/W_{K\cap H}} K\overline{A^+}wH.$$

Since all the parabolic subgroups $w^{-1}Pw$ contain $\mathfrak{a} = \mathfrak{a}_1 \times \mathfrak{a}_2 \times \mathfrak{a}_3$ it follows from the first part of the proof that $w^{-1}PwH$ is open, and hence the definition 2.3 is satisfied. $\square$

Note, however, that the geometric argument fails in higher dimension than 2: In $\mathbb{R}^3$ we could have the three points $(1,0,0)$, $(0,0,0)$ and $(-1,0,0)$ and as lines we could have the 3 coordinate axes. It is quite obvious that it is impossible to move isometrically the 3 points to the three lines. That seems to put the restriction on the $\mathfrak{a}_i$'s that they should span a space of dimension 2. That is what we will assume in the example below. It can be shown by a direct calculation in the case $G' = SO_e(3,1)$ that the KAH-decomposition fails, if the $\mathfrak{a}_i$'s span a 3-dimensional space!

Theorem 2.5. *For $G' = SO_e(n,1)$ the pair $(G' \times G' \times G', G')$ is strongly spherical.*

Proof. First we note that $SO_e(2,1)$ is locally isomorphic to $SL(2,\mathbb{R})$, hence this case is already captured by the example above. So we need only consider $SO_e(n,1)$ for $n \geq 3$. We can use induction to reduce to the case $n = 2$.

The proof follows the same line of reasoning as in the $SL(2,\mathbb{R})$ case. Let $K' = SO(n)$ sitting in the upper left corner of $SO_e(n,1)$ and $Z' := G'/K'$. Putting $o := e_{n+1} \in \mathbb{R}^{n+1}$, then the map $gK' \longmapsto g\cdot o$ sends Z' bijectively to $\{z \in \mathbb{R}^{n+1} \mid z_1^2 + ... + z_n^2 - z_{n+1}^2 = -1\}$. This is our model of Z'.

From the Cartan decomposition we have $\mathfrak{s}' \cong \mathbb{R}^n$ where the column $X \in \mathbb{R}^n$ can be embedded into $\mathfrak{so}(n,1)$ as the symmetric matrix

$$\begin{pmatrix} 0 & X \\ X^T & 0 \end{pmatrix}.$$

Now, put $G := G' \times G' \times G'$ and $H := \mathrm{Diag}(G')$ and $K := K' \times K' \times K'$. Pick nonzero $X_1, X_2, X_3 \in \mathfrak{s}'$ which span a 2-dimensional subspace in $\mathfrak{s}$. Let $A_i := \exp(\mathbb{R}X_i)$ and $A := A_1 \times A_2 \times A_3$. The claim is that we can use this A for the KAH-decomposition. Or rather for the equivalent HAK-decomposition.

Following the idea from the $SL(2,\mathbb{R})$-case, an equivalent statement of the HAK-decomposition is that for all triples $(z_1,z_2,z_3) \in Z' \times Z' \times Z'$, there exists $g \in G'$ and $a_i \in A_i$ such that $g \cdot z_i = a_i \cdot o$. Note that K' acts like $SO(n)$ on $Z' \cong \mathfrak{s}' \cong \mathbb{R}^n$, and hence with an element of K' we can rotate the 2-dimensional subspace spanned by X_1, X_2, X_3 so that it is embedded in $\mathfrak{s}' \cong \mathbb{R}^n$ as $\{(0, \cdots, 0, *, *)\}$.

As mentioned, our strategy is to reduce to $n = 2$, so put $\widetilde{G}' = SO_e(2,1)$ which embeds into $SO(n,1)$ as

$$\begin{pmatrix} I_{n-2} & 0 \\ 0 & \widetilde{G}' \end{pmatrix}$$

Note that $\widetilde{G}'$ contains A_1, A_2 and A_3. Put $\widetilde{G} = \widetilde{G}' \times \widetilde{G}' \times \widetilde{G}'$. By the $\widetilde{H}A\widetilde{K}$-decomposition of $\widetilde{G}$ it suffices to show that $G = H\widetilde{G}K$. In other words we need to show that for all triples $(z_1, z_2, z_3) \in Z' \times Z' \times Z'$ there exists $g \in G'$ such that $g \cdot z_1, g \cdot z_2, g \cdot z_3$ are all in $G' \cdot o$, i.e. viewed as vectors in $\mathbb{R}^{n+1}$ they have 0 in the first $n-2$ coordinates.

Since G' acts transitively on Z' we can pick g_1 so that $g_1 \cdot z_1 = o$. Since K' is transitive on the unit ball in $\mathbb{R}^n$, we can pick $k_2 \in K'$ in such a way that $k_2 g_1 \cdot z_2$ is of the form $(0, \ldots, 0, *)$ (note that k_2 stabilizes $o = g_1 \cdot z_1$). Finally we can pick $k_3 \in SO(n-1)$ (sitting in $SO(n)$ in the upper left corner) so that $k_3 k_2 g_1 \cdot z_3$ is of the form $(0, \ldots, 0, *, *)$ (note that k_3 stabilizes both $g_1 \cdot z_1$ and $k_2 g_1 \cdot z_2$). Put $g = k_3 k_2 g_1$ and we are done proving that it is polar.

Now to prove that it is also spherical: we use induction, and since and since $SO_e(2,1)$ is locally isomorphic to $SL(2,\mathbb{R})$ (in particular they have the same Lie algebras), we already know it to be true in this case. Thus we consider $SO_e(n,1)$ for $n \geq 3$ (where $SO_e(m,1)$ for $m < n$ sits in the lower right corner of $SO_e(n,1)$). To ease notation let

$$\mathfrak{g}_n := \mathfrak{so}(n,1) \oplus \mathfrak{so}(n,1) \oplus \mathfrak{so}(n,1)$$

and let $\mathfrak{h}_n$ be the diagonal in that.

Let $\mathfrak{a}_1$, $\mathfrak{a}_2$ and $\mathfrak{a}_3$ be 1-dimensional subalgebras of $\mathfrak{s}'$ which together span a 2-dimensional subspace. W.l.o.g. we may assume that $\mathfrak{a}_i = \mathbb{R}H_i$ where

$$H_i := \begin{pmatrix} 0 & \widetilde{q}_i \\ \widetilde{q}_i & 0 \end{pmatrix}. \tag{2.1}$$

and

$$\widetilde{q}_i := \begin{pmatrix} 0 \\ q_i \end{pmatrix}.$$

for q_1, q_2 and q_3 some unit vectors in $\mathbb{R}^{n-1}$ which span a subspace of dimension 2. Let $\mathfrak{p}_i = \mathfrak{m}_i \oplus \mathfrak{a}_i \oplus \mathfrak{n}_i$ be the corresponding parabolic subalgebra. Our goal is it to show that

$$\mathfrak{g}_n = (\mathfrak{p}_1 \times \mathfrak{p}_2 \times \mathfrak{p}_3) + \mathfrak{h}_n$$

Our induction assumption is that this holds for $n-1$ and hence it suffices to show that

$$\mathfrak{g}_n = (\mathfrak{p}_1 \times \mathfrak{p}_2 \times \mathfrak{p}_3) + \mathfrak{h}_n + \mathfrak{g}_{n-1}$$

since $\mathfrak{p}_1 \times \mathfrak{p}_2 \times \mathfrak{p}_3 \subseteq \mathfrak{g}_{n-1}$. From the Bruhat decomposition we know that $\mathfrak{g}_n = (\mathfrak{p}_1 \times \mathfrak{p}_2 \times \mathfrak{p}_3) \oplus (\overline{\mathfrak{n}}_1 \times \overline{\mathfrak{n}}_2 \times \overline{\mathfrak{n}}_3)$ so it suffices to show that $\overline{\mathfrak{n}}_1 \times \overline{\mathfrak{n}}_2 \times \overline{\mathfrak{n}}_3 \subseteq (\mathfrak{p}_1 \times \mathfrak{p}_2 \times \mathfrak{p}_3) + \mathfrak{h}_n + \mathfrak{g}_{n-1}$. Actually it is enough to verify that $\overline{\mathfrak{n}}_1 \subseteq \mathfrak{p}_1 + \mathfrak{p}_2 \cap \mathfrak{p}_3 + \mathfrak{so}(n-1,1)$ (and similarly for $\overline{\mathfrak{n}}_2$ and $\overline{\mathfrak{n}}_3$) for that being the case, an $X \in \overline{\mathfrak{n}}_1$ can be written as $X = A + B + C$, and then

$$(X,0,0) = (A,-B,-B) + (B,B,B) + (C,0,0)$$

where $(A,-B,-B) \in \mathfrak{p}_1 \times \mathfrak{p}_2 \times \mathfrak{p}_3$, $(B,B,B) \in \mathfrak{h}_n$ and $(C,0,0) \in \mathfrak{g}_{n-1}$.

Now to prove that $\overline{\mathfrak{n}}_1 \subseteq \mathfrak{p}_1 + \mathfrak{p}_2 \cap \mathfrak{p}_3 + \mathfrak{so}(n-1,1)$ actually holds. First we note that $\mathfrak{m} + \mathfrak{n} = \{X \in \mathfrak{so}(n,1) \mid X \begin{pmatrix} \widetilde{q} \\ 1 \end{pmatrix} = 0\}$ (to ease the notation we leave out the subscript i for the moment, since it holds for any unit vector $q \in \mathbb{R}^n$). First we check that the dimensions agree. To that end, consider the map $\mathfrak{so}(n,1) \longrightarrow \mathbb{R}^{n+1}$ given by $X \longmapsto X \begin{pmatrix} \widetilde{q} \\ 1 \end{pmatrix}$. Since q has norm 1, we can find $A \in SO(n)$ such that $A\widetilde{q} = (0,\dots,0,1)^T$, so if we put $\widetilde{A} := \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$ then

$$\widetilde{A} \begin{pmatrix} \widetilde{q} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 1 \end{pmatrix} =: v_0.$$

It is easy to check that $\mathfrak{so}(n,1)v_0 = \{w \in \mathbb{R}^{n+1} \mid w_n = w_{n+1}\}$ and hence of dimension n. Thus we get

$$\mathfrak{so}(n,1) \begin{pmatrix} \widetilde{q} \\ 1 \end{pmatrix} = \widetilde{A}(\mathfrak{so}(n,1)v_0)$$

in other words, the image of the map $\mathfrak{so}(n,1) \longrightarrow \mathbb{R}^{n+1}$ has dimension n. Thus the kernel has dimension $\frac{1}{2}n(n-1)$ which is exactly equal to $\dim \mathfrak{m} + \dim \mathfrak{n}$ (use the $\overline{N}MAN$-decomposition along with the fact that $\dim A = 1$). Hence we only have to check that $\mathfrak{m}$ and $\mathfrak{n}$ lie in the kernel. A general matrix in $\mathfrak{so}(n,1)$ has the form $X = \begin{pmatrix} A & b \\ b^T & 0 \end{pmatrix}$ for $A \in \mathfrak{so}(n)$ and $b \in \mathbb{R}^n$. A quick calculation shows

$$[H, X] = \begin{pmatrix} \widetilde{q}b^T - b\widetilde{q}^T & -A\widetilde{q} \\ \widetilde{q}^T & 0 \end{pmatrix} \tag{2.2}$$

when H is of the form (2.1).

If $X \in \mathfrak{k}$, then $b = 0$ and the calculation above shows that $X \in \mathfrak{m}$ (i.e. $[H, X] = 0$) if and only if $A\widetilde{q} = 0$ which is equivalent to

$$X \begin{pmatrix} \widetilde{q} \\ 1 \end{pmatrix} = 0$$

and hence X is in the kernel.

Now for $\mathfrak{n}$: The condition $[H, X] = X$ leads to $A = \widetilde{q}b^T - b\widetilde{q}^T$ and $b = -A\widetilde{q}$, hence

$$b = -A\widetilde{q} = -(\widetilde{q}b^T - b\widetilde{q}^T)\widetilde{q} = -\widetilde{q}b^T\widetilde{q} + b$$

and hence $\widetilde{q}b^T\widetilde{q} = 0$. Using that $\|\widetilde{q}\| = 1$ we obtain $b^T\widetilde{q} = \widetilde{q}^T\widetilde{q}b^T\widetilde{q} = 0$, thus $X \in \mathfrak{n}$ implies

$$X \begin{pmatrix} \widetilde{q} \\ 1 \end{pmatrix} = \begin{pmatrix} A\widetilde{q} + b \\ b^T\widetilde{q} \end{pmatrix} = 0.$$

All in all we have $\mathfrak{m}+\mathfrak{n} \subseteq \{X \in \mathfrak{so}(n,1) \,|\, X \begin{pmatrix} \widetilde{q} \\ 1 \end{pmatrix} = 0\}$ and for dimension reasons as outlined above, the inclusion is actually an identity.

So now we know how the parabolic generated by q behaves.

Now we consider $\overline{\mathfrak{n}}_1$ which is known to have dimension $n-1$. But since $\mathfrak{a}_1$ from which it is formed lies inside $\mathfrak{so}(n-1,1)$, we know that $\overline{\mathfrak{n}}_1 \cap \mathfrak{so}(n-1,1)$ has dimension $n-2$. The matrix

$$Y = \begin{pmatrix} 0 & q_1^T & 1 \\ -q_1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

is in $\overline{\mathfrak{n}}_1$ and not in $\mathfrak{so}(n-1,1)$. Thus we are done, if we can show that $Y \in \mathfrak{p}_1 + \mathfrak{p}_2 \cap \mathfrak{p}_3$. Define

$$U := \begin{pmatrix} 0 & q_1^T - r^T & c \\ -(q_1 - r) & 0 & 0 \\ c & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad V := \begin{pmatrix} 0 & r^T & 1-c \\ -r & 0 & 0 \\ 1-c & 0 & 0 \end{pmatrix}$$

for some $r \in \mathbb{R}^{n-1}$ and some $c \in \mathbb{R}$. Clearly, $Y = U + V$. Since the q_i's span a space of dimension 2, we can find λ_i's such that $\lambda_1 q_1 + \lambda_2 q_2 + \lambda_3 q_3 = 0$ and such that $\lambda_1 + \lambda_2 + \lambda_3 \neq 0$. Using the characterization of the parabolic, we see that $U \in \mathfrak{p}_1$ and $V \in \mathfrak{p}_2 \cap \mathfrak{p}_3$ if and only if

$$\begin{aligned} r \cdot q_1 &= 1 + c \\ r \cdot q_2 &= -1 + c \\ r \cdot q_3 &= -1 + c \end{aligned}$$

This system of equations must have a solution r, if

$$\lambda_1(1 + c) + \lambda_2(-1 + c) + \lambda_3(-1 + c) = 0,$$

i.e. if

$$c = \frac{\lambda_1 - \lambda_2 - \lambda_3}{\lambda_1 + \lambda_2 + \lambda_3}.$$

So with this c and r, we can split Y into two components lying in $\mathfrak{p}_1$ and $\mathfrak{p}_2 \cap \mathfrak{p}_3$ respectively.

By the same reasoning as in the end of the proof of Theorem 2.4 it follows that Definition 2.3 is satisfied (in the previous proof we only used that $SL(2, \mathbb{R})$ has rang 1). $\square$

All examples of triple spaces so far have the common feature that G' is a group of rank 1. It is still unknown if there are groups of higher rank for which the triple space is strongly spherical (or just spherical and/or polar).

On the other hand it is also clear that not every triple space is strongly spherical, or even spherical. A necessary condition for a triple space $G' \times G' \times G'/G'$ to be spherical is of course that $\dim \mathfrak{p} + \dim \mathfrak{h} \geq \dim \mathfrak{g} = 3 \dim \mathfrak{g}'$. Writing $\mathfrak{h} = \mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{a}' \oplus \mathfrak{n}'$ and $\mathfrak{p}' = \mathfrak{m}' \oplus \mathfrak{a}' \oplus \mathfrak{n}'$, we see that the condition can be restated as $3 \dim \mathfrak{m}' + \dim \mathfrak{a}' + \dim \mathfrak{n}' - 2 \dim \mathfrak{k}' \geq 0$. And using that $\dim \mathfrak{k}' = \dim \mathfrak{n}' + \dim \mathfrak{m}'$ we obtain the necessary condition

$$\dim \mathfrak{m}' + \dim \mathfrak{a}' - \dim \mathfrak{n}' \geq 0.$$

Below this number has been calculated for various real simple Lie groups (the relevant information for the different groups can be found in Appendix C of [22]):

- $SL(n, \mathbb{R})$: $\frac{1}{2}(-n^2 + 3n - 2)$
- $Sp(2n, \mathbb{R})$: $n - n^2$.

- $SU(p,q)$, for $1 \leq p \leq q$: $3p + p^2 + q^2 - 4pq - 1$
- $SO_e(2p, 2q+1)$, for $1 \leq p \leq q$: $2p^2 + 2q^2 - 8pq + p + q$
- $SO_e(2p, 2q+1)$, for $p > q \geq 0$: $2p^2 + 2q^2 - 8pq - 5p + 5q + 3$
- $SO_e(2p+1, 2q+1)$, for $0 \leq p \leq q$: $2p^2 + 2q^2 - 8pq + 3p - 3q + 1$
- $SO_e(2p, 2q)$, for $1 \leq p \leq q$: $2p^2 + 2q^2 - 8pq + 5p - q$.

From this list a lot of groups can be ruled out as building blocks for spherical triple spaces. For example, among the groups $SL(n, \mathbb{R})$ only $SL(2, \mathbb{R})$ works! Among all the groups $SU(1,q)$, which are all of rank 1, the above inequality is satisfied for $q \neq 2$, i.e. the rank 1 group $SU(1,2)$ cannot act as a building block for a spherical triple space.

Triple spaces are examples of so-called *Gross-Prasad spaces* [2]:

Definition 2.6 (**Gross-Prasad space**). Given a symmetric pair (G,H), the space $(G \times H)/H$ (where we view H as the diagonal subgroup $\{(h,h) \,|\, h \in H\}$ of $G \times H$) is called a *Gross-Prasad space* provided the pair $(G \times H, H)$ is spherical and of polar type.

In [24] some examples of strongly spherical Gross-Prasad spaces are given, namely those coming from the pairs $(GL(n+1, \mathbb{R}), GL(n, \mathbb{R}))$ as well as $(O(p, q+1), O(p,q))$, $(U(p, q+1), U(p,q))$ and $(Sp(p, q+1), Sp(p,q))$. We will briefly return to these spaces when discussing invariant differential operators later in this chapter.

2.2 Uniqueness in the Polar Decomposition

If G/H is a homogeneous space of polar type, so that every element $g \in G$ allows a decomposition $g = kah$, it is of interest to know to which extend the components in this decomposition are unique. More specifically, we are interested in uniqueness of the A-component, and hence we ask: if $KaH = Ka'H$ what is the relation between a and a'? Let's turn it around and assume $HaK = Ha'K$, then we may view aK and $a'K$ as two triples (a_1K', a_2K', a_3K') and $(a_1'K', a_2'K', a_3'K')$ in $G/K = G'/K' \times G'/K' \times G'/K'$, and $HaK = Ha'K$ is now the equality of two H-orbits in G/K. These orbits are equal if and

[2] The name stems from the work of Benedict Gross and Dipendra Prasad who in [16] considered branching from $SO_N \times SO_{N-1}$ (over a local field) to the diagonally embedded subgroup SO_{N-1}.

only if there exists $h = (g, g, g) \in H$ such that $haK = a'K$, i.e. $g \in G'$ is an isometry of G'/K' which maps the triple (a_1K', a_2K', a_3K') to the triple $(a_1'K', a_2'K', a_3'K')$.

An obvious non-uniqueness is caused by the normalizer $N_{K\cap H}(\mathfrak{a})$ of $\mathfrak{a}$ in $K \cap H$, which acts on A by conjugation. In the case of a symmetric space, it is known (see [18], Prop. 7.1.3) that the A component of every $g \in G$ is unique up to such conjugation. For our current triple spaces the description of which elements in A generate the same $K \times H$ orbit appears to be more complicated, unless $\mathfrak{a}_1 = \mathfrak{a}_2 \perp \mathfrak{a}_3$ (see further remarks on this at the end of this section).

Theorem 2.7. *Let G/H be the triple space with G' either $SL(2,\mathbb{R})$ or $SO_e(n,1)$, and let $\mathfrak{a} = \mathfrak{a}_1 \times \mathfrak{a}_2 \times \mathfrak{a}_3$ be a maximally abelian subalgebra with $\mathfrak{a}_1 = \mathfrak{a}_2 \perp \mathfrak{a}_3$. Let $a = (a_1, a_2, a_3) \in A$ with $a_1 \neq a_2$ and let $a' = (a_1', a_2', a_3') \in A$. Then $KaH = Ka'H$ if and only if a and a' are conjugate by $N_{K\cap H}(\mathfrak{a})$.*

We first determine explicitly which pairs of elements $a, a' \in A$ are $N_{K\cap H}(\mathfrak{a})$-conjugate when $\mathfrak{a}_1 = \mathfrak{a}_2 \perp \mathfrak{a}_3$.

Lemma 2.8. *Let $\mathfrak{a}$ be as above. Then $a, a' \in A$ are conjugate by $N_{K\cap H}(\mathfrak{a})$ if and only if*

1) $(a_1', a_2') = (a_1, a_2)^{\pm 1}$ *and* $a_3' = a_3^{\pm 1}$ *if* $n > 2$

2) $(a_1', a_2', a_3') = (a_1, a_2, a_3)^{\pm 1}$ *if* $n = 2$.

Proof. The normalizer $N_{K\cap H}(\mathfrak{a})$ consists of all the diagonal elements $k = (k_0, k_0, k_0) \in G$ for which

$$k_0 \in N_{K'}(\mathfrak{a}_1) \cap N_{K'}(\mathfrak{a}_2) \cap N_{K'}(\mathfrak{a}_3).$$

As elements $a_j, a_j' \in A_j$ are $N_{K'}(\mathfrak{a}_j)$-conjugate if and only if $a_j' = a_j^{\pm 1}$, only the pairs mentioned under (1) can be conjugate when $\mathfrak{a}_1 = \mathfrak{a}_2$.

Let $\delta, \varepsilon = \pm 1$. For the groups $G' = SL(2,\mathbb{R})$ or $G' = SO_e(n,1)$ the adjoint representation $K \longrightarrow SO(\mathfrak{s}')$ is surjective. If $n > 2$ then there exists a transformation in $\mathrm{SO}(\mathfrak{s}')$ which acts by δ on $\mathfrak{a}_1 = \mathfrak{a}_2$ and by ε on $\mathfrak{a}_3$. Its preimages in K' conjugate (a_1, a_2, a_3) to $(a_1^\delta, a_2^\delta, a_3^\varepsilon)$. When $n = 2$ such a transformation exists if and only if $\delta = \varepsilon$. The lemma follows. $\square$

The following lemmas are used in the proof of Theorem 2.7. Here G' can be any semisimple group with Cartan decomposition $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{s}'$.

Lemma 2.9. *Let $X, U \in \mathfrak{s}'$. Then $\exp X \exp U \exp X \in \exp \mathfrak{s}'$.*

Proof. Let θ denote the Cartan involution and note that the product $\exp(tX)\exp(tU)\exp(tX)$ belongs to $S := \{g \in G' \mid \theta(g) = g^{-1}\}$ for all $t \in [0,1]$. It is easily seen that $k \exp Y \in S$ implies $k^2 = e$ for $k \in K_0$ and $Y \in \mathfrak{s}'$, and since e is isolated in the set of elements of order 2 it follows that $\exp \mathfrak{s}'$ is the identity component of S. Hence $\exp X \exp U \exp X \in \exp \mathfrak{s}'$. □

Lemma 2.10. *Let $\mathfrak{a}' \subset \mathfrak{s}'$ be a one-dimensional subspace and let $A' = \exp \mathfrak{a}'$.*

1) *If $g \in \exp \mathfrak{s}'$ and $ga_0 \in a_0' K'$ for some $a_0, a_0' \in A'$, then $g = a_0' a_0^{-1}$.*

2) *If $g \in G'$ and $ga_1, ga_2 \in A'K'$ for some $a_1, a_2 \in A'$ with $a_1 \neq a_2$ then $g \in N_{K'}(\mathfrak{a}')A'$.*

Proof. (1) It follows from $ga_0 \in a_0'K'$ that $a_0 g a_0 \in a_0 a_0' K'$. Since $a_0 g a_0 \in \exp \mathfrak{s}'$ by Lemma 2.9, it follows from uniqueness of the Cartan decomposition that $a_0 g a_0 = a_0 a_0'$ and thus $g = a_0' a_0^{-1}$.

(2) Put $z_0 = eK'$, then $A'.z_0$ is a geodesic in G'/K'. Since g maps two distinct points on $A'.z_0$ into $A'.z_0$, it maps the entire geodesic onto itself, and hence so does g^{-1}. In particular $g^{-1}.z_0 \in A'K'$, that is, $g = k_0 a_0$ for some $k_0 \in K'$, $a_0 \in A'$. It follows for all $a \in A'$ that

$$k_0 a k_0^{-1} = g a_0^{-1} a k_0^{-1} \in gA'K' = A'K'.$$

As $k_0 a k_0^{-1} \in \exp \mathfrak{s}'$, uniqueness of the Cartan decomposition implies $k_0 a k_0^{-1} \in A'$, i.e. $k_0 \in N_{K'}(\mathfrak{a}')$. □

Lemma 2.11. *Let $\mathfrak{a}_1, \mathfrak{a}_3 \subset \mathfrak{s}'$ be one-dimensional subspaces with $\mathfrak{a}_1 \perp \mathfrak{a}_3$ and let $A_1 = \exp \mathfrak{a}_1$, $A_3 = \exp \mathfrak{a}_3$. If $g \in N_{K'}(\mathfrak{a}_1)A_1$ and $ga_3 \in a_3'K'$ for some $a_3, a_3' \in A_3$, not both equal to e, then $g \in N_{K'}(\mathfrak{a}_1) \cap N_{K'}(\mathfrak{a}_3)$.*

Proof. We may assume $a_3' \neq e$, as otherwise we interchange it with a_3 and replace g by g^{-1}. We consider the geodesic triangle in G'/K' formed by the geodesics

$$L_1 := A_1.z_0, \quad L_2 := A_3.z_0, \quad L_3 := gA_3.z_0.$$

The vertices are

$$D_3 := z_0, \quad D_2 := g.z_0, \quad D_1 := ga_3.z_0 = a_3'.z_0.$$

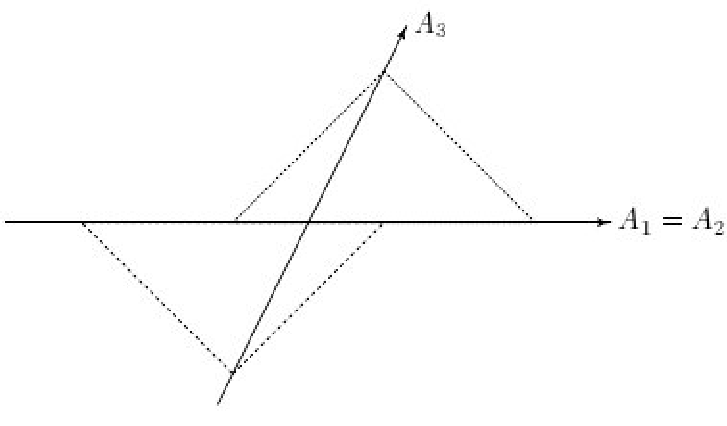
As L_1 and L_2 intersect orthogonally, angle D_3 is right. The isometry g maps L_1 to itself and L_2 to L_3. Hence L_1 and L_3 also intersect orthogonally and angle D_2 is right. As the sectional curvature of G'/K' is non-positive, it is impossible for a proper triangle to have two right angles. As $L_1 \neq L_2$ and $D_3 \neq D_1$ we conclude $D_3 = D_2$ and $L_3 = L_2$. It follows that $g \in K'$ and by Lemma 2.10 (2) that $g \in N_{K'}(\mathfrak{a}_3)$. $\square$

Proof of Theorem 2.7. Assume $KaH = Ka'H$. Then $Kah = Ka'$ for some $h = (g, g, g) \in H$. Applying Lemma 2.10 (2) to the first two coordinates of $Kah = Ka'$ we conclude that $g \in N_{K'}(\mathfrak{a}_1)A_1$.

If a'_3 and a_3 are not both e, we can apply Lemma 2.11 to the last coordinate and conclude $g \in N_{K'}(\mathfrak{a}_1) \cap N_{K'}(\mathfrak{a}_3)$. Hence $h \in N_{K\cap H}(\mathfrak{a})$, and we conclude that $a' = h^{-1}ah$.

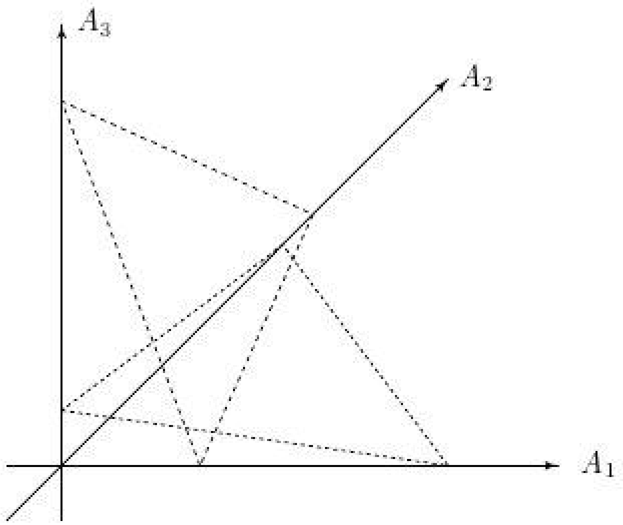
If $a'_3 = a_3 = e$ it follows from the third coordinate that $g \in K'$. Hence $g \in N_{K'}(\mathfrak{a}_1)$ and $a' = a$ or $a' = a^{-1}$. $\square$

As promised some remarks in the case where we do not assume $\mathfrak{a}_1 = \mathfrak{a}_2 \perp \mathfrak{a}_3$. Assume $\mathfrak{s}$ to have dimension at least 3, and suppose that we have $\mathfrak{a}_1 = \mathfrak{a}_2$ not perpendicular to $\mathfrak{a}_3$. In the Euclidean setting, this would look like the following (the plane symbolizing the space spanned by $\exp \mathfrak{a}_1$ and $\exp \mathfrak{a}_3$):



We see from the figure that we have two triples which can be mapped to each other by an isometry which does not preserve the origin (since it takes place in a three dimensional space, it is the composition of a rotation around the $A_1 = A_2$ axis and a translation). In particular this would correspond to an isometry that cannot be conjugation by an element in $N_{K\cap H}(\mathfrak{a})$.

If we skip the other requirement, that $\mathfrak{a}_1$ be equal to $\mathfrak{a}_2$ (but still require $\mathfrak{a}_1$, $\mathfrak{a}_2$ and $\mathfrak{a}_3$ to span a 2-dimensional subspace), we could get a situation like the following (again in the Euclidean setting).



Again we see that we can move the one triple into the other by an isometry that does not preserve the origin, i.e. cannot be a simple conjugation by an element in $N_{K\cap H}(\mathfrak{a})$.

2.3 Plancherel Decomposition

In this section we will study triple spaces in full generality (with G' being a semisimple group), in the next chapter we will restrict further to the case where $G' = SL(2, \mathbb{R})$.

Let K' be a maximally compact subgroup of G', then $K := K' \times K' \times K'$ is a maximally compact subgroup of G.

For an irreducible symmetric pair (G, H) (i.e. a pair such that the symmetric space cannot be written as a product of strictly smaller symmetric spaces) where H is connected, it is known that H is a maximal connected subgroup in the sense, that if H_0 is another connected subgroup such that $H \subseteq H_0 \subseteq G$, then H_0 is either H or G. In the triple case, this is no longer true, albeit almost

Proposition 2.12. *Let H_0 be a connected subgroup satisfying $H \subseteq H_0 \subseteq G$, then H_0 is either H, G or one of the following 3 groups*

$$\begin{aligned} H_1 &:= \{(x, g, g) \mid x, g \in G'\} \\ H_2 &:= \{(g, x, g) \mid x, g \in G'\} \\ H_3 &:= \{(g, g, x) \mid x, g \in G'\}. \end{aligned}$$

PROOF. Consider for $i = 1, 2, 3$ the projection $p_i : G' \times G' \times G' \longrightarrow G' \times G'$ which leaves out the i'th component. The image $p_i(H_0)$ is a connected subgroup of $G' \times G'$ containing the diagonal. Since $(G' \times G', G')$ is a symmetric pair, there are only two possibilities, $p_i(H_0)$ equals the diagonal or everything. That basically leaves us four different cases to check.

First case: If $p_i(H_0)$ equals the diagonal for all i, then $H_0 = H$.

Second case: If $p_i(H_0)$ equals $G' \times G'$ and the two others equal the diagonal: this can't occur.

Third case: If $p_i(H_0)$ equals the diagonal for some i and the two others equal $G' \times G'$, then $H_0 = H_i$.

Fourth case: If $p_i(H_0) = G' \times G'$ for all i, then $H_0 = G$. □

In certain situations it can be useful to have a concrete model for the triple space, just as in the group case, where the group itself is a model for the space $G' \times G'/G'$. The map

$$[g_1, g_2, g_3] \longmapsto (g_1 g_3^{-1}, g_2 g_3^{-1})$$

is easily seen to be a well-defined diffeomorphism $\varphi : G/H \xrightarrow{\sim} G' \times G'$ with inverse

$$(g_1, g_2) \longmapsto [g_1, g_2, e].$$

Note, however, that this concrete model is rather non-canonical: we singled out the third component. Of course, this component is nothing special, we could just as well have picked the first or the second. In the following we will use this model mostly for calculations, and to the extend possible we will try to formulate results without reference to the model.

G/H has a natural G-action $\rho_g = \rho_{(g_1,g_2,g_3)}$ given by componentwise left multiplication. Under the diffeomorphism φ above this is turned into the following G-action on $G' \times G'$:

$$(g_1, g_2, g_3) \cdot (g_1', g_2') = (g_1 g_1'(g_3)^{-1}, g_2 g_2'(g_3)^{-1})$$

which we also denote ρ. We can write it a little more compactly,

$$\varphi \circ \rho_{(g_1,g_2,g_3)} = \ell_{(g_1,g_2)} \circ r_{(g_3,g_3)} \circ \varphi. \tag{2.3}$$

where $\ell_{(g_1,g_2)}$ and $r_{(g_3,g_3)}$ are the left and right actions of $G' \times G'$ on itself (cf. Section 1.4).

Proposition 2.13. *Let G' be a semisimple group. Then the triple space G/H has a G-invariant measure (unique up to a scalar), and under the diffeomorphism φ above, this measure coincides (up to scalar multiplication) with the Haar measure on $G' \times G'$. In particular we have the following isomorphisms of G-spaces*

$$L^2(G/H) \cong L^2(G' \times G') \cong L^2(G')\widehat{\otimes}L^2(G') \tag{2.4}$$

where the first isomorphism is $f \longmapsto f \circ \varphi^{-1}$.

Proof. Since G and H are both semisimple, they are unimodular, hence an invariant measure on G/H exists.

Let μ denote the Haar measure on $G' \times G'$, then $(\varphi^{-1})_*(\mu)$ by (2.3) satisfies

$$\rho_{(g_1,g_2,g_3)*}(\varphi^{-1})_*(\mu) = (\varphi^{-1})_* \circ \ell_{(g_1,g_2)*} \circ r_{(g_3,g_3)*}(\mu) = (\varphi^{-1})_*(\mu)$$

by left and right invariance of μ. Thus $(\varphi^{-1})_*(\mu)$ is G-invariant and therefore must equal the G-invariant measure on G/H (up to a nonzero scalar).

The last claim of the proposition now easily follows. □

Our goal in the following is to develop a Plancherel decomposition for triple spaces. First we consider a compact example, to see what we can expect.

Example 2.14. Let's consider the space compact triple space

$$SU(2) \times SU(2) \times SU(2)/SU(2).$$

The Plancherel formula for a compact homogenous space K/M tells us that

$$L^2(K/M) = \bigoplus_{\pi \in \widehat{K}_M} \mathrm{Hom}(V_\pi, V_\pi^M).$$

In this case where $K = SU(2) \times SU(2) \times SU(2)$, the dual $\widehat{K}$ consists of all representations of the form $\pi_1 \times \pi_2 \times \pi_3$ on $V_1 \otimes V_2 \otimes V_3$. To find the M-fixed vectors we make the following general observation: assume $G_1 \subseteq G_2$ are two arbitrary Lie groups and that we have representations π_i of G_i on V_i. Then we can form a representation π of $G_1 \times G_2$ on $\mathrm{Hom}(V_1, V_2)$ by

$$\pi(g_1, g_2)(T) := \pi_2(g_2) \circ T \circ \pi_1(g_1)^{-1}.$$

Viewing G_1 as the diagonal in $G_1 \times G_2$, the invariant vectors in $\mathrm{Hom}(V_1, V_2)$ are simply $\mathrm{Hom}_{G_1}(V_1, V_2)$, i.e. the set of intertwining operators between π_1 and $\pi_2|_{G_2}$. Moreover

$$\mathrm{Hom}(V_1, V_2) \cong V_1^* \otimes V_2$$

as $G_1 \times G_2$-spaces and thus

$$(V_1^* \otimes V_2)^{G_1} \cong \mathrm{Hom}_{G_1}(V_1, V_2).$$

We can apply this to the following situation: assume we have a group G and 3 finite-dimensional G-representations V_1, V_2 and V_3. Put $G_2 := G \times G$ and let $G_1 = G$ be sitting in G_2 as the diagonal. Consider now the triple product $V_1 \otimes V_2 \otimes V_3$ under the action of $G \times G \times G$, and write it as $V_1 \otimes (V_2 \otimes V_3)$ under the action of $G \times (G \times G)$. Viewing G as the diagonal in $G \times G$ we are in the situation as before and we get

$$(V_1 \otimes V_2 \otimes V_3)^G \cong \mathrm{Hom}_G(V_1^*, V_2 \otimes V_3).$$

Note, that the representation $V_2 \otimes V_3$ on the right is the restriction *to the diagonal*, i.e. it is to be considered a representation of G and not of $G \times G$ (in which case it would have been irreducible and the Hom_G space would have been either 0 or 1-dimensional)!

Obviously, there is nothing special about V_1, even though we have singled it out here. We have a natural isomorphism $V_1 \otimes V_2 \otimes V_3 \xrightarrow{\sim} V_2 \otimes (V_1 \otimes V_3)$ and hence a natural isomorphism $\mathrm{Hom}_G(V_1^*, V_2 \otimes V_3) \xrightarrow{\sim} \mathrm{Hom}_G(V_2^*, V_1 \otimes V_3)$ which are of course also isomorphic to $\mathrm{Hom}_G(V_3^*, V_1 \otimes V_2)$.

Now we return to $SU(2)$. Given a representation $V_1 \otimes V_2 \otimes V_3$, in order to find the space $(V_1 \otimes V_2 \otimes V_3)^{SU(2)}$ (the dimension of which ultimately equals the multiplicity of $\pi_1 \times \pi_2 \times \pi_3$ in the Plancherel decomposition of $L^2(K/M)$) we have to decompose $\pi_1 \otimes \pi_2$ according to Clebsch-Gordan and find the multiplicity of π_3^* in this decomposition.

Recall that the irreducible representations of $SU(2)$ are ρ_n for $n \in \mathbb{Z}_{\geq 0}$ of dimension $n + 1$ and that $\rho_n \otimes \rho_m$ decomposes according to

$$\rho_n \otimes \rho_m = \rho_{|n-m|} \oplus \rho_{|n-m|+2} \oplus \cdots \oplus \rho_{n+m-2} \oplus \rho_{n+m}. \tag{2.5}$$

Moreover, all these representations are self-dual. From the discussion above we infer that $\rho_n \times \rho_m \times \rho_k$ occurs in the Plancherel decomposition if and only if $k \in \{|n-m|, |n-m|+2, \dots, n+m\}$, and they do so with multiplicity 1.

A concrete example: does the representation $\rho_5 \times \rho_7 \times \rho_1$ occur in the decomposition? Well, according to our recipe, we form the tensor product $\rho_7 \otimes \rho_1$ and decompose according to Clebsch-Gordan:

$$\rho_5 \otimes \rho_1 = \rho_4 \oplus \rho_6.$$

We see that ρ_1 does not occur in this, and hence that $\rho_5 \times \rho_1 \times \rho_1$ does not occur in the Plancherel decomposition. Then what about, say $\rho_5 \times \rho_7 \times \rho_{12}$. Again we decompose

$$\rho_7 \otimes \rho_{12} \cong \rho_5 \oplus \rho_7 \oplus \cdots \rho_{17} \oplus \rho_{19}$$

and see that ρ_5 does occur, and hence $\rho_5 \times \rho_7 \times \rho_{12}$ is present in the Plancherel decomposition.

This is the idea we want to generalize to non-compact triple spaces.

Recall from the section on Fourier transforms that $L^2(G')$ can be decomposed as

$$L^2(G') \cong \int_{\widehat{G'}}^{\oplus} \mathcal{H}_\pi \widehat{\otimes} \overline{\mathcal{H}}_\pi d\mu(\pi)$$

and that the bi-regular representation $T_{G'}$ of $G' \times G'$ on $L^2(G')$ is decomposed as

$$T_{G'} \sim \int_{\widehat{G'}}^{\oplus} (\pi \times \pi^*) d\mu(\pi).$$

Replacing G' by $G' \times G'$ we have a bi-regular representation $T_{G' \times G'}$ of $G' \times G' \times G' \times G'$ on $L^2(G' \times G')$ which is given by

$$\begin{aligned} T_{G' \times G'}(g_1, g_2, g_3, g_4) f(g, g') &= L_{G' \times G'}(g_1, g_2) \circ R_{G' \times G'}(g_3, g_4) f(g, g') \\ &= f(g_1^{-1} g g_3, g_2^{-1} g' g_4) \end{aligned}$$

The action of $G' \times G' \times G'$ on G/H gives a representation ρ of $G' \times G' \times G'$ on $L^2(G/H)$. Under the isomorphism $L^2(G/H) \cong L^2(G' \times G')$ this representation is simply (cf. (2.3))

$$\rho(g_1, g_2, g_3) f(g, g') = f(g_1^{-1} g g_3, g_2^{-1} g' g_3).$$

I.e. if we inject $G' \times G' \times G'$ into $G' \times G' \times G' \times G'$ by

$$(g_1, g_2, g_3) \longmapsto (g_1, g_2, g_3, g_3),$$

ρ is simply the restriction of $T_{G'\times G'}$ to $G'\times G'\times G'$. Decomposing $T_{G'\times G'}$ we get

$$T_{G'\times G'} \sim \int_{\widehat{G}'\times\widehat{G}'}^{\oplus} (\pi_1\times\pi_2)\times(\pi_1\times\pi_2)^* d\mu(\pi_1)d\mu(\pi_2).$$

Restricting to $G'\times G'\times G'$ (which comes down to restricting $(\pi_1\times\pi_2)^*$ to the diagonal of $G'\times G'$) and replacing π_i by π_i^* (this is allowed, since π is in $\widehat{G}'$ if and only if π^* is) we get

$$\rho \sim \int_{\widehat{G}'\times\widehat{G}'}^{\oplus} (\pi_1^*\times\pi_2^*)\times(\pi_1\otimes\pi_2)d\mu(\pi_1)d\mu(\pi_2). \tag{2.6}$$

Decomposing the tensor product $\pi_1\otimes\pi_2 = \int_{\widehat{G}'} \pi_3^{\oplus m_{\pi_3}} d\nu_{\pi_1,\pi_2}(\pi_3)$ (cf. Theorem 1.22) we arrive at

Theorem 2.15. *The decomposition of ρ into irreducibles is*

$$\begin{aligned}\rho &\sim \int_{\widehat{G}'\times\widehat{G}'}^{\oplus} \pi_1^*\times\pi_2^*\times\Big(\int_{\widehat{G}'}^{\oplus} \pi_3^{\oplus m_{\pi_3}}\, d\nu_{\pi_1,\pi_2}(\pi_3)\Big)d\mu(\pi_1)d\mu(\pi_2)\\ &\sim \int_{\widehat{G}'\times\widehat{G}'\times G'}^{\oplus} (\pi_1^*\times\pi_2^*\times\pi_3)^{\oplus m_{\pi_3}}\, d\nu_{\pi_1,\pi_2}(\pi_3)d\mu(\pi_2)d\mu(\pi_1)\end{aligned} \tag{2.7}$$

where μ is the Plancherel measure on $\widehat{G}'$.

This is the general version of the observation from the compact case above. In effect, the result states that the triple product $\pi_1^*\times\pi_2^*\times\pi_3$ "occurs" in the decomposition of ρ if and only if π_3 "occurs" in the decomposition of the tensor product $\pi_1\otimes\pi_2$, and with the same multiplicity.

Just as for groups, we can talk about *discrete series representations* for a homogenous space:

Definition 2.16 (Discrete Series). A representation $(\pi,\mathcal{H}_\pi)\in\widehat{G}_H$ is called a *discrete series representation* for G/H if $\mathrm{Hom}_G(\mathcal{H}_\pi, L^2(G/H))\neq\{0\}$.

It is clear from the theorem above, that $\pi_1\times\pi_2\times\pi_3$ is a discrete series representation for the triple space G/H, if and only if π_1 and π_2 are discrete series representations with π_3^* sitting discretely in $\pi_1\otimes\pi_2$, and it is clear that the multiplicity of $\pi_1\times\pi_2\times\pi_3$ in $L^2(G/H)$ equals that of π_3 in $\pi_1^*\otimes\pi_2^*$. If this multiplicity is finite we can write down an explicit isomorphism

$$\Phi : \mathrm{Hom}_{G'}(\mathcal{H}_3, \mathcal{H}_1\widehat{\otimes}\mathcal{H}_2) \xrightarrow{\ \sim\ } \mathrm{Hom}_G(\mathcal{H}_1^*\widehat{\otimes}\mathcal{H}_2^*\widehat{\otimes}\mathcal{H}_3, L^2(G/H))$$

given by

$$\Phi(S)(\varphi_1 \otimes \varphi_2 \otimes v_3)[g_1, g_2, g_3] := \big(\pi_1^*(g_1^{-1})\varphi_1 \otimes \pi_2^*(g_2^{-1})\varphi_2\big)\, S(\pi_3(g_3^{-1})v_3). \quad (2.8)$$

2.4 Invariant Differential Operators

In this section we investigate the algebra of invariant differential operators $\mathbb{D}(G/H)$ first for a general triple space. More specifically, our first concern will be, to determine, for which of the strongly spherical spaces listed previously in this chapter, the algebra of invariant differential operators is commutative. It turns out to be the case for only a few of them.

Definition 2.17. Let G be a Lie group and let K be a compact subgroup. The pair (G, K) is called a *Gelfand pair* if for any irreducible unitary representation π of G, it holds that the multiplicity of the trivial representation of K in $\pi|_K$ is at most 1.

In the older literature, the terminology will often be used that K is a *spherical subgroup* of G.

As the following proposition states there are other (equivalent) ways of defining Gelfand pairs. We took the above one as the defining property since it is the one which is best suited for our needs.

Proposition 2.18. *For a pair (G, K) of a Lie group and a compact subgroup K, the following are equivalent*

1) *(G, K) is a Gelfand pair.*

2) *The algebra $\mathbb{D}(G/K)$ of left G-invariant differential operators on G/K is commutative.*

3) *The convolution algebra $L^1(K\backslash G/K)$ is commutative.*

The equivalence between 1) and 2) is proved in Proposition 6.3.1 in [12] and the equivalence between 2) and 3) is stated in Theorem 4 in [11].

Well known examples are non-compact Riemannian symmetric pairs (G, K) where G is a reductive Lie group and K is a maximally compact subgroup. Because of the duality between compact and non-compact Riemannian symmetric spaces, also compact Riemannian symmetric pairs (U, K) are Gelfand pairs.

Definition 2.19. A pair (G, K) of a Lie group G and a compact subgroup K is called a *strong Gelfand pair* provided that $(G \times K, \Delta K)$ (where ΔK means the K-diagonal in $G \times K$) is a Gelfand pair.

Proposition 2.20. *A pair (G, K) where G is connected and K is connected and compact, is a strong Gelfand pair if and only if for any $\pi \in \widehat{G}$ and any $\delta \in \widehat{K}$, the multiplicity of δ in $\pi|_K$ is at most 1.*

Proof. If $\pi \in \widehat{G}$ and $\delta \in \widehat{K}$, we note the identities [3]

$$\mathcal{H}_{\pi\times\delta}^{\Delta K} = (\mathcal{H}_\pi \widehat{\otimes} \mathcal{H}_\delta)^{\Delta K} = \mathrm{Hom}_K(\delta^*, \pi|_K)$$

where δ^* is the dual of δ. This leads to the following string of identities of dimensions:

$$\begin{aligned}\dim \mathrm{Hom}_K(1, (\pi \times \delta)|_{\Delta K}) &= \dim \mathrm{Hom}_K(\pi \times \delta|_{\Delta K}, 1) = \dim \mathcal{H}_{\pi\times\delta}^{\Delta K} \\ &= \dim \mathrm{Hom}_K(\delta^*, \pi|_K).\end{aligned}$$

Since any irreducible unitary representation of $G \times K$ is of the form $\pi \times \delta$, it follows that $(G \times K, \Delta K)$ is a Gelfand pair, if and only if the multiplicity of δ^* in $\pi|_K$ is at most 1. □

From the proposition it is clear that any strong Gelfand pair is in particular a Gelfand pair, hence justifying the name. Note that if (G, K) and (G', K') are strong Gelfand pairs, then so is $(G \times G', K \times K')$, simply because, if $\pi|_K = \bigoplus_{\delta\in\widehat{K}} \delta^{\oplus m_\delta}$ and $\pi'|_{K'} = \bigoplus_{\tau\in\widehat{K'}} \tau^{\oplus m_\tau}$ for irreducible representations π and π' of G and G' respectively, then $m_\delta, m_\tau \leq 1$ and hence

$$(\pi \times \pi')|_{K\times K'} = \bigoplus_{(\delta,\tau)\in\widehat{K}\times\widehat{K'}} (\delta \times \tau)^{\oplus m_\delta m_\tau}$$

where $m_\delta m_\tau \leq 1$. Thus irreducible representations of $G \times G'$ decompose without multiplicity when restricted to $K \times K'$.

In the case where G itself is compact, the strong Gelfand pairs have been classified by Krämer in [23] (he uses the terminology that K is a multiplicity free subgroup in G). The first two results of his paper relevant to us are the following (Proposition 2 slightly reformulated and Corollary 2 in *loc.cit.*):

[3] We use the notation that $\pi \times \delta$ is the representation of $G \times K$ on $\mathcal{H}_\pi \widehat{\otimes} \mathcal{H}_\delta$ (the completed Hilbert tensor product) given by $(\pi \times \delta)(g, k)v \otimes w = \pi(g)v \otimes \delta(k)w$.

Lemma 2.21. *Let (G, K) and (G', K') be pairs where G and G' are compact and connected, and where K and K' are closed and connected subgroups and assume we have a local isomorphism $G \longrightarrow G'$ which maps K to K', then (G, K) is a strong Gelfand pair if and only if (G', K') is a strong Gelfand pair.*

Lemma 2.22. *Let G be a compact connected and simple Lie group and K a closed subgroup and K_0 the identity component, then (G, K) is a strong Gelfand pair if and only if (G, K_0) is a strong Gelfand pair.*

The following is the main theorem of Krämer's paper. Note that point 4) is a consequence of the preceding two lemmas.

Theorem 2.23 (Krämer). *Let G be a compact connected Lie group and K a closed subgroup whose identity component we denote K_0.*

1) *If G is simple, then (G, K) is a strong Gelfand pair if and only if up to covering, either*
   - *the pair is of the form $(SU(n), U(n-1))$ for $n \geq 2$, or*
   - *$G = SO(n)$ and $K_0 = SO(n-1)$ for $n \geq 3$, $n \neq 4$, or*
   - *$G = SO(8)$ and $K_0 = \mathrm{Spin}(7)$, or*
   - *trivially $G = K$.*
2) *If G is semisimple, then (G, K) is a strong Gelfand pair if and only if there exist strong Gelfand pairs $(G_1, K_1), \ldots, (G_n, K_n)$ with G_i simple and K_i connected, and there exists a number m, such that we have a local isomorphism $G \xrightarrow{\sim} G_1 \times \cdots \times G_n \times SO(4)^{\times m}$ which maps K_0 to $K_1 \times \cdots \times K_n \times SO(3)^{\times m}$ where $SO(3)$ sits inside $SO(4)$ (which is locally isomorphic to $SO(3) \times SO(3)$) as the diagonal.*
3) *If $G = G_s \times T$ where G_s is semisimple and T is a torus, then (G, K) is a strong Gelfand pair if and only if $(G_s, p(K))$ is a strong Gelfand pair (where $p : G \longrightarrow G_s$ is the projection).*
4) *If $q : \widetilde{G} \longrightarrow G$ is a connected covering of G, then (G, K) is a strong Gelfand pair, if and only if $(\widetilde{G}, q^{-1}(K))$ is a strong Gelfand pair.*

This theorem enables us to determine for each pair (G, K) (where G is compact and connected, and K is a closed subgroup) if it is a strong Gelfand pair or not: As G is compact, its Lie algebra is automatically reductive, hence is of the form $\mathfrak{g} = \mathfrak{g}_s \oplus Z_{\mathfrak{g}}$, i.e. is a direct sum of a semisimple part $\mathfrak{g}_s = [\mathfrak{g}, \mathfrak{g}]$ and the center. Let G_s be the analytic subgroup of G corresponding to $\mathfrak{g}_s$, then we have a covering map $\pi : G_s \times T \longrightarrow G$ by $(g, z) \longmapsto gz$. Thus (G, K) is a strong Gelfand pair if and only if $(G_s, q(\pi^{-1}(K)))$ is a strong Gelfand pair.

Corollary 2.24. *The only strong Gelfand pair of the form $(G_0 \times G_0, G_0)$ with G_0 compact, connected and simple is $(SO(3) \times SO(3), SO(3))$. Consequently, $(SO(n) \times SO(n), SO(n))$ is a strong Gelfand pair, if and only if $n = 3$ or $n = 4$.*

Proof. $(SO(3) \times SO(3), SO(3))$ is a strong Gelfand pair as it is locally isomorphic to the Gelfand pair $(SO(4), SO(3))$. Also $(SO(4) \times SO(4), SO(4))$ is a strong Gelfand since it is locally isomorphic to the product of the strong Gelfand pair $(SO(3) \times SO(3), SO(3))$ with itself. The only if statement of this corollary follows from Krämer's list of semisimple strong Gelfand pairs above. □

Recall that given a real Lie algebra $\mathfrak{g}$, we denote by $U(\mathfrak{g}_{\mathbb{C}})$ the universal enveloping algebra of the complexification of $\mathfrak{g}$ and we identify this with the algebra $\mathbb{D}(G)$ of left-invariant differential operators on G. If H is a closed subgroup of G we can extend the adjoint action of H on $\mathfrak{g}$ to an algebra action of H on $U(\mathfrak{g}_{\mathbb{C}})$ by

$$\mathrm{Ad}(h) X_1 \cdots X_n := (\mathrm{Ad}(h) X_1) \cdots (\mathrm{Ad}(h) X_n).$$

Let $U(\mathfrak{g}_{\mathbb{C}})_k$ denote the set of elements of $U(\mathfrak{g})$ of degree at most k. By the PBW-Theorem this is a finite-dimensional space, and it is clear, that it is preserved by $\mathrm{Ad}(h)$. Thus Ad is a finite-dimensional representation of H on $U(\mathfrak{g}_{\mathbb{C}})_k$. This of course has a derived representation of $\mathfrak{h}$:

$$\begin{aligned}
\mathrm{Ad}_*(X)(X_1 \cdots X_n) &= \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}(\exp(tX))(X_1 \cdots X_n) \\
&= \frac{d}{dt}\Big|_{t=0} (\mathrm{Ad}(\exp(tX))X_1) \cdots (\mathrm{Ad}(\exp(tX))X_n) \\
&= \sum_{j=1}^{n} X_1 \cdots \Big(\frac{d}{dt}\Big|_{t=0} \mathrm{Ad}(\exp(tX))X_j \Big) \cdots X_n \\
&= \sum_{j=1}^{n} X_1 \cdots [X, X_j] \cdots X_n.
\end{aligned}$$

We extend this representation to $U(\mathfrak{g}_{\mathbb{C}}) = \bigcup U(\mathfrak{g}_{\mathbb{C}})_k$ and denote it ad:

$$\begin{aligned}\operatorname{ad}(X)(X_1 \cdots X_n) &= \sum_{j=1}^{n} X_1 \cdots (\operatorname{ad}(X)X_j) \cdots X_n \\ &= X(X_1 \cdots X_n) - (X_1 \cdots X_n)X\end{aligned}$$

where the last equality follows from a simple computation involving commutators. The first expression tells us that $\operatorname{ad}(X)$ is a derivation of $U(\mathfrak{g}_{\mathbb{C}})$ and the second, that we can view $\operatorname{ad}(X)$ as a commutator, just as in the Lie algebra situation. Since $U(\mathfrak{g}_{\mathbb{C}})$ is a complex algebra, we can without problems extend ad to a representation of $\mathfrak{h}_{\mathbb{C}}$.

By $U(\mathfrak{g}_{\mathbb{C}})^H$ we denote the subset of elements of $U(\mathfrak{g}_{\mathbb{C}})$ which are invariant under $\operatorname{Ad}(h)$ for all $h \in H$, and by $U(\mathfrak{g}_{\mathbb{C}})^{\mathfrak{h}}$ the set of elements which are mapped to 0 by $\operatorname{ad}(X)$ for all $X \in \mathfrak{h}$, i.e.

$$U(\mathfrak{g}_{\mathbb{C}})^{\mathfrak{h}} = \bigcap_{X \in \mathfrak{h}} \ker \operatorname{ad}(X).$$

$U(\mathfrak{g}_{\mathbb{C}})^{\mathfrak{h}_{\mathbb{C}}}$ is defined similarly.

Lemma 2.25. *If H is a connected subgroup of G, it follows that $U(\mathfrak{g}_{\mathbb{C}})^H = U(\mathfrak{g}_{\mathbb{C}})^{\mathfrak{h}} = U(\mathfrak{g}_{\mathbb{C}})^{\mathfrak{h}_{\mathbb{C}}}$.*

Proof. The last identity is obvious and holds whether H is connected or not.

For the first identity, the inclusion "$\subseteq$" follows easily by the above, since ad is the derived representation of Ad. For the reverse inclusion "$\supseteq$" we need the connectivity assumption, for then any $h \in H$ can be written as a finite product $h = \exp X_1 \cdots \exp X_n$ for $X_j \in \mathfrak{h}$ so we may assume w.l.o.g. that $h = \exp X_1$. If $X \in U(\mathfrak{g}_{\mathbb{C}})^{\mathfrak{h}}$, then

$$\operatorname{Ad}(\exp X_1)X = \exp(\operatorname{ad}(X_1))X = (I + (\operatorname{ad} X_1) + \tfrac{1}{2}(\operatorname{ad} X_1)^2 + \cdots)X = X,$$

i.e. $X \in U(\mathfrak{g}_{\mathbb{C}})^H$. □

Lemma 2.26. *If G and G' are Lie groups with closed connected subgroups H and H' respectively, for which there is a Lie algebra isomorphism $\varphi : \mathfrak{g}_{\mathbb{C}} \xrightarrow{\sim} \mathfrak{g}'_{\mathbb{C}}$ between the complexifications of the Lie algebras of G and G' and such that $\varphi(\mathfrak{h}_{\mathbb{C}}) = \varphi(\mathfrak{h}'_{\mathbb{C}})$, then $\mathbb{D}(G/H)$ and $\mathbb{D}(G'/H')$ are isomorphic as algebras.*

PROOF. For any homogenous space, we have an algebra isomorphism

$$\mathbb{D}(G/H) \xrightarrow{\sim} U(\mathfrak{g}_{\mathbb{C}})^H/(U(\mathfrak{g}_{\mathbb{C}})^H \cap U(\mathfrak{g}_{\mathbb{C}})\mathfrak{h}) \tag{2.9}$$

and by the lemma above the latter equals $U(\mathfrak{g}_{\mathbb{C}})^{\mathfrak{h}_{\mathbb{C}}}/(U(\mathfrak{g}_{\mathbb{C}})^{\mathfrak{h}_{\mathbb{C}}} \cap U(\mathfrak{g}_{\mathbb{C}})\mathfrak{h}_{\mathbb{C}})$.

The Lie algebra isomorphism φ induces an algebra isomorphism $\overline{\varphi} : U(\mathfrak{g}_{\mathbb{C}}) \xrightarrow{\sim} U(\mathfrak{g}'_{\mathbb{C}})$, and it requires but a small calculation to verify that $\overline{\varphi}(U(\mathfrak{g}_{\mathbb{C}})^{\mathfrak{h}_{\mathbb{C}}}) = U(\mathfrak{g}'_{\mathbb{C}})^{\mathfrak{h}'_{\mathbb{C}}}$ and $\overline{\varphi}(U(\mathfrak{g}_{\mathbb{C}})\mathfrak{h}_{\mathbb{C}}) = U(\mathfrak{g}'_{\mathbb{C}})\mathfrak{h}'_{\mathbb{C}}$. Thus $\overline{\varphi}$ induces an algebra isomorphism

$$U(\mathfrak{g}_{\mathbb{C}})^{\mathfrak{h}_{\mathbb{C}}}/(U(\mathfrak{g}_{\mathbb{C}})^{\mathfrak{h}_{\mathbb{C}}} \cap U(\mathfrak{g}_{\mathbb{C}})\mathfrak{h}_{\mathbb{C}}) \xrightarrow{\sim} U(\mathfrak{g}'_{\mathbb{C}})^{\mathfrak{h}'_{\mathbb{C}}}/(U(\mathfrak{g}'_{\mathbb{C}})^{\mathfrak{h}'_{\mathbb{C}}} \cap U(\mathfrak{g}'_{\mathbb{C}})\mathfrak{h}'_{\mathbb{C}})$$

which, by the remarks at the beginning of the proof, is the desired algebra isomorphism $\mathbb{D}(G/H) \xrightarrow{\sim} \mathbb{D}(G'/H')$. □

Let us apply this to triple spaces. If G' is a compact Lie group, it follows from the above discussion of Gelfand pairs that $(G' \times G' \times G', G')$ (where G' is viewed as the diagonal) is a Gelfand pair if and only if $(G' \times G', G')$ is a strong Gelfand pair.

Any linear semisimple Lie group G' has a complexification $G'^{\mathbb{C}}$ which in turn has a compact real form $\widetilde{G}'$, in other words, for any linear semisimple Lie group G', there is a compact Lie group $\widetilde{G}'$ such that the complexification of the two Lie algebras are isomorphic. We call $\widetilde{G}'$ the *compact form* of G'. By the above lemma, commutativity of $\mathbb{D}(G/H)$ follows if and only if $(\widetilde{G}' \times \widetilde{G}', \widetilde{G}')$ is a strong Gelfand pair. Thus we get

Theorem 2.27. *Let G/H be a triple space as above with G' connected and semisimple. The algebra $\mathbb{D}(G/H)$ is commutative if and only if the compact form of G' is locally isomorphic to $SO(3)$ or $SO(4)$. In particular, among the groups $G' = SO_e(n,1)$, only $SO_e(2,1)$ and $SO_e(3,1)$ have corresponding triple spaces with commuting invariant differential operators.*

Our second application will be to Gross-Prasad spaces. These have been studied in [24] where it was found that Gross-Prasad spaces from the pairs

- $(GL(n+1,\mathbb{F}), GL(n,\mathbb{F}))$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}$,
- $(O(p,q+1), O(p,q))$ for $p+q \geq 2$,
- $(U(p,q+1), U(p,q))$ for $p+q \geq 2$,
- $(Sp(p,q+1), Sp(p,q))$ for $p+q \geq 2$

are spherical and polar.

Theorem 2.28. *For all the Gross-Prasad spaces in the list above except* $(Sp(p,q+1)\times Sp(p,q))/Sp(p,q)$, *it holds that the algebra of invariant differential operators is commutative.*

PROOF. Again it follows from Lemma 2.26 that if (G',H') is a compact form of (G,H), then $\mathbb{D}(G\times H/H)\cong\mathbb{D}(G'\times H'/H')$. And it follows from Proposition 2.18 that $\mathbb{D}(G'\times H'/H')$ is commutative if and only if (G',H') is a strong Gelfand pair.

The pair $(GL(n+1,\mathbb{R}),GL(n,\mathbb{R}))$ has the compact form $(U(n+1),U(n))$. The map $q: SU(n+1)\times U(1)\longrightarrow U(n+1)$ given by $(A,z)\longmapsto zA$ is a covering map (since it is a group homomorphism whose corresponding Lie algebra map is an isomorphism). If we view $U(n)$ as sitting in $U(n+1)$ in the lower right corner, then

$$q^{-1}(U(n))=\Big\{(\begin{pmatrix} z^{-1} & 0\\ 0 & z^{-1}A'\end{pmatrix},z)\;\Big|\;A'\in U(n)\;z^{n+1}=\det A'\Big\}.$$

The projection of this set onto $SU(n+1)$ is therefore

$$p(q^{-1}(U(n)))=\Big\{\begin{pmatrix} z^{-1} & 0\\ 0 & z^{-1}A'\end{pmatrix}\;\Big|\;A'\in U(n)\;z^{n+1}=\det A'\Big\}$$

which is just the standard embedding of $U(n)$ into $SU(n+1)$. Since $(SU(n+1),U(n))$ is a strong Gelfand pair, it follows (by Krämer's theorem and Lemma 2.22) that $(U(n+1),U(n))$ is a strong Gelfand pair. Hence the algebra of differential operators for the Gross-Prasad space of the pair $(GL(n+1,\mathbb{R}),GL(n,\mathbb{R}))$ is commutative.

Similarly, the pair $(U(p,q+1),U(p,q))$ has compact form $(U(p+q+1),U(p+q))$, and this is a strong Gelfand pair by the same arguments as above.

The pair $(O(p,q+1),O(p,q))$ has compact form $(O(p+q+1),O(p+q))$ which is locally isomorphic to the strong Gelfand pair $(SO(p+q+1),SO(p+q))$.

Finally $(Sp(p,q+1),Sp(p,q))$ has compact form $(Sp(p+q+1),Sp(p+q))$ which is not a strong Gelfand pair. □

CHAPTER 3

A SPECIAL CASE

3.1 Invariant Differential Operators

We continue our description of the algebra $\mathbb{D}(G/H)$ from the previous chapter, now only for the triple space with $G' = SL(2,\mathbb{R})$. We prove that this is an abelian algebra in 3 generators.

In this section we will work a lot with $U(\mathfrak{g}_{1\mathbb{C}} \oplus \mathfrak{g}_{2\mathbb{C}})$, and the following lemma will make this space slightly easier to handle.

Lemma 3.1. *We have an algebra isomorphism*

$$\mathbb{D}(G_1 \times G_2) = U(\mathfrak{g}_{1\mathbb{C}} \oplus \mathfrak{g}_{2\mathbb{C}}) \xrightarrow{\sim} U(\mathfrak{g}_{1\mathbb{C}}) \otimes U(\mathfrak{g}_{2\mathbb{C}}) = \mathbb{D}(G_1) \otimes \mathbb{D}(G_2)$$

which maps

$$(X_1, 0) \cdots (X_n, 0)(0, Y_1) \cdots (0, Y_m) \longmapsto (X_1 \cdots X_n) \otimes (Y_1 \cdots Y_m)$$

Proof. Consider the map $\mathfrak{g}_1 \oplus \mathfrak{g}_2 \longrightarrow U(\mathfrak{g}_{1\mathbb{C}}) \otimes U(\mathfrak{g}_{2\mathbb{C}})$ given by $(X, Y) \longmapsto X \otimes 1 + 1 \otimes Y$. It is easily checked that this map is linear and satisfies the bracket relation. Thus it factorizes as an algebra homomorphism through $U(\mathfrak{g}_{1\mathbb{C}} \oplus \mathfrak{g}_{2\mathbb{C}})$. This is the desired map. It is easily seen to map a PBW-basis to a PBW-basis, and thus it is an algebra isomorphism. $\square$

We use the model $G/H \cong G' \times G'$. The G-action on $G' \times G'$ is simply that the two first copies act on each of the arguments from the left and the third copy acts diagonally from the right. Thus we immediately see that we have the following string of inclusions

$$\mathbb{Z}(G' \times G') \subseteq \mathbb{D}(G/H) \subseteq \mathbb{D}(G' \times G') \tag{3.1}$$

where $\mathbb{Z}(G' \times G')$ is the algebra of bi-invariant differential operators on $G' \times G'$, which happens to be equal to the center of $\mathbb{D}(G' \times G') \cong U(\mathfrak{g}'_{\mathbb{C}} \oplus \mathfrak{g}'_{\mathbb{C}})$. In our present example, all inclusions will be strict

Consider the standard basis for $\mathfrak{sl}(2, \mathbb{R})$

$$H_0 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad X_0 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Y_0 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

(we add subscripts 0 to avoid confusion between H_0 as a matrix and H as a subgroup), which satisfies the usual commutation relations

$$[X_0, Y_0] = H_0, \qquad [H_0, X_0] = 2X_0, \qquad [H_0, Y_0] = -2Y_0.$$

Consider the inclusions (3.1). $\mathbb{Z}(G' \times G') \subseteq \mathbb{D}(G/H)$ and $\mathbb{Z}(G' \times G') \cong \mathbb{Z}(G') \otimes \mathbb{Z}(G')$ (it is a standard fact from algebra, that the center of a tensor product of algebras is the tensor product of the centers), and in this specific case of $G' = SL(2, \mathbb{R})$, the center $\mathbb{Z}(G')$ is generated by the Casimir element , which relative the the basis for $\mathfrak{sl}(2, \mathbb{R})$ above takes the following form: $\omega = \frac{1}{2}H_0^2 + X_0Y_0 + Y_0X_0 = \frac{1}{2}H_0^2 + H_0 + 2X_0Y_0$. Consequently, we have at least two generators for the algebra $\mathbb{D}(G/H)$, namely $\omega \otimes 1$ and $1 \otimes \omega$. Furthermore, it can be checked that the element

$$\Omega := \tfrac{1}{2}H_0 \otimes H_0 + X_0 \otimes Y_0 + Y_0 \otimes X_0$$

is actually an element of $\mathbb{D}(G/H) \cong U(\mathfrak{g}'_{\mathbb{C}} \oplus \mathfrak{g}'_{\mathbb{C}})^H$ without being an element of the center $\mathbb{Z}(G' \times G')$ (it doesn't commute with $H_0 \otimes 1$ for instance). In fact, $\omega \otimes 1$ and $1 \otimes \omega$ and Ω constitute a basis for $U(\mathfrak{g}'_{\mathbb{C}} \oplus \mathfrak{g}'_{\mathbb{C}})^H_2$. Thus we have a third generator and in particular the first inclusion in (3.1) is strict. Also the other inclusion is strict: $X_0 \otimes Y_0^2$ for example, is an element in $\mathbb{D}(G' \times G') = U(\mathfrak{g}'_{\mathbb{C}} \oplus \mathfrak{g}'_{\mathbb{C}})$ which is not in $U(\mathfrak{g}'_{\mathbb{C}} \oplus \mathfrak{g}'_{\mathbb{C}})^H = U(\mathfrak{g}'_{\mathbb{C}} \oplus \mathfrak{g}'_{\mathbb{C}})^{\mathfrak{h}}$ (it is not $\mathrm{ad}(H_0)$-invariant - in order to be so, the sum of the powers of X_0 should equal the sum of the powers of Y_0).

That leaves us the following question: are there any more generators for $\mathbb{D}(G/H)$ other than $\omega \otimes 1$, $1 \otimes \omega$ and Ω? The answer turns out to be no, as we now explain. First, note that $\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and hence by complexification $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{so}(4,\mathbb{C})$. Thus $U(\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})) \cong U(\mathfrak{so}(4,\mathbb{C}))$. Furthermore $\mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{su}(2)_\mathbb{C} \cong \mathfrak{so}(3,\mathbb{C})$. Thus we have

$$\mathbb{D}(G/H) = U(\mathfrak{g}'_\mathbb{C} \oplus \mathfrak{g}'_\mathbb{C})^{\mathfrak{h}_\mathbb{C}} = U(\mathfrak{so}(4,\mathbb{C}))^{\mathfrak{so}(3)}.$$

This particular algebra has been studied in [4] and the statement is that it is generated by $\mathbb{Z}(SO(4))$ and by $\mathbb{Z}(SO(3))$ (the centers of $U(\mathfrak{so}(4,\mathbb{C}))$ and $U(\mathfrak{so}(3,\mathbb{C}))$ respectively). $SO(4)$ has rank 2 and $SO(3)$ has rang 1 and hence in total there are 3 generators for $\mathbb{D}(G/H)$. Wrapping up:

Theorem 3.2. *$\mathbb{D}(G/H)$ is a commutative algebra in 3 generators. Viewed as a subalgebra of $\mathbb{D}(G' \times G') \cong \mathbb{D}(G') \otimes \mathbb{D}(G')$, the generators are $\omega \otimes 1$, $1 \otimes \omega$ and Ω.*

One drawback of this viewpoint is that it depends on the specific realization of G/H as $G' \times G'$, which is non-canonical since we singled out the third component. A slightly different and more invariant viewpoint is the following. For any homogenous space G/H we have an algebra homomorphism

$$\overline{p} : \mathbb{D}_H(G) \longrightarrow \mathbb{D}(G/H)$$

given by $(\overline{p}(D)f) \circ p = D(f \circ p)$ where $p : G \longrightarrow G/H$ is the projection and where $\mathbb{D}_H(G)$ is the algebra of differential operators which are left G-invariant and right H-invariant (this algebra is isomorphic to $U(\mathfrak{g}_\mathbb{C})^H$). In particular we can restrict $\overline{p}$ to $\mathbb{Z}(G) \subseteq \mathbb{D}_H(G)$ and study how the generators of $\mathbb{Z}(G)$ are mapped to $\mathbb{D}(G/H)$. Both algebras are abelian in 3 generators, so if we can show that it maps generators to generators, it follows that it is an algebra isomorphism. The identification of $\mathbb{D}(G/H)$ with $\mathbb{Z}(G' \times G' \times G')$ is more canonical than the identification with $U(\mathfrak{g}'_\mathbb{C} \oplus \mathfrak{g}'_\mathbb{C})^{\mathfrak{h}}$.

To formalize this, let $\varphi : G/H \xrightarrow{\sim} G' \times G'$ be the G-equivariant diffeomorphism $[g_1, g_2, g_3] \longmapsto (g_1 g_3^{-1}, g_2 g_3^{-1})$. The induced map on algebras of differential operators $\overline{\varphi} : \mathbb{D}(G/H) \longrightarrow \mathbb{D}_{G'}(G' \times G')$ given by

$$(\overline{\varphi}(D)f)(g_1, g_2) = D(f \circ \varphi)(\varphi^{-1}(g_1, g_2))$$

is a realization of the identification $\mathbb{D}(G/H) \cong \mathbb{D}_{G'}(G' \times G')$. Composing $\overline{p}$ and $\overline{\varphi}$ we obtain the map $\overline{\varphi} \circ \overline{p} : \mathbb{Z}(G' \times G' \times G') \longrightarrow \mathbb{D}_{G'}(G' \times G')$ which is given by

$$(\overline{\varphi} \circ \overline{p})(D)f(\varphi(g_0 H)) = D(f \circ \varphi \circ p)(g_0). \tag{3.2}$$

Note how the right-hand side makes sense, for any $D \in \mathbb{D}(G)$, whereas the left-hand side only makes sense as an invariant differential operator on functions on G/H if $D \in \mathbb{D}_H(G)$.

We want to calculate the action of this map on the three generators $\omega \otimes 1 \otimes 1$, $1 \otimes \omega \otimes 1$ and $1 \otimes 1 \otimes \omega$ of $\mathbb{Z}(G' \times G' \times G') \cong \mathbb{Z}(G') \otimes \mathbb{Z}(G') \otimes \mathbb{Z}(G')$. It is not hard to see that $\omega \otimes 1 \otimes 1$ and $1 \otimes \omega \otimes 1$ are mapped to $\omega \otimes 1$ and $1 \otimes \omega$ respectively. So the only problem is calculating $(\overline{\varphi} \circ \overline{p})(1 \otimes 1 \otimes \omega)$. We calculate the right hand side of (3.2) term by term. First $1 \otimes 1 \otimes H_0^2$:

$$\begin{aligned}
(1 \otimes 1 \otimes H_0^2)&(f \circ \varphi \circ p)(g_1, g_2, e) \\
&= \frac{\partial^2}{\partial t \partial s}\bigg|_{(0,0)} (f \circ \varphi \circ p)(g_1, g_2, \exp((s+t)H_0)) \\
&= \frac{\partial^2}{\partial t \partial s}\bigg|_{(0,0)} f(g_1 \exp(-(s+t)H_0), g_2 \exp(-(s+t)H_0)).
\end{aligned}$$

For a function h of the form $(s,t) \longmapsto f(s+t, s+t)$ we get from the chain rule

$$\frac{\partial^2 h}{\partial t \partial s}(0,0) = \frac{\partial^2 f}{\partial x_1^2}(0,0) + \frac{\partial^2 f}{\partial x_2^2}(0,0) + 2\frac{\partial^2 f}{\partial x_1 \partial x_2}(0,0).$$

Applied to the expression above we get

$$(1 \otimes 1 \otimes H_0^2)(f \circ \varphi \circ p)(g_1, g_2, e) = (H_0^2 \otimes 1 + 1 \otimes H_0^2 + 2H_0 \otimes H_0)f(g_1, g_2)$$

in other words $(\overline{\varphi} \circ \overline{p})(1 \otimes 1 \otimes H_0^2) = H_0^2 \otimes 1 + 1 \otimes H_0^2 + 2H_0 \otimes H_0$.

Then what about $(\overline{\varphi} \circ \overline{p})(1 \otimes 1 \otimes X_0 Y_0)$?

$$\begin{aligned}
(1 \otimes 1 \otimes &X_0 Y_0)(f \circ \varphi \circ p)(g_1, g_2, e) \\
&= \frac{\partial^2}{\partial t \partial s}\bigg|_{(0,0)} (f \circ \varphi \circ p)(g_1, g_2, \exp(tX_0)\exp(sY_0)) \\
&= \frac{\partial^2}{\partial s \partial t}\bigg|_{(0,0)} f(g_1 \exp(-sY_0)\exp(-tX_0), g_2 \exp(-sY_0)\exp(-tX_0)).
\end{aligned}$$

The t-derivative must be $(-X_0 \otimes 1 - 1 \otimes X_0)f(g_1 \exp(-sY_0), g_2 \exp(-sY_0))$ and the s-derivative of this must be

$$(-Y_0 \otimes 1 - 1 \otimes Y_0)(-X_0 \otimes 1 - 1 \otimes X_0)f(g_1, g_2)$$

in other words

$$(\overline{\varphi} \circ \overline{p})(1 \otimes 1 \otimes X_0 Y_0) = Y_0 X_0 \otimes 1 + Y_0 \otimes X_0 + X_0 \otimes Y_0 + 1 \otimes Y_0 X_0$$

and similarly

$$(\overline{\varphi} \circ \overline{p})(1 \otimes 1 \otimes Y_0X_0) = X_0Y_0 \otimes 1 + X_0 \otimes Y_0 + Y_0 \otimes X_0 + 1 \otimes X_0Y_0.$$

From these formulas we conclude that

$$(\overline{\varphi} \circ \overline{p})(1 \otimes 1 \otimes \omega) = \omega \otimes 1 + 1 \otimes \omega + 2\Omega \tag{3.3}$$

and this in combination with $1 \otimes \omega$ and $\omega \otimes 1$ is a complete set of generators for $\mathbb{D}_{G'}(G' \times G')$. Since $\overline{\varphi}$ is already an algebra homomorphism we conclude

Theorem 3.3. *The map $\overline{p} : \mathbb{Z}(G' \times G' \times G') \longrightarrow \mathbb{D}(G/H)$ is an algebra isomorphism.*

In the previous chapter, we noted that there were only two triple spaces (known so far) which have commuting algebras of invariant differential operators, namely triple spaces with $G' = SL(2,\mathbb{R})$ and $G' = SO_e(3,1)$.

For the second space, we have

$$\mathbb{D}(G/H) = U(\mathfrak{so}(4,\mathbb{C}) \oplus \mathfrak{so}(4,\mathbb{C}))^{\mathfrak{so}(4)}$$

and as $\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$, we can write this as

$$U(\mathfrak{so}(3,\mathbb{C})^{\oplus 4})^{\mathfrak{so}(3)\oplus\mathfrak{so}(3)}$$

where $(X,Y) \in \mathfrak{so}(3) \oplus \mathfrak{so}(3) =: \mathfrak{h}$ sits inside $\mathfrak{so}(4,\mathbb{C}) \oplus \mathfrak{so}(4,\mathbb{C}) \cong \mathfrak{so}(3,\mathbb{C})^{\oplus 4}$ as (X,Y,X,Y). We can also embed $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$ in $\mathfrak{so}(4,\mathbb{C}) \oplus \mathfrak{so}(4,\mathbb{C}) \cong \mathfrak{so}(3,\mathbb{C})^{\oplus 4}$ as $(X,Y) \longmapsto (X,X,Y,Y)$, i.e. a componentwise embedding of $\mathfrak{so}(3)$ into $\mathfrak{so}(4)$ (as the diagonal). We call this subalgebra $\mathfrak{h}'$. It is clear that the Lie algebra automorphism $\mathfrak{so}(3,\mathbb{C})^{\oplus 4} \longrightarrow \mathfrak{so}(3,\mathbb{C})^{\oplus 4}$ which flips the second and third component, maps $\mathfrak{h}$ to $\mathfrak{h}'$ and thus gives an isomorphism

$$\mathbb{D}(G/H) \cong U(\mathfrak{so}(4,\mathbb{C}) \oplus \mathfrak{so}(4,\mathbb{C}))^{\mathfrak{h}'}.$$

By (1.15) (in the special case where δ_1 and δ_2 are both the trivial representation) we get that the space of $K_1 \times K_2$-invariant vectors in a tensor product is the tensor product of the spaces of K_1 and K_2-invariant vectors for the individual representations. We can apply this to the above setting with $K_1 = K_2 = SO(3)$ acting on the finite-dimensional spac $U(\mathfrak{so}(4,\mathbb{C}))_N$ (the set of elements of degree at most N). Then we get

$$\begin{aligned} U(\mathfrak{so}(4,\mathbb{C}) \oplus \mathfrak{so}(4,\mathbb{C}))_{2N}^{\mathfrak{h}'} &= [U(\mathfrak{so}(4,\mathbb{C}))_N \otimes U(\mathfrak{so}(4,\mathbb{C}))_N]^{SO(3)\times SO(3)} \\ &\cong U(\mathfrak{so}(4,\mathbb{C}))_N^{SO(3)} \otimes U(\mathfrak{so}(4,\mathbb{C}))_N^{SO(3)}. \end{aligned}$$

As this holds for all N, we conclude

$$U(\mathfrak{so}(4,\mathbb{C}) \oplus \mathfrak{so}(4,\mathbb{C}))^{\mathfrak{h}'} \cong U(\mathfrak{so}(4,\mathbb{C}))^{SO(3)} \otimes U(\mathfrak{so}(4,\mathbb{C}))^{SO(3)}$$

and hence:

Theorem 3.4. *For the triple space $(G' \times G' \times G')/G'$ with $G' = SO_e(3,1)$, it holds that*

$$\mathbb{D}(G' \times G' \times G')/G') \cong \mathbb{D}(SL(2,\mathbb{R})^{\times 3}/SL(2,\mathbb{R}))^{\otimes 2}$$

hence the algebra of invariant differential operators is a commutative algebra in 6 generators.

This shows, however, that the identification $\mathbb{D}(G/H) \cong \mathbb{Z}(G)$ which was true for the $SL(2,\mathbb{R})$-triple space, is not true in general, since $\mathbb{Z}(SO_e(3,1)^{\times 3})$ is an abelian algebra in only 3 generators.

3.2 Action on Matrix Coefficients

Now we return to the triple space G/H with $G' = SL(2,\mathbb{R})$. Next goal is to see how invariant differential operators act on irreducible matrix coefficients and to show that they are eigenfunctions for the invariant differential operators. This is analogous to the result for reductive symmetric spaces, where it was shown by van den Ban in [2], Theorem 1.5.

Theorem 3.5. *For any $D \in \mathbb{D}(G/H)$ and any $\pi \in \widehat{G}^H$, then*

$$DM_{\eta,v} = M_{\pi(D)\eta,v} = \chi_\pi(D)M_{\eta,v} \tag{3.4}$$

holds for all $v \in \mathcal{H}_\pi^\infty$ and $\eta \in \mathcal{H}_\pi^{-\infty,H}$ where χ_π is the infinitesimal character of π.

Proof. The entire proof is based on the fact, that we can identify $\mathbb{D}(G/H)$ with $\mathbb{Z}(G)$ as shown in the previous section. For any such D we know that $\pi(D) : \mathcal{H}_\pi^\infty \longrightarrow \mathcal{H}_\pi^\infty$ is just multiplication by $\chi_\pi(D)$, the infinitesimal character evaluated in D. By duality, $\pi(X)$ acts by the same scalar on $\mathcal{H}_\pi^{-\infty,H}$. This explains the last identity in (3.4). To verify the first, we view $M_{\eta,v}$ as a right H-invariant function on G and compute for $X \in \mathfrak{g}$:

$$(XM_{\eta,v})(g) = \frac{d}{dt}\bigg|_{t=0} \eta(\pi(\exp(-tX)g^{-1})v) = (X\eta)(\pi(g^{-1})v) = M_{X\eta,v}(g).$$

This shows the identity. $\square$

In general, if $\mathfrak{h}_{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$, we recall the Harish-Chandra isomorphism (1.9)

$$\gamma : \mathbb{Z}(G) \xrightarrow{\sim} U(\mathfrak{h}_{\mathbb{C}})^W$$

from $\mathbb{Z}(G)$ onto Weyl-invariant polynomials over $\mathfrak{h}_{\mathbb{C}}^*$. For any $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$ we get a character of $\mathbb{Z}(G)$ by composing it with the Harish-Chandra isomorphism

$$\chi_\lambda(D) := \lambda(\gamma(D))$$

and all characters of $\mathbb{Z}(G)$ are of this form (cf. Theorem 1.17).

Getting back to $SL(2,\mathbb{R})$: the holomorphic discrete series representation T_m has Harish-Chandra parameter $m+1$ (the m is the Blattner parameter, i.e. the lowest K-weight for the representation) and the anti-holomorphic discrete series representation T_{-n} has Harish-Chandra parameter $-n-1$. Thus

$$\chi_{T_m}(D) = \gamma(D)(m \pm 1)^1.$$

As the infinitesimal character satisfies

$$\chi_{\pi_1 \times \pi_2}(D_1 \otimes D_2) = \chi_{\pi_1}(D_1) + \chi_{\pi_2}(D_2)$$

we get in general

$$\chi_{T_n \times T_m \times T_k}(D_1 \otimes D_2 \otimes D_3) = \gamma(D_1)(n \pm 1) + \gamma(D_2)(m \pm 1) + \gamma(D_3)(k \pm 1).$$

In the following example we verify the above considerations by a direct computation.

Example 3.6. Let us confirm these deductions by a concrete hands-on example where we consider the representation $T_{-n} \times T_{-m} \times T_{n+m}$ which is a discrete series representation for G/H, since T_{n+m} sits as a direct summand in $T_{-n}^* \otimes T_{-m}^* = T_n \otimes T_m$. From [27] we know that $\mathrm{Hom}_{G'}(T_{n+m}, T_n \otimes T_m)$ is 1-dimensional. From the proof in [27] of the decomposition of $T_n \otimes T_m$ we know that both T_{n+m} and $T_n \otimes T_m$ have unique (up to a scalar) K-weight vectors of weight $n+m$, namely ψ_{n+m}^0 and $\psi_n^0 \otimes \psi_m^0$ respectively. Thus any $S \in \mathrm{Hom}_{G'}(\mathcal{H}_{n+m}, \mathcal{H}_n \widehat{\otimes} \mathcal{H}_m)$ must map ψ_{n+m}^0 to a multiple of $\psi_n^0 \otimes \psi_m^0$. We scale S such that it actually maps ψ_{n+m}^0 to $\psi_n^0 \otimes \psi_m^0$.

Under the isomorphism (2.8)

$$\Phi : \mathrm{Hom}_{G'}(\mathcal{H}_{n+m}, \mathcal{H}_n \widehat{\otimes} \mathcal{H}_m) \xrightarrow{\sim} \mathrm{Hom}_G(\mathcal{H}_{-n} \widehat{\otimes} \mathcal{H}_{-m} \widehat{\otimes} \mathcal{H}_{n+m}, L^2(G/H)),$$

[1] Here the $\pm$ should be interpreted so that it is $+$ when $m > 0$ and $-$ when $m < 0$.

$\Phi(S)$ maps $\psi^0_{-n}\otimes\psi^0_{-m}\otimes\psi^0_{n+m}$ to the following function in $L^2(G/H)$:

$$\begin{aligned}\Phi(S)(\psi^0_{-n}&\otimes\psi^0_{-m}\otimes\psi^0_{n+m})[g_1,g_2,g_3]\\&=\big(T_{-n}(g_1^{-1})\psi^0_{-n}\otimes T_{-m}(g_2^{-1})\psi^0_{-m}\big)S(T_{n+m}(g_3^{-1})\psi^0_{n+m}).\end{aligned}$$

By the identification $G/H\cong G'\times G'$, $(g_1,g_2)\sim[g_1,g_2,e]$ it suffices to look at

$$\begin{aligned}f(g_1,g_2)&:=\Phi(S)(\psi^0_{-n}\otimes\psi^0_{-m}\otimes\psi^0_{n+m})(g_1,g_2)\\&=\big[T_{-n}(g_1^{-1})\psi^0_{-n}\otimes T_{-m}(g_2^{-1})\psi^0_{-m}\big]S(\psi^0_{n+m})\\&=[T_{-n}(g_1^{-1})\psi^0_{-n}\otimes T_{-m}(g_2^{-1})\psi^0_{-m}](\psi^0_n\otimes\psi^0_m)\\&=(T_{-n}(g_1^{-1})\psi^0_{-n})(\psi^0_n)\cdot(T_{-m}(g_2^{-1})\psi^0_{-m})(\psi^0_m).\end{aligned}$$

We know that $\psi^0_{-n}=\overline{\psi^0_n}$, and

$$T_{-n}(g_1^{-1})\overline{\psi^0_n}=\overline{T_n(g_1^{-1})\psi^0_n},$$

and we know how this acts on the vector ψ^0_n, namely by

$$\overline{T_n(g_1^{-1})\psi^0_n}(\psi^0_n)=\langle T_n(g_1^{-1})\psi^0_n,\psi^0_n\rangle.$$

Thus we end up with the function

$$f(g_1,g_2)=\langle T_n(g_1^{-1})\psi^0_n,\psi^0_n\rangle\langle T_m(g_2^{-1})\psi^0_m,\psi^0_m\rangle.$$

Now, let us see how a differential operator of the form $X\otimes 1$ acts on this function

$$\begin{aligned}(X\otimes 1)f(g_1,g_2)&=\frac{d}{dt}\bigg|_{t=0}\langle T_n(\exp(-tX)g_1^{-1})\psi^0_n,\psi^0_n\rangle\langle T_m(g_2^{-1})\psi^0_m,\psi^0_m\rangle\\&=\langle -T_n(X)T_n(g_1^{-1})\psi^0_n,\psi^0_n\rangle\langle T_m(g_2^{-1})\psi^0_m,\psi^0_m\rangle\\&=\langle T_n(g_1^{-1})\psi^0_n,T_n(X)\psi^0_n\rangle\langle T_m(g_2^{-1})\psi^0_m,\psi^0_m\rangle.\end{aligned}$$

Thus the X-action is moved to the other side of the bracket. Acting with $\Omega=\frac{1}{2}T\otimes T+X_+\otimes X_-+X_-\otimes X_+$ (written in the basis (1.21)) we get

$$\begin{aligned}(\Omega f)(g_1,g_2)=&-\tfrac{1}{2}\langle T_n(g_1^{-1})\psi^0_n,T_n(T)\psi^0_n\rangle\langle T_m(g_2^{-1})\psi^0_m,T_m(T)\psi^0_m\rangle\\&+\langle T_n(g_1^{-1})\psi^0_n,T_n(X_+)\psi^0_n\rangle\langle T_m(g_2^{-1})\psi^0_m,T_m(X_-)\psi^0_m\rangle\\&+\langle T_n(g_1^{-1})\psi^0_n,T_n(X_-)\psi^0_n\rangle\langle T_m(g_2^{-1})\psi^0_m,T_m(X_+)\psi^0_m\rangle\end{aligned}$$

and since the two last terms are 0 and $T_n(T)\psi_n^0 = in\psi_n^0$ we get

$$\Omega f = \tfrac{1}{2}nmf.$$

This constant is in agreement with the general considerations above: from (3.3) we can identify Ω with

$$\frac{1}{2}(1 \otimes 1 \otimes \omega - \omega \otimes 1 \otimes 1 - 1 \otimes \omega \otimes 1).$$

In [21] p. 220 the action of the Harish-Chandra isomorphism on the Casimir element of $SL(2,\mathbb{R})$ is calculated to be

$$\gamma(\omega) = \tfrac{1}{2}h^2 - \tfrac{1}{2}$$

and thus we get

$$\begin{aligned}\chi_{T_{-n}\times T_{-m}\times T_{n+m}}(\Omega) &= \tfrac{1}{2}\left(\gamma(\omega)(n+m+1) - \gamma(\omega)(-n-1) - \gamma(\omega)(-m-1)\right)\\ &= \tfrac{1}{2}\big((\tfrac{1}{2}(n+m+1)^2 - \tfrac{1}{2}) - (\tfrac{1}{2}(n+1)^2 - \tfrac{1}{2}) - (\tfrac{1}{2}(m+1)^2 - \tfrac{1}{2})\big)\\ &= \tfrac{1}{2}nm.\end{aligned}$$

3.3 Finite-Dimensional Spherical Representations

Sometimes it can be beneficial to be able to embed a homogenous space G/H into a finite-dimensional vector space, giving a concrete model of the space. This is possible if the group G admits a so-called *strongly H-spherical representation*:

Definition 3.7. Let G be a Lie group and H a subgroup. A representation $(\pi, \mathcal{H}_\pi)$ of G is called *H-spherical* if there exists a non-trivial vector $v \in \mathcal{H}_\pi$ such that $\pi(h)v = v$ for all $h \in H$. The space of such vectors is denoted $\mathcal{H}_\pi^H$.

The representation is called *strongly H-spherical* if there exists $v \in \mathcal{H}_\pi^H$ for which the fixed-point group $\{g \in G \mid \pi(g)v = v\}$ equals H.

Clearly, if π is a finite-dimensional strongly H-spherical representation of G, and v a vector with fixed-point group H, then clearly the map $g \longmapsto \pi(g)v$ descends to an embedding $G/H \longrightarrow V_\pi$, providing us with a model of G/H in V_π.

In this section we determine the finite-dimensional irreducible H-spherical representations of G. The set of these we denote by $\widehat{G}_H^{\mathrm{fd}}$ (even though they are not unitary). But first, let's recall the corresponding for results for Riemannian symmetric spaces (Helgason).

By Weyl's unitary trick, all finite-dimensional irreducible representations of a semisimple group G (for which the complexification exists and is simply connected) are in 1-1 correspondence with irreducible finite-dimensional complex $\mathfrak{g}_{\mathbb{C}}$-modules, which are again classified by their highest weights. To be more precise, let $\mathfrak{h}_{\mathbb{C}}$ be a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$ and let $\Sigma(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ be the corresponding root system. An element $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$ is called *dominant* , if

$$\langle \lambda, \alpha \rangle \geq 0$$

for all $\alpha \in \Sigma^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$, and λ is called *algebraically integral* if

$$2\frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$$

for all $\alpha \in \Sigma(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. However, by Proposition 4.15 in [21] it suffices to show this condition for the simple roots only. Irreducible finite-dimensional $\mathfrak{g}_{\mathbb{C}}$-modules are - through their highest weights - in bijective correspondence with dominant integral elements. Helgason sought out the ones, corresponding to K-spherical representations of G (cf. [22], Thm. 8.49):

Theorem 3.8 (**Helgason**). *Let G be as above, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ the Cartan decomposition and $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ a maximally non-compact Cartan subalgebra ($\mathfrak{a} \subseteq \mathfrak{s}$ is maximally abelian and $\mathfrak{t} \subseteq Z_{\mathfrak{k}}(\mathfrak{a})$ is maximally abelian). The K-spherical representations of G are the π_λ's for which the highest weight $\lambda \in \mathfrak{h}_{\mathbb{C}}$ of the corresponding $\mathfrak{g}_{\mathbb{C}}$-module satisfies*

1) $\lambda|_{\mathfrak{t}} = 0$,

2) $\frac{\langle \lambda|_{\mathfrak{a}}, \beta \rangle}{\langle \beta, \beta \rangle} \in \mathbb{Z}$ *for all restricted roots* $\beta \in \Sigma(\mathfrak{g}, \mathfrak{a})$.

The conditions can be paraphrased as follows: λ is a functional on $\mathfrak{a}$ for which $\frac{1}{2}\lambda$ is an integral element for the root system $\Sigma(\mathfrak{g}, \mathfrak{a})$.

For any root system Σ, with a given set of simple roots $\{\alpha_1, \ldots, \alpha_n\}$ we can define *fundamental weights* $\omega_1, \ldots, \omega_n$ by the requirement

$$2\frac{\langle \omega_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = \delta_{ij}.$$

These are clearly integral elements and actually the $\mathbb{Z}$-span of them equals the set of integral elements. If the fundamental weights are defined relative to the root system $\Sigma(\mathfrak{g},\mathfrak{a})$ from Helgason's theorem, it is clear that the K-spherical representations are parametrized by the lattice

$$\Lambda_K := \mathbb{Z}(2\omega_1) + \cdots + \mathbb{Z}(2\omega_n).$$

In the special case of $\mathfrak{sl}(2,\mathbb{R})$, (since it is split), $\mathfrak{a} := \mathbb{R}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ is a maximally non-compact Cartan subalgebra. The roots can be identified with ± 2 (in the sense, that they are the functionals on $\mathfrak{a}$ mapping $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ to ± 2). From that it is clear that the fundamental weight is 1, and hence that the dominant integral weights are just $\mathbb{Z}_{\geq 0}$ and for each such k we have a highest weight representation π_k of dimension $k+1$. π_0 is the trivial one and π_1 is the defining representation. Note that these representations are self-dual: $\pi^* = \pi$. The finite-dimensional irreducible representations of the triple product group $SL(2,\mathbb{R})^{\times 3}$ are parametrized by triples (n,m,k) with $n,m,k \in \mathbb{Z}_{\geq 0}$.

Theorem 3.9. *The set $\widehat{G}_H^{fd}$ is in 1-1 correspondence with the lattice $\Lambda_H := \mathbb{N}_0\mu_1 + \mathbb{N}_0\mu_2 + \mathbb{N}_0\mu_3$ where $\mu_1 = (1,1,0)$, $\mu_2 = (1,0,1)$ and $\mu_3 = (0,1,1)$.*

Proof. From the discussion in Example 2.14 we know exactly which finite-dimensional irreducible representations have non-trivial H-fixed vectors, namely $\pi_n \times \pi_m \times \pi_k$ where $|n-m| \leq k \leq n+m$ and where k has the same parity as $n+m$.

A general element in Λ_H is of the form $(\lambda_1,\lambda_2,\lambda_3) = (x+y, x+z, y+z)$ for positive integers x, y and z, and it is quite easy to see that this triple satisfies the double inequality above. Moreover $\lambda_1+\lambda_2-\lambda_3 = 2x$, i.e. λ_3 has the same parity as $\lambda_1+\lambda_2$. This shows that any element in Λ_H is the highest weight of an H-spherical finite-dimensional irreducible representation.

On the other hand given a triple $(\lambda_1,\lambda_2,\lambda_3)$ satisfying $|\lambda_1-\lambda_2| \leq \lambda_3 \leq \lambda_1+\lambda_2$, then

$$x := \frac{1}{2}(\lambda_1+\lambda_2-\lambda_3), \qquad y := \frac{1}{2}(\lambda_1-\lambda_2+\lambda_3), \qquad z := \frac{1}{2}(-\lambda_1,\lambda_2,\lambda_3)$$

are actually integers and $(\lambda_1,\lambda_2,\lambda_3) = x(1,1,0) + y(1,0,1) + z(0,1,1)$, i.e. $(\lambda_1,\lambda_2,\lambda_3) \in \Lambda_H$. $\square$

Combining this theorem with the theorem of Helgason we get the highest weights corresponding to representations having both H- and K-fixed vectors:

Corollary 3.10. *The finite-dimensional irreducible representations having both K-invariant vectors and H-invariant vectors are parametrized by the weight lattice*

$$\Lambda_{K,H} := \mathbb{N}_0\nu_1 + \mathbb{N}_0\nu_2 + \mathbb{N}_0\nu_3,$$

where $\nu_1 = 2\mu_1 = (2,2,0)$, $\nu_2 = 2\mu_2 = (2,0,2)$ and $\nu_3 = 2\mu_3 = (0,2,2)$.

Note however, that none of the H-spherical irreducible representations are strongly H-spherical: the condition that k should have the same parity as $n+m$ excludes the possibility that they are all odd. Assume n to be even, then $(\pi_n \times \pi_m \times \pi_k)(-1,1,1)$ is the identity, and hence any H-fixed vector is also fixed under $(-1,1,1)$ which is not in H.

Example 3.11. Here is an example of a (necessarily non-irreducible) strongly H-spherical representation: $\pi := (\pi_1 \times \pi_1^* \times \pi_0) \oplus (\pi_0 \times \pi_1 \times \pi_1^*)$ acting on $(V_{\pi_1} \otimes V_{\pi_1}^* \otimes \mathbb{C}) \oplus (\mathbb{C} \otimes V_{\pi_1} \otimes V_{\pi_1}^*)$. Under the isomorphism $V_{\pi_1} \otimes V_{\pi_1}^* \cong \mathrm{Hom}_{\mathbb{C}}(V_{\pi_1}, V_{\pi_1})$ the representation $(\pi_1 \times \pi_1^*)(g_1, g_2)$ corresponds to the representation $\pi'(g_1,g_2)T = \pi_1(g_2) \circ T \circ \pi_1(g_1^{-1})$. The group elements fixing the identity in $\mathrm{Hom}_{\mathbb{C}}(V_{\pi_1}, V_{\pi_1})$ are exactly (g,g). Now let v be the element in $V_{\pi_1} \otimes V_{\pi_1}^*$ corresponding to the identity map. Then $(v \otimes 1, 1 \otimes v)$ is an element in the representation space of π, and we note that $(\pi_1 \times \pi_1^* \times \pi_0)(g_1, g_2, g_3)$ fixes $v \otimes 1$ if and only if $g_1 = g_2$. Similarly, $(\pi_0 \times \pi_1 \times \pi_1^*)(g_1, g_2, g_3)$ fixes $1 \otimes v$ if and only if $g_2 = g_3$. All in all $\pi(g_1, g_2, g_3)$ fixes $(v \otimes 1, 1 \otimes v)$ if and only if $g_1 = g_2 = g_3$. In other words, the fixed point group of that element is exactly equal to H.

3.4 The Polar Decomposition

In the theory of symmetric spaces root systems are used to define regular elements. However, in our case, we don't have a root system at hand. Well, of course we do have $\Sigma(\mathfrak{g}, \mathfrak{a})$, but the problem with this root system is, that it has nothing to do with the subgroup H (unlike in the symmetric case, where one considers the root systems $\Sigma(\mathfrak{g}, \mathfrak{a}_q)$ or $\Sigma(\mathfrak{g}^+, \mathfrak{a}_q)$ which depend on σ). Therefore we begin this section by giving an alternative characterization of regular elements in the Riemannian symmetric case and use this as motivation for the definition of regular elements in our case. We then set out to determine the set of such elements.

But first, we consider the Riemannian symmetric case: The set of regular elements of $\mathfrak{a}$ is defined as

$$\mathfrak{a}^{\mathrm{reg}} := \mathfrak{a} \setminus \bigcup_{\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a})} \ker \alpha.$$

Recall that $\mathfrak{m}$ equals $Z_{\mathfrak{k}}(\mathfrak{a})$ by definition.

Proposition 3.12. *An element $H \in \mathfrak{a}$ is regular if and only if* $(\mathrm{Ad}(\exp H)\mathfrak{k}) \cap \mathfrak{k} = \mathfrak{m}$ *which on the other hand is equivalent to* $\mathrm{Ad}(\exp H)\mathfrak{k} + \mathfrak{a} + \mathfrak{k} = \mathfrak{g}$.

Proof. Let $a := \exp H$. We note first, that $\mathfrak{m}$ always sits in $(\mathrm{Ad}(a)\mathfrak{k}) \cap \mathfrak{k}$ since $\mathrm{Ad}(a)X = X$ for $X \in \mathfrak{m}$. But of course, the intersection $\mathrm{Ad}(a)\mathfrak{k} \cap \mathfrak{k}$ may be bigger (for instance if $a = e$). However, for dimension reason, the only case in which $\mathrm{Ad}(\exp H)\mathfrak{k} + \mathfrak{a} + \mathfrak{k}$ can fill up $\mathfrak{g}$, is when this intersection is minimal: from the Iwasawa decomposition and root space decomposition, we easily obtain $\dim \mathfrak{k} = \dim \mathfrak{n} + \dim \mathfrak{m}$, therefore the dimension of $(\mathrm{Ad}(a)\mathfrak{k}) + \mathfrak{a} + \mathfrak{k}$ is at most

$$2 \dim \mathfrak{k} + \dim \mathfrak{a} - \dim \mathfrak{m} = \dim \mathfrak{g}.$$

which is only attained in the case of minimal intersection. This verifies the last equivalence.

Assume now that $\alpha(H) = 0$ for some root α and let $X_\alpha \neq 0$ be a root vector. Then obviously $X_\alpha + \theta X_\alpha \in \mathfrak{k}$ and $\mathrm{Ad}(a)X_\alpha = X_\alpha$ and $\mathrm{Ad}(a)(\theta X_\alpha) = \theta X_\alpha$ (since $\theta X_\alpha = X_{-\alpha}$ - a root vector for the root $-\alpha$). Thus $X_\alpha + \theta X_\alpha \in \mathrm{Ad}(a)\mathfrak{k}$ as well. Clearly this is not in $\mathfrak{m}$. In other words if H is not regular, the intersection is non-minimal.

For the converse implication, we note that

$$\mathfrak{k} = \mathfrak{m} + \{X + \theta X \mid X \in \mathfrak{n}\}$$

for dimension reasons (the formula for $\dim \mathfrak{k}$ above). Now we assume $(\mathrm{Ad}(a)\mathfrak{k}) \cap \mathfrak{k}$ to contain an element X which is not in $\mathfrak{m}$. Since it is in particular in $\mathfrak{k}$ we can assume it of the form $X_0 + \sum_{\alpha \in \Sigma^+(\mathfrak{g},\mathfrak{a})}(X_\alpha + \theta X_\alpha)$ with $X_0 \in \mathfrak{m}$ and X_α an element of $\mathfrak{g}_\alpha$ for which at least one X_α is non-zero. Note that $\theta X_\alpha \in \mathfrak{g}_{-\alpha}$. Then

$$\begin{aligned}\mathrm{Ad}(a)X &= X_0 + \sum_{\alpha \in \Sigma^+(\mathfrak{g},\mathfrak{a})} (\exp(\mathrm{ad}(H))X_\alpha + \exp(\mathrm{ad}(H))\theta X_\alpha) \\ &= X_0 + (e^{\alpha(H)}X_\alpha + e^{-\alpha(H)}\theta X_\alpha)\end{aligned}$$

and since this is also in $\mathfrak{k}$, we must have

$$\begin{aligned}0 &= \sum_{\alpha \in \Sigma^+(\mathfrak{g},\mathfrak{a})} [(e^{\alpha(H)}X_\alpha + e^{-\alpha(H)}\theta X_\alpha) - \theta(e^{\alpha(H)}X_\alpha + e^{-\alpha(H)}\theta X_\alpha)] \\ &= \sum_{\alpha \in \Sigma^+(\mathfrak{g},\mathfrak{a})} 2\sinh(\alpha(H))X_\alpha - \sum_{\alpha \in \Sigma^+(\mathfrak{g},\mathfrak{a})} 2\sinh(\alpha(H))\theta X_\alpha\end{aligned}$$

and since the root spaces $\mathfrak{g}_\alpha$ are linearly independent, we get $\alpha(H) = 0$ for all α where $X_\alpha \neq 0$. Hence H is not regular. This proves the first equivalence. $\square$

In the semisimple symmetric case, a similar result holds. Recall from Section 1.10 that we defined $\mathfrak{a}_q^{\mathrm{reg}}$ to be the complement in $\mathfrak{a}_q$ of the union of kernels of roots in the smaller root system $\Sigma(\mathfrak{g}^+, \mathfrak{a}_q)$. Then the following holds (cf. [31] Lemma 1.5):

Proposition 3.13. *If $a \in A_q^{\mathrm{reg}}$, then there is a decomposition*

$$\mathfrak{g} = \mathrm{Ad}(a)\mathfrak{k} + \mathfrak{a}_q + \mathfrak{h}.$$

Now, let as usual K be a threefold product of $K' = SO(2)$. From our study of polar decompositions in the Sections 2.1 and 2.2, we know that we need $A = A_1 \times A_2 \times A_3$ to be such that two of the groups are equal and orthogonal to the other, in order to obtain a polar decomposition with minimal non-uniqueness. The condition is fulfilled for $A = A_1 \times A_2 \times A_1$ where

$$A_1 = \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \,\middle|\, t \in \mathbb{R} \right\} = \exp\left(\mathbb{R} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right)$$
$$A_2 = \left\{ \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \,\middle|\, t \in \mathbb{R} \right\} = \exp\left(\mathbb{R} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) = SO_e(1,1)$$

and $A = A_2 \times A_1 \times A_2$. The propositions above and the fact that $\mathfrak{m} = \{0\}$ motivates the following definition

Definition 3.14. An element $a \in A$ is called *regular* if $\mathrm{Ad}(a)\mathfrak{k} \cap (\mathfrak{a} \oplus \mathfrak{h})$ is non-trivial. An element in A which is not regular is called *singular*.

The main result in this subsection is the determination of the regular elements as well as a "regular" polar decomposition of G and an explicit formula for the Haar measure, when pulled back along the decomposition. Some notation: let $t = (t_1, t_2, t_3)$ and define

$$a_t := \left(\begin{pmatrix} \cosh t_1 & \sinh t_1 \\ \sinh t_1 & \cosh t_1 \end{pmatrix}, \begin{pmatrix} e^{t_2} & 0 \\ 0 & e^{-t_2} \end{pmatrix}, \begin{pmatrix} \cosh t_3 & \sinh t_3 \\ \sinh t_3 & \cosh t_3 \end{pmatrix} \right)$$

which is a typical element in A (in fact $t \longmapsto a_t$ is the exponential map $\mathfrak{a} \cong \mathbb{R}^3 \xrightarrow{\sim} A$).

Proposition 3.15. *The element a_t is regular, precisely when $t_1 \neq t_3$. Equivalently, a_t is singular precisely when $t_1 = t_3$.*

Note that the set of singular elements is a hypersurface in A.

Proof. a_t being of the form above, and $X_\theta \in \mathfrak{k}$ of the form

$$X_\theta = \left(\begin{pmatrix} 0 & -\theta_1 \\ \theta_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\theta_2 \\ \theta_2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\theta_3 \\ \theta_3 & 0 \end{pmatrix} \right)$$

we calculate

$$\begin{aligned} \operatorname{Ad}(a_t)X_\theta = \Big(&\theta_1 \begin{pmatrix} \sinh(2t_1) & -\cosh(2t_1) \\ \cosh(2t_1) & -\sinh(2t_1) \end{pmatrix}, \theta_2 \begin{pmatrix} 0 & -e^{2t_2} \\ e^{-2t_2} & 0 \end{pmatrix}, \\ &\theta_3 \begin{pmatrix} \sinh(2t_3) & -\cosh(2t_3) \\ \cosh(2t_3) & -\sinh(2t_3) \end{pmatrix} \Big). \end{aligned} \tag{3.5}$$

A general element of $\mathfrak{a} \oplus \mathfrak{h}$ is of the form

$$\left(\begin{pmatrix} a & b+\alpha \\ c+\alpha & -a \end{pmatrix}, \begin{pmatrix} a+\beta & b \\ c & -a-\beta \end{pmatrix}, \begin{pmatrix} a & b+\gamma \\ c+\gamma & -a \end{pmatrix} \right) \tag{3.6}$$

for some $a, b, c, \alpha, \beta, \gamma \in \mathbb{R}$. The question that we ask is: for which t is it possible to bring $\operatorname{Ad}(a_t)X_\theta$ on the form (3.6) (by choosing θ properly)? By comparing the diagonal elements in the first and the third matrix we get the equation

$$\theta_1 \sinh(2t_1) = \theta_3 \sinh(2t_3).$$

The difference between the antidiagonal elements in all three matrices should also be the same, i.e.

$$\theta_1 \cosh(2t_1) = \theta_2 \cosh(2t_2) = \theta_3 \cosh(2t_3).$$

All in all we get the following system of equations

$$\begin{aligned} \theta_1 \sinh(2t_1) - \theta_3 \sinh(2t_3) &= 0 \\ \theta_1 \cosh(2t_1) - \theta_3 \cosh(2t_3) &= 0 \\ \theta_1 \cosh(2t_1) - \theta_2 \cosh(2t_2) &= 0 \end{aligned}$$

in the variables θ_i (the t_i's are fixed). The determinant of this equation system is (use a hyperbolic addition formula) $\cosh(2t_2)\sinh(2(t_3 - t_1))$. Thus if this determinant is non-zero, i.e. if $t_1 \neq t_3$, then we have only the trivial solution $\theta_1 = \theta_2 = \theta_3 = 0$ and the overlap $\operatorname{Ad}(a_t)\mathfrak{k} \cap (\mathfrak{a} \oplus \mathfrak{h})$ is trivial.

If $t_1 = t_3$, we get a non-trivial solution $\theta_2 = 1$ and $\theta_1 = \theta_3 = \frac{\cosh(2t_2)}{\cosh(2t_1)}$, it is easy to see that $\mathrm{Ad}(a_t)X_\theta$ is in $\mathfrak{a} \oplus \mathfrak{h}$ with

$$\begin{aligned}
a = \beta &= \cosh(2t_2)\tanh(2t_1) \\
b &= -e^{2t_2} \\
c &= e^{-2t_2} \\
\alpha = \gamma_1 &= -\cosh(2t_2) + e^{2t_2}
\end{aligned}$$

This proves the proposition. $\square$

By $\mathfrak{a}^{\mathrm{reg}}$ we denote the set of regular elements in $\mathfrak{a}$, and similarly by $A^{\mathrm{reg}} := \exp(\mathfrak{a}^{\mathrm{reg}}) = \{(a_1, a_2, a_3) \mid a_1 \neq a_3\}$ the set of regular elements in A. Denote by G^+ the set $KA^{\mathrm{reg}}H$ inside G and let X^+ be the image of G^+ in G/H.

It is clear from the proposition that A^{reg} consists of two connected components. These will serve as our "Weyl chambers". We now fix the chamber $A^+ := \{a_t \mid t_1 > t_3\}$.

Theorem 3.16. *It holds that $G^+ = \{(g_1, g_2, g_3) \in G | g_1 g_3^{-1} \notin K'\}$. Hence G^+ is open and dense in G and X^+ is open and dense in G/H. Put $M := \{\pm(e, e, e)\}$. Then the map*

$$\Phi : K/M \times A^+ \xrightarrow{\sim} X^+$$

given by $(kM, a) \longmapsto kaH$ is a diffeomorphism.

Proof. From Theorem 2.4 we know that the pair (G, H) is polar, i.e. we have $G = KAH$. Hence for each triple (g_1, g_2, g_3) we can find k_1, k_2, k_3 and a_1, a_2, a_3 as well as $h = (g_0, g_0, g_0)$ such that

$$\begin{aligned}
g_1 &= k_1 a_1 g_0 \\
g_2 &= k_2 a_2 g_0 \\
g_3 &= k_3 a_3 g_0
\end{aligned}$$

from which we see that $g_1 g_3^{-1} = k_1 a_1 a_3^{-1} k_3^{-1}$. Obviously this is in K' if and only if $a_1 = a_3$ (the "if" statement is trivial and the "only if" statement follows from uniqueness of the A-part in the $K'AK'$-decomposition).

Next we check that Φ is injective. For this we need the fact, that $(SL(2,\mathbb{R}), A_2)$ is a symmetric pair (the symmetric space $SL(2,\mathbb{R})/SO_e(1,1)$ is locally isomorphic to the hyperbolic space $SO_e(2,1)/SO_e(1,1)$). Hence we have a decomposition

$$G' = K' A_1 A_2. \tag{3.7}$$

As $W_{K\cap H} = \{e\}$ (cf. Example 1.48: $SL(2,\mathbb{R})/SO_e(1,1)$ is locally isomorphic to $SO_e(2,1)/SO_e(1,1)$ - in particular anything that has to do with root spaces and Weyl groups is the same for these two spaces) and $Z_{K'\cap A_2}(A_1) = \{e\}$ (even $K' \cap A_2 = \{e\}$) the $K'A_1A_2$-decomposition of any element in G' is actually unique (the irregular elements are in the set $K'A_1$, i.e. $a_2 = e$, and decompositions in $K'A_1$ are unique, since $K' \cap A_1 = \{e\}$).

In the first of the equations above we eliminate g_0, insert it in the following two and multiply the first equation by k_1 to obtain

$$\begin{aligned} k_2 a_2 a_1^{-1} &= g_2 g_1^{-1} k_1 \\ k_3 a_3 a_1^{-1} k_1^{-1} &= g_3 g_1^{-1}. \end{aligned}$$

In the second equation (note, that $a_1, a_3 \in A_2$) we now fix k_1, k_3 and $a := a_3 a_1^{-1}$ according to the $K'A_2K'$-decomposition of $SL(2,\mathbb{R})$. k_1 is unique up to multiplication by an element in $M' = \{\pm e\}$ (cf. Theorem 7.39 in [22]), and a is unique up to inverse. In the first equation we fix k_2, $a_2 \in A_1$ and $a_1 \in A_2$ according to the decomposition (3.7). As noted above, these choices are unique. Now we can either pick $a_3 = aa_1$ or $a_3 = a^{-1}a_1$ and since $a \neq e$, we see that we can pick a_3 in a unique way such that $(a_1, a_2, a_3) \in A^+$. We also see, that if we change k_1 to $-k_1$ we have also to change k_2 to $-k_2$ and k_3 to $-k_3$. In other words $k = (k_1, k_2, k_3)$ is unique up to M (which happens to be equal to $Z_{K\cap H}(A)$).

To check that Φ, which is already smooth and bijective, is a diffeomorphism, it suffices to calculate the determinant of the differential

$$d\Phi_{(kM,a_t)} : T_{(kM,a_t)}(K/M \times A) \longrightarrow T_{ka_tH}G/H.$$

We identify the tangent space $T_{(kM,a_t)}(K/M \times A^+) = T_{kM}K/M \times T_{a_t}A^+$ with $\mathfrak{k} \oplus \mathfrak{a}$ via the identification of $X \in \mathfrak{k}$ and $Y \in \mathfrak{a}$ with

$$\frac{d}{ds}\Big|_{s=0} k\exp(sX)M \qquad \text{resp.} \qquad \frac{d}{ds}\Big|_{s=0} a_t \exp(sY).$$

and if we identify G/H with $G' \times G'$, we may identify $T_{gH}G/H$ with $\mathfrak{g}' \oplus \mathfrak{g}'$.

To calculate the determinant, we first fix orthonormal bases for $\mathfrak{k}\oplus\mathfrak{a}$ and $\mathfrak{g}'\oplus\mathfrak{g}'$, determine the matrix of the differential w.r.t. this basis and then calculate the determinant. Now put

$$X := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad Y := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad Z := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then

$$\{(X,0,0),(0,X,0),(0,0,X)\} \quad \text{and} \quad \{(Z,0,0),(0,Y,0),(0,0,Z)\}$$

are orthonormal bases for $\mathfrak{k}$ and $\mathfrak{a}$ respectively and

$$\{(X,0),(Y,0),(Z,0),(0,X),(0,Y),(0,Z)\}$$

is an orthonormal basis for $\mathfrak{g}' \oplus \mathfrak{g}'$. The matrix representation of $d\Phi_{(kM,a_t)}$ w.r.t. these bases is:

$$\begin{pmatrix} \cosh(2(t_1-t_3)) & 0 & -1 & 0 & 0 & 0 \\ -\sinh(2(t_1-t_3))\cos(2\theta_3) & 0 & 0 & -\sin(2\theta_3) & 0 & \sin(2\theta_3) \\ -\sinh(2(t_1-t_3))\sin(2\theta_3) & 0 & 0 & \cos(2\theta_3) & 0 & -\cos(2\theta_3) \\ 0 & \cosh(2t_2)\cosh(2t_3) & -1 & 0 & \sinh(2t_3) & 0 \\ 0 & f_1(t_2,t_3,\theta_3) & 0 & 0 & \cosh(2t_3)\cos(2\theta_3) & \sin(2\theta_3) \\ 0 & f_2(t_2,t_3,\theta_3) & 0 & 0 & \cosh(2t_3)\sin(2\theta_3) & -\cos(2\theta_3) \end{pmatrix}$$

where

$$\begin{aligned} f_1(t_2,t_3,\theta_3) &:= -\sinh(2t_2)\sin(2\theta_3) + \sinh(2t_3)\cosh(2t_2)\cos(2\theta_3) \\ f_2(t_2,t_3,\theta_3) &:= \sinh(2t_2)\cos(2\theta_3) + \sinh(2t_3)\cosh(2t_2)\sin(2\theta_3). \end{aligned}$$

The determinant of this matrix can be calculated to be

$$\det(d\Phi_{(kM,a_t)}) = \sinh(2(t_1-t_3))\cosh(2t_2)$$

which is non-zero for $t_1 \neq t_3$, in particular when $a_t \in A^+$. This proves that Φ is a diffeomorphism. □

We can use Φ to pull-back the invariant measure μ on G/H to $K/M \times A^+$ (the image measure of μ under Φ^{-1}). From the transformation theorem for integrals, it is known that

$$\Phi^*(\mu) = |\det(D\Phi)| dkda$$

and hence from the computation in the proof above we get

Corollary 3.17. *For any integrable function f on G/H we have*

$$\int_{G/H} f(gH)d\mu(gH) = \int_{K/M}\int_{A^+} f(\Phi(kM,a_t))\sinh(2(t_1-t_3))\cosh(2t_2)dkda_t.$$

One thing to note is that the determinant of the Jacobian does not depend on k. This is not surprising as can be seen as follows: Á priori we have

$$\Phi^*(\mu) = J(k,a)dkda$$

but since $\Phi^{-1}(k_0 gH) = (k_0, e)\Phi^{-1}(gH)$ we get

$$\ell^*_{(k_0,e)}\Phi^*\mu = \Phi^*\ell^*_{k_0}\mu = \Phi^*\mu$$

and as $\ell^*_{(k_0,e)}(J(k,a)dkda) = J(k_0^{-1}k, a)dkda$ we conclude that J must be k-independent.

3.5 Failure of Convexity

One of the important building blocks in the theory of symmetric spaces, is the convexity theorem of Erik van den Ban (see [1]), the content of which is as follows (see also Section 1.10): Let $\mathfrak{a}_q$ be a maximally abelian subspace of $\mathfrak{s}\cap\mathfrak{q}$ and let $\mathfrak{a}$ be a σ-stable maximally abelian subspace of $\mathfrak{s}$ extending $\mathfrak{a}_q$. It has a decomposition $\mathfrak{a} = (\mathfrak{a}\cap\mathfrak{h})\oplus\mathfrak{a}_q$ and we denote by $Q : \mathfrak{a} \longrightarrow \mathfrak{a}_q$ the corresponding projection. Letting $A := \exp\mathfrak{a}$ and picking a positive system for the root system $\Sigma(\mathfrak{g}, \mathfrak{a})$ we consider the corresponding Iwasawa decomposition $G = KAN$. We denote by $\mathscr{H} : G \longrightarrow \mathfrak{a}$ the Iwasawa projection given uniquely by $g = k\exp(\mathscr{H}(g))n$. The convexity gives a complete description of the image of the map $F_a : H \longrightarrow \mathfrak{a}_q$ where $a \in \exp(\mathfrak{a}_q)$ is fixed and $F_a(h) := Q(\mathscr{H}(ah))$.

First, we will briefly describe the ingredients in the expression. We consider the decomposition $\mathfrak{g} = \mathfrak{g}^+\oplus\mathfrak{g}^-$ where $\mathfrak{g}^+ := (\mathfrak{k}\cap\mathfrak{h})\oplus(\mathfrak{s}\cap\mathfrak{q})$ and $\mathfrak{g}^- := (\mathfrak{k}\cap\mathfrak{q})\oplus(\mathfrak{s}\cap\mathfrak{h})$ are the plus and minus 1 eigenspaces of the involution $\sigma\circ\theta$. As $(\mathfrak{k}\cap\mathfrak{h})\oplus(\mathfrak{s}\cap\mathfrak{q})$ is the Cartan decomposition of $\mathfrak{g}^+$, it follows that $\Sigma(\mathfrak{g}^+, \mathfrak{a}_q)$ is a root system and the corresponding Weyl group is denoted $W_{K\cap H}$. Moreover the root spaces $\mathfrak{g}_\alpha$ for α in the bigger root system $\Sigma(\mathfrak{g}, \mathfrak{a}_q)$ are invariant under $\sigma\circ\theta$, thus they split in $\mathfrak{g}_\alpha^+\oplus\mathfrak{g}_\alpha^-$ where $\mathfrak{g}_\alpha^\pm := \mathfrak{g}_\alpha\cap\mathfrak{g}^\pm$. By Σ_-^+ we understand the set of positive roots $\alpha\in\Sigma^+(\mathfrak{g}, \mathfrak{a}_q)$ for which $\mathfrak{g}_\alpha^-\neq 0$. For each root α there is a corresponding co-root H_α, uniquely determined by the condition that H_α is orthogonal to the kernel of α and $\alpha(H_\alpha) = 1$. Let $\Gamma(\Sigma_-^+)$ be the cone

$$\Gamma(\Sigma_-^+) := \bigoplus_{\alpha\in\Sigma_-^+}\mathbb{R}_{\geq 0}H_\alpha.$$

The statement of the convexity theorem is now the following

$$F_a(H) = \operatorname{conv}(W_{K\cap H}\log a) + \Gamma(\Sigma_-^+).$$

Here conv means the convex hull of the set $W_{K\cap H}\log a$. We note, in particular, that the cone is independent of a and equals $F_e(H)$, and that the entire image is a convex set.

A similar result is not to be expected in this more general setting. The triple $SL(2,\mathbb{R})$-space will provide us with a counterexample. As the subgroup A we take

$$A := \left\{ \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix}, \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2^{-1} \end{pmatrix}, \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \right\}.$$

This is the split component of the parabolic subgroup $P = P_1' \times P_2' \times P_3'$ where P_1', P_2' and P_3' are the parabolics leaving invariant the lines spanned by $e_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $e_3 := \frac{\sqrt{2}}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ respectively.

Now, let $\mathscr{H} : G \longrightarrow \mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \mathfrak{a}_3$ denote the Iwasawa projection and pick some $a = (a_1, a_2, a_3) \in A$. We want the study the image of the map $F_a : H \longrightarrow \mathfrak{a}$ given by

$$F_a(g) := \mathscr{H}(a_1 g, a_2 g, a_3 g)$$

(there is no $\mathfrak{a}_q$ to project down to: we need all three dimensions of A to obtain a full KAH-decomposition of G).

First we need to compute the Iwasawa projection. Put

$$H_0 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{and} \qquad T := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

so that $\mathfrak{a} = \mathbb{R}H_0 \oplus \mathbb{R}H_0 \oplus \mathbb{R}T$ then if

$$a = (a_1, a_2, a_3) = (\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix}, \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2^{-1} \end{pmatrix}, \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix})$$

then $(\log \|a_1 e_1\|, \log \|a_2 e_2\|, \log \|a_3 e_3\|) = (\lambda_1, -\lambda_2, t)$ and if $g = k_i a_i n_i$ is the Iwasawa decomposition of g according to $G' = K' A_i' N_i'$, then (since $N_i e_i = e_i$ and since k_i is an orthogonal transformation)

$$\mathscr{H}(g_1, g_2, g_3) = (\log(\|g_1 e_1\|) H_0, -\log(\|g_2 e_2\|) H_0, \log(\|g_3 e_3\|) T).$$

In particular, if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we get more explicitly

$$\mathscr{H}(g,g,g) = \big(\tfrac{1}{2}\log(a^2+c^2) H_0, -\tfrac{1}{2}\log(b^2+d^2) H_0, \tfrac{1}{2}\log(\tfrac{1}{2}(a+b)^2 + \tfrac{1}{2}(c+d)^2) T\big).$$

By some simple arithmetics we see that if $(\frac{1}{2}\log(x) H_0, -\frac{1}{2}\log(y) H_0, \frac{1}{2}\log(z) T)$ is in the image $\mathscr{H}(H)$ (i.e. equals $\mathscr{H} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for some a, b, c and d, then

$xy - 1 = (ab + cd)^2 \geq 0$ and we have $z = \frac{1}{2}(a^2 + b^2 + c^2 + d^2) + ab + cd$ which equals either

$$z = \frac{x+y}{2} + \sqrt{xy - 1}$$

or

$$z = \frac{x+y}{2} - \sqrt{xy - 1}$$

depending on the sign of $ab + cd$. We conclude that the image $\mathscr{H}(H)$ lies inside the following set:

$$\{(\tfrac{1}{2}\log(x)H_0, -\tfrac{1}{2}\log(y)H_0, \tfrac{1}{2}\log(\tfrac{1}{2}(x+y) \pm \sqrt{xy-1})T) \mid x, y > 0,\ xy \geq 1\}.$$

Conversely, assume we have an element in the above set. We want to see that it is of the form $\mathscr{H}(g, g, g)$ i.e. we want to solve the system of equations

$$\begin{aligned} x &= a^2 + c^2 \\ y &= b^2 + d^2 \\ 1 &= ad - bc \end{aligned}$$

given x and y, subject to the constraints $x, y > 0$ and $xy \geq 1$. This is equivalent to finding two vectors $\begin{pmatrix} a \\ c \end{pmatrix}$ and $\begin{pmatrix} d \\ -b \end{pmatrix}$ satisfying that the lengths are $\sqrt{x}$ and $\sqrt{y}$ respectively, and that the inner product is 1. The last condition says that the angle θ between these two vectors should satisfy

$$\cos\theta = \frac{1}{\sqrt{xy}}$$

and the condition $xy \geq 1$ gives that the right-hand side is ≤ 1. Thus the equation can always be solved (albeit highly non-uniquely). We therefore conclude that the image $\mathscr{H}(H)$ equals

$$\{(\tfrac{1}{2}\log(x)H_0, -\tfrac{1}{2}\log(y)H_0, \tfrac{1}{2}\log(\tfrac{1}{2}(x+y) \pm \sqrt{xy-1})T) \mid x, y > 0,\ xy \geq 1\}.$$

or equivalently

$$\{(sH_0, tH_0, \tfrac{1}{2}\log(\tfrac{1}{2}(e^{2s} + e^{-2t}) \pm \sqrt{e^{2(s-t)} - 1})T) \mid s, t \in \mathbb{R},\ s - t \geq 0\}.$$

Obviously, this set is not convex, and it is not a cone as in the symmetric case. The chances of obtaining a general convexity theorem don't seem too good.

3.6 Plancherel Decomposition

The first point here is to describe what happens when we take the outer product of two principal series representations. We consider a semisimple group G with maximally compact subgroup K and a parabolic subgroup $P = MAN$ (not necessarily minimal). If $\xi \in \widehat{M}$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, we denote by $(\pi_{\xi,\lambda}, \mathcal{H}_{\xi,\lambda})$ the corresponding principal series representation of G on $\mathcal{H}_{\xi,\lambda}$. If G' is another reductive group and $P' = M'A'N'$ a parabolic subgroup, then $P \times P'$ is a parabolic subgroup of $G \times G'$ with Langlands decomposition

$$P \times P' = (M \times M')(A \times A')(N \times N').$$

The claim is that

$$\pi_{\xi,\lambda} \times \pi_{\xi',\lambda'} = \pi_{\xi \times \xi',(\lambda,\lambda')} \tag{3.8}$$

as representations of $G \times G'$.

Let's set up some machinery to prove this

Definition 3.18 (Hilbert Bundle). A *Hilbert bundle* over a connected topological space X is a topological space $\mathcal{H}_X$ along with a continuous map $p : \mathcal{H}_X \longrightarrow X$ so that

1) Each fiber $p^{-1}(x)$ has a vector space structure.

2) The subspace topology from E on the fiber can be generated by an inner product.

3) There exists an open cover (U_i) of X and a Hilbert space $\mathcal{H}$ and a homeomorphism

$$p^{-1}(U_i) \xrightarrow{\sim} U_i \times \mathcal{H}$$

whose restriction to a fiber is a unitary map $p^{-1}(x) \xrightarrow{\sim} \mathcal{H}$.

Assume now that the space X comes equipped with a Radon measure. We will focus on the space of L^2-sections $L^2(X, \mathcal{H}_X)$ of a Hilbert bundle and in order to analyze this, the following result will be useful [2].

Theorem 3.19 (Kuiper). *Any infinite-dimensional Hilbert bundle over a compact base space is trivial, i.e. isomorphic to a product bundle $X \times \mathcal{H}$ for some fixed Hilbert space, $\mathcal{H}$.*

[2]The usual statement of Kuiper's Theorem is that the set of homotopy classes of maps $[X, GL(\mathcal{H})]$ where X is compact, $\mathcal{H}$ is an infinite-dimensional Hilbert space and $GL(\mathcal{H})$ is the space of invertible bounded isomorphisms of $\mathcal{H}$ equipped with the norm topology, is a singleton. That the following statement follows from this is remarked in [6] p. 63.

In particular L^2-sections of such a vector bundle are just $\mathcal{H}$-valued L^2-function on X, i.e.

$$L^2(X, \mathcal{H}_X) = L^2(X, \mathcal{H}) = L^2(X)\widehat{\otimes}\mathcal{H}. \tag{3.9}$$

In the case of a finite-dimensional Hilbert bundle , the situation is of course more complicated, since it is by no means true that any vector bundle over a compact space is trivial (just consider the Möbius bundle over the circle). However, in this case (3.9) is still true, as we now show.

Lemma 3.20. *Let $p : E \longrightarrow X$ be a topological vector bundle over a compact topological space X. There exists a measurable bundle isomorphism $\Phi : E \longrightarrow X \times \mathbb{C}^n$ (i.e. a measurable map with measurable inverse) whose restrictions to the fibers are unitary maps.*

Proof. We assume the vector bundle to be complex. The real case is similar. Pick a finite family of local, continuous trivializations

$$\Phi_i : p^{-1}(U_i) \xrightarrow{\sim} U_i \times \mathbb{C}^n, \qquad i = 1, \dots, N$$

about which we may assume that they are fiberwise unitary (if they are not, simply perform a Gram-Schmidt orthonormalization). Define

$$V_1 := U_1, \quad V_2 := U_2 \setminus U_1, \quad \dots \quad V_k := U_k \setminus (U_1 \cup \dots \cup U_{k-1}), \dots$$

$V_1, \dots, V_N$, is a division of X into disjoint measurable sets. Consequently, $p^{-1}(V_i)$ is a measurable subset of E. We can restrict Φ_i to $p^{-1}(V_i)$ to obtain a measurable map $\Phi_i : p^{-1}(V_i) \longrightarrow V_i \times \mathbb{C}^n$ with measurable inverse. Thus we define $\Phi : E \longrightarrow X \times \mathbb{C}^n$ by

$$\Phi(e) := \begin{cases} \Phi_1(e), & e \in p^{-1}(V_1) \\ \vdots & \\ \Phi_N(e), & e \in p^{-1}(V_N) \end{cases}.$$

This is a measurable map with inverse

$$\Phi^{-1}(x, v) = \begin{cases} \Phi_1^{-1}(x, v), & x \in V_1 \\ \vdots & \\ \Phi_N^{-1}(x, v), & x \in V_N \end{cases}$$

which is also measurable. They are clearly both bundle maps. This proves the lemma. $\square$

Lemma 3.21. *If E is an n-dimensional complex vector bundle over the compact base manifold X then the measurable bundle isomorphism from above gives rise to a unitary isomorphism $\widetilde{\Phi} : L^2(X,E) \xrightarrow{\sim} L^2(X,\mathbb{C}^n) \cong L^2(X) \otimes \mathbb{C}^n$, more specifically, the map is given by*

$$(\widetilde{\Phi} f)(x) := p_2 \circ \Phi(f(x)) \qquad \textit{almost everywhere}$$

where $p_2 : X \times \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is the projection on the second factor. The inverse is given by

$$\widetilde{\Phi}^{-1}(x) = \Phi^{-1}(x, f(x)) \qquad \textit{almost everywhere.}$$

These are easily checked to be inverses of each other. Unitarity of $\widetilde{\Phi}$ follows from fiberwise unitarity of Φ.

Now given two Hilbert bundles (finite or infinite-dimensional), we form the *outer tensor* product $\mathcal{H}_X \boxtimes \mathcal{H}_Y$ over the base space $X \times Y$ where the fiber over (x,y) is $\mathcal{H}_x \widehat{\otimes} \mathcal{H}_y$ [3]. We now consider the L^2-space of such a tensor product

Proposition 3.22. *If $\mathcal{H}_X$ and $\mathcal{H}_Y$ are two Hilbert bundles over compact base spaces, then the map $L^2(X,\mathcal{H}_X)\widehat{\otimes}L^2(Y,\mathcal{H}_Y) \longrightarrow L^2(X \times Y, \mathcal{H}_X \boxtimes \mathcal{H}_Y)$ which sends $s_1 \otimes s_2$ to the section $(x,y) \longmapsto s_1(x) \otimes s_2(y)$ is a unitary isomorphism.*

Proof. It is easy to check that the map is actually unitary. In particular it is injective and has closed image. Thus we only need to check that the image is dense.

Assume that e_n and f_m are orthonormal bases for $L^2(X,\mathcal{H}_X)$ and $L^2(Y,\mathcal{H}_Y)$ respectively. The claim is that the sections

$$(x,y) \longmapsto e_n(x) \otimes f_m(y)$$

form an orthonormal basis for $L^2(X \times Y, \mathcal{H}_X \boxtimes \mathcal{H}_Y)$.

Trivialize the bundles by $\Phi : \mathcal{H}_X \longrightarrow X \times \mathcal{H}$ and $\Phi' : \mathcal{H}_Y \longrightarrow Y \times \mathcal{H}'$ (in the infinite-dimensional case, this can be done continuously by Kuiper's Theorem, in the finite-dimensional case, this can be done measurably by the lemma above). If (v_i) and (w_j) are orthonormal bases of $\mathcal{H}$ and $\mathcal{H}'$ respectively, we can form global L^2-sections by

$$\varphi_i(x) := \Phi^{-1}(x, v_i) \qquad \text{and} \qquad \psi_j(y) := \Phi'^{-1}(y, w_j).$$

Obviously $(\varphi_i(x))$ and $(\psi_j(y))$ are orthonormal bases of the fibers $\mathcal{H}_x$ and $\mathcal{H}_y$. For this it is important that the base spaces are compact.

[3] This is opposed to the usual tensor product of two bundles over X, which is the pullback of the outer tensor product from $X \times X$ to X along the diagonal embedding map.

Now assume that $g \in L^2(X \times Y, \mathcal{H}_X \boxtimes \mathcal{H}_Y)$ satisfies $\langle e_n \otimes f_m, g \rangle = 0$, i.e.

$$\int_X \int_Y \langle e_n(x) \otimes f_m(y), g(x,y) \rangle \, dydx = 0$$

for all n and m. We can expand $g(x,y) \in \mathcal{H}_x \widehat{\otimes} \mathcal{H}_y$ in $\varphi_i(x) \otimes \psi_j(y)$ to obtain

$$g(x,y) = \sum_{i,j} a_{ij}(x,y) \varphi_i(x) \otimes \psi_j(y)$$

for some coefficients $a_{ij}(x,y)$ which are L^2-functions. Hence the equation becomes

$$\begin{aligned} 0 &= \int_X \int_Y \Big\langle e_n(x) \otimes f_m(y), \sum_{ij} a_{ij}(x,y) \varphi_i(x) \otimes \psi_j(y) \Big\rangle \, dxdy \\ &= \int_X \langle e_n(x), \varphi_i(x) \rangle \Big(\int_Y \Big\langle f_m(y), \sum_{ij} a_{ij}(x,y) \psi_j(y) \Big\rangle dy \Big) dx. \end{aligned}$$

Since functions of the form $x \longmapsto \langle e_n(x), \varphi_i(x) \rangle$ span $L^2(X)$, it follows that the function

$$x \longmapsto \int_Y \Big\langle f_m(y), \sum_{ij} a_{ij}(x,y) \psi_j(y) \Big\rangle dy$$

is zero as an L^2-function. Since (f_m) is an orthonormal basis of $L^2(Y, \mathcal{H}_Y)$, it means that for almost all x, the section

$$y \longmapsto \sum_{ij} a_{ij}(x,y) \psi_j(y)$$

is zero as an L^2-section. Thus it follows that a_{ij} are all zero as L^2-functions, and hence that $g = 0$ in $L^2(X \times Y, \mathcal{H}_X \boxtimes \mathcal{H}_Y)$. $\square$

In the case of trivial line bundles, we recover the well-known fact that $L^2(X \times Y) \cong L^2(X) \widehat{\otimes} L^2(Y)$.

We can formulate the inducing construction in this setting. Let $P = MAN$ be some parabolic subgroup (not necessarily minimal). Then the flag variety $G/P = K/(M \cap K)$ is compact and

$$P \hookrightarrow G \longrightarrow G/P$$

realizes G as the total space of a principal P-bundle over G/P. Given $\xi \in \widehat{M}$ and $\lambda \in \mathfrak{a}'_{\mathbb{C}}$ we have a representation (ξ, λ) of P on $\mathcal{H}_\xi$ by $man \longmapsto \xi(m)a^{\lambda+\rho}$. Hence we can form the associated Hilbert bundle

$$G \times_{(\xi,\lambda)} \mathcal{H}_\xi \longrightarrow G/P$$

consisting of the equivalence classes $[g, v]$ under the relation

$$(g, v) \sim (gp, (\xi, \lambda)(p^{-1})v)$$

for $p \in P$. This is a homogenous Hilbert bundle, when we define the G-action by $g_0 \cdot [g, v] = [g_0 g, v]$. The Hilbert space for the induced representation $\pi_{\xi,\lambda}$ is $\mathcal{H}_{\pi,\xi} \cong L^2(G/P, G \times_{(\xi,\lambda)} \mathcal{H}_\xi)$, the space of L^2-sections of the associated vector bundle above, and the action is given by [4]

$$(\pi_{\xi,\lambda}(g_0)s)(gP) = g_0 \cdot (s(g_0^{-1} gP)).$$

The situation we want to consider is the following: we have two parabolic subgroups $P_1 = M_1 A_1 N_1$ and $P_2 = M_2 A_2 N_2$ of G and G' respectively, and hence a parabolic subgroup

$$P_1 \times P_2 = (M_1 \times M_2)(A_1 \times A_2)(N_1 \times N_2)$$

of $G \times G'$. Given $\xi_i \in \widehat{M_i}$ and $\lambda_i \in (\mathfrak{a}_i^*)_{\mathbb{C}}$ we have $\xi_1 \times \xi_2 \in (M_1 \times M_2)^\wedge$ and $(\lambda_1, \lambda_2) \in (\mathfrak{a}_1 \oplus \mathfrak{a}_2)^*_{\mathbb{C}}$ and we want to verify that the principal series representation $\pi_{\xi_1 \times \xi_2, (\lambda_1, \lambda_2)}$ of $G \times G'$ is equivalent to $\pi_{\xi_1,\lambda_1} \times \pi_{\xi_2,\lambda_2}$.

$\pi_{\xi_1 \times \xi_2, (\lambda_1, \lambda_2)}$ is defined on the Hilbert space of L^2-sections of the vector bundle

$$(G \times G') \times_{(\xi_1 \times \xi_2, (\lambda_1, \lambda_2))} \mathcal{H}_{\xi_1} \widehat{\otimes} \mathcal{H}_{\xi_2} \tag{3.10}$$

over $G/P_1 \times G'/P_2 = (G \times G')/(P_1 \times P_2)$. The map

$$(G \times G') \times_{(\xi_1 \times \xi_2, (\lambda_1, \lambda_2))} \mathcal{H}_{\xi_1} \widehat{\otimes} \mathcal{H}_{\xi_2} \longrightarrow (G \times_{(\xi_1,\lambda_1)} \mathcal{H}_{\xi_1}) \boxtimes (G' \times_{(\xi_2,\lambda_2)} \mathcal{H}_{\xi_2}) \tag{3.11}$$

given by $[g_1, v_1] \otimes [g_2, v_2] \longmapsto [(g_1, g_2), v_1 \otimes v_2]$ (which is well-defined) can be seen to be a bundle isomorphism. Thus from the theory above, we get a natural Hilbert space isomorphism

$$L^2((G \times G') \times_{(\xi_1 \times \xi_2, (\lambda_1, \lambda_2))} \mathcal{H}_{\xi_1} \widehat{\otimes} \mathcal{H}_{\xi_2}) \xrightarrow{\sim} L^2(G \times_{(\xi_1,\lambda_1)} \mathcal{H}_{\xi_1}) \widehat{\otimes} L^2(G' \times_{(\xi_2,\lambda_2)} \mathcal{H}_{\xi_2}) \tag{3.12}$$

[4] More on this can be found in the lecture notes "Induced representations and the Langlands classification" by Erik van den Ban.

by composing a section with the bundle isomorphism, in other words

$$\mathcal{H}_{\xi_1\times\xi_2,(\lambda_1,\lambda_2)} \cong \mathcal{H}_{\xi_1,\lambda_1}\widehat{\otimes}\mathcal{H}_{\xi_2,\lambda_2}.$$

It is easy to see that isomorphism respects the $G \times G'$-action, and we may therefore conclude

Theorem 3.23. *The unitary map* (3.12) *realizes the equivalence*

$$\pi_{\xi_1\times\xi_2,(\lambda_1,\lambda_2)} \cong \pi_{\xi_1,\lambda_1} \times \pi_{\xi_2,\lambda_2}.$$

We want to use this result along with (2.7) to calculate the Plancherel measure (up to absolute continuity) for the $SL(2,\mathbb{R})$-triple space. The Plancherel measure for the group is an important ingredient in (2.7), so first we recall the Plancherel decomposition for $SL(2,\mathbb{R})$ [5]:

$$\begin{aligned} L^2(SL(2,\mathbb{R})) \cong & \bigoplus_{k=2}^{\infty}(\mathcal{H}_k \oplus \mathcal{H}_{-k})\widehat{\otimes}(\mathcal{H}_{-k}\oplus\mathcal{H}_k) \\ & \oplus \int_{i\mathbb{R}^+}^{\oplus}\mathcal{H}_{1,\lambda}\widehat{\otimes}\mathcal{H}_{1,-\lambda}d\lambda \oplus \int_{i\mathbb{R}^+}^{\oplus}\mathcal{H}_{-1,\lambda}\widehat{\otimes}\mathcal{H}_{-1,-\lambda}d\lambda \end{aligned} \tag{3.13}$$

where the measure $d\lambda$ on $i\mathbb{R}^+$ is the standard Lebesgue measure.

From (2.7) we see that we need to decompose the tensor product of two representations occurring in the Plancherel decomposition of $SL(2,\mathbb{R})$. Such a decomposition of representations is provided by [27] Theorems 4.6, 7.1 and 7.3 respectively:

$$\pi_{\xi_1,\lambda_1}\otimes\pi_{\xi_2,\lambda_2} \cong \Big(\int_{i\mathbb{R}^+}^{\oplus}\pi^{\oplus 2}_{\xi_1\xi_2,\lambda}d\lambda\Big) \oplus \bigoplus_{|k|\geq 2, k\equiv\xi_1\xi_2} T_k \tag{3.14}$$

where the notation $k\equiv\xi$ means that the sum is taken over even k if ξ is trivial and over odd k if ξ is non-trivial. For a tensor product of a principal series and a discrete series, the decomposition is

$$\pi_{\xi,\lambda}\otimes T_n \cong \Big(\int_{i\mathbb{R}^+}^{\oplus}\pi_{(-1)^n\xi,\lambda}\Big) \oplus \bigoplus_k T_k \tag{3.15}$$

[5]See [21] p. 42.

where the last summation is over $|k| \geq 2$ with the same sign as n and which satisfy $k \equiv (-1)^n \xi$. Finally, for a tensor product of discrete series representations we have (for $n, m \geq 2$)

$$\begin{aligned} T_n \otimes T_m &\cong \bigoplus_{k=0}^{\infty} T_{n+m+2k} \\ T_{-n} \otimes T_{-m} &\cong \bigoplus_{k=0}^{\infty} T_{-n-m-2k} \\ T_{-m} \otimes T_n &\cong \left(\int_{i\mathbb{R}^+}^{\oplus} \pi_{\xi,\lambda} d\lambda \right) \oplus \bigoplus_k T_k \end{aligned}$$

where $\xi = (-1)^{n-m}$ and where the summation is over k which satisfy $2 \leq |k| \leq |n-m|$ and have the same sign and parity (i.e. even/odd) as $n - m$.

We can now write down a list of the representations occurring in the Plancherel formula for G/H. We divide them into levels according to the number of $SL(2,\mathbb{R})$ principal series in them. The zeroth level (i.e. the discrete series for G/H) consists of

$$\begin{aligned} T_{-n} \times T_{-m} \times T_{n+m+2k} &\qquad \text{for } k \geq 0 \\ T_n \times T_m \times T_{-n-m-2k} &\qquad \text{for } k \geq 0 \\ T_n \times T_{-m} \times T_k & \end{aligned} \tag{3.16}$$

where in the last, k has the same parity and sign as $n - m$ and where $2 \leq |k| \leq |n-m|$. The Plancherel measure on this part of $\widehat{G}_H$ is just the counting measure and all the representations occur with multiplicity 1.

The next level contains the following representations

$$\begin{aligned} T_m \times T_{-n} \times \pi_{(-1)^{n-m},\lambda} &\qquad \text{for } |n|, |m| \geq 2,\ \lambda \in i\mathbb{R}^+ \\ T_{-n} \times \pi_{(-1)^{n-m},\lambda} \times T_m &\qquad \text{for } |n|, |m| \geq 2,\ \lambda \in i\mathbb{R}^+ \\ \pi_{(-1)^{n-m},\lambda} \times T_{-n} \times T_m &\qquad \text{for } |n|, |m| \geq 2,\ \lambda \in i\mathbb{R}^+. \end{aligned} \tag{3.17}$$

We can view this subset of $\widehat{G}_H$ as a countable set of half-lines, and the Plancherel measure on each of these half-lines is simply the usual Lebesgue measure, and all the representation have multiplicity 1.

The second level looks as follows

$$\begin{aligned}
&\pi_{\xi_1,\lambda_1} \times \pi_{\xi_2,\lambda_2} \times T_k && \text{where } |k| \geq 2 \text{ and } k \equiv \xi_1\xi_2,\ \lambda_1, \lambda_2 \in i\mathbb{R}^+ \\
&\pi_{\xi_1,\lambda_1} \times T_k \times \pi_{\xi_2,\lambda_2} && \text{where } |k| \geq 2 \text{ and } k \equiv \xi_1\xi_2,\ \lambda_1, \lambda_2 \in i\mathbb{R}^+ \\
&T_k \times \pi_{\xi_1,\lambda_1} \times \pi_{\xi_2,\lambda_2} && \text{where } |k| \geq 2 \text{ and } k \equiv \xi_1\xi_2,\ \lambda_1, \lambda_2 \in i\mathbb{R}^+.
\end{aligned} \tag{3.18}$$

This subset of $\widehat{G}_H$, we may view as a countable collection of quadrants and the Plancherel measure on these is the usual Lebesgue measure. Again, all of these representations have multiplicity 1.

Finally, the third and last level - the *most continuous part* looks as follows

$$\pi_{\xi_1,\lambda_1} \times \pi_{\xi_2,\lambda_2} \times \pi_{\xi_1\xi_2,\lambda_3} \qquad \text{for } \lambda_1, \lambda_2, \lambda_3 \in i\mathbb{R}^+. \tag{3.19}$$

We view this as an octant, and the Plancherel measure here equals the Lebesgue measure. All representations in the most continuous part occur with multiplicity 2.

In the following we want to analyze further the connection between the multiplicities and the number of open orbits of parabolic subgroups.

3.7 Orbits of Parabolic Subgroups

As minimal parabolic subgroup P' in $SL(2,\mathbb{R})$ we pick the subgroup of upper triangular matrices. This is, up to conjugation within $SL(2,\mathbb{R})$ the unique minimal parabolic subgroup in $SL(2,\mathbb{R})$. Thus, in $G = G' \times G' \times G'$ the list of conjugacy classes of parabolic subgroups is

$$P' \times P' \times P' \tag{3.20}$$

minimal of dimension 6,

$$G' \times P' \times P', \quad P' \times G' \times P', \quad P' \times P' \times G' \tag{3.21}$$

of dimension 7

$$G' \times G' \times P', \quad G' \times P' \times G', \quad P' \times G' \times G' \tag{3.22}$$

of dimension 8 and

$$G' \times G' \times G' \tag{3.23}$$

of maximal dimension 9.

By Theorem 3.23 the representations (3.16) are trivially induced from the parabolic (3.23), the representations (3.17) are induced from (3.22), the representations (3.18) are induced from (3.21) and the representations (3.19) are induced from (3.22).

The parabolic subgroups act on G/H. We want to analyze the orbit structure of this action. Since (G,H) is a spherical pair, we know at least that it admits open orbits. For the further analysis we introduce the functions $c : SL(2,\mathbb{R}) \longrightarrow \mathbb{R}$ and $d : SL(2,\mathbb{R}) \longrightarrow \mathbb{R}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto c \qquad \text{resp.} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto d$$

and

$$B := \{g \in SL(2,\mathbb{R}) \mid c(g) \neq 0\}.$$

The Bruhat-decomposition of $SL(2,\mathbb{R})$ states that $SL(2,\mathbb{R})$ is the disjoint union of P and $B = \overline{N}MAN = MANwMAN$ where $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is a representative of the non-trivial Weyl group element in $SL(2,\mathbb{R})$.

In the following theorem we make use of our "non-canonical" model $G/H \cong G' \times G'$. However, later in the case of P minimal, we shall give a more invariant characterization [6].

Theorem 3.24. *Consider a parabolic P acting on $G/H \cong G' \times G'$. If P is of the form* (3.23) *or* (3.22) *there is exactly one orbit, namely G/H itself.*

If P is of the form (3.21)*, then there are two orbits, one open, namely $G' \times B$ and one of lower dimension, namely $G' \times P'$.*

If P is of the form (3.20) *there are two open orbits, namely*

$$\mathcal{O}_1 := \{(g_1,g_2) \in G' \times G' \mid c(g_1), c(g_2) \neq 0,\ \tfrac{d(g_1)}{c(g_1)} < \tfrac{d(g_2)}{c(g_2)}\}$$
$$\mathcal{O}_2 := \{(g_1,g_2) \in G' \times G' \mid c(g_1), c(g_2) \neq 0,\ \tfrac{d(g_1)}{c(g_1)} > \tfrac{d(g_2)}{c(g_2)}\}$$

3 orbits of dimension 5, namely

$$\mathcal{O}_3 := B \times P'$$
$$\mathcal{O}_4 := P' \times B$$
$$\mathcal{O}_5 := \{(g_1,g_2) \in B \times B \mid g_1 g_2^{-1} \in P'\}$$

[6] A similar result as the following has been obtained in [25] Proposition 1.2. The result there is for minimal P but for more general groups than just $G' = SL(2,\mathbb{R})$. Instead of considering P-orbits in G/H they consider H-orbits in G/P but that of course amounts to the same thing.

and one orbit of dimension 4, namely $\mathcal{O}_6 := P' \times P'$. *Furthermore*

$$\partial\mathcal{O}_1 = \partial\mathcal{O}_2 = \mathcal{O}_3 \cup \mathcal{O}_4 \cup \mathcal{O}_5 \cup \mathcal{O}_6$$

$$\partial\mathcal{O}_3 = \partial\mathcal{O}_4 = \partial\mathcal{O}_5 = \mathcal{O}_6.$$

PROOF. If $P = G' \times G' \times G'$, it is clear that the only orbit is G/H.

Now consider the parabolic $P' \times G' \times G'$. This action is transitive on G/H, since the point $[g_1, g_2, g_3]$ can be reached from $[e, e, e]$ via $(e, g_1^{-1}g_2, g_1^{-1}g_3) \in P' \times G' \times G'$. Similar of course with the other parabolics in (3.22).

For the parabolic $G' \times P' \times P'$ acting on $G' \times G'$, the claim is that we have 2 orbits, namely $G' \times P'$ and $G' \times B$. These orbits contain the points (e, e) and (e, w). We can reach any point (g, p) in $G' \times P'$ from (e, e) by acting with (g, p, e). Similarly, we can reach any point $(g_1, g_2) \in G' \times B$ from (e, w): simply by picking p_1 and p_2 in P' such that $p_1 w p_2^{-1} = g_2$ and $g_0 \in G'$ such that $g_0 p_2^{-1} = g_1$. Then $(g_0, p_1, p_2) \cdot (e, w) = (g_1, g_2)$. Similarly for the parabolics $P' \times G' \times P'$ and $P' \times P' \times G'$.

Now we come to the minimal parabolic $P = P' \times P' \times P'$. Note that $G' \times G' = (P' \times P') \cup (B \times P') \cup (P' \times B) \cup (B \times B)$ by the Bruhat decomposition of G'. Note also that $B \times B = \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_5$, since we can reformulate the condition on $\mathcal{O}_5$ to $\frac{d(g_1)}{c(g_1)} = \frac{d(g_2)}{c(g_2)}$. Therefore the sets $\mathcal{O}_1, \ldots, \mathcal{O}_6$ fill up all of $G' \times G'$. Hence we just have to prove that they are actually orbits, i.e. that the action of P is transitive on them. This is clear in the case of the orbit $P' \times P'$.

Now consider $\mathcal{O}_3 = B \times P'$. It is easily checked to be invariant under P. It contains (w, e). Given $(g, p) \in B \times P'$, we can find p_1 and p_3 in P' such that $g = p_1 w p_3^{-1}$. Put $p_2 := pp_3$, then $(g, p) = (p_1, p_2, p_3) \cdot (w, e)$. Hence $B \times P'$ is an orbit. Similarly with $P' \times B$.

Next we consider $\mathcal{O}_5$. This set contains (w, w). Again it is easy to see that it is P-invariant. For transitivity: let $(g_1, g_2) \in \mathcal{O}_5$ be given, and find p_1 and p_3 in P' such that $g_1 = p_1 w p_3^{-1}$ and put $p_2 := (g_2 g_1^{-1}) p_1$, then $(p_1, p_2, p_3) \cdot (w, w) = (g_1, g_2)$.

Finally, we come to the orbit $\mathcal{O}_1$. The set contains the point $\left(w, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}\right)$. Let (g_1, g_2) be some arbitrary element in $\mathcal{O}_1$, then $\frac{d(g_1)}{c(g_1)} < \frac{d(g_2)}{c(g_2)}$. We define

$$t := \sqrt{\frac{d(g_2)}{c(g_2)} - \frac{d(g_1)}{c(g_1)}} \qquad \text{and} \qquad x := -\frac{d(g_1)}{c(g_1)t} \qquad \text{and} \qquad p_0 := \begin{pmatrix} t & x \\ 0 & t^{-1} \end{pmatrix}$$

and from that we get

$$\frac{d(g_1p_0)}{c(g_1p_0)} = t^{-2}\Big(xt + \frac{d(g_1)}{c(g_1)}\Big) = 0 \qquad \text{and} \qquad \frac{d(g_2p_0)}{c(g_2p_0)} = t^{-2}\Big(xt + \frac{d(g_2)}{c(g_2)}\Big) = 1.$$

In other words, by acting with $(e, e, p_0^{-1}) \in P$ we can bring (g_1, g_2) on the following form

$$g_1p_0 = \begin{pmatrix} a_1 & b_1 \\ c_1 & 0 \end{pmatrix} \qquad \text{and} \qquad g_2p_0 = \begin{pmatrix} a_2 & b_2 \\ c_2 & c_2 \end{pmatrix}$$

with $c_1, c_2 \neq 0$. Now we see that

$$\begin{pmatrix} c_1 & -a_1 \\ 0 & c_1^{-1} \end{pmatrix} g_1p_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} c_2 & -a_2 \\ 0 & c_2^{-1} \end{pmatrix} g_2p_0 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

and this means that we can bring any element in $\mathcal{O}_1$ to $(w, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix})$ by acting with an element in P. Hence the P-action is transitive on $\mathcal{O}_1$. The same proof with obvious modifications, works for $\mathcal{O}_2$. $\square$

If $[x_0]$ is a point in a P-orbit $\mathcal{O}$ in G/H, then the stabilizer for the P-action at that point is easily seen to be $P \cap x_0Hx_0^{-1}$. Consequently, we have a diffeomorphism

$$P/P \cap x_0Hx_0^{-1} \xrightarrow{\sim} \mathcal{O} \tag{3.24}$$

by mapping $[p] \longmapsto p \cdot x_0$. In the case of a symmetric space, if P is a $\theta\sigma$-invariant parabolic subgroup we get a Langlands-like decomposition of this quotient as well:

Lemma 3.25. *Let (G, H) be a symmetric pair and $P \subseteq G$ a $\sigma\theta$-invariant parabolic subgroup. Then for each $w \in \mathcal{W}^P$ we have a diffeomorphism*

$$(M_P/M_P \cap wHw^{-1}) \times A_{P\mathfrak{q}} \times N_P \xrightarrow{\sim} P/P \cap wHw^{-1} \tag{3.25}$$

by mapping

$$([m], a, n) \longmapsto [man].$$

Proof. Assume first that $w = e$. The map is well-defined, for if $[m] = [m']$, i.e. $m'm^{-1} \in H$, then $m'an(man)^{-1} = m'm^{-1} \in H$, i.e. $[man] = [m'an]$. It is surjective by the Langlands decomposition and it is injective for the following reason: if $([m], a, n)$ is mapped to $[e]$, then $man \in H$, meaning that

$man\sigma(n)^{-1}\sigma(a)^{-1}\sigma(m)^{-1} = e$. Using the fact that $\sigma(a) = a^{-1}$ and the normalization properties of M_P and A_P we get

$$man\sigma(n)^{-1}\sigma(a)^{-1}\sigma(m)^{-1} = m\sigma(m)^{-1}a^2n'$$

for some $n' \in N_P$. By uniqueness of the Langlands decomposition we get $m\sigma(m)^{-1} = e$ and $a^2 = e$, which implies $m \in M_P \cap H$ and $a = e$. From this it follows that $n \in N_P \cap H = \{e\}$ and hence the map is injective.

We can reduce $P/P \cap wHw^{-1}$, to the above case, since (G, wHw^{-1}) is a symmetric pair under the involution $\sigma^w := C_w \circ \sigma \circ C_{w^{-1}}$ [7] of G, and P is a $\sigma^w\theta$-invariant parabolic. Equivalently, wPw^{-1} is a $\sigma\theta$-invariant parabolic. This is for the following reason: The Lie algebra of the parabolic P splits in a Langlands decomposition $\mathfrak{m}_P \oplus \mathfrak{a}_P \oplus \mathfrak{n}_P$. Consider $\mathfrak{a}_P$ which by definition is contained in $\mathfrak{s}$. We can extend it to a maximally abelian subspace $\mathfrak{a}$ in $\mathfrak{s}$ containing $\mathfrak{a}_\mathfrak{q}$. We can choose w in such a way that $\mathrm{Ad}(w)$ preserves both $\mathfrak{a} \cap \mathfrak{q} = \mathfrak{a}_\mathfrak{q}$ and $\mathfrak{a} \cap \mathfrak{h} := \mathfrak{a}_\mathfrak{h}$ [8].

First, we consider $\mathfrak{a}_P$ which splits into $\mathfrak{a}_{P_\mathfrak{h}} \oplus \mathfrak{a}_{P_\mathfrak{q}}$. The fact that σ preserves $\mathfrak{a}_\mathfrak{h}$ and $\mathfrak{a}_\mathfrak{q}$ translates into the fact that $\mathrm{Ad}(w)|_{\mathfrak{a}_P}$ commutes with σ. Since $\mathrm{Ad}(w)$ also commutes with θ (as w is chosen in K), it commutes with $\sigma\theta$. Thus, as $\mathfrak{a}_P$ is $\sigma\theta$-invariant, so is $\mathrm{Ad}(w)\mathfrak{a}_P$. In particular it also follows that $\mathrm{Ad}(w)\mathfrak{a}_{P_\mathfrak{q}}$ is $\sigma\theta$-invariant. From this it easily follows that $\mathfrak{m}_P \oplus \mathfrak{a}_P$ which is the centralizer of $\mathfrak{a}_{P_\mathfrak{q}}$ in $\mathfrak{g}$ is $\sigma\theta$-invariant. So we are only left with $\mathfrak{n}_P$. A trivial calculation shows that if

$$\mathfrak{n}_P = \bigoplus_{\alpha \in \Gamma} \mathfrak{g}_\alpha$$

then

$$\mathrm{Ad}(w)\mathfrak{n}_P = \bigoplus_{\alpha \in \Gamma} \mathfrak{g}_{w \cdot \alpha}.$$

Again, θ commutes with $\mathrm{Ad}(w)$, and a quick calculation reveals that $[H, \sigma \circ \mathrm{Ad}(w)X_\alpha] = -(w \cdot \alpha)(H)(\sigma \circ \mathrm{Ad}(w)X_\alpha)$, i.e. that sigma maps $\mathfrak{g}_{w\cdot\alpha}$ to $\mathfrak{g}_{-w\cdot\alpha}$. This is equal to the action of θ, and hence $\mathfrak{n}_P$ is $\sigma\theta$-invariant. □

In the symmetric space case, this is important since it allows us to effectively reduce to $M_P/(M_P \cap wHw^{-1})$ and combined with the fact that $M_P/(M_P \cap wHw^{-1})$ is again a symmetric pair, an induction argument can be applied.

Inspired by this, we might anticipate a relation like the following

$$P/P \cap x_0Hx_0^{-1} = M_P/M_P \cap x_0Hx_0^{-1} \times A_P/A_P \cap x_0Hx_0^{-1} \times N_P/N_P \cap x_0Hx_0^{-1}$$

[7] C_w is conjugation with w.

[8] This is Lemma 1.1 in [3].

to hold in our setting. The following example, however, shows that this cannot be the case in general. Take as P the parabolic $G' \times G' \times P'$. Then $P \cap H = \{(p,p,p) \mid p \in P'\}$ has dimension 2, and hence $P/P \cap H$ has dimension 6. But as $M_P = G' \times G' \times M'$ and $M_P \cap H = \{\pm(1,1,1)\}$ then $M_P/M_P \cap H$ alone has dimension 6. As $A_P = 1 \times 1 \times A'$, $N_P = 1 \times 1 \times N'$ and $A_P \cap H = N_P \cap H = \{e\}$, then $A_P/A_P \cap H$ and $N_P/N_P \cap H$ both have dimension 1, and we see that the above identity cannot be satisfied.

Moreover, it is no longer true, that $(M_P, M_P \cap x_0 H x_0^{-1})$ is a spherical pair: With the same parabolic as above: a minimal parabolic P_0 in $M_P = G' \times G' \times M'$ has dimension 4, but as $M_P \cap H$ has dimension 0, it is impossible for $P_0 \times (M_P \cap H)$ to have an open orbit in M_P (which should be 6-dimensional).

3.8 H-Invariant Distribution Vectors

First a useful lemma

Lemma 3.26. *Let G be a Lie group and H a closed subgroup such that G/H has an invariant measure. Let η be a distribution on G/H satisfying $L_g\eta = f(g)\eta$ for some $f \in C^\infty(G)$. Then f is a homomorphism $f : G \longrightarrow \mathbb{R} \setminus \{0\}$, constant 1 on H, and η equals the smooth function $g \longmapsto cf(g)^{-1}$ for some constant c.*

Proof. First we assume that $H = \{e\}$ and that η is left invariant, i.e. $L_g\eta = \eta$ for all $g \in G$. We calculate (for φ a test function)

$$\begin{aligned} X\eta(\varphi) &= -\eta(X\varphi) = -\eta\Big(\lim_{t\to 0} \frac{L_{\exp(-tX)}\varphi - \varphi}{t}\Big) \\ &= -\lim_{t\to 0} \eta\Big(\frac{L_{\exp(-tX)}\varphi - \varphi}{t}\Big) = -\lim_{t\to 0}\Big(\frac{L_{\exp(tX)}\eta - \eta}{t}(\varphi)\Big) = 0. \end{aligned}$$

From this formula it follows immediately that a left invariant η satisfies $X\eta = 0$. Consequently it is also in the kernel of every invariant differential operator on G. Pick an inner product on T_eG and extend it to a G-invariant Riemannian metric on G. Then the Laplace-Beltrami operator constructed through this metric, is G-invariant and elliptic. As η is in the kernel of it, η must be a smooth function, and a constant one even.

For non-trivial H: Assume $\eta \in \mathscr{D}'(G/H)^G$. We lift η to a distribution $\eta^\sharp$ on G by defining $\eta^\sharp(\varphi) := \eta(\varphi^\sharp)$ where

$$\varphi^\sharp(gH) := \int_H \varphi(gh)dh$$

for $\varphi \in C_c^\infty(G)$. It is easy to see that $\eta^\sharp$ is a G-invariant distribution on G. By the first part of the proof, it is a constant. This means that its action on functions is just the integral times a constant. For a function $\varphi \in C_c^\infty(G/H)$ we can (by [19] Lemma I.1.10) assume it to be of the form $\varphi = \psi^\sharp$ for $\psi \in C_c^\infty(G)$ and hence we get (assuming that the measures on G, H and G/H are properly normalized)

$$\begin{aligned}\eta(\varphi) = \eta(\psi^\sharp) = c\int_G \psi(g)dg &= c\int_{G/H}\Big(\int_H \psi(gh)dh\Big)d(gH)\\ = c\int_{G/H}\psi^\sharp(gH)d(gH).\end{aligned}$$

This shows that η is constant.

In the more general case, it is easy to see (by multiplicativity of $g \longmapsto L_g$) that f is a homomorphism $f : G \longrightarrow \mathbb{R}^*$ (ifη happens to be the zero distribution, we can pick f to be constant 1). In particular f is everywhere non-zero. Now consider the distribution $f\eta$. It is easily checked that $L_g(f\eta) = (L_g f)(L_g\eta) = f(g^{-1})ff(g)\eta = f\eta$, i.e. is left-invariant, hence constant cf. the first part of the proof. Consequently $\eta = cf^{-1}$. □

Now, consider a principal series representation $\pi_{P,\xi,\lambda}$ occurring in the Plancherel decomposition of G/H. Then $\mathcal{H}_{P,\xi,\lambda}^{-\infty,H} \neq 0$ and any such $\eta \in \mathcal{H}_{P,\xi,\lambda}^{-\infty,H} \neq 0$ can be viewed as a $\mathcal{H}_\xi^{-\infty}$-valued distribution on G/H satisfying left P-equivariance, in the sense that

$$L_{man}\eta = a^{\lambda+\rho}\xi(m)\eta.$$

From the lemma above, it follows that η restricts to a smooth function on the open P-orbits in G/H (since an open P-orbit is of the form $P/P \cap x_0Hx_0^{-1}$ - it has an invariant measure, since it is an open subset of G/H). Recall the diffeomorphism (3.24). On an orbit $\mathcal{O}$ containing x_0, η satisfies $\eta(p\cdot x_0) = \eta(x_0)$, and by the equivariance of η:

$$(\xi,\lambda)(p)\eta(x_0) = \eta(x_0)$$

in other words, $\eta(x_0) \in \mathcal{H}_\xi^{-\infty}$ should be invariant under the representation (ξ,λ). In particular $\eta(x_0) \in \mathcal{H}_\xi^{-\infty,M_P\cap x_0Hx_0^{-1}}$ and $a^{\lambda+\rho} = 1$ for all $a \in A_P \cap x_0Hx_0^{-1}$.

For symmetric spaces it holds in general that $\mathcal{H}_\pi^{-\infty,H}$ is finite-dimensional when $\pi \in \widehat{G}$ (cf. [2] Lemma 3.3) and if $\pi = \pi_{P,\xi,\lambda}$ is a principal series representation for a generic λ (meaning for $\lambda \in \mathfrak{a}_{q\mathbb{C}}^*$ outside a countable union of certain hyperplanes), then a distribution vector $\eta \in \mathcal{H}_{P,\xi,\lambda}^{-\infty,H}$ is actually completely determined by its valued on the open P-orbits in G/H (this is Corollary 5.3 in [3] when P is minimal parabolic, in the non-minimal case this is due to Brylinski and Delorme in [7]).

In our case, we can prove a similar result, namely that for generic λ, the H-fixed distribution vectors are determined by their values on the open orbits. Our main tool in the following will be a theorem of Bruhat which we first describe in full generality (following the delineation in [3] Appendix A which is for distributions with values in a finite-dimensional vector space - the infinite-dimensional version is treated in [8] Appendix A2). Later we restrict to the situation we need.

Consider a manifold M on which a group G acts smoothly. Assume furthermore that we have a finite-dimensional representation τ of G on a vector space V. By $\mathscr{D}'(M,V)$ we denote the space of V-valued distributions on M, i.e. the space of continuous linear maps $\mathscr{D}(M) \longrightarrow V$, and we denote by $\mathscr{D}'(M,\tau)$ the subspace of $\mathscr{D}'(M,V)$ of distributions η which satisfy $L_g\eta = \tau(g^{-1})\eta$ (here $L_g\eta$ is the dual of the left action on smooth functions on M, just as in the lemma above in the case of M being G itself). For a G-orbit $\mathcal{O}$ in M, we let $\mathscr{D}_k'(M,\mathcal{O},\tau)$ denote the space of distributions $\eta \in \mathscr{D}'(M,\tau)$ of distribution order at most k which satisfy

$$\mathcal{O} \cap \operatorname{supp} \eta \ \text{ open in } \operatorname{supp} \eta.$$

In order to exclude the trivial case, where $\mathcal{O} \cap \operatorname{supp} \eta$ is open in $\operatorname{supp} \eta$ because it is empty, we define

$$\mathscr{D}_k'(\mathcal{O},\tau) := \mathscr{D}_k'(M,\mathcal{O},\tau)/\{\eta \in \mathscr{D}_k'(M,\mathcal{O},\tau) \mid \operatorname{supp} \eta \cap \mathcal{O} = \emptyset\}.$$

The Bruhat theorem is exactly the tool we need to determine this dimension. In order to state the theorem, we need some more notation. Fix a point x_0 in the orbit $\mathcal{O}$ and let G_{x_0} denote the stabilizer at this point. G_{x_0} acts in a natural way on $T_{x_0}M$, and it maps the subspace $T_{x_0}\mathcal{O}$ to itself. In other words, we have a representation of G_{x_0} on the quotient $T_{x_0}M/T_{x_0}\mathcal{O}$. By taking the derivative, we obtain a representation of the Lie algebra $\mathfrak{g}_{x_0}$ on the quotient $T_{x_0}M/T_{x_0}\mathcal{O}$. For a given $H \in \mathfrak{g}_{x_0}$ we denote by $\gamma_1(H),\dots,\gamma_m(H)$ (where m is the codimension of $\mathcal{O}$ in M) the eigenvalues of the corresponding endomorphism of $T_{x_0}M/T_{x_0}\mathcal{O}$.

Theorem 3.27 (**Bruhat**). *If there exists $H \in \mathfrak{g}_{x_0}$, such that for all eigenvalues μ of $\tau(H)$ and for all $\nu \in \mathbb{N}_0^m$ with $|\nu| := \nu_1 + \cdots + \nu_m \le k$, it holds that*

$$\sum_{i=1}^{m} (\nu_i + 1)\gamma_i(H) + \mu \neq 0, \tag{3.26}$$

then $\mathscr{D}'_k(\mathcal{O},\pi) = 0$.

Now, let's apply it to our setting. The quotient G/H will play the role as the manifold M, on which the minimal parabolic $P = P' \times P' \times P'$ is acting. As we have seen in the previous section, there are 6 orbits for this action. Consider now the unitary principal series $\pi = \pi_{P,\xi,\lambda} = \pi_{\xi_1,\lambda_1} \times \pi_{\xi_2,\lambda_2} \times \pi_{\xi_3,\lambda_3}$ (i.e. $\lambda_j \in i\mathbb{R}$). Taking τ to be the character (ξ,λ) of P, we see that $\mathcal{H}_\pi^{-\infty,H}$ is exactly equal to the space $\mathscr{D}'(G/H,\tau)$. Since $\mathcal{H}_\xi^{-\infty} = \mathcal{H}_\xi = \mathbb{C}$, these are $\mathbb{C}$-valued distributions.

For simplicity of the exposition we shall assume in the following that ξ is the trivial character of M. The effect of a different ξ just an overall sign change, that will not affect the analysis.

The strategy is now to apply the Bruhat theorem to the non-open orbits one by one. First the orbit $\mathcal{O}_3$. This orbit contains the point $x_0 := [w,e,e]$ and the stabilizer at that point is

$$P_{x_0} = P \cap x_0 H x_0^{-1} = \left\{ \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \;\middle|\; a \in \mathbb{R}^* \right\}.$$

For notational convenience we denote an element as above by $p_a := (p_a'^{-1}, p_a', p_a')$. Then the representation $(1,\lambda)$ restricted to P_{x_0} is just

$$(1,\lambda)(p_a) = |a|^{-\lambda_1 - 1} |a|^{\lambda_2 + 1} |a|^{\lambda_3 + 1} = |a|^{-\lambda_1 + \lambda_2 + \lambda_3}.$$

If we put

$$X_0 := \left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right)$$

then the Lie algebra of the stabilizer is $\mathfrak{p}_{x_0} = \mathbb{R}X_0$. For the corresponding Lie algebra representation, we see

$$(1,\lambda)(X_0) = -\lambda_1 + \lambda_2 + \lambda_3 + 1,$$

and this obviously has the eigenvalue $-\lambda_1 + \lambda_2 + \lambda_3 + 1$.

On the other hand we have the action of the stabilizer on the quotient $T_{x_0}(G/H)/T_{x_0}\mathcal{O}_3$. For these calculations we use the model $G/H = G' \times G'$. The point $x_0 = [w,e,e]$ is identified with (w,e) and we see

$$T_{x_0}(G/H) \cong T_w G' \times T_e G' = ((L_w)_* T_e G') \times T_e G',$$

and the action of p_a on this quotient is

$$p_a \cdot ((L_w)_* X_1, X_2) = (p_a'^{-1} w X_1 p_a'^{-1}, \operatorname{Ad}(p_a') X_2) = ((L_w)_* \operatorname{Ad}(p_a') X_1, \operatorname{Ad}(p_a') X_2).$$

The derived action of X_0 then is

$$X_0 \cdot ((L_w)_* X_1, X_2) = \Big((L_w)_* \operatorname{ad} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} X_1, \operatorname{ad} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} X_2 \Big).$$

It is clear that the quotient $T_{x_0}(G/H)/T_{x_0}\mathcal{O}_3$ is spanned by

$$\Big(0, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\Big)$$

and the action of X_0 on this is

$$X_0 \cdot \Big(0, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\Big) = \Big(0, \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix}\Big)$$

meaning that X_0 acts by the eigenvalue -2.

The left-hand side of (3.26) for $H = X_0$ now reads

$$-2(\nu+1) - \lambda_1 + \lambda_2 + \lambda_3 + 1$$

for some $\nu \in \mathbb{N}_0$ less than or equal to k. As $-\lambda_j$ are all imaginary, this expression can't possibly be 0, unless $-\lambda_1+\lambda_2+\lambda_3 = 0$. But even in that case, we would be left with $-2(\nu+1)+1$ which cannot be zero. Thus we conclude that there can be no H-fixed distribution vector for the principal series representation $\pi_{P,\xi,\lambda}$ which is supported solely on the orbit $\mathcal{O}_3$.

The calculations for the orbits $\mathcal{O}_4$ and $\mathcal{O}_5$ are similar. They contain the points $[e,w,e]$ and $[e,e,w]$ respectively and the left-hand side of (3.26) in this case reads

$$-2(\nu+1) + \lambda_1 - \lambda_2 + \lambda_3 + 1 \qquad \text{and} \qquad -2(\nu+1) + \lambda_1 + \lambda_2 - \lambda_3 + 1.$$

Again, as long as λ_j is imaginary, this can never be 0, so we conclude that there can be no distribution vectors supported on $\mathcal{O}_4$ and $\mathcal{O}_5$.

This leaves us with the closed orbit $\mathcal{O}_6$. We retain the assumption that $\xi = 1$. In the model $G/H = G' \times G'$ this is just $P' \times P'$ and it contains the point (e,e) corresponding to $[e,e,e] := x_0$. The stabilizer in P of this point is easily seen to be

$$P_{x_0} = \{(p,p,p) \mid p \in P'\}$$

whose Lie algebra is given by

$$\mathfrak{p}_{x_0} = \{(X,X,X) \mid X \in \mathfrak{p}'\}.$$

The restriction of $(1, \lambda)$ to P_{x_0} is now given by

$$(1, \lambda)(p, p, p) = |a|^{\lambda_1+\lambda_2+\lambda_3+3}$$

when $p = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$. To determine the derived representation we fix the following basis elements for $\mathfrak{p}_{x_0}$:

$$X_1 := \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$$
$$X_2 := \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)$$

and a simple calculation gives

$$(1, \lambda)(X_1) = \lambda_1 + \lambda_2 + \lambda_3 + 3,$$
$$(1, \lambda)(X_2) = 0.$$

For the tangent space, we have

$$T_{x_0} G/H = T_{(e,e)}(G' \times G') = \mathfrak{g}' \oplus \mathfrak{g}'$$

and the action of P_{x_0} hereon is simply

$$(p, p, p) \cdot (X, Y) = (\mathrm{Ad}(p)X, \mathrm{Ad}(p)Y)$$

and consequently the derived action of $\mathfrak{p}_{x_0}$ is

$$(Z, Z, Z) \cdot (X, Y) = ([Z, X], [Z, Y]).$$

As basis for the quotient $T_{x_0}(G/H)/T_{x_0}\mathcal{O}_6$ we pick

$$X_0 := \left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, 0\right) \qquad \text{and} \qquad Y_0 = \left(0, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right)$$

and then it is clear (basically since X_0 and Y_0 are root vectors for the root -2) that

$$(X_1, X_1, X_1) \cdot X_0 = -2X_0 \qquad \text{and} \qquad (X_1, X_1, X_1) \cdot Y_0 = -2Y_0.$$

Thus (X_1, X_1, X_1) acts with the eigenvalue -2 and therefore the left-hand side of (3.26) reads

$$-2(\nu_1 + \nu_2) + \lambda_1 + \lambda_2 + \lambda_3 - 1.$$

Again, as long as λ_j is imaginary, this can never be 0. So we draw the first conclusion [9].

Theorem 3.28. *Let $\eta \in \mathcal{H}^{-\infty,H}_{P,\xi,\lambda}$ be an H-fixed distribution vector for a unitary principal series representation $\pi_{P,\xi,\lambda}$ with P minimal. Then η is completely determined by its values on the union of the open orbits. More precisely, given two points $x_i \in \mathcal{O}_i$, $i = 1, 2$ the evaluation map $ev : \mathcal{H}^{-\infty,H}_{P,\xi,\lambda} \longrightarrow \mathbb{C}^2$ given by $\eta \longmapsto (\eta(x_1), \eta(x_2))$ is injective. In particular the dimension of $\mathcal{H}^{-\infty,H}_{P,\xi,\lambda}$ is at most 2.*

The last statement follows from the fact, that the distribution vector is uniquely determined on an open orbit by its value at a single point, and the fact that there are two open orbits, when P is minimal.

However, so far we didn't use the full strength of the Bruhat theorem. We only used it for λ imaginary. If we allow the λ-parameters to take non-imaginary values, we see that the condition (3.26) is actually satisfied for all k, when the λ's satisfy the following 4 relations.

$$-\lambda_1 + \lambda_2 + \lambda_3 \notin 1 + 2\mathbb{N}_0 \tag{3.27}$$
$$\lambda_1 - \lambda_2 + \lambda_3 \notin 1 + 2\mathbb{N}_0 \tag{3.28}$$
$$\lambda_1 + \lambda_2 - \lambda_3 \notin 1 + 2\mathbb{N}_0 \tag{3.29}$$
$$\lambda_1 + \lambda_2 + \lambda_3 \notin 1 + 2\mathbb{N}_0 \tag{3.30}$$

In other words, for $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{C}^3$ outside a union of countably many hyperplanes, the conclusions in the theorem above are still valid.

We can exploit this to draw similar conclusions for representations induced from higher parabolics. Specifically, let us consider the following representation $\pi := T_n \times \pi_2 \times \pi_3$ where $\pi_2 = \pi_{\xi_2,\lambda_2}$ and $\pi_3 = \pi_{\xi_3,\lambda_3}$ are some unitary principal series representations and $n \geq 2$. The representation π is induced from the higher parabolic $G' \times P' \times P'$. By (1.25) we have a G-map

$$\mathcal{H}^\infty_{\xi_0,-n} \longrightarrow \mathcal{H}^\infty_{-n} \oplus \mathcal{H}^\infty_n$$

where $\xi_0 = -(-1)^n$, and this map is surjective with finite-dimensional kernel. If π_2 and π_3 denote two arbitrary unitary principal series representations of $\mathrm{SL}(2,\mathbb{R})$, we get a surjective G-map

$$\mathcal{H}^\infty_{\xi_0,-n} \widehat{\otimes} \mathcal{H}^\infty_{\pi_2} \widehat{\otimes} \mathcal{H}^\infty_{\pi_3} \longrightarrow (\mathcal{H}^\infty_{-n} \oplus \mathcal{H}^\infty_n) \widehat{\otimes} \mathcal{H}^\infty_{\pi_2} \widehat{\otimes} \mathcal{H}^\infty_{\pi_3}$$

[9] The following result has also been obtained in [25], in Section 2 and 3, for $G' = SO_e(1,n)$ where they refer to elements in $\mathcal{H}^{-\infty,H}_{P,\xi,\lambda}$ as invariant trilinear functionals. Since the number of open orbits in their case is 1, they actually prove uniqueness of invariant trilinear functionals.

which in turn induces an injective linear map

$$\mathrm{Hom}_H((\mathcal{H}_{-n}^\infty \oplus \mathcal{H}_n^\infty)\widehat{\otimes}\mathcal{H}_{\pi_2}^\infty\widehat{\otimes}\mathcal{H}_{\pi_3}^\infty, \mathbb{C}) \longrightarrow \mathrm{Hom}_H(\mathcal{H}_{\xi_0,-n}^\infty\widehat{\otimes}\mathcal{H}_{\pi_2}^\infty\widehat{\otimes}\mathcal{H}_{\pi_3}^\infty, \mathbb{C}).$$

Upon noting that $\mathcal{H}_\pi^{-\infty,H} = \mathrm{Hom}_H(\mathcal{H}_\pi^\infty, \mathbb{C})$ and invoking Proposition 1.36, we obtain

$$\mathcal{H}_{T_{-n}\times\pi_2\times\pi_3}^{-\infty,H} \oplus \mathcal{H}_{T_n\times\pi_2\times\pi_3}^{-\infty,H} \subseteq \mathcal{H}_{\pi_{\xi,-n}\times\pi_2\times\pi_3}^{-\infty,H}.$$

Complex conjugation gives an isomorphism between $\mathcal{H}_{T_{-n}\times\pi_2\times\pi_3}^{-\infty,H}$ and $\mathcal{H}_{T_n\times\pi_2\times\pi_3}^{-\infty,H}$. Thus in the cases where $\mathcal{H}_{\pi_{\xi_0,-n}\times\pi_2\times\pi_3}^{-\infty,H}$ is 2-dimensional, we must have that $\mathcal{H}_{T_{\pm n}\times\pi_2\times\pi_3}^{-\infty,H}$ is 1-dimensional. But we see that the triple $(-n, \lambda_1, \lambda_2)$ satisfies the conditions above, if $\lambda_2 \neq \lambda_3$ or if $\lambda_2 = \lambda_3$ and n is even. We conclude

Theorem 3.29. *If $\lambda_2 \neq \lambda_3$ or if $\lambda_2 = \lambda_3$ and n is even, then $\mathcal{H}_{T_{\pm n}\times\pi_2\times\pi_3}^{-\infty,H}$ is at most 1-dimensional. The same is true for $\mathcal{H}_{\pi_1\times T_{\pm n}\times\pi_3}^{-\infty,H}$ and $\mathcal{H}_{\pi_1\times\pi_2\times T_{\pm n}}^{-\infty,H}$*

At the next level we have representations of the form $T_n \times T_m \times \pi_3$ for some unitary principal series representation $\pi_3 = \pi_{\xi_3,\lambda_3}$. We can apply the same trick. Letting $\xi_0 = -(-1)^n$ and $\xi_0' = -(-1)^m$ we have a surjective map

$$\mathcal{H}_{\xi_0,-n}\widehat{\otimes}\mathcal{H}_{\xi_0',-m}^\infty\widehat{\otimes}\mathcal{H}_{\pi_3}^\infty \longrightarrow (\mathcal{H}_n^\infty \oplus \mathcal{H}_{-n}^\infty)\widehat{\otimes}(\mathcal{H}_m^\infty \oplus \mathcal{H}_{-m}^\infty)\widehat{\otimes}\mathcal{H}_{\pi_3}^\infty.$$

By the same reasoning as above (plus restriction) we obtain injective maps

$$\mathcal{H}_{T_n\times T_{-m}\times\pi_3}^{-\infty,H} \oplus \mathcal{H}_{T_{-n}\times T_m\times\pi_3}^{-\infty,H} \hookrightarrow \mathcal{H}_{\pi_{\xi,-n}\times\pi_{\xi',-m}\times\pi_3}^{-\infty,H}$$

and

$$\mathcal{H}_{T_n\times T_m\times\pi_3}^{-\infty,H} \oplus \mathcal{H}_{T_{-n}\times T_{-m}\times\pi_3}^{-\infty,H} \hookrightarrow \mathcal{H}_{\pi_{\xi,-n}\times\pi_{\xi',-m}\times\pi_3}^{-\infty,H}.$$

Since λ_3 is imaginary and non-zero, we see that the triple $(-n, -m, \lambda_3)$ always satisfies all the three conditions above, and as complex conjugation gives isomorphisms

$$\mathcal{H}_{T_n\times T_{-m}\times\pi_3}^{-\infty,H} \xrightarrow{\sim} \mathcal{H}_{T_{-n}\times T_m\times\pi_3}^{-\infty,H}$$

and

$$\mathcal{H}_{T_{-n}\times T_{-m}\times\pi_3}^{-\infty,H} \xrightarrow{\sim} \mathcal{H}_{T_n\times T_m\times\pi_3}^{-\infty,H}$$

we arrive at

Theorem 3.30. *For any representation of the form $\pi = T_n \times T_m \times \pi_3$ or $\pi = T_n\times\pi_2\times T_m$ or $\pi = \pi_1\times T_n\times T_m$ with π_i unitary irreducible and $|n|, |m| \geq 2$, it holds that $\mathcal{H}_\pi^{-\infty,H}$ is of dimension at most 1.*

We can summarize this by saying, that outside the discrete series, the set of irreducible unitary representations π for which it *does not hold* that the dimension of $\mathcal{H}_\pi^{-\infty,H}$ equals the number of open orbits in G/H of the corresponding parabolic, is a set of Plancherel measure 0.

Unfortunately, the method does not apply to the discrete series. Then we would have to consider triples (n, m, k) of integers and here there are lot of examples where the conditions above are *not* satisfied.

3.9 The Most Continuous Part

In this section, we study the H-fixed distribution vectors of the unitary principal series representations induced from a minimal parabolic. Here we are able to show that the space of H-fixed distribution vectors is actually *equal* to 2.

However, before we can do that we need to put up some machinery. We begin by giving a more invariant characterization of the orbits under the minimal parabolic subgroup (without referring to the model $G/H \cong G' \times G'$). First we define the following functions on G/H:

$$\begin{aligned}
\psi_1([g_1, g_2, g_3]) &:= c(g_2 g_3^{-1})\\
\psi_2([g_1, g_2, g_3]) &:= c(g_1 g_3^{-1})\\
\psi_3([g_1, g_2, g_3]) &:= c(g_1 g_2^{-1}).
\end{aligned}$$

Then we see that we can characterize the minimal P-orbits along the following lines

$$\begin{array}{lll}
\mathcal{O}_1 : & \varepsilon_1\psi_1,\ \varepsilon_2\psi_2,\ \varepsilon_3\psi_3 > 0, & \text{when } \varepsilon_1\varepsilon_2\varepsilon_3 = 1\\
\mathcal{O}_2 : & \varepsilon_1\psi_1,\ \varepsilon_2\psi_2,\ \varepsilon_3\psi_3 > 0, & \text{when } \varepsilon_1\varepsilon_2\varepsilon_3 = -1\\
\mathcal{O}_3 : & \psi_1 = 0, \quad \psi_2 \neq 0, \quad \psi_3 \neq 0 & \\
\mathcal{O}_4 : & \psi_1 \neq 0, \quad \psi_2 = 0, \quad \psi_3 \neq 0 & \\
\mathcal{O}_5 : & \psi_1 \neq 0, \quad \psi_2 \neq 0, \quad \psi_3 = 0 & \\
\mathcal{O}_6 : & \psi_1 = \psi_2 = \psi_3 = 0. &
\end{array}$$

In the following we concentrate on the two open orbits $\mathcal{O}_1$ and $\mathcal{O}_2$. Defining $\gamma_1 := \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ and $\gamma_2 := \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$, then (w, γ_i) lies in $\mathcal{O}_i$ (where $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$). It is easy to see that the stabilizer of the P-action at (w, γ_i) is

$\pm(1,1,1)$ i.e. equals $M \cap H$. Hence we have a diffeomorphism

$$\begin{aligned} M/(M\cap H) \times A \times N &\xrightarrow{\sim} \mathcal{O}_i \\ ([m], a, n) &\longmapsto man \cdot (w, \gamma_i). \end{aligned} \tag{3.31}$$

From this it is clear, that if $\eta \in \mathcal{H}^{-\infty,H}_{P,\xi,\lambda}$, then $\eta(w,\lambda_i) \in \mathcal{H}^{M\cap H}_{\xi}$, meaning that we only need to consider $\xi \in \widehat{M}^{M\cap H}$, which are $\xi = (\xi_1,\xi_2,\xi_3)$ (viewed as an element in $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$) which satisfy $\xi_1\xi_2\xi_3 = 1$.

Composing the inverse of the above diffeomorphism with the projection onto the M-part resp. the A-part gives us maps

$$\begin{aligned} \mathbf{m} : \mathcal{O}_i &\longrightarrow M/(M\cap H) \\ \mathbf{a} : \mathcal{O}_i &\longrightarrow A \cong \mathbb{R}^3_+ \end{aligned}$$

and the individual components of the map $\mathbf{a}$ are denoted $\mathbf{a}_i$, $i = 1,2,3$. Note that these maps depend on the choice of base points. We now set out to compute these maps. The first thing to note is that the maps ψ_i are left $N = N' \times N' \times N'$-invariant, simply because an upper triangular matrix leaves the c-component unchanged. Therefore, since MA normalizes N we see

$$\psi_i(man\cdot x) = \psi_i(n'ma\cdot x) = \psi_i(ma\cdot x).$$

Thus we may neglect N. An element of $M'A'$ is a diagonal matrix of the form $\mathrm{Diag}(ma,(ma)^{-1})$ where $a \in]0,\infty[$ and $m = \pm 1$. Thus we consider a point in $\mathcal{O}_1$ of the form

$$x := (\begin{pmatrix} m_1a_1 & 0 \\ 0 & (m_1a_1)^{-1} \end{pmatrix}, \begin{pmatrix} m_2a_2 & 0 \\ 0 & (m_2a_2)^{-1} \end{pmatrix}, \begin{pmatrix} m_3a_3 & 0 \\ 0 & (m_3a_3)^{-1} \end{pmatrix})\cdot[w,\gamma_1,e] \tag{3.32}$$

with $a_1,a_2,a_3 > 0$ and $m_1,m_2,m_3 = \pm 1$. A simple calculation gives

$$\psi_1(x) = (m_2a_2m_3a_3)^{-1}, \quad \psi_2(x) = (m_1a_1m_3a_3)^{-1}, \quad \psi_3(x) = (m_1a_1m_2a_2)^{-1}$$

(note that $\psi_1(x)\psi_2(x)^{-1}\psi_3(x)^{-1}$ and the two other expressions are automatically positive). Solving this system of equations gives us the following formulas for the a-components

$$\mathbf{a}_1(x) = \sqrt{\psi_1(x)\psi_2(x)^{-1}\psi_3(x)^{-1}} \quad , \quad \mathbf{a}_2(x) = \sqrt{\psi_1(x)^{-1}\psi_2(x)\psi_3(x)^{-1}}$$

$$\mathbf{a}_3(x) = \sqrt{\psi_1(x)^{-1}\psi_2(x)^{-1}\psi_3(x)}.$$

According to the remarks above, these formulas hold for any point in $\mathcal{O}_1$ and not just for those of the form (3.32).

On $\mathcal{O}_2$ relative to the point $(w,\gamma_2)=[w,\gamma_2,e]$ we have similar formulas

$$\mathbf{a}_1(x)=\sqrt{-\psi_1(x)\psi_2(x)^{-1}\psi_3(x)^{-1}} \quad , \quad \mathbf{a}_2(x)=\sqrt{-\psi_1(x)^{-1}\psi_2(x)\psi_3(x)^{-1}}$$
$$\mathbf{a}_3(x)=\sqrt{-\psi_1(x)^{-1}\psi_2(x)^{-1}\psi_3(x)}.$$

As in the symmetric case we want to examine the connection between the open orbits and the principal series representations. In the previous section we showed that the space $\mathcal{H}^{-\infty,H}_{P,\xi,\lambda}$ was at most 2-dimensional. We devote the rest of this section to prove the result promised at the beginning, namely that the dimension of this space is equal to 2. We do it concretely by writing down an explicit inverse to the evaluation map $\mathrm{ev}:\mathcal{H}^{-\infty,H}_{P,\xi,\lambda}\longrightarrow\mathbb{C}^2$: Given $\eta=(\eta_1,\eta_2)\in\mathbb{C}^2$, $\xi\in\widehat{M}^{M\cap H}$ as well as $\lambda=(\lambda_1,\lambda_2,\lambda_3)\in\mathfrak{a}^*_{\mathbb{C}}\times\mathfrak{a}^*_{\mathbb{C}}\times\mathfrak{a}^*_{\mathbb{C}}$ we can construct an H-invariant distribution vector as the function

$$j(P,\xi,\lambda,\eta)(x):=\begin{cases}\xi(\mathbf{m}(x))\mathbf{a}_1(x)^{\lambda_1+1}\mathbf{a}_2(x)^{\lambda_2+1}\mathbf{a}_3(x)^{\lambda_3+1}\eta_i & \text{if } x\in\mathcal{O}_1\cup\mathcal{O}_2\\ 0 & \text{if } x\notin\mathcal{O}_1\cup\mathcal{O}_2\end{cases}.$$

Here $\lambda_i\in\mathfrak{a}^*_{\mathbb{C}}$ is identified with an element in $\mathbb{C}$.

Lemma 3.31. *The function $j(P,\xi,\lambda,\eta):G/H\longrightarrow\mathbb{C}$ as defined above is a continuous function on G/H if λ satisfies*

$$\begin{aligned}\mathrm{Re}(\lambda_1-\lambda_2-\lambda_3-1)&>0,\\ \mathrm{Re}(-\lambda_1+\lambda_2-\lambda_3-1)&>0,\\ \mathrm{Re}(-\lambda_1-\lambda_2+\lambda_3-1)&>0.\end{aligned} \tag{3.33}$$

This is an open set in $\mathfrak{a}^*_{\mathbb{C}}\cong\mathbb{C}^3$ and consists of the λ's where $\mathrm{Re}\,\lambda$ lies in the translated cone

$$\Gamma:=\mathbb{R}_-\begin{pmatrix}1\\1\\0\end{pmatrix}+\mathbb{R}_-\begin{pmatrix}1\\0\\1\end{pmatrix}+\mathbb{R}_-\begin{pmatrix}0\\1\\1\end{pmatrix}-\begin{pmatrix}1\\1\\1\end{pmatrix}.$$

Proof. Substituting in the definition of $j(P,\xi,\lambda,\eta)$ the formulas for $\mathbf{a}_j$ plus defining

$$\begin{aligned}\sigma_1&:=\tfrac{1}{2}(\lambda_1-\lambda_2-\lambda_3-1)\\ \sigma_2&:=\tfrac{1}{2}(-\lambda_1+\lambda_2-\lambda_3-1)\\ \sigma_3&:=\tfrac{1}{2}(-\lambda_1-\lambda_2+\lambda_3-1)\end{aligned}$$

we get on $\mathcal{O}_i$ (for $i = 1, 2$):

$$j(P,\xi,\lambda,\eta)(x) = \xi(\mathbf{m}(x))|\psi_1(x)|^{\sigma_1}|\psi_2(x)|^{\sigma_2}|\psi_3(x)|^{\sigma_3}\eta_i.$$

We want this expression to go to 0, when we converge to the boundary. Recall that the boundary of $\mathcal{O}_1 \cup \mathcal{O}_2$ is $\mathcal{O}_3 \cup \mathcal{O}_4 \cup \mathcal{O}_5 \cup \mathcal{O}_6$, and from the characterization of these orbits earlier in this section we see that converging to a point on the boundary implies the functions ψ_1, ψ_2 and ψ_3 converge to some finite numbers [10], at least one of which is zero. When the λ's satisfy the conditions in the lemma, then also the expression above converges to zero. The continuity now follows. □

The problem with the above lemma, is that it only guarantees continuity of $j(P,\xi,\lambda,\eta)$ in a sector in $\mathfrak{a}^*_{\mathbb{C}}$ which does *not* contain the imaginary "axis" $i\mathfrak{a}^*$. In the theory for symmetric spaces, this problem is dealt with by a meromorphic extension of $\lambda \longmapsto j(P,\xi,\lambda,\eta)$ to all of $\mathfrak{a}^*_{\mathbb{C}}$.

However, in our case we can do with less. A function need not be continuous in order to be a distribution, if suffices that it be locally integrable. This is actually the case for $j(P,\xi,\lambda,\eta)$ for λ on the imaginary axis

Theorem 3.32. *The function $j(P,\xi,\lambda,\eta)$ on G/H is locally integrable when $\lambda \in i\mathfrak{a}^*$, in particular $j(P,\xi,\lambda,\eta) \in \mathcal{H}^{-\infty,H}_{P,\xi,\lambda}$ for $\lambda \in i\mathfrak{a}^*$.*

Proof. Since $j(P,\xi,\lambda,\eta)$ is smooth on $\mathcal{O}_1 \cup \mathcal{O}_2$, it is locally integrable around any of these points. So we only have to show local integrability around points on the boundary of $\mathcal{O}_1 \cup \mathcal{O}_2$. First, from the expression of $j(P,\xi,\lambda,\eta)$ from the proof above with λ imaginary, we see that

$$|j(P,\xi,\lambda,\eta)(x)| = |\psi_1(x)|^{-\frac{1}{2}}|\psi_2(x)|^{-\frac{1}{2}}|\psi_3(x)|^{-\frac{1}{2}}|\eta_i|$$

on $\mathcal{O}_i$, i.e. λ-independent. First we assume $x \in \mathcal{O}_3$, then $\psi_2(x), \psi_3(x) \neq 0$, so we can chose a neighborhood around x where these are non-zero. The problem is with $\psi_1(x) = 0$. To analyze $|\psi_1|^{-\frac{1}{2}}$ around this point we use again our model $G/H \cong G' \times G'$. Thus

$$|\psi_1(g_1,g_2)|^{-\frac{1}{2}} = |c(g_1)|^{-\frac{1}{2}}$$

where $c(g_1)$ is the lower left entry of the matrix g_1. Writing $g_1 = kna = kan'$ according to the $K'N'A'$ and $K'A'N'$ decomposition respectively, and letting

[10] As opposed to the **a**-functions which can diverge to ∞.

U_1 and U_2 be open neighborhoods around g_1 and g_2 respectively (U_2 of total measure 1) we get (assuming for simplicity of calculations that $|\eta_i| = 1$)

$$\int_{G'\times G'} 1_{U_1\times U_2}|\psi_1(g_1,g_2)|^{-\frac{1}{2}}dg_1dg_2 = \int_{K'\times N'\times A'} 1_{U_1}|c(kna)|^{-\frac{1}{2}}dadndk$$

and since $|c(kna)|^{-\frac{1}{2}} = |c(kan')|^{-\frac{1}{2}} = |c(ka)|^{-\frac{1}{2}}$ we get

$$\int_{G'\times G'} 1_{U_1\times U_2}|\psi_1(g_1,g_2)|^{-\frac{1}{2}}dg_1dg_2 = \int_{K'\times N'\times A'} 1_{U_1}|c(ka)|^{-\frac{1}{2}}dadndk.$$

Putting $a = a_t = \mathrm{Diag}(e^t, e^{-t})$ and $k = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ the expression equals

$$\int_N \int_0^{2\pi} \int_{\mathbb{R}} 1_V|\sin\theta|^{-\frac{1}{2}}e^{-\frac{1}{2}t}dtd\theta dn$$

where V is the preimage in $N' \times [0,2\pi[\times\mathbb{R}$ of U_1 under the natural map $N' \times [0,2\pi[\times\mathbb{R} \longrightarrow SL(2,\mathbb{R})$. As $|\sin\theta| \geq \frac{1}{2}|\theta|$ for sufficiently small θ, we have $|\sin\theta|^{-\frac{1}{2}} \leq \sqrt{2}|\theta|^{-\frac{1}{2}}$ and since

$$\int_{-\varepsilon}^{\varepsilon} |\theta|^{-\frac{1}{2}}d\theta = 4\sqrt{\varepsilon} < \infty$$

we see that the integral above converges provided U_1 is chosen small enough. Thus $j(P,\xi,\lambda,\eta)$ is integrable around any point $x \in \mathcal{O}_3$. The argument runs similarly for $x \in \mathcal{O}_4$ or $x \in \mathcal{O}_5$.

Therefore only the case of a point in $\mathcal{O}_6$ remains. So we have to investigate the function

$$|j(P,\xi,\lambda,\eta)(x)| = |\psi_1(x)|^{-\frac{1}{2}}|\psi_2(x)|^{-\frac{1}{2}}|\psi_3(x)|^{-\frac{1}{2}}|\eta_i| \tag{3.34}$$

around a point where $\psi_1(x) = \psi_2(x) = \psi_3(x) = 0$. We may assume that $|\eta_i| = 1$, and employing our model $G/H = G' \times G'$ we consider the function

$$(g_1,g_2) \longmapsto |c(g_2)|^{-\frac{1}{2}}|c(g_1)|^{-\frac{1}{2}}|c(g_1g_2^{-1})|^{-\frac{1}{2}}.$$

We write $g_i = a_in_ik_i$ (for $i = 1,2$) and since the c-function is both left and right N-invariant, we get

$$\begin{aligned} &|c(a_2n_2k_2)|^{-\frac{1}{2}}|c(a_1n_1k_1)|^{-\frac{1}{2}}|c(a_1n_1k_1k_2^{-1}n_2^{-1}a_2^{-1})|^{-\frac{1}{2}} \\ &\qquad = |c(a_2k_2)|^{-\frac{1}{2}}|c(a_1k_1)|^{-\frac{1}{2}}|c(a_1k_1k_2^{-1}a_2^{-1}|^{-\frac{1}{2}} \end{aligned}$$

and if $a_i = \begin{pmatrix} e^{t_i} & 0 \\ 0 & e^{-t_i} \end{pmatrix}$ and $k_i = \begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix}$, then

$$k_1 k_2^{-1} = \begin{pmatrix} \cos(\theta_1-\theta_2) & -\sin(\theta_1-\theta_2) \\ \sin(\theta_1-\theta_2) & \cos(\theta_1-\theta_2) \end{pmatrix}$$

so that $c(k_1k_2^{-1}) = \sin(\theta_1-\theta_2)$ and the expression above becomes

$$|\sin\theta_1|^{-\frac{1}{2}}|\sin\theta_2|^{-\frac{1}{2}}|\sin(\theta_1-\theta_2)|^{-\frac{1}{2}}e^{t_1+t_2}. \tag{3.35}$$

We need to analyze this around $(\theta_1,\theta_2)=(0,0)$. The exponential factor is locally bounded, so that we need not worry about. As above, when $|\theta_i|$ is sufficiently small, we may assume $|\sin\theta_i|^{-\frac{1}{2}} \leq \sqrt{2}|\theta_i|^{-\frac{1}{2}}$. Thus we can estimate (again in a proper neighborhood of $(0,0)$) the expression (3.35) to be bounded by

$$|\theta_1|^{-\frac{1}{2}}|\theta_2|^{-\frac{1}{2}}|\theta_1-\theta_2|^{-\frac{1}{2}}$$

(times some constant which we leave out). We want to integrate this expression over a ball of sufficiently small radius ε in the $\theta_1\theta_2$-plane, and switch to polar coordinates (r,φ):

$$\begin{aligned}&\iint_{B_\varepsilon(0)} |\theta_1|^{-\frac{1}{2}}|\theta_2|^{-\frac{1}{2}}|\theta_1-\theta_2|^{-\frac{1}{2}}d\theta_1 d\theta_2 \qquad (3.36)\\ &\qquad = \int_0^\varepsilon r^{-\frac{1}{2}}dr \int_{-\pi}^{\pi} |\cos\varphi\sin\varphi(\cos\varphi-\sin\varphi)|^{-\frac{1}{2}}d\varphi.\end{aligned}$$

The first integral is convergent, so no worry here. For the second integral, we have 7 problematic points where the integrand diverges to infinity: 0, π and 2π (where $\sin\varphi = 0$), $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ where $\cos\varphi = 0$ and finally $\frac{\pi}{4}$ and $\frac{5\pi}{4}$ where $\cos\varphi - \sin\varphi = 0$. Around 0 we can estimate as we did above, namely $|\sin\varphi|^{-\frac{1}{2}} \leq \sqrt{2}|\varphi|^{-\frac{1}{2}}$ and $|\varphi|^{-\frac{1}{2}}$ is integrable around 0. Thus 0 is no problem. Similar arguments work for the points π, 2π, $\frac{\pi}{2}$ and $\frac{3\pi}{2}$. Around $\frac{\pi}{4}$ we can estimate as follows: $|\cos\varphi-\sin\varphi| \geq \frac{1}{2}|\varphi-\frac{\pi}{4}|$ and consequently $|\cos\varphi-\sin\varphi|^{-\frac{1}{2}} \leq |\varphi-\frac{\pi}{4}|^{-\frac{1}{2}}$ and the right-hand side is integrable around $\frac{\pi}{4}$. This means that the integral (3.36) converges and that (3.34) is locally integrable around points in $\mathcal{O}_6$. $\square$

Corollary 3.33. *When $\lambda \in i\mathfrak{a}^*$, then the map $\mathbb{C}^2 \ni \eta \longmapsto j(P,\xi,\lambda,\eta) \in \mathcal{H}^{-\infty,H}_{P,\xi,\lambda}$ is an inverse to the evaluation map* ev. *In particular $\mathcal{H}^{-\infty,H}_{P,\xi,\lambda}$ is 2-dimensional.*

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INDEX

DECOMPOSITION THEOREMS FOR TRIPLE SPACES

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Abstract. A triple space is a homogeneous space G/H where $G = G_0 \times G_0 \times G_0$ is a threefold product group and $H \simeq G_0$ the diagonal subgroup of G. This paper concerns the geometry of the triple spaces with $G_0 = \mathrm{SL}(2, \mathbb{R})$, $\mathrm{SL}(2, \mathbb{C})$ or $\mathrm{SO}_e(n, 1)$ for $n \geq 2$. We determine the abelian subgroups $A \subset G$ for which there is a polar decomposition $G = KAH$, and we determine for which minimal parabolic subgroups $P \subset G$, the orbit PH is open in G/H.

Date: December 20, 2012.
2000 *Mathematics Subject Classification.* 22F30, 22E46.
Key words and phrases. triple space, polar decomposition, spherical.
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1. Introduction

Let G_0 be a real reductive group and let $G = G_0 \times G_0 \times G_0$ and $H = \mathrm{diag}(G_0)$. The corresponding homogeneous space G/H is called a *triple space*. Triple spaces are examples of non-symmetric homogeneous spaces, as there is no involution of G with fixed point group H. It is interesting in the non-symmetric setting to explore properties, which play an important role for the harmonic analysis of symmetric spaces. In this paper we examine the geometric structure of some triple spaces from this point of view.

One important structural result for symmetric spaces is the polar decomposition $G = KAH$. Here $K \subset G$ is a maximal compact subgroup, and $A \subset G$ is abelian. Polar decomposition for a Riemannian symmetric space G/K is due to Cartan, and it was generalized to reductive symmetric spaces in the form $G = KAH$ by Flensted-Jensen [2].

For triple spaces in general, the sum of the dimensions of K, A and H can be strictly smaller than the dimension of G, which obviously prevents $G = KAH$. Here we are interested in the triple spaces with

$$G_0 = \mathrm{SL}(2,\mathbb{R}),\ \mathrm{SL}(2,\mathbb{C}),\ \mathrm{SO}_e(n,1) \quad (n = 2,3,\dots) \tag{1.1}$$

for which there is no obstruction by dimensions. In Theorem 3.2 we show that indeed these spaces admit a polar decomposition as above, and we determine precisely for which maximal split abelian subgroups A the decomposition is valid. For the simplest choice of group A we describe the indeterminateness of the A-component for a given element in G, and we identify the invariant measure on G/H in these coordinates.

Another important structural result for a Riemannian symmetric space G/K is the fact (closely related to Iwasawa decomposition) that minimal parabolic subgroups P act transitively. For non-Riemannian symmetric spaces there is no transitive action of P, but it is an important result, due to Wolf [7], that P has an orbit on G/H which is open. In Proposition 6.1 we verify that this is the case also for the spaces in (1.1), and we determine precisely for which minimal parabolic subgroups P the orbit through the origin is open.

By combining these results we conclude in Corollary 6.4 that there exist maximal split abelian subgroups A for which $G = KAH$ and for which PH is open for all minimal parabolic subgroups P with $P \supset A$, a property which plays an important role in [5].

An interesting observation (which surprised us) is that in some cases there are also maximal split abelian subgroups A for which PH is open for all minimal parabolic subgroups P with $P \subset A$, but for which the polar decomposition fails (see Remark 6.5).

The fact that the triple space of $\mathrm{SL}(2,\mathbb{C})$ admits open P-orbits follows from [4] p. 152. A homogeneous space of algebraic groups over $\mathbb{C}$ with an open Borel orbit is said to be spherical, cf [1], and the spaces we consider may be seen as prototypes of spherical spaces over $\mathbb{R}$.

In a final section we introduce an infinitesimal version of the polar decomposition, and show that in the current setting it is valid if and only if the global polar decomposition $G = KAH$ is valid.

The harmonic analysis on $\mathrm{SL}(2,\mathbb{R})$ is an essential example for understanding the harmonic analysis on general reductive groups. We expect the triple spaces considered here to serve similarly for the harmonic analysis on non-symmetric homogeneous spaces, which is yet to be developed.

2. Notation

Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{s}_0$ be a Cartan decomposition of the Lie algebra $\mathfrak{g}_0$ of G_0, and put

$$\mathfrak{k} = \mathfrak{k}_0 \times \mathfrak{k}_0 \times \mathfrak{k}_0, \quad \mathfrak{s} = \mathfrak{s}_0 \times \mathfrak{s}_0 \times \mathfrak{s}_0,$$

then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ is also a Cartan decomposition. The maximal abelian subspaces of $\mathfrak{s}$ have the form

$$\mathfrak{a} = \mathfrak{a}_1 \times \mathfrak{a}_2 \times \mathfrak{a}_3 \tag{2.1}$$

with three maximal abelian subspaces $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3$ in $\mathfrak{s}_0$.

If for each j we let $A_j = \exp \mathfrak{a}_j$ and choose a positive system for the roots of $\mathfrak{a}_j$, then with $G_0 = K_0 A_j N_j$ for $j = 1, 2, 3$ we obtain the Iwasawa decomposition $G = KAN$ where

$$K = K_0 \times K_0 \times K_0, \quad A = A_1 \times A_2 \times A_3, \quad N = N_1 \times N_2 \times N_3.$$

Likewise we obtain the minimal parabolic subgroup

$$P = P_1 \times P_2 \times P_3 = MAN$$

where $M = M_1 \times M_2 \times M_3$ and each $P_j = M_j A_j N_j$ is a minimal parabolic subgroup of G_0.

3. Polar decomposition

Let G/H be a homogeneous space of a reductive group G, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ be a Cartan decomposition of the Lie algebra of G. A decomposition of G of the form

$$G = KAH, \tag{3.1}$$

with $A = \exp \mathfrak{a}$, for an abelian subspace $\mathfrak{a} \subset \mathfrak{s}$, is said to be a *polar* decomposition. If such a decomposition exists, then the homogeneous space G/H is said to be of *polar type* (see [5]).

The fact that symmetric spaces are of polar type implies in particular that every double space $G/H = (G_0 \times G_0)/\operatorname{diag}(G_0)$ with G_0 a real reductive group admits a polar decomposition. Here we can take

$$\mathfrak{a} = \mathfrak{a}_0 \times \mathfrak{a}_0$$

for a maximal abelian subspace $\mathfrak{a}_0 \subset \mathfrak{s}_0$ (in fact, it would suffice to take already the antidiagonal of $\mathfrak{a}_0 \times \mathfrak{a}_0$). Then A has the form $A_1 \times A_2$ with $A_1 = A_2$. In contrast, triple spaces do not admit $G = KAH$ for $A = A_1 \times A_2 \times A_3$ if $A_1 = A_2 = A_3$:

Lemma 3.1. *Let G/H be the triple space of a non-compact semisimple Lie group G_0. Let $\mathfrak{a}_0 \subset \mathfrak{s}_0$ be maximal abelian and let $A = A_0 \times A_0 \times A_0$. Then KAH is a proper subset of G.*

Proof. Let $a_0 \in A_0$ be a regular element. We claim that a triple $(g_1, g_2, g_3) = (g_1, a_0, e)$ belongs to KAH only if $g_1 \in K_0A_0$. Assume $g_i = k_i a_i g$ for $i = 1, 2, 3$ with $k_i \in K_0$, $A_i \in A_0$ and $g \in G_0$. From

$$a_0 = g_2 g_3^{-1} = k_2 a_2 a_3^{-1} k_3^{-1}$$

we deduce that $k_2 = k_3$, and from the regularity of a_0 we then deduce that k_3 belongs to the normalizer $N_{K_0}(\mathfrak{a}_0)$ (see [3], Thm. 7.39). Then

$$g_1 = g_1 g_3^{-1} = k_1 a_1 a_3^{-1} k_3^{-1} \in K_0 A_0.$$

The lemma follows immediately. ◻

It was observed in [5] that the triple spaces for the groups considered in (1.1) are of polar type. In the following theorem we determine, for these groups, all the maximal abelian subspaces $\mathfrak{a}$ of $\mathfrak{g}$ for which (3.1) holds.

Theorem 3.2. *Let G_0 be one of groups* (1.1) *and $\mathfrak{a} \subset \mathfrak{s}$ as in* (2.1). *Then $G = KAH$ if and only if $\mathfrak{a}_1 + \mathfrak{a}_2 + \mathfrak{a}_3$ has dimension two in $\mathfrak{g}_0$.*
In particular, G/H is of polar type.

We shall approach $G = KAH$ by a geometric argument. Let $Z_0 = G_0/K_0$ be the Riemannian symmetric space associated with G_0, and let $z_0 = eK_0 \in Z_0$ denote its origin. Recall that (up to covering) G_0 is the identity component of the group of isometries of Z_0. Then it is easily seen that $G = KAH$ is equivalent to the following:

Property 3.3. *For every triple (z_1, z_2, z_3) of points $z_j \in Z_0$ there exist a triple (y_1, y_2, y_3) of points $y_j \in Z_0$ with $y_j \in A_j z_0$ for each j, and an isometry $g \in G_0$ such that $gz_j = y_j$ for $j = 1, 2, 3$.*

In order to illustrate the idea of proof, let us first state and prove a Euclidean analogue.

Proposition 3.4. *Let $\ell_1, \ell_2, \ell_3 \subset \mathbb{R}^n$ be lines through the origin O. The following statements are equivalent*

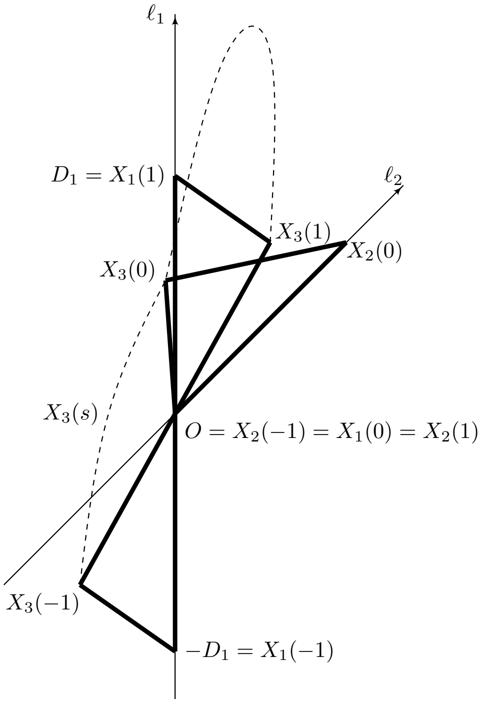
(1) $\dim(\ell_1 + \ell_2 + \ell_3) = 2$

(2) *For every triple of points $z_1, z_2, z_3 \in \mathbb{R}^n$ there exists a rigid motion g of $\mathbb{R}^n$ with $g(z_j) \in \ell_j$ for each $j = 1, 2, 3$.*

Proof. (1)⇒(2). Since the group of rigid motions is transitive on the 2-planes in $\mathbb{R}^n$, we may assume that z_1, z_2 and z_3 belong to the subspace spanned by the lines. This reduces the proof to the case $n = 2$.

We shall assume the z_j are distinct as otherwise the result is easily seen. Furthermore, as at most two of the lines are identical, let us assume that $\ell_1 \neq \ell_2$. Let d denote the distance between z_1 and z_2, and consider the set $\mathfrak{X}$ of pairs (X_1, X_2) of points $X_1 \in \ell_1$ and $X_2 \in \ell_2$ with distance d from each other. Let D_1 be a point on ℓ_1 with distance d to the origin, then (D_1, O) and $(-D_1, O)$ belong to $\mathfrak{X}$, and it follows from the geometry that we can connect these points by a continuous curve $s \mapsto (X_1(s), X_2(s))$ in $\mathfrak{X}$, say with $s \in [-1, 1]$. For example, we can arrange that first $X_1(s)$ moves from $-D_1$ to O along ℓ_1, while at the same time $X_2(s)$ moves along ℓ_2 at distance d from $X_1(s)$. Then $X_2(s)$ moves from O to a point $D_2 \in \ell_2$ at distance d from O. After that, $X_1(s)$ moves from O to D_1, while $X_2(s)$ moves back from D_2 to O.

When s passes through the interval $[-1, 1]$, the line segment from $X_1(s)$ to $X_2(s)$ slides with its endpoints on the two lines. We define $X_3(s)$ such that the three points form a triangle congruent to the one formed by z_1, z_2 and z_3. In other words, for each $s \in [-1, 1]$ there exists a unique rigid motion g_s of $\mathbb{R}^n$ for which $g_s(z_1) = X_1(s)$ and $g_s(z_2) = X_2(s)$. We let $X_3(s) = g_s(z_3)$. See the following figure.



As $X_1(s)$ and $X_2(s)$ depend continuously on s, then so does g_s (in the standard topology of the group of rigid motions) and hence also $X_3(s)$. Since $X_1(\pm 1)$ are opposite points while $X_2(\pm 1) = O$, the points $X_3(\pm 1)$ must be opposite as well. Since $s \mapsto X_3(s)$ is a continuous curve that connects two opposite points, it intersects with every line through O. Let $s \in [-1, 1]$ be a parameter value for which $X_3(s) \in \ell_3$. Now g_s is the desired rigid motion.

(2)⇒(1). Note that a rigid motion maps affine lines to affine lines. If $\dim(\ell_1 + \ell_2 + \ell_3) = 1$ then $\ell_1 = \ell_2 = \ell_3$, and it is clear that only triples of points which are positioned in a common affine line can be brought into it by a rigid motion. Hence $\dim(\ell_1 + \ell_2 + \ell_3) = 1$ is excluded.

Let z_1, z_2, z_3 be an arbitrary triple of distinct points located on a common affine line ℓ, and let g be a rigid motion which brings these points into the ℓ_j. Then O can be one of the points $g(z_j)$, or not. In the first case, say if $g(z_1) = O$, it follows that ℓ_2 and ℓ_3 are both equal to $g(\ell)$, since each of these lines have two points in common with $g(\ell)$.

Hence $\dim(\ell_1+\ell_2+\ell_3) \le 2$. In the second case, the line $g(\ell)$ together with O spans a 2-dimensional subspace of $\mathbb{R}^n$, which contains all the lines ℓ_j. Hence again $\dim(\ell_1+\ell_2+\ell_3) \le 2$. $\square$

We proceed with the proof of Theorem 3.2.

Proof. Note that $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{C})$ are locally isomorphic to $\mathrm{SO}_e(2,1)$ and $\mathrm{SO}_e(3,1)$, respectively. The centers of $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{C})$ belong to K, and hence $G = KAH$ will hold for the triple spaces of these groups if and only if it holds for the triple spaces of their adjoint groups. Thus it suffices to consider $G_0 = \mathrm{SO}(n,1)$ with $n \ge 2$.

The elements in $\mathfrak{so}(n,1)$ have the form

$$X = \begin{pmatrix} A & b \\ b^t & 0 \end{pmatrix} \tag{3.2}$$

where $A \in \mathfrak{so}(n)$ and $b \in \mathbb{R}^n$, and $\mathfrak{s}_0$ consists of the elements with $A = 0$.

Assume first that $\mathfrak{a}_1+\mathfrak{a}_2+\mathfrak{a}_3$ is 2-dimensional. By transitivity of the action of $K_0 = \mathrm{SO}(n)$ on the 2-dimensional subspaces of $\mathbb{R}^n$ we may assume that $\mathfrak{a}_1+\mathfrak{a}_2+\mathfrak{a}_3$ consists of the matrices X as above with $A = 0$ and b non-zero only in the last two coordinates. Hence $\mathfrak{a}_1+\mathfrak{a}_2+\mathfrak{a}_3$ is contained in the $\mathfrak{so}(2,1)$-subalgebra in the lower right corner of $\mathfrak{so}(n,1)$. It follows that $\exp(\mathfrak{a}_1+\mathfrak{a}_2+\mathfrak{a}_3).z_0$ is a 2-dimensional totally geodesic submanifold of Z_0.

Let $z_1, z_2, z_3 \in Z_0$ be given. Every triple of points in Z_0 belongs to a 2-dimensional totally geodesic submanifold Z_0' of Z_0. For example, in the model of Z_0 as a one-sheeted hyperboliod in $\mathbb{R}^{n+1}$, we can obtain Z_0' as the intersection of Z_0 with a 3-dimensional subspace of $\mathbb{R}^{n+1}$ containing the three points. Since G_0 is transitive on geodesic submanifolds, we may assume that z_1, z_2, z_3 are contained in the submanifold generated by $\mathfrak{a}_1+\mathfrak{a}_2+\mathfrak{a}_3$. We have thus essentially reduced to the case $n = 2$, and shall assume $n = 2$ from now on.

We proceed exactly as in the Euclidean case and produce a pair of points $X_1(s)$ and $X_2(s)$ on the geodesic lines $\exp(\mathfrak{a}_1).z_0$ and $\exp(\mathfrak{a}_2).z_0$, respectively. The two points are chosen so that they have the same non-Euclidean distance from each other as z_1 and z_2, and they depend continuously on $s \in [-1,1]$. Moreover, $X_1(-1)$ and $X_1(1)$ are symmetric with respect to z_0, while $X_2(-1) = X_2(1) = z_0$. As Z_0 is two-point homogeneous, there exists for each $s \in [-1,1]$ a unique isometry $g_s \in G_0$ such that $g_s(z_j) = X_j(s)$ for $j = 1,2$. As before, a value of s, where the continuous curve $s \mapsto g_s(z_3)$ intersects $\exp(\mathfrak{a}_3)$, produces the desired isometry g_s of Property 3.3. Hence $G = KAH$.

We return to the case $n \geq 2$ and assume conversely that $G = KAH$. It follows from Lemma 3.1 that $\dim(\mathfrak{a}_1 + \mathfrak{a}_2 + \mathfrak{a}_3) > 1$. We want to exclude $\dim(\mathfrak{a}_1 + \mathfrak{a}_2 + \mathfrak{a}_3) = 3$. Again we follow the Euclidean proof and select an arbitrary triple of distinct points z_1, z_2, z_3 on a single geodesic γ in Z_0. Then there is $g \in G_0$ such that $gz_j = y_j$ for some $y_j \in \exp(\mathfrak{a}_j).z_0$, for $j = 1, 2, 3$. If one of the y_j's, say y_1, is z_0, then $\exp(\mathfrak{a}_2).z_0 = \exp(\mathfrak{a}_3).z_0 = g(\gamma)$ and hence $\mathfrak{a}_2 = \mathfrak{a}_3$. Otherwise, the geodesic $g(\gamma)$ is contained, together with O, in a 2-dimensional totally geodesic submanifold of Z_0. This submanifold necessarily contains the geodesic $\exp(\mathfrak{a}_j).z_0$ for each j. Hence $\dim(\mathfrak{a}_1 + \mathfrak{a}_2 + \mathfrak{a}_3) \leq 2$. □

4. Uniqueness

If G/H is a homogeneous space of polar type, so that every element $g \in G$ allows a decomposition $g = kah$, it is of interest to know to which extend the components in this decomposition are unique. An obvious non-uniqueness is caused by the normalizer $N_{K\cap H}(\mathfrak{a})$ of $\mathfrak{a}$ in $K \cap H$, which acts on A by conjugation. In the case of a symmetric space, it is known (see [6], Prop. 7.1.3) that the A component of every $g \in G$ is unique up to such conjugation. For our current triple spaces the description of which elements in A generate the same $K \times H$ orbit appears to be more complicated, unless $\mathfrak{a}_1 = \mathfrak{a}_2 \perp \mathfrak{a}_3$.

Theorem 4.1. *Let G/H be the triple space with G_0 as in* (1.1)*, and let $\mathfrak{a}$ be as in* (2.1) *with $\mathfrak{a}_1 = \mathfrak{a}_2 \perp \mathfrak{a}_3$. Let $a = (a_1, a_2, a_3) \in A$ with $a_1 \neq a_2$ and let $a' = (a'_1, a'_2, a'_3) \in A$. Then $KaH = Ka'H$ if and only if a and a' are conjugate by $N_{K\cap H}(\mathfrak{a})$.*

We first determine explicitly which pairs of elements $a, a' \in A$ are $N_{K\cap H}(\mathfrak{a})$-conjugate when $\mathfrak{a}_1 = \mathfrak{a}_2 \perp \mathfrak{a}_3$.

Lemma 4.2. *Let $\mathfrak{a}$ be as above. Then $a, a' \in A$ are conjugate by $N_{K\cap H}(\mathfrak{a})$ if and only if*

(1) $(a'_1, a'_2) = (a_1, a_2)^{\pm 1}$ *and* $a'_3 = a_3^{\pm 1}$ *if* $n > 2$
(2) $(a'_1, a'_2, a'_3) = (a_1, a_2, a_3)^{\pm 1}$ *if* $n = 2$.

Proof. The normalizer $N_{K\cap H}(\mathfrak{a})$ consists of all the diagonal elements $k = (k_0, k_0, k_0) \in G$ for which

$$k_0 \in N_{K_0}(\mathfrak{a}_1) \cap N_{K_0}(\mathfrak{a}_2) \cap N_{K_0}(\mathfrak{a}_3).$$

As elements $a_j, a'_j \in A_j$ are $N_{K_0}(\mathfrak{a}_j)$-conjugate if and only if $a'_j = a_j^{\pm 1}$, only the pairs mentioned under (1) can be conjugate when $\mathfrak{a}_1 = \mathfrak{a}_2$.

Let $\delta, \epsilon = \pm 1$. For the groups in (1.1) the adjoint representation is surjective $K_0 \to \mathrm{SO}(\mathfrak{s}_0)$. If $n > 2$ then there exists a transformation in $\mathrm{SO}(\mathfrak{s}_0)$ which acts by δ on $\mathfrak{a}_1 = \mathfrak{a}_2$ and by ϵ on $\mathfrak{a}_3$. Its preimages in K_0

conjugate (a_1, a_2, a_3) to $(a_1^\delta, a_2^\delta, a_3^\epsilon)$. When $n = 2$ such a transformation exists if and only if $\delta = \epsilon$. The lemma follows. $\square$

The following lemmas are used in the proof of Theorem 4.1. Here G_0 can be any real reductive group with Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{s}_0$.

Lemma 4.3. *Let* $X, U \in \mathfrak{s}_0$. *Then* $\exp X \exp U \exp X \in \exp \mathfrak{s}_0$.

Proof. Let θ denote the Cartan involution and note that the product $\exp(tX) \exp(tU) \exp(tX)$ belongs to $S = \{g \in G_0 \mid \theta(g) = g^{-1}\}$ for all $t \in [0, 1]$. It is easily seen that $k \exp Y \in S$ implies $k^2 = e$ for $k \in K_0$ and $Y \in \mathfrak{s}_0$, and since e is isolated in the set of elements of order 2 it follows that $\exp \mathfrak{s}_0$ is the identity component of S. Hence $\exp X \exp U \exp X \in \exp \mathfrak{s}_0$. $\square$

Lemma 4.4. *Let* $\mathfrak{a}_0 \subset \mathfrak{s}_0$ *be a one-dimensional subspace and let* $A_0 = \exp \mathfrak{a}_0$.

(1) *If* $g \in \exp \mathfrak{s}_0$ *and* $ga_0 \in a_0' K_0$ *for some* $a_0, a_0' \in A_0$, *then* $g = a_0' a_0^{-1}$.
(2) *If* $g \in G_0$ *and* $ga_1, ga_2 \in A_0 K_0$ *for some* $a_1, a_2 \in A_0$ *with* $a_1 \neq a_2$ *then* $g \in N_{K_0}(\mathfrak{a}_0) A_0$.

Proof. (1) It follows from $ga_0 \in a_0' K_0$ that $a_0 g a_0 \in a_0 a_0' K_0$. Since $a_0 g a_0 \in \exp \mathfrak{s}_0$ by Lemma 4.3, it follows from uniqueness of the Cartan decomposition that $a_0 g a_0 = a_0 a_0'$ and thus $g = a_0' a_0^{-1}$.

(2) Put $z_0 = eK_0$, then $A_0.z_0$ is a geodesic in G_0/K_0. Since g maps two distinct points on $A_0.z_0$ into $A_0.z_0$, it maps the entire geodesic onto itself, and hence so does g^{-1}. In particular $g^{-1}.z_0 \in A_0 K_0$, that is, $g = k_0 a_0$ for some $k_0 \in K_0$, $a_0 \in A_0$. It follows for all $a \in A_0$ that

$$k_0 a k_0^{-1} = g a_0^{-1} a k_0^{-1} \in g A_0 K_0 = A_0 K_0.$$

As $k_0 a k_0^{-1} \in \exp \mathfrak{s}_0$, uniqueness of the Cartan decomposition implies $k_0 a k_0^{-1} \in A_0$, i.e. $k_0 \in N_{K_0}(\mathfrak{a}_0)$. $\square$

Lemma 4.5. *Let* $\mathfrak{a}_1, \mathfrak{a}_3 \subset \mathfrak{s}_0$ *be one-dimensional subspaces with* $\mathfrak{a}_1 \perp \mathfrak{a}_3$ *and let* $A_1 = \exp \mathfrak{a}_1$, $A_3 = \exp \mathfrak{a}_3$. *If* $g \in N_{K_0}(\mathfrak{a}_1) A_1$ *and* $ga_3 \in a_3' K_0$ *for some* $a_3, a_3' \in A_3$, *not both equal to* e, *then* $g \in N_{K_0}(\mathfrak{a}_1) \cap N_{K_0}(\mathfrak{a}_3)$.

Proof. We may assume $a_3' \neq e$, as otherwise we interchange it with a_3 and replace g by g^{-1}. We consider the geodesic triangle in G_0/K_0 formed by the geodesics

$$L_1 := A_1.z_0, \quad L_2 := A_3.z_0, \quad L_3 := gA_3.z_0.$$

The vertices are

$$D_3 := z_0, \quad D_2 := g.z_0, \quad D_1 := ga_3.z_0 = a_3'.z_0.$$

As L_1 and L_2 intersect orthogonally, angle D_3 is right. The isometry g maps L_1 to itself and L_2 to L_3. Hence L_1 and L_3 also intersect orthogonally and angle D_2 is right. As the sectional curvature of G_0/K_0 is ≤ 0, it is impossible for a proper triangle to have two right angles. As $L_1 \neq L_2$ and $D_3 \neq D_1$ we conclude $D_3 = D_2$ and $L_3 = L_2$. It follows that $g \in K_0$ and by Lemma 4.4 (2) that $g \in N_{K_0}(\mathfrak{a}_3)$. $\square$

Proof of Theorem 4.1. Assume $KaH = Ka'H$. Then $Kah = Ka'$ for some $h = (g, g, g) \in H$. Applying Lemma 4.4 (2) to the first two coordinates of $Kah = Ka'$ we conclude that $g \in N_{K_0}(\mathfrak{a}_1)A_1$.

If a_3' and a_3 are not both e, we can apply Lemma 4.5 to the last coordinate and conclude $g \in N_{K_0}(\mathfrak{a}_1) \cap N_{K_0}(\mathfrak{a}_3)$. Hence $h \in N_{K\cap H}(\mathfrak{a})$, and we conclude that $a' = h^{-1}ah$.

If $a_3' = a_3 = e$ it follows from the third coordinate that $g \in K_0$. Hence $g \in N_{K_0}(\mathfrak{a}_1)$ and $a' = a$ or $a' = a^{-1}$. $\square$

Remark 4.6. When $\dim \mathfrak{s}_0 = 2$ the assumption in Theorem 4.1 and Lemmas 4.2, 4.5, that $\mathfrak{a}_1 = \mathfrak{a}_2 \perp \mathfrak{a}_3$, can be relaxed to $\mathfrak{a}_1 = \mathfrak{a}_2 \neq \mathfrak{a}_3$ with unchanged conclusions. This follows from the fact that in a two dimensional space the only proper orthogonal transformations which normalize a one-dimensional subspace are $\pm I$. Hence $N_{K_0}(\mathfrak{a}_1) = N_{K_0}(\mathfrak{a}_3)$ in this case.

5. A formula for the invariant measure

In a situation where there is uniqueness (up to some well-described isomorphism), it is of interest to explicitly determine the invariant measure with respect to the KAH-coordinates.

For any triple space G/H of a unimodular Lie group G_0 we note that the map

$$G_0 \times G_0 \to G/H, \quad (g_1, g_2) \mapsto (g_1, g_2, e)H \tag{5.1}$$

is a $G_0 \times G_0$-equivariant diffeomorphism. Accordingly the invariant measure on G/H identifies with the Haar measure on $G_0 \times G_0$.

For $G_0 = \mathrm{SO}_e(n, 1)$ we define $X \in \mathfrak{so}(n, 1)$ by (3.2) with $A = 0$ and $b = e_n$, and $Y \in \mathfrak{so}(n, 1)$ similarly with $A = 0$ and $b = e_1$. Let $\mathfrak{a}_1 = \mathfrak{a}_2 = \mathbb{R}X$ and $\mathfrak{a}_3 = \mathbb{R}Y$, then $\mathfrak{a}_3 \perp \mathfrak{a}_1$. Let

$$a_t = \exp(tX) \in A_1 = A_2, \quad b_s = \exp(sY) \in A_3.$$

Lemma 5.1. *Let G/H be the triple space of $G_0 = \mathrm{SO}_e(n, 1)$ and let $\mathfrak{a}_1 = \mathfrak{a}_2$ and $\mathfrak{a}_3$ be as above. Consider the polar coordinates*

$$K \times \mathbb{R}^3 \ni (k, t_1, t_2, s) \mapsto (k_1 a_{t_1}, k_2 a_{t_2}, k_3 b_s)H \tag{5.2}$$

on G/H. The invariant measure dz of G/H can be normalized so that in these coordinates

$$dz = J(t_1, t_2, s)\, dk\, dt_1\, dt_2\, ds \tag{5.3}$$

where dk is Haar measure, dt_1, dt_2, ds Lebesgue measure, and where

$$J(t_1, t_2, s) = |\sinh^{n-1}(t_1 - t_2) \sinh^{n-2}(s) \cosh(s)|.$$

Proof. On $G_0 \times G_0$ we use the formula (see [6], Thm. 8.1.1) for integration in KAH coordinates for the symmetric space $G_0 \times G_0/\operatorname{diag}(G_0) = G_0$. The map

$$(K_0 \times K_0) \times A_0 \times G_0 \to G_0 \times G_0$$

defined by

$$(k, a_t, g) \mapsto (k_1 a_{t/2} g, k_2 a_{-t/2} g)$$

is a parametrization (up to the sign of t), and the Haar measure on $G_0 \times G_0$ writes as

$$|\sinh^{n-1}(t)|\, dk_1\, dk_2\, dt\, dg\,. \tag{5.4}$$

Further we decompose the diagonal copy of G_0 by means of the HAK coordinates for the symmetric space $G_0/(\mathrm{SO}(n-1) \times A_1)$, where $\mathrm{SO}(n-1)$ is located in the upper left corner of G_0. Note that the subgroup A_3 serves as the 'A' in this decomposition. In the coordinates

$$K_0 \times A_3 \times \mathrm{SO}(n-1) \times A_1 \to G_0, \quad (k_3, b_s, m, a_u) \mapsto a_u m b_s k_3$$

we obtain (again using [6], Thm. 8.1.1),

$$dg = |\sinh^{n-2}(s) \cosh(s)|\, dk_3\, db_s\, dm\, du\,. \tag{5.5}$$

Combining (5.4) and (5.5), we have the coordinates

$$(k_1 a_{u+t/2} m b_s k_3, k_2 a_{u-t/2} m b_s k_3)$$

on $G_0 \times G_0$, with Jacobian $|\sinh^{n-1}(t) \sinh^{n-2}(s) \cosh(s)|$. As the subgroup $\mathrm{SO}(n-1)$ centralizes A_1, the integration over m is swallowed by the integrations over k_1 and k_2. Changing coordinates u, t to $t_1 = u + t/2$ and $t_2 = u - t/2$ we find $t = t_1 - t_2$.

Finally we apply (5.1) so that the above coordinates correspond to

$$(k_1, k_2, k_3)(a_{t_1}, a_{t_2}, b_{-s}) \operatorname{diag}(G_0)$$

This proves (5.3). □

6. Spherical decomposition

A decomposition of $\mathfrak{g}$ of the form

$$\mathfrak{g} = \mathfrak{p} + \mathfrak{h} \tag{6.1}$$

with $\mathfrak{p}$ a minimal parabolic subalgebra is said to be a *spherical* decomposition. If such a decomposition exists, then the homogeneous space G/H is said to be of *spherical type* (see [5]).

Note that with $\mathfrak{g}_0 = \mathfrak{so}(n,1)$ we have (see (6.4) and (6.5))

$$\dim \mathfrak{p} + \dim \mathfrak{h} - \dim \mathfrak{g} = \tfrac{1}{2}(n^2 - 5n + 6) \geq 0.$$

In particular spherical decompositions will be direct sums if $n = 2, 3$.

It was observed in [5] that the triple spaces for the groups considered in (1.1) are of spherical type. In the following we determine for each n all the minimal parabolic subalgebras $\mathfrak{p}$ for which (6.1) holds.

Proposition 6.1. *Let G_0 be one of the groups* (1.1) *and let $\mathfrak{p} = \mathfrak{p}_1 \times \mathfrak{p}_2 \times \mathfrak{p}_3$ a minimal parabolic subalgebra. Then $\mathfrak{g} = \mathfrak{p} + \mathfrak{h}$ holds if and only if $\mathfrak{p}_1$, $\mathfrak{p}_2$ and $\mathfrak{p}_3$ are distinct.*

In particular, the triple space G/H is of spherical type for all groups G_0 in (1.1).

We prepare by the following lemma.

Lemma 6.2. *Let $U_1, U_2, U_3 \subset V$ be subspaces of a vector space V. Put*

$$U := U_1 \times U_2 \times U_3 \subset X := V \times V \times V,$$

and $Y := \operatorname{diag}(V) \subset X$. Then $X = U + Y$ if and only if

$$V = U_1 + (U_2 \cap U_3) = U_2 + (U_3 \cap U_1) = U_3 + (U_1 \cap U_2). \tag{6.2}$$

Proof. Assume first that $X = U + Y$ and let $v \in V$ be given. Writing

$$(v, 0, 0) = (u_1, u_2, u_3) + \operatorname{diag}(w)$$

we see that $w = -u_2 = -u_3 \in U_2 \cap U_3$, and hence $v = u_1 + w \in U_1 + (U_2 \cap U_3)$. The other two statements in (6.2) follow similarly.

Conversely, we assume (6.2) and let $x = (x_1, x_2, x_3) \in X$ be given. We decompose x_1, x_2 and x_3 according to the three decompositions in (6.2), that is,

$$\begin{aligned} x_1 &= u_1 + t_1, \qquad && u_1 \in U_1,\ t_1 \in U_2 \cap U_3 \\ x_2 &= u_2 + t_2, && u_2 \in U_2,\ t_2 \in U_3 \cap U_1 \\ x_3 &= u_3 + t_3, && u_3 \in U_3,\ t_3 \in U_1 \cap U_2. \end{aligned}$$

Then

$$x = (u_1 - t_2 - t_3, u_2 - t_1 - t_3, u_3 - t_1 - t_2) + \operatorname{diag}(t_1 + t_2 + t_3)$$

is a decomposition of the desired form $U + Y$. $\square$

Remark 6.3. In fact, it is easily seen that any two of the decompositions of V in (6.2) together imply the third.

Proof of Proposition 6.1. It suffices to consider $G_0 = \mathrm{SO}_0(n,1)$ because of the local isomorphisms.

If for example $\mathfrak{p}_1 = \mathfrak{p}_2$ then $\mathfrak{p}_1 + (\mathfrak{p}_2 \cap \mathfrak{p}_3) = \mathfrak{p}_1$. Hence $\mathfrak{p}_1 + (\mathfrak{p}_2 \cap \mathfrak{p}_3)$ is proper in $\mathfrak{g}_0$ and it follows from Lemma 6.2 that $\mathfrak{g} = \mathfrak{p} + \mathfrak{h}$ fails to hold. This implies one direction of the first statement.

For the other direction it follows from Lemma 6.2 that it suffices to prove

$$\mathfrak{g}_0 = \mathfrak{p}_1 + (\mathfrak{p}_2 \cap \mathfrak{p}_3)$$

for all triples of distinct parabolics in $\mathfrak{so}(n,1)$. We shall do this by proving

$$\dim \mathfrak{g}_0 = \dim \mathfrak{p}_1 + \dim(\mathfrak{p}_2 \cap \mathfrak{p}_3) - \dim(\mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{p}_3). \tag{6.3}$$

We find

$$\dim \mathfrak{g}_0 = \dim \mathfrak{so}(n,1) = \tfrac{1}{2}(n^2+n), \tag{6.4}$$

and claim that

$$\dim \mathfrak{p}_1 = \tfrac{1}{2}(n^2 - n + 2) \tag{6.5}$$

$$\dim(\mathfrak{p}_1 \cap \mathfrak{p}_2) = \tfrac{1}{2}(n^2 - 3n + 4) \tag{6.6}$$

$$\dim(\mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{p}_3) = \tfrac{1}{2}(n^2 - 5n + 6). \tag{6.7}$$

The equations (6.4)-(6.7) imply (6.3).

The parabolic subalgebras $\mathfrak{p}$ of $\mathfrak{so}(n,1)$ are the normalizers of the isotropic lines in $\mathbb{R}^{n+1}$, that is, the one-dimensional subspaces of the form $L_q = \mathbb{R}(q,1)$ where $q \in \mathbb{R}^n$ with $\|q\| = 1$.

Recall that all elements in $\mathfrak{so}(n,1)$ have the form (3.2) with $A \in \mathfrak{so}(n)$ and $b \in \mathbb{R}^n$. It follows that $X \in \mathfrak{p}$ if and only if

$$Aq + b = (b \cdot q)q. \tag{6.8}$$

Let us prove (6.5). Let q_1 be the unit vector such that $\mathfrak{p}_1$ is the stabilizer of L_{q_1}, and extend q_1 to a basis $q_1, \dots, q_n$ for $\mathbb{R}^n$. For $b \in \mathbb{R}^n$ we let $x_1 = (b \cdot q_1)q_1 - b$ and we observe that $x_1 \cdot q_1 = 0$. According to (6.8) the matrix X of (3.2) belongs to $\mathfrak{p}_1$ if and only if $Aq_1 = x_1$. In order to satisfy that we can define an $n \times n$ matrix A by

$$Aq_i \cdot q_j := \begin{cases} x_1 \cdot q_j & \text{for } i = 1 \\ -x_1 \cdot q_i & \text{for } j = 1 \\ a_{ij} & \text{for } i,j > 1 \end{cases} \tag{6.9}$$

with arbitrary antisymmetric entries in the last line. Then $A \in \mathfrak{so}(n)$ and $Aq_1 = x_1$. The degree of freedom for each b is

$$\dim \mathfrak{so}(n-1) = \tfrac{1}{2}(n-1)(n-2),$$

and hence $\dim \mathfrak{p}_1 = n + \frac{1}{2}(n-1)(n-2) = \frac{1}{2}(n^2 - n + 2)$ as asserted.

Next we prove (6.6). Let q_1, q_2 be the unit vectors such that $\mathfrak{p}_i$ is the stabilizer of L_{q_i}. By assumption $q_1 \neq q_2$. For the element X of (3.2) to be in $\mathfrak{p}_1 \cap \mathfrak{p}_2$ we need that (6.8) is satisfied in both cases, that is,

$$Aq_i = x_i, \quad (i = 1, 2). \tag{6.10}$$

where $x_i = (b \cdot q_i)q_i - b$. Now

$$x_2 \cdot q_1 + x_1 \cdot q_2 = (q_1 \cdot q_2 - 1)(b \cdot (q_1 + q_2)).$$

Note that $q_1 \cdot q_2 < 1$ since $q_1 \neq q_2$. As $A \in \mathfrak{so}(n)$ we conclude that

$$b \cdot (q_1 + q_2) = 0$$

since otherwise (6.10) would lead to contradiction.

Conversely, let $b \in \mathbb{R}^n$ be such that $b \cdot (q_1 + q_2) = 0$ and define x_1, x_2 by $x_i = (b \cdot q_i)q_i - b$. Then $x_i \cdot q_j = -x_j \cdot q_i$ for all pairs $i, j \leq 1, 2$. We extend q_1, q_2 to a basis and define an $n \times n$ matrix A by

$$Aq_i \cdot q_j = \begin{cases} x_i \cdot q_j & \text{for } i = 1, 2 \\ -x_j \cdot q_i & \text{for } j = 1, 2 \\ a_{ij} & \text{for } i, j > 2 \end{cases} \tag{6.11}$$

with arbitrary antisymmetric entries in the last line. Then $A \in \mathfrak{so}(n)$ and (6.10) holds. The degree of freedom for each b is

$$\dim \mathfrak{so}(n-2) = \tfrac{1}{2}(n-2)(n-3)$$

and hence $\dim(\mathfrak{p}_1 \cap \mathfrak{p}_2) = n - 1 + \frac{1}{2}(n-2)(n-3) = \frac{1}{2}(n^2 - 3n + 4)$ as asserted.

Finally, to prove (6.7) assume that X in (3.2) belongs to $\mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{p}_3$. As above, it follows that

$$b \cdot (q_1 + q_2) = b \cdot (q_1 + q_3) = b \cdot (q_2 + q_3) = 0$$

which implies that $b \cdot q_i = 0$ for $i = 1, 2, 3$. If this is satisfied by b, the condition (6.8) simplifies to

$$Aq_i = -b, \qquad i = 1, 2, 3. \tag{6.12}$$

We first assume that q_1, q_2, q_3 are linearly independent and extend to a basis as before. Given $b \in \mathbb{R}^n$ such that $b \cdot q_i = 0$ for $i = 1, 2, 3$ we define A by

$$Aq_i \cdot q_j = \begin{cases} -b \cdot q_j & \text{for } i = 1,2,3 \\ b \cdot q_i & \text{for } j = 1,2,3 \\ a_{ij} & \text{for } i,j > 3 \end{cases} \tag{6.13}$$

with arbitrary antisymmetric entries in the last line. The degree of freedom for each b is

$$\dim \mathfrak{so}(n-3) = \tfrac{1}{2}(n-3)(n-4)$$

and hence $\dim(\mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{p}_3) = n - 3 + \frac{1}{2}(n-3)(n-4) = \frac{1}{2}(n^2 - 5n + 6)$ as asserted.

Next we assume linear dependence of q_1, q_2, q_3. This implies a further obstruction to b. In fact, let $\lambda_1 q_1 + \lambda_2 q_2 + \lambda_3 q_3 = 0$ be a non-trivial relation, then it follows from (6.12) that $(\lambda_1 + \lambda_2 + \lambda_3)b = 0$. Since q_1, q_2, q_3 are assumed to be distinct unit vectors the sum of the λ's cannot be zero, and we conclude that $b = 0$. Thus in this case the only freedom comes from the choice of A. That can be chosen arbitrarily from the annihilator in $\mathfrak{so}(n)$ of the space spanned by the three q's. We obtain $\dim(\mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{p}_3) = \dim \mathfrak{so}(n-2) = \frac{1}{2}(n^2 - 5n + 6)$ as before. This concludes the proof of (6.7).

In particular, if $\mathfrak{a}_1$, $\mathfrak{a}_2$ and $\mathfrak{a}_3$ are all different, then $\mathfrak{g} = \mathfrak{p} + \mathfrak{h}$ for every parabolic subalgebra $\mathfrak{p}$ above $\mathfrak{a} = \mathfrak{a}_1 \times \mathfrak{a}_2 \times \mathfrak{a}_3$. Hence G/H is of spherical type. □

Corollary 6.4. *There exists a maximal abelian subspace* $\mathfrak{a} \subset \mathfrak{s}$ *for which both*

(i) *the polar decomposition* (3.1) *is valid, and*
(ii) *the spherical decomposition* (6.1) *is valid for all minimal parabolic subalgebras containing* $\mathfrak{a}$.

Proof. Let $\mathfrak{a}_j \subset \mathfrak{s}_0$ for $j = 1,2,3$ be mutually different and with a two-dimensional sum. It follows from Theorem 3.2 and Proposition 6.1 that $\mathfrak{a} = \mathfrak{a}_1 \times \mathfrak{a}_2 \times \mathfrak{a}_3$ satisfies (i) and (ii). □

Remark 6.5. The properties of a reductive homogeneous space G/H that it is of polar type, respectively of spherical type, appear to be closely related. However, the relation is not as strong as one might hope, because the conditions on $\mathfrak{a}$ are different in Theorem 3.2 and Proposition 6.1. In particular, there exist maximal abelian subspaces $\mathfrak{a} \subset \mathfrak{g}$ which fulfill (ii) but not (i), namely the 'most generic' ones, for which $\dim(\mathfrak{a}_1 + \mathfrak{a}_2 + \mathfrak{a}_3) = 3$.

7. Infinitesimal polar decomposition

Here we consider an infinitesimal version of the polar decomposition $G = KAH$. Let G/H be a homogeneous space of a reductive group G, and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ be a Cartan decomposition.

Definition 7.1. *A decomposition of the form*

$$\mathfrak{s} = \mathrm{Ad}(K \cap H)\mathfrak{a} + \mathfrak{s} \cap \mathfrak{h} \tag{7.1}$$

with an abelian subspace $\mathfrak{a} \subset \mathfrak{s}$ is called a polar decomposition.

If there exists such a decomposition of $\mathfrak{s}$ then we say that G/H is infinitesimally polar.

Here

$$\mathrm{Ad}(K \cap H)\mathfrak{a} = \{\mathrm{Ad}(k)X \mid k \in K \cap H, X \in \mathfrak{a}\}.$$

Note that this need not be a vector subspace of $\mathfrak{s}$.

If G/H is a symmetric space, then we can choose the Cartan decomposition so that $\mathfrak{k}$ and $\mathfrak{s}$ are stable under the involution σ that determines G/H. If $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ denotes the decomposition of $\mathfrak{g}$ in $+1$ and -1 eigenspaces for σ, then $\mathfrak{s} = \mathfrak{s} \cap \mathfrak{q} + \mathfrak{s} \cap \mathfrak{h}$. If furthermore $\mathfrak{a}_q$ is a maximal abelian subspace of $\mathfrak{s} \cap \mathfrak{q}$, then it is known that $\mathfrak{s} \cap \mathfrak{q} = \mathrm{Ad}(K \cap H)\mathfrak{a}_q$ and hence (7.1) follows.

The following lemma suggests that there is a close connection between polar decomposability and infinitesimally polar decomposability.

Lemma 7.2. *Let G_0 be one of groups* (1.1) *and let $\mathfrak{a} = \mathfrak{a}_1 \times \mathfrak{a}_2 \times \mathfrak{a}_3$. Then the infinitesimal polar decomposition* (7.1) *holds if and only if $\dim(\mathfrak{a}_1 + \mathfrak{a}_2 + \mathfrak{a}_3) = 2$.*

Proof. For the triple spaces, the polar decomposition (7.1) interprets to the statement that for every triple of points $Z_1, Z_2, Z_3 \in \mathfrak{s}_0$ there exist $k \in K_0$, $T \in \mathfrak{s}_0$ and $X_j \in \mathfrak{a}_j$ $(j = 1, 2, 3)$ such that $Z_j = \mathrm{Ad}(k)X_j + T$. As the maps $X \mapsto \mathrm{Ad}(k)X + T$ with $k \in K_0$ and $T \in \mathfrak{s}_0$ are exactly the rigid motions of $\mathfrak{s}_0$, this lemma is precisely the content of Proposition 3.4. □

Combining the lemma with Theorem 3.2 we see that for our triple spaces the infinitesimal polar decomposition holds with a given $\mathfrak{a}$ if and only if the global polar decomposition $G = KAH$ holds for the corresponding $A = \exp \mathfrak{a}$.

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