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Dissertation

Contributions to the structure theory of non-simple C^*-algebras

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Abstract

This thesis is mainly concerned with classification results for non-simple purely infinite C^*-algebras, specifically Cuntz–Krieger algebras and graph C^*-algebras, and continuous fields of Kirchberg algebras. In Article A, we perform some computations concerning projective dimension in filtrated K-theory. In joint work with Sara Arklint and Takeshi Katsura, we provide a range result complementing Gunnar Restorff's classification theorem for Cuntz–Krieger algebras (Article B) and we investigate reduction of filtrated K-theory for C^*-algebras of real rank zero, thereby obtaining a characterization of Cuntz–Krieger algebras with primitive ideal space of accordion type (Article C). In Article D, we establish a universal coefficient theorem computing Eberhard Kirchberg's ideal-related KK-groups over a finite space for algebras with vanishing boundary maps. This result is used to classify certain continuous fields of Kirchberg algebras in Article F. A stronger result for one-parameter continuous fields is obtained in joint work with Marius Dadarlat (Article E). In Article G, we compute Stefan Schwede's n-order for certain triangulated categories of C^*-algebras.

The following is a Danish translation of the abstract as required by the rules of the University of Copenhagen.

Resumé

Denne afhandling omhandler klassifikationsresultater for ikke-simple, rent uendelige C^*-algebraer, specielt Cuntz–Kriegeralgebraer og graf C^*-algebraer, samt kontinuerte felter af Kirchbergalgebraer. I Artikel A udfører vi diverse udregninger, som omhandler projektiv dimension i filtreret K-teori. I samarbejde med Sara Arklint og Takeshi Katsura opnår vi et billederesultat som komplimenterer Gunnar Restorffs klassifikationssætning for Cuntz–Kriegeralgebraer (Artikel B), og vi udforsker reduktion af filtreret K-teori for C^*-algebraer af reel rang nul, og dermed får vi en karakterisering af Cuntz–Kriegeralgebraer, hvis primitive idealrum er et harmonikarum (Artikel C). I Artikel D etablerer vi en universel koefficientsætning, som udregner Eberhard Kirchbergs idealrelaterede KK-grupper over endelige rum for algebraer, hvis randafbildninger forsvinder. Dette resultat benyttes til at klassificere visse kontinuerte felter af Kirchbergalgebraer i Artikel F. Et stærkere resultat for én-parameter kontinuerte felter opnås i samarbejde med Marius Dadarlat (Artikel E). I Artikel G udregner vi Stefan Schwedes n-orden for visse triangulerede kategorier af C^*-algebraer.

Preface

The present thesis contains research material obtained during the three-year period of my PhD studies at the University of Copenhagen from October 2010 to September 2013. The results are presented in cumulative form as appendices. In this section, I will provide some context concerning this work, along the way mentioning also ongoing projects on which I spent time during my PhD studies. The content of the appended articles will be described in more detail in the summary.

I started doing mathematical research in 2009, when I began working towards my Diplom thesis under supervision of Ralf Meyer at the University of Göttingen. The goal was to elaborate on Meyer's joint work with Ryszard Nest to decide, given a finite T_0-space X, whether there exists a feasible invariant for C^*-algebras over X computing Kirchberg's ideal-related Kasparov theory groups under appropriate bootstrap assumptions in terms of a short exact universal coefficient sequence. This question is particularly interesting in the light of Eberhard Kirchberg's classification theorem for non-simple strongly purely infinite separable nuclear C^*-algebras. In fact, it is no exaggeration to say that Kirchberg's result is the main motivation for almost all of my work to date.

My Diplom thesis resulted in a joint paper with Manuel Köhler titled *Universal coefficient theorems for C^*-algebras over finite topological spaces* in which we prove that the invariant called *filtrated* K*-theory* serves the desired purpose if and only if the space X is a disjoint union of so-called *accordion spaces.* The proof of the main result in that paper is rather ad hoc; as I shall describe below, I plan to provide a more conceptual proof using more sophisticated methods in the future.

During a masterclass in Copenhagen, Köhler told Søren Eilers about our joint work. This lead Eilers to invite us to give a presentation in the operator algebra seminar in Copenhagen; he also encouraged me to come to Copenhagen for PhD studies under joint supervision with Ryszard Nest. The primary goal was to use the machinery of homological algebra in triangulated categories to obtain further classification results for non-simple purely infinite C^*-algebras, possibly by exploiting specific properties (K-theoretic or otherwise) of graph C^*-algebras. A strong motivation for this endeavor was Gunnar Restorff's classification theorem for Cuntz–Krieger algebras of real rank zero. Some computations in this direction, mainly negative in outcome, were performed in the paper *Projective dimension in filtrated* K*-theory.*

Upon arrival in Copenhagen, I started working with Sara Arklint and Takeshi Katsura. We wrote a joint paper, which was later split into two parts. In *The* K*-theoretical range of Cuntz–Krieger algebras* we provide a range-of-invariant result for Restorff's classification. In *Reduction of filtered* K*-theory and a characterization of Cuntz–Krieger algebras*, we investigate how filtrated K-theory can be simplified without losing information if the input C^*-algebra is assumed to have real rank zero; this shows in particular that there exists no phantom Cuntz–Krieger algebra whose primitive ideal space is a disjoint union of accordion spaces (or, for that matter, any four-point spaces except the pseudocircle).

Following a suggestion of Søren Eilers, I investigated the classification problem for C^*-algebras over a finite space X with *vanishing boundary maps.* In the paper *Kirchberg X-algebras with real rank zero and intermediate cancellation*, I establish a universal coefficient theorem in this context and, building on the range result with Arklint and Katsura, show that no phantom Cuntz–Krieger algebra with vanishing boundary maps exists. The homology theory XK used for the universal coefficient theorem in this setting takes values in $\mathbb{Z}/2$-graded integral representations of the poset X (here X is considered as a poset via the specialization preorder); it comprises the $\mathbb{Z}/2$-graded K-theory groups of certain distinguished ideals in the algebra together with the group homomorphisms induced by all ideal inclusions among them.

I spent most of the year 2012 abroad, visiting Marius Dadarlat at Purdue University and Ralf Meyer at the University of Göttingen. While I stayed in Göttingen, my son Oskar was born; he and his mother would later accompany me for my remaining study time in Copenhagen.

When I arrived at Purdue, Marius Dadarlat suggested that the universal coefficient theorem for C^*-algebras over accordion spaces might be useful to study ideal-related KK-theory over the unit interval. His intuition was correct and resulted in a classification theorem discussed in our joint paper *One-parameter continuous fields of Kirchberg algebras with rational* K*-theory.*

A similar approach is used in my paper *Classification of certain continuous fields of Kirchberg algebras*, where the unit interval is replaced by an arbitrary finite-dimensional compact metrizable topological space—at the expense of the strong additional assumption of vanishing boundary maps. For fields with finitely generated K-theory, I provide a range result in this context.

During my stay in Göttingen, Ralf Meyer made me aware of the existence of module spectra representing (equivariant) K-theory groups of C^*-algebras and suggested that it could be useful to achieve something similar for C^*-algebras over finite spaces using poset diagrams of module spectra. This idea indeed turned out to provide a highly useful tool for analysing the structure of the bootstrap class in ideal-related KK-theory.

Specifically, I intend to use this approach to establish generalized Bernstein–Gelfand–Ponomarev reflection functors identifying the bootstrap categories in ideal-related KK-theory for certain pairs of finite spaces. This will imply that the universal coefficient theorem over accordion spaces reduces to the totally ordered case already considered by Meyer and Nest; it will also greatly reduce the amount of computation necessary to establish universal coefficient theorems over certain non-accordion spaces. (Generalizing an example of Meyer–Nest, I found universal coefficient theorems over *some* (four-point) non-accordion spaces in my Diplom thesis. Due to relations with quiver and poset representations, we expect that there are universal coefficient theorems involving "invariants of finite type" for many more finite spaces, including all spaces whose Hasse diagram is a simply laced Dynkin diagram.)

In *Homotopy-theoretic* E*-theory and n-order*, I use the above-mentioned spectral models to compute Stefan Schwede's n-order for certain triangulated categories arising from C^*-algebras. In the ongoing project *Algebraic models in rational equivariant* KK*-theory*, I use results of Brooke Shipley and intrinsic formality to establish explicit algebraic models for rationalizations of some such triangulated categories. As a consequence of this, for an arbitrary finite T_0-space X, the *-isomorphism types over X of stable Kirchberg X-algebras whose simple subquotients have rational K-theory groups and satisfy the universal coefficient theorem are in natural bijection with the 2-periodic quasi-isomorphism types of 2-periodic complexes of countable rational poset representations of X.

In order to establish universal coefficient theorems, one needs length-1 projective resolutions. But in many interesting situations one only has resolutions of length 2. Luckily, something still can be done under this assumption. This has been known to topologists since the work of Aldridge K. Bousfield in the eighties; it involves refining the given homology theory by remembering a canonical obstruction class in a certain Ext^2-group. In the ongoing project *Circle actions on C^*-algebras up to* KK*-equivalence and some other cases* with Ralf Meyer, we translate Bousfield's approach to the general setup of triangulated categories and homological ideals and establish classification results in circle-equivariant KK-theory and in KK-theory over so-called *unique path spaces*. In particular, we show that the gauge actions on two Cuntz–Krieger algebras are KK-equivalent if and only if their defining 0-1-matrices are shift equivalent over the integers. We also determine when gauge actions on Nekrashevych's C^*-algebras associated with certain hyperbolic rational functions are KK-equivalent; the resulting classification turns out to be very coarse when compared to conjugacy of the involved dynamical systems. Using the previously mentioned reflection functors, the result for unique path spaces may be extended to cover all spaces with five points or less. This leaves the six-point pseudosphere as a minimal unsolved case.

In July 2013, Sara Arklint and I organized a masterclass on classification of non-simple purely infinite C^*-algebras in Copenhagen. The main speakers were Marius Dadarlat and Ralf Meyer. On this occasion, Ralf and I realized that $\mathrm{XK}\big(C^*(E)\big)$ has a length-2 projective resolution for every graph C^*-algebra $C^*(E)$ over an *arbitrary* finite T_0-space X such that all distinguished ideals are gauge-invariant and that the resulting obstruction class in the group $\mathrm{Ext}^2\Big(\mathrm{XK}_0\big(C^*(E)\big), \mathrm{XK}_1\big(C^*(E)\big)\Big)$ has an explicit representative given by the dual Pimsner–Voiculescu sequence

$$0 \to \mathrm{XK}_1\big(C^*(E)\big) \to \mathrm{XK}_0\big(C^*(E)^{\mathbb{T}}\big) \to \mathrm{XK}_0\big(C^*(E)^{\mathbb{T}}\big) \to \mathrm{XK}_0\big(C^*(E)\big) \to 0.$$

As a result, we obtain in particular a classification of purely infinite graph C^*-algebras with finitely many ideals up to stable isomorphism (and up to actual isomorphism when we restrict to unital algebras and add the unit class in $\mathrm{K}_0\big(C^*(E)\big)$ to the invariant). I hope that this approach might also shed light on the question of existence of phantom Cuntz–Krieger algebras.

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I am deeply grateful to my advisors, Søren Eilers and Ryszard Nest, for their professional and friendly supervision; this includes trusting me to make the best out of the vast freedom that I was granted. I would also like to thank Marius Dadarlat and Ralf Meyer for their hospitality and inspiration during my visits, and my other coworkers, Sara Arklint and Takeshi Katsura, for the enjoyable teamwork. The present version of the thesis incorporates a number of corrections suggested by Efren Ruiz for which I thank him warmly.

During the past three years, I have benefited from discussions with a variety of mathematicians. The attempt to write down a complete list escalated quickly and had to be abandoned. I do, however, wish to point out my greatest appreciation of the generosity in sharing mathematical ideas that I have experienced in many individuals. More explicit acknowledgements may be found in the appended and upcoming articles.

Above all, I wish to thank my wife, Ina Schachtschneider, for her love and for making all this possible, and my son Oskar for radiating so much joy and happiness. Our last year together in Copenhagen has been wonderful. I am also most thankful to my parents and my brother for their enduring support and for all those things that ever seemed to be a matter of course.

Rasmus Moritz Bentmann
Copenhagen, September 2013

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Introduction

The Elliott Program

The attempt to classify nuclear separable C^*-algebras using K-theoretic invariants (together with the actions of possible traces on them) has become known as the *Elliott Program* – and the sentiment that this could be a viable task as the *Elliott Conjecture*. We will describe the history of this endeavor in somewhat greater detail than necessary to merely provide context for the author's work. The beginnings of this story lie in Glimm's classification of uniformly hyperfinite (UHF) C^*-algebras [**27**] and, more generally, Elliott's classification of approximately finite dimensional (AF) C^*-algebras [**21**]. The invariant in this case can be expressed in terms of the K_0-group equipped with the additional structure of the *dimension range* [**60**, §7.3]. A corresponding description of the range of the invariant was given by Effros, Handelman and Shen [**20**]. The proofs of these classification results crucially rely on the so-called *intertwining argument*, which has turned out to be applicable in much greater generality. In the course of time, classification results were established for various inductive limit classes, such as $\mathrm{A}\mathbb{T}$-algebras of real rank zero [**22**] and simple unital approximately homogeneous (AH) C^*-algebras with either slow dimension growth and real rank zero [**12**, **23**, **28**] or with very slow dimension growth [**24**, **29**]. On the side of purely infinite rather than stably finite C^*-algebras, meanwhile, Kirchberg [**34**] and Phillips [**48**] classified the class of Kirchberg algebras satisfying the universal coefficient theorem based on approaches using Kasparov's bivariant K-theory [**32**]. For more detailed surveys on the history of the classification program for nuclear C^*-algebras, we refer to [**25**, **56**, **59**].

In the meantime, it has become apparent from the work of Rørdam [**57**] and Toms [**64**–**66**] that additional assumptions – even for simple unital AH-algebras – are necessary for a general result along the lines of the Elliott Program. There exists a whole range of regularity properties for C^*-algebras with various flavors and intricate relations. According to a conjecture of Toms and Winter [**68**], *$\mathcal{Z}$-absorption*, Blackadar's *strict comparison of positive elements* and Winter–Zacharias's *finite nuclear dimension* are equivalent for unital separable simple infinite-dimensional nuclear C^*-algebras. Furthermore, it is expected that these properties characterize those C^*-algebras that are classifiable in the sense of the Elliott Conjecture [**68**]. Both conjectures have been confirmed for the class of unital separable simple AH-algebras by the work of Lin, Toms and Winter [**37**, **68**, **73**, **75**]. Rørdam proved that $\mathcal{Z}$-stability implies strict comparison for unital simple exact C^*-algebras [**58**]. Winter showed, using Robert's notion of *n-comparison* [**52**], that finite nuclear dimension implies $\mathcal{Z}$-stability for unital separable simple infinite-dimensional C^*-algebras [**75**]. Under certain size assumptions on the trace simplex, it was proved for C^*-algebras as in the Toms–Winter Conjecture that strict comparison implies $\mathcal{Z}$-stability by Matui–Sato [**39**], Sato [**62**], Toms–White–Winter [**69**] and Kirchberg–Rørdam [**36**]. It follows from results of Winter [**75**] and Toms [**67**] that a unital simple separable approximately subhomogeneous (ASH) C^*-algebra is $\mathcal{Z}$-stable if and only if it has slow dimensions growth; using [**38**, **72**, **76**], this implies that unital simple separable ASH-algebras with slow dimension growth in which projections separate traces are classified by their graded ordered K-theory. Recently, Matui and Sato showed that, for C^*-algebras as in the Toms–Winter Conjecture that are quasidiagonal and have a unique trace, the conditions strict comparison, $\mathcal{Z}$-stability and finite decomposition rank are equivalent; moreover, they verify the Elliott Conjecture for the class of C^*-algebras satisfying all these assumptions and the universal coefficient theorem [**40**].

The C^*-algebra $\mathcal{Z}$ is the so-called *Jiang–Su algebra* [**31**, **61**]; it is simple, unital, infinite-dimensional, stably finite, strongly self-absorbing in the sense of Toms–Winter [**70**] and KK-equivalent to the C^*-algebra $\mathbb{C}$ of complex numbers. A C^*-algebra is called *$\mathcal{Z}$-stable* or *$\mathcal{Z}$-absorbing* if $A \otimes \mathcal{Z} \cong A$. Winter characterized $\mathcal{Z}$ as the *initial* strongly self-absorbing C^*-algebra [**74**]. Hence tensorial absorption of any strongly self-absorbing C^*-algebra different from $\mathcal{Z}$ gives rise to a regularity property stronger than $\mathcal{Z}$-stability. Other prominent examples of strongly self-absorbing C^*-algebras are the Cuntz algebras $\mathcal{O}_2$ and $\mathcal{O}_\infty$ [**9**], two unital nuclear purely infinite simple C^*-algebras KK-equivalent to 0 and $\mathbb{C}$, respectively. An $\mathcal{O}_2$-absorbing C^*-algebra is trivial in any K-theoretical sense and, indeed, Kirchberg showed that two separable nuclear stable $\mathcal{O}_2$-absorbing C^*-algebras are isomorphic once they have the same ideal structure [**33**]. It was shown by Kirchberg and Rørdam that the more general property of $\mathcal{O}_\infty$-stability is equivalent, for separable nuclear stable C^*-algebras, to being what they call *strongly purely infinite* [**35**]. Such a C^*-algebra cannot have any traces; in fact, by a result of Rørdam, a separable nuclear $\mathcal{Z}$-stable C^*-algebra absorbs $\mathcal{O}_\infty$ if and only if it is traceless [**58**]. An intermediate element between $\mathcal{O}_2$ and $\mathcal{O}_\infty$ in the hierarchy of strongly self-absorbing C^*-algebras is given by the tensor product $\mathcal{O}_\infty \otimes \mathbb{M}_\mathbb{Q}$, where $\mathbb{M}_\mathbb{Q}$ denotes the universal UHF-algebra. Since $\mathrm{K}_*(\mathbb{M}_\mathbb{Q}) \cong \mathbb{Q} \oplus 0$, the Künneth formula [**53**] shows that $\mathbb{M}_\mathbb{Q}$-absorption entails a general rationality in K-theory.

Kirchberg's classification theorem

The classification problem for separable nuclear $\mathcal{O}_\infty$-absorbing C^*-algebras amounts to an important special case of the more general problem for $\mathcal{Z}$-stable C^*-algebras, where traces may complicate matters. A *simple* separable nuclear $\mathcal{O}_\infty$-absorbing C^*-algebra is called a *Kirchberg algebra* [**56**]. The celebrated Kirchberg–Phillips classification theorem states that a KK-equivalence between stable Kirchberg algebras lifts to a *-isomorphism [**34**, **48**]. In order to lift isomorphisms on K-theory to KK-equivalences, one has to make the additional assumption that the algebras belong to the bootstrap class of Rosenberg and Schochet [**53**]. This class is defined to be the smallest class of separable nuclear C^*-algebras containing the C^*-algebra $\mathbb{C}$ of complex numbers and being closed under countable inductive limits, extensions and KK-equivalence. It is designed such that its members satisfy the *universal coefficient theorem*, of which the aforementioned lifting result is a simple corollary (see [**5**, Proposition 23.10.1]).

The following framework from [**43**] is convenient for the discussion of classification problems for non-simple C^*-algebras and, in particular, invariants whose definition explicitly involves the collection of ideals of the input C^*-algebra. We fix a second countable space X and consider *C^*-algebras over X*, that is, C^*-algebras A together with a continuous map $\varphi\colon \mathrm{Prim}(A) \to X$, where $\mathrm{Prim}(A)$ denotes the primitive ideal space of A. Every open subset of X gives rise to a *distinguished ideal* in A. The analogue of simplicity in this context is *tightness*, which asks the map φ to be a homeomorphism. There are natural equivariance conditions for *-homomorphisms and KK-cycles (as well as their homotopies) in this setting. In particular, there is a bivariant theory $\mathrm{KK}(X)$ with the expected properties (see [**6, 33, 43**]), which we call *Kirchberg's ideal-related Kasparov theory*. We propose to call a *tight* separable nuclear $\mathcal{O}_\infty$-absorbing C^*-algebra over X a *Kirchberg X-algebra*. Kirchberg proved a generalization of the Kirchberg–Phillips theorem that, in our terminology, reads as follows.

Theorem (Kirchberg [**33**]). *A $\mathrm{KK}(X)$-equivalence of stable Kirchberg X-algebras lifts to a *-isomorphism over X.*

For the purposes of this thesis, we may treat this theorem as a black box. It leaves the following crucial question of rather topological nature.

Question. *When are two given separable C^*-algebras over X $\mathrm{KK}(X)$-equivalent?*

Answering this question requires some sort of generalization of the universal coefficient theorem, of course again under appropriate bootstrap assumptions. A natural first case to consider is the one where X is finite. In his thesis [**6**], Bonkat established a universal coefficient theorem for C^*-algebras over the two-point Sierpiński space or, equivalently, for extensions of C^*-algebras in terms of the K-theoretic six-term exact sequence, hence recovering, in combination with Kirchberg's result, Rørdam's earlier classification of stable extensions of UCT Kirchberg algebras [**55**]. Restorff gave a generalization of this result for the three-point T_0-space with totally ordered topology [**51**].

An indication of what an appropriate invariant for C^*-algebras over general finite spaces might look like comes from Restorff's classification of purely infinite, not necessarily simple Cuntz–Krieger algebras [**50**]. The resulting invariant goes under various names, such as *filtrated* K*-theory*, *filtered* K*-theory*, *ideal-related* K*-theory* and K*-web*. Meyer and Nest proved that this invariant indeed suffices to obtain a universal coefficient theorem if the topology of X is totally ordered and that it is inadequate if X is a certain four-point space [**45**]. Generalizing this work, Köhler and the author showed that filtrated K-theory is a suitable invariant if and only if every connected component of X is a so-called accordion space [**4**]. For their problematic four-point space, Meyer and Nest showed how to refine filtrated K-theory in order to establish a universal coefficient theorem; their method was applied to some other four-point spaces in the authors final year project [**3**]. Exhausting this refinement approach remains to be done. A universal coefficient theorem for $\mathrm{KK}(X)$-theory over an arbitrary finite space X is proved in Article D for C^*-algebras over X with vanishing boundary maps.

Cuntz–Krieger algebras

In [**10, 11**], Cuntz and Krieger introduced a class of C^*-algebras—nowadays called *Cuntz–Krieger algebras*—associated with certain dynamical systems, namely shift spaces of finite type. They proved that the stable isomorphism class of their C^*-algebra is an invariant of the shift space up to flow equivalence. The Cuntz–Krieger algebras form an interesting class of not necessarily simple, nuclear and often purely infinite C^*-algebras (a Cuntz–Krieger algebra is purely infinite if and only if it has finitely many ideals). Rørdam classified the simple Cuntz–Krieger algebras in terms of the K_0-group [**54**] and Huang classified the Cuntz–Krieger algebras with one proper non-zero ideal using the Cuntz invariant [**30**]. Building on work of Boyle and Huang in symbolic dynamics [**7**], Restorff proved the following classification theorem.

Theorem (Restorff [**50**]). *Reduced filtered* K*-theory is a complete stable isomorphism invariant for purely infinite Cuntz–Krieger algebras.*

In Article B we prove a range-of-invariant result complementing this classification theorem. As stated, Restorff's theorem does not imply that isomorphisms on the invariant necessarily lift to stable isomorphisms of C^*-algebras. It also leaves the following question raised by Eilers, see [**1**] for an overview:

QUESTION. *Do phantom Cuntz–Krieger algebras exist?*

A less catchy formulation of the question is as follows: let A be a unital real-rank-zero Kirchberg X-algebra for a finite space X such that every simple subquotient S of A satisfies the universal coefficient theorem, has free K_1-group and fulfills the relation $\operatorname{rank} \mathrm{K}_0(S) = \operatorname{rank} \mathrm{K}_1(S) < \infty$; then, is A necessarily a Cuntz–Krieger algebra? These questions could potentially be answered by an approach using Kirchberg's classification theorem. We provide partial answers in Articles C and D. One conclusion of Article A is that one straightforward approach to the problem—trying to establish a universal coefficient theorem for Cuntz–Krieger algebras by establishing length-1 projective resolutions in filtrated K-theory—fails.

Continuous fields

Another interesting classification problem is the one of continuous fields of Kirchberg algebras over a finite-dimensional compact metrizable space. Continuous fields of C^*-algebras [**19**, **26**], or *C^*-bundles*, are generally far from locally trivial; hence complicated invariants will be needed to capture their potential complexity. The assumption of finite-dimensionality is crucial: there is a non-trivial unital separable continuous field over the Hilbert cube with constant fiber $\mathcal{O}_2$ [**14**].

While some recent classification results in this setting rely on the full force of Kirchberg's theorem, there are also alternative approaches. The zero-dimensional case has been studied quite conclusively by Dadarlat and Pasnicu [**18**]. It follows from their work that continuous fields of stable UCT Kirchberg algebras over zero-dimensional spaces are classified by *filtrated* K*-theory with coefficients*. This result is also a consequence of the more recent universal multicoefficient theorem of Dadarlat and Meyer [**17**] (in combination with Kirchberg's classification). Using approximations by so-called elementary fields of semiprojective Kirchberg algebras, Dadarlat and Elliott classified separable continuous fields over the unit interval—*one-parameter* continuous fields—of stable UCT Kirchberg algebras under certain K-theoretical assumptions on the fibers (torsion-free K_d and vanishing K_{d+1}) in [**15**]. A complementing range result is deduced in [**16**]. In [**13**], Dadarlat proves automatic/conditional (local) triviality theorems for continuous fields of (stabilized) Cuntz algebras over an arbitrary finite-dimensional compact metrizable space. In Articles E and F, based on Kirchberg's theorem, we provide classification results for continuous fields of Kirchberg algebras under certain K-theoretical assumptions.

Triangulated categories

Triangulated categories were introduced by Verdier [**71**] and Puppe [**49**] in order to describe the basic formal structure of derived categories and the stable homotopy category, respectively. This structure is also present on the category of separable C^*-algebras and KK-classes, as well as on equivariant generalizations of this category [**42**, **43**]. Inspired by this analogy, one may seek to carry over notions like derived functors or the Adams spectral sequence to the general setting of triangulated categories. Building on work of Beligiannis [**2**] and Christensen [**8**], Meyer and Nest developed a theory of (relative) homological algebra in triangulated categories [**41**, **44**] providing a convenient unifying framework for universal coefficient theorems and several other problems in noncommutative topology; we use it in Articles A and D.

As indicated above, triangulated categories arise in various mathematical fields. It turns out that triangulated categories of algebraic origin have some instrinsic properties distinguishing them from many examples of topological or pathological nature [**46**]. To quantify this phenomenon, Schwede has constructed an invariant called *n-order* for every natural number n [**63**]. It is interesting to ask for the n-order of (equivariant) KK- or E-theory. In Article G we do some computations to this effect, but we must restrict to certain triangulated subcategories. Our result says that, as far as the n-order is concerned, the categories we investigate behave like algebraic triangulated categories.

Summary

For the reader's convenience, we concisely state in this section the main results of the articles contained in this thesis. To keep this overview as brief as possible, we refer to the articles for formal definitions.

Article A: Projective dimension in filtrated K-theory

Tautologically, filtrated K-theory takes values in the category of graded modules over the ring $\mathcal{NT}$ of natural transformations acting on the collection of its various entry functors. We assume that this ring decomposes as $\mathcal{NT} = \mathcal{NT}_{\mathrm{nil}} \rtimes \mathcal{NT}_{\mathrm{ss}}$ into a nilpotent part $\mathcal{NT}_{\mathrm{nil}}$ and a semi-simple part $\mathcal{NT}_{\mathrm{ss}}$; we do not know a finite space for which this fails. Generalizing results of Meyer and Nest, we show:

Proposition. *An $\mathcal{NT}$-module M has a projective resolution of length $n \in \mathbb{N}$ if and only if the abelian group $\mathrm{Tor}_n^{\mathcal{NT}}(\mathcal{NT}_{\mathrm{ss}}, M)$ is free and the abelian group $\mathrm{Tor}_{n+1}^{\mathcal{NT}}(\mathcal{NT}_{\mathrm{ss}}, M)$ vanishes.*

This result allows for explicit computations that imply the following.

Corollary. *The projective dimension of filtrated* K*-theory over finite spaces is, in general, not bounded by* 2*. On Cuntz–Krieger algebras, it is, in general, not bounded by* 1.

The first conclusion shows that there is no Bousfield-type refinement for filtrated K-theory yielding classification over every finite space. The second conclusion may be regarded as negative evidence for a universal coefficient theorem for Cuntz–Krieger algebras in terms of filtrated K-theory.

Article B: The K-theoretical range of Cuntz–Krieger algebras (with Sara Arklint and Takeshi Katsura)

In this article, we formalize the target category of Restorff's reduced filtered K-theory functor $\mathrm{FK}_{\mathcal{R}}$ as modules over a certain graded ring $\mathcal{R}$. We determine the range of $\mathrm{FK}_{\mathcal{R}}$ on purely infinite Cuntz–Krieger algebras in terms of exactness, freeness and rank conditions. In combination with Restorff's classification theorem, our result reads as follows.

Corollary. *Let X be a finite T_0-space. The functor $\mathrm{FK}_{\mathcal{R}}$ induces a bijection between the set of stable isomorphism classes of tight purely infinite Cuntz–Krieger algebras over X and the set of isomorphism classes of exact $\mathcal{R}$-modules M such that, for all $x \in X$, the abelian group $M(x_1)$ is free, the abelian groups $M(x_1)$ and $M(\widetilde{x}_0)$ are finitely generated, and the rank of $M(x_1)$ coincides with the rank of the cokernel of the map $i\colon M(\widetilde{\partial} x_0) \to M(\widetilde{x}_0)$.*

We also show that there are no restrictions on the class in K_0 of the unit of the Cuntz–Krieger algebra. There is a version of our result for purely infinite graph C^*-algebras.

Article C: Reduction of filtered K-theory and a characterization of Cuntz–Krieger algebras (with Sara Arklint and Takeshi Katsura)

In this article, we show that, under suitable assumptions, reduced filtrated K-theory contains as much information as concrete filtrated K-theory. More specifically, we show the following.

THEOREM. *Let A and B be C^*-algebras of real rank zero over an EBP space X. Assume that $\mathrm{K}_1\big(A(x)\big)$ and $\mathrm{K}_1\big(B(x)\big)$ are free abelian groups for all points $x \in X$. Then any isomorphism $\mathrm{FK}_{\mathcal{R}}(A) \cong \mathrm{FK}_{\mathcal{R}}(B)$ lifts to an isomorphism in concrete filtrated* K*-theory.*

Every accordion space is an EBP space. Combining the theorem with the range result from Article B and the universal coefficient theorem for C^*-algebras over accordion spaces, we obtain:

COROLLARY. *The primitive ideal space of a phantom Cuntz–Krieger algebra cannot be a disjoint union of accordion spaces.*

In fact, the same conclusion can be made also for the five four-point non-accordion spaces, for which universal coefficient theorems have been established.

Article D: Kirchberg X-algebras with real rank zero and intermediate cancellation

In this article, we consider C^*-algebras over a finite space X which have vanishing boundary maps. A Kirchberg X-algebra has vanishing boundary maps if and only if it has real rank zero and intermediate cancellation (a cancellation property formulated in a similar way as weak cancellation). We establish the following universal coefficient theorem.

THEOREM. *Let A and B be separable C^*-algebras over X. Assume that A belongs to the bootstrap class $\mathcal{B}(X)$ and has vanishing boundary maps. Then there is a natural short exact sequence of $\mathbb{Z}/2$-graded abelian groups*

$$\mathrm{Ext}^1\big(\mathrm{XK}(A)[1], \mathrm{XK}(B)\big) \rightarrowtail \mathrm{KK}_*(X; A, B) \twoheadrightarrow \mathrm{Hom}\big(\mathrm{XK}(A), \mathrm{XK}(B)\big).$$

As a result, we obtain the following corollary of Kirchberg's classification theorem:

COROLLARY. *The functor* XK *is a strongly complete stable isomorphism invariant for Kirchberg X-algebras in $\mathcal{B}(X)$ with real rank zero and intermediate cancellation.*

Together with the range result from Article B, this can be used to show the following.

COROLLARY. *A phantom Cuntz–Krieger algebra cannot have intermediate cancellation.*

Article E: One-parameter continuous fields of Kirchberg algebras with rational K-theory (with Marius Dadarlat)

We define a version of filtrated K-theory for C^*-algebras over the unit interval and show the following.

Theorem. *Let A and B be separable continuous fields over the unit interval whose fibers are stable Kirchberg algebras that satisfy the universal coefficient theorem and have rational* K*-theory groups. Then any isomorphism of filtrated* K*-theory* $\mathrm{FK}(A) \cong \mathrm{FK}(B)$ *lifts to a* $C[0,1]$*-linear* **-isomorphism.*

Besides Kirchberg's classification theorem, we use the universal coefficient theorem for C^*-algebras over accordion spaces and results of Dadarlat and Meyer relating E-theory over $[0,1]$ with the corresponding version of KK-theory and with E-theory groups over finite approximating spaces of $[0,1]$.

Article F: Classification of certain continuous fields of Kirchberg algebras

A similar approach as in Article E, but now based on the universal coefficient theorem in Article D, gives the following result; while the base space X is only required to be finite-dimensional, the K-theoretic assumptions are stronger.

Theorem. *Let A and B be separable continuous fields over a finite-dimensional compact metrizable topological space X whose fibers are stable Kirchberg algebras that satisfy the universal coefficient theorem and have rational* K*-theory groups. Assume that A and B have vanishing boundary maps. Then any isomorphism of* K*-theory cosheaves* $\mathbb{O}\mathrm{K}(A) \cong \mathbb{O}\mathrm{K}(B)$ *lifts to a* $C(X)$*-linear* **-isomorphism.*

We also provide a partial range description accompanying the above classification.

Proposition. *Let X be a finite-dimensional compact metrizable topological space X. Let M be a flabby cosheaf of $\mathbb{Z}/2$-graded $\mathbb{Q}$-vector spaces on X such that $M(X)$ is finite-dimensional. Then M is a direct sum of a finite number of skyscraper cosheaves and* $M \cong \mathbb{O}\mathrm{K}(A)$ *for a continuous field A as in the previous theorem.*

Article G: Homotopy-theoretic E-theory and n-order

Let $\mathcal{SW}_{\mathrm{bu}}$ be the thick triangulated subcategory of connective E-theory generated by the C^*-algebra of complex numbers. Let $\mathcal{B}_{\mathrm{E}}^G$ be the $\aleph_0$-localizing subcategory of G-equivariant E-theory for a compact group G generated by the C^*-algebra of complex numbers equipped with the trivial action. Let $\mathcal{B}_{\mathrm{E}}(X)$ be the bootstrap category in E-theory over a finite space X. We compute Schwede's n-order for these triangulated categories.

Theorem. *The triangulated categories* $\mathcal{SW}_{\mathrm{bu}}$, $\mathcal{B}_{\mathrm{E}}^G$ *and* $\mathcal{B}_{\mathrm{E}}(X)$ *have infinite n-order for every $n \in \mathbb{N}$.*

Our proof makes use of results due to Lawson and Angeltveit.

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Articles

On the following pages, we have attached the seven articles that constitute the research material for this thesis.

A. Projective dimension in filtrated K-theory.

The article will appear in Operator Algebra and Dynamics, Springer Proceedings in Mathematics & Statistics, Vol. 58. It is included on pages 17–32 in its most recent preprint version from January 2013 that is publicly available at `arxiv.org/abs/1210.4785v2`.

B. The K-theoretical range of Cuntz–Krieger algebras.

The article is co-authored with Sara Arklint and Takeshi Katsura. It is included on pages 33–48 in its most recent preprint version from November 2013 that is publicly available at `arxiv.org/abs/1309.7162v2`.

C. Reduction of filtered K-theory and a characterization of Cuntz–Krieger algebras.

The article is co-authored with Sara Arklint and Takeshi Katsura. It is included on pages 49–83 in its most recent preprint version from November 2013 that is publicly available at `arxiv.org/abs/1301.7223v3`.

D. Kirchberg X-algebras with real rank zero and intermediate cancellation.

The article is included on pages 85–100 in its most recent preprint version from November 2013 that is publicly available at `arxiv.org/abs/1301.6652v3`.

E. One-parameter continuous fields of Kirchberg algebras with rational K-theory.

The article is co-authored with Marius Dadarlat. It is included on pages 101–112 in its most recent preprint version from November 2013 that is publicly available at `arxiv.org/abs/1306.1691v2`.

F. Classification of certain continuous fields of Kirchberg algebras.

The article is included on pages 113–120 in its most recent preprint version from November 2013 that is publicly available at `arxiv.org/abs/1308.2126v2`.

G. Homotopy-theoretic E-theory and n-order.

The article will appear in the Journal of Homotopy and Related Structures. It is included on pages 121–129. A preprint version from May 2013 is publicly available at `arxiv.org/abs/1302.6924v2`.

PROJECTIVE DIMENSION IN FILTRATED K-THEORY

RASMUS BENTMANN

Abstract. Under mild assumptions, we characterise modules with projective resolutions of length $n \in \mathbb{N}$ in the target category of filtrated K-theory over a finite topological space in terms of two conditions involving certain Tor-groups. We show that the filtrated K-theory of any separable C^*-algebra over any topological space with at most four points has projective dimension 2 or less. We observe that this implies a universal coefficient theorem for rational equivariant KK-theory over these spaces. As a contrasting example, we find a separable C^*-algebra in the bootstrap class over a certain five-point space, the filtrated K-theory of which has projective dimension 3. Finally, as an application of our investigations, we exhibit Cuntz-Krieger algebras which have projective dimension 2 in filtrated K-theory over their respective primitive spectrum.

1. Introduction

A far-reaching classification theorem in [7] motivates the computation of Eberhard Kirchberg's ideal-related Kasparov groups $\mathrm{KK}(X; A, B)$ for separable C^*-algebras A and B over a non-Hausdorff topological space X by means of K-theoretic invariants. We are interested in the specific case of finite spaces here. In [9,10], Ralf Meyer and Ryszard Nest laid out a theoretic framework that allows for a generalisation of Jonathan Rosenberg's and Claude Schochet's universal coefficient theorem [16] to the equivariant setting. Starting from a set of generators of the equivariant bootstrap class, they define a homology theory with a certain universality property, which computes $\mathrm{KK}(X)$-theory via a spectral sequence. In order for this *universal coefficient* spectral sequence to degenerate to a short exact sequence, it remains to be checked *by hand* that objects in the range of the homology theory admit projective resolutions of length 1 in the Abelian target category.

Generalising earlier results from [3,10,15] the verification of the above-mentioned condition for *filtrated* K-*theory* was achieved in [2] for the case that the underlying space is a disjoint union of so-called accordion spaces. A finite connected T_0-space X is an accordion space if and only if the directed graph corresponding to its specialisation pre-order is a Dynkin quiver of type A. Moreover, it was shown in [2,10] that, if X is a finite T_0-space which is not a disjoint union of accordion spaces, then the projective dimension of filtrated K-theory over X is *not* bounded by 1 and objects in the equivariant bootstrap class are *not* classified by filtrated K-theory. The assumption of the separation axiom T_0 is not a loss of generality in this context (see [11, §2.5]).

There are two natural approaches to tackle the problem arising for non-accordion spaces: one can either try to refine the invariant—this has been done with some success in [10] and [1]; or one can hold onto the invariant and try to establish projective resolutions of length 1 on suitable subcategories or localisations of the category $\mathfrak{KK}(X)$, in which X-equivariant KK-theory is organised. The latter is the course we pursue in this note. We state our results in the next section.

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2. Statement of Results

The definition of filtrated K-theory and related notation are recalled in §3.

Proposition 1. *Let X be a finite topological space. Assume that the ideal $\mathcal{NT}_{\mathrm{nil}} \subset \mathcal{NT}^*(X)$ is nilpotent and that the decomposition $\mathcal{NT}^*(X) = \mathcal{NT}_{\mathrm{nil}} \rtimes \mathcal{NT}_{\mathrm{ss}}$ holds. Fix $n \in \mathbb{N}$. For an $\mathcal{NT}^*(X)$-module M, the following assertions are equivalent:*

(i) *M has a projective resolution of length n.*
(ii) *The Abelian group $\mathrm{Tor}_n^{\mathcal{NT}^*(X)}(\mathcal{NT}_{\mathrm{ss}}, M)$ is free and the Abelian group $\mathrm{Tor}_{n+1}^{\mathcal{NT}^*(X)}(\mathcal{NT}_{\mathrm{ss}}, M)$ vanishes.*

The basic idea of this paper is to compute the Tor-groups above by writing down projective resolutions for the fixed right-module $\mathcal{NT}_{\mathrm{ss}}$.

Let Z_m be the $(m+1)$-point space on the set $\{1, 2, \ldots, m+1\}$ such that $Y \subseteq Z_m$ is open if and only if $Y \ni m+1$ or $Y = \emptyset$. A C^*-algebra over Z_m is a C^*-algebra A with a distinguished ideal such that the corresponding quotient decomposes as a direct sum of m orthogonal ideals. Let S be the set $\{1, 2, 3, 4\}$ equipped with the topology $\{\emptyset, 4, 24, 34, 234, 1234\}$, where we write $24 := \{2, 4\}$ etc. A C^*-algebra over S is a C^*-algebra together with two distinguished ideals which need not satisfy any further conditions; see [11, Lemma 2.35].

Proposition 2. *Let X be a topological space with at most 4 points. Let $M = \mathrm{FK}(A)$ for some C^*-algebra A over X. Then M has a projective resolution of length 2 and $\mathrm{Tor}_2^{\mathcal{NT}^*}(\mathcal{NT}_{\mathrm{ss}}, M) = 0$.*

Moreover, we can find explicit formulas for $\mathrm{Tor}_1^{\mathcal{NT}^*}(\mathcal{NT}_{\mathrm{ss}}, M)$; for instance, $\mathrm{Tor}_1^{\mathcal{NT}^*(Z_3)}(\mathcal{NT}_{\mathrm{ss}}, M)$ is isomorphic to the homology of the complex

$$\bigoplus_{j=1}^{3} M(j4) \xrightarrow{\begin{pmatrix} i & -i & 0 \\ -i & 0 & i \\ 0 & i & -i \end{pmatrix}} \bigoplus_{k=1}^{3} M(1234 \setminus k) \xrightarrow{(i\ i\ i)} M(1234)\,. \tag{1}$$

A similar formula holds for the space S; see (6).

The situation simplifies if we consider *rational* $\mathrm{KK}(X)$-theory, whose morphism groups are given by $\mathrm{KK}(X; A, B) \otimes \mathbb{Q}$; see [6]. This is a $\mathbb{Q}$-linear triangulated category which can be constructed as a localisation of $\mathfrak{KK}(X)$; the corresponding localisation of filtrated K-theory is given by $A \mapsto \mathrm{FK}(A) \otimes \mathbb{Q}$ and takes values in the category of modules over the $\mathbb{Q}$-linear category $\mathcal{NT}^*(X) \otimes \mathbb{Q}$.

Proposition 3. *Let X be a topological space with at most 4 points. Let A and B be C^*-algebras over X. If A belongs to the equivariant bootstrap class $\mathcal{B}(X)$, then there is a natural short exact universal coefficient sequence*

$$\begin{aligned} \mathrm{Ext}^1_{\mathcal{NT}^*(X)\otimes\mathbb{Q}}\big(\mathrm{FK}_{*+1}(A) \otimes \mathbb{Q}, \mathrm{FK}_*(B) \otimes \mathbb{Q}\big) &\rightarrowtail \mathrm{KK}_*(X; A, B) \otimes \mathbb{Q} \\ &\twoheadrightarrow \mathrm{Hom}_{\mathcal{NT}^*(X)\otimes\mathbb{Q}}\big(\mathrm{FK}_*(A) \otimes \mathbb{Q}, \mathrm{FK}_*(B) \otimes \mathbb{Q}\big)\,. \end{aligned}$$

In [6], a long exact sequence is constructed which in our setting, by the above proposition, reduces the computation of $\mathrm{KK}_*(X; A, B)$, up to extension problems, to the computation of a certain torsion theory $\mathrm{KK}_*(X; A, B; \mathbb{Q}/\mathbb{Z})$.

The next proposition says that the upper bound of 2 for the projective dimension in Proposition 2 does not hold for all finite spaces.

Proposition 4. *There is an $\mathcal{NT}^*(Z_4)$-module M of projective dimension 2 with free entries and $\operatorname{Tor}_2^{\mathcal{NT}^*}(\mathcal{NT}_{\mathrm{ss}}, M) \neq 0$. The module $M \otimes_{\mathbb{Z}} \mathbb{Z}/k$ has projective dimension 3 for every $k \in \mathbb{N}_{\geq 2}$. Both M and $M \otimes_{\mathbb{Z}} \mathbb{Z}/k$ can be realised as the filtrated* K*-theory of an object in the equivariant bootstrap class $\mathcal{B}(X)$.*

As an application of Proposition 2 we investigate in §10 the obstruction term $\operatorname{Tor}_1^{\mathcal{NT}^*}\big(\mathcal{NT}_{\mathrm{ss}}, \mathrm{FK}(A)\big)$ for certain Cuntz-Krieger algebras with four-point primitive ideal spaces. We find:

Proposition 5. *There is a Cuntz-Krieger algebra with primitive ideal space homeomorphic to Z_3 which fulfills Cuntz's condition* (II) *and has projective dimension* 2 *in filtrated* K*-theory over Z_3. The analogous statement for the space S holds as well.*

The relevance of this observation lies in the following: *if* Cuntz-Krieger algebras *had* projective dimension at most 1 in filtrated K-theory over their primitive ideal space, this would lead to a strengthened version of Gunnar Restorff's classification result [14] with a proof avoiding reference to results from symbolic dynamics.

3. Preliminaries

Let X be a finite topological space. A subset $Y \subseteq X$ is called *locally closed* if it is the difference $U \setminus V$ of two open subsets U and V of X; in this case, U and V can always be chosen such that $V \subseteq U$. The set of locally closed subsets of X is denoted by $\mathbb{LC}(X)$. By $\mathbb{LC}(X)^*$, we denote the set of *non-empty, connected* locally closed subsets of X.

Recall from [11] that a *C^*-algebra over X* is pair (A, ψ) consisting of a C^*-algebra A and a continuous map $\psi\colon \operatorname{Prim}(A) \to X$. A C^*-algebra (A, ψ) over X is called *tight* if the map ψ is a homeomorphism. A C^*-algebra (A, ψ) over X comes with *distinguished subquotients* $A(Y)$ for every $Y \in \mathbb{LC}(X)$.

There is an appropriate version $\mathrm{KK}(X)$ of bivariant K-theory for C^*-algebras over X (see [7, 11]). The corresponding category, denoted by $\mathfrak{KK}(X)$, is equipped with the structure of a triangulated category (see [12]); moreover, there is an equivariant analogue $\mathcal{B}(X) \subseteq \mathfrak{KK}(X)$ of the bootstrap class [11].

Recall that a triangulated category comes with a class of distinguished candidate triangles. An *anti-distinguished* triangle is a candidate triangle which can be obtained from a distinguished triangle by reversing the sign of one of its three morphisms. Both distinguished and anti-distinguished triangles induce long exact Hom-sequences.

As defined in [10], for $Y \in \mathbb{LC}(X)$, we let $\mathrm{FK}_Y(A) := \mathrm{K}_*\big(A(Y)\big)$ denote the $\mathbb{Z}/2$-graded K-group of the subquotient of A associated to Y. Let $\mathcal{NT}(X)$ be the $\mathbb{Z}/2$-graded pre-additive category whose object set is $\mathbb{LC}(X)$ and whose space of morphisms from Y to Z is $\mathcal{NT}_*(X)(Y, Z)$ – the $\mathbb{Z}/2$-graded Abelian group of all natural transformations $\mathrm{FK}_Y \Rightarrow \mathrm{FK}_Z$. Let $\mathcal{NT}^*(X)$ be the full subcategory with object set $\mathbb{LC}(X)^*$. We often abbreviate $\mathcal{NT}^*(X)$ by $\mathcal{NT}^*$.

Every open subset of a locally closed subset of X gives rise to an extension of distinguished subquotients. The corresponding natural maps in the associated six-term exact sequence yield morphisms in the category $\mathcal{NT}$, which we briefly denote by i, r and δ.

A (*left-*)*module* over $\mathcal{NT}(X)$ is a grading-preserving, additive functor from $\mathcal{NT}(X)$ to the category $\mathfrak{Ab}^{\mathbb{Z}/2}$ of $\mathbb{Z}/2$-graded Abelian groups. A morphism of $\mathcal{NT}(X)$-modules is a natural transformation of functors. Left-modules over $\mathcal{NT}^*(X)$ are defined similarly. By $\mathfrak{Mod}\big(\mathcal{NT}^*(X)\big)_{\mathrm{c}}$ we denote the category of countable $\mathcal{NT}^*(X)$-modules.

Filtrated K*-theory* is the functor $\mathfrak{KK}(X) \to \mathfrak{Mod}\big(\mathcal{NT}^*(X)\big)_{\mathrm{c}}$ taking a C^*-algebra A over X to the collection $\big(\mathrm{K}_*(A(Y))\big)_{Y\in\mathbb{LC}(X)^*}$ with the obvious $\mathcal{NT}^*(X)$-module structure.

Let $\mathcal{NT}_{\mathrm{nil}} \subset \mathcal{NT}^*$ be the ideal generated by all natural transformations between different objects, and let $\mathcal{NT}_{\mathrm{ss}} \subset \mathcal{NT}^*$ be the subgroup spanned by the identity transformations id_Y^Y for objects $Y \in \mathbb{LC}(X)^*$. The subgroup $\mathcal{NT}_{\mathrm{ss}}$ is in fact a subring of $\mathcal{NT}^*$ isomorpic to $\mathbb{Z}^{\mathbb{LC}(X)^*}$. We say that $\mathcal{NT}^*$ decomposes as semi-direct product $\mathcal{NT}^* = \mathcal{NT}_{\mathrm{nil}} \rtimes \mathcal{NT}_{\mathrm{ss}}$ if $\mathcal{NT}^*$ as an Abelian group is the inner direct sum of $\mathcal{NT}_{\mathrm{nil}}$ and $\mathcal{NT}_{\mathrm{ss}}$; see [2, 10]. We do not know if this fails for any finite space.

We define *right-modules* over $\mathcal{NT}^*(X)$ as *contravariant*, grading-preserving, additive functors $\mathcal{NT}^*(X) \to \mathfrak{Ab}^{\mathbb{Z}/2}$. If we do not specify between left and right, then we always mean left-modules. The subring $\mathcal{NT}_{\mathrm{ss}} \subset \mathcal{NT}^*$ is regarded as an $\mathcal{NT}^*$-right-module by the obvious action: The ideal $\mathcal{NT}_{\mathrm{nil}} \subset \mathcal{NT}^*$ acts trivially, while $\mathcal{NT}_{\mathrm{ss}}$ acts via right-multiplication in $\mathcal{NT}_{\mathrm{ss}} \cong \mathbb{Z}^{\mathbb{LC}(X)^*}$. For an $\mathcal{NT}^*$-module M, we set $M_{\mathrm{ss}} := M/\mathcal{NT}_{\mathrm{nil}} \cdot M$.

For $Y \in \mathbb{LC}(X)^*$ we define the *free* $\mathcal{NT}^*$*-left-module on* Y by $P_Y(Z) := \mathcal{NT}(Y, Z)$ for all $Z \in \mathbb{LC}(X)^*$ and similarly for morphisms $Z \to Z'$ in $\mathcal{NT}^*$. Analogously, we define the *free* $\mathcal{NT}^*$*-right-module on* Y by $Q_Y(Z) := \mathcal{NT}(Z, Y)$ for all $Z \in \mathbb{LC}(X)^*$. An $\mathcal{NT}^*$-left/right-module is called *free* if it is isomorphic to a direct sum of degree-shifted free left/right-modules on objects $Y \in \mathbb{LC}(X)^*$. It follows directly from Yoneda's Lemma that free $\mathcal{NT}^*$-left/right-modules are projective.

An $\mathcal{NT}$-module M is called *exact* if the $\mathbb{Z}/2$-graded chain complexes

$$\cdots \to M(U) \xrightarrow{i_U^Y} M(Y) \xrightarrow{r_Y^{Y\setminus U}} M(Y \setminus U) \xrightarrow{\delta_{Y\setminus U}^U} M(U)[1] \to \cdots$$

are exact for all $U, Y \in \mathbb{LC}(X)$ with U open in Y. An $\mathcal{NT}^*$-module M is called *exact* if the corresponding $\mathcal{NT}$-module is exact (see [2]).

We use the notation $C \in\in \mathcal{C}$ to denote that C is an object in a category $\mathcal{C}$.

In [10], the functors FK_Y are shown to be representable, that is, there are objects $\mathcal{R}_Y \in\in \mathfrak{KK}(X)$ and isomorphisms of functors $\mathrm{FK}_Y \cong \mathrm{KK}(X; \mathcal{R}_Y, \sqcup)$. We let $\widehat{\mathrm{FK}}$ denote the stable *co*homological functor on $\mathfrak{KK}(X)$ represented by the same set of objects $\{\mathcal{R}_Y \mid Y \in \mathbb{LC}(X)^*\}$; it takes values in $\mathcal{NT}^*$-*right*-modules. We warn that $\mathrm{KK}(X; A, \mathcal{R}_Y)$ does not identify with the K-homology of $A(Y)$. By Yoneda's lemma, we have $\mathrm{FK}(\mathcal{R}_Y) \cong P_Y$ and $\widehat{\mathrm{FK}}(\mathcal{R}_Y) \cong Q_Y$.

We occasionally use terminology from [9, 10] concerning homological algebra in $\mathfrak{KK}(X)$ relative to the ideal $\mathfrak{I} := \ker(\mathrm{FK})$ of morphisms in $\mathfrak{KK}(X)$ inducing trivial module maps on FK. An object $A \in\in \mathfrak{KK}(X)$ is called $\mathfrak{I}$*-projective* if $\mathfrak{I}(A, B) = 0$ for every $B \in\in \mathfrak{KK}(X)$. We recall from [9] that FK restricts to an equivalence of categories between the subcategories of $\mathfrak{I}$-projective objects in $\mathfrak{KK}(X)$ and of projective objects in $\mathfrak{Mod}\big(\mathcal{NT}^*(X)\big)_{\mathrm{c}}$. Similarly, the functor $\widehat{\mathrm{FK}}$ induces a contravariant equivalence between the $\mathfrak{I}$-projective objects in $\mathfrak{KK}(X)$ and projective $\mathcal{NT}^*$-*right*-modules.

4. Proof of Proposition 1

Recall the following result from [10].

Lemma 1 ([10, Theorem 3.12])**.** *Let X be a finite topological space. Assume that the ideal $\mathcal{NT}_{\mathrm{nil}} \subset \mathcal{NT}^*(X)$ is nilpotent and that the decomposition $\mathcal{NT}^*(X) = \mathcal{NT}_{\mathrm{nil}} \rtimes \mathcal{NT}_{\mathrm{ss}}$ holds. Let M be an $\mathcal{NT}^*(X)$-module. The following assertions are equivalent:*

(1) *M is a free $\mathcal{NT}^*(X)$-module.*

(2) *M is a projective $\mathcal{NT}^*(X)$-module.*
(3) *M_{ss} is a free Abelian group and* $\mathrm{Tor}_1^{\mathcal{NT}^*(X)}(\mathcal{NT}_{\mathrm{ss}}, M) = 0$.

Now we prove Proposition 1. We consider the case $n = 1$ first. Choose an epimorphism $f\colon P \twoheadrightarrow M$ for some projective module P, and let K be its kernel. M has a projective resolution of length 1 if and only if K is projective. By Lemma 1, this is equivalent to K_{ss} being a free Abelian group and $\mathrm{Tor}_1^{\mathcal{NT}^*}(\mathcal{NT}_{\mathrm{ss}}, K) = 0$. We have $\mathrm{Tor}_1^{\mathcal{NT}^*}(\mathcal{NT}_{\mathrm{ss}}, K) = 0$ if and only if $\mathrm{Tor}_2^{\mathcal{NT}^*}(\mathcal{NT}_{\mathrm{ss}}, M) = 0$ because these groups are isomorphic. We will show that K_{ss} is free if and only if $\mathrm{Tor}_1^{\mathcal{NT}^*}(\mathcal{NT}_{\mathrm{ss}}, M)$ is free. The extension $K \rightarrowtail P \twoheadrightarrow M$ induces the following long exact sequence:

$$0 \to \mathrm{Tor}_1^{\mathcal{NT}^*}(\mathcal{NT}_{\mathrm{ss}}, M) \to K_{\mathrm{ss}} \to P_{\mathrm{ss}} \to M_{\mathrm{ss}} \to 0 \,.$$

Assume that K_{ss} is free. Then its subgroup $\mathrm{Tor}_1^{\mathcal{NT}^*}(\mathcal{NT}_{\mathrm{ss}}, M)$ is free as well. Conversely, if $\mathrm{Tor}_1^{\mathcal{NT}^*}(\mathcal{NT}_{\mathrm{ss}}, M)$ is free, then K_{ss} is an extension of free Abelian groups and thus free. Notice that P_{ss} is free because P is projective. The general case $n \in \mathbb{N}$ follows by induction using an argument based on syzygies as above. This completes the proof of Proposition 1.

5. Free Resolutions for $\mathcal{NT}_{\mathrm{ss}}$

The $\mathcal{NT}^*$-right-module $\mathcal{NT}_{\mathrm{ss}}$ decomposes as a direct sum $\bigoplus_{Y \in \mathbb{LC}(X)^*} S_Y$ of the simple submodules S_Y which are given by $S_Y(Y) \cong \mathbb{Z}$ and $S_Y(Z) = 0$ for $Z \neq Y$. We obtain

$$\mathrm{Tor}_n^{\mathcal{NT}^*}(\mathcal{NT}_{\mathrm{ss}}, M) = \bigoplus_{Y \in \mathbb{LC}(X)^*} \mathrm{Tor}_n^{\mathcal{NT}}(S_Y, M) \,.$$

Our task is then to write down projective resolutions for the $\mathcal{NT}^*$-right-modules S_Y. The first step is easy: we map Q_Y onto S_Y by mapping the class of the identity in $Q_Y(Y)$ to the generator of $S_Y(Y)$. Extended by zero, this yields an epimorphism $Q_Y \twoheadrightarrow S_Y$.

In order to surject onto the kernel of this epimorphism, we use the indecomposable transformations in $\mathcal{NT}^*$ whose range is Y. Denoting these by $\eta_i\colon W_i \to Y$, $1 \leq i \leq n$, we obtain the two step resolution

$$\bigoplus_{i=1}^n Q_{W_i} \xrightarrow{(\,\eta_1\ \eta_2\ \cdots\ \eta_n\,)} Q_Y \twoheadrightarrow S_Y \,.$$

In the notation of [10], the map $\bigoplus_{i=1}^n Q_{W_i} \to Q_Y$ corresponds to a morphism $\phi\colon \mathcal{R}_Y \to \bigoplus_{i=1}^n \mathcal{R}_{W_i}$ of $\mathfrak{I}$-projectives in $\mathfrak{KK}(X)$. If the mapping cone C_ϕ of ϕ is again $\mathfrak{I}$-projective, the distinguished triangle $\Sigma C_\phi \to \mathcal{R}_Y \xrightarrow{\phi} \bigoplus_{i=1}^n \mathcal{R}_{W_i} \to C_\phi$ yields the projective resolution

$$\cdots \to Q_Y \to Q_\phi[1] \to \bigoplus_{i=1}^n Q_{W_i}[1] \to Q_Y[1] \to Q_\phi \to \bigoplus_{i=1}^n Q_{W_i} \to Q_Y \twoheadrightarrow S_Y \ ,$$

where $Q_\phi = \mathrm{FK}(C_\phi)$. We denote periodic resolutions like this by

$$\begin{tikzcd} Q_\phi \arrow[r] & \bigoplus_{i=1}^n Q_{W_i} \arrow[r] & Q_Y \arrow[ll, bend right, "\circ"'] \arrow[r] & S_Y \end{tikzcd} \,.$$

If the mapping cone C_ϕ is not $\mathfrak{I}$-projective, the situation has to be investigated individually. We will see examples of this in §7 and §9. The resolutions we construct in these cases exhibit a certain six-term periodicity as well. However, they begin with a finite number of "non-periodic steps" (one in §7 and two in §9), which can be considered as a symptom of the deficiency of the invariant filtrated K-theory over non-accordion spaces from the homological viewpoint. We remark without proof that the mapping cone of the morphism $\phi\colon \mathcal{R}_Y \to \bigoplus_{i=1}^n \mathcal{R}_{W_i}$ is $\mathfrak{I}$-projective for every $Y \in \mathbb{LC}(X)^*$ if and only if X is a disjoint union of accordion spaces.

6. Tensor Products with Free Right-Modules

Lemma 2. *Let M be an $\mathcal{NT}^*$-left-module. There is an isomorphism $Q_Y \otimes_{\mathcal{NT}^*} M \cong M(Y)$ of $\mathbb{Z}/2$-graded Abelian groups which is natural in $Y \in\in \mathcal{NT}^*$.*

Proof. This is a simple consequence of Yoneda's lemma and the tensor-hom adjunction. ☐

Lemma 3. *Let $\Sigma\mathcal{R}_{(3)} \xrightarrow{\gamma} \mathcal{R}_{(1)} \xrightarrow{\alpha} \mathcal{R}_{(2)} \xrightarrow{\beta_*} \mathcal{R}_{(3)}$ be a distinguished or anti-distinguished triangle in $\mathfrak{KK}(X)$, where $\mathcal{R}_{(i)} = \bigoplus_{j=1}^{m_i} \mathcal{R}_{Y_j^i} \oplus \bigoplus_{k=1}^{n_i} \Sigma\mathcal{R}_{Z_k^i}$ for $1 \leq i \leq 3$, $m_i, n_i \in \mathbb{N}$ and $Y_j^i, Z_k^i \in \mathbb{LC}(X)^*$. Set $Q_{(i)} = \widehat{\mathrm{FK}}(\mathcal{R}_{(i)})$. If $M = \mathrm{FK}(A)$ for some $A \in\in \mathfrak{KK}(X)$, then the induced sequence*

$$
\begin{tikzcd}
Q_{(1)} \otimes_{\mathcal{NT}^*} M \arrow[r, "\alpha^*\otimes \mathrm{id}_M"] & Q_{(2)} \otimes_{\mathcal{NT}^*} M \arrow[r, "\beta^*\otimes \mathrm{id}_M"] & Q_{(3)} \otimes_{\mathcal{NT}^*} M \arrow[d, "\gamma^*\otimes \mathrm{id}_M"] \\
Q_{(3)} \otimes_{\mathcal{NT}^*} M[1] \arrow[u, "\gamma^*\otimes \mathrm{id}_M[1]"] & Q_{(2)} \otimes_{\mathcal{NT}^*} M[1] \arrow[l, "\beta^*\otimes \mathrm{id}_M[1]"] & Q_{(1)} \otimes_{\mathcal{NT}^*} M[1] \arrow[l, "\alpha^*\otimes \mathrm{id}_M[1]"]
\end{tikzcd} \tag{2}
$$

is exact.

Proof. Using the previous lemma and the representability theorem, we naturally identify $Q_{(i)} \otimes_{\mathcal{NT}^*} M \cong \mathrm{KK}(X; \mathcal{R}_{(i)}, A)$. Since, in triangulated categories, distinguished or anti-distinguished triangles induce long exact Hom-sequences, the sequence (2) is thus exact. ☐

7. Proof of Proposition 2

We may restrict to connected T_0-spaces. In [11], a list of isomorphism classes of connected T_0-spaces with three or four points is given. If X is a disjoint union of accordion spaces, then the assertion follows from [2]. The remaining spaces fall into two classes:

(1) all connected non-accordion four-point T_0-spaces except for the pseudocircle;
(2) the pseudocircle (see §7.2).

The spaces in the first class have the following in common: If we fix two of them, say X, Y, then there is an ungraded isomorphism $\Phi\colon \mathcal{NT}^*(X) \to \mathcal{NT}^*(Y)$ between the categories of natural transformations on the respective filtrated K-theories such that the induced equivalence of ungraded module categories

$$\Phi^*\colon \mathfrak{Mod}^{\mathrm{ungr}}\big(\mathcal{NT}^*(Y)\big)_{\mathrm{c}} \to \mathfrak{Mod}^{\mathrm{ungr}}\big(\mathcal{NT}^*(X)\big)_{\mathrm{c}}$$

restricts to a bijective correspondence between exact ungraded $\mathcal{NT}^*(Y)$-modules and exact ungraded $\mathcal{NT}^*(X)$-modules. Moreover, the isomorphism Φ restricts to isomorphisms from $\mathcal{NT}_{\mathrm{ss}}(X)$ onto $\mathcal{NT}_{\mathrm{ss}}(Y)$ and from $\mathcal{NT}_{\mathrm{nil}}(X)$ onto $\mathcal{NT}_{\mathrm{nil}}(Y)$. In particular, the assertion holds for X if and only if it holds for Y.

The above is a consequence of the investigations in [1, 2, 10]; the same kind of relation was found in [2] for the categories of natural transformations associated to accordion spaces with the same number of points. As a consequence, it suffices to verify the assertion for one representative of the first class—we choose Z_3—and for the pseudocircle.

7.1. **Resolutions for the space Z_3.** We refer to [10] for a description of the category $\mathcal{NT}^*(Z_3)$, which in particular implies, that the space Z_3 satisfies the conditions of Proposition 1. Using the extension triangles from [10, (2.5)], the procedure described in §5 yields the following projective resolutions induced by distinguished triangles as in Lemma 3:

$$\begin{tikzcd} Q_1[1] \arrow[r] & Q_4 \arrow[r] & Q_{14} \arrow[ll, bend right, "\circ" marking] \end{tikzcd} \to S_{14}\ , \quad \text{and similarly for } S_{24},\ S_{34};$$

$$\begin{tikzcd} Q_{1234}[1] \arrow[r] & Q_1[1] \oplus Q_2[1] \oplus Q_3[1] \arrow[r] & Q_4 \arrow[ll, bend right, "\circ" marking] \end{tikzcd} \to S_4\ ;$$

$$\begin{tikzcd} Q_{234} \arrow[r] & Q_{1234} \arrow[r] & Q_1 \arrow[ll, bend right, "\circ" marking] \end{tikzcd} \to S_1\ , \quad \text{and similarly for } S_2,\ S_3.$$

Next we will deal with the modules S_{jk4}, where $1 \le j < k \le 3$. We observe that there is a Mayer-Vietoris type exact sequence of the form

$$\begin{tikzcd} Q_4 \arrow[r] & Q_{j4} \oplus Q_{k4} \arrow[r] & Q_{jk4} \arrow[ll, bend right, "\circ" marking] \end{tikzcd}\ . \tag{3}$$

Lemma 4. *The candidate triangle $\Sigma\mathcal{R}_4 \to \mathcal{R}_{jk4} \to \mathcal{R}_{j4} \oplus \mathcal{R}_{k4} \to \mathcal{R}_4$ corresponding to the periodic part of the sequence* (3) *is distinguished or anti-distinguished* (*depending on the choice of signs for the maps in* (3)).

Proof. We give the proof for $j = 1$ and $k = 2$. The other cases follow from cyclicly permuting the indices 1, 2 and 3. We denote the morphism $\mathcal{R}_{124} \to \mathcal{R}_{14} \oplus \mathcal{R}_{24}$ by φ and the corresponding map $Q_{14} \oplus Q_{24} \to Q_{124}$ in (3) by φ^*. It suffices to check that $\widehat{\mathrm{FK}}(\mathrm{Cone}_\varphi)$ and Q_4 correspond, possibly up to a sign, to the same element in $\mathrm{Ext}^1_{\mathcal{NT}^*(Z_3)^{\mathrm{op}}}\big(\ker(\varphi^*), \mathrm{coker}(\varphi^*)[1]\big)$. We have $\mathrm{coker}(\varphi^*) \cong S_{124}$ and an extension $S_{124}[1] \rightarrowtail Q_4 \twoheadrightarrow \ker(\varphi^*)$. Since $\mathrm{Hom}(Q_4, S_{124}[1]) \cong S_{124}(4)[1] = 0$ and $\mathrm{Ext}^1(Q_4, S_{124}[1])$ because Q_4 is projective, the long exact Ext-sequence yields $\mathrm{Ext}^1\big(\ker(\varphi^*), \mathrm{coker}(\varphi^*)[1]\big) \cong \mathrm{Hom}(S_{124}[1], S_{124}[1]) \cong \mathbb{Z}$. Considering the sequence of transformations $3 \xrightarrow{\delta} 124 \xrightarrow{i} 1234 \xrightarrow{r} 3$, it is straight-forward to check that such an extension corresponds to one of the generators $\pm 1 \in \mathbb{Z}$ if and only if its underlying module is exact. This concludes the proof because both $\widehat{\mathrm{FK}}(\mathrm{Cone}_\varphi)$ and Q_4 are exact. □

Hence we obtain the following projective resolutions induced by distinguished or anti-distinguished triangles as in Lemma 3:

$$\begin{tikzcd} Q_4 \arrow[r] & Q_{j4} \oplus Q_{k4} \arrow[r] & Q_{jk4} \arrow[ll, bend right, "\circ" marking] \end{tikzcd} \to S_{jk4}\ .$$

To summarize, by Lemma 3, $\mathrm{Tor}_n^{\mathcal{NT}^*}(S_Y, M) = 0$ for $Y \neq 1234$ and $n \ge 1$.

As we know from [10], the subset 1234 of Z_3 plays an exceptional role. In the notation of [10] (with the direction of the arrows reversed because we are dealing with *right*-modules), the kernel of the homomorphism $Q_{124} \oplus Q_{134} \oplus Q_{234} \xrightarrow{(\, i\ i\ i\,)} Q_{1234}$ is of the form

$$\begin{tikzcd}
 & \mathbb{Z} \arrow[dl] & 0 \arrow[l] \arrow[dl] & & \mathbb{Z}[1] \arrow[dl] & \\
\mathbb{Z}^2 & \mathbb{Z} \arrow[l] & 0 \arrow[ul] \arrow[dl] & 0 \arrow[l] \arrow[ul] \arrow[dl] & \mathbb{Z}[1] \arrow[l] & \mathbb{Z}^2 \arrow[l, "\circ" marking] \arrow[ul, "\circ" marking] \arrow[dl, "\circ" marking] \\
 & \mathbb{Z} \arrow[ul] & 0 \arrow[l] \arrow[ul] & & \mathbb{Z}[1] \arrow[ul] &
\end{tikzcd}\ .$$

It is the image of the module homomorphism

$$Q_{14} \oplus Q_{24} \oplus Q_{34} \xrightarrow{\begin{pmatrix} i & -i & 0 \\ -i & 0 & i \\ 0 & i & -i \end{pmatrix}} Q_{124} \oplus Q_{134} \oplus Q_{234}, \tag{4}$$

the kernel of which, in turn, is of the form

$$\begin{tikzcd}
 & 0 \arrow[dl] & \mathbb{Z}[1] \arrow[l] \arrow[dl] & & \mathbb{Z}[1] \arrow[dl] & \\
\mathbb{Z} & 0 \arrow[l] & \mathbb{Z}[1] \arrow[ul] \arrow[dl] & \mathbb{Z}[1]^3 \arrow[l] \arrow[ul] \arrow[dl] & \mathbb{Z}[1] \arrow[l] & \mathbb{Z} . \arrow[l, "\circ" marking] \arrow[ul, "\circ" marking] \arrow[dl, "\circ" marking] \\
 & 0 \arrow[ul] & \mathbb{Z}[1] \arrow[l] \arrow[ul] & & \mathbb{Z}[1] \arrow[ul] &
\end{tikzcd}$$

A surjection from $Q_4 \oplus Q_{1234}[1]$ onto this module is given by $\begin{pmatrix} i & i & i \\ \delta^{14}_{1234} & 0 & 0 \end{pmatrix}$, where $\delta^{14}_{1234} := \delta^{14}_3 \circ r^3_{1234}$. The kernel of this homomorphism has the form

$$\begin{tikzcd}
 & \mathbb{Z}[1] \arrow[dl] & \mathbb{Z}[1] \arrow[l] \arrow[dl] & & 0 \arrow[dl] & \\
\mathbb{Z}[1] & \mathbb{Z}[1] \arrow[l] & \mathbb{Z}[1] \arrow[ul] \arrow[dl] & 0 \arrow[l] \arrow[ul] \arrow[dl] & 0 \arrow[l] & 0 . \arrow[l, "\circ" marking] \arrow[ul, "\circ" marking] \arrow[dl, "\circ" marking] \\
 & \mathbb{Z}[1] \arrow[ul] & \mathbb{Z}[1] \arrow[l] \arrow[ul] & & 0 \arrow[ul] &
\end{tikzcd}$$

This module is isomorphic to $\mathrm{Syz}_{1234}[1]$, where $\mathrm{Syz}_{1234} := \ker(Q_{1234} \twoheadrightarrow S_{1234})$. Therefore, we end up with the projective resolution

(5)

$$\begin{tikzcd}
Q_4 \oplus Q_{1234}[1] \arrow[r] & Q_{14} \oplus Q_{24} \oplus Q_{34} \arrow[r] & Q_{124} \oplus Q_{134} \oplus Q_{234} \arrow[r] \arrow[ll, bend right, "\circ" marking] & Q_{1234} \arrow[r] & S_{1234} .
\end{tikzcd}$$

The homomorphism from $Q_{124} \oplus Q_{134} \oplus Q_{234}$ to $Q_4 \oplus Q_{1234}[1]$ is given by $\begin{pmatrix} 0 & 0 & -\delta^4_{234} \\ i & i & i \end{pmatrix}$, where $\delta^4_{234} := \delta^4_2 \circ r^2_{234}$.

Lemma 5. *The candidate triangle in $\mathfrak{KK}(X)$ corresponding to the periodic part of the sequence* (5) *is distinguished or anti-distinguished* (*depending on the choice of signs for the maps in* (5)).

Proof. The argument is analogous to the one in the proof of Lemma 4. Again, we consider the group $\mathrm{Ext}^1_{\mathcal{NT}^*(Z_3)^{\mathrm{op}}}\bigl(\ker(\varphi^*), \mathrm{coker}(\varphi^*)[1]\bigr)$ where φ^* now denotes the map (4). We have $\mathrm{coker}(\varphi^*) \cong \mathrm{Syz}_{1234}$ and an extension $Q_4 \rightarrowtail \ker(\varphi^*) \twoheadrightarrow S_{1234}[1]$. Using long exact sequences, we obtain

$$\begin{aligned}
\mathrm{Ext}^1\bigl(\ker(\varphi^*), \mathrm{coker}(\varphi^*)[1]\bigr) &\cong \mathrm{Ext}^1(S_{1234}[1], \mathrm{Syz}_{1234}[1]) \\
&\cong \mathrm{Hom}(S_{1234}[1], S_{1234}[1]) \cong \mathbb{Z}.
\end{aligned}$$

Again, an extension corresponds to a generator if and only if its underlying module is exact. □

By the previous lemma and §6, computing the tensor product of this complex with M and taking homology shows that $\mathrm{Tor}_n^{\mathcal{NT}^*}(\mathcal{NT}_{\mathrm{ss}}, M) = 0$ for $n \geq 2$ and that $\mathrm{Tor}_1^{\mathcal{NT}^*}(\mathcal{NT}_{\mathrm{ss}}, M)$ is equal to $\mathrm{Tor}_1^{\mathcal{NT}^*}(S_{1234}, M)$ and isomorphic to the homology of the complex (1).

Example 1. *For the filtrated* K*-module with projective dimension* 2 *constructed in* [10, §5] *we get* $\mathrm{Tor}_1^{\mathcal{NT}^*}(\mathcal{NT}_{\mathrm{ss}}, M) \cong \mathbb{Z}/k$.

Remark 1. *As explicated in the beginning of this section, the category $\mathcal{NT}^*(S)$ corresponding to the four-point space S defined in the introduction is isomorphic in an appropriate sense to the category $\mathcal{NT}^*(Z_3)$. As has been established in* [1], *the indecomposable morphisms in $\mathcal{NT}^*(S)$ are organised in the diagram*

$$\begin{tikzcd}
 & 12 \arrow[r, "\delta", circ] \arrow[dr, "r"] & 34 \arrow[dr, "i"] & & 2 \arrow[dr, "i"] & \\
123 \arrow[ur, "r"] \arrow[r, "\delta", circ] \arrow[dr, "r"'] & 4 \arrow[ur, "i"] \arrow[dr, "i"'] & 1 \arrow[r, "\delta", circ] & 234 \arrow[ur, "r"] \arrow[r, "i"] \arrow[dr, "r"'] & 1234 \arrow[r, "r"] & 123\,. \\
 & 13 \arrow[r, "\delta", circ] \arrow[ur, "r"'] & 24 \arrow[ur, "i"'] & & 3 \arrow[ur, "i"'] &
\end{tikzcd}$$

In analogy to (1), *we have that $\mathrm{Tor}_1^{\mathcal{NT}^*(S)}(\mathcal{NT}_{\mathrm{ss}}, M)$ is isomorphic to the homology of the complex*

$$(6) \quad M(12)[1] \oplus M(4) \oplus M(13)[1] \xrightarrow{\begin{pmatrix} \delta & -r & 0 \\ -i & 0 & i \\ 0 & r & -\delta \end{pmatrix}} M(34) \oplus M(1)[1] \oplus M(24) \xrightarrow{(\,i\;\delta\;i\,)} M(234)\,,$$

where $M = \mathrm{FK}(A)$ for some separable C^*-algebra A over X.

7.2. **Resolutions for the pseudocircle.** Let $C_2 = \{1, 2, 3, 4\}$ with the partial order defined by $1 < 3$, $1 < 4$, $2 < 3$, $2 < 4$. The topology on C_2 is thus given by $\{\emptyset, 3, 4, 34, 134, 234, 1234\}$. Hence the non-empty, connected, locally closed subsets are

$$\mathbb{LC}(C_2)^* = \{3, 4, 134, 234, 1234, 13, 14, 23, 24, 124, 123, 1, 2\}\,.$$

The partial order on C_2 corresponds to the directed graph

$$\begin{tikzcd}
\overset{4}{\bullet} \arrow[r] \arrow[dr] & \overset{2}{\bullet} \\
\underset{3}{\bullet} \arrow[r] \arrow[ur] & \underset{1\,.}{\bullet}
\end{tikzcd}$$

The space C_2 is the only T_0-space with at most four points with the property that its order complex (see [10, Definition 2.6]) is not contractible; in fact, it is homeomorphic to the circle $\mathbb{S}^1$. Therefore, by the representability theorem [10, §2.1] we find

$$\mathcal{NT}_*(C_2, C_2) \cong \mathrm{KK}_*(X; \mathcal{R}_{C_2}, \mathcal{R}_{C_2}) \cong \mathrm{K}_*\big(\mathcal{R}_{C_2}(C_2)\big) \cong \mathrm{K}^*\left(\mathbb{S}^1\right) \cong \mathbb{Z} \oplus \mathbb{Z}[1]\,,$$

that is, there are non-trivial odd natural transformations $\mathrm{FK}_{C_2} \Rightarrow \mathrm{FK}_{C_2}$. These are generated, for instance, by the composition $C_2 \xrightarrow{r} 1 \xrightarrow{\delta} 3 \xrightarrow{i} C_2$. This follows from the description of the category $\mathcal{NT}^*(C_2)$ below. Note that $\delta_{C_2}^{C_2} \circ \delta_{C_2}^{C_2}$ vanishes because it factors through $r_{13}^1 \circ i_3^{13} = 0$.

Figure 1 displays a set of indecomposable transformations generating the category $\mathcal{NT}^*(C_2)$ determined in [1, §6.3.2], where also a list of relations generating the relations in the category $\mathcal{NT}^*(C_2)$ can be found. From this, it is straightforward to verify that the space C_2 satisfies the conditions of Proposition 1.

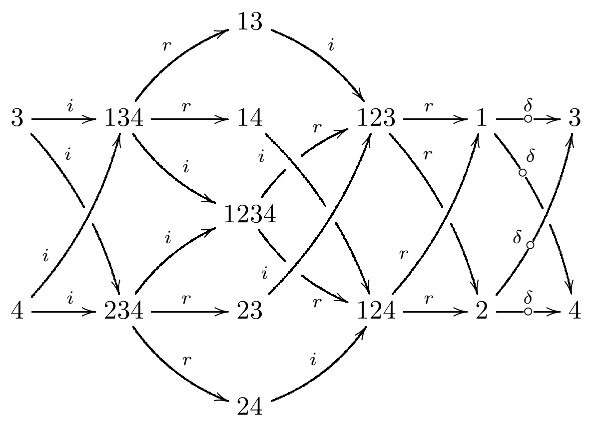

FIGURE 1. Indecomposable natural transformations in $\mathcal{NT}^*(C_2)$

Proceeding as described in §5, we find projective resolutions of the following form (we omit explicit descriptions of the boundary maps):

$$\begin{tikzcd} Q_{123}[1] \arrow[r] & Q_1[1] \oplus Q_2[1] \arrow[r] & Q_3 \arrow[ll, bend right, "\circ" marking] \end{tikzcd} \to S_3\ , \quad \text{and similarly for } S_4;$$

$$\begin{tikzcd} Q_1[1] \arrow[r] & Q_3 \oplus Q_4 \arrow[r] & Q_{134} \arrow[ll, bend right, "\circ" marking] \end{tikzcd} \to S_{134}\ , \quad \text{and similarly for } S_{234};$$

$$\begin{tikzcd} Q_4 \arrow[r] & Q_{134} \arrow[r] & Q_{13} \arrow[ll, bend right, "\circ" marking] \end{tikzcd} \to S_{13}\ , \quad \text{and similarly for } S_{14},\ S_{23},\ S_{24};$$

$$\begin{tikzcd} Q_3 \oplus Q_4 \arrow[r] & Q_{134} \oplus Q_{234} \arrow[r] & Q_{1234} \arrow[l, bend right, "\circ" marking] \end{tikzcd} \to S_{1234}\ ;$$

$$\begin{tikzcd} Q_4 \oplus Q_{123}[1] \arrow[r] & Q_{134} \oplus Q_{234} \arrow[r] & Q_{1234} \oplus Q_{13} \oplus Q_{23} \arrow[ll, bend right, "\circ" marking] \end{tikzcd} \to Q_{123} \to S_{123}\ ,$$

and similarly for S_{124};

$$\begin{tikzcd} Q_{234} \oplus Q_1[1] \arrow[r] & Q_{1234} \oplus Q_{23} \oplus Q_{24} \arrow[r] & Q_{123} \oplus Q_{124} \arrow[ll, bend right, "\circ" marking] \end{tikzcd} \to Q_1 \to S_1\ ,$$

and similarly for S_2. Again, the periodic part of each of these resolutions is induced by an extension triangle, a Mayer-Vietoris triangle as in Lemma 4 or a more exotic (anti-)distinguished triangle as in Lemma 5 (we omit the analogous computation here).

We get $\mathrm{Tor}_1^{\mathcal{NT}^*}(S_Y, M) = 0$ for every $Y \in \mathbb{LC}(C_2)^* \setminus \{123, 124, 1, 2\}$, and $\mathrm{Tor}_n^{\mathcal{NT}^*}(S_Y, M) = 0$ for all $Y \in \mathbb{LC}(C_2)^*$ and $n \geq 2$. Therefore,

$$\mathrm{Tor}_1^{\mathcal{NT}^*}(\mathcal{NT}_{\mathrm{ss}}, M) \cong \bigoplus_{Y \in \{123,124,1,2\}} \mathrm{Tor}_1^{\mathcal{NT}^*}(S_Y, M)\ .$$

The four groups $\mathrm{Tor}_1^{\mathcal{NT}^*}(S_Y, M)$ with $Y \in \{123, 124, 1, 2\}$ can be described explicitly as in §7.1 using the above resolutions. This finishes the proof of Proposition 2.

8. Proof of Proposition 3

We apply the Meyer-Nest machinery to the homological functor $\mathrm{FK} \otimes \mathbb{Q}$ on the triangulated category $\mathfrak{KK}(X) \otimes \mathbb{Q}$. We need to show that every $\mathcal{NT}^* \otimes \mathbb{Q}$ module of the form $M = \mathrm{FK}(A) \otimes \mathbb{Q}$ has a projective resolution of length 1. It is easy to see that analogues of Propositions 1 and 2 hold. In particular, the term $\mathrm{Tor}_2^{\mathcal{NT}^* \otimes \mathbb{Q}}(\mathcal{NT}_{\mathrm{ss}} \otimes \mathbb{Q}, M)$ always vanishes. Here we use that $\mathbb{Q}$ is a flat $\mathbb{Z}$-module, so that tensoring with $\mathbb{Q}$ turns projective $\mathcal{NT}^*$-module resolutions into projective $\mathcal{NT}^* \otimes \mathbb{Q}$-module resolutions. Moreover, the freeness condition for the $\mathbb{Q}$-module $\mathrm{Tor}_1^{\mathcal{NT}^*\otimes\mathbb{Q}}(\mathcal{NT}_{\mathrm{ss}} \otimes \mathbb{Q}, M)$ is empty since $\mathbb{Q}$ is a field.

9. Proof of Proposition 4

The computations to determine the category $\mathcal{NT}^*(Z_4)$ are very similar to those for the category $\mathcal{NT}^*(Z_3)$ which were carried out in [10]. We summarise its structure in Figure 2. The relations in $\mathcal{NT}^*(Z_4)$ are generated by the following:

- the hypercube with vertices $5, 15, 25, \ldots, 12345$ is a commuting diagram;
- the following compositions vanish:

$$1235 \xrightarrow{i} 12345 \xrightarrow{r} 4\ , \quad 1245 \xrightarrow{i} 12345 \xrightarrow{r} 3\ ,$$

$$1345 \xrightarrow{i} 12345 \xrightarrow{r} 2\ , \quad 2345 \xrightarrow{i} 12345 \xrightarrow{r} 1\ ,$$

$$1 \xrightarrow{\delta} 5 \xrightarrow{i} 15\ , \quad 2 \xrightarrow{\delta} 5 \xrightarrow{i} 25\ , \quad 3 \xrightarrow{\delta} 5 \xrightarrow{i} 35\ , \quad 4 \xrightarrow{\delta} 5 \xrightarrow{i} 45\ ;$$

- the sum of the four maps $12345 \to 5$ via 1, 2, 3, and 4 vanishes.

This implies that the space Z_4 satisfies the conditions of Proposition 1.

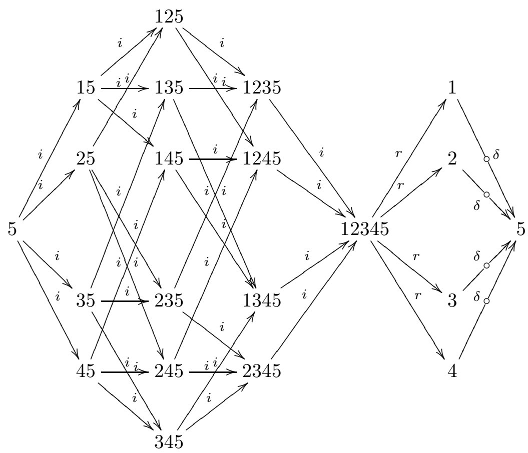

Figure 2. Indecomposable natural transformations in $\mathcal{NT}^*(Z_4)$

In the following, we will define an exact $\mathcal{NT}^*$-left-module M and compute $\mathrm{Tor}_2^{\mathcal{NT}^*}(S_{12345}, M)$. By explicit computation, one finds a projective resolution of the simple $\mathcal{NT}^*$-right-module S_{12345} of the following form (again omitting explicit formulas for the boundary maps):

$$Q_5 \oplus \bigoplus_{1\le i\le 4} Q_{12345\setminus i}[1] \longrightarrow \bigoplus_{1\le l\le 4} Q_{l5} \oplus Q_{12345}[1] \longrightarrow \bigoplus_{1\le j<k\le 4} Q_{jk5}$$

$$\longrightarrow \bigoplus_{1\le i\le 4} Q_{12345\setminus i} \longrightarrow Q_{12345} \longrightarrow S_{12345}.$$

Notice that this sequence is periodic as a cyclic six-term sequence except for the first *two* steps.

Consider the exact $\mathcal{NT}^*$-left-module M defined by the exact sequence

$$0 \to P_{12345} \xrightarrow{\begin{pmatrix} i \\ i \\ i \\ i \end{pmatrix}} \bigoplus_{1\leq i\leq 4} P_{12345\setminus i} \xrightarrow{\begin{pmatrix} i & -i & 0 & 0 \\ -i & 0 & i & 0 \\ 0 & i & -i & 0 \\ i & 0 & 0 & -i \\ 0 & -i & 0 & i \\ 0 & 0 & i & -i \end{pmatrix}} \bigoplus_{1\leq j<k\leq 4} P_{jk5} \twoheadrightarrow M\ . \tag{7}$$

We have $\bigoplus_{1\leq l\leq 4} M(l5) \oplus M(12345)[1] \cong 0 \oplus \mathbb{Z}^3$, $\bigoplus_{1\leq j<k\leq 4} M(jk5) \cong \mathbb{Z}^6$, and $M(5) \oplus \bigoplus_{1\leq i\leq 4} M(12345 \setminus i)[1] \cong \mathbb{Z}[1] \oplus \mathbb{Z}[1]^8$. Since

$$\begin{tikzcd}
\bigoplus\limits_{1\leq l\leq 4} M(l5) \oplus M(12345)[1] \arrow[rr] & & \bigoplus\limits_{1\leq j<k\leq 4} M(jk5) \arrow[d, "\circ"] \\
& & M(5) \oplus \bigoplus\limits_{1\leq i\leq 4} M(12345\setminus i)[1] \arrow[ull]
\end{tikzcd}$$

is exact, a rank argument shows that the map

$$\bigoplus_{1\leq l\leq 4} M(l5) \oplus M(12345)[1] \to \bigoplus_{1\leq j<k\leq 4} M(jk5)$$

is zero. On the other hand, the kernel of the map

$$\bigoplus_{1\leq j<k\leq 4} M(jk5) \xrightarrow{\begin{pmatrix} i & -i & 0 & i & 0 & 0 \\ -i & 0 & i & 0 & -i & 0 \\ 0 & i & -i & 0 & 0 & i \\ 0 & 0 & 0 & -i & i & -i \end{pmatrix}} \bigoplus_{1\leq i\leq 4} M(12345\setminus i)$$

is non-trivial; it consists precisely of the elements in

$$\bigoplus_{1\leq j<k\leq 4} M(jk5) \cong \bigoplus_{1\leq j<k\leq 4} \mathbb{Z}[\mathrm{id}_{jk5}^{jk5}]$$

which are multiples of $([\mathrm{id}_{jk5}^{jk5}])_{1\leq j<k\leq 4}$. This shows $\mathrm{Tor}_2^{\mathcal{NT}^*}(S_{12345}, M) \cong \mathbb{Z}$. Hence, by Proposition 1, the module M has projective dimension at least 2. On the other hand, (7) is a resolution of length 2. Therefore, the projective dimension of M is exactly 2.

Let $k \in \mathbb{N}_{\geq 2}$ and define $M_k = M \otimes_{\mathbb{Z}} \mathbb{Z}/k$. Since $\mathrm{Tor}_2^{\mathcal{NT}^*}(S_{12345}, M_k) \cong \mathbb{Z}/k$ is non-free, Proposition 1 shows that M_k has at least projective dimension 3. On the other hand, if we abbreviate the resolution (7) for M by

$$0 \to P^{(5)} \xrightarrow{\alpha} P^{(4)} \xrightarrow{\beta} P^{(3)} \twoheadrightarrow M\ , \tag{8}$$

a projective resolution of length 3 for M_k is given by

$$0 \to P^{(5)} \xrightarrow{\begin{pmatrix} k \\ \alpha \end{pmatrix}} P^{(5)} \oplus P^{(4)} \xrightarrow{\begin{pmatrix} \alpha & -k \\ 0 & \beta \end{pmatrix}} P^{(4)} \oplus P^{(3)} \xrightarrow{(\beta\ k)} P^{(3)} \twoheadrightarrow M_k\ ,$$

where k denotes multiplication by k.

It remains to show that the modules M and M_k can be realised as the filtrated K-theory of objects in $\mathcal{B}(X)$. It suffices to prove this for the module M since tensoring with the Cuntz algebra $\mathcal{O}_{k+1}$ then yields a separable C^*-algebra with filtrated K-theory M_k by the Künneth Theorem.

The projective resolution (8) can be written as

$$0 \to \mathrm{FK}(P^2) \xrightarrow{\mathrm{FK}(f_2)} \mathrm{FK}(P^1) \xrightarrow{\mathrm{FK}(f_1)} \mathrm{FK}(P^0) \twoheadrightarrow M,$$

because of the equivalence of the category of projective $\mathcal{NT}^*$-modules and the category of $\mathfrak{I}$-projective objects in $\mathfrak{KK}(X)$. Let N be the cokernel of the module map $\mathrm{FK}(f_2)$. Using [10, Theorem 4.11], we obtain an object $A \in\in \mathcal{B}(X)$ with $\mathrm{FK}(A) \cong N$. We thus have a commutative diagram of the form

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{FK}(P^2) \arrow[r, "\mathrm{FK}(f_2)"] & \mathrm{FK}(P^1) \arrow[rr, "\mathrm{FK}(f_1)"] \arrow[dr] & & \mathrm{FK}(P^0) \arrow[r, two heads] & M . \\
& & & \mathrm{FK}(A) \arrow[ur, tail, "\gamma"] & &
\end{tikzcd}$$

Since A belongs to the bootstrap class $\mathcal{B}(X)$ and $\mathrm{FK}(A)$ has a projective resolution of length 1, we can apply the universal coefficient theorem to lift the homomorphism γ to an element $f \in \mathrm{KK}(X; A, P^0)$. Now we can argue as in the proof of [10, Theorem 4.11]: since f is $\mathfrak{I}$-monic, the filtrated K-theory of its mapping cone is isomorphic to $\operatorname{coker}(\gamma) \cong M$. This completes the proof of Proposition 4.

10. Cuntz-Krieger Algebras with Projective Dimension 2

In this section we exhibit a Cuntz-Krieger algebra A which is a tight C^*-algebra over the space Z_3 and for which the odd part of $\mathrm{Tor}_1^{\mathcal{NT}^*(Z_3)}\big(\mathcal{NT}_{\mathrm{ss}}, \mathrm{FK}(A)\big)$—denoted $\mathrm{Tor}_1^{\mathrm{odd}}$ in the following—is not free. By Proposition 2 this C^*-algebra has projective dimension 2 in filtrated K-theory.

In the following we will adhere to the conventions for graph algebras and adjacency matrices from [4]. Let E be the finite graph with vertex set $E^0 = \{v_1, v_2, \ldots, v_8\}$ and edges corresponding to the adjacency matrix

$$\begin{pmatrix} B_4 & 0 & 0 & 0 \\ X_1 & B_1 & 0 & 0 \\ X_2 & 0 & B_2 & 0 \\ X_3 & 0 & 0 & B_3 \end{pmatrix} := \begin{pmatrix} \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} & 0 & 0 & 0 \\ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} & 0 & 0 \\ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & 0 & \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} & 0 \\ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & 0 & 0 & \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} \end{pmatrix} . \tag{9}$$

Since this is a finite graph with no sinks and no sources, the associated graph C^*-algebra $C^*(E)$ is in fact a Cuntz-Krieger algebra (we can replace E with its *edge graph*; see [13, Remark 2.8]). Moreover, the graph E is easily seen to fulfill condition (K) because every vertex is the base of two or more simple cycles. As a consequence, the adjacency matrix of the edge graph of E fulfills condition (II) from [5]. In fact, condition (K) is designed as a generalisation of condition (II): see, for instance, [8].

Applying [13, Theorem 4.9]—and carefully translating between different graph algebra conventions—we find that the ideals of $C^*(E)$ correspond bijectively and in an inclusion-preserving manner to the open subsets of the space Z_3. By [11, Lemma 2.35], we may turn A into a tight C^*-algebra over Z_3 by declaring $A(\{4\}) = I_{\{v_1,v_2\}}$, $A(\{1,4\}) = I_{\{v_1,v_2,v_3,v_4\}}$, $A(\{2,4\}) = I_{\{v_1,v_2,v_5,v_6\}}$ and $A(\{3,4\}) = I_{\{v_1,v_2,v_7,v_8\}}$, where I_S denotes the ideal corresponding to the saturated hereditary subset S.

It is known how to compute the six-term sequence in K-theory for an extension of graph C^*-algebras: see [4]. Using this and Proposition 2, $\mathrm{Tor}_1^{\mathrm{odd}}$ is the homology of the complex

$$\ker(\phi_0) \xrightarrow{\left(\begin{smallmatrix} i & -i & 0 \\ -i & 0 & i \\ 0 & i & -i \end{smallmatrix}\right)} \ker(\phi_1) \xrightarrow{(i\ i\ i)} \ker(\phi_2) , \tag{10}$$

$$\text{where}\quad \phi_0 = \operatorname{diag}\left(\begin{pmatrix} B_4' & X_1^t \\ 0 & B_1' \end{pmatrix}, \begin{pmatrix} B_4' & X_2^t \\ 0 & B_2' \end{pmatrix}, \begin{pmatrix} B_4' & X_3^t \\ 0 & B_3' \end{pmatrix} \right) , \quad \phi_2 = \begin{pmatrix} B_4' & X_1^t & X_2^t & X_3^t \\ 0 & B_1' & 0 & 0 \\ 0 & 0 & B_2' & 0 \\ 0 & 0 & 0 & B_3' \end{pmatrix} ,$$

$$\phi_1 = \operatorname{diag}\left(\begin{pmatrix} B_4' & X_1^t & X_2^t \\ 0 & B_1' & 0 \\ 0 & 0 & B_2' \end{pmatrix}, \begin{pmatrix} B_4' & X_1^t & X_3^t \\ 0 & B_1' & 0 \\ 0 & 0 & B_3' \end{pmatrix}, \begin{pmatrix} B_4' & X_2^t & X_3^t \\ 0 & B_2' & 0 \\ 0 & 0 & B_3' \end{pmatrix} \right) ,$$

and $B_4' = B_4^t - \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 2 & 2 \\ 2 & 2 \end{smallmatrix}\right)$ and $B_j' = B_j^t - \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 2 & 1 \\ 2 & 1 \end{smallmatrix}\right)$ for $1 \leq j \leq 3$. We obtain a commutative diagram

$$\begin{tikzcd}
\ker(\phi_0) \arrow[r, tail] \arrow[d, "f_K"] & (\mathbb{Z}^{\oplus 2})^{\oplus(2\cdot 3)} \arrow[r, two heads, "\phi_0"] \arrow[d, "f"] & \operatorname{im}(\phi_0) \arrow[d, "f_I"] \\
\ker(\phi_1) \arrow[r, tail] \arrow[d, "g_K"] & (\mathbb{Z}^{\oplus 2})^{\oplus(3\cdot 3)} \arrow[r, two heads, "\phi_1"] \arrow[d, "g"] & \operatorname{im}(\phi_1) \arrow[d, "g_I"] \\
\ker(\phi_2) \arrow[r, tail] & (\mathbb{Z}^{\oplus 2})^{\oplus(4\cdot 1)} \arrow[r, two heads, "\phi_2"] & \operatorname{im}(\phi_2) ,
\end{tikzcd} \tag{11}$$

where f and g have the block forms

$$f = \begin{pmatrix} \mathrm{id} & 0 & -\mathrm{id} & 0 & 0 & 0 \\ 0 & \mathrm{id} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{id} & 0 & 0 \\ -\mathrm{id} & 0 & 0 & 0 & \mathrm{id} & 0 \\ 0 & -\mathrm{id} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathrm{id} \\ 0 & 0 & \mathrm{id} & 0 & -\mathrm{id} & 0 \\ 0 & 0 & 0 & \mathrm{id} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mathrm{id} \end{pmatrix} , \qquad g = \begin{pmatrix} \mathrm{id} & 0 & 0 & \mathrm{id} & 0 & 0 & \mathrm{id} & 0 & 0 \\ 0 & \mathrm{id} & 0 & 0 & \mathrm{id} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{id} & 0 & 0 & 0 & 0 & \mathrm{id} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathrm{id} & 0 & 0 & \mathrm{id} \end{pmatrix} ,$$

and $f_K := f|_{\ker(\phi_0)}$, $f_I := f|_{\operatorname{im}(\phi_0)}$, $g_K := g|_{\ker(\phi_1)}$, $g_I := g|_{\operatorname{im}(\phi_1)}$. Notice that f and g are defined in a way such that the restrictions $f|_{\ker(\phi_0)}$ and $g|_{\ker(\phi_1)}$ are exactly the maps from (10) in the identification made above.

We abbreviate the above short exact sequence of cochain complexes (11) as $K_\bullet \rightarrowtail Z_\bullet \twoheadrightarrow I_\bullet$. The part $\mathrm{H}^0(Z_\bullet) \to \mathrm{H}^0(I_\bullet) \to \mathrm{H}^1(K_\bullet) \to \mathrm{H}^1(Z_\bullet)$ in the corresponding long exact homology sequence can be identified with

$$\ker(f) \xrightarrow{\phi_0} \ker(f_I) \to \frac{\ker(g_K)}{\operatorname{im}(f_K)} \to 0 .$$

Hence

$$\mathrm{Tor}_1^{\mathrm{odd}} \cong \frac{\ker(g_K)}{\operatorname{im}(f_K)} \cong \frac{\ker(f_I)}{\phi_0\big(\ker(f)\big)} \cong \frac{\ker(f) \cap \operatorname{im}(\phi_0)}{\phi_0\big(\ker(f)\big)} .$$

We have $\ker(f) = \{(v, 0, v, 0, v, 0) \mid v \in \mathbb{Z}^2\} \subset (\mathbb{Z}^{\oplus 2})^{\oplus(2\cdot 3)}$.

From the concrete form (9) of the adjacency matrix, we find that $\ker(f) \cap \operatorname{im}(\phi_0)$ is the free cyclic group generated by $(1, 1, 0, 0, 1, 1, 0, 0, 1, 1, 0, 0)$, while $\phi_0\big(\ker(f)\big)$ is the subgroup generated by $(2, 2, 0, 0, 2, 2, 0, 0, 2, 2, 0, 0)$. Hence $\mathrm{Tor}_1^{\mathrm{odd}} \cong \mathbb{Z}/2$ is not free.

Now we briefly indicate how to construct a similar counterexample for the space S. Consider the integer matrix

$$\begin{pmatrix} B_4 & 0 & 0 & 0 \\ X_{43} & B_3 & 0 & 0 \\ X_{42} & 0 & B_2 & 0 \\ X_{41} & X_{31} & X_{21} & B_1 \end{pmatrix} := \begin{pmatrix} (3) & 0 & 0 & 0 \\ (2) & (3) & 0 & 0 \\ (2) & 0 & (3) & 0 \\ \begin{pmatrix} 2 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \end{pmatrix} .$$

The corresponding graph F fulfills condition (K) and has no sources or sinks. The associated graph C^*-algebra $C^*(F)$ is therefore a Cuntz-Krieger algebra satisfying condition (II). It is easily read from the block structure of the edge matrix that the primitive ideal space of $C^*(F)$ is homeomorphic to S. We are going to compute the even part of $\mathrm{Tor}_1^{\mathcal{NT}^*(S)}\big(\mathcal{NT}_{\mathrm{ss}}, \mathrm{FK}(C^*(F))\big)$. Since the nice computation methods from the previous example do not carry over, we carry out a more ad hoc calculation.

By Remark 1, the even part of our Tor-term is isomorphic to the homology of the complex

$$\begin{tikzcd}
\ker\begin{pmatrix} B_2' & X_{21}^t \\ 0 & B_1' \end{pmatrix} \arrow[r, "\begin{pmatrix} X_{42}^t & X_{41}^t \\ 0 & X_{31}^t \end{pmatrix}", "\circ"'] \arrow[dr, "-r"' near start] & \operatorname{coker}\begin{pmatrix} B_4' & X_{43}^t \\ 0 & B_3' \end{pmatrix} \arrow[dr, "i"] & \\
\operatorname{coker}(B_4') \arrow[ur, "-i" near start] \arrow[dr, "i"' near start] & \ker(B_1') \arrow[r, "\begin{pmatrix} X_{41}^t \\ X_{31}^t \\ X_{21}^t \end{pmatrix}", "\circ"'] & \operatorname{coker}\begin{pmatrix} B_4' & X_{43}^t & X_{42}^t \\ 0 & B_3' & 0 \\ 0 & 0 & B_2' \end{pmatrix}, \\
\ker\begin{pmatrix} B_3' & X_{31}^t \\ 0 & B_1' \end{pmatrix} \arrow[r, "\circ", "-\begin{pmatrix} X_{43}^t & X_{41}^t \\ 0 & X_{21}^t \end{pmatrix}"'] \arrow[ur, "r" near start] & \operatorname{coker}\begin{pmatrix} B_4' & X_{42}^t \\ 0 & B_2' \end{pmatrix} \arrow[ur, "i"] &
\end{tikzcd}$$

where column-wise direct sums are taken. Here $B_1' = B_1^t - \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ and $B_j' = B_j^t - \big(1\big) = \big(2\big)$ for $2 \le j \le 4$. This complex can be identified with

$$\mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z} \xrightarrow{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -2 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}} (\mathbb{Z}/2)^2 \oplus \mathbb{Z} \oplus (\mathbb{Z}/2)^2 \xrightarrow{\begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}} (\mathbb{Z}/2)^3 ,$$

the homology of which is isomorphic to $\mathbb{Z}/2$; a generator is given by the class of $(0, 1, 1, 0, 1) \in (\mathbb{Z}/2)^2 \oplus \mathbb{Z} \oplus (\mathbb{Z}/2)^2$. This concludes the proof of Proposition 5.

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THE K-THEORETICAL RANGE OF CUNTZ–KRIEGER ALGEBRAS

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Abstract. We augment Restorff's classification of purely infinite Cuntz–Krieger algebras by describing the range of his invariant on purely infinite Cuntz–Krieger algebras. We also describe its range on purely infinite graph C^*-algebras with finitely many ideals, and provide 'unital' range results for purely infinite Cuntz–Krieger algebras and unital purely infinite graph C^*-algebras.

1. Introduction

Cuntz–Krieger algebras form a class of C^*-algebras closely related to symbolic dynamics [7, 8]. Based on this relationship, classification results for purely infinite Cuntz–Krieger algebras by K-theoretical invariants have been established by Mikael Rørdam in the simple case [16] and by Gunnar Restorff in the case of finitely many ideals [15].

For simple Cuntz–Krieger algebras, the K_0-group suffices for classification (because the K_1-group can be identified with the free part of the K_0-group). Moreover, it is known that every finitely generated abelian group arises as the K_0-group of some simple Cuntz–Krieger algebra [17].

The invariant in Restorff's classification theorem for non-simple purely infinite Cuntz–Krieger algebras is called *reduced filtered* K-*theory*; we denote it by $\mathrm{FK}_{\mathcal{R}}$. Being an almost precise analogue of the K-web of Boyle and Huang [3] in the world of C^*-algebras, it comprises the K_0-groups of certain distinguished ideals and the K_1-groups of all simple subquotients, along with the action of certain natural maps.

The first aim of this article is to clarify the definition of the target category of reduced filtered K-theory. We define a certain pre-additive category $\mathcal{R}$ such that $\mathrm{FK}_{\mathcal{R}}$ becomes a functor to $\mathcal{R}$-modules in a natural way. Our second aim is then to determine the class of $\mathcal{R}$-modules that arise (up to isomorphism) as the reduced filtered K-theory of some (tight, purely infinite) Cuntz–Krieger algebra. This involves a natural exactness condition, as well as some conditions that translate well-known K-theoretical properties of purely infinite Cuntz–Krieger algebras.

A Cuntz–Krieger algebra is purely infinite if and only if it has finitely many ideals, and if and only if it has real rank zero [11]. For a C^*-algebra with real rank zero, the exponential map in the K-theoretical six-term exact sequence for every inclusion of subquotients vanishes [5]. This fact is crucial to our definitions and results, in this article and in the companion article [1].

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Our work is based on the article [10] of Eilers, Katsura, Tomforde and West who characterized the six-term exact sequences in K-theory of Cuntz–Krieger algebras with a unique non-trivial proper ideal. We obtain our result by a careful inductive application of the result in [10].

Combining our range result for purely infinite Cuntz–Krieger algebras with Restorff's classification theorem, we obtain an explicit natural description of the set of stable isomorphism classes of purely infinite Cuntz–Krieger algebras, completing the picture in a way previously known only in the simple case and the one-ideal case.

Anticipating potential future classification results generalizing Restorff's theorem, we also provide a range result for purely infinite graph C^*-algebras with finitely many ideals. Cuntz–Krieger algebras may be viewed as a specific type of graph C^*-algebras, namely those arising from finite graphs with no sources [2]. Indeed, our range result is established by graph C^*-algebraic methods. All purely infinite graph C^*-algebras have real rank zero [11], so that the abovementioned K-theoretical particularities that make our approach work are still present in this more general setting.

Finally, we equip reduced filtered K-theory with a unit class and establish corresponding range results for purely infinite Cuntz–Krieger algebras and unital purely infinite graph C^*-algebras. In [1], this is used to give an "external characterization" of purely infinite Cuntz–Krieger algebras under some conditions on the ideal structure.

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2. Preliminaries

We follow the notation and definition for graph C^*-algebras of Iain Raeburn's monograph [14]; this is also our reference for basic facts about graph C^*-algebras. All graphs are assumed to be countable and to satisfy Condition (K), hence all considered graph C^*-algebras are separable and of real rank zero. In this article, matrices act from the right and the composite of maps $A \xrightarrow{f} B \xrightarrow{g} C$ is denoted by fg. The category of abelian groups is denoted by $\mathfrak{Ab}$, the category of $\mathbb{Z}/2$-graded abelian groups by $\mathfrak{Ab}^{\mathbb{Z}/2}$. We let $\mathbb{Z}_+$ denote the set of non-negative integers. When S is a set, we use the symbol M_S to indicate the set of square matrices whose rows and columns are indexed by elements in S.

2.1. **Finite spaces.** Throughout the article, let X be a finite T_0-space, that is, a finite topological space, in which no two different points have the same open neighbourhood filter. For a subset Y of X, we let $\overline{Y}$ denote the closure of Y in X, and we let $\overline{\partial}Y$ denote the boundary $\overline{Y} \setminus Y$ of Y. Since X is a finite space, there exists a smallest open subset $\widetilde{Y}$ of X containing Y. We let $\widetilde{\partial}Y$ denote the set $\widetilde{Y} \setminus Y$.

For $x, y \in X$ we write $x \leq y$ when $\overline{\{x\}} \subseteq \overline{\{y\}}$, and $x < y$ when $x \leq y$ and $x \neq y$. We write $y \to x$ when $x < y$ and no $z \in X$ satisfies $x < z < y$. The following lemma is straightforward to verify.

Lemma 2.1. *For an element $x \in X$, the following hold:*

(1) *An element $y \in X$ satisfies $y \to x$ if and only if y is a closed point of $\widetilde{\partial}\{x\}$.*
(2) *We have $\widetilde{\partial}\{x\} = \bigcup_{y \to x} \widetilde{\{y\}}$, and consequently $\widetilde{\partial}\{x\}$ is open.*
(3) *An element $y \in X$ satisfies $x \leq y$ if and only if there exists a finite sequence $(z_k)_{k=1}^n$ in X such that $z_{k+1} \to z_k$ for $k = 1, \ldots, n-1$ where $z_1 = x$, $z_n = y$.*

We call a sequence $(z_k)_{k=1}^n$ as in Lemma 2.1(3) a *path* from y to x. We denote by $\mathrm{Path}(y, x)$ the set of paths from y to x. Thus Lemma 2.1(3) can be rephrased as follows: two points $x, y \in X$ satisfy $x \leq y$ if and only if there exists a path from y to x. Such a path is not unique in general. Two points $x, y \in X$ satisfy $y \to x$ if and only if (x, y) is a path from y to x; in this case, there are no other paths from y to x.

2.2. **C^*-algebras over finite spaces.** Recall from [13], that a *C^*-algebra A over X* is a C^*-algebra A equipped with a continuous map $\mathrm{Prim}(A) \to X$ or, equivalently, an infima- and suprema-preserving map $\mathbb{O}(X) \to \mathbb{I}(A), U \mapsto A(U)$ mapping open subsets in X to (closed, two-sided) ideals in A (in particular, one has $A(\emptyset) = 0$ and $A(X) = A$). The C^*-algebra A is called *tight* over X if this map is a lattice isomorphism. A *-homomorphism $\varphi\colon A \to B$ for C^*-algebras A and B over X is called *X-equivariant* if $\varphi\bigl(A(U)\bigr) \subseteq B(U)$ for all $U \in \mathbb{O}(X)$. The category of C^*-algebras over X and X-equivariant *-homomorphisms is denoted by $\mathfrak{C}^*\mathfrak{alg}(X)$.

Let $\mathbb{LC}(X)$ denote the set of locally closed subsets of X, that is, subsets of the form $U \setminus V$ with U and V open subsets of X satisfying $V \subseteq U$. For $Y \in \mathbb{LC}(X)$, and $U, V \in \mathbb{O}(X)$ satisfying that $Y = U \setminus V$ and $U \supseteq V$, we define $A(Y)$ as $A(Y) = A(U)/A(V)$, which up to natural isomorphism is independent of the choice of U and V (see [13, Lemma 2.15]). For a C^*-algebra A over X, the $\mathbb{Z}/2$-graded abelian group $\mathrm{FK}_Y^*(A)$ is defined as $\mathrm{K}_*\bigl(A(Y)\bigr)$ for all $Y \in \mathbb{LC}(X)$. Thus FK_Y^* is a functor from $\mathfrak{C}^*\mathfrak{alg}(X)$ to the category $\mathfrak{Ab}^{\mathbb{Z}/2}$ of $\mathbb{Z}/2$-graded abelian groups; compare [12, §2].

Definition 2.2. Let $Y \in \mathbb{LC}(X)$, $U \subseteq Y$ be open and set $C = Y \setminus U$. A pair (U, C) obtained in this way is called a *boundary pair*. The natural transformations occuring in the six-term exact sequence in K-theory for the distinguished subquotient inclusion associated to $U \subseteq Y$ are denoted by i_U^Y, r_Y^C and δ_C^Y:

$$\begin{tikzcd} \mathrm{FK}_U^* \arrow[rr, "i_U^Y"] & & \mathrm{FK}_Y^* . \arrow[dl, "r_Y^C"] \\ & \mathrm{FK}_C^* \arrow[ul, "\circ", "\delta_C^U"'] & \end{tikzcd}$$

Lemma 2.3. *Let (U, C) be a boundary pair and let $V \subseteq U$ be an open subset. The following relations hold:*

(1) $\delta_C^U i_U^Y = 0$;
(2) $i_V^U i_U^Y = i_V^Y$.

Proof. The first statement follows from the exactness of the six-term sequence in K-theory. The second statement already holds for the ideal inclusions inducing the relevant maps on K-theory. □

3. Reduced filtered K-theory

In this section we introduce the functor $\mathrm{FK}_{\mathcal{R}}$ which (for real-rank-zero C^*-algebras) is equivalent to the reduced filtered K-theory defined by Gunnar Restorff in [15].

Definition 3.1. Let $\mathcal{R}$ denote the universal pre-additive category generated by objects $x_1, \widetilde{\partial}x_0, \widetilde{x}_0$ for all $x \in X$ and morphisms $\delta_{x_1}^{\widetilde{\partial}x_0}$ and $i_{\widetilde{\partial}x_0}^{\widetilde{x}_0}$ for all $x \in X$, and $i_{\widetilde{y}_0}^{\widetilde{\partial}x_0}$ when $y \to x$, subject to the relations

$$\delta_{x_1}^{\widetilde{\partial}x_0} i_{\widetilde{\partial}x_0}^{\widetilde{x}_0} = 0 \tag{3.2}$$

$$i_p i_{\widetilde{y(p)}_0}^{\widetilde{\partial}x_0} = i_q i_{\widetilde{y(q)}_0}^{\widetilde{\partial}x_0} \tag{3.3}$$

for all $x \in X$, all $y \in X$ satisfying $y > x$, and all paths $p, q \in \mathrm{Path}(y, x)$, where for a path $p = (z_k)_{k=1}^n$ in $\mathrm{Path}(y, x)$, we define $y(p) = z_2$, and

$$i_p = i_{\widetilde{z_n}_0}^{\widetilde{\partial}z_{n-1_0}} i_{\widetilde{\partial}z_{n-1_0}}^{\widetilde{z_{n-1}}_0} \cdots i_{\widetilde{z_3}_0}^{\widetilde{\partial}z_{2_0}} i_{\widetilde{\partial}z_{2_0}}^{\widetilde{z_2}_0}.$$

Here subscripts indicate domains of morphisms and superscripts indicate codomains.

Definition 3.4 (*Reduced filtered* K*-theory*)**.** The functor

$$\mathrm{FK}_{\mathcal{R}} \colon \mathfrak{C}^*\mathfrak{alg}(X) \to \mathfrak{Mod}(\mathcal{R})$$

is defined as follows: For a C^*-algebra A over X and $x \in X$, we define

$$\begin{aligned} \mathrm{FK}_{\mathcal{R}}(A)(x_1) &= \mathrm{FK}^1_{\{x\}}(A) \\ \mathrm{FK}_{\mathcal{R}}(A)(\widetilde{\partial}x_0) &= \mathrm{FK}^0_{\widetilde{\partial}\{x\}}(A) \\ \mathrm{FK}_{\mathcal{R}}(A)(\widetilde{x}_0) &= \mathrm{FK}^0_{\widetilde{\{x\}}}(A) \end{aligned}$$

using the notation from §2.1, and for a morphism η in $\mathcal{R}$, we set $\mathrm{FK}_{\mathcal{R}}(A)(\eta)$ to be the corresponding map constructed in Definition 2.2. On an X-equivariant *-homomorphism, $\mathrm{FK}_{\mathcal{R}}$ acts in the obvious way dictated by its entry functors FK_Y. It follows from Lemma 2.3 that the functor $\mathrm{FK}_{\mathcal{R}}$ indeed takes values in $\mathcal{R}$-modules.

Remark 3.5. We would like to make the reader aware of the following slight subtlety. It may happen that $\widetilde{\partial}\{x\} = \widetilde{\{y\}}$ for two points $x, y \in X$. But, if M is an exact $\mathcal{R}$-module in the sense of the definition below, then the map $i_{\widetilde{y}_0}^{\widetilde{\partial}x_0} \colon M(\widetilde{y}_0) \to M(\widetilde{\partial}x_0)$ is an isomorphism. More generally, if $\widetilde{\partial}\{x\}$ decomposes as a disjoint union $\bigsqcup_{i=1}^n \widetilde{\{y_i\}}$, then there is an isomorphism $\bigoplus_{i=1}^n M(\widetilde{y_i}_0) \to M(\widetilde{\partial}x_0)$.

Definition 3.6. For an element x in X, let $\mathrm{DP}(x)$ denote the set of pairs of distinct paths (p, q) in X to x and from some common element which is denoted by $s(p, q)$. An $\mathcal{R}$-module M is called *exact* if the sequences

$$M(x_1) \xrightarrow{\delta_{x_1}^{\widetilde{\partial}x_0}} M(\widetilde{\partial}x_0) \xrightarrow{i_{\widetilde{\partial}x_0}^{\widetilde{x}_0}} M(\widetilde{x}_0) \tag{3.7}$$

$$\bigoplus_{(p,q)\in \mathrm{DP}(x)} M(\widetilde{s(p,q)}_0) \xrightarrow{(i_p - i_q)} \bigoplus_{y \to x} M(\widetilde{y}_0) \xrightarrow{(i_{\widetilde{y}_0}^{\widetilde{\partial}x_0})} M(\widetilde{\partial}x_0) \longrightarrow 0 \tag{3.8}$$

are exact for all $x \in X$.

Lemma 3.9. *Let A be a C^*-algebra over X with real rank zero. Let Y be an open subset of X and let $(U_i)_{i\in I}$ be an open covering of Y satisfying $U_i \subseteq Y$ for all $i \in I$. Then the following sequence is exact:*

$$\bigoplus_{i,j\in I} \mathrm{FK}^0_{U_i\cap U_j}(A) \xrightarrow{(i^{U_i}_{U_i\cap U_j} - i^{U_j}_{U_i\cap U_j})} \bigoplus_{i\in I} \mathrm{FK}^0_{U_i}(A) \xrightarrow{(i^{Y}_{U_i})} \mathrm{FK}^0_Y(A) \longrightarrow 0.$$

Proof. Using an inductive argument as in [4, Proposition 1.3], we can reduce to the case that I has only two elements. In this case, exactness follows from a straightforward diagram chase using the exact six-term sequences of the involved ideal inclusions. Here we use that the exponential map $\mathrm{FK}^0_{V\setminus U}(A) \to \mathrm{FK}^1_U(A)$ vanishes for every closed subset U of a locally closed subset V of X if A has real rank zero [5, Theorem 3.14]. □

Corollary 3.10. *Let A be a C^*-algebra over X with real rank zero. Then* $\mathrm{FK}_{\mathcal{R}}(A)$ *is an exact $\mathcal{R}$-module.*

Proof. We verify the exactness of the desired sequences in $\mathrm{FK}_{\mathcal{R}}(A)$. The sequence (3.7) is exact since it is part of the six-term sequence associated to the open inclusion $\widetilde{\partial}\{x\} \subseteq \widetilde{\{x\}}$. To prove exactness of the sequence (3.8), we apply the previous lemma to the covering $(\widetilde{\{y\}})_{y\to x}$ of $Y = \widetilde{\partial}\{x\}$ and get the exact sequence

$$\bigoplus_{y\to x,\, y'\to x} \mathrm{FK}^0_{\widetilde{\{y\}}\cap\widetilde{\{y'\}}}(A) \xrightarrow{\left(i^{\widetilde{\{y\}}}_{\widetilde{\{y\}}\cap\widetilde{\{y'\}}} - i^{\widetilde{\{y'\}}}_{\widetilde{\{y\}}\cap\widetilde{\{y'\}}}\right)} \bigoplus_{y\to x} \mathrm{FK}^0_{\widetilde{\{y\}}}(A) \xrightarrow{\left(i^{\widetilde{\partial}\{x\}}_{\widetilde{\{y\}}}\right)} \mathrm{FK}^0_{\widetilde{\partial}\{x\}}(A) \longrightarrow 0.$$

Another application of the previous lemma shows that $\bigoplus_{(p,q)\in \mathrm{DP}(x)} \mathrm{FK}^0_{\widetilde{s(p,q)}}(A)$ surjects onto $\bigoplus_{y\to x,\, y'\to x} \mathrm{FK}^0_{\widetilde{\{y\}}\cap\widetilde{\{y'\}}}(A)$ in a way making the obvious triangle commute. This establishes the exact sequence (3.8). □

4. Range of reduced filtered K-theory

In this section, we determine the range of reduced filtered K-theory with respect to the class purely infinite graph C^*-algebras and, by specifying appropriate additional conditions, on the subclass of purely infinite Cuntz–Krieger algebras. First, we recall relevant definitions and properties of graph C^*-algebras, and explain how one can determine, for a graph E, whether the graph C^*-algebra $C^*(E)$ can be regarded as a (tight) C^*-algebra over a given finite space X. We also introduce a formula from [6] for calculating reduced filtered K-theory of a graph C^*-algebra using the adjacency matrix of its defining graph. Finally, Proposition 4.7 in conjunction with Theorem 4.8 consitutes the desired range-of-invariant result.

Definition 4.1. Let $E = (E^0, E^1, s, r)$ be a countable directed graph. The *graph algebra* $C^*(E)$ is defined as the universal C^*-algebra generated by a set of mutually orthogonal projections $\{p_v \mid v \in E^0\}$ and a set $\{s_e \mid e \in E^1\}$ of partial isometries satisfying the relations

- $s_e^* s_f = 0$ if $e, f \in E^1$ and $e \neq f$,

- $s_e^* s_e = p_{s(e)}$ for all $e \in E^1$,
- $s_e s_e^* \leq p_{r(e)}$ for all $e \in E^1$, and,
- $p_v = \sum_{e \in r^{-1}(v)} s_e s_e^*$ for all $v \in E^0$ with $0 < |r^{-1}(v)| < \infty$.

If (and only if) E is finite and has no sources, then $C^*(E)$ is a Cuntz–Krieger algebra, see [2].

Definition 4.2. Let E be a directed graph. An edge $e \in E^1$ in E is called a *cycle* if $s(e) = r(e)$. A vertex $v \in E^0$ in E is called *regular* if $r^{-1}(v)$ is finite and nonempty. A vertex $v \in E^0$ in E is called a *breaking vertex* with respect to the saturated hereditary subset H if $s(r^{-1}(v)) \cap H$ is infinite and $s(r^{-1}(v)) \setminus H$ is finite and nonempty.

If all vertices in E support two cycles, then $C^*(E)$ is purely infinite, see [11, Theorem 2.3].

Definition 4.3. Let E be a directed graph. For vertices v, w in E, we write $v \geq w$ if there is a path in E from v to w.

Let H be a subset of E^0. The subset H is called *saturated* if $s(r^{-1}(v)) \subseteq H$ implies $v \in H$ for all regular vertices v in E. When H is saturated, we let I_H denote the ideal in $C^*(E)$ generated by $\{p_v \mid v \in H\}$.

A subset H of E^0 is called *hereditary* if for all $w \in H$ and $v \in E^0$, $v \geq w$ implies $v \in H$.

Definition 4.4. Let E be a directed graph. We say that E satisfies *Condition* (K) if for all edges $v \in E^0$ in E, either there is no path of positive length in E from v to v or there are at least two distinct paths of positive length in E from v to v. We call E *row-finite* when $r^{-1}(v)$ is finite for all $v \in E^0$.

When E satisfies Condition (K) and no saturated hereditary subsets in E^0 have breaking vertices, then the map $H \mapsto I_H$ defines a lattice isomorphism between the saturated hereditary subsets in E^0 and the ideals in $C^*(E)$, see [9, Theorem 3.5].

4.1. **Calculating reduced filtered K-theory of a graph C^*-algebra.** Let E be a countable graph and assume that all vertices in E are regular and support at least two cycles. Then E satisfies Condition (K) and has no breaking vertices, so since all subsets of E^0 are saturated, the map $H \mapsto I_H$ defines a lattice isomorphism from the hereditary subsets of E^0 to the ideals of $C^*(E)$. Given a map $\psi\colon E^0 \to X$ satisfying $\psi\big(s(e)\big) \geq \psi\big(r(e)\big)$ for all $e \in E^1$, we may therefore define a structure on $C^*(E)$ as a C^*-algebra over X by $U \mapsto I_{\psi^{-1}(U)}$ for $U \in \mathbb{O}(X)$.

Assume that such a map ψ is given, that is, that $C^*(E)$ is a C^*-algebra over X. Define for each subset $F \subseteq X$ a matrix $D_F \in M_{\psi^{-1}(F)}(\mathbb{Z}_+)$ as $D_F = A_F - 1$, where $A_F(v, w)$ is defined for $v, w \in \psi^{-1}(F)$ by

$$A_F(v, w) = \#\{e \in E^1 \mid r(e) = v, s(e) = w\},$$

the number of edges in E from w to v; here 1 denotes the identity matrix. For subsets $S_1, S_2 \subseteq F$, we let $D_F|_{S_1}^{S_2}$ denote the $S_1 \times S_2$ matrix given by $D_F|_{S_1}^{S_2}(s_1, s_2) = D_F(s_1, s_2)$ for all $s_1 \in S_1$ and $s_2 \in S_2$.

Note that a given map $\psi\colon E^0 \to X$ turns $C^*(E)$ into a C^*-algebra over X if and only if $D_X|_{\psi^{-1}(y)}^{\psi^{-1}(z)}$ vanishes when $y \not\leq z$. And if furthermore $D_X|_{\psi^{-1}(y)}^{\psi^{-1}(z)}$ is non-zero whenever $y < z$, then $C^*(E)$ is tight over X (this condition for tightness is sufficient but not necessary).

Let a map $\psi\colon E^0 \to X$ satisfying $\psi\big(s(e)\big) \geq \psi\big(r(e)\big)$ for all $e \in E^1$ be given. Then $\mathrm{FK}_{\mathcal{R}}\big(C^*(E)\big)$ can be computed in the following way. Let $Y \in \mathbb{LC}(X)$ and $U \in \mathbb{O}(Y)$ be given, and define $C = Y \setminus U$. Then by [6], the six-term exact sequence induced by $C^*(E)(U) \hookrightarrow C^*(E)(Y) \twoheadrightarrow C^*(E)(C)$ is naturally isomorphic to the sequence

$$\begin{tikzcd} \operatorname{coker} D_U \arrow[r] & \operatorname{coker} D_Y \arrow[r] & \operatorname{coker} D_C \arrow[d, "0"] \\ \ker D_C \arrow[u, "{\left[D_Y|^{\psi^{-1}(U)}_{\psi^{-1}(C)}\right]}"] & \ker D_Y \arrow[l] & \ker D_U \arrow[l] \end{tikzcd} \tag{4.5}$$

induced, via the Snake Lemma, by the commuting diagram

$$\begin{tikzcd} \mathbb{Z}^{\psi^{-1}(U)} \arrow[r] \arrow[d, "D_U"] & \mathbb{Z}^{\psi^{-1}(Y)} \arrow[r] \arrow[d, "D_Y"] & \mathbb{Z}^{\psi^{-1}(C)} \arrow[d, "D_C"] \\ \mathbb{Z}^{\psi^{-1}(U)} \arrow[r] & \mathbb{Z}^{\psi^{-1}(Y)} \arrow[r] & \mathbb{Z}^{\psi^{-1}(C)}. \end{tikzcd}$$

A more general formula is given in [6] for the case where E is not row-finite. This will be needed in §5.

When calculating the reduced filtered K-theory of a graph C^*-algebra, we will denote the maps in the sequence (4.5) by ι, π, and Δ, indexed as the natural transformations i, r, and δ in Definition 2.2. For a path p in X, a composite ι_p of natural transformations is defined as in Definition 3.1.

4.2. **Range of reduced filtered K-theory for graph C^*-algebras.** The following theorem by Søren Eilers, Mark Tomforde, James West and the third named author, determines the range of filtered K-theory over the two-point space $\{1, 2\}$ with $2 \to 1$. To apply it in the proof of Theorem 4.8, we quote it here reformulated for matrices acting from the right (thereby changing column-finiteness to row-finiteness, etc.).

Theorem 4.6 ([10, Propositions 4.3 and 4.7])**.** *Let*

$$\begin{tikzcd} G_1 \arrow[r, "\varepsilon"] & G_2 \arrow[r, "\gamma"] & G_3 \arrow[d, "0"] \\ F_3 \arrow[u, "\delta"] & F_2 \arrow[l, "\gamma'"] & F_1 \arrow[l, "\varepsilon'"] \end{tikzcd}$$

be an exact sequence $\mathcal{E}$ of abelian groups with F_1, F_2, F_3 free. Suppose that there exist row-finite matrices $A \in M_{n_1,n_1'}(\mathbb{Z})$ and $B \in M_{n_3,n_3'}(\mathbb{Z})$ for some $n_1, n_1', n_3, n_3' \in \{1, 2, \ldots, \infty\}$ with isomorphisms

$$\alpha_1\colon \operatorname{coker} A \to G_1, \quad \beta_1\colon \ker A \to F_1,$$

$$\alpha_3\colon \operatorname{coker} B \to G_3, \quad \beta_3\colon \ker B \to F_3.$$

Then there exist a row-finite matrix $Y \in M_{n_3,n_1'}(\mathbb{Z})$ and isomorphisms

$$\alpha_2\colon \operatorname{coker}\begin{pmatrix} A & 0 \\ Y & B \end{pmatrix} \to G_2, \quad \beta_2\colon \ker\begin{pmatrix} A & 0 \\ Y & B \end{pmatrix} \to F_2$$

such that the tuple $(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$ *gives an isomorphism of complexes from the exact sequence*

$$\begin{tikzcd} \operatorname{coker} A \arrow[r, "I"] & \operatorname{coker}\begin{pmatrix} A & 0 \\ Y & B \end{pmatrix} \arrow[r, "P"] & \operatorname{coker} B \arrow[d, "0"] \\ \operatorname{coker} B \arrow[u, "{[Y]}"] & \operatorname{coker}\begin{pmatrix} A & 0 \\ Y & B \end{pmatrix} \arrow[l, "P'"] & \operatorname{coker} A, \arrow[l, "I'"] \end{tikzcd}$$

where the maps I, I' *and* P, P' *are induced by the obvious inclusions and projections, to the exact sequence* $\mathcal{E}$.

If there exist an $A' \in M_{n_1', n_1}(\mathbb{Z})$ *such that* $A'A - 1 \in M_{n_1', n_1'}(\mathbb{Z}_+)$, *then* Y *can be chosen such that* $Y \in M_{n_3, n_1'}(\mathbb{Z}_+)$. *If furthermore a row-finite matrix* $Z \in M_{n_3, n_1'}(\mathbb{Z})$ *is given, then* Y *can be chosen such that* $Y - Z \in M_{n_3, n_1'}(\mathbb{Z}_+)$.

Proposition 4.7. *Let* A *be a purely infinite graph* C^**-algebra over* X. *Then* $\mathrm{FK}_{\mathcal{R}}(A)$ *is an exact* $\mathcal{R}$*-module, and* $\mathrm{FK}^1_{\{x\}}(A)$ *is free for all* $x \in X$. *If* A *is a purely infinite Cuntz–Krieger algebra over* X, *then, for all* $x \in X$, *the groups* $\mathrm{FK}^1_{\{x\}}(A)$ *and* $\mathrm{FK}^0_{\widetilde{\{x\}}}(A)$ *are furthermore finitely generated, and the rank of* $\mathrm{FK}^1_{\{x\}}(A)$ *coincides with the rank of the cokernel of the map* $i_{\partial\{x\}}^{\widetilde{\{x\}}} \colon \mathrm{FK}^0_{\partial\{x\}}(A) \to \mathrm{FK}^0_{\widetilde{\{x\}}}(A)$.

Proof. Exactness of $\mathrm{FK}_{\mathcal{R}}(A)$ is stated in Corollary 3.10. The group $\mathrm{FK}^1_{\{x\}}(A)$ is free for all $x \in X$ since the K_1-group of a graph C^*-algebra is free, and a subquotient of a real-rank-zero graph C^*-algebra is Morita equivalent to a graph C^*-algebra [14, Theorem 4.9].

Assume that A is a Cuntz–Krieger algebra. For any Cuntz–Krieger algebra B, $\mathrm{K}_*(B)$ is finitely generated and $\operatorname{rank}\mathrm{K}_0(B) = \operatorname{rank}\mathrm{K}_1(B)$. Since a subquotient of a purely infinite Cuntz–Krieger algebra is stably isomorphic to a Cuntz–Krieger algebra, the groups $\mathrm{FK}^1_{\{x\}}(A)$ and $\mathrm{FK}^0_{\widetilde{\{x\}}}(A)$ are finitely generated and $\operatorname{rank}\mathrm{FK}^1_{\{x\}}(A) = \operatorname{rank}\mathrm{FK}^0_{\{x\}}(A)$ for all $x \in X$. Since A has real rank zero, the sequence

$$\mathrm{FK}^0_{\partial\{x\}}(A) \xrightarrow{i_{\partial\{x\}}^{\widetilde{\{x\}}}} \mathrm{FK}^0_{\widetilde{\{x\}}}(A) \xrightarrow{r_{\widetilde{\{x\}}}^{\{x\}}} \mathrm{FK}^0_{\{x\}}(A) \to 0$$

is exact by [5, Theorem 3.14]. Hence

$$\operatorname{rank}\mathrm{FK}^1_{\{x\}}(A) = \operatorname{rank}\Big(\operatorname{coker}\big(i_{\partial\{x\}}^{\widetilde{\{x\}}} \colon \mathrm{FK}^0_{\partial\{x\}}(A) \to \mathrm{FK}^0_{\widetilde{\{x\}}}(A)\big)\Big). \qquad \square$$

Combining Proposition 4.7 with Theorem 4.8, one obtains a complete description of the range of reduced filtered K-theory on purely infinite tight graph C^*-algebras over X, and on purely infinite tight Cuntz–Krieger algebras over X.

Theorem 4.8. *Let* M *be an exact* $\mathcal{R}$*-module with* $M(x_1)$ *free for all* $x \in X$. *Then there exists a countable graph* E *such that all vertices in* E *are regular and support at least two cycles, the* C^**-algebra* $C^*(E)$ *is tight over* X *and* $\mathrm{FK}_{\mathcal{R}}\big(C^*(E)\big)$ *is isomorphic to* M. *By construction* $C^*(E)$ *is purely infinite.*

The graph E *can be chosen to be finite if (and only if)* $M(x_1)$ *and* $M(\widetilde{x}_0)$ *are finitely generated and the rank of* $M(x_1)$ *coincides with the rank of the cokernel of*

$i\colon M(\widetilde{\partial}x_0) \to M(\widetilde{x}_0)$ for all $x \in X$. If E is chosen finite, then by construction $C^*(E)$ is a Cuntz–Krieger algebra.

Proof. For each $x \in X$, we may choose, by [10, Proposition 3.3], a countable, non-empty set V_x, a matrix $D_x \in M_{V_x}(\mathbb{Z}_+)$ and isomorphisms $\varphi_{x_1}\colon M(x_1) \to \ker D_x$ and $\varphi_{x_0}\colon M(x_0) \to \operatorname{coker} D_x$, where $M(x_0) := \operatorname{coker}(M(\widetilde{\partial}x_0) \xrightarrow{i} M(\widetilde{x}_0))$. Define $r^{x_0}_{\widetilde{x}_0}\colon M(\widetilde{x}_0) \to M(x_0)$ as the cokernel map. Given a matrix D, we let $E(D)$ denote the graph with adjacency matrix D^t. We may furthermore assume that all vertices in the graph $E(1 + D_x)$ are regular and support at least two cycles. If $M(x_1)$ and $M(\widetilde{x}_0)$ are finitely generated and the rank of $M(x_1)$ coincides with the rank of the cokernel of $i\colon M(\widetilde{\partial}x_0) \to M(\widetilde{x}_0)$, then the set V_x can be chosen to be finite.

For each $y, z \in X$ with $y \neq z$ we desire to construct a matrix $H_{yz}\colon \mathbb{Z}^{V_z} \to \mathbb{Z}^{V_y}$ with non-negative entries satisfying that H_{yz} is non-zero if and only if $y > z$, and satisfying that for each $x \in X$ there exist isomophisms $\varphi_{\widetilde{\partial}x_0}$ and $\varphi_{\widetilde{x}_0}$ making the diagrams

(4.9)
$$\begin{tikzcd}
M(\widetilde{\partial}x_0) \arrow[rr, "i^{\widetilde{x}_0}_{\widetilde{\partial}x_0}"] \arrow[dr, "\varphi_{\widetilde{\partial}x_0}"] & & M(\widetilde{x}_0) \arrow[rr, "r^{x_0}_{\widetilde{x}_0}"] \arrow[d, "\varphi_{\widetilde{x}_0}"] & & M(x_0) \arrow[dl, "\varphi_{x_0}"] \\
& \operatorname{coker} D_{\widetilde{\partial}\{x\}} \arrow[r, "\iota^{\widetilde{\{x\}}}_{\widetilde{\partial}\{x\}}"] & \operatorname{coker} D_{\widetilde{\{x\}}} \arrow[r, "\pi^{\{x\}}_{\widetilde{\{x\}}}"] & \operatorname{coker} D_x \arrow[d, "0"] & \\
& \ker D_x \arrow[u, "D_{\widetilde{\{x\}}}|^{\varphi^{-1}(\widetilde{\partial}\{x\})}_{\varphi^{-1}(x)}"] & \ker D_{\widetilde{\{x\}}} \arrow[l, "\pi^{\{x\}}_{\widetilde{\{x\}}}"] & \ker D_{\widetilde{\partial}\{x\}} \arrow[l, "\iota^{\widetilde{\{x\}}}_{\widetilde{\partial}\{x\}}"] & \\
M(x_1) \arrow[uuu, "\delta^{\widetilde{\partial}x_0}_{x_1}"] \arrow[ur, "\varphi_{x_1}"] & & & &
\end{tikzcd}$$

and

(4.10)
$$\begin{tikzcd}
M(\widetilde{y}_0) \arrow[rr, "i^{\widetilde{\partial}x_0}_{\widetilde{y}_0}"] \arrow[dr, "\varphi_{\widetilde{y}_0}"] & & M(\widetilde{\partial}x_0) \arrow[d, "\varphi_{\widetilde{\partial}x_0}"] & \\
& \operatorname{coker} D_{\widetilde{\{y\}}} \arrow[r, "\iota^{\widetilde{\partial}\{x\}}_{\widetilde{\{y\}}}"] & \operatorname{coker} D_{\widetilde{\partial}\{x\}} \arrow[r, "\pi^{\widetilde{\partial}\{x\}\setminus\widetilde{\{y\}}}_{\widetilde{\partial}\{x\}}"] & \operatorname{coker} D_{\widetilde{\partial}\{x\}\setminus\widetilde{\{y\}}} \arrow[d, "0"] \\
& \ker D_{\widetilde{\partial}\{x\}\setminus\widetilde{\{y\}}} \arrow[u, "D_{\widetilde{\partial}\{x\}}|^{\varphi^{-1}(\widetilde{\{y\}})}_{\varphi^{-1}(\widetilde{\partial}\{x\}\setminus\widetilde{\{y\}})}"] & \ker D_{\widetilde{\partial}\{x\}} \arrow[l, "\pi^{\widetilde{\partial}\{x\}\setminus\widetilde{\{y\}}}_{\widetilde{\partial}\{x\}}"] & \ker D_{\widetilde{\{y\}}} \arrow[l, "\iota^{\widetilde{\partial}\{x\}}_{\widetilde{\{y\}}}"]
\end{tikzcd}$$

commute when $y \to x$, where $D_F \in M_{V_F}(\mathbb{Z}_+)$ for each $F \subseteq X$ is defined as

$$D_F(v, w) = \begin{cases} D_x(v, w) & v, w \in V_x \\ H_{yz}(v, w) & v \in V_y, w \in V_x, x \neq y \end{cases}$$

with $V_F = \bigcup_{y \in F} V_y$. The constructed graph $E(D_X + 1)$ then has the desired properties.

We proceed by a recursive argument, by adding to an open subset an open point in the complement. Given $U \in \mathbb{O}(X)$, assume that for all $z, y \in U$, the matrices H_{yz} and isomorphisms $\varphi_{\widetilde{\partial} y_0}$ and $\varphi_{\widetilde{y}_0}$ have been defined and satisfy that the diagrams (4.9) and (4.10) commute for all $x, y \in U$ with $y \to x$. Let x be an open point in $X \setminus U$ and let us construct isomorphisms $\varphi_{\widetilde{\partial} x_0}$ and $\varphi_{\widetilde{x}_0}$, and for all $y \in \widetilde{\partial}\{x\}$ non-zero matrices H_{yx}, making the diagrams (4.9) and (4.10) commute.

Consider the commuting diagram

$$\begin{tikzcd}
\bigoplus\limits_{(p,q)\in \mathrm{DP}(x)} M(\widetilde{s(p,q)}_0) \arrow[r, "\begin{pmatrix} i_p & -i_q \end{pmatrix}"] \arrow[d, "(\varphi_{\widetilde{s(p,q)}_0})"] & \bigoplus\limits_{y\to x} M(\widetilde{y}_0) \arrow[r, "(i^{\widetilde{\partial} x_0}_{\widetilde{y}_0})"] \arrow[d, "(\varphi_{\widetilde{y}_0})"] & M(\widetilde{\partial} x_0) \arrow[r] \arrow[d, dotted, "(\varphi_{\widetilde{\partial} x_0})"] & 0 \\
\bigoplus\limits_{(p,q)\in \mathrm{DP}(x)} \operatorname{coker} D_{\{\widetilde{s(p,q)}\}} \arrow[r, "\begin{pmatrix} \iota_p & -\iota_q \end{pmatrix}"] & \bigoplus\limits_{y\to x} \operatorname{coker} D_{\widetilde{\{y\}}} \arrow[r, "(\iota^{\widetilde{\partial}\{x\}}_{\widetilde{\{y\}}})"] & \operatorname{coker} D_{\widetilde{\partial}\{x\}} \arrow[r] & 0.
\end{tikzcd}$$

The top row is exact by exactness of M, and the bottom row is exact by exactness of $\mathrm{FK}\big(C^*(E(1 + D_{\widetilde{\partial}\{x\}}))\big)$. An isomorphism $\varphi_{\widetilde{\partial} x_0} \colon M(\widetilde{\partial} x_0) \to \operatorname{coker} D_{\widetilde{\partial}\{x\}}$ is therefore induced. By construction, (4.10) commutes for all $y \to x$.

Now consider the commuting diagram

$$\begin{tikzcd}
M(\widetilde{\partial} x_0) \arrow[rr, "i^{\widetilde{x}_0}_{\widetilde{\partial} x_0}"] \arrow[dr, "\varphi_{\widetilde{\partial} x_0}"] & & M(\widetilde{x}_0) \arrow[rr, "r^{x_0}_{\widetilde{x}_0}"] \arrow[d, equal] & & M(x_0) \arrow[dl, "\varphi_{x_0}"'] \\
& \operatorname{coker} D_{\widetilde{\partial}\{x\}} \arrow[r] & M(\widetilde{x}_0) \arrow[r] & \operatorname{coker} D_x \arrow[d, "0"] & \\
& \ker D_x \arrow[u] & F \arrow[l] & \ker D_{\widetilde{\partial}\{x\}} \arrow[l] & \\
M(x_1) \arrow[uuu, "\delta^{\widetilde{\partial} x_0}_{x_1}"] \arrow[ur, "\varphi_{x_1}"] & & & &
\end{tikzcd}$$

with the maps in the inner sequence being the unique maps making the squares commute, and where a free group F and maps into and out of it have been chosen so that the inner six-term sequence is exact. Apply Theorem 4.6 to the inner six-term exact sequence to get non-zero matrices H_{yx} for all $y \in \widetilde{\partial}\{x\}$ realizing the sequence, that is, making (4.9) commute.

Finally, we note that the constructed graph algebra $C^*(E(D_X + 1))$ is purely infinite by [11, Theorem 2.3] since all vertices in $E(D_X + 1)$ are regular and support two cycles. Since the graph $E(D_X + 1)$ has no sinks or sources, the graph algebra $C^*(E(D_X + 1))$ is a Cuntz–Krieger algebra when $E(D_X + 1)$ is finite. $\square$

Combining the previous theorem with Restorff's classification of purely infinite Cuntz–Krieger algebras [15], we obtain the following description of stable isomorphism classes of purely infinite Cuntz–Krieger algebras.

Corollary 4.11. *The functor* $\mathrm{FK}_{\mathcal{R}}$ *induces a bijection between the set of stable isomorphism classes of tight purely infinite Cuntz–Krieger algebras over* X *and the* *set of isomorphism classes of exact $\mathcal{R}$-modules M such that, for all $x \in X$, $M(x_1)$ is free, $M(x_1)$ and $M(\widetilde{x}_0)$ are finitely generated, and the rank of $M(x_1)$ coincides with the rank of the cokernel of the map $i_{\widetilde{\partial} x_0}^{\widetilde{x}_0}\colon M(\widetilde{\partial} x_0) \to M(\widetilde{x}_0)$.*

5. Unital reduced filtered K-theory

Anticipating a generalization of the main result in [15] accounting for actual isomorphisms rather than stable isomorphisms, we also provide a 'unital' version of our range result. Depending on the space X, the group $\mathrm{K}_0(A)$ may not be part of the invariant $\mathrm{FK}_{\mathcal{R}}(A)$. This slightly complicates the definition of unital reduced filtered K-theory.

For $x, x' \in X$, we let $\inf(x, x')$ denote the set $\{y \in X \mid y \to x, y \to x'\}$.

Definition 5.1. The category $\mathfrak{Mod}(\mathcal{R})^{\mathrm{pt}}$ of *pointed $\mathcal{R}$-modules* is defined to have objects given by pairs (M, m) where M is an $\mathcal{R}$-module and

$$m \in \operatorname{coker}\left(\bigoplus_{\substack{x,x'\in X\\ y\in\inf(x,x')}} M(\widetilde{y}_0) \xrightarrow{\begin{pmatrix} i_{\widetilde{y}_0}^{\widetilde{\partial} x_0} i_{\widetilde{\partial} x_0}^{\widetilde{x}_0} & -i_{\widetilde{y}_0}^{\widetilde{\partial} x'_0} i_{\widetilde{\partial} x'_0}^{\widetilde{x}'_0}\end{pmatrix}} \bigoplus_{x\in X} M(\widetilde{x}_0)\right),$$

and a morphism $\varphi\colon (M, m) \to (N, n)$ is an $\mathcal{R}$-module homomorphism from M to N whose induced map on the cokernels sends m to n.

Lemma 5.2. *Let A be a unital C^*-algebra over X of real rank zero, and let $U \in \mathbb{O}(X)$. Then the sequence*

$$\bigoplus_{\substack{x,x'\in U\\ y\in\inf(x,x')}} \mathrm{FK}^0_{\widetilde{\{y\}}}(A) \xrightarrow{\begin{pmatrix} i_{\widetilde{\{y\}}}^{\widetilde{\{x\}}} & -i_{\widetilde{\{y\}}}^{\widetilde{\{x'\}}}\end{pmatrix}} \bigoplus_{x\in U} \mathrm{FK}^0_{\widetilde{\{x\}}}(A) \xrightarrow{(i_{\widetilde{\{x\}}}^{U})} \mathrm{FK}^0_U(A) \to 0$$

is exact.

Proof. This follows from a twofold application of Lemma 3.9 using that U is covered by $(\widetilde{\{x\}})_{x\in U}$ and that $\widetilde{\{x\}} \cap \widetilde{\{x'\}}$ is covered by $(\widetilde{\{y\}})_{y\in\inf(x,x')}$. □

Definition 5.3. Let A be a unital C^*-algebra over X with real rank zero. The *unital reduced filtered* K*-theory* $\mathrm{FK}^{\mathrm{unit}}_{\mathcal{R}}(A)$ is defined as the pointed $\mathcal{R}$-module $(\mathrm{FK}_{\mathcal{R}}(A), u(A))$ where $u(A)$ is the unique element in

$$\operatorname{coker}\left(\bigoplus_{\substack{x,x'\in X\\ y\in\inf(x,x')}} \mathrm{FK}^0_{\widetilde{\{y\}}}(A) \xrightarrow{\begin{pmatrix} i_{\widetilde{\{y\}}}^{\widetilde{\{x\}}} & -i_{\widetilde{\{y\}}}^{\widetilde{\{x'\}}}\end{pmatrix}} \bigoplus_{x\in X} \mathrm{FK}^0_{\widetilde{\{x\}}}(A)\right)$$

that is mapped to $[1_A]$ in $\mathrm{K}_0(A)$ by the map induced by the family $\big(\mathrm{FK}^0_{\widetilde{\{x\}}}(A) \xrightarrow{(i_{\widetilde{\{x\}}}^{X})} \mathrm{FK}^0_X(A)\big)_{x\in X}$, see Lemma 5.2.

Lemma 5.4. *Let A and B be C^*-algebras over X of real rank zero, and let $U \in \mathbb{O}(X)$. Let a family of isomorphisms $\varphi^0_{\widetilde{\{x\}}}\colon \mathrm{FK}^0_{\widetilde{\{x\}}}(A) \to \mathrm{FK}^0_{\widetilde{\{x\}}}(B)$, $x \in U$ be given and assume that $\varphi^0_{\widetilde{\{y\}}} i_{\widetilde{\{y\}}}^{\widetilde{\{x\}}} = i_{\widetilde{\{y\}}}^{\widetilde{\{x\}}} \varphi^0_{\widetilde{\{x\}}}$ holds for all $x, y \in U$ with $y \to x$. Then*

$(\varphi^0_{\widetilde{\{x\}}})_{x\in U}$ can be uniquely extended to a family of isomorphisms $\varphi^0_Y\colon \mathrm{FK}^0_Y(A)\to \mathrm{FK}^0_Y(B)$, $Y\in\mathbb{LC}(U)$, that commute with the natural transformations i and r.

Proof. We may assume that $U = X$. The part of the construction in the proof [1, Theorem 5.17] that involves only groups in even degree makes sense when X is an arbitrary finite T_0-space and implies the present claim as a corollary. □

For a unital graph C^*-algebra $C^*(E)$, the class of the unit $[1_{C^*(E)}]$ in $\mathrm{K}_0\big(C^*(E)\big)$ is sent, via the canonical isomorphism $\mathrm{K}_0\big(C^*(E)\big)\to\operatorname{coker} D_E$, to the class $[\mathbb{1}] = \begin{pmatrix}1 & 1 & \cdots & 1\end{pmatrix} + \operatorname{im} D_E$, where $1 + D_E^t$ denotes the adjacency matrix for E, [18, Theorem 2.2]. Using this and the formula of [6], see §4.1, the unital reduced filtered K-theory of a unital graph C^*-algebra can be calculated. A graph C^*-algebra $C^*(E)$ is unital if and only if its underlying graph E has finitely many vertices. So by the formula of [6], a unital graph C^*-algebra and its subquotients always have finitely generated K-theory.

Theorem 5.5. *Let X be a finite T_0-space, and let (M,m) be an exact pointed $\mathcal{R}$-module. Assume that for all $x\in X$, $M(x_1)$ is a free abelian group,*

$$\operatorname{coker}(M(\widetilde{\partial}x_0)\xrightarrow{i^{\widetilde{x}_0}_{\widetilde{\partial}x_0}} M\big(\widetilde{x}_0\big))$$

is finitely generated, and $\operatorname{rank} M(x_1)\le \operatorname{rank}\operatorname{coker}(M(\widetilde{\partial}x_0)\xrightarrow{i^{\widetilde{x}_0}_{\widetilde{\partial}x_0}} M\big(\widetilde{x}_0\big))$.

Then there exists a countable graph E such that all vertices in E support at least two cycles, the set E^0 is finite, the C^*-algebra $C^*(E)$ is tight over X and the pointed $\mathcal{R}$-module $\mathrm{FK}^{\mathrm{unit}}_{\mathcal{R}}\big(C^*(E)\big)$ is isomorphic to (M,m). By construction $C^*(E)$ is unital and purely infinite.

The graph E can be chosen such that all of its vertices are regular if (and only if) the rank of $M(x_1)$ coincides with the rank of the cokernel of $i^{\widetilde{x}_0}_{\widetilde{\partial}x_0}\colon M(\widetilde{\partial}x_0)\to M(\widetilde{x}_0)$ for all $x\in X$. If E is chosen to have regular vertices, then by construction $C^*(E)$ is a Cuntz–Krieger algebra.

Proof. The proof is carried out by the same strategy as the proof of Theorem 4.8. However, we here construct graphs that may have singular vertices. We refer to [6] for the formula for calculating six-term exact sequences in K-theory for such graph C^*-algebras.

By Theorem 4.8 there exists a C^*-algebra B of real rank zero with $M\cong \mathrm{FK}_{\mathcal{R}}(B)$, so we may assume that $M = \mathrm{FK}_{\mathcal{R}}(B)$. Since, by Lemma 5.2,

$$\operatorname{coker}\Big(\bigoplus_{y\in\inf(x,x')} M(\widetilde{y}_0)\xrightarrow{\begin{pmatrix} i^{\widetilde{x}_0}_{\widetilde{y}_0} & -i^{\widetilde{x}'_0}_{\widetilde{y}_0}\end{pmatrix}} \bigoplus_{x\in X} M(\widetilde{x}_0,0)\Big)$$

is isomorphic to $\mathrm{FK}^0_X(B)=\mathrm{K}_0(B)$, we may identify m with its image in $\mathrm{FK}^0_X(B)$ but note that this image may not be $[1_B]$.

The construction is similar to the construction in the proof of Theorem 4.8. Let x_1 be an open point in X and define $U_1=\{x_1\}$. Define U_k recursively by choosing an open point x_k in $X\setminus U_{k-1}$ and defining $U_k = U_{k-1}\cup\{x_k\}$. Let C_k denote the largest subset of U_k that is closed in X. Observe that

$$C_k = X\setminus \bigcup_{y\in \mathrm{CP}(X)\setminus U_k} \widetilde{\{y\}}$$

where $\mathrm{CP}(X)$ denotes the closed points in X. And observe that if x_k is closed in X then $C_k \setminus C_{k-1} \subseteq \widetilde{\{x_k\}}$ and $C_k \setminus \widetilde{\{x_k\}} = C_{k-1}$, and otherwise $C_k = C_{k-1}$. Define for all closed subsets C of X the element m_C of $\mathrm{FK}^0_C(B)$ as the image of m under $r\colon \mathrm{FK}^0_X(B) \to \mathrm{FK}^0_C(B)$.

For each x not closed in X, choose by Proposition 3.6 of [10] a graph E_x that is transitive, has finitely many vertices that all support at least two cycles, and such that $\mathrm{K}_1\big(C^*(E_x)\big)$ is isomorphic to $\mathrm{FK}^1_{\{x\}}(B)$ and $\mathrm{K}_0\big(C^*(E_x)\big)$ is isomorphic to $\mathrm{FK}^0_{\{x\}}(B)$. Define $V_{\{x\}} = E^0_x$ and $V'_{\{x\}} = (E^0_x)_{\mathrm{reg}}$, let $D_{\{x\}} \in M_{V_{\{x\}}}(\mathbb{Z}_+ \cup \{\infty\})$ such that $1 + D^t_{\{x\}}$ is the adjacency matrix for E_x, and let $D'_{\{x\}}$ denote the $V'_{\{x\}} \times V_{\{x\}}$ matrix defined by $D'_{\{x\}}(v,w) = D_{\{x\}}(v,w)$. If $\operatorname{rank}\mathrm{FK}^1_{\{x\}}(B) = \operatorname{rank}\mathrm{FK}^0_{\{x\}}(B)$ then $V_{\{x\}} = V'_{\{x\}}$. Let isomorphisms $\varphi^1_{\{x\}}\colon \mathrm{FK}^1_{\{x\}}(B) \to \ker D'_{\{x\}}$ and $\varphi^0_{\{x\}}\colon \mathrm{FK}^0_{\{x\}}(B) \to \operatorname{coker} D'_{\{x\}}$ be given. For x closed in X we may by Proposition 3.6 of [10] choose E_x and $\varphi^0_{\{x\}}$ such that furthermore $\varphi^0_{\{x\}}(m_{\{x\}}) = [\mathbb{1}]$.

Define
$$V_U = \bigcup_{x\in U} V_{\{x\}}, V'_U = \bigcup_{x\in U} V'_{\{x\}}$$
for all $U \in \mathbb{O}(X)$. As in the proof of Theorem 4.8 we wish to construct for all $x, y \in X$ with $x \neq y$, $V'_{\{y\}} \times V_{\{x\}}$ matrices H'_{yx} over $\mathbb{Z}_+$ with $H'_{yx} \neq 0$ if and only if $y > x$. When having constructed such H'_{yx}, we construct a $V_{\{y\}} \times V_{\{x\}}$ matrix H_{yx} over $\mathbb{Z}_+ \cup \{\infty\}$ by
$$H_{yx}(v,w) = \begin{cases} H'_{yx}(v,w) & v \in V'_{\{y\}} \\ \infty & v \in V_y \setminus V'_{\{y\}} \text{ and } y > x \\ 0 & v \in V_y \setminus V'_{\{y\}} \text{ and } y < x. \end{cases}$$
We then define for $U \in \mathbb{O}(X)$, a $V'_U \times V_U$ matrix D'_U over $\mathbb{Z}_+$ and a matrix $D_U \in M_{V_U}(\mathbb{Z}_+ \cup \{\infty\})$ by
$$D'_U(v,w) = \begin{cases} D'_{\{x\}}(v,w) & v \in V'_{\{x\}}, w \in V_{\{x\}} \\ H'_{yx}(v,w) & v \in V'_{\{y\}}, w \in V_{\{x\}} \end{cases}$$
and
$$D_U(v,w) = \begin{cases} D_{\{x\}}(v,w) & v, w \in V_{\{x\}} \\ H_{yx}(v,w) & v \in V_{\{y\}}, w \in V_{\{x\}}. \end{cases}$$

For a matrix D, we denote by $E(D)$ the graph with adjacency matrix D^t. Since the graph $E(1 + D_{\{x\}})$ has a simple graph C^*-algebra, it cannot contain breaking vertices. By construction, neither will $E(1 + D_X)$. So $E(1 + D_X)$ will be tight over X as in the proof of Theorem 4.8. The matrices $(H'_{yx_k})_{y \in \tilde{\partial}\{x_k\}}$ will be constructed recursively over $k \in \{1, \dots, n\}$ so that the following holds: For each $x \in X$, there are isomorphisms making the diagram

$$\begin{tikzcd} \mathrm{FK}^1_{\{x\}}(B) \arrow[r, "\delta^{\tilde{\partial}\{x\}}_{\{x\}}"] \arrow[d, "\varphi^1_{\{x\}}", "\cong"'] & \mathrm{FK}^0_{\tilde{\partial}\{x\}}(B) \arrow[r, "i^{\widetilde{\{x\}}}_{\tilde{\partial}\{x\}}"] \arrow[d, "\varphi^0_{\tilde{\partial}\{x\}}", "\cong"'] & \mathrm{FK}^0_{\widetilde{\{x\}}}(B) \arrow[d, "\varphi^0_{\widetilde{\{x\}}}", "\cong"'] \\ \ker D'_{\{x\}} \arrow[r, "\Delta^{\tilde{\partial}\{x\}}_{\{x\}}"] & \operatorname{coker} D'_{\tilde{\partial}\{x\}} \arrow[r, "\iota^{\widetilde{\{x\}}}_{\tilde{\partial}\{x\}}"] & \operatorname{coker} D'_{\widetilde{\{x\}}} \end{tikzcd} \tag{5.6}$$

commute and satisfying for all $y \to x$ that

$$(5.7)\qquad \begin{tikzcd} \mathrm{FK}^0_{\widetilde{\{y\}}}(B) \arrow[r, "i^{\widetilde{\partial}\{x\}}_{\widetilde{\{y\}}}"] \arrow[d, "\cong"', "\varphi^0_{\widetilde{\{y\}}}"] & \mathrm{FK}^0_{\widetilde{\partial}\{x\}}(B) \arrow[d, "\cong"', "\varphi^0_{\widetilde{\partial}\{x\}}"] \\ \operatorname{coker} D'_{\widetilde{\{y\}}} \arrow[r, "\iota^{\widetilde{\partial}\{x\}}_{\widetilde{\{y\}}}"] & \operatorname{coker} D'_{\widetilde{\partial}\{x\}} \end{tikzcd}$$

commutes, and for all $k \in \{1, \ldots, n\}$ that the isomorphism

$$\varphi^0_{C_k}\colon\ \mathrm{FK}^0_{C_K}(B) \to \operatorname{coker} D'_{C_k}$$

induced by $(\varphi^0_{\widetilde{\{x\}}})_{x\in X}$, see Lemma 5.4, sends m_{C_k} to $[\mathbb{1}]$.

Assume that the matrices $(H'_{yx_i})_{y\in\widetilde{\partial}\{x_i\}}$ have been constructed for all $i < k$. Then isomorphisms $(\varphi^0_Y)_{Y\in\mathbb{LC}(U_{k-1})}$ are induced by Lemma 5.4.

Assume that x_k is a closed point in X. Since $D_{C_k\setminus\{x_k\}}$ is already defined, we may define $1_{C_k\setminus\widetilde{\{x_k\}}}$ as the element in $\mathbb{Z}^{V_{C_k\setminus\{x_k\}}}$ with

$$1_{C_k\setminus\widetilde{\{x_k\}}}(i) = \begin{cases} 1 & \text{if } i \in V_{C_k\setminus\widetilde{\{x_k\}}} \\ 0 & \text{if } i \in V_{(C_k\cap\widetilde{\{x_k\}})\setminus\{x_k\}}. \end{cases}$$

Define $\widetilde{m}_{C_k\setminus\{x_k\}}$ as the preimage of $1_{C_k\setminus\widetilde{\{x_k\}}}$ under the isomorphism $\varphi^0_{C_k\setminus\{x_k\}}$. Notice that $m_{C_k} r^{C_{k-1}}_{C_k}\varphi^0_{C_{k-1}} = [\mathbb{1}]$ and that, since C_{k-1} is closed in $C_k\setminus\{x_k\}$,

$$\begin{aligned} \widetilde{m}_{C_k\setminus\{x_k\}} i^{C_k}_{C_k\setminus\{x_k\}} r^{C_{k-1}}_{C_k}\varphi^0_{C_{k-1}} &= \widetilde{m}_{C_k\setminus\{x_k\}} r^{C_{k-1}}_{C_k\setminus\{x_k\}}\varphi^0_{C_{k-1}} \\ &= \widetilde{m}_{C_k\setminus\{x_k\}}\varphi^0_{C_k\setminus\{x_k\}} r^{C_{k-1}}_{C_k\setminus\{x_k\}} \\ &= 1_{C_k\setminus\widetilde{\{x_k\}}} r^{C_{k-1}}_{C_k\setminus\{x_k\}} = [\mathbb{1}], \end{aligned}$$

so by injectivity of the map $\varphi^0_{C_{k-1}}$, the element $m_{C_k} - \widetilde{m}_{C_k\setminus\{x_k\}} i^{C_k}_{C_k\setminus\{x_k\}}$ lies in $\ker r^{C_{k-1}}_{C_k} = \operatorname{im} i^{C_k}_{C_k\cap\widetilde{\{x_k\}}}$. Choose $\widetilde{m}_{C_k\cap\widetilde{\{x_k\}}}$ in $\mathrm{FK}^0_{C_k\cap\widetilde{\{x_k\}}}(B)$ such that

$$m_{C_k} = m_{C_k\cap\widetilde{\{x_k\}}} i^{C_k}_{C_k\cap\widetilde{\{x_k\}}} + \widetilde{m}_{C_k\setminus\{x_k\}} i^{C_k}_{C_k\setminus\{x_k\}}.$$

Choose $\widetilde{m}_{\widetilde{\{x_k\}}}$ in $\mathrm{FK}^0_{\widetilde{\{x_k\}}}(B)$ such that

$$\widetilde{m}_{\widetilde{\{x_k\}}} r^{C_k\cap\widetilde{\{x_k\}}}_{\widetilde{\{x_k\}}} = \widetilde{m}_{C_k\cap\widetilde{\{x_k\}}}.$$

Consider the diagram

$$(5.8)\qquad \begin{tikzcd} \mathrm{FK}^1_{\{x_k\}}(B) \arrow[r, "\delta^{\widetilde{\partial}\{x_k\}}_{\{x_k\}}"] \arrow[d, "\cong"', "\varphi^1_{\{x_k\}}"] & \mathrm{FK}^0_{\widetilde{\partial}\{x_k\}}(B) \arrow[r, "i^{\widetilde{\{x_k\}}}_{\widetilde{\partial}\{x_k\}}"] \arrow[d, "\cong"', "\varphi^0_{\widetilde{\partial}\{x_k\}}"] & \mathrm{FK}^0_{\widetilde{\{x_k\}}}(B) \arrow[r, "r^{\{x_k\}}_{\widetilde{\{x_k\}}}"] \arrow[d, dotted, "\varphi^0_{\widetilde{\{x_k\}}}"] & \mathrm{FK}^0_{\{x_k\}}(B) \arrow[d, "\cong"', "\varphi^0_{\{x_k\}}"] \\ \ker D'_{x_k} \arrow[r] & \operatorname{coker} D'_{\widetilde{\partial}\{x_k\}} \arrow[r, dotted] & \operatorname{coker} D'_{\widetilde{\{x\}}} \arrow[r, dotted] & \operatorname{coker} D'_{\{x_k\}}, \end{tikzcd}$$

where $\widetilde{m}_{\widetilde{\{x_k\}}}$ is mapped to $m_{\{x_k\}}$ which by $\varphi^0_{\{x_k\}}$ is mapped to $[\mathbb{1}]$. As in the proof of Theorem 4.8 we apply Theorem 4.6 to construct $D'_{\widetilde{\{x_k\}}}$ from $D'_{\{x_k\}}$ and $D'_{\widetilde{\partial}\{x_k\}}$

by constructing nonzero matrices H_{yx_k} for all $y \in \widetilde{\partial}\{x_k\}$. By Proposition 4.8 of [10] we may furthermore achieve that $m_{\widetilde{\{x_k\}}}\varphi_{\widetilde{\{x_k\}}} = [\mathbb{1}]$.

That (5.6) and (5.7) hold for x_k follows immediately from the construction. To verify that the map $\varphi^0_{C_k}$ induced by $(\varphi^0_{\widetilde{\{y\}}})_{y \in U_k}$ satisfies $m_{C_k}\varphi^0_{C_k} = [\mathbb{1}]$, observe that the map $\varphi^0_{C_k \cap \widetilde{\{x_k\}}}$ induced by $\varphi^0_{\widetilde{\{x_k\}}}$ will map $\widetilde{m}_{C_k \cap \widetilde{\{x_k\}}}$ to $[\mathbb{1}]$, and consider the commuting diagram

$$\begin{tikzcd}[column sep=huge]
\mathrm{FK}^0_{C_k \setminus \{x_k\}}(B) \oplus \mathrm{FK}^0_{C_k \cap \widetilde{\{x_k\}}}(B) \arrow[r, "\begin{pmatrix} i^{C_k}_{C_k \setminus \{x_k\}} \\ i^{C_k}_{C_k \cap \widetilde{\{x_k\}}} \end{pmatrix}"] \arrow[d, "\cong"', "\varphi^0_{C_k \setminus \{x_k\}} \oplus \varphi^0_{C_k \cap \widetilde{\{x_k\}}}"] & \mathrm{FK}^0_{C_k}(B) \arrow[d, "\cong"', "\varphi^0_{C_k}"] \\
\operatorname{coker} D'_{C_k \setminus \{x_k\}} \oplus \operatorname{coker} D'_{C_k \cap \widetilde{\{x_k\}}} \arrow[r, "\begin{pmatrix} \iota^{C_k}_{C_k \setminus \{x_k\}} \\ \iota^{C_k}_{C_k \cap \widetilde{\{x_k\}}} \end{pmatrix}"] & \operatorname{coker} D'_{C_k}.
\end{tikzcd}$$

Since $(\widetilde{m}_{C_k \setminus \{x_k\}}, \widetilde{m}_{C_k \cap \widetilde{\{x_k\}}})$ is mapped to m_{C_k} by $\begin{pmatrix} i^{C_k}_{C_k \setminus \{x_k\}} \\ i^{C_k}_{C_k \cap \widetilde{\{x_k\}}} \end{pmatrix}$ and to $(1_{C_k \setminus \widetilde{\{x_k\}}}, [\mathbb{1}])$ by $\varphi^0_{C_k \setminus \{x_k\}} \oplus \varphi^0_{C_k \cap \widetilde{\{x_k\}}}$, we see by commutativity of the diagram that $m_{C_k}\varphi^0_{C_k} = [\mathbb{1}]$.

For k with x_k not closed in X, C_k equals C_{k-1} and a construction similar to the one in the proof of Theorem 4.8 applies. As in the proof of Theorem 4.8, Proposition 4.7 of [10] allows us to make sure that $H'_{yx_k} \neq 0$ when $y \in \widetilde{\partial}\{x_k\}$.

Finally, we note that the constructed graph algebra $C^*(E(D_X + 1))$ is purely infinite by [11, Theorem 2.3] since all vertices in $E(D_X + 1)$ support two cycles and $E(D_X + 1)$ has no breaking vertices. Since the graph $E(D_X + 1)$ has no sinks or sources, the graph algebra $C^*(E(D_X + 1))$ is a Cuntz–Krieger algebra when $E(D_X + 1)$ is finite. □

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REDUCTION OF FILTERED K-THEORY AND A CHARACTERIZATION OF CUNTZ–KRIEGER ALGEBRAS

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Abstract. We show that filtered K-theory is equivalent to a substantially smaller invariant for all real-rank-zero C^*-algebras with certain primitive ideal spaces – including the infinitely many so-called accordion spaces for which filtered K-theory is known to be a complete invariant. As a consequence, we give a characterization of purely infinite Cuntz–Krieger algebras whose primitive ideal space is an accordion space.

1. Introduction

The Cuntz and Cuntz–Krieger algebras are historically and in general of great importance for our understanding of simple and non-simple purely infinite C^*-algebras as they were not only the first constructed examples of such but are also very tangible due to the combinatorial nature of their construction [12]. Cuntz–Krieger algebras arise from shifts of finite type and it has been shown that they are exactly the graph C^*-algebras $C^*(E)$ arising from finite directed graphs E with no sources [5]. Using the Kirchberg–Phillips classification theorem [16, 22], the Cuntz algebras and simple Cuntz–Krieger algebras can be identified, up to isomorphism, as the unital UCT Kirchberg algebras with a specific type of K-theory [11, 25]. A similar characterization for non-simple, purely infinite Cuntz–Krieger algebras and, more generally, of unital graph C^*-algebras of this type is desirable.

A Kirchberg X-algebra is a purely infinite, nuclear, separable C^*-algebra with primitive ideal space homeomorphic to X (in a specified way). When X is a so-called accordion space, see Definition 2.1, the invariant *filtered* K-*theory* FK is a strongly complete invariant for stable Kirchberg X-algebras with simple subquotients in the bootstrap class [8, 17, 19]. In particular, filtered K-theory is complete for purely infinite graph C^*-algebras with primitive ideal space of accordion type, and the main goal of this paper is to use this to achieve a *characterization* in the sense of the previous paragraph of such purely infinite Cuntz–Krieger algebras and graph C^*-algebras. Since a Cuntz–Krieger algebra is purely infinite if and only if it has real rank zero (and more generally, a purely infinite graph C^*-algebra always has real rank zero [15]), we will specifically investigate filtered K-theory for C^*-algebras of real rank zero.

In the companion paper [3], we determine the range of *reduced* filtered K-theory with respect to purely infinite Cuntz–Krieger algebras and graph C^*-algebras.

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This invariant was originally defined by Gunnar Restorff [23], who used it to give an "internal" classification of purely infinite Cuntz–Krieger algebras, inspired by work of Mikael Rørdam [26] and work of Mike Boyle and Danrun Huang on dynamical systems [9]. In the present note, we show that under some assumptions on the primitive ideal space—which are satisfied for accordion spaces—the invariants filtered K-theory and reduced filtered K-theory are in a certain sense equivalent when restricted to purely infinite graph C^*-algebras.

To be more precise, we show that isomorphisms on the reduced filtered K-theory of purely infinite graph C^*-algebras over so-called EBP spaces lift to isomorphisms on *concrete* filtered K-theory – this invariant may be considered as a more explicit model of filtered K-theory: the two are known to coincide for many spaces but not in general (compare Remark 5.16). Along the way, we introduce filtered K-theory *restricted to the canonical base*, denoted $\mathrm{FK}_{\mathcal{B}}$, and show that, for real-rank-zero C^*-algebras over an EBP space, isomorphisms on $\mathrm{FK}_{\mathcal{B}}$ lift to isomorphisms on concrete filtered K-theory.

For accordion spaces, our results furnish one-to-one correspondences, induced by the different variants of filtered K-theory, between purely infinite graph C^*-algebras respectively unital purely infinite graph C^*-algebras or purely infinite Cuntz–Krieger algebras on the one hand, and certain types of modules in the respective target categories on the other hand. In particular, we obtain the desired characterization of purely infinite Cuntz–Krieger algebras with accordion spaces as primitive ideal spaces:

Theorem 1.1. *Let A be a C^*-algebra whose primitive ideal space is an accordion space. Then A is a purely infinite Cuntz–Krieger algebra if and only if A satisfies the following:*

- *A is unital, purely infinite, nuclear, separable, and of real rank zero,*
- *for all ideals I and J of A with $I \subseteq J$ and J/I simple, the quotient J/I belongs to the bootstrap class, the group $\mathrm{K}_*(J/I)$ is finitely generated, the group $\mathrm{K}_1(J/I)$ is free and $\operatorname{rank}\mathrm{K}_1(J/I) = \operatorname{rank}\mathrm{K}_0(J/I)$.*

In the terms introduced by the first named author in [2], our Theorem 1.1 states that there is no *phantom Cuntz–Krieger algebra* whose primitive ideal space is an accordion space. It is an open question whether this holds for all finite primitive ideal spaces.

1.1. **Historical account.** By a seminal result of Eberhard Kirchberg, $\mathrm{KK}(X)$-equivalences between stable Kirchberg X-algebras, that is, stable, tight, $\mathcal{O}_\infty$-absorbing, nuclear, separable C^*-algebras over a space X, lift to X-equivariant *-isomorphisms. In [19], Ralf Meyer and Ryszard Nest established a Universal Coefficient Theorem computing the equivariant bivariant theory $\mathrm{KK}(X)$ from filtered K-theory under the assumption that the topology of X is finite and totally ordered. As a result, for such spaces X, isomorphisms on filtered K-theory between stable Kirchberg X-algebras with simple subquotients in the bootstrap class lift to X-equivariant *-isomorphisms. This result was generalized in [8] by the second-named author and Manuel Köhler to the case of so-called accordion spaces. Building on these results, Søren Eilers, Gunnar Restorff, and Efren Ruiz classified in [13] certain classes of real-rank-zero (not necessarily purely infinite) graph C^*-algebras using *ordered* filtered K-theory.

On the other hand, Meyer–Nest and the second-named author have constructed counterexamples to the analogous classification statement over all six four-point non-accordion connected T_0-spaces. More precisely, for each of these spaces X, they exhibit two non-$\mathrm{KK}(X)$-equivalent Kirchberg X-algebras with simple subquotients in the bootstrap class whose filtered K-theory is isomorphic (see [6, 19]).

Despite this obstruction, it had previously been shown by Gunnar Restorff in [23] that filtered K-theory—in fact reduced filtered K-theory—is a complete invariant for purely infinite Cuntz–Krieger algebras. Any finite T_0-space, in particular the six problematic four-point spaces mentioned above, can be realized as the primitive ideal space of a purely infinite Cuntz–Krieger algebra. Unfortunately, Restorff's result only gives an *internal* classification of Cuntz–Krieger algebras and admits no conclusion concerning when a given Cuntz–Krieger algebra is stably isomorphic to a given purely infinite, nuclear, separable C^*-algebra with the same ideal structure and filtered K-theory.

In [4], Gunnar Restorff, Efren Ruiz, and the first-named author noted that, for five of the six problematic four-point spaces, the constructed counterexamples to classification do *not* have real rank zero. They went on to show that for four of these spaces X, filtered K-theory is in fact a complete invariant for Kirchberg X-algebras of real rank zero with simple subquotients in the bootstrap class. The four-point non-accordion space for which the constructed counterexample does have real rank zero will be denoted by $\mathcal{D}$.

It is a general property of Cuntz–Krieger algebras that the K_1-group of every subquotient is free. The same is true, more generally, for graph C^*-algebras. We observe that, for real-rank-zero C^*-algebras over $\mathcal{D}$ satisfying this K-theoretic condition, isomorphisms on the reduced filtered K-theory lift to $\mathrm{KK}(\mathcal{D})$-equivalences (see Proposition 7.17). There are therefore no known counterexamples to classification by filtered K-theory of Kirchberg X-algebras with simple subquotients in the bootstrap class that have the K-theory of a real-rank-zero graph C^*-algebra.

1.2. **Organization of the paper.** After fixing some basic conventions and definitions in Section 2, we introduce filtered K-theory FK and *concrete* filtered K-theory $\mathrm{FK}_{\mathcal{ST}}$ in Section 3. Section 4 contains some basic definitions and facts concerning sheaves and cosheaves.

In Section 5, filtered K-theory restricted to the canonical base $\mathrm{FK}_{\mathcal{B}}$ is defined for spaces with the unique path property. We introduce EBP spaces and show that the concrete filtered K-theory $\mathrm{FK}_{\mathcal{ST}}(A)$ of a real-rank-zero C^*-algebra A over an EBP space is completely determined by the filtered K-theory restricted to the canonical base $\mathrm{FK}_{\mathcal{B}}(A)$, see Corollary 5.19.

In Section 6, reduced filtered K-theory $\mathrm{FK}_{\mathcal{R}}$ is defined, and it is shown in Section 7 that the concrete filtered K-theory $\mathrm{FK}_{\mathcal{ST}}(A)$ of a real-rank-zero C^*-algebra A over an EBP space satisfying that all subquotients have free K_1-groups can be recovered from the reduced filtered K-theory $\mathrm{FK}_{\mathcal{R}}(A)$, see Corollary 7.15. This is of particular interest because of the range results from [3] for (unital) reduced filtered K-theory on (unital) purely infinite graph C^*-algebras, see Theorem 6.12 (and 8.10). In order to proceed from reduced to concrete filtered K-theory in Section 7, an "intermediate" invariant is introducted, which serves only technical purposes.

In Sections 8 and 9, unital filtered K-theory and ordered filtered K-theory are treated. The most complete results in our framework are possible for C^*-algebras with primitive ideal spaces of accordion type; these are summarized in Section 10.

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2. Notation

In this article, matrices act from the right and the composite of maps $A \xrightarrow{f} B \xrightarrow{g} C$ is denoted by fg. The category of abelian groups is denoted by $\mathfrak{Ab}$, the category of $\mathbb{Z}/2$-graded abelian groups by $\mathfrak{Ab}^{\mathbb{Z}/2}$.

Let X be a finite T_0-space. For a subset Y of X, we let $\overline{Y}$ denote the closure of Y in X, and let $\overline{\partial}Y$ denote the boundary $\overline{Y} \setminus Y$ of Y. Since X is a finite space, there exists a smallest open subset $\widetilde{Y}$ of X containing Y. We let $\widetilde{\partial}Y$ denote the set $\widetilde{Y} \setminus Y$. For $x, y \in X$ we write $x \leq y$ when $\overline{\{x\}} \subseteq \overline{\{y\}}$, and $x < y$ when $x \leq y$ and $x \neq y$. We write $y \to x$ when $x < y$ and no $z \in X$ satisfies $x < z < y$. A *path* from y to x is a sequence $(z_k)_{k=1}^n$ such that $z_{k+1} \to z_k$ for $k = 1, \ldots, n-1$ and $z_1 = x$, $z_n = y$. We let $\mathrm{Path}(y, x)$ denote the set of paths from y to x.

Definition 2.1. An *accordion space* is a T_0-space $X = \{x_1, \ldots, x_n\}$ such that for every $k = 1, 2, \ldots, n-1$ either $x_k \to x_{k+1}$ or $x_k \leftarrow x_{k+1}$ holds and such that $x_k \to x_l$ does not hold for any k, l with $|k - l| \neq 1$.

For instance, if X is linear, that is, if $X = \{x_1, \ldots, x_n\}$ with $x_n \to \cdots \to x_2 \to x_1$, then X is an accordion space.

3. Filtered K-theory

In this section filtered K-theory and concrete filtered K-theory are defined. Some properties of objects in their target categories are introduced.

A *C^*-algebra A over X* is (equivalently given by) a C^*-algebra A equipped with an infima- and suprema-preserving map $\mathbb{O}(X) \to \mathbb{I}(A), U \mapsto A(U)$ mapping open subsets in X to (closed, two-sided) ideals in A (in particular it holds that $A(\emptyset) = 0$ and $A(X) = A$). The C^*-algebra A is called *tight* over X if the map is a lattice-isomorphism. A *-homomorphism $\varphi\colon A \to B$ for C^*-algebras A and B over X is called *X-equivariant* if $\varphi\big(A(U)\big) \subseteq B(U)$ for all $U \in \mathbb{O}(X)$. Let $\mathbb{LC}(X)$ denote the set of locally closed subsets of X, that is, subsets of the form $U \setminus V$ with U and V open subsets of X satisfying $V \subseteq U$. For $Y \in \mathbb{LC}(X)$, and $U, V \in \mathbb{O}(X)$ satisfying that $Y = U \setminus V$ and $U \supseteq V$, we define $A(Y)$ as the subquotient $A(Y) = A(U)/A(V)$, which up to natural isomorphism is independent of the choice of U and V (see [20, Lemma 2.15]).

Definition 3.1. A tight, $\mathcal{O}_\infty$-absorbing, nuclear, separable C^*-algebra over X is called a *Kirchberg X-algebra*.

Let $\mathfrak{KK}(X)$ be the additive category whose objects are separable C^*-algebras over X and whose set of morphisms from A to B is the Kasparov group $\mathrm{KK}_0(X; A, B)$ defined by Kirchberg (see [20, Section 3] for details). For a C^*-algebra A over X, a $\mathbb{Z}/2$-graded abelian group $\mathrm{FK}^*_Y(A)$ is defined as $\mathrm{K}_*\big(A(Y)\big)$ for all $Y \in \mathbb{LC}(X)$. Thus FK^*_Y is an additive funtor from $\mathfrak{KK}(X)$ to the category $\mathfrak{Ab}^{\mathbb{Z}/2}$ of $\mathbb{Z}/2$-graded abelian groups. Ralf Meyer and Ryszard Nest constructed in [19] C^*-algebras R_Y over X satisfying that the functors FK^*_Y and $\mathrm{KK}_*(X; R_Y, -)$ are naturally isomorphic.

In their definition of filtered K-theory FK^*, Meyer–Nest consider the $\mathbb{Z}/2$-graded pre-additive category $\mathcal{NT}_*$ with objects $\mathbb{LC}(X)$ and morphisms

$$\mathrm{Nat}_*(\mathrm{FK}^*_Y, \mathrm{FK}^*_Z) \cong \mathrm{KK}_*(X; R_Z, R_Y)$$

between Y and Z, where $\mathrm{Nat}_*(\mathrm{FK}^*_Y, \mathrm{FK}^*_Z)$ denotes the set of graded natural transformations from the functor FK^*_Y to the functor FK^*_Z. The target category of FK^* is the category $\mathfrak{Mod}(\mathcal{NT}_*)^{\mathbb{Z}/2}$ of graded modules over $\mathcal{NT}_*$, that is, $\mathbb{Z}/2$-graded additive functors $\mathcal{NT}_* \to \mathfrak{Ab}^{\mathbb{Z}/2}$. Hence $\mathrm{FK}^*(A)$ consists of the groups $\mathrm{FK}^*_Y(A)$ together with the natural transformations $\mathrm{FK}^*_Y(A) \to \mathrm{FK}^*_Z(A)$.

For reasons of notation we will often find it convenient to consider instead the pre-additive category $\mathcal{NT}$ with objects $\mathbb{LC}(X) \times \{0, 1\}$ and morphisms between (Y, j) and (Z, k) given by natural transformations

$$\mathrm{Nat}(\mathrm{FK}^j_Y\, \mathrm{FK}^k_Z) \cong \mathrm{KK}_0(X; \Sigma^k R_Z, \Sigma^j R_Y),$$

where $\mathrm{FK}^j_Y(A)$ denotes $\mathrm{K}_j\big(A(Y)\big)$ for $j = 0, 1$ and Σ denotes suspension (with $\Sigma^0 A = A$). Let $\mathfrak{Mod}(\mathcal{NT})$ denote the category of modules over $\mathcal{NT}$, that is, additive functors $\mathcal{NT} \to \mathfrak{Ab}$.

Given a graded $\mathcal{NT}_*$-module M, we define an $\mathcal{NT}$-module $D(M)$ as follows: we set $D(M)(Y, i) = M(Y)_i$ for $(Y, i) \in \mathbb{LC}(X) \times \{0, 1\}$; for a morphism $f\colon (Y, i) \to (Z, j)$ in $\mathcal{NT}$, we define $D(M)(f)\colon D(M)(Y, i) \to D(M)(Z, j)$ as the composite

$$M(Y)_i \hookrightarrow M(Y)_* \xrightarrow{M(f)} M(Z)_* \twoheadrightarrow M(Z)_j.$$

It is straightforward to check that this yields a functor $D\colon \mathfrak{Mod}(\mathcal{NT}_*)^{\mathbb{Z}/2} \to \mathfrak{Mod}(\mathcal{NT})$. In fact, D is an equivalence of categories—an inverse can be defined by a direct sum construction. Consequently, we define the functor $\mathrm{FK}\colon \mathfrak{KK}(X) \to \mathfrak{Mod}(\mathcal{NT})$ as the composite $\mathrm{FK} = D \circ \mathrm{FK}^*$.

Definition 3.2. Let $Y \in \mathbb{LC}(X)$, $U \subseteq Y$ be open in Y, and set $C = Y \setminus U$. A pair (U, C) obtained in this way is called a *boundary pair*. The natural transformations occuring in the six-term exact sequence in K-theory for the distinguished subquotient inclusion associated to $U \subseteq Y$ are denoted by i^Y_U, r^C_Y and δ^Y_C:

$$\begin{tikzcd}
\mathrm{FK}_U \arrow[rr, "i^Y_U"] & & \mathrm{FK}_Y \arrow[dl, "r^C_Y"] \\
& \mathrm{FK}_C \arrow[ul, "\delta^U_C"] &
\end{tikzcd}$$

These elements i^Y_U, r^C_Y and δ^Y_C correspond to the $\mathrm{KK}(X)$-classes of the *-homomorphisms $R_Y \twoheadrightarrow R_U$, $R_C \hookrightarrow R_Y$, and the extention $R_C \hookrightarrow R_Y \twoheadrightarrow R_U$, see [19]. These elements of $\mathcal{NT}_*$ satisfy the following relations.

Proposition 3.3. *In the category $\mathcal{NT}_*$, the following relations hold.*

(1) *For every $Y \in \mathbb{LC}(X)$,*

$$i^Y_Y = r^Y_Y = \mathrm{id}_Y .$$

(2) *If $Y, Z \in \mathbb{LC}(X)$ are topologically disjoint, then $Y \cup Z \in \mathbb{LC}(X)$ and*

$$r^Y_{Y\cup Z} i^{Y\cup Z}_Y + r^Z_{Y\cup Z} i^{Y\cup Z}_Z = \mathrm{id}_{Y\cup Z} .$$

(3) *For $Y \in \mathbb{LC}(X)$ and open subsets $U \subseteq V \subseteq Y$,*

$$i_U^V i_V^Y = i_U^Y.$$

(4) *For $Y \in \mathbb{LC}(X)$ and closed subsets $C \subseteq D \subseteq Y$,*

$$r_Y^D r_D^C = r_Y^C.$$

(5) *For $Y \in \mathbb{LC}(X)$, an open subset $U \subseteq Y$ and a closed subset $C \subseteq Y$,*

$$i_U^Y r_Y^C = r_U^{U\cap C} i_{U\cap C}^C.$$

(6) *For a boundary pair (U, C) in X and an open subset $C' \subseteq C$, (U, C') is a boundary pair and we have*

$$i_{C'}^C \delta_C^U = \delta_{C'}^U.$$

(7) *For a boundary pair (U, C) in X and a closed subset $U' \subseteq U$, (U', C) is a boundary pair and we have*

$$\delta_C^U r_U^{U'} = \delta_C^{U'}.$$

(8) *For $Y, Z, W \in \mathbb{LC}(X)$ such that $Y \cup W \in \mathbb{LC}(X)$ containing Y, W as closed subsets, $Z \cup W \in \mathbb{LC}(X)$ containing Z, W as open subsets, and $W \subseteq Y \cup Z$, we have*

$$\delta_Y^{W\setminus Y} i_{W\setminus Y}^Z = r_Y^{W\setminus Z} \delta_{W\setminus Z}^Z.$$

Proof. We only prove (8), because the other relations can be proved similarly and more easily (their proofs can be found in [6, Section 3.2]).

Let us take $Y, Z, W \in \mathbb{LC}(X)$ as in (8). Let us also take a C^*-algebra A over X. Since both Y and W are closed subsets of $Y \cup W \in \mathbb{LC}(X)$, $Y \cap W$ is closed both in Y and in W. Therefore we have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & A(W\setminus Y) \arrow[r] \arrow[d, equal] & A(Y\cup W) \arrow[r] \arrow[d] & A(Y) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A(W\setminus Y) \arrow[r] & A(W) \arrow[r] & A(Y\cap W) \arrow[r] & 0.
\end{tikzcd}$$

Since both Z and W are open subsets of $Z \cup W \in \mathbb{LC}(X)$, $Z \cap W$ is open both in Z and in W. Therefore we have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & A(Z\cap W) \arrow[r] \arrow[d] & A(W) \arrow[r] \arrow[d] & A(W\setminus Z) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A(Z) \arrow[r] & A(Z\cup W) \arrow[r] & A(W\setminus Z) \arrow[r] & 0.
\end{tikzcd}$$

From $W \subseteq Y \cup Z$, we get $W \setminus Y \subseteq Z \cap W$ and $W \setminus Z \subseteq Y \cap W$. Since $W \setminus Y$ is open in W, we see that $W \setminus Y$ is open in $Z \cap W$. Similarly, $W \setminus Z$ is closed in $Y \cap W$. Hence we get a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & A(W\setminus Y) \arrow[r] \arrow[d] & A(W) \arrow[r] \arrow[d, equal] & A(Y\cap W) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A(Z\cap W) \arrow[r] & A(W) \arrow[r] & A(W\setminus Z) \arrow[r] & 0.
\end{tikzcd}$$

By combining these three diagrams, we obtain a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & A(W\setminus Y) \arrow[r] \arrow[d] & A(Y\cup W) \arrow[r] \arrow[d] & A(Y) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A(Z) \arrow[r] & A(Z\cup W) \arrow[r] & A(W\setminus Z) \arrow[r] & 0.
\end{tikzcd}$$

From this digram, we get a commutative diagram

$$\begin{tikzcd}
\mathrm{K}_*\big(A(Y\cup W)\big) \arrow[r, "r"] \arrow[d, "ri"] & \mathrm{K}_*\big(A(Y)\big) \arrow[r, "\delta"] \arrow[d, "r"] & \mathrm{K}_*\big(A(W\setminus Y)\big) \arrow[r, "i"] \arrow[d, "i"] & \mathrm{K}_*\big(A(Y\cup W)\big) \arrow[d, "ri"] \\
\mathrm{K}_*\big(A(Z\cup W)\big) \arrow[r, "r"] & \mathrm{K}_*\big(A(W\setminus Z)\big) \arrow[r, "\delta"] & \mathrm{K}_*\big(A(Z)\big) \arrow[r, "i"] & \mathrm{K}_*\big(A(Z\cup W)\big).
\end{tikzcd}$$

Now (8) follows from the commutativity of the middle square of this natural diagram. □

Remark 3.4. From Proposition 3.3(2), we see that the empty set $\emptyset$ is a zero object in $\mathcal{NT}_*$ (because initial objects in pre-additive categories are also terminal). From this and other relations in Proposition 3.3, we can conclude that compositions of consecutive maps in six-term sequences associated to relatively open subset inclusions vanish.

Remark 3.5. We usually denote the even and the odd component of the element i_U^Y in $\mathcal{NT}_*$ defined in Definition 3.2 simply by i_U^Y. Often, sub- and superscripts are suppressed when clear from context. Similar comments apply to r and δ.

Definition 3.6. Let $\mathcal{ST}_*$ be the universal $\mathbb{Z}/2$-graded pre-additive category whose set of objects is $\mathbb{LC}(X)$ and whose set of morphisms are generated by elements as in Definition 3.2 with the relations as in Proposition 3.3. Let $\mathcal{ST}$ be the corresponding pre-additive category with object set $\mathbb{LC}(X)\times\{0,1\}$.

By Proposition 3.3, we have a canonical additive functor $\mathcal{ST}\to\mathcal{NT}$. This functor has been shown to be an isomorphism in all examples which have been investigated—including accordion spaces and all four-point spaces (see [6,19]). However there is an example Q of a finite T_0-space for which the functor $\mathcal{ST}\to\mathcal{NT}$ seems to be non-faithful (see Remark 5.16). For such spaces one would need to modify the definition of the category $\mathcal{ST}$, but we do not pursue this problem in this paper.

Let $\mathfrak{F}_{\mathcal{ST}}\colon \mathfrak{Mod}(\mathcal{NT})\to\mathfrak{Mod}(\mathcal{ST})$ be the functor induced by the canonical functor $\mathcal{ST}\to\mathcal{NT}$.

Definition 3.7. We define *concrete filtered* K*-theory* $\mathrm{FK}_{\mathcal{ST}}\colon \mathfrak{KK}(X)\to\mathfrak{Mod}(\mathcal{ST})$ as the composition $\mathfrak{F}_{\mathcal{ST}}\circ\mathrm{FK}$.

Remark 3.8. As noted above, filtered K-theory FK and concrete filtered K-theory $\mathrm{FK}_{\mathcal{ST}}$ coincide for accordion spaces and all four-point spaces.

Definition 3.9. An $\mathcal{NT}$-module M is called *exact* if, for all $Y \in \mathbb{LC}(X)$ and $U \in \mathbb{O}(Y)$, the sequence

$$\begin{tikzcd} M(U,0) \arrow[r, "i"] & M(Y,0) \arrow[r, "r"] & M(Y \setminus U, 0) \arrow[d, "\delta"] \\ M(Y \setminus U, 1) \arrow[u, "\delta"] & M(Y,1) \arrow[l, "r"] & M(U,1) \arrow[l, "i"] \end{tikzcd}$$

is exact. An $\mathcal{NT}$-module M is called *real-rank-zero-like* if, for all $Y \in \mathbb{LC}(X)$ and $U \in \mathbb{O}(Y)$, the map $\delta\colon M(Y \setminus U, 0) \to M(U,1)$ vanishes.

In the same way, we define exact $\mathcal{ST}$-modules and real-rank-zero-like $\mathcal{ST}$-modules.

Remark 3.10. For a C^*-algebra A over X, the module $\mathrm{FK}(A)$ is exact. It follows from [7, Lemma 3.4] that, if A is tight over X, then $\mathrm{FK}(A)$ is real-rank-zero-like if and only if the underlying C^*-algebra of A is K_0-*liftable* in the sense of Pasnicu–Rørdam [21]. By [18, Proposition 4], all real-rank-zero C^*-algebras are K_0-liftable. By Theorem 4.2 and Example 4.8 of [21], a tight, purely infinite C^*-algebra A over X has real rank zero if and only if $\mathrm{FK}(A)$ is real-rank-zero-like. Analogous remarks apply with $\mathrm{FK}_{\mathcal{ST}}(A)$ in place of $\mathrm{FK}(A)$.

The following theorem is the basis for the corollaries obtained in Section 10.

Theorem 3.11 ([8, 17, 19])**.** *Let X be an accordion space. The canonical functor $\mathcal{ST} \to \mathcal{NT}$ is an isomorphism. Moreover, if A and B are stable Kirchberg X-algebras with all simple subquotients in the bootstrap class, then any isomorphism $\mathrm{FK}(A) \to \mathrm{FK}(B)$ lifts to an X-equivariant *-isomorphism $A \to B$.*

4. Sheaves

In this section we introduce sheaves and cosheaves and recall that it suffices to specify them on a basis for the topology.

Let X be an arbitrary topological space. Let $\mathbb{B}$ be a basis for the topology on X. We note that the set $\mathbb{O}$ of all open subsets is the largest basis for the topology on X. We also note that for a finite space X, the collection $\big\{\widetilde{\{x\}} \mid x \in X\big\}$ is an example of a basis. The set $\mathbb{B}$ is a category whose morphisms are inclusions.

Definition 4.1. A *covering* of a set $U \in \mathbb{B}$ is a collection $\{U_j\}_{j\in J} \subseteq \mathbb{B}$ such that $U_j \subseteq U$ for all $j \in J$ and $\bigcup_{j\in J} U_j = U$. A *presheaf on* $\mathbb{B}$ is a contravariant functor $M\colon \mathbb{B} \to \mathfrak{Ab}$. It is a *sheaf on* $\mathbb{B}$ if, for every $U \in \mathbb{B}$, every covering $\{U_j\}_{j\in J} \subseteq \mathbb{B}$ of U, and all coverings $\{U_{jkl}\}_{l\in L_{jk}} \subseteq \mathbb{B}$ of $U_j \cap U_k$, the sequence

$$0 \longrightarrow M(U) \xrightarrow{\left(M(i_U^{U_j})\right)} \prod_{j\in J} M(U_j) \xrightarrow{\left(M(i_{U_j}^{U_{jkl}}) - M(i_{U_k}^{U_{jkl}})\right)} \prod_{j,k\in J} \prod_{l\in L_{jk}} M(U_{jkl}) \tag{4.2}$$

is exact. A morphism for sheaves is a natural transformation of functors. We denote by $\mathfrak{Sh}(\mathbb{B})$ the category of sheaves on $\mathbb{B}$.

If $\mathbb{B}$ is closed under intersection (for example if $\mathbb{B} = \mathbb{O}$), then the definition of sheaf can be replaced with the exactness of the sequence

$$0 \longrightarrow M(U) \xrightarrow{\left(M(i_U^{U_j})\right)} \prod_{j\in J} M(U_j) \xrightarrow{\left(M(i_{U_j}^{U_j\cap U_k})-M(i_{U_k}^{U_j\cap U_k})\right)} \prod_{j,k\in J} M(U_j\cap U_k)$$

for all $U \in \mathbb{B}$ and every covering $\{U_j\}_{j\in J} \subseteq \mathbb{B}$ of U.

Lemma 4.3. *For a basis $\mathbb{B}$ for the topology on X, the restriction functor $\mathfrak{Sh}(\mathbb{O}) \to \mathfrak{Sh}(\mathbb{B})$ is an equivalence of categories.*

Proof. This is a well-known fact in algebraic geometry (see, for instance the encyclopedic treatment in [27, Lemma 009O]). We confine ourselves on mentioning that (4.2) provides a formula for computing $M(U)$ for an arbitrary open subset U. □

Definition 4.4. A *precosheaf on* $\mathbb{B}$ is a covariant functor $M\colon \mathbb{B} \to \mathfrak{Ab}$. It is a *cosheaf on* $\mathbb{B}$ if, for every $U \in \mathbb{B}$, every covering $\{U_j\}_{j\in J} \subseteq \mathbb{B}$ of U, and all coverings $\{U_{jkl}\}_{l\in L_{jk}} \subseteq \mathbb{B}$ of $U_j \cap U_k$, the sequence

$$\bigoplus_{j,k\in J}\bigoplus_{l\in L_{jk}} M(U_{jkl}) \xrightarrow{\left(M(i_{U_{jkl}}^{U_j})-M(i_{U_{jkl}}^{U_k})\right)} \bigoplus_{j\in J} M(U_j) \xrightarrow{\left(M(i_{U_j}^{U})\right)} M(U) \longrightarrow 0. \tag{4.5}$$

is exact. A morphism for cosheaves is a natural transformation of functors. We denote by $\mathfrak{CoSh}(\mathbb{B})$ the category of cosheaves on $\mathbb{B}$.

Similarly to the case of sheaves, if $\mathbb{B}$ is closed under intersection, the definition of cosheaf can be replaced with the exactness of the sequence

(4.6)

$$\bigoplus_{j,k\in J} M(U_j\cap U_k) \xrightarrow{\left(M(i_{U_j\cap U_k}^{U_j})-M(i_{U_j\cap U_k}^{U_k})\right)} \bigoplus_{j\in J} M(U_j) \xrightarrow{\left(M(i_{U_j}^{U})\right)} M(U) \longrightarrow 0.$$

for $U \in \mathbb{B}$ and a covering $\{U_j\}_{j\in J} \subseteq \mathbb{B}$ of U.

Lemma 4.7. *The restriction functor $\mathfrak{CoSh}(\mathbb{O}) \to \mathfrak{CoSh}(\mathbb{B})$ is an equivalence of categories.*

Proof. This statement is the dual of Lemma 4.3 and follows in an analogous way. Again, (4.5) can be used to compute $M(U)$ for an arbitrary open subset U. □

With regard to the next section we remark that every finite T_0-space (more generally every Alexandrov space) comes with canonical bases for the open subsets, namely $\bigl\{\widetilde{\{x\}} \mid x \in X\bigr\}$, and for the closed subsets: $\bigl\{\overline{\{x\}} \mid x \in X\bigr\}$.

Lemma 4.8. *Let X be a finite T_0-space and let S be a pre(co)sheaf on the basis $\mathbb{B} = \bigl\{\widetilde{\{x\}} \mid x \in X\bigr\}$. Then S is a (co)sheaf.*

Proof. This follows from the observation that in the basis $\mathbb{B}$ there are no non-trivial coverings, that is, if $\mathcal{U}$ is a covering of U, then $U \in \mathcal{U}$. □

5. Filtered K-theory restricted to the canonical base

In this section, the functor $\mathrm{FK}_{\mathcal{B}}$ and the notions of unique path spaces and EBP spaces are introduced. The following lemma is straightforward to verify.

Lemma 5.1. *For a finite T_0-space X the following conditions are equivalent.*

- *For all $x, y \in X$, there is at most one path from y to x.*
- *There are no elements a, b, c, d in X with $a < b < d$, $a < c < d$ and neither $b \leq c$ nor $c \leq b$.*
- *For all $x, y \in X$ with $x \to y$, we have $\widetilde{\{x\}} \cup \overline{\{y\}} \in \mathbb{LC}(X)$.*
- *For every boundary pair (U, C), the pair $(\widetilde{U}, \overline{C})$ is a boundary pair.*
- *For all $x \in X$, $\widetilde{\partial}\{x\} = \bigsqcup_{y \to x} \widetilde{\{y\}}$.*
- *For all $x \in X$, $\overline{\partial}\{x\} = \bigsqcup_{y \leftarrow x} \overline{\{y\}}$.*

Definition 5.2. A finite T_0-space X is called a *unique path space* if it satisfies the equivalent conditions specified in Lemma 5.1.

Let X be a unique path space.

Definition 5.3. Let $\mathcal{B}$ denote the universal pre-additive category generated by objects $\overline{x}_1$, $\widetilde{x}_0$ for all $x \in X$ and morphisms $r_{\overline{x}_1}^{\overline{y}_1}$, $\delta_{\overline{y}_1}^{\widetilde{x}_0}$ and $i_{\widetilde{x}_0}^{\widetilde{y}_0}$ when $x \to y$, subject to the relations

$$\sum_{x \to y} r_{\overline{x}_1}^{\overline{y}_1} \delta_{\overline{y}_1}^{\widetilde{x}_0} = \sum_{z \to x} \delta_{\overline{x}_1}^{\widetilde{z}_0} i_{\widetilde{z}_0}^{\widetilde{x}_0} \tag{5.4}$$

for all $x \in X$.

Lemma 5.5. *In the category $\mathcal{ST}$, we have the relation*

$$\sum_{x \to y} r_{\overline{\{x\}}}^{\overline{\{y\}}} \delta_{\overline{\{y\}}}^{\widetilde{\{x\}}} = \sum_{z \to x} \delta_{\overline{\{x\}}}^{\widetilde{\{z\}}} i_{\widetilde{\{z\}}}^{\widetilde{\{x\}}}$$

for all $x \in X$.

Proof. Since X is a unique path space, the collections $\left(\overline{\{y\}}\right)_{x \to y}$ and $\left(\widetilde{\{z\}}\right)_{z \to x}$ are disjoint, respectively. Hence the desired relation simplifies to

$$r_{\overline{\{x\}}}^{\overline{\partial}\{x\}} \delta_{\overline{\partial}\{x\}}^{\widetilde{\{x\}}} = \delta_{\overline{\{x\}}}^{\widetilde{\partial}\{x\}} i_{\widetilde{\partial}\{x\}}^{\widetilde{\{x\}}},$$

which follows from Proposition 3.3(8) by setting $Y = \overline{\{x\}}$, $Z = \widetilde{\{x\}}$ and $W = \overline{\{x\}} \cup \widetilde{\{x\}}$. $\square$

Definition 5.6. The previous lemma allows us to define an additive functor $\mathcal{B} \to \mathcal{ST}$ by $\overline{x}_1 \mapsto (\overline{\{x\}}, 1)$ and $\widetilde{x}_0 \mapsto (\widetilde{\{x\}}, 0)$, and in the obvious way on morphisms. Let

$$\mathfrak{F}_{\mathcal{B}} \colon \mathfrak{Mod}(\mathcal{ST}) \to \mathfrak{Mod}(\mathcal{B})$$

denote the induced functor. Define *filtered* K*-theory restricted to the canonical base*, $\mathrm{FK}_{\mathcal{B}} \colon \mathfrak{KK}(X) \to \mathfrak{Mod}(\mathcal{B})$, as the composition of $\mathrm{FK}_{\mathcal{ST}}$ with $\mathfrak{F}_{\mathcal{B}}$.

Remark 5.7. The invariant $\mathrm{FK}_{\mathcal{B}}$ is only defined for unique path spaces because the boundary map $\delta_{\overline{\{y\}}}^{\widetilde{\{x\}}}$ only exists when $\overline{\{y\}} \cup \widetilde{\{x\}}$ belongs to $\mathbb{LC}(X)$. We also point out that the invariant $\mathrm{FK}_{\mathcal{B}}$ can only be expected to be very useful for spaces such that the relation (5.14) holds for all boundary pairs (U, C).

Definition 5.8. A $\mathcal{B}$-module M is called *exact* if the sequence

$$M(\overline{x}_1) \xrightarrow{\begin{pmatrix} r_{\overline{x}_1}^{\overline{y}_1} & -\delta_{\overline{x}_1}^{\widetilde{z}_0} \end{pmatrix}} \bigoplus_{x \to y} M(\overline{y}_1) \oplus \bigoplus_{z \to x} M(\widetilde{z}_0) \xrightarrow{\begin{pmatrix} \delta_{\overline{y}_1}^{\widetilde{x}_0} \\ i_{\widetilde{z}_0}^{\widetilde{x}_0} \end{pmatrix}} M(\widetilde{x}_0) \tag{5.9}$$

is exact for all $x \in X$.

Lemma 5.10. *If M is an exact $\mathcal{ST}$-module, then $\mathfrak{F}_{\mathcal{B}}(M)$ is an exact $\mathcal{B}$-module. In particular, if A is a C^*-algebra over X, then the $\mathcal{B}$-module $\mathrm{FK}_{\mathcal{B}}(A)$ is exact.*

Proof. Using again that the collections $\left(\overline{\{y\}}\right)_{x \to y}$ and $\left(\widetilde{\{z\}}\right)_{z \to x}$ are respectively disjoint, it suffices to prove exactness of the sequence

$$M(\overline{\{x\}}, 1) \xrightarrow{\begin{pmatrix} r_{\{x\}}^{\overline{\partial}\{x\}} & -\delta_{\{x\}}^{\widetilde{\partial}\{x\}} \end{pmatrix}} M(\overline{\partial}\{x\}, 1) \oplus M(\widetilde{\partial}\{x\}, 0) \xrightarrow{\begin{pmatrix} \delta_{\overline{\partial}\{x\}}^{\widetilde{\{x\}}} \\ i_{\widetilde{\partial}\{x\}}^{\widetilde{\{x\}}} \end{pmatrix}} M(\widetilde{\{x\}}, 0),$$

which follows from a diagram chase through the commutative diagram

$$\begin{tikzcd}
M(\{x\}, 1) \arrow[r] \arrow[d, equal] & M(\overline{\{x\}}, 1) \arrow[r] \arrow[d, "\circ"] & M(\overline{\partial}\{x\}, 1) \arrow[r, "\circ"] \arrow[d, "\circ"] & M(\{x\}, 0) \arrow[d, equal] \\
M(\{x\}, 1) \arrow[r, "\circ"] & M(\widetilde{\partial}\{x\}, 0) \arrow[r] & M(\widetilde{\{x\}}, 0) \arrow[r] & M(\{x\}, 0)
\end{tikzcd}$$

whose rows are exact. □

Definition 5.11. Let X be a finite T_0-space. A boundary pair (U, C) in X is called *elementary* if U and C are connected, U is open, C is closed and if, moreover, $U \subseteq \widetilde{C}$ and $C \subseteq \overline{U}$.

Definition 5.12. A unique path space X is called an *EBP space* if every elementary boundary pair (U, C) in X is of the form $(\widetilde{\{x\}}, \overline{\{y\}})$ for two points x and y in X with $x \to y$.

Lemma 5.13. *Let X be an EBP space, and let (U, C) be a boundary pair in X. Then the following relation holds in the category $\mathcal{ST}_*$:*

$$\delta_C^U = \sum_{x \to y, x \in U, y \in C} r_C^{\overline{\{y\}} \cap C} \, i_{\overline{\{y\}} \cap C}^{\overline{\{y\}}} \, \delta_{\overline{\{y\}}}^{\widetilde{\{x\}}} \, r_{\widetilde{\{x\}}}^{\widetilde{\{x\}} \cap U} \, i_{\widetilde{\{x\}} \cap U}^{U}. \tag{5.14}$$

Proof. We would like to show the relation (5.14) for a boundary pair (U, C) in X. The proof goes by the induction on the number $|U \cup C|$ of elements of $U \cup C$. If either U or C is empty, then both sides of (5.14) are 0. This takes care of the case $|U \cup C| = 0$. Suppose for a natural number n, we have shown (5.14) for all boundary pairs (U, C) with $|U \cup C| \leq n$, and take a boundary pair (U, C) with $|U \cup C| = n + 1$, arbitrarily. We are going to show (5.14) for this pair. If either U or

C is empty, again both sides of (5.14) are zero. So we may assume that both U and C are non-empty. Suppose U is not connected, and choose two non-empty open and closed subsets U_1 and U_2 of U such that $U = U_1 \sqcup U_2$. Then for $i = 1, 2$, (U_i, C) is a boundary pair with $|U_i \cup C| \leq n$. Thus by the assumption of the induction, both (U_1, C) and (U_2, C) satisfy (5.14). Hence by (2), (7) and (3) of Proposition 3.3 we have

$$\begin{aligned}
\delta^U_C &= \delta^U_C (r^{U_1}_U i^U_{U_1} + r^{U_2}_U i^U_{U_2}) \\
&= \delta^{U_1}_C i^U_{U_1} + \delta^{U_2}_C i^U_{U_2} \\
&= \Big(\sum_{x \to y, x \in U_1, y \in C} r^{\overline{\{y\}} \cap C}_C \; i^{\overline{\{y\}}}_{\overline{\{y\}} \cap C} \; \delta^{\widetilde{\{x\}}}_{\overline{\{y\}}} \; r^{\widetilde{\{x\}} \cap U_1}_{\widetilde{\{x\}}} \; i^{U_1}_{\widetilde{\{x\}} \cap U_1} \Big) i^U_{U_1} \\
&\quad + \Big(\sum_{x \to y, x \in U_2, y \in C} r^{\overline{\{y\}} \cap C}_C \; i^{\overline{\{y\}}}_{\overline{\{y\}} \cap C} \; \delta^{\widetilde{\{x\}}}_{\overline{\{y\}}} \; r^{\widetilde{\{x\}} \cap U_2}_{\widetilde{\{x\}}} \; i^{U_2}_{\widetilde{\{x\}} \cap U_2} \Big) i^U_{U_2} \\
&= \sum_{x \to y, x \in U, y \in C} r^{\overline{\{y\}} \cap C}_C \; i^{\overline{\{y\}}}_{\overline{\{y\}} \cap C} \; \delta^{\widetilde{\{x\}}}_{\overline{\{y\}}} \; r^{\widetilde{\{x\}} \cap U}_{\widetilde{\{x\}}} \; i^{U}_{\widetilde{\{x\}} \cap U}
\end{aligned}$$

since we for $x \in U_i$ have $\widetilde{\{x\}} \cap U_1 = \widetilde{\{x\}} \cap U$ because $U_i \subseteq U$ is open. This shows (5.14) for (U, C). Thus we may now assume U is connected. In a very similar way, we get (5.14) using the assumption of the induction if C is not connected. Thus we may assume C is connected. Next suppose we have $U \not\subseteq \widetilde{C}$. Set $U' = U \cap \widetilde{C}$ which is a proper open subset of U. The pair (U', C) is a boundary pair because $U' \cup C = (U \cup C) \cap \widetilde{C} \in \mathbb{LC}(X)$. We have $\delta^U_C = \delta^{U'}_C i^U_{U'}$ by applying (8) of Proposition 3.3 for $Y = C$, $Z = U$ and $W = U' \cup C$. Since $|U' \cup C| \leq n$, we get by the assumption of the induction that

$$\begin{aligned}
\delta^U_C &= \delta^{U'}_C i^U_{U'} \\
&= \Big(\sum_{x \to y, x \in U', y \in C} r^{\overline{\{y\}} \cap C}_C \; i^{\overline{\{y\}}}_{\overline{\{y\}} \cap C} \; \delta^{\widetilde{\{x\}}}_{\overline{\{y\}}} \; r^{\widetilde{\{x\}} \cap U'}_{\widetilde{\{x\}}} \; i^{U'}_{\widetilde{\{x\}} \cap U'} \Big) i^U_{U'} \\
&= \sum_{x \to y, x \in U, y \in C} r^{\overline{\{y\}} \cap C}_C \; i^{\overline{\{y\}}}_{\overline{\{y\}} \cap C} \; \delta^{\widetilde{\{x\}}}_{\overline{\{y\}}} \; r^{\widetilde{\{x\}} \cap U}_{\widetilde{\{x\}}} \; i^{U}_{\widetilde{\{x\}} \cap U}
\end{aligned}$$

since $x \to y$, $x \in U$ and $y \in C$ imply $x \in U'$, and we have $\widetilde{\{x\}} \cap U = \widetilde{\{x\}} \cap U'$. This shows (5.14) for (U, C). Thus we may now assume $U \subseteq \widetilde{C}$. In a very similar way, we get (5.14) using the assumption of the induction if $C \not\subseteq \overline{U}$. Thus we may assume $C \subseteq \overline{U}$.

It remains to show (5.14) for a boundary pair (U, C) such that U and C are connected, $U \subseteq \widetilde{C}$ and $C \subseteq \overline{U}$. To this end, we use the assumption of the lemma. Take such a pair (U, C). Since X is a unique path space, the pair $(\widetilde{U}, \overline{C})$ is a boundary pair by Lemma 5.1. It is not difficult to see that the pair $(\widetilde{U}, \overline{C})$ is elementary. Hence by the assumption of the lemma, there exist $x \in \widetilde{U}$ and $y \in \overline{C}$ such that $\widetilde{U} = \widetilde{\{x\}}$, $\overline{C} = \overline{\{y\}}$ and $x \to y$. By (6) and (7) of Proposition 3.3, we get

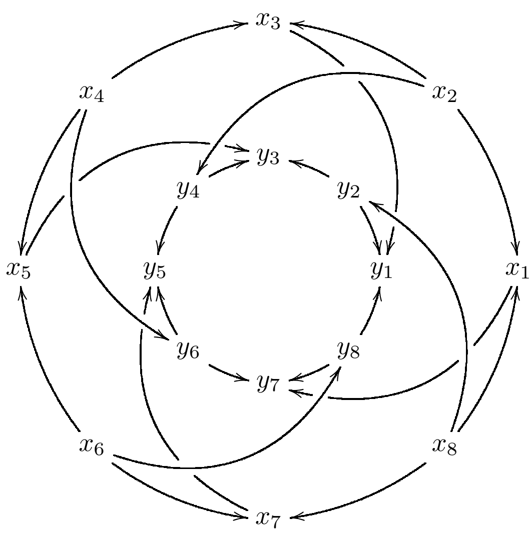
$$\delta^U_C = i^{\overline{C}}_C \delta^{\widetilde{U}}_{\overline{C}} r^U_{\widetilde{U}} = i^{\overline{\{y\}}}_C \delta^{\widetilde{\{x\}}}_{\overline{\{y\}}} r^U_{\widetilde{\{x\}}}.$$

It remains to prove that (x, y) is the only pair satisfying $x \to y$, $x \in U$ and $y \in C$. First note that $\widetilde{U} = \widetilde{\{x\}}$ implies $x \in U$, and also that $\overline{C} = \overline{\{y\}}$ implies $y \in C$. Now take $u \in U$ and $c \in C$ with $u \to c$. Since $U \subseteq \widetilde{\{x\}}$ and $C \subseteq \overline{\{y\}}$, there exist a path from u to x, and a path from y to c. These two paths together with the arrow $x \to y$ give us a path from u to c. Since X is a unique path space, this path should coincide with the arrow $u \to c$. Hence we get $u = x$ and $c = y$. This finishes the proof. $\square$

Lemma 5.15. *Let X be a finite T_0-space. Assume that the directed graph associated to X is a* forrest*, that is, it contains no undirected cycles. Then X is an EBP space.*

Proof. It is clear that, if the directed graph associated to X is a forrest, then X is a unique path space. Let us take an elementary boundary pair (U, C). Choose a minimal element $x \in U$. Since $U \subseteq \widetilde{C}$, there is $y \in C$ with $x > y$. We can, moreover, assume that $x \to y$ because $U \cup C$ is locally closed and x is minimal in U. Since U is open and C is closed, we have $\widetilde{\{x\}} \subseteq U$ and $\overline{\{y\}} \subseteq C$. We will show that these inclusions are equalities using the fact that X is a forrest. Take $u \in U$ arbitrarily. Since $U \subseteq \widetilde{C}$, there exists an element $c \in C$ such that $u > c$. Thus we have a path from u to c. Since both U and C are connected, there exist undirected paths from u to x and from y to c. These two paths give us an undirected path from u to c through the arrow $x \to y$. This path should coincide with the directed path from u to c because X contains no undirected cycles. Hence we get a path from u to x. This shows $u \in \widetilde{\{x\}}$, and therefore we get $U = \widetilde{\{x\}}$. In a similar manner, we get $C = \overline{\{y\}}$. $\square$

Remark 5.16. The above lemma applies, in particular, to accordion spaces. The conclusion of Lemma 5.13 can also be verified for various unique path spaces which are not forrests—the smallest example being the so-called pseudocircle with four points. Consider, however, the sixteen-point space Q defined by the directed graph



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Then Q is a unique path space that is not an EBP space because the subsets $U = \{x_1, x_2, \ldots, x_8\}$ and $C = \{y_1, y_2, \ldots, y_8\}$ give an elementary boundary pair (U, C) that does not satisfy $U = \widetilde{\{x\}}$ nor $C = \overline{\{y\}}$ for any $x, y \in X$. A simple computation shows that the boundary decomposition (5.14) of δ_C^U indeed holds in the category $\mathcal{NT}_*$. However, we believe that it does not hold in the category $\mathcal{ST}_*$.

The following theorem has two important consequences. Firstly, as stated in Corollary 5.19, it implies that for real-rank-zero C^*-algebras, isomorphisms on $\mathrm{FK}_{\mathcal{B}}$ lift to isomorphisms on $\mathrm{FK}_{\mathcal{ST}}$. By Theorem 3.11, $\mathrm{FK}_{\mathcal{ST}}$ is strongly complete for stable Kirchberg X-algebras when X is an accordion space. Secondly, by Lemma 5.6 of [8], if X is an accordion space, any exact $\mathcal{NT}$-module is of the form $\mathrm{FK}(A)$ for some Kirchberg X-algebra A, so any exact $\mathcal{B}$-module is of the form $\mathrm{FK}_{\mathcal{B}}(A)$ for some Kirchberg X-algebra A of real rank zero. This second consequence is useful for constructing examples of Kirchberg X-algebras.

Theorem 5.17. *Let X be an EBP space. The functor*

$$\mathfrak{F}_{\mathcal{B}}\colon \mathfrak{Mod}(\mathcal{ST}) \to \mathfrak{Mod}(\mathcal{B})$$

restricts to an equivalence between the category of exact real-rank-zero-like $\mathcal{ST}$-modules and the category of exact $\mathcal{B}$-modules.

A proof of this theorem is given after the following remark and corollary.

Remark 5.18. The proof of Theorem 5.17 given below works in fact not only for EBP spaces but more generally for unique path spaces for which the relation (5.14) holds in the category $\mathcal{ST}$ for all boundary pairs (U, C), see Lemma 5.13 and Remark 5.16.

Corollary 5.19. *Let A and B be C^*-algebras of real rank zero over an EBP space X. Then for any homomorphism $\varphi\colon \mathrm{FK}_{\mathcal{B}}(A) \to \mathrm{FK}_{\mathcal{B}}(B)$, there exists a unique homomorphism $\Phi\colon \mathrm{FK}_{\mathcal{ST}}(A) \to \mathrm{FK}_{\mathcal{ST}}(B)$ such that $\mathfrak{F}_{\mathcal{B}}(\Phi) = \varphi$. If φ is an isomorphism, then so is Φ.*

Proof of Theorem 5.17. We will explicitly define a functor from the category of exact $\mathcal{B}$-modules to the category of exact real-rank-zero-like $\mathcal{ST}$-modules.

Let an exact $\mathcal{B}$-module N be given. We will define an $\mathcal{ST}$-module M. We begin in the obvious way: For $x \in X$, let $M(\overline{\{x\}}, 1) = N(\overline{x}_1)$ and $M(\widetilde{\{x\}}, 0) = N(\widetilde{x}_0)$. Similarly, for $x \to y$, we define the even component of $i_{\widetilde{\{x\}}}^{\widetilde{\{y\}}}$ to be $i_{\widetilde{x}_0}^{\widetilde{y}_0}$, the odd component of $r_{\overline{\{x\}}}^{\overline{\{y\}}}$ to be $r_{\overline{x}_1}^{\overline{y}_1}$, and the odd-to-even component of $\delta_{\overline{\{y\}}}^{\widetilde{\{x\}}}$ to be $\delta_{\overline{y}_1}^{\widetilde{x}_0}$. This makes sure that, finally, we will have $\mathfrak{F}_{\mathcal{B}}(M) = N$. Also, we of course define $\delta_C^U\colon M(C, 0) \to M(U, 1)$ to be zero for every boundary pair (U, C) so that M will be real-rank-zero-like.

For $x \geq y$, let $x \to x_1 \to x_2 \to \cdots \to x_n \to y$ be the unique path from x to y. Define the even component of $i_{\widetilde{\{x\}}}^{\widetilde{\{y\}}}$ to be the composition $i_{\widetilde{x}_0}^{\widetilde{x_1}_0} i_{\widetilde{x_1}_0}^{\widetilde{x_2}_0} \cdots i_{\widetilde{x_n}_0}^{\widetilde{y}_0}$ and the odd component of $r_{\overline{\{x\}}}^{\overline{\{y\}}}$ as the composition $r_{\overline{x}_1}^{\overline{x_1}_1} r_{\overline{x_1}_1}^{\overline{x_2}_1} \cdots r_{\overline{x_n}_1}^{\overline{y}_1}$. In case of $x = y$, this specifies to $i_{\widetilde{x}_0}^{\widetilde{x}_0} = \mathrm{id}_{M(\widetilde{\{x\}}, 0)}$ and $r_{\overline{x}_1}^{\overline{x}_1} = \mathrm{id}_{M(\overline{\{x\}}, 1)}$. If we have $x \to y$, then these definitions coincide with the ones we gave before.

We observe that the groups $M(\widetilde{\{x\}}, 0)$ with the maps $i_{\widetilde{\{x\}}}^{\widetilde{\{y\}}}$ define a precosheaf on $\mathbb{B} = \left\{ \widetilde{\{x\}} \mid x \in X \right\}$. By Lemma 4.8 it is in fact a cosheaf. We can therefore apply Lemma 4.7 and obtain groups $M(U, 0)$ for all sets U and maps $i_U^V\colon M(U, 0) \to M(V, 0)$ for open sets $U \subseteq V$ which fulfill the relations (1) and (3) in Proposition 3.3.

For an arbitrary locally closed subset $Y \in \mathbb{LC}(X)$ we write $Y = V \setminus U$ with open sets $U \subseteq V$ and define $M(Y, 0)$ as the cokernel of the map $i_U^V\colon M(U, 0) \to M(V, 0)$.

That this definition does not depend on the choice of U and V can be seen in a way similar to the proof of [20, Lemma 2.15] using that pushouts of abelian groups preserve cokernels. We obtain maps $r_V^Y\colon M(V,0)\to M(Y,0)$ for every open set V with relatively closed subset $Y\subseteq V$ such that the following holds: If $Y\in\mathbb{LC}(X)$ can be written as differences $V_i\setminus U_i$ of open sets for $i\in\{1,2\}$ such that $U_1\subseteq U_2$ and $V_1\subseteq V_2$, then the diagram

$$\begin{tikzcd} M(U_1,0) \arrow[r, "i"] \arrow[d, "i"] & M(V_1,0) \arrow[r, two heads, "r"] \arrow[d, "i"] & M(Y,0) \arrow[d, equal] \\ M(U_2,0) \arrow[r, "i"] & M(V_2,0) \arrow[r, two heads, "r"] & M(Y,0) \end{tikzcd} \tag{5.20}$$

commutes.

For a relatively open subset $U\subseteq Y\in\mathbb{LC}(X)$ we obtain a unique map $i_U^Y\colon M(U,0)\to M(Y,0)$ using the diagram

$$\begin{tikzcd} M(\widetilde{\partial}U,0) \arrow[r, "i"] \arrow[d, "i"] & M(\widetilde{U},0) \arrow[r, two heads, "r"] \arrow[d, "i"] & M(U,0) \arrow[d, dotted, "i"] \\ M(\widetilde{\partial}Y,0) \arrow[r, "i"] & M(\widetilde{Y},0) \arrow[r, two heads, "r"] & M(Y,0). \end{tikzcd} \tag{5.21}$$

It is easy to check that this map coincides with the previously defined one in case Y is open.

We find that, for $Y_i\in\mathbb{LC}(X)$ with $Y_1\subseteq Y_2$ open, and $Y_i=V_i\setminus U_i$ for $i\in\{1,2\}$ and open sets U_i, V_i such that $U_1\subseteq U_2$ and $V_1\subseteq V_2$, the diagram

$$\begin{tikzcd} M(U_1,0) \arrow[r, "i"] \arrow[d, "i"] & M(V_1,0) \arrow[r, two heads, "r"] \arrow[d, "i"] & M(Y_1,0) \arrow[d, "i"] \\ M(U_2,0) \arrow[r, "i"] & M(V_2,0) \arrow[r, two heads, "r"] & M(Y_2,0) \end{tikzcd} \tag{5.22}$$

commutes. We know this already for the left-hand square. For the right-hand square, it can be seen as follows: since V_1 is covered by U_1 and $\widetilde{Y_1}$, it suffices to check commutativity on the images $i_{U_1}^{V_1}\big(M(U_1,0)\big)$ and $i_{\widetilde{Y_1}}^{V_1}\big(M(\widetilde{Y_1},0)\big)$. On $i_{U_1}^{V_1}\big(M(U_1,0)\big)$ both compositions vanish. On the image of $M(\widetilde{Y_1},0)$, commutativity follows from (5.20) and (5.21) considering the diagram

$$\begin{tikzcd} M(\widetilde{Y_1},0) \arrow[r, "i"'] \arrow[d, "i"] \arrow[rr, bend left, "r"] & M(V_1,0) \arrow[r, two heads, "r"'] \arrow[d, "i"] & M(Y_1,0) \arrow[d, "i"] \\ M(\widetilde{Y_2},0) \arrow[r, "i"] \arrow[rr, bend right, "r"'] & M(V_2,0) \arrow[r, two heads, "r"] & M(Y_2,0). \end{tikzcd}$$

Now let $Y \in \mathbb{LC}(X)$, let U be a relatively open subset of Y and let $C = Y \setminus U$. Consider the diagram

$$\begin{tikzcd}
M(\widetilde{\partial}U,0) \arrow[r, "i"] \arrow[d, "i"] & M(\widetilde{U},0) \arrow[r, two heads, "r"] \arrow[d, "i"] & M(U,0) \arrow[d, "i"] \\
M(\widetilde{\partial}Y,0) \arrow[r, "i"] \arrow[d, two heads, "r"] & M(\widetilde{Y},0) \arrow[r, two heads, "r"] \arrow[d, two heads, "r"] & M(Y,0) \arrow[d, two heads, dotted] \\
M(\widetilde{\partial}Y \setminus \widetilde{\partial}U,0) \arrow[r, "i"] & M(\widetilde{Y} \setminus \widetilde{U},0) \arrow[r, two heads, "r"] & M(C,0),
\end{tikzcd} \tag{5.23}$$

whose solid squares commute and whose rows and solid columns are exact. A diagram chase shows that there is a unique surjective map $r_Y^C\colon M(Y,0) \to M(C,0)$, as indicated by the dotted arrow, making the bottom-right square commute and making the right-hand column exact at $M(Y,0)$. Again, we can easily check that this map coincides with the previously defined one in case Y is open.

We have now defined the even part of the module M completely. It is straightforward to check the relations (3) and (4) in Proposition 3.3. We will now prove that the relation (5) holds as well.

For this purpose, fix $Y \in \mathbb{LC}(X)$, let $U \subseteq Y$ be open and let $C \subseteq Y$ be closed. Consider the diagram

$$\begin{tikzcd}
M(\widetilde{U},0) \arrow[r, two heads, "r"] \arrow[d, "i"] & M(U,0) \arrow[r, "r"] \arrow[d, "i"] & M(U \cap C,0) \arrow[d, "i"] \\
M(\widetilde{Y},0) \arrow[r, "r"] & M(Y,0) \arrow[r, "r"] & M(C,0)
\end{tikzcd}$$

We would like to prove that the right-hand square commutes. The left-hand square commutes by definition of the map i_U^Y. Since $\widetilde{U} \cap C = U \cap C$, we can therefore assume without loss of generality that U and Y are open. Commutativity then follows from (5.22).

Next, we will convince ourselves that the relation (2) in Proposition 3.3 holds on the even part of M. Let $W = Y \sqcup Z$ be a topologically disjoint union of subsets $Y, Z \in \mathbb{LC}(X)$. Fix $w \in M(W,0)$. Then $(w - wr_W^Z i_Z^W) r_W^Z = 0$ as $i_Z^W r_W^Z = \mathrm{id}_Z$. Hence there is $y \in M(Y,0)$ with $y i_Y^W = w - wr_W^Z i_Z^W$. Applying r_W^Y shows $y = wr_W^Y$ as $i_Z^W r_W^Y = 0$. We get

$$w(r_W^Y i_Y^W + r_W^Z i_Z^W) = y i_Y^W + w r_W^Z i_Z^W = w.$$

We have shown that $r_W^Y i_Y^W + r_W^Z i_Z^W = \mathrm{id}_W$ as desired.

We have defined all even groups for the desired module M and the action of all transformations between them. We have checked all relations only involving transformations between even groups and verified exactness of $M(C,0) \to M(Y,0) \to M(U,0)$ for every boundary pair $Y = U \cup C$.

We intend to do the same for the odd part of the module M in an analogous way. We start out with the given data consisting of the groups $M(\overline{\{x\}},1)$ and the maps $r_{\overline{x}_1}^{\overline{y}_1}$, $x \to y$, extend this to a sheaf on the basis $\{\overline{\{x\}} \mid x \in X\}$ of closed sets and apply Lemma 4.3. Observing that every locally closed subset of X can be written as a difference of two nested closed sets and using the functoriality of the kernel of a group homomorphism, we define groups $M(\overline{Y},1)$ for all $Y \in \mathbb{LC}(X)$ and actions for all transformations between these odd groups. Using arguments analogous to the ones above, we can verify the relations (1) to (5) in Proposition 3.3 on the odd part of M.

It remains to define the odd-to-even components of the boundary maps δ_C^U for all boundary pairs (U,C), which has only been done in the special case $U = \widetilde{\{x\}}$, $C = \overline{\{y\}}$ with $x \to y$. Our general definition for $\delta_C^U \colon M(C,1) \to M(U,0)$ is

$$\delta_C^U = \sum_{x\to y, x\in U, y\in C} r_C^{\overline{\{y\}}\cap C}\, i^{\overline{\{y\}}}_{\overline{\{y\}}\cap C}\, \delta^{\widetilde{\{x\}}}_{\overline{\{y\}}}\, r^{\widetilde{\{x\}}\cap U}_{\widetilde{\{x\}}}\, i^U_{\widetilde{\{x\}}\cap U}. \tag{5.24}$$

Our next aim is to verify the relations (6), (7) and (8) in Proposition 3.3. We begin with relation (6). Let (U,C) be a boundary pair and let $C' \subseteq C$ be relatively open. We have by the relations (3) and (5) that

$$\begin{aligned} i_{C'}^C\, \delta_C^U &= i_{C'}^C \Big(\sum_{x\to y, x\in U, y\in C} r_C^{\overline{\{y\}}\cap C}\, i^{\overline{\{y\}}}_{\overline{\{y\}}\cap C}\, \delta^{\widetilde{\{x\}}}_{\overline{\{y\}}}\, r^{\widetilde{\{x\}}\cap U}_{\widetilde{\{x\}}}\, i^U_{\widetilde{\{x\}}\cap U} \Big) \\ &= \sum_{x\to y, x\in U, y\in C} r_{C'}^{\overline{\{y\}}\cap C'}\, i^{\overline{\{y\}}}_{\overline{\{y\}}\cap C'}\, \delta^{\widetilde{\{x\}}}_{\overline{\{y\}}}\, r^{\widetilde{\{x\}}\cap U}_{\widetilde{\{x\}}}\, i^U_{\widetilde{\{x\}}\cap U}. \end{aligned}$$

Since C' is relatively open in C, $\overline{\{y\}} \cap C'$ is empty unless $y \in C'$. Therefore, the above sum equals

$$\delta_{C'}^U = \sum_{x\to y, x\in U, y\in C'} r_{C'}^{\overline{\{y\}}\cap C'}\, i^{\overline{\{y\}}}_{\overline{\{y\}}\cap C'}\, \delta^{\widetilde{\{x\}}}_{\overline{\{y\}}}\, r^{\widetilde{\{x\}}\cap U}_{\widetilde{\{x\}}}\, i^U_{\widetilde{\{x\}}\cap U}.$$

This shows relation (6). The relation (7) follows similarly.

Next we will check relation (8). Let $Y, Z, W \in \mathbb{LC}(X)$ such that $Y \cup W \in \mathbb{LC}(X)$ containing Y, W as closed subsets, $Z \cup W \in \mathbb{LC}(X)$ containing Z, W as open subsets, and $W \subseteq Y \cup Z$. For each $x \in Z$ and $y \in Y$ with $x \to y$, we define $\gamma_{x,y} \colon M(Y,1) \to M(Z,0)$ by

$$\gamma_{x,y} = r_Y^{\overline{\{y\}}\cap Y}\, i^{\overline{\{y\}}}_{\overline{\{y\}}\cap Y}\, \delta^{\widetilde{\{x\}}}_{\overline{\{y\}}}\, r^{\widetilde{\{x\}}\cap Z}_{\widetilde{\{x\}}}\, i^Z_{\widetilde{\{x\}}\cap Z}.$$

Since $W \setminus Y$ is an open subset of Z (see the proof of Proposition 3.3), we have $\widetilde{\{x\}} \cap (W \setminus Y) = \widetilde{\{x\}} \cap Z$ for each $x \in W \setminus Y$. We also have $y \in W$ if $y \in Y$ satisfies $x \to y$ for some $x \in W \setminus Y$ because $W \subseteq Y \cup W$ is closed. Therefore, by the relation (4) we get

$$\begin{aligned} \delta_Y^{W\setminus Y}\, i^Z_{W\setminus Y} &= \Big(\sum_{x\to y, x\in W\setminus Y, y\in Y} r_Y^{\overline{\{y\}}\cap Y}\, i^{\overline{\{y\}}}_{\overline{\{y\}}\cap Y}\, \delta^{\widetilde{\{x\}}}_{\overline{\{y\}}}\, r^{\widetilde{\{x\}}\cap (W\setminus Y)}_{\widetilde{\{x\}}}\, i^{W\setminus Y}_{\widetilde{\{x\}}\cap (W\setminus Y)} \Big) i^Z_{W\setminus Y} \\ &= \sum_{x\to y, x\in W\setminus Y, y\in W\cap Y} r_Y^{\overline{\{y\}}\cap Y}\, i^{\overline{\{y\}}}_{\overline{\{y\}}\cap Y}\, \delta^{\widetilde{\{x\}}}_{\overline{\{y\}}}\, r^{\widetilde{\{x\}}\cap Z}_{\widetilde{\{x\}}}\, i^Z_{\widetilde{\{x\}}\cap Z} \\ &= \sum_{(x,y)\in\Lambda_1} \gamma_{x,y} \end{aligned}$$

where we set

$$\Lambda_1 = \{(x,y) \mid x \to y,\ x \in W \setminus Y,\ y \in W \cap Y\}.$$

In a similar way using the facts that $W \setminus Z$ is a closed subset of Y and that Z is an open subset of $Y \cup Z$, we get

$$r_Y^{W\setminus Z}\delta_{W\setminus Z}^{Z} = \sum_{(x,y)\in\Lambda_2} \gamma_{x,y}$$

where we set

$$\Lambda_2 = \{(x,y) \mid x \to y,\ x \in W \cap Z,\ y \in W \setminus Z\}.$$

If we set

$$\begin{aligned}\Lambda_1' &= \{(x,y) \mid x \to y,\ x \in W \cap Y \cap Z,\ y \in W \cap Y\},\\ \Lambda_2' &= \{(x,y) \mid x \to y,\ x \in W \cap Z,\ y \in W \cap Y \cap Z\}\end{aligned}$$

then we have

$$\{(x,y) \mid x \to y,\ x \in W \cap Z,\ y \in W \cap Y\} = \Lambda_1 \sqcup \Lambda_1' = \Lambda_2 \sqcup \Lambda_2'$$

because $W \subseteq Y \cup Z$ implies $(W \cap Z) \setminus Y = W \setminus Y$ and $(W \cap Y) \setminus Z = W \setminus Z$. Therefore in order to show the equality $\delta_Y^{W\setminus Y} i_{W\setminus Y}^{Z} = r_Y^{W\setminus Z}\delta_{W\setminus Z}^{Z}$, it suffices to show

$$\sum_{(x,y)\in\Lambda_1'} \gamma_{x,y} = \sum_{(x,y)\in\Lambda_2'} \gamma_{x,y}.$$

For each $p \in W \cap Y \cap Z$, we get

$$\sum_{y\leftarrow p} r_{\overline{\{p\}}}^{\overline{\{y\}}}\ \delta_{\overline{\{y\}}}^{\widetilde{\{p\}}} = \sum_{x\to p} \delta_{\overline{\{p\}}}^{\widetilde{\{x\}}}\ i_{\widetilde{\{x\}}}^{\widetilde{\{p\}}}$$

from the definition of $\mathcal{B}$-modules. Multiplying from the left with $r_Y^{\overline{\{p\}}\cap Y}\ i_{\overline{\{p\}}\cap Y}^{\overline{\{p\}}}$ and from the right with $r_{\widetilde{\{p\}}}^{\widetilde{\{p\}}\cap Z}\ i_{\widetilde{\{p\}}\cap Z}^{Z}$, and summing up over $p \in W \cap Y \cap Z$, we get

$$\begin{aligned}&\sum_{p\in W\cap Y\cap Z} r_Y^{\overline{\{p\}}\cap Y}\ i_{\overline{\{p\}}\cap Y}^{\overline{\{p\}}}\Big(\sum_{y\leftarrow p} r_{\overline{\{p\}}}^{\overline{\{y\}}}\ \delta_{\overline{\{y\}}}^{\widetilde{\{p\}}}\Big) r_{\widetilde{\{p\}}}^{\widetilde{\{p\}}\cap Z}\ i_{\widetilde{\{p\}}\cap Z}^{Z}\\ &\qquad = \sum_{p\in W\cap Y\cap Z} r_Y^{\overline{\{p\}}\cap Y}\ i_{\overline{\{p\}}\cap Y}^{\overline{\{p\}}}\Big(\sum_{x\to p} \delta_{\overline{\{p\}}}^{\widetilde{\{x\}}}\ i_{\widetilde{\{x\}}}^{\widetilde{\{p\}}}\Big) r_{\widetilde{\{p\}}}^{\widetilde{\{p\}}\cap Z}\ i_{\widetilde{\{p\}}\cap Z}^{Z}.\end{aligned}$$

By the relations (3), (4) and (5), we get

$$\begin{aligned}&\sum_{p\in W\cap Y\cap Z}\sum_{y\leftarrow p} r_Y^{\overline{\{y\}}\cap Y}\ i_{\overline{\{y\}}\cap Y}^{\overline{\{y\}}}\ \delta_{\overline{\{y\}}}^{\widetilde{\{p\}}} r_{\widetilde{\{p\}}}^{\widetilde{\{p\}}\cap Z}\ i_{\widetilde{\{p\}}\cap Z}^{Z}\\ &\qquad = \sum_{p\in W\cap Y\cap Z}\sum_{x\to p} r_Y^{\overline{\{p\}}\cap Y}\ i_{\overline{\{p\}}\cap Y}^{\overline{\{p\}}} \delta_{\overline{\{p\}}}^{\widetilde{\{x\}}}\ r_{\widetilde{\{x\}}}^{\widetilde{\{x\}}\cap Z}\ i_{\widetilde{\{x\}}\cap Z}^{Z}.\end{aligned} \tag{5.25}$$

Since Y is locally closed, the conditions $p \in W \cap Y \cap Z$, $y \leftarrow p$ and $\overline{\{y\}} \cap Y \neq \emptyset$ imply $y \in Y$. This further implies $y \in W$ because $W \subseteq Y \cup W$ is closed. Hence the left-hand side of (5.25) equals $\sum_{(x,y)\in\Lambda_1'} \gamma_{x,y}$. In a similar way, we can see that the right-hand side of (5.25) equals $\sum_{(x,y)\in\Lambda_2'} \gamma_{x,y}$. Thus we have proven the relation (8), and this finishes the verification of all relations in Proposition 3.3.

Hence, M is indeed an $\mathcal{ST}$-module. To see that M is exact, it remains to show that the sequences $M(C,1) \xrightarrow{\delta_C^U} M(U,0) \xrightarrow{i_U^Y} M(Y,0)$ and $M(Y,1) \xrightarrow{r_Y^C} M(C,1) \xrightarrow{\delta_C^U} M(U,0)$ are exact for all boundary pairs (U,C) with $Y = U \cup C$.

Fix an element $x \in X$ and consider the commutative diagram

$$\begin{tikzcd}
M(\{x\},1) \arrow[r, "i"] \arrow[d, equal] & M(\overline{\{x\}},1) \arrow[r, "r"] \arrow[d, "\circ" marking] & M(\overline{\partial}\{x\},1) \arrow[d, "\circ" marking] \\
M(\{x\},1) \arrow[r, "\circ" marking] & M(\widetilde{\partial}\{x\},0) \arrow[r, "i"] & M(\widetilde{\{x\}},0)
\end{tikzcd}$$

Using exactness of the upper row and the fact that N was an exact $\mathcal{B}$-module, a diagram chase shows that the bottom row is exact. In a similar way, we see that the sequence

$$M(\overline{\{x\}},1) \to M(\overline{\partial}\{x\},0) \to M(\{x\},0).$$

is exact for every $x \in X$.

Next, let $Y \in \mathbb{LC}(X)$ and let $x \in Y$ be a closed point. Then $Y \cap \widetilde{\{x\}}$ is relatively closed in $\widetilde{\{x\}}$ because Y is locally closed. A diagram chase in the commutative diagram

$$\begin{tikzcd}
 & M(\widetilde{\partial}\{x\} \setminus (Y \cap \widetilde{\partial}\{x\}),0) \arrow[r, equal] \arrow[d, "i"] & M(\widetilde{\{x\}} \setminus (Y \cap \widetilde{\{x\}}),0) \arrow[d, "i"] \\
M(\{x\},1) \arrow[r, "\circ" marking] \arrow[d, equal] & M(\widetilde{\partial}\{x\},0) \arrow[r, "i"] \arrow[d, "r"] & M(\widetilde{\{x\}},0) \arrow[d, "r"] \\
M(\{x\},1) \arrow[r, "\circ" marking] & M(Y \cap \widetilde{\partial}\{x\},0) \arrow[r, "i"] & M(Y \cap \widetilde{\{x\}},0),
\end{tikzcd}$$

whose columns and top row are exact, yields exactness of the bottom row. By a diagram chase in the commutative diagram

$$\begin{tikzcd}
M(\{x\},1) \arrow[r, "\circ" marking] \arrow[d, equal] & M(Y \cap \widetilde{\partial}\{x\},0) \arrow[r, "i"] \arrow[d, "i"] & M(Y \cap \widetilde{\{x\}},0) \arrow[d, "i"] \\
M(\{x\},1) \arrow[r, "\circ" marking] & M(Y \setminus \{x\},0) \arrow[r, "i"] & M(Y,0)
\end{tikzcd}$$

using the exact cosheaf sequence (4.6) for the covering $(Y \setminus \{x\}, Y \cap \widetilde{\{x\}})$ of Y we obtain exactness of the bottom row. Notice that, using a further diagram chase, it is not hard to deduce the exactness of the cosheaf sequence for a relatively open covering of a locally closed set from the open case.

We have established the exactness of the sequence $M(C,1) \xrightarrow{\delta_C^U} M(U,0) \xrightarrow{i_U^Y} M(Y,0)$ for all boundary pairs (U,C) with C a singleton. Analogously, we find that $M(Y,1) \xrightarrow{r_Y^C} M(C,1) \xrightarrow{\delta_C^U} M(U,0)$ is exact whenever U is a singleton.

We will proceed by an inductive argument. Let $n \geq 1$ be a natural number and assume that exactness of the sequence $M(C,1) \xrightarrow{\delta_C^U} M(U,0) \xrightarrow{i_U^Y} M(Y,0)$ is proven for all boundary pairs (U,C) for which C has at most n elements. Let (U,C) be a boundary pair such that C has $n+1$ elements. Write $Y = U \cup C$. Let $p \in C$ be a maximal point and set $U' = U \cup \{p\}$, $C' = C \setminus \{p\}$. Then (U', C') is a boundary pair. A diagram chase in the commutative diagram

$$\begin{tikzcd}
M(\{p\},1) \arrow[r, "i"] \arrow[d, equal] & M(C,1) \arrow[r, "r"] \arrow[d, "\circ" marking] & M(C',1) \arrow[r, "\circ" marking] \arrow[d, "\circ" marking] & M(\{p\},0) \arrow[d, equal] \\
M(\{p\},1) \arrow[r, "\circ" marking] & M(U,0) \arrow[r, "i"] \arrow[d, "i"] & M(U',0) \arrow[r, "r"] \arrow[d, "i"] & M(\{p\},0) \\
& M(Y,0) \arrow[r, equal] & M(Y,0), &
\end{tikzcd}$$

whose rows and third column are exact, shows exactness of the second column. Again, exactness of $M(Y,1) \xrightarrow{r_Y^C} M(C,1) \xrightarrow{\delta_C^U} M(U,0)$ for all boundary pairs follows in a analogous manner. We conclude that M is an exact $\mathcal{ST}$-module.

Summing up, we have associated an exact real-rank-zero-like $\mathcal{ST}$-module with every exact $\mathcal{B}$-module. By a routine argument, this assignment extends uniquely to a functor G from the category of exact $\mathcal{B}$-modules to the category of exact real-rank-zero-like $\mathcal{ST}$-modules. Let F be the restriction of the functor $\mathfrak{F}_{\mathcal{B}}$ to the category of exact real-rank-zero-like $\mathcal{ST}$-modules. Then the composition GF is equal to the identity functor on the category of exact $\mathcal{B}$-modules. It remains to show that FG is naturally isomorphic to the identity functor on the category of exact real-rank-zero-like $\mathcal{ST}$-modules.

Let M be an exact real-rank-zero-like $\mathcal{ST}$-module. We will construct a natural $\mathcal{ST}$-module isomorphism $\eta_M \colon M \to (FG)(M)$. For $x \in X$ we have $M(\widetilde{\{x\}},0) = (FG)(M)(\widetilde{\{x\}},0)$ and $M(\overline{\{x\}},1) = (FG)(M)(\overline{\{x\}},1)$. Hence we set $\eta_M(\widetilde{\{x\}},0) = \mathrm{id}_{M(\widetilde{\{x\}},0)}$ and $\eta_M(\overline{\{x\}},1) = \mathrm{id}_{M(\overline{\{x\}},1)}$. Using the universal property of kernels and cokernels we obtain natural group homomorphisms $f_Y \colon M(Y,1) \to (FG)(M)(Y,1)$ and $g_Y \colon (FG)(M)(Y,0) \to M(Y,0)$ for every $Y \in \mathbb{LC}(X)$. An application of the Five Lemma shows that these are in fact isomorphisms. We can therefore define $\eta_M(Y,1) = f_Y$ and $\eta_M(Y,0) = (g_Y)^{-1}$.

Finally, we check that this collection of maps constitutes an $\mathcal{ST}$-module homomorphism, that is, the group homomorphism $\eta_M \colon M \to (FG)(M)$ intertwines the actions of the category $\mathcal{ST}$ on M and on $(FG)(M)$. By construction this is true for the transformations $(i_{\widetilde{\{x\}}}^{\widetilde{\{y\}}},0)$, $(r_{\overline{\{x\}}}^{\overline{\{y\}}},1)$ and $\delta_{\overline{\{y\}}}^{\widetilde{\{x\}}}$ for all $x,y \in X$ with $x \to y$. By Lemma 4.3 and Lemma 4.7 it is also true for the transformation $(i_U^V,0)$ for all open subset U,V of X with $U \subseteq V$ and for $(r_C^D,1)$ for all closed subsets C,D of X with $D \subseteq C$.

Let $V \subseteq X$ be open and let $Y \subseteq V$ be relatively closed. Since $(r_V^Y,0)$ was defined as a natural projection onto a cokernel, our assertion holds for this transformation as well. Consequently, by (5.21) the assertion also follows for the transformation $(i_U^Y,0)$ for $Y \in \mathbb{LC}(X)$ and $U \subseteq Y$ relatively open. Finally (5.23) implies the assertion for the transformation r_Y^C with $Y \in \mathbb{LC}(X)$ and $C \subseteq Y$ relatively closed. We have shown that η intertwines the actions of all even transformations on the 0-parts of M and $(FG)(M)$. By analogous arguments the same follows for the actions of all even transformations on the 1-parts of M and $(FG)(M)$.

Our last step is to consider the action of a boundary transformation δ_C^U for a boundary pair (U,C). Since M and $(FG)(M)$ are real-rank-zero-like the 0-to-1 component of δ_C^U acts trivially on both modules. We have already seen that the assertion is true for the 1-to-0 component of δ_C^U in the specific case that $(U, C) = (\widetilde{\{x\}}, \overline{\{y\}})$ with $x \to y$. The general case then follows from (5.24) since X is an EBP space. $\square$

6. Reduced filtered K-theory

Let X be an arbitrary finite T_0-space. We recall some definitions and facts from [3]. In [23], Gunnar Restorff introduced reduced filtered K-theory $\mathrm{FK}_{\mathcal{R}}$ and showed that it classifies purely infinite Cuntz–Krieger algebras up to stable isomorphism. In [3], the range of reduced filtered K-theory is established with respect to purely infinite Cuntz–Krieger algebras.

Definition 6.1 ([3, Definition 3.1])**.** Let $\mathcal{R}$ denote the universal pre-additive category generated by objects $x_1, \widetilde{\partial} x_0, \widetilde{x}_0$ for all $x \in X$ and morphisms $\delta_{x_1}^{\widetilde{\partial} x_0}$ and $i_{\widetilde{\partial} x_0}^{\widetilde{x}_0}$ for all $x \in X$, and $i_{\widetilde{y}_0}^{\widetilde{\partial} x_0}$ when $y \to x$, subject to the relations

$$\delta_{x_1}^{\widetilde{\partial} x_0} i_{\widetilde{\partial} x_0}^{\widetilde{x}_0} = 0 \tag{6.2}$$

$$i_p i_{\widetilde{y(p)}_0}^{\widetilde{\partial} x_0} = i_q i_{\widetilde{y(q)}_0}^{\widetilde{\partial} x_0} \tag{6.3}$$

for all $x \in X$, all $y \in X$ satisfying $y > x$, and all paths $p, q \in \mathrm{Path}(y, x)$, where for a path $p = (z_k)_{k=1}^n$ in $\mathrm{Path}(y, x)$, we define $y(p) = z_2$, and

$$i_p = i_{\widetilde{z_n}_0}^{\widetilde{\partial} z_{n-1_0}} i_{\widetilde{\partial} z_{n-1_0}}^{\widetilde{z_{n-2}}_0} \cdots i_{\widetilde{z}_{3_0}}^{\widetilde{\partial} z_{2_0}} i_{\widetilde{\partial} z_{2_0}}^{\widetilde{z}_{2_0}}.$$

Definition 6.4. It is easy to see that the relations in $\mathcal{ST}$ corresponding to (6.2) and (6.3) hold. We can thus define an additive functor $\mathcal{R} \to \mathcal{ST}$ by $x_1 \mapsto (\{x\}, 1)$, $\widetilde{\partial} x_0 \mapsto (\widetilde{\partial}\{x\}, 0)$ and $\widetilde{x}_0 \mapsto (\widetilde{\{x\}}, 0)$, and in the obvious way on morphisms. Let $\mathfrak{F}_{\mathcal{R}} \colon \mathfrak{Mod}(\mathcal{ST}) \to \mathfrak{Mod}(\mathcal{R})$ denote the induced functor. Define *reduced filtered* K*-theory*, $\mathrm{FK}_{\mathcal{R}}$ as the composition of $\mathrm{FK}_{\mathcal{ST}}$ with $\mathfrak{F}_{\mathcal{R}}$.

An equivalent definition of the functor $\mathrm{FK}_{\mathcal{R}}$ is given in [3, Definition 3.4].

Definition 6.5 ([3, Definition 3.6])**.** An $\mathcal{R}$-module M is called *exact* if the sequences

$$M(x_1) \xrightarrow{\delta} M(\widetilde{\partial} x_0) \xrightarrow{i} M(\widetilde{x}_0) \tag{6.6}$$

$$\bigoplus_{(p,q) \in \mathrm{DP}(x)} M(\widetilde{s(p,q)}_0) \xrightarrow{(i_p - i_q)} \bigoplus_{y \to x} M(\widetilde{y}_0) \xrightarrow{(i_{\widetilde{y}_0}^{\widetilde{\partial} x_0})} M(\widetilde{\partial} x_0) \longrightarrow 0 \tag{6.7}$$

are exact for all $x \in X$, where $\mathrm{DP}(x)$ denotes the set of pairs of distinct paths (p, q) to x and from some common element which is denoted $s(p, q)$.

The following lemma is a generalization of [3, Lemma 3.9]. We omit the proof as the same technique applies here.

Lemma 6.8. *Let M be an exact real-rank-zero-like $\mathcal{ST}$-module. Let Y be an open subset of X and let $(U_i)_{i \in I}$ be an open covering of Y. Then the following sequence is exact:*

$$\bigoplus_{i,j \in I} M(U_i \cap U_j, 0) \xrightarrow{(i_{U_i \cap U_j}^{U_i} - i_{U_i \cap U_j}^{U_j})} \bigoplus_{i \in I} M(U_i, 0) \xrightarrow{(i_{U_i}^{Y})} M(Y, 0) \longrightarrow 0.$$

Corollary 6.9. *Let M be an exact real-rank-zero-like $\mathcal{ST}$-module and set $N = \mathfrak{F}_{\mathcal{R}}(M)$. Then N is an exact $\mathcal{R}$-module.*

Remark 6.10. If X is a unique path space, then the set $\mathrm{DP}(x)$ is empty for every $x \in X$. Hence, for an exact $\mathcal{R}$-module M, the map $(i_{\widetilde{y}_0}^{\widetilde{\partial} x_0})\colon \bigoplus_{y \to x} M(\widetilde{y}_0) \to M(\widetilde{\partial} x_0)$ is an isomorphism. In this sense, the groups $M(\widetilde{\partial} x_0)$ are redundant for an exact $\mathcal{R}$-module in case X is a unique path space.

By combining the following Proposition 6.11 and Theorem 6.12, one may obtain a complete description of the range of reduced filtered K-theory for purely infinite graph C^*-algebras and Cuntz–Krieger algebras.

Proposition 6.11 ([3, Proposition 4.7])**.** *Let A be a purely infinite graph C^*-algebra over X. Then $\mathrm{FK}_{\mathcal{R}}(A)$ is an exact $\mathcal{R}$-module, and $\mathrm{FK}^1_{\{x\}}(A)$ is free for all $x \in X$.*

If A is a purely infinite Cuntz–Krieger algebra over X, then furthermore $\mathrm{K}_1(A(x))$ and $\mathrm{K}_0(A(\widetilde{\{x\}}))$ are finitely generated, and the rank of $\mathrm{K}_1(A(x))$ coincides with the rank of the cokernel of the map $i\colon \mathrm{K}_0(A(\widetilde{\partial}\{x\}) \to \mathrm{K}_0(A(\widetilde{\{x\}}))$, for all $x \in X$.

Theorem 6.12 ([3, Theorem 4.8])**.** *Let M be an exact $\mathcal{R}$-module with $M(x_1)$ free for all $x \in X$. Then there exists a countable graph E satisfying that all vertices in E are regular and support at least two cycles, that $C^*(E)$ is tight over X and that $\mathrm{FK}_{\mathcal{R}}\big(C^*(E)\big)$ is isomorphic to M. By construction $C^*(E)$ is purely infinite.*

The graph E can be chosen to be finite if (and only if) $M(x_1)$ and $M(\widetilde{x}_0)$ are finitely generated, and the rank of $M(x_1)$ coincides with the rank of the cokernel of $i\colon M(\widetilde{\partial} x_0) \to M(\widetilde{x}_0)$, for all $x \in X$. If E is chosen finite, then by construction $C^*(E)$ is a Cuntz–Krieger algebra.

In Corollary 7.16, we combine this range-of-invariant theorem with the isomorphism lifting result from the next section.

7. An intermediate invariant

In this section, we define one more invariant, which, in a sense, can be thought of as a union or join of reduced filtered K-theory $\mathrm{FK}_{\mathcal{R}}$ and filtered K-theory restricted to canonical base $\mathrm{FK}_{\mathcal{B}}$. It functions as an intermediate invariant towards concrete filtered K-theory $\mathrm{FK}_{\mathcal{ST}}$.

Let X be a unique path space.

Definition 7.1. Let $\mathcal{BR}$ denote the universal pre-additive category generated by objects x_1, $\overline{x}_1$, $\widetilde{x}_0$ for all $x \in X$ and morphisms $i_{x_1}^{\overline{x}_1}$ for all $x \in X$ and $r_{\overline{x}_1}^{\overline{y}_1}$, $\delta_{\overline{y}_1}^{\widetilde{x}_0}$ and $i_{\widetilde{x}_0}^{\widetilde{y}_0}$ when $x \to y$, subject to the relations

$$\sum_{x \to y} r_{\overline{x}_1}^{\overline{y}_1} \delta_{\overline{y}_1}^{\widetilde{x}_0} = \sum_{z \to x} \delta_{\overline{x}_1}^{\widetilde{z}_0} i_{\widetilde{z}_0}^{\widetilde{x}_0} \tag{7.2}$$

for all $x \in X$ and

$$i_{x_1}^{\overline{x}_1} r_{\overline{x}_1}^{\overline{y}_1} = 0 \tag{7.3}$$

when $x \to y$.

As before, there is a canonical additive functor $\mathcal{BR} \to \mathcal{ST}$, inducing a functor $\mathfrak{F}_{\mathcal{BR}}\colon \mathfrak{Mod}(\mathcal{ST}) \to \mathfrak{Mod}(\mathcal{BR})$. Define $\mathrm{FK}_{\mathcal{BR}}$ as the composition of $\mathrm{FK}_{\mathcal{ST}}$ with $\mathfrak{F}_{\mathcal{BR}}$.

The category $\mathcal{B}$ embeds into the category $\mathcal{BR}$, and a forgetful functor $\mathfrak{Mod}(\mathcal{BR}) \to \mathfrak{Mod}(\mathcal{B})$ is induced. We define an additive functor $\mathfrak{F}_{\mathcal{BR},\mathcal{R}}\colon \mathfrak{Mod}(\mathcal{BR}) \to \mathfrak{Mod}(\mathcal{R})$ by

$$M(\widetilde{\partial}x_0) = \bigoplus_{y \to x} M(\widetilde{y}_0)$$

and $\delta_{x_1}^{\widetilde{\partial}x_0} = (i_{x_1}^{\overline{x}_1}\delta_{\overline{x}_1}^{\widetilde{y}_0})$ and otherwise in the obvious way.

Definition 7.4. A $\mathcal{BR}$-module M is called *exact* if the sequences

$$M(\overline{x}_1) \xrightarrow{\begin{pmatrix} r_{\overline{x}_1}^{\overline{y}_1} & -\delta_{\overline{x}_1}^{\widetilde{z}_0}\end{pmatrix}} \bigoplus_{x\to y} M(\overline{y}_1) \oplus \bigoplus_{z\to x} M(\widetilde{z}_0) \xrightarrow{\begin{pmatrix} \delta_{\overline{y}_1}^{\widetilde{x}_0} \\ i_{\widetilde{z}_0}^{\widetilde{x}_0}\end{pmatrix}} M(\widetilde{x}_0) \tag{7.5}$$

$$0 \to M(x_1) \xrightarrow{i_{x_1}^{\overline{x}_1}} M(\overline{x}_1) \xrightarrow{(r_{\overline{x}_1}^{\overline{y}_1})} \bigoplus_{x\to y} M(\overline{y}_1) \tag{7.6}$$

are exact for all $x \in X$.

Lemma 7.7. *Let M be an exact real-rank-zero-like $\mathcal{ST}$-module. Then $\mathfrak{F}_{\mathcal{BR}}(M)$ is an exact $\mathcal{BR}$-module.*

Proof. The proof is similar to the proof of Lemma 5.10. □

Theorem 7.8. *Assume that X is a unique path space. Let M and N be exact $\mathcal{BR}$-modules with $M(x_1)$ and $N(x_1)$ free for all non-open points $x \in X$, and let $\varphi\colon \mathfrak{F}_{\mathcal{BR},\mathcal{R}}(M) \to \mathfrak{F}_{\mathcal{BR},\mathcal{R}}(N)$ be an $\mathcal{R}$-module homomorphism. Then there exists a (not necessarily unique) $\mathcal{BR}$-module homomorphism $\Phi\colon M \to N$ such that $\mathfrak{F}_{\mathcal{BR},\mathcal{R}}(\Phi) = \varphi$. If φ is an isomorphism then, by construction, so is Φ.*

Proof. For $x \in X$, we define $\Phi_{x_1} = \varphi_{x_1}$ and $\Phi_{\widetilde{x}_0} = \varphi_{\widetilde{x}_0}$. In the following, we will define $\Phi_{\overline{x}_1}$ by induction on the partial order of X in a way such that the relations

$$r_{\overline{x}_1}^{\overline{y}_1}\Phi_{\overline{y}_1} = \Phi_{\overline{x}_1} r_{\overline{x}_1}^{\overline{y}_1}, \tag{7.9}$$

$$\delta_{\overline{x}_1}^{\widetilde{z}_0}\Phi_{\widetilde{z}_0} = \Phi_{\overline{x}_1}\delta_{\overline{x}_1}^{\widetilde{z}_0} \tag{7.10}$$

$$i_{x_1}^{\overline{x}_1}\Phi_{\overline{x}_1} = \Phi_{x_1} i_{x_1}^{\overline{x}_1} \tag{7.11}$$

hold for all y with $x \to y$ and all z with $z \to x$. For closed points $x \in X$, we set

$$\Phi_{\overline{x}_1} = i_{x_1}^{\overline{x}_1}\varphi_{x_1}\left(i_{x_1}^{\overline{x}_1}\right)^{-1}.$$

Here we have used that, by exactness of (7.6), $i_{x_1}^{\overline{x}_1}$ is invertible as there is no y with $x \to y$. While the condition (7.9) is empty, (7.10) is guaranteed by φ being an $\mathcal{R}$-module homomorphism, and (7.11) holds by construction.

Now fix an element $w \in X$ and assume that $\Phi_{\overline{x}_1}$ is defined for all $x < w$ in a way such that (7.9) and (7.10) hold. Using the exact sequence (7.6) and the freeness of $\bigoplus\limits_{w\to x} M(\overline{w}_1)$, we can choose a free subgroup $V \subseteq M(\overline{w}_1)$ such that $M(\overline{w}_1)$ decomposes as an inner direct sum

$$M(\overline{w}_1) = V \oplus M(w_1)\cdot i_{w_1}^{\overline{w}_1}.$$

We will define $\Phi_{\overline{w}_1}$ by specifying the two restrictions $\Phi_{\overline{w}_1}|_V$ and $\Phi_{\overline{w}_1}|_{M(w_1)\cdot i_{w_1}^{\overline{w}_1}}$. Consider the diagram

(7.12)

$$\begin{tikzcd}
V \arrow[r, tail] \arrow[dr, dotted] & M(\overline{x}_1) \arrow[r, "\left(r_{\overline{x}_1}^{\overline{y}_1},\, -\delta_{\overline{x}_1}^{\widetilde{z}_0}\right)"] & \bigoplus\limits_{x\to y} M(\overline{y}_1) \oplus \bigoplus\limits_{z\to x} M(\widetilde{z}_0) \arrow[r, "\begin{pmatrix}\delta_{\overline{y}_1}^{\widetilde{x}_0}\\ i_{\widetilde{z}_0}^{\widetilde{x}_0}\end{pmatrix}"] \arrow[d, "\left((\Phi_{\overline{y}_1}),(\Phi_{\widetilde{z}_0})\right)"] & M(\widetilde{x}_0) \arrow[d, "\Phi_{\widetilde{x}_0}"] \\
& N(\overline{x}_1) \arrow[r, "\left(r_{\overline{x}_1}^{\overline{y}_1},\, -\delta_{\overline{x}_1}^{\widetilde{z}_0}\right)"'] & \bigoplus\limits_{x\to y} N(\overline{y}_1) \oplus \bigoplus\limits_{z\to x} N(\widetilde{z}_0) \arrow[r, "\begin{pmatrix}\delta_{\overline{y}_1}^{\widetilde{x}_0}\\ i_{\widetilde{z}_0}^{\widetilde{x}_0}\end{pmatrix}"'] & N(\widetilde{x}_0)
\end{tikzcd}$$

By assumption, the bottom row of this diagram is exact, the top row is exact in $\bigoplus\limits_{x\to y} M(\overline{y}_1) \oplus \bigoplus\limits_{z\to x} M(\widetilde{z}_0)$, and the right-hand square commutes. We can therefore choose a homomorphism $\Phi_{\overline{x}_1}|_V\colon V \to N(\overline{x}_1)$ such that the left-hand pentagon commutes.

By exactness of (7.6), $i_{x_1}^{\overline{x}_1}$ is injective. Its corestriction onto its image $M(x_1)\cdot i_{x_1}^{\overline{x}_1}$ is thus an isomorphism. We may therefore define the restriction $\Phi_{\overline{x}_1}|_{M(x_1)\cdot i_{x_1}^{\overline{x}_1}}$ in the unique way that makes the following diagram commute:

$$\begin{tikzcd}
M(x_1) \arrow[r, "i_{x_1}^{\overline{x}_1}"] \arrow[d, "\varphi_{x_1}"] & M(x_1)\cdot i_{x_1}^{\overline{x}_1} \arrow[d, "\Phi_{\overline{x}_1}|_{M(x_1)\cdot i_{x_1}^{\overline{x}_1}}"] \\
N(x_1) \arrow[r, "i_{x_1}^{\overline{x}_1}"] & N(x_1)\cdot i_{x_1}^{\overline{x}_1}
\end{tikzcd} \tag{7.13}$$

We have to check that $\Phi_{\overline{w}_1} = (\Phi_{\overline{w}_1}|_V, \Phi_{\overline{w}_1}|_{M(w_1)\cdot i_{w_1}^{\overline{w}_1}})$ fulfills (7.9) and (7.10) (with x replaced with w). This is true on V because of the commutativity of the left-hand side of (7.12). It is also true on the second summand: by (7.3), both sides of (7.9) vanish on this subgroup; (7.10) follows again from φ being an $\mathcal{R}$-module homomorphism; and (7.11) holds by construction. This completes the induction step.

The claim, that Φ is an isomorphism whenever φ is, follows from a repeated application of the Five Lemma. □

Corollary 7.14. *Assume that X is an EBP space. Let M and N be exact, real-rank-zero-like $\mathcal{ST}$-modules with $M(\{x\},1)$ and $N(\{x\},1)$ free for all non-open points $x \in X$, and let $\varphi\colon \mathfrak{F}_{\mathcal{R}}(M) \to \mathfrak{F}_{\mathcal{R}}(N)$ be an $\mathcal{R}$-module homomorphism. Then there exists a (not necessarily unique) $\mathcal{ST}$-module homomorphism $\Phi\colon M \to N$ satisfying $\mathfrak{F}_{\mathcal{R}}(\Phi) = \varphi$. If φ is an isomorphism then, by construction, so is Φ.*

Proof. Combine Theorems 7.8 and 5.17. □

Corollary 7.15. *Let A and B be C^*-algebras of real rank zero over an EBP space X, and assume that $\mathrm{K}_1\big(A(x)\big)$ and $\mathrm{K}_1\big(B(x)\big)$ are free abelian groups for all non-open points $x \in X$. Then for any homomorphism $\varphi\colon \mathrm{FK}_{\mathcal{R}}(A) \to \mathrm{FK}_{\mathcal{R}}(B)$, there exist a (not necessarily unique) homomorphism $\Phi\colon \mathrm{FK}_{\mathcal{ST}}(A) \to \mathrm{FK}_{\mathcal{ST}}(B)$ for which $\mathfrak{F}_{\mathcal{R}}(\Phi) = \varphi$. If φ is an isomorphism then, by construction, so is Φ.*

Corollary 7.16. *Let A be a C^*-algebra over X with real rank zero, and assume that $\mathrm{K}_1\big(A(x)\big)$ is free for all $x \in X$. Then there exists a purely infinite graph C^*-algebra $C^*(E)$ that is tight over X and satisfies $\mathrm{FK}_{\mathcal{R}}\big(C^*(E)\big) \cong \mathrm{FK}_{\mathcal{R}}(A)$. If X is an EBP space, then automatically $\mathrm{FK}_{\mathcal{ST}}\big(C^*(E)\big) \cong \mathrm{FK}_{\mathcal{ST}}(A)$.*

If furthermore for all $x \in X$, the group $\mathrm{K}_*\big(A(x)\big)$ is finitely generated and $\operatorname{rank}\mathrm{K}_1\big(A(x)\big) = \operatorname{rank}\mathrm{K}_0\big(A(x)\big)$, *then $C^*(E)$ can be chosen to be a purely infinite Cuntz–Krieger algebra.*

Proof. Combine Theorem 6.12 with Corollary 6.9 and Corollary 7.15. □

7.1. **The particular case of the four-point space $\mathcal{D}$.** Consider the space $\mathcal{D} = \{1,2,3,4\}$ defined by $4 \to 3, 4 \to 2, 3 \to 1, 2 \to 1$. The space $\mathcal{D}$ is not a unique path space. The second-named author showed in [6] that there exists a finite refinement FK' of filtered K-theory FK given by adding a C^*-algebra $R_{1\setminus 4}$ over $\mathcal{D}$ to the collection $(R_Y)_{Y\in\mathbb{LC}(\mathcal{D})^*}$ of representing objects, creating a larger category $\mathcal{NT}'$. By [6, Theorem 6.2.14], isomorphisms on the refined filtered K-theory FK' lift to $\mathrm{KK}(\mathcal{D})$-equivalences, and thereby (using [17]) to $\mathcal{D}$-equivariant *-isomorphisms, for stable Kirchberg $\mathcal{D}$-algebras with all simple subquotients in the bootstrap class. However, there exist two non-isomorphic stable Kirchberg $\mathcal{D}$-algebras A and B with real rank zero and simple subquotients in the bootstrap class such that $\mathrm{FK}(A) \cong \mathrm{FK}(B)$, see [4, 6].

Proposition 7.17. *Let A and B be C^*-algebras over $\mathcal{D}$, assume that A and B have real rank zero, and assume that $\mathrm{K}_1\big(A(x)\big)$ and $\mathrm{K}_1\big(B(x)\big)$ are free abelian groups for all $x \in \{1,2,3\}$. Then any homomorphism $\varphi\colon \mathrm{FK}_{\mathcal{R}}(A) \to \mathrm{FK}_{\mathcal{R}}(B)$ extends (non-uniquely) to a homomorphism $\Phi\colon \mathrm{FK}'(A) \to \mathrm{FK}'(B)$. If φ is an isomorphism, then Φ is by construction an isomorphism.*

Proof. By Section 6.2.5 of [6], the refined filtered K-theory FK' consists of the following groups and maps:

$$\begin{tikzcd}
 & 12 \arrow[dr, "f_{12}"] & & 34 \arrow[dr, "i"] & & 3 \arrow[dr, "i"] & \\
123 \arrow[ur, "r"] \arrow[r, "\delta"] \arrow[dr, "r"'] & 4 \arrow[r, "f_4"] & 1\setminus 4 \arrow[ur, "f_{34}"] \arrow[r, "f_1"] \arrow[dr, "f_{24}"'] & 1 \arrow[r, "\delta"] & 234 \arrow[ur, "r"] \arrow[r, "i"] \arrow[dr, "r"'] & 1234 \arrow[r, "r"] & 123 \\
 & 13 \arrow[ur, "f_{13}"'] & & 24 \arrow[ur, "i"'] & & 2 \arrow[ur, "i"'] &
\end{tikzcd}$$

The proof of [4, Lemma 3.4] applies to the space $\mathcal{D}$, hence the two triangles

$$\begin{tikzcd}
\mathrm{FK}_{234}(A) \arrow[rr, "r^3_{234} i^{123}_3 \delta^4_{123} f_4"] & & \mathrm{FK}_{1\setminus 4}(A) \arrow[dl, "\circ (f^{34}, f^1, f^{24})"] \\
& \mathrm{FK}_{34}(A) \oplus \mathrm{FK}_1(A)[1] \oplus \mathrm{FK}_{23}(A) \arrow[ul, "i^{234}_{34} + \delta^{234}_1 + i^{234}_{24}"] &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{FK}_{1\setminus 4}(A) \arrow[rr, "f^1 \delta^{234}_1 r^3_{234} i^{123}_3 \circ"] & & \mathrm{FK}_{123}(A) \arrow[dl, "(r^{12}_{123}, \delta^4_{123}, r^{13}_{123})"] \\
& \mathrm{FK}_{12}(A) \oplus \mathrm{FK}_4(A)[1] \oplus \mathrm{FK}_{13}(A) \arrow[ul, "f_{12} + f_4 + f_{13}"] &
\end{tikzcd}$$

are exact. Since A is of real rank zero, the maps

$$\mathrm{FK}^0_{123}(A) \xrightarrow{\delta^4_{123}} \mathrm{FK}^1_4(A), \quad \mathrm{FK}^0_{12}(A) \xrightarrow{\delta^{34}_{12}} \mathrm{FK}^1_{34}(A),$$

$$\mathrm{FK}^0_{13}(A) \xrightarrow{\delta^{24}_{13}} \mathrm{FK}^1_{24}(A), \quad \mathrm{FK}^0_1(A) \xrightarrow{\delta^{234}_1} \mathrm{FK}^1_{234}(A)$$

vanish by Proposition 4 of [18]. So for C^*-algebras over $\mathcal{D}$ of real rank zero, the invariant FK' with the group $1 \setminus 4_0$ and its related maps omitted, consists of the following groups and maps:

$$\begin{tikzcd}[column sep=small]
& 24_1 \arrow[dr] & & 2_1 \arrow[dr] & & 12_1 \arrow[dr] & & 34_0 \arrow[dr] & & 3_0 \arrow[dr] & & 13_0 \arrow[dr] & \\
4_1 \arrow[ur] \arrow[dr] & & 234_1 \arrow[ur] \arrow[r] \arrow[dr] & 1234_1 \arrow[r] & 123_1 \arrow[ur] \arrow[r] \arrow[dr] & 4_0 \arrow[r] & 1 \setminus 4_1 \arrow[ur] \arrow[r] \arrow[dr] & 1_1 \arrow[r] & 234_0 \arrow[ur] \arrow[r] \arrow[dr] & 1234_0 \arrow[r] & 123_0 \arrow[ur] \arrow[dr] & & 1_0 \\
& 34_1 \arrow[ur] & & 3_1 \arrow[ur] & & 13_1 \arrow[ur] & & 24_0 \arrow[ur] & & 2_0 \arrow[ur] & & 12_0 \arrow[ur] &
\end{tikzcd}$$

The reduced filtered K-theory $\mathrm{FK}_{\mathcal{R}}$ consists of the sequences $3_1 \to 4_0 \to 34_0$, $2_1 \to 4_0 \to 24_0$, $1_1 \to 234_0 \to 1234_0$ together with the maps $34_0 \to 234_0$ and $24_0 \to 234_0$ and the group 4_1.

We will now construct $\Phi = (\Phi^*_Y)_{Y \in \mathbb{LC}(\mathcal{D})^* \cup \{1\setminus 4\}}$ from φ. Define $\Phi^1_{\{x\}} = \varphi_{x_1}$, $\Phi^0_{\widetilde{\partial x}} = \varphi_{\widetilde{\partial x}_0}$, and $\Phi^0_{\widetilde{\{x\}}} = \varphi_{\widetilde{x}_0}$ for all $x \in \mathcal{D}$. For $Y \in \{3, 2, 123, 13, 12, 1\}$, the maps Φ^0_Y are constructed as the induced maps on cokernels, as in the proof of Theorem 5.17.

Since $\mathrm{FK}^1_1(A)$ is free and the sequence

$$0 \longrightarrow \mathrm{FK}^0_4(A) \xrightarrow{f_4} \mathrm{FK}^1_{1\setminus 4}(A) \xrightarrow{f^1} \mathrm{FK}^1_1(A) \longrightarrow 0$$

is exact, we can find a free subgroup $V_{1\setminus 4}$ of $\mathrm{FK}^1_{1\setminus 4}(A)$ for which $\operatorname{im} f_4 \oplus V_{1\setminus 4} = \mathrm{FK}^1_{1\setminus 4}(A)$. Consider the commuting diagram

$$\begin{tikzcd}
V_{1\setminus 4} \arrow[r, "(f^{34}, f^1, f^{24})"] \arrow[d, dotted] & \mathrm{FK}^0_{34}(A) \oplus \mathrm{FK}^1_1(A) \oplus \mathrm{FK}^0_{24}(A) \arrow[r, "i^{234}_{34} + \delta^{234}_1 + i^{234}_{24}"] \arrow[d, "\Phi^0_{34} \oplus \Phi^1_1 \oplus \Phi^0_{24}"] & \mathrm{FK}^0_{234}(A) \arrow[d, "\Phi^0_{234}"] \\
\mathrm{FK}_{1\setminus 4}(B) \arrow[r, "(f^{34}, f^1, f^{24})"] & \mathrm{FK}^0_{34}(B) \oplus \mathrm{FK}^1_1(B) \oplus \mathrm{FK}^0_{24}(B) \arrow[r, "i^{234}_{34} + \delta^{234}_1 + i^{234}_{24}"] & \mathrm{FK}^0_{234}(B).
\end{tikzcd}$$

Since the bottom row is exact and the top row is a complex, and due to freeness of $V_{1\setminus 4}$, we may choose a map $\psi\colon V_{1\setminus 4} \to \mathrm{FK}^1_{1\setminus 4}(B)$ that makes the left square of the diagram commute. Define $\Phi^1_{1\setminus 4}$ on $\operatorname{im} f_4 \oplus V_{1\setminus 4}$ as $\Phi^0_4 + \psi$. By construction,

$$\Phi^1_{1\setminus 4} f_4 = f_4 \Phi^0_4, \quad f^1 \Phi^1_{1\setminus 4} = \Phi^1_1 f^1, \quad f^{34}\Phi^1_{1\setminus 4} = \Phi^0_{34} f^{34}, \quad f^{24}\Phi^1_{1\setminus 4} = \Phi^0_{24} f^{24},$$

and by the Five Lemma, the homomorphism $\Phi^1_{1\setminus 4}$ is an isomorphism if φ is an isomorphism.

Similarly, to construct Φ^1_{12}, use exactness of the sequence

$$0 \longrightarrow \mathrm{FK}^1_2(A) \xrightarrow{i_2^{12}} \mathrm{FK}_{12}(A) \xrightarrow{r^1_{12}} \mathrm{FK}^1_1(A) \xrightarrow{\delta^2_1} \mathrm{FK}^0_2(A)$$

and freeness of $\mathrm{FK}^1_1(A)$ to choose a free subgroup V_{12} of $\mathrm{FK}^1_{12}(A)$ for which $\operatorname{im} i_2^{12} \oplus V_{12} = \mathrm{FK}^1_{12}(A)$. Consider the commuting diagram

$$\begin{tikzcd}
V_{12} \arrow[r, "f_{12}"] \arrow[d, dotted] & \mathrm{FK}^1_{1\setminus 4}(A) \arrow[r, "f^{24}"] \arrow[d, "\Phi^1_{1\setminus 4}"] & \mathrm{FK}^0_{24}(A) \arrow[d, "\Phi^0_{24}"] \\
\mathrm{FK}^1_{12}(B) \arrow[r, "f_{12}"] & \mathrm{FK}^1_{1\setminus 4}(B) \arrow[r, "f^{24}"] & \mathrm{FK}^0_{24}(B).
\end{tikzcd}$$

Using exactness of the bottom row and that the top row is a complex, the map Φ^1_{12} can be constructed so that

$$\Phi^1_{12} i_2^{12} = i_2^{12} \Phi^1_2, \quad f_{12}\Phi^1_{12} = \Phi^1_{1\setminus 4} f_{12}.$$

Again due to the Five Lemma, Φ^1_{12} is an isomorphism if φ is. The maps Φ^1_{13}, Φ^1_{123}, Φ^1_{1234}, Φ^1_{234}, Φ^1_{34}, and Φ^1_{24} are constructed similarly and in the specified order.

Finally, the group $\mathrm{FK}^0_{1\setminus 4}(A)$ is naturally isomorphic to

$$\begin{aligned}
&\operatorname{coker}\bigl(\mathrm{FK}^0_{123}(A) \xrightarrow{(r^{12}_{123},\delta^4_{123},r^{13}_{123})} \mathrm{FK}^0_{12}(A) \oplus \mathrm{FK}^1_4(A) \oplus \mathrm{FK}^1_{13}(A)\bigr) \\
&= \mathrm{FK}^1_4(A) \oplus \operatorname{coker}\bigl(\mathrm{FK}^0_{123}(A) \xrightarrow{(r^{12}_{123},r^{13}_{123})} \mathrm{FK}^0_{12}(A) \oplus \mathrm{FK}^1_{13}(A)\bigr)
\end{aligned}$$

whose second summand, due to real rank zero, is naturally isomorphic to $\mathrm{FK}^0_1(A)$ Therefore, by defining $\Phi^0_{1\setminus 4}$ as the map induced by $\Phi^1_4 \oplus \Phi^0_1$, Φ becomes a $\mathcal{NT}'$-morphism. □

Corollary 7.18. *Let A and B be C^*-algebras over $\mathcal{D}$. Assume that A and B have real rank zero, that $\mathrm{K}_1\bigl(A(x)\bigr)$ and $\mathrm{K}_1\bigl(B(x)\bigr)$ are free abelian groups for all $x \in \{1,2,3\}$, and that A and B are in the bootstrap class of Meyer–Nest. Then any isomorphism $\mathrm{FK}_{\mathcal{R}}(A) \to \mathrm{FK}_{\mathcal{R}}(B)$ lifts to a $\mathrm{KK}(\mathcal{D})$-equivalence.*

Proof. Combine Proposition 7.17 with [6, Theorem 6.2.14]. □

Corollary 7.19. *Let A and B be stable Kirchberg $\mathcal{D}$-algebras of real rank zero, assume that $\mathrm{K}_1\bigl(A(x)\bigr)$ and $\mathrm{K}_1\bigl(B(x)\bigr)$ are free abelian groups for all $x \in \{1,2,3\}$, and assume that $A(x)$ and $B(x)$ are in the bootstrap class for all $x \in \mathcal{D}$. Then any isomorphism $\mathrm{FK}_{\mathcal{R}}(A) \to \mathrm{FK}_{\mathcal{R}}(B)$ lifts to a $\mathcal{D}$-equivariant *-isomorphism $A \to B$.*

Proof. Combine Proposition 7.17 with [6, Theorem 6.2.15]. □

8. Unital filtered K-theory

In [24, 2.1], Gunnar Restorff and Efren Ruiz showed that if a functor F (that factors through the functor K_0) strongly classifies a certain type of class of C^*-algebras up to stable isomorphism, then the functor $A \mapsto \big(F(A), [1_A] \in \mathrm{K}_0(A)\big)$ classifies unital, properly infinite C^*-algebras in the class up to isomorphism. A version with slightly generalized assumptions of this so-called *meta-theorem* may be found in [14] as Theorem 3.3. With these generalized assumptions, the theorem applies to filtered K-theory FK over accordion spaces X with respect to Kirchberg X-algebras with simple subquotients in the bootstrap class.

Let X be an arbitrary finite T_0-space. For $x, x' \in X$, we let $\inf(x, x')$ denote the set $\{y \in X \mid y \to x, y \to x'\}$.

Definition 8.1. The category $\mathfrak{Mod}(\mathcal{ST})^{\mathrm{pt}}$ of *pointed $\mathcal{ST}$-modules* is defined to have objects (M, m) where M is a $\mathcal{ST}$-module and $m \in M(X, 0)$, and morphisms $\varphi\colon (M, m) \to (N, n)$ that are $\mathcal{ST}$-morphisms with $\varphi(m) = n$.

The category $\mathfrak{Mod}(\mathcal{B})^{\mathrm{pt}}$ of *pointed $\mathcal{B}$-modules* is defined similarly with objects (M, m) where M is a $\mathcal{B}$-module and

$$m \in \operatorname{coker}\left(\bigoplus_{x,x'\in X,\, y\in\inf(x,x')} M(\widetilde{y}_0) \xrightarrow{\left(i_{\widetilde{y}_0}^{\widetilde{x}_0} \quad -i_{\widetilde{y}_0}^{\widetilde{x}'_0}\right)} \bigoplus_{x\in X} M(\widetilde{x}_0) \right),$$

and a morphism $\varphi\colon (M, m) \to (N, n)$ is a $\mathcal{B}$-morphism whose induced map on the cokernels sends m to n.

Similarly, the categories $\mathfrak{Mod}(\mathcal{BR})^{\mathrm{pt}}$ and $\mathfrak{Mod}(\mathcal{R})^{\mathrm{pt}}$ of *pointed $\mathcal{BR}$-modules* respectively *pointed $\mathcal{R}$-modules* are defined.

Definition 8.2. A pointed $\mathcal{ST}$-module (M, m) is called *exact* if M is an exact $\mathcal{ST}$-module, and *real-rank-zero-like* if M is real-rank-zero-like. Similary, a pointed $\mathcal{B}$-module, $\mathcal{BR}$-module, or $\mathcal{R}$-module (M, m) is called *exact* if M is exact.

Lemma 8.3. *Let M be an exact real-rank-zero-like $\mathcal{ST}$-module. Then the sequence*

$$\bigoplus_{x,x'\in X,\, y\in\inf(x,x')} M(\widetilde{\{y\}}, 0) \xrightarrow{\left(i_{\widetilde{\{y\}}}^{\widetilde{\{x\}}} \quad -i_{\widetilde{\{y\}}}^{\widetilde{\{x\}}'}\right)} \bigoplus_{x\in X} M(\widetilde{\{x\}}, 0) \xrightarrow{(i_{\widetilde{\{x\}}}^{X})} M(X, 0) \to 0$$

is exact.

Proof. By Lemma 6.8 the horizontal row of the following commuting diagram is exact:

$$\begin{tikzcd}
\bigoplus_{x,x'\in X} M(\widetilde{\{x\}} \cap \widetilde{\{x'\}}, 0) \arrow[r, "\left(i_{\widetilde{\{x\}}\cap\widetilde{\{x'\}}}^{\widetilde{\{x\}}} \quad -i_{\widetilde{\{x\}}\cap\widetilde{\{x'\}}}^{\widetilde{\{x\}}'}\right)"] & \bigoplus_{x\in X} M(\widetilde{\{x\}}, 0) \arrow[r] & M(X, 0) \arrow[r] & 0 \\
\bigoplus_{\substack{x,x'\in X \\ y\in\inf(x,x')}} M(\widetilde{\{y\}}, 0) \arrow[u, "\left(i_{\widetilde{\{y\}}}^{\widetilde{\{x\}}\cap\widetilde{\{x'\}}}\right)"] \arrow[ur, "\left(i_{\widetilde{\{y\}}}^{\widetilde{\{x\}}} \quad -i_{\widetilde{\{y\}}}^{\widetilde{\{x\}}'}\right)"'] & & &
\end{tikzcd}$$

Furthermore, since for any pair $x, x' \in X$ the collection $(\widetilde{\{y\}})_{y\in\inf(x,x')}$ covers $\widetilde{\{x\}} \cap \widetilde{\{x'\}}$, we see by Lemma 6.8 that the vertical map in the diagram is surjective. This establishes the desired result. $\square$

Definition 8.4. Let A be a unital C^*-algebra over a finite T_0-space X. Its *unital concrete filtered* K*-theory* $\mathrm{FK}^{\mathrm{unit}}_{\mathcal{ST}}(A)$ is defined as the pointed $\mathcal{ST}$-module $(\mathrm{FK}_{\mathcal{ST}}(A), [1_A])$.

If A has real rank zero, then its *unital reduced filtered* K*-theory* $\mathrm{FK}^{\mathrm{unit}}_{\mathcal{R}}(A)$ is defined as the pointed $\mathcal{R}$-module $(\mathrm{FK}_{\mathcal{R}}(A), u(A))$ where $u(A)$ is the unique element in

$$\operatorname{coker}\left(\bigoplus_{x,x'\in X,\, y\in\inf(x,x')} \mathrm{FK}^0_{\widetilde{\{y\}}}(A) \xrightarrow{\begin{pmatrix} i_{\widetilde{y}_0}^{\widetilde{x}_0} & -i_{\widetilde{y}_0}^{\widetilde{x}'_0}\end{pmatrix}} \bigoplus_{x\in X} \mathrm{FK}^0_{\widetilde{\{x\}}}(A)\right)$$

that is mapped to $[1_A]$ in $\mathrm{K}_0(A)$ by the map induced by the family $\big(\mathrm{FK}^0_{\widetilde{\{x\}}}(A) \xrightarrow{i^X_{\widetilde{\{x\}}}} \mathrm{FK}^0_X(A)\big)_{x\in X}$, see Lemma 8.3.

If A has real rank zero and X is a unique path space, then its *unital filtered* K*-theory restricted to the canonical base* $\mathrm{FK}^{\mathrm{unit}}_{\mathcal{B}}(A)$ is defined similarly.

By Lemma 8.3, we may view the forgetful functor $\mathfrak{F}_{\mathcal{B}}\colon \mathfrak{Mod}(\mathcal{ST}) \to \mathfrak{Mod}(\mathcal{B})$ as a functor from *pointed* exact real-rank-zero-like $\mathcal{ST}$-modules to *pointed* exact $\mathcal{B}$-modules and immediately obtain the following pointed version of Theorem 5.17:

Proposition 8.5. *For every EBP space X, the forgetful functor from exact pointed real-rank-zero-like $\mathcal{ST}$-modules to exact pointed $\mathcal{B}$-modules is an equivalence of categories.*

Proposition 8.6. *Assume that X is a unique path space. Let (M, m) and (N, n) be exact pointed $\mathcal{BR}$-modules with $M(x_1)$ and $N(x_1)$ free for all non-open points $x \in X$, and let $\varphi\colon \mathfrak{F}_{\mathcal{BR},\mathcal{R}}(M) \to \mathfrak{F}_{\mathcal{BR},\mathcal{R}}(N)$ be a pointed $\mathcal{R}$-module homomorphism. Then there exists a (not necessarily unique) pointed $\mathcal{BR}$-module homomorphism $\Phi\colon M \to N$ satisfying $\mathfrak{F}_{\mathcal{BR},\mathcal{R}}(\Phi) = \varphi$, and if φ is an isomorphism, then Φ is by construction an isomorphism.*

Proof. This follows from Theorem 7.8 since the groups $M(\widetilde{x}_0)$ are not forgotten by $\mathfrak{F}_{\mathcal{BR},\mathcal{R}}$. $\square$

Corollary 8.7. *Let X be an accordion space, and let A and B be unital Kirchberg X-algebras of real rank zero with all simple subquotients in the bootstrap class. Then any isomorphism $\mathrm{FK}^{\mathrm{unit}}_{\mathcal{B}}(A) \to \mathrm{FK}^{\mathrm{unit}}_{\mathcal{B}}(B)$ lifts to an X-equivariant *-isomorphism $A \to B$.*

Proof. This follows from Theorem 3.3 in [14] together with Theorem 3.11 and Corollary 5.19. $\square$

Corollary 8.8. *Let X be an accordion space, and let A and B be unital Kirchberg X-algebras of real rank zero with all simple subquotients in the bootstrap class. Assume that $\mathrm{K}_1\big(A(x)\big)$ and $\mathrm{K}_1\big(B(x)\big)$ are free abelian groups for all $x \in X$. Then any isomorphism $\mathrm{FK}^{\mathrm{unit}}_{\mathcal{R}}(A) \to \mathrm{FK}^{\mathrm{unit}}_{\mathcal{R}}(B)$ lifts to an X-equivariant *-isomorphism $A \to B$.*

Proof. This follows from Theorem 3.3 in [14] together with Theorem 3.11 and Corollary 7.14. □

Remark 8.9. There exist, up to homeomorphism, precisely four contractible unique path spaces with four points that are not accordion spaces. For all these spaces, the categories $\mathcal{NT}$ and $\mathcal{ST}$ coincide. In [4], Gunnar Restorff, Efren Ruiz and the first-named author showed that if X is one of these spaces, then FK is a complete invariant for stable Kirchberg X-algebras of real rank zero. Therefore, Corollaries 8.7 and 8.8 also hold for these spaces. Furthermore, the proof of Proposition 7.17 also applies to $\mathrm{FK}_{\mathcal{R}}^{\mathrm{unit}}$ and unital C^*-algebras, hence Corollary 8.8 also holds for the space $\mathcal{D}$.

We now recall the unital version of the range result from [3].

Theorem 8.10 ([3, Theorem 5.5])**.** *Let X be a finite T_0-space, and let (M,m) be an exact pointed $\mathcal{R}$-module. Assume that for all $x \in X$, $M(x_1)$ is a free abelian group,*

$$\operatorname{coker}\big(M(\widetilde{\partial}x_0) \xrightarrow{i_{\widetilde{\partial}x_0}^{\widetilde{x}_0}} M(\widetilde{x}_0)\big)$$

is finitely generated, and $\operatorname{rank} M(x_1) \leq \operatorname{rank}\operatorname{coker}\big(M(\widetilde{\partial}x_0) \xrightarrow{i_{\widetilde{\partial}x_0}^{\widetilde{x}_0}} M(\widetilde{x}_0)\big)$.

Then there exists a countable graph E satisfying that all vertices in E support at least two cycles, that E^0 is finite, that $C^*(E)$ is tight over X, and that $\mathrm{FK}_{\mathcal{R}}^{\mathrm{unit}}\big(C^*(E)\big)$ is isomorphic to (M,m). By construction $C^*(E)$ is unital and purely infinite.

The graph E can be chosen to have only regular vertices if (and only if) the rank of $M(x_1)$ coincides with the rank of the cokernel of $i\colon M(\widetilde{\partial}x_0) \to M(\widetilde{x}_0)$ for all $x \in X$. If E is chosen to have only regular vertices, then by construction $C^*(E)$ is a Cuntz–Krieger algebra.

Corollary 8.11. *Let X be a finite T_0-space and let A be a unital C^*-algebra over X of real rank zero. Assume for all $x \in X$ that $\mathrm{K}_1\big(A(x)\big)$ is free, $\mathrm{K}_0\big(A(x)\big)$ is finitely generated, and* $\operatorname{rank}\mathrm{K}_1\big(A(x)\big) \leq \operatorname{rank}\mathrm{K}_0\big(A(x)\big)$.

Then there exists a countable graph E for which $C^*(E)$ is unital, purely infinite, and tight over X such that $\mathrm{FK}_{\mathcal{R}}^{\mathrm{unit}}\big(C^*(E)\big) \cong \mathrm{FK}_{\mathcal{R}}^{\mathrm{unit}}(A)$. If X is an EBP space, then automatically $\mathrm{FK}_{\mathcal{ST}}^{\mathrm{unit}}\big(C^*(E)\big) \cong \mathrm{FK}_{\mathcal{ST}}^{\mathrm{unit}}(A)$.

If furthermore $\operatorname{rank}\mathrm{K}_1\big(A(x)\big) = \operatorname{rank}\mathrm{K}_0\big(A(x)\big)$ *for all $x \in X$, then E can be chosen such that $C^*(E)$ is a purely infinite Cuntz–Krieger algebra.*

Corollary 8.12. *Let X be an accordion space, and let $I \hookrightarrow A \twoheadrightarrow B$ be an extension of C^*-algebras. Assume that A is unital and tight over X.*

Then A is a purely infinite Cuntz–Krieger algebra if and only if

- *I is stably isomorphic to a purely infinite Cuntz–Krieger algebra,*
- *B is a purely infinite Cuntz–Krieger algebra,*
- *the exponential map $\mathrm{K}_0(B) \to \mathrm{K}_1(I)$ vanishes.*

Proof. Recall that Cuntz–Krieger algebras are purely infinite if and only if they have real rank zero. Assume that A is a purely infinite Cuntz–Krieger algebra. It is well-known that then B is also a purely infinite Cuntz–Krieger algebra and I is stably isomorphic to one. By Theorem 4.2 of [21], $\mathrm{K}_0(B) \to \mathrm{K}_1(I)$ vanishes since A has real rank zero and therefore is K_0-liftable, see Remark 3.10.

Now, assume that B is a purely infinite Cuntz–Krieger algebra, that I is stably isomorphic to one, and that the map $\mathrm{K}_0(B) \to \mathrm{K}_1(I)$ vanishes. By Theorem 4.3 of [28], A is $\mathcal{O}_\infty$-absorbing since B and I are. Since B and I are K_0-liftable and $\mathrm{K}_0(B) \to \mathrm{K}_1(I)$ vanishes, A is also K_0-liftable (that is, $\mathrm{FK}(A)$ is real-rank-zero-like) by [7, Proposition 3.5]. So by pure infiniteness of A it therefore follows from Theorem 4.2 of [21] that A has real rank zero. For all $x \in X$, $\mathrm{K}_1\big(A(x)\big)$ is free since B and I are stably isomorphic to Cuntz–Krieger algebras. So by Theorem 8.10 there exists a real-rank-zero Cuntz–Krieger algebra C that is tight over X and has $\mathrm{FK}^{\mathrm{unit}}_{\mathcal{R}}(A) \cong \mathrm{FK}^{\mathrm{unit}}_{\mathcal{R}}(C)$. By Corollary 8.8, A and C are isomorphic. □

Remark 8.13. Corollary 8.12 holds in fact for all spaces X for which $\mathrm{FK}^{\mathrm{unit}}_{\mathcal{R}}$ is a complete invariant for unital Kirchberg X-algebras A where $A(x)$ is in the bootstrap class and $\mathrm{K}_1\big(A(x)\big)$ is free for all $x \in X$, see Remark 8.9.

9. Ordered filtered K-theory

The notion of ordered filtered K-theory was introduced by Søren Eilers, Gunnar Restorff, and Efren Ruiz in [13] to classify certain (not necessarily purely infinite) graph C^*-algebras of real rank zero. We hope that the results in this section will be useful for future work in this direction.

Recall that for a C^*-algebra A, a class in $\mathrm{K}_0(A)$ of the from $[p]_0$ for a projection p in $M_n(A)$ for some $n \in \mathbb{N}$ is called *positive*. The *positive cone* $\mathrm{K}_0(A)^+$ consists of all positive elements in $\mathrm{K}_0(A)$. For two C^*-algebras A and B, a group homomorphism $\varphi\colon \mathrm{K}_0(A) \to \mathrm{K}_0(B)$ is called *positive* if $\varphi(\mathrm{K}_0(A)^+) \subseteq \mathrm{K}_0(B)^+$, and a group isomorphism $\varphi\colon \mathrm{K}_0(A) \to \mathrm{K}_0(B)$ is called an *order isomorphism* if $\varphi(\mathrm{K}_0(A)^+) = \mathrm{K}_0(B)^+$.

Note that for a finite topological space X, a locally closed subset Y of X, and an open subset U of Y, the maps $i_U^Y\colon \mathrm{K}_0\big(A(U)\big) \to \mathrm{K}_0\big(A(Y)\big)$ and $r_Y^{Y\setminus U}\colon \mathrm{K}_0\big(A(Y)\big) \to \mathrm{K}_0\big(A(Y\setminus U)\big)$ are positive.

Definition 9.1. For C^*-algebras A and B over a finite topological space X, an $\mathcal{ST}$-module homomorphism $\varphi\colon \mathrm{FK}_{\mathcal{ST}}(A) \to \mathrm{FK}_{\mathcal{ST}}(B)$ is called *positive* if the induced maps $\mathrm{FK}^0_Y(A) \to \mathrm{FK}^0_Y(B)$ are positive for all $Y \in \mathbb{LC}(X)$, and an $\mathcal{ST}$-module isomorphism $\mathrm{FK}_{\mathcal{ST}}(A) \to \mathrm{FK}_{\mathcal{ST}}(B)$ is called an *order isomorphism* if the induced isomomorphisms are order isomorphisms. For the reduced versions $\mathrm{FK}_{\mathcal{R}}$, $\mathrm{FK}_{\mathcal{B}}$, and $\mathrm{FK}_{\mathcal{BR}}$ of filtered K-theory, analogous definitions apply.

We are indebted to Mikael Rørdam for the elegant proof of the following lemma.

Lemma 9.2. *Let A be a real-rank-zero C^*-algebra and let I and J be* (*closed, two-sided*) *ideals in A satisfying $I + J = A$. Then any projection p in A can be written as $p = q + q'$ with a projection q in I and a projection q' in J.*

Proof. Let p a projection in A be given and write $p = a + b$ with $a \in I$ and $b \in J$. We may assume that $a = pap$ and $b = pbp$. As A has real rank zero, the hereditary subalgebra pIp has an approximate unit of projections, so there exists a projection q in pIp satisfying $\|a - aq\| < 1$. Since $q = pqp$, $q \le p$ and we may define a projection q' as $q' = p - q$. It remains to prove $q' \in J$. We have

$$\|q' - q'bq'\| = \|q'(p-b)q'\| = \|q'a(p-q)\| \le \|q'\|\|a - aq\| < 1.$$

Since $q'bq' \in J$, the image of q' in the quotient A/J is a projection of norm strictly less than 1. Since such a projection is 0, we get $q' \in J$. □

The following theorem is a version of Corollary 5.19 taking the order into account.

Theorem 9.3. *Let X be an EBP space, and let A and B be C^*-algebras over X of real rank zero. Then for any order isomorphism $\varphi\colon \mathrm{FK}_{\mathcal{B}}(A) \to \mathrm{FK}_{\mathcal{B}}(B)$ there is a unique order isomorphism $\Phi\colon \mathrm{FK}_{\mathcal{ST}}(A) \to \mathrm{FK}_{\mathcal{ST}}(B)$ satisfying $\mathfrak{F}_{\mathcal{B}}(\Phi) = \varphi$.*

Proof. By Corollary 5.19, Φ is an isomorphism if and only if φ is. Assume that φ is an order isomorphism, and let us show first for $Y \in \mathbb{O}(X)$ and then for $Y \in \mathbb{LC}(X)$ that Φ^0_Y is an order isomorphism.

For U an open subset of X, the following diagram has commuting squares and its rows are exact by Lemmas 4.7 and 4.8.

$$\begin{tikzcd}
\bigoplus\limits_{y\in\inf(x,x')} \mathrm{FK}^0_{\widetilde{\{y\}}}(A) \arrow[r, "\begin{pmatrix} i^{\widetilde{\{x\}}}_{\widetilde{\{y\}}} & -i^{\widetilde{\{x\}}'}_{\widetilde{\{y\}}} \end{pmatrix}"] \arrow[d, "(\varphi^0_{\widetilde{\{y\}}})"] & \bigoplus\limits_{x\in U} \mathrm{FK}^0_{\widetilde{\{x\}}}(A) \arrow[r, "(i^U_{\widetilde{\{x\}}})"] \arrow[d, "(\varphi^0_{\widetilde{\{x\}}})"] & \mathrm{FK}^0_U(A) \arrow[r] \arrow[d, "\Phi^0_U"] & 0 \\
\bigoplus\limits_{y\in\inf(x,x')} \mathrm{FK}^0_{\widetilde{\{y\}}}(B) \arrow[r, "\begin{pmatrix} i^{\widetilde{\{x\}}}_{\widetilde{\{y\}}} & -i^{\widetilde{\{x\}}'}_{\widetilde{\{y\}}} \end{pmatrix}"] & \bigoplus\limits_{x\in U} \mathrm{FK}^0_{\widetilde{\{x\}}}(B) \arrow[r, "(i^U_{\widetilde{\{x\}}})"] & \mathrm{FK}^0_U(B) \arrow[r] & 0
\end{tikzcd}$$

Since $\big(A(\widetilde{\{x\}})\big)_{x\in U}$ is a finite collection of ideals in $A(U)$, we see by Lemma 9.2 that the map $(i^U_{\widetilde{\{x\}}})\colon \bigoplus\limits_{x\in U} \mathrm{K}_0(A(\widetilde{\{x\}}) \to \mathrm{K}_0\big(A(U)\big)$ surjects $\bigoplus\limits_{x\in U} \mathrm{K}_0(A(\widetilde{\{x\}})^+$ onto $\mathrm{K}_0\big(A(U)\big)^+$. Similarly, the map $(i^U_{\widetilde{\{x\}}})\colon \bigoplus\limits_{x\in U} \mathrm{K}_0(B(\widetilde{\{x\}}) \to \mathrm{K}_0\big(B(U)\big)$ surjects $\bigoplus\limits_{x\in U} \mathrm{K}_0(B(\widetilde{\{x\}})^+$ onto $\mathrm{K}_0\big(B(U)\big)^+$. A simple diagram chase therefore shows that Φ^0_U is an order isomorphism since the map $\varphi^0_{\widetilde{\{x\}}}$ is an order isomorphism for all $x \in U$.

For a locally closed subset Y of X, choose open subsets U and V of X satisfying $V \subseteq U$ and $U \setminus V = Y$. Then Φ^0_U is an order isomorphism. Consider the following diagram with exact rows and commuting squares.

$$\begin{tikzcd}
\mathrm{FK}^0_V(A) \arrow[r, "i^U_V"] \arrow[d, "\Phi^0_V"] & \mathrm{FK}^0_U(A) \arrow[r, "r^Y_U"] \arrow[d, "\Phi^0_U"] & \mathrm{FK}^0_Y(A) \arrow[r] \arrow[d, "\Phi^0_Y"] & 0 \\
\mathrm{FK}^0_V(B) \arrow[r, "i^U_V"] & \mathrm{FK}^0_U(B) \arrow[r, "r^Y_U"] & \mathrm{FK}^0_Y(B) \arrow[r] & 0
\end{tikzcd}$$

In [10, Theorem 3.14], Lawrence G. Brown and Gert K. Pedersen showed that given an extension $I \hookrightarrow C \twoheadrightarrow C/I$ of C^*-algebras, the C^*-algebra C has real rank zero if and only if I and C/I have real rank zero and projections in C/I lift to projections in C. Thus, since A and therefore $M_n \otimes A(U)$ for all n has real rank zero, the map $i^Y_U\colon \mathrm{K}_0\big(A(U)\big) \to \mathrm{K}_0\big(A(Y)\big)$ surjects $\mathrm{K}_0\big(A(U)\big)^+$ onto $\mathrm{K}_0\big(A(Y)\big)^+$. Similarly, the map $i^Y_U\colon \mathrm{K}_0\big(B(U)\big) \to \mathrm{K}_0\big(B(Y)\big)$ surjects $\mathrm{K}_0\big(B(U)\big)^+$ onto $\mathrm{K}_0\big(B(Y)\big)^+$. A simple diagram chase therefore shows that Φ^0_Y is an order isomorphism. $\square$

We have the following ordered analogs of Theorem 7.8 and Corollary 7.15.

Theorem 9.4. *Let X be a unique path space, and let A and B be C^*-algebras over X of real rank zero. Assume that $\mathrm{K}_1\big(A(\{x\})\big)$ and $\mathrm{K}_1\big(B(\{x\})\big)$ are free abelian groups for all non-open points $x \in X$. Then for any order isomorphism $\varphi\colon \mathrm{FK}_{\mathcal{R}}(A) \to \mathrm{FK}_{\mathcal{R}}(B)$ there exists a (not necessarily unique) order isomorphism $\Phi\colon \mathrm{FK}_{\mathcal{BR}}(A) \to \mathrm{FK}_{\mathcal{BR}}(B)$ that satisfies $\mathfrak{F}_{\mathcal{BR},\mathcal{R}}(\Phi) = \varphi$.*

Proof. Since the functor $\mathfrak{F}_{\mathcal{BR},\mathcal{R}}$ only forgets K_1-groups, the desired follows immediately from Theorem 7.8. □

Corollary 9.5. *Let X be an EBP space, and let A and B be C^*-algebras over X of real rank zero. Assume that $\mathrm{K}_1\big(A(\{x\})\big)$ and $\mathrm{K}_1\big(B(\{x\})\big)$ are free abelian groups for all $x \in X$. Then for any order isomorphism $\varphi\colon \mathrm{FK}_{\mathcal{R}}(A) \to \mathrm{FK}_{\mathcal{R}}(B)$ there exists a order isomorphism $\Phi\colon \mathrm{FK}_{\mathcal{ST}}(A) \to \mathrm{FK}_{\mathcal{ST}}(B)$ that satisfies $\mathfrak{F}_{\mathcal{R}}(\Phi) = \varphi$.*

Proof. Combine the previous two theorems. □

10. Corollaries for accordion spaces

We summarize our results in the most satisfying case of accordion spaces. By combining Theorems 3.11, 5.17, 6.12 and Corollaries 6.9, 7.14 in the stable case and Proposition 8.5, Corollaries 6.9, 8.7, 8.8, and Theorem 8.10 in the unital case, we obtain the following characterization of purely infinite graph C^*-algebras, and of purely infinite Cuntz–Krieger algebras. In the first list, we use that the stabilization of a graph C^*-algebra is again a graph C^*-algebra by [1, Proposition 9.8(3)].

Corollary 10.1. *Let X be an accordion space. The different versions of filtered* K*-theory introduced in this article induce bijections between the sets of isomorphism classes of objects in the following three lists, respectively.*

List 1:

- *tight, stable, purely infinite graph C^*-algebras over X,*
- *stable Kirchberg X-algebras A of real rank zero with all simple subquotients in the bootstrap class satisfying that $\mathrm{K}_1\big(A(\{x\})\big)$ is free for all $x \in X$,*
- *countable, exact, real-rank-zero-like $\mathcal{NT}$-modules M with $M(\{x\}, 1)$ free for all $x \in X$,*
- *countable, exact $\mathcal{B}$-modules M with $M(x_1)$ free for all $x \in X$,*
- *countable, exact $\mathcal{R}$-modules M with $M(\overline{x}_1)$ free for all $x \in X$.*

List 2:

- *tight, unital, purely infinite graph C^*-algebras over X,*
- *unital Kirchberg X-algebras A of real rank zero, with all simple subquotients in the bootstrap class such that, for all $x \in X$, the group $\mathrm{K}_1\big(A(\{x\})\big)$ is free and*
$$\operatorname{rank} \mathrm{K}_1\big(A(\{x\})\big) \leq \operatorname{rank} \mathrm{K}_0\big(A(\{x\})\big) < \infty,$$
- *countable, exact, real-rank-zero-like pointed $\mathcal{NT}$-modules M such that, for all $x \in X$, the group $M(\{x\}, 1)$ is free and*
$$\operatorname{rank}\big(M(\{x\}, 1)\big) \leq \operatorname{rank}\big(M(\{x\}, 0)\big) < \infty,$$
- *countable, exact pointed $\mathcal{B}$-modules M such that, for all $x \in X$, the group $M(x_1)$ is free and*
$$\operatorname{rank}\big(M(x_1)\big) \leq \operatorname{rank}\Big(\operatorname{coker}\big(\bigoplus_{y \to x} M(\widetilde{y}_0) \to M(\widetilde{x}_0)\big)\Big) < \infty,$$

- *isomorphism classes of countable, exact pointed $\mathcal{R}$-modules M such that, for all $x \in X$, the group $M(\overline{x}_1)$ is free and*
$$\operatorname{rank}\big(M(x_1)\big) \leq \operatorname{rank}\Big(\operatorname{coker}\big(M(\widetilde{\partial} x_0) \to M(\widetilde{x}_0)\big)\Big) < \infty.$$

List 3:

- *tight, purely infinite Cuntz–Krieger algebras over X,*
- *unital Kirchberg X-algebras A of real rank zero, with all simple subquotients in the bootstrap class such that, for all $x \in X$, the group $\mathrm{K}_1\big(A(\{x\})\big)$ is free and*
$$\operatorname{rank} \mathrm{K}_1\big(A(\{x\})\big) = \operatorname{rank} \mathrm{K}_0\big(A(\{x\})\big) < \infty,$$
- *countable, exact, real-rank-zero-like pointed $\mathcal{NT}$-modules M such that, for all $x \in X$, the group $M(\{x\},1)$ is free and*
$$\operatorname{rank}\big(M(\{x\},1)\big) = \operatorname{rank}\big(M(\{x\},0)\big) < \infty,$$
- *countable, exact pointed $\mathcal{B}$-modules M such that, for all $x \in X$, the group $M(x_1)$ is free and*
$$\operatorname{rank}\big(M(x_1)\big) = \operatorname{rank}\Big(\operatorname{coker}\big(\bigoplus_{y \to x} M(\widetilde{y}_0) \to M(\widetilde{x}_0)\big)\Big) < \infty,$$
- *countable, exact pointed $\mathcal{R}$-modules M such that, for all $x \in X$, the group $M(\overline{x}_1)$ is free and*
$$\operatorname{rank}\big(M(x_1)\big) = \operatorname{rank}\Big(\operatorname{coker}\big(M(\widetilde{\partial} x_0) \to M(\widetilde{x}_0)\big)\Big) < \infty.$$

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KIRCHBERG X-ALGEBRAS WITH REAL RANK ZERO AND INTERMEDIATE CANCELLATION

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Abstract. A universal coefficient theorem is proved for C^*-algebras over an arbitrary finite T_0-space X which have vanishing boundary maps. Under bootstrap assumptions, this leads to a complete classification of unital/stable real-rank-zero Kirchberg X-algebras with intermediate cancellation. Range results are obtained for (unital) purely infinite graph C^*-algebras with intermediate cancellation and Cuntz–Krieger algebras with intermediate cancellation. Permanence results for extensions of these classes follow.

1. Introduction

Since Eberhard Kirchberg's groundbreaking classification theorem for non-simple $\mathcal{O}_\infty$-absorbing nuclear C^*-algebras [16], much effort has gone into the task of deciding when two separable C^*-algebras over a topological space X are $\mathrm{KK}(X)$-equivalent. This is a hard task even when X is a finite space. The usual way to go is to prove equivariant versions of the *universal coefficient theorem* of Rosenberg and Schochet [27]. For *some* spaces, such have been established in [3, 4, 7, 21, 24]. In [5], a complete classification in purely algebraic terms of objects in the equivariant bootstrap class $\mathcal{B}(X) \subset \mathfrak{KK}(X)$ up to $\mathrm{KK}(X)$-equivalence is given under the assumption that X is a so-called *unique path space*. Nevertheless, it seems fair to state that, for *most* finite spaces, no classification is available at the present time.

In this note we establish a universal coefficient theorem computing the groups $\mathrm{KK}_*(X; A, B)$ which holds for all finite T_0-spaces X—but only under certain K-theoretical assumptions on A. More precisely, we have to ask that the boundary maps in all six-term exact sequences arising from inclusions of distinguished ideals vanish. If A is separable, purely infinite and tight over X, this condition is equivalent to A having real rank zero and the following non-stable K-theory property suggested to us by Mikael Rørdam: if p and q are projections in A which generate the same ideal and which give rise to the same element in $\mathrm{K}_0(A)$, then p and q are Murray-von Neumann equivalent. This property has been considered earlier by Lawrence G. Brown [9]. Since the property is stronger than Brown-Pedersen's weak cancellation property and weaker than Rieffel's strong cancellation property (compare [11]), it is referred to as *intermediate cancellation*.

The invariant appearing in our universal coefficient theorem, denoted by XK, is relatively simple: for a point $x \in X$, let U_x denote its minimal open neighbourhood. Then $\mathrm{XK}(A)$ consists of the collection $\{\mathrm{K}_*\big(A(U_x)\big) \mid x \in X\}$ together with the natural maps induced by the ideal inclusions $A(U_x) \hookrightarrow A(U_y)$ for $U_x \subseteq U_y$. Hence $\mathrm{XK}(A)$ can be regarded as a representation of the partially ordered set X with values in countable $\mathbb{Z}/2$-graded Abelian groups. Equivalently, we may view $\mathrm{XK}(A)$ as a countable $\mathbb{Z}/2$-graded module over the integral incidence algebra $\mathbb{Z}X$ of X. The fact that the ring $\mathbb{Z}X$ itself is ungraded allows us to show that the universal

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coefficient sequence for $\mathrm{KK}_*(X; A, B)$ splits if both A and B have vanishing boundary maps and that an object in the equivariant bootstrap class $\mathcal{B}(X)$ with vanishing boundary maps is $\mathrm{KK}(X)$-equivalent to a commutative C^*-algebra over X.

A Kirchberg X-algebra is a nuclear purely infinite separable tight C^*-algebra over X. Combining our universal coefficient theorem with Kirchberg's theorem, we find that the invariant XK strongly classifies stable real-rank-zero Kirchberg X-algebras with intermediate cancellation and simple subquotients in the bootstrap class up to *-isomorphism over X.

We also describe the range of the invariant XK on this class of C^*-algebras over X, but only in the case that X is a unique path space. To this aim, we use a second invariant denoted by $\mathbb{O}\mathrm{K}$. It is defined similarly to XK but it contains the K-groups of *all* distinguished ideals. The target category of $\mathbb{O}\mathrm{K}$ is the category of precosheaves on the topology of X with values in countable $\mathbb{Z}/2$-graded Abelian groups. It turns out that the range of $\mathbb{O}\mathrm{K}$ on the class of stable real-rank-zero Kirchberg X-algebras with intermediate cancellation and simple subquotients in the bootstrap class consists precisely of those precosheaves which satisfy a certain cosheaf condition and have injective structure maps; following Bredon [8], we call these *flabby cosheaves*.

Appealing to the so-called meta theorem [14, Theorem 3.3], we can achieve strong classification also in the unital case. The invariant in this case, denoted by $\mathbb{O}\mathrm{K}^+$, consists of the functor $\mathbb{O}\mathrm{K}$ together with the unit class in the K_0-group of the whole C^*-algebra.

We apply our results to the classification programme of (purely infinite) graph C^*-algebras. Here real rank zero comes for free, as do separability, nuclearity and bootstrap assumptions. We determine the range of the invariant $\mathbb{O}\mathrm{K}$ on the class of purely infinite tight graph C^*-algebras over X with intermediate cancellation. We also determine the range of the invariant $\mathbb{O}\mathrm{K}^+$ on the class of unital purely infinite tight graph C^*-algebras over X with intermediate cancellation and on the class of tight Cuntz–Krieger algebras over X with intermediate cancellation. Here we use a result from [2] that allows to construct graph C^*-algebras with prescribed K-theory data.

As an application, we show that the class of Cuntz–Krieger algebras with intermediate cancellation is, in a suitable sense, stable under extensions (see Theorem 8.4 for the precise statement). A similar result is obtained in [2, Corollary 9.15], but under different assumptions: in [2] we make assumptions on the primitive ideal space to make the classification machinery work; in *this* article we use intermediate cancellation to achieve that. Similar permanence results hold for (unital) purely infinite graph C^*-algebras with intermediate cancellation.

2. Preliminaries

Throughout, let X be an arbitrary finite T_0-space. A subset of X is *locally closed* if it is the difference of two open subsets of X. Every point $x \in X$ possesses a smallest open neighbourhood denoted by U_x. The *specialization preorder* on X is the partial order defined such that $x \leq y$ if and only if $U_y \subseteq U_x$. For two points $x, y \in X$, there is an arrow from y to x in the Hasse diagram associated to the specialization preorder on X if and only if y is a closed point in $U_x \setminus \{x\}$; in this case we write $y \to x$. We say that X is a *unique path space* if every pair of points in X is connected by at most one directed path in the Hasse diagram associated to the specialization preorder on X.

A *C^*-algebra over X* is a pair (A, ψ) consisting of a C^*-algebra A and a continuous map $\psi\colon \mathrm{Prim}(A) \to X$. The pair (A, ψ) is called *tight* if the map ψ is a homeomorphism. We usually omit the map ψ in order to simplify notation.

There is a lattice isomorphism between the open subsets in $\mathrm{Prim}(A)$ and the ideals in A. Hence every open subset U of X gives rise to a *distinguished ideal* $A(U)$ in A. A **-homomorphism over* X is a *-homomorphism mapping distinguished ideals into corresponding distinguished ideals. We obtain the category $\mathfrak{C}^*\mathfrak{alg}(X)$ of C^*-algebras over X and *-homomorphisms over X. Any locally closed subset Y of X determines a *distinguished subquotient* $A(Y)$ of A. There is a natural way to regard the subquotient $A(Y)$ as a C^*-algebra over Y. For a point $x \in X$, we let $i_x\mathbb{C}$ denote the C^*-algebra over X given by the C^*-algebra of complex numbers $\mathbb{C}$ with the map $\mathrm{Prim}(\mathbb{C}) \to X$ taking the unique primitive ideal in $\mathbb{C}$ to x. For more details on C^*-algebras over topological spaces, see [22].

Eberhard Kirchberg developed a version of Kasparov's KK-theory for separable C^*-algebras over X in [16] denoted by $\mathrm{KK}(X)$. In [22], Ralf Meyer and Ryszard Nest establish basic properties of the resulting category $\mathfrak{KK}(X)$, describe a natural triangulated category structure on it, and give an appropriate definition of the equivariant bootstrap class $\mathcal{B}(X) \subset \mathfrak{KK}(X)$: it is the smallest triangulated subcategory of $\mathfrak{KK}(X)$ that contains the object set $\{i_x\mathbb{C} \mid x \in X\}$ and is closed under countable direct sums. The usual bootstrap class in $\mathfrak{KK}$ of Rosenberg and Schochet is denoted by $\mathcal{B}$. The translation functor on $\mathfrak{KK}(X)$ is given by suspension and denoted by Σ. The category $\mathfrak{KK}(X)$ is tensored over $\mathfrak{KK}$; in particular, we can talk about the stabilization $A \otimes \mathbb{K}$ of an object A in $\mathfrak{KK}(X)$. Here $\mathbb{K}$ denotes the C^*-algebra of compact operators on some countably infinite-dimensional Hilbert space.

For an object M in a $\mathbb{Z}/2$-graded category, we write M_0 for the even part, M_1 for the odd part and $M[1]$ for the shifted object. If N is an object in the ungraded category, we let $N[i]$ denote the corresponding graded object concentrated in degree i. We write $C \in\in \mathcal{C}$ to denote that C is an object in a category $\mathcal{C}$.

3. Vanishing boundary maps

In this section, we introduce two K-theoretical conditions for C^*-algebras over X that are sufficient, as we shall see later, to obtain a universal coefficient theorem. We provide alternative formulations of these conditions for separable purely infinite tight C^*-algebras over X.

Given a C^*-algebra A over X and open subsets $U \subseteq V \subseteq X$, we have a six-term exact sequence

$$\begin{tikzcd} \mathrm{K}_1\big(A(U)\big) \arrow[r] & \mathrm{K}_1\big(A(V)\big) \arrow[r] & \mathrm{K}_1\big(A(V)/A(U)\big) \arrow[d, "\partial_1"] \\ \mathrm{K}_0\big(A(V)/A(U)\big) \arrow[u, "\partial_0"] & \mathrm{K}_0\big(A(V)\big) \arrow[l] & \mathrm{K}_0\big(A(U)\big). \arrow[l] \end{tikzcd} \tag{3.1}$$

Definition 3.2. Let A be a C^*-algebra over X. We say that A *has vanishing index maps* if the map $\partial_1\colon \mathrm{K}_1\big(A(V)/A(U)\big) \to \mathrm{K}_0\big(A(U)\big)$ vanishes for all open subsets $U \subseteq V \subseteq X$. Similarly, we say that A *has vanishing exponential maps* if the map $\partial_0\colon \mathrm{K}_0\big(A(V)/A(U)\big) \to \mathrm{K}_1\big(A(U)\big)$ vanishes for all open subsets $U \subseteq V \subseteq X$. We say that A *has vanishing boundary maps* if it has vanishing index maps and vanishing exponential maps.

Remarks 3.3. If A is a tight C^*-algebra over X then A has vanishing exponential maps if and only if the underlying C^*-algebra of A is K_0-liftable in the sense of [26, Definition 3.1].

In the definition above, we could replace the subset $V \subseteq X$ with X, but to us the definition seems more natural as it stands.

Another, a priori *stronger* condition consists in the vanishing of all boundary maps arising from inclusions of distinguished *subquotients*. The following lemma shows that this assumption is in fact equivalent to the one in our definition.

Lemma 3.4. *Let $Y \subseteq X$ be locally closed. Let $U \subseteq Y$ be relatively open. Write $C = Y \setminus U$. Let A be a C^*-algebra over X with vanishing index/exponential maps. Then the index/exponential map corresponding to the extension $A(U) \rightarrowtail A(Y) \twoheadrightarrow A(C)$ vanishes, too.*

Proof. Write $Y = V \setminus W$ as the difference of two open subsets $W \subseteq V \subseteq X$. Consider the morphism of extensions of distinguished subquotients

$$\begin{tikzcd} A(V \setminus C) \arrow[r, tail] \arrow[d] & A(V) \arrow[r, two heads] \arrow[d] & A(C) \arrow[d, equal] \\ A(U) \arrow[r, tail] & A(Y) \arrow[r, two heads] & A(C). \end{tikzcd}$$

The first extension has vanishing index/exponential map by assumption. By naturality, the same follows for the second extension. □

Proposition 3.5. *Let $U \subseteq X$ be an open subset and write $C = X \setminus U$. Let A be a C^*-algebra over X. Then A has vanishing index maps if and only if the following hold:*

- *$A(U) \in\in \mathfrak{C}^*\mathfrak{alg}(U)$ has vanishing index maps,*
- *$A(C) \in\in \mathfrak{C}^*\mathfrak{alg}(C)$ has vanishing index maps,*
- *the index map $\mathrm{K}_1\big(A(C)\big) \to \mathrm{K}_0\big(A(U)\big)$ vanishes.*

An analogous statement holds for vanishing exponential maps.

Proof. We will only prove the statement for index maps, the case of exponential maps being entirely analogous. By the previous lemma, the three conditions are necessary. To show that they are also sufficient, we consider an open subset $V \subseteq X$. It suffices to check that the map $\mathrm{K}_0\big(A(V)\big) \to \mathrm{K}_0\big(A(X)\big)$ is injective. We consider the morphism of extensions of distinguished subquotients

$$\begin{tikzcd} A(U \cap V) \arrow[r, tail] \arrow[d, tail] & A(U) \arrow[r, two heads] \arrow[d, tail] & A\big(U \setminus (U \cap V)\big) \arrow[d, equal] \\ A(V) \arrow[r, tail] & A(U \cup V) \arrow[r, two heads] & A\big((U \setminus (U \cap V)\big). \end{tikzcd}$$

By the first condition, the upper extension has vanishing index map. By naturality, so has the second. Hence the map $\mathrm{K}_0\big(A(V)\big) \to \mathrm{K}_0\big(A(U \cup V)\big)$ is injective. By the second and third condition, the composition

$$\mathrm{K}_1\big(A(X)\big) \to \mathrm{K}_1\big(A(C)\big) \to \mathrm{K}_1\big(A(X \setminus (U \cup V))\big)$$

is surjective. By the six-term exact sequence, the map $\mathrm{K}_0\big(A(U \cup V)\big) \to \mathrm{K}_0\big(A(X)\big)$ is thus injective. The result follows. □

Corollary 3.6. *Let A be a C^*-algebra over X. Then A has vanishing index/exponential maps if and only if the index/exponential map of the extension*

$$A(U_x \setminus \{x\}) \rightarrowtail A(U_x) \twoheadrightarrow A(\{x\})$$

vanishes for every point $x \in X$.

Proof. Again, we will only prove the statement for index maps. The condition is clearly necessary. In order to prove sufficiency, we choose a filtration

$$\emptyset = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_\ell = X,$$

of X by open subsets V_j such that $V_j \setminus V_{j-1} = \{x_j\}$ is a singleton for all $j = 1, \dots, \ell$. By naturality of the index map, the condition implies that the index map of the extension

$$A(V_{j-1}) \rightarrowtail A(V_j) \twoheadrightarrow A(\{x_j\})$$

vanishes for all $j = 1, \dots, \ell$. A repeated application of Proposition 3.5 gives the desired result, because a C^*-algebra over the one-point space automatically has vanishing index maps. □

Now we turn to the description of separable purely infinite tight C^*-algebras over X with vanishing boundary maps.

Proposition 3.7. *A separable purely infinite tight C^*-algebra over X has vanishing exponential maps if and only if its underlying C^*-algebra has real rank zero.*

Proof. This is a special case of [26, Theorem 4.2] because X is a quasi-compact space; see also [26, Example 4.8]. □

The following definition has been suggested to us by Mikael Rørdam; it has been considered earlier by Lawrence G. Brown [9].

Definition 3.8. A C^*-algebra A has *intermediate cancellation* if the following holds: if p and q are projections in A which generate the same ideal and which give rise to the same element in $\mathrm{K}_0(A)$, then $p \sim q$ (that is, the projections p and q are Murray-von Neumann equivalent).

Lemma 3.9. *Let A be a separable purely infinite C^*-algebra with finite ideal lattice. Then*

$$\mathrm{K}_0(A) = \{[p] \mid p \text{ is a full projection in } A\}.$$

Moreover, if p and q are full projections in A with $[p] = [q]$ in $\mathrm{K}_0(A)$, then $p \sim q$.

Proof. It follows from [17, Theorem 4.16], that every non-zero projection in A is properly infinite. The lemma thus follows from [25, Proposition 4.1.4] because A contains a full projection by [26, Proposition 2.7]. □

Proposition 3.10. *A separable purely infinite tight C^*-algebra over X has vanishing index maps if and only if its underlying C^*-algebra has intermediate cancellation.*

Proof. By [17, Proposition 4.3], every ideal in A is purely infinite. The proposition follows from applying Lemma 3.9 to every ideal of A. □

Corollary 3.11. *Let $I \rightarrowtail A \twoheadrightarrow B$ be an extension of C^*-algebras. Assume that A is separable, purely infinite and has finite ideal lattice. Then A has intermediate cancellation if and only if the following hold:*

- *I has intermediate cancellation,*
- *B has intermediate cancellation,*
- *the index map $\mathrm{K}_1(B) \to \mathrm{K}_0(I)$ vanishes.*

Proof. Combine Propositions 3.5 and 3.10. □

Remark 3.12. The analogue of Corollary 3.11 with real rank zero replacing intermediate cancellation and the exponential map $\mathrm{K}_0(B) \to \mathrm{K}_1(I)$ replacing the index map $\mathrm{K}_1(B) \to \mathrm{K}_0(I)$ is well-known and holds in much greater generality; see [10,19].

4. Representations and cosheaves

In this section, we introduce two K-theoretical invariants for C^*-algebras over X that are well-adapted to algebras with vanishing boundary maps. First we define their target categories.

We associate the following two partially ordered sets to X:

- the set X itself, equipped with the specialization preorder;
- the collection $\mathbb{O}(X)$ of open subsets of X, partially ordered by inclusion.

The map $X^{\mathrm{op}} \to \mathbb{O}(X)$, $x \mapsto U_x$ is an embedding of partially ordered sets. Here X^{op} denotes the set X with reversed partial ordering. For the following definition, recall that every partially ordered set can be viewed as a category such that $\mathrm{Hom}(x, y)$ has one element, denoted by i_x^y, if $x \leq y$ and zero elements otherwise.

Definition 4.1. Let $\mathfrak{Ab}_{\mathrm{c}}^{\mathbb{Z}/2}$ be the category of countable $\mathbb{Z}/2$-graded Abelian groups. A *representation* of X is a covariant functor $X^{\mathrm{op}} \to \mathfrak{Ab}_{\mathrm{c}}^{\mathbb{Z}/2}$. A *precosheaf* on $\mathbb{O}(X)$ is a covariant functor $\mathbb{O}(X) \to \mathfrak{Ab}_{\mathrm{c}}^{\mathbb{Z}/2}$. A precosheaf $M\colon \mathbb{O}(X) \to \mathfrak{Ab}_{\mathrm{c}}^{\mathbb{Z}/2}$ is a *cosheaf* if, for every $U \in \mathbb{O}(X)$ and every open covering $\{U_j\}_{j\in J}$ of U, the sequence

$$\bigoplus_{j,k\in J} M(U_j \cap U_k) \xrightarrow{\left(M(i_{U_j\cap U_k}^{U_j})-M(i_{U_j\cap U_k}^{U_k})\right)} \bigoplus_{j\in J} M(U_j) \xrightarrow{\left(M(i_{U_j}^{U})\right)} M(U) \longrightarrow 0$$

is exact. Letting morphisms be natural transformations of functors, we define the category $\mathfrak{Rep}(X)$ of representations of X, the category $\mathfrak{PreCoSh}\big(\mathbb{O}(X)\big)$ of precosheaves over $\mathbb{O}(X)$ and the category $\mathfrak{CoSh}\big(\mathbb{O}(X)\big)$ of cosheaves over $\mathbb{O}(X)$.

The notion of cosheaf was introduced by Bredon [8]. Just like sheaves, cosheaves are determined by their behaviour on a basis. This is made precise in the following definition and lemma.

Definition 4.2. Let $\mathrm{Res}\colon \mathfrak{CoSh}\big(\mathbb{O}(X)\big) \to \mathfrak{Rep}(X)$ be the *restriction* functor given by

$$\mathrm{Res}(M)(x) = M(U_x), \quad \mathrm{Res}(M)(i_x^y) = M\left(i_{U_x}^{U_y}\right).$$

Let $\mathrm{Colim}\colon \mathfrak{Rep}(X) \to \mathfrak{CoSh}\big(\mathbb{O}(X)\big)$ be the functor that extends a representation M of X to a cosheaf on $\mathbb{O}(X)$ in a way such that $\big(\mathrm{Colim}(M)\big)(U)$ is given by the cokernel of the map

$$\bigoplus_{x,y\in U} \bigoplus_{z\in U_x\cap U_y} M(z) \xrightarrow{\left(M(i_z^x)-M(i_z^y)\right)} \bigoplus_{x\in U} M(x)$$

and $\mathrm{Colim}(M)(i_U^V)$ is induced by the obvious inclusions $\bigoplus_{x\in U} M(x) \subseteq \bigoplus_{x\in V} M(x)$ and $\bigoplus_{x,y\in U} \bigoplus_{z\in U_x\cap U_y} M(z) \subseteq \bigoplus_{x,y\in V} \bigoplus_{z\in U_x\cap U_y} M(z)$. We call $\mathrm{Colim}(M)$ the *associated cosheaf* of the representation M.

Lemma 4.3. *The functor* Colim *indeed takes values in cosheaves on* $\mathbb{O}(X)$. *The functors* Res *and* Colim *are mutually inverse equivalences of categories.*

Proof. The corresponding statements for sheaves are well-known: see, for instance, [29, Lemmas 009N and 009O]. Our dual version for cosheaves is a straight-forward analogue. Notice that $\{U_x \mid x \in X\}$ is a basis for the topology on X with the special property that every covering of an open set in it must contain this open set. Hence every precosheaf on this basis is already a cosheaf. □

Definition 4.4. The *integral incidence algebra* $\mathbb{Z}X$ of X is the free Abelian group generated by elements i_x^y for all pairs (x, y) with $y \leq x$ equipped with the unique bilinear multiplication such that $i_z^w i_x^y$ equals i_x^w if $y = z$ and otherwise is zero. By $\mathfrak{Mod}(\mathbb{Z}X)$, we denote the category of countable $\mathbb{Z}/2$-graded left-modules over $\mathbb{Z}X$.

The categories $\mathfrak{Rep}(X)$ and $\mathfrak{Mod}(\mathbb{Z}X)$ are canonically equivalent; we will identify them tacitly. For every point $x \in X$, we have a projective module $P^x := \mathbb{Z}X \cdot i_x^x$ in $\mathfrak{Mod}(\mathbb{Z}X)$ associated to the idempotent element i_x^x. Its entries are given by

$$(P^x)(y) = \begin{cases} \mathbb{Z}[0] \cdot i_x^y & \text{if } y \leq x \\ 0 & \text{otherwise} \end{cases}$$

and the map $(P^x)(i_y^z)$ for $y \geq z$ is an isomorphism if $x \geq y$ and zero otherwise.

Definition 4.5 ([8, §1])**.** A cosheaf on $\mathbb{O}(X)$ is called *flabby* if all its structure maps are injective.

The following is our key-lemma towards the universal coefficient theorem.

Lemma 4.6. *Let M be a representation of X such that the associated cosheaf* $\mathrm{Colim}(M)$ *on $\mathbb{O}(X)$ is flabby. Then M has a projective resolution of length* 1.

Proof. As before, we may choose a filtration

$$\emptyset = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_\ell = X,$$

of X by open subsets V_j such that $V_j \setminus V_{j-1} = \{x_j\}$ is a singleton for all $j = 1, \ldots, \ell$. For $V \in \mathbb{O}(X)$ we define a representation $P_V M$ of X by

$$(P_V M)(x) = \mathrm{Colim}(M)(V \cap U_x).$$

Since $\mathrm{Colim}(M)$ is flabby, we obtain a filtration

$$0 = P_{V_0} M \subseteq P_{V_1} M \subseteq \cdots \subseteq P_{V_\ell} M = M.$$

It follows from the so-called Horseshoe Lemma that an extension of modules with projective resolutions of length 1 also has a projective resolution of length 1. Hence it remains to show that the subquotients $Q^j := P_{V_j} M / P_{V_{j-1}} M$ in our filtration have resolutions of length 1.

Let us describe the modules Q^j explicitly. If $x_j \notin U_x$, then we have

$$(P_{V_j} M)(x) = \mathrm{Colim}(M)(V_j \cap U_x) = \mathrm{Colim}(M)(V_{j-1} \cap U_x) = (P_{V_{j-1}} M)(x),$$

so that $Q^j(x) = 0$. Now we assume $x_j \in U_x$. We fix $y \in X$ with $x \in U_y$ and abbreviate $C := \mathrm{Colim}(M)$. Since C is a cosheaf, we have a pushout diagram

$$\begin{tikzcd} C(V_{j-1} \cap U_x) \arrow[r] \arrow[d] & C(V_j \cap U_x) \arrow[d] \\ C(V_{j-1} \cap U_y) \arrow[r] & C(V_j \cap U_y). \end{tikzcd}$$

Since pushouts preserve cokernels, we obtain that the map $Q^j(x) \to Q^j(y)$ is an isomorphism. In conclusion, we may identify $Q^j \cong P^{x_j} \otimes G^j$, where G^j is some countable $\mathbb{Z}/2$-graded Abelian group. A projective resolution of length 1 for Q^j can thus be obtained by tensoring the projective module P^{x_j} with a resolution of G^j. □

Now we turn to the definition of our K-theoretical invariants.

Definition 4.7. We define a functor $\mathrm{XK}\colon \mathfrak{KK}(X) \to \mathfrak{Rep}(X) \cong \mathfrak{Mod}(\mathbb{Z}X)$ as follows: set

$$\mathrm{XK}(A)(x) = \mathrm{K}_*\big(A(U_x)\big)$$

and let $\mathrm{XK}(A)(i_x^y)$ be the map induced by the ideal inclusion $A(U_x) \hookrightarrow A(U_y)$.

Similarly, we define $\mathbb{O}\mathrm{K}\colon \mathfrak{KK}(X) \to \mathfrak{PreCoSh}\big(\mathbb{O}(X)\big)$ by $\mathbb{O}\mathrm{K}(A)(U) = \mathrm{K}_*\big(A(U)\big)$ and let the structure maps be the homomorphisms induced by the ideal inclusions.

We have an identity of functors $\mathrm{Res} \circ \mathbb{O}\mathrm{K} = \mathrm{XK}$.

Lemma 4.8. *A C^*-algebra A over X has vanishing boundary maps if and only if $\mathbb{O}\mathrm{K}(A)$ is a flabby cosheaf.*

Proof. Suppose that A has vanishing boundary maps. By an inductive argument as in [8, Proposition 1.3], it suffices to verify the cosheaf condition for all coverings consisting of two open sets. This case reduces to the Mayer-Vietoris sequence. The six-term exact sequence (3.1) shows that $\mathbb{O}\mathrm{K}(A)$ is flabby.

Conversely, if $\mathbb{O}\mathrm{K}(A)$ is a flabby cosheaf, the six-term exact sequence (3.1) shows that A has vanishing boundary maps. □

It follows from Lemma 4.3 that, on the full subcategory of C^*-algebras over X with vanishing boundary maps, we have a natural isomorphism $\mathrm{Colim} \circ \mathrm{XK} \cong \mathbb{O}\mathrm{K}$.

Remark 4.9. Instead of working with K-theory groups of distinguished ideals, we could define similar invariants in terms of K-theory groups of distinguished quotients. This would not make a difference for the universal coefficient theorem in the next section. However, our choice of definition interacts more nicely with the invariant $\mathrm{FK}_{\mathcal{R}}$ that we will use in §7.

For reference in future work, we record the following lemma.

Lemma 4.10. *Let A be a C^*-algebra over X with vanishing boundary maps such that the Abelian group $\mathrm{K}_*\big(A(Y)\big)$ is free for every locally closed subset $Y \subseteq X$. Then $\mathrm{XK}(A)$ is projective.*

Proof. By Lemma 4.8, the cosheaf $\mathbb{O}\mathrm{K}(A)$ is flabby. We follow the proof of Lemma 4.6. Our freeness assumption implies that the Abelian groups G coming up in the proof are free: the six-term exact sequence shows that

$$G^j = \mathrm{K}_*\big(A(U_x)\big)/\mathrm{K}_*\big(A(U_x \setminus \{x\})\big) \cong \mathrm{K}_*\big(A(\{x\})\big).$$

Hence $\mathrm{XK}(A)$ is an iterated extension of projective modules and thus itself projective. □

5. A universal coefficient theorem

In this section, we establish a universal coefficient theorem for C^*-algebras over X with vanishing boundary maps. We discuss the splitting of the resulting short exact sequence and the realization of objects in the bootstrap class as commutative algebras.

We describe how the invariant XK fits into the framework for homological algebra in triangulated categories developed by Meyer and Nest in [20]. The set-up is given by the triangulated category $\mathfrak{KK}(X)$ and the stable homological ideal $\mathfrak{I} := \ker(\mathrm{XK})$, the kernel of XK on morphisms. Using the adjointness relation

$$\mathrm{KK}_*(X; i_x\mathbb{C}, A) \cong \mathrm{KK}_*\big(\mathbb{C}, A(U_x)\big) \cong \mathrm{K}_*\big(A(U_x)\big) \tag{5.1}$$

from [22, Proposition 3.13] and machinery from [20], one can easily show the following (a slightly more detailed account for the particular example at hand is given in [5, §4]):

- the ideal $\mathfrak{I}$ has enough projective objects,
- the functor XK is the *universal* $\mathfrak{I}$-exact stable homological functor,
- A belongs to $\mathcal{B}(X)$ if and only if $\mathrm{KK}_*(X; A, B) = 0$ for all $\mathfrak{I}$-contractible B.

These facts allow us to apply the abstract universal coefficient theorem [20, Theorem 66] to our concrete setting. We abbreviate $\mathcal{A} := \mathfrak{Mod}(\mathbb{Z}X)$.

Theorem 5.2. *Let A and B be separable C^*-algebras over X. Assume that A belongs to $\mathcal{B}(X)$ and has vanishing boundary maps. Then there is a natural short exact sequence of $\mathbb{Z}/2$-graded Abelian groups*

$$\mathrm{Ext}^1_{\mathcal{A}}\big(\mathrm{XK}(A)[1], \mathrm{XK}(B)\big) \rightarrowtail \mathrm{KK}_*(X; A, B) \twoheadrightarrow \mathrm{Hom}_{\mathcal{A}}\big(\mathrm{XK}(A), \mathrm{XK}(B)\big). \tag{5.3}$$

Proof. By [20, Theorem 66], we only have to check that $\mathrm{XK}(A)$ has a projective resolution of length 1. This follows from Lemmas 4.8 and 4.6. □

Corollary 5.4. *Let A and B be separable C^*-algebras over X. Assume that A and B belong to $\mathcal{B}(X)$ and that A has vanishing boundary maps. Then every isomorphism $\mathrm{XK}(A) \cong \mathrm{XK}(B)$ in $\mathcal{A}$ can be lifted to a $\mathrm{KK}(X)$-equivalence.*

Proof. Since A has vanishing boundary maps, the module $\mathrm{XK}(A) \cong \mathrm{XK}(B)$ has a projective resolution of length 1 by Lemmas 4.8 and 4.6. Hence the result follows from the universal coefficient theorem [20, Theorem 66] by a standard argument; see, for instance, [6, Proposition 23.10.1] or [21, Corollary 4.6]. □

Proposition 5.5. *Let A and B be separable C^*-algebras over X. Assume that A belongs to $\mathcal{B}(X)$ and that A and B have vanishing boundary maps. Then the short exact sequence* (5.3) *splits* (*unnaturally*).

Proof. For this result, it is crucial that the ring $\mathbb{Z}X$ itself is ungraded. We can thus imitate the proof from [6, §23.11]: we have direct sum decompositions $\mathrm{XK}(A) \cong M_0 \oplus M_1[1]$ and $\mathrm{XK}(B) \cong N_0 \oplus N_1[1]$ where M_i and N_i are ungraded $\mathbb{Z}X$-modules of projective dimension at most 1. By a simple argument based on the universality of the functor XK (compare [21, Theorem 4.8]), we can find objects A_i and B_i in $\mathcal{B}(X)$ such that $\mathrm{XK}(A_i) \cong M_i[0]$ and $\mathrm{XK}(B_i) \cong N_i[0]$ for $i \in \{0, 1\}$. By Corollary 5.4, there is a (non-canonical) $\mathrm{KK}(X)$-equivalence $A \cong A_1 \oplus \Sigma A_2$. Using the universal coefficient theorem, we can find an element $f \in \mathrm{KK}_0(X; B, B_1 \oplus \Sigma B_2)$ inducing an isomorphism $\mathrm{XK}(B) \cong \mathrm{XK}(B_1 \oplus \Sigma B_2)$. By the definition of XK, the element f induces isomorphisms $\mathrm{KK}(X; i_x\mathbb{C}, B) \cong \mathrm{KK}(X; i_x\mathbb{C}, B_1 \oplus \Sigma B_2)$ for all $x \in X$. The usual bootstrap argument shows that f induces isomorphisms $\mathrm{KK}(X; D, B) \cong \mathrm{KK}(X; D, B_1 \oplus \Sigma B_2)$ for every object D in $\mathcal{B}(X)$. We may thus replace A by $A_1 \oplus \Sigma A_2$ and B by $B_1 \oplus \Sigma B_2$. Hence the sequence (5.3) decomposes as a direct sum of four sequences in which, for degree reasons, either the left-hand or the right-hand term vanishes, making the construction of a splitting trivial. □

Proposition 5.6. *Let A be a separable C^*-algebra over X with vanishing boundary maps. Then there is a commutative C^*-algebra C over X such that $\mathrm{XK}(A) \cong \mathrm{XK}(C)$. The spectrum of C may be chosen to be at most three-dimensional. If $\mathrm{XK}(A)$ is finitely generated, the spectrum of C may be chosen to be a finite complex of dimension at most three.*

Proof. It is straight-forward to generalize the argument from [6, Corollary 23.10.3]. Using that modules split into even and odd part, a suspension argument reduces to the case that $\mathrm{XK}(A)$ vanishes in degree zero. Choose a projective resolution

$$0 \to P_1 \xrightarrow{f} P_0 \to \mathrm{XK}(A) \to 0$$

such that $P_i = \bigoplus_{x\in X} \bigoplus_{\mathbb{N}} (P^x \oplus P^x[1])$. Setting $D_i = \bigoplus_{x\in X} \bigoplus_{\mathbb{N}} \big(i_x\mathbb{C} \otimes C(S^1)\big)$, we have $P_i \cong \mathrm{XK}(D_i)$. Then there is a *-homomorphism $\varphi\colon D_1 \to D_0$ over X inducing the map f. The mapping cone of φ has the desired properties. In the finitely generated case, it clearly suffices to use finite direct sums instead of countable ones. □

Corollary 5.7. *Let A be a separable C^*-algebra over X with vanishing boundary maps. Then A belongs to the bootstrap class $\mathcal{B}(X)$ if and only if A is $\mathrm{KK}(X)$-equivalent to a commutative C^*-algebra over X.*

Proof. If A is a commutative C^*-algebra over X then it is nuclear and the subquotient $A(\{x\})$ belongs to the bootstrap class $\mathcal{B}$ for every $x \in X$. Hence A belongs to $\mathcal{B}(X)$ by [22, Corollary 4.13]. Since $\mathcal{B}(X)$ is closed under $\mathrm{KK}(X)$-equivalence, one implication follows. The converse implication follows from Proposition 5.6 and Corollary 5.4. ☐

Remark 5.8. The stable homological functor $\mathbb{O}\mathrm{K}$ does not fit into this framework as nicely: if the space X is sufficiently complicated then $\mathbb{O}\mathrm{K}$ is not universal for its kernel on morphisms because it has "hidden symmetries." More precisely, there are natural transformations among the K-theoretical functors comprised by the invariant $\mathbb{O}\mathrm{K}$, the action of which is not part of the definition of $\mathbb{O}\mathrm{K}$ (compare [21, §2.1]).

6. Classification of certain Kirchberg X-algebras

In this section, we use our universal coefficient theorem to obtain classification results for Kirchberg X-algebras with vanishing boundary maps.

Definition 6.1. A C^*-algebra over X is a *Kirchberg X-algebra* if it is tight, nuclear, purely infinite and separable.

Theorem 6.2. *Let A and B be stable real-rank-zero Kirchberg X-algebras with intermediate cancellation and simple subquotients in the bootstrap class $\mathcal{B}$. Then every isomorphism $\mathrm{XK}(A) \cong \mathrm{XK}(B)$ can be lifted to a *-isomorphism over X. Consequently, every isomorphism $\mathbb{O}\mathrm{K}(A) \cong \mathbb{O}\mathrm{K}(B)$ can be lifted to a *-isomorphism over X.*

Proof. By Propositions 3.7 and 3.10, the algebras A and B have vanishing boundary maps. Hence the first claim follows from Corollary 5.4 together with Kirchberg's classification theorem [16]. Recall that a nuclear C^*-algebra belongs to $\mathcal{B}(X)$ if and only if the fibre $A(\{x\})$ belongs to $\mathcal{B}$ for every $x \in X$ by [22, Corollary 4.13]. Notice also that stable nuclear purely infinite C^*-algebras with real rank zero are $\mathcal{O}_\infty$-absorbing by [18, Corollary 9.4]. The second claim follows from the equivalence in Lemma 4.3. ☐

Next, we establish a range result for the invariant $\mathbb{O}\mathrm{K}$ on stable real-rank-zero Kirchberg X-algebra with intermediate cancellation. For this, we need to assume that X is a unique path space.

Theorem 6.3. *Assume that X is a unique path space. Let M be a flabby cosheaf on $\mathbb{O}(X)$. Then there is a stable real-rank-zero Kirchberg X-algebra with intermediate cancellation and simple subquotients in the bootstrap class $\mathcal{B}$ such that $\mathbb{O}\mathrm{K}(A) \cong M$.*

Proof. Since M is a flabby cosheaf, its restriction $\mathrm{Res}(M) \in\in \mathfrak{Rep}(X)$ has a projective resolution of length 1 by Lemma 4.6. A simple argument as in [21, Theorem 4.8] shows that there is a separable C^*-algebra A over X in the bootstrap class $\mathcal{B}(X)$ with $\mathrm{XK}(A) \cong \mathrm{Res}(M)$. By [22, Corollary 5.5], we may assume that A is a stable Kirchberg X-algebra with simple subquotients in $\mathcal{B}$.

Since X is a unique path space, the set $U_x \setminus \{x\}$ is the disjoint union of the sets U_y, where y is a closed point in $U_x \setminus \{x\}$. Hence the map $\mathrm{K}_*\bigl(A(U_x \setminus \{x\})\bigr) \to \mathrm{K}_*\bigl(A(U_x)\bigr)$ identifies with the map $M(U_x \setminus \{x\}) \to M(U_x)$ because K-theory preserves direct sums and cosheaves take disjoint unions to direct sums. Since M is flabby by assumption, Corollary 3.6 therefore shows that A has vanishing boundary maps. Thus A has real rank zero and intermediate cancellation by Propositions 3.7 and 3.10 and we have $\mathbb{O}\mathrm{K}(A) \cong \mathrm{Colim}\bigl(\mathrm{XK}(A)\bigr) \cong \mathrm{Colim}\bigl(\mathrm{Res}(M)\bigr) \cong M$. ☐

Corollary 6.4. *Assume that X is a unique path space. The functors* $\mathbb{O}\mathrm{K}$ *and* XK *implement bijections of isomorphism classes of*

- *stable real-rank-zero Kirchberg X-algebras with intermediate cancellation and simple subquotients in the bootstrap class $\mathcal{B}$,*
- *flabby cosheaves on $\mathbb{O}(X)$,*
- *representations of X whose associated cosheaf is flabby.*

Proof. Denote the three sets above by (Kirchberg), (Cosheaves) and (Representations), respectively. We have maps induced by functors as indicated in the following commutative diagram.

$$\begin{tikzcd}
 & \text{(Kirchberg)} \arrow[dl, bend right, "\mathbb{O}\mathrm{K}"'] \arrow[dr, bend left, "\mathrm{XK}"] & \\
\text{(Cosheaves)} \arrow[rr, bend right, "\mathrm{Res}"'] & & \text{(Representations)} \arrow[ll, bend right, "\mathrm{Colim}"']
\end{tikzcd}$$

We observed in §4 that the functors Res and Colim induce mutually inverse bijections. By Theorem 6.2, the functor XK induces an injective map. By Theorem 6.3, the functor $\mathbb{O}\mathrm{K}$ induces a surjective map. Hence all four maps are bijective. ☐

Now we enhance our invariant in order to obtain a classification result in the unital case.

Definition 6.5. A *pointed cosheaf* on $\mathbb{O}(X)$ is a cosheaf M on $\mathbb{O}(X)$ together with a distinguished element $m \in M(X)_0$. A morphism of pointed cosheaves is a morphism of cosheaves preserving the distinguished element. The category of pointed cosheaves on $\mathbb{O}(X)$ is denoted by $\mathfrak{CoSh}\big(\mathbb{O}(X)\big)^+$.

Definition 6.6. Let $\mathfrak{KK}(X)^+$ denote the full subcategory of $\mathfrak{KK}(X)$ consisting of all unital separable C^*-algebras over X. We define a functor $\mathbb{O}\mathrm{K}^+ \colon \mathfrak{KK}(X)^+ \to \mathfrak{CoSh}\big(\mathbb{O}(X)\big)^+$ by

$$\mathbb{O}\mathrm{K}^+(A) = \big(\mathbb{O}\mathrm{K}(A), [1_A]\big).$$

Corollary 6.7. *Let A and B be unital real-rank-zero Kirchberg X-algebras with intermediate cancellation and simple subquotients in the bootstrap class $\mathcal{B}$. Then every isomorphism $\mathbb{O}\mathrm{K}^+(A) \cong \mathbb{O}\mathrm{K}^+(B)$ can be lifted to a *-isomorphism over X.*

Proof. This is a consequence of the strong stable classification result in Theorem 6.2 using the so-called meta theorem [14, Theorem 3.3]. ☐

7. Cosheaves arising as invariants of graph C^*-algebras

In this section, we provide range results for the invariants $\mathbb{O}\mathrm{K}$ and $\mathbb{O}\mathrm{K}^+$ on purely infinite tight graph C^*-algebra over X with intermediate cancellation. For definitions and general facts concerning graph C^*-algebras we refer to [23]. The Cuntz–Krieger algebras introduced in [12, 13] are in particular unital graph C^*-algebras; when using the word Cuntz–Krieger algebra we implicitly assume that the underlying square matrix satisfies Cuntz's condition (II), which ensures that the algebra is purely infinite.

Definition 7.1. A *tight graph C^*-algebra over X* is a graph C^*-algebra $C^*(E)$ equipped with a homeomorphism $\mathrm{Prim}\big(C^*(E)\big) \to X$. A *tight Cuntz–Krieger algebra over X* is defined analogously.

We point out that a purely infinite tight graph C^*-algebra over X is in particular a real-rank-zero Kirchberg X-algebras with simple subquotients in the bootstrap class $\mathcal{B}$ (see [23, Remark 4.3] and [15, §2]). Hence the classification results in the previous section apply to purely infinite tight graph C^*-algebras over X with intermediate cancellation. We obtain the following corollary.

Corollary 7.2. *Let A and B be purely infinite tight graph C^*-algebras over X with intermediate cancellation. If $\mathbb{O}\mathrm{K}(A) \cong \mathbb{O}\mathrm{K}(B)$, then A is stably isomorphic to B. If A and B are unital and $\mathbb{O}\mathrm{K}^+(A) \cong \mathbb{O}\mathrm{K}^+(B)$, then A is isomorphic to B.*

It is now natural to ask which (pointed) cosheaves arise as the invariant of a (unital) purely infinite tight graph C^*-algebra over X with intermediate cancellation.

Definition 7.3. A flabby cosheaf M on $\mathbb{O}(X)$ is said to have *free quotients in odd degree* if the quotient $M(V)_1/M(U)_1$ is free for all open subsets $U \subseteq V \subseteq X$. We say that M has *finite ordered ranks* if, for all $U \in \mathbb{O}(X)$,

$$\operatorname{rank} M(U)_1 \leq \operatorname{rank} M(U)_0 < \infty.$$

Similarly, we say that M has *finite equal ranks* if $\operatorname{rank} M(U)_1 = \operatorname{rank} M(U)_0 < \infty$ for all $U \in \mathbb{O}(X)$. A pointed cosheaf is called *flabby* if the underlying cosheaf is flabby. A flabby pointed cosheaf has one of the three properties above if this is the case for the underlying cosheaf.

We will use the invariant $\mathrm{FK}_{\mathcal{R}}$ for C^*-algebras over X from [2].

Definition 7.4 ([2, Definition 6.1])**.** An $\mathcal{R}$-module N is a collection of Abelian groups $N(\{x\})_1$, $N(U_x)_0$ and $N(U_x \setminus \{x\})_0$ for $x \in X$ together with group homomorphisms $\delta_{\{x\}}^{U_x\setminus\{x\}}\colon N(\{x\})_1 \to N(U_x \setminus \{x\})_0$ and $i_{U_x\setminus\{x\}}^{U_x}\colon N(U_x \setminus \{x\})_0 \to N(U_x)_0$ for $x \in X$ and $i_{U_y}^{U_x\setminus\{x\}}\colon N(U_y)_0 \to N(U_x \setminus \{x\})_0$ for all pairs (x, y) with $y \to x$ such that certain relations are fulfilled. A homomorphism of $\mathcal{R}$-modules is a collection of group homomorphisms making all squares commute.

There is a notion of *exactness* for $\mathcal{R}$-modules (see [2, Definition 6.5]) and our notation suggests an obvious K-theoretical functor $\mathrm{FK}_{\mathcal{R}}$ from $\mathfrak{KK}(X)$ to exact $\mathcal{R}$-modules (see [2, Definition 6.4 and Corollary 6.9]). Notice that, for $U_x \subseteq U_y$, we can obtain the map $\mathrm{K}_0\big(A(U_x)\big) \to \mathrm{K}_0\big(A(U_y)\big)$ by composing maps that are part of the invariant $\mathrm{FK}_{\mathcal{R}}(A)$.

Theorem 7.5. *A flabby cosheaf on $\mathbb{O}(X)$ is isomorphic to $\mathbb{O}\mathrm{K}\big(C^*(E)\big)$ for some purely infinite tight graph C^*-algebra $C^*(E)$ over X with intermediate cancellation if and only if it has free quotients in odd degree.*

Proof. It is well-known that graph C^*-algebras have free K_1-groups. Since (gauge-invariant) ideals in graph C^*-algebras are themselves graph C^*-algebras by [28], it follows that $\mathbb{O}\mathrm{K}\big(C^*(E)\big)$ has free quotients in odd degree if $C^*(E)$ is a purely infinite tight graph C^*-algebra over X.

Conversely, let M be a flabby cosheaf on $\mathbb{O}(X)$ that has free quotients in odd degree. We associate to M an $\mathcal{R}$-module N in the following way: for $x \in X$, set $N(U_x)_0 = M(U_x)_0$, $N(U_x \setminus \{x\})_0 = M(U_x \setminus \{x\})_0$ and let $N(\{x\})_1$ be the quotient of $M(U_x)_1$ by $M(U_x \setminus \{x\})_1$. The maps $i_{U_x\setminus\{x\}}^{U_x}$ and $i_{U_y}^{U_x\setminus\{x\}}$ for N are defined to be the even parts of the identically denoted maps for M. The homomorphisms $\delta_{\{x\}}^{U_x\setminus\{x\}}$ are defined to be the zero homomorphisms.

To check that this really defines an $\mathcal{R}$-module, one has to verify the relations (6.2) and (6.3) in [2]. This is straight-forward: the relation (6.2) is fulfilled because we have defined the maps $\delta_{\{x\}}^{U_x\setminus\{x\}}$ as zero maps; the relation (6.3) follows from the fact that the composition $M(U) \xrightarrow{i_U^V} M(V) \xrightarrow{i_V^W} M(W)$ is equal to $M(U) \xrightarrow{i_U^W} M(W)$ for all open subsets $U \subseteq V \subseteq W \subseteq X$. We observe that the $\mathcal{R}$-module N is exact:

the exactness of the sequence (6.7) in [2] follows from the fact that M is a cosheaf; the sequence (6.6) in [2] is exact because M is flabby.

Since M has free quotients in odd degree, [2, Theorem 8.2] implies that there is a purely infinite tight graph C^*-algebra $C^*(E)$ over X such that $\mathrm{FK}_{\mathcal{R}}\big(C^*(E)\big) \cong N$. Since $C^*(E)$ has real rank zero, it has vanishing exponential maps, so that the K_0-groups of its ideals form an (ungraded) cosheaf on $\mathbb{O}(X)$. This cosheaf coincides with the even part of M on the basis of minimal open neighbourhoods of points. Since cosheaves are determined by their restriction to a basis, the (ungraded) cosheaves M_0 and $\mathbb{O}\mathrm{K}\big(C^*(E)\big)_0$ are isomorphic. Since M is flabby this shows that $C^*(E)$ has vanishing index maps and therefore intermediate cancellation.

Exploiting freeness of the K_1-groups and vanishing of boundary maps, we obtain isomorphisms

$$\mathrm{K}_1\big(C^*(E)(U)\big) \cong \bigoplus_{x\in U} \mathrm{K}_1\big(C^*(E)(\{x\})\big)$$

for all open subsets $U \subseteq X$ such that, under this identification, the homomorphisms induced by the ideal inclusions correspond to the obvious subgroup inclusions. Analogously, we have isomorphisms $M(U)_1 \cong \bigoplus_{x\in U} N(\{x\})_1$ for all open subsets $U \subseteq X$ because M is flabby and has free quotients in odd degree. Hence $\mathbb{O}\mathrm{K}\big(C^*(E)\big)_1 \cong M_1$. It follows that $\mathbb{O}\mathrm{K}\big(C^*(E)\big) \cong M$ as desired. □

For the proof of the next result, we need to recall that there is a notion of *pointed* $\mathcal{R}$-module (see [2, Definition 9.1]) and a functor $\mathrm{FK}^+_{\mathcal{R}}$ from $\mathfrak{KK}(X)^+$ to pointed $\mathcal{R}$-modules.

Theorem 7.6. *A flabby pointed cosheaf on $\mathbb{O}(X)$ is isomorphic to $\mathbb{O}\mathrm{K}^+\big(C^*(E)\big)$ for some unital purely infinite tight graph C^*-algebra $C^*(E)$ over X with intermediate cancellation if and only if it has free quotients in odd degree and finite ordered ranks.*

Proof. Again, the well-known formulas for the K-theory of graph C^*-algebras show that $\mathbb{O}\mathrm{K}\big(C^*(E)\big)$ has free quotients in odd degree and finite ordered ranks if $C^*(E)$ is a unital purely infinite tight graph C^*-algebra over X. Conversely, to a given flabby pointed cosheaf (M, m) we associate an exact pointed $\mathcal{R}$-module (N, n) as in the previous proof. Our assumptions on M then guarantee that we can apply [2, Theorem 9.11] to obtain a unital purely infinite tight graph C^*-algebra $C^*(E)$ over X such that there is an isomorphism of pointed $\mathcal{R}$-modules $\mathrm{FK}^+_{\mathcal{R}}\big(C^*(E)\big) \cong (N, n)$. An argument as in the previous proof shows that $\mathbb{O}\mathrm{K}^+\big(C^*(E)\big) \cong (M, m)$ and that $C^*(E)$ has intermediate cancellation. □

Theorem 7.7. *A flabby pointed cosheaf on $\mathbb{O}(X)$ is isomorphic to $\mathbb{O}\mathrm{K}^+(\mathcal{O}_A)$ for some tight Cuntz–Krieger algebra $\mathcal{O}_A$ over X with intermediate cancellation if and only if it has free quotients in odd degree and finite equal ranks.*

Proof. The K-theory formulas for graph C^*-algebras imply that the cosheaf $\mathbb{O}\mathrm{K}(\mathcal{O}_A)$ has finite equal degrees if $\mathcal{O}_A$ is a tight Cuntz–Krieger algebra over X. Conversely, $\mathbb{O}\mathrm{K}(\mathcal{O}_A)$ having finite equal ranks implies that $\mathrm{FK}_{\mathcal{R}}(A)$ meets the additional conditions in [2, Theorem 8.2] that guarantee that the graph E in the previous proof can be chosen finite (it has no sinks or sources by construction). □

8. Extensions of Cuntz–Krieger algebras

In this section, we establish a permanence property of Cuntz–Krieger algebras with intermediate cancellation with respect to extensions.

Definition 8.1 ([1, Definition 1.1])**.** A C^*-algebra A over X *looks like a Cuntz–Krieger algebra* if A is a unital real-rank-zero Kirchberg X-algebra with simple subquotients in the bootstrap class $\mathcal{B}$ such that, for all $x \in X$, the group $\mathrm{K}_1\big(A(\{x\})\big)$ is free and $\operatorname{rank} \mathrm{K}_0\big(A(\{x\})\big) = \operatorname{rank} \mathrm{K}_1\big(A(\{x\})\big) < \infty$.

A C^*-algebra A over X that satisfies these conditions but is stable rather than unital is said to *look like a stabilized Cuntz–Krieger algebra*.

The following result generalizes the observation in [1, Corollary 2.4], which is concerned with Cuntz–Krieger algebras with trivial K-theory.

Corollary 8.2. *Let A be a C^*-algebra over X that looks like a Cuntz–Krieger algebra and has intermediate cancellation. Then A is *-isomorphic over X to a tight Cuntz–Krieger algebra over X with intermediate cancellation.*

Proof. Let B be a C^*-algebra over X with intermediate cancellation that looks like a Cuntz–Krieger algebra. Repeated use of the six-term exact sequence shows that $\mathbb{O}\mathrm{K}(B)$ has free quotients in odd degree and finite equal ranks. By Theorem 7.7, there is a tight Cuntz–Krieger algebra $\mathcal{O}_A$ over X with intermediate cancellation such that $\mathbb{O}\mathrm{K}^+(B) \cong \mathbb{O}\mathrm{K}^+(\mathcal{O}_A)$. By Corollary 6.7, we have $B \cong \mathcal{O}_A$. ◻

Corollary 8.3. *Let A be a C^*-algebra over X that looks like a stabilized Cuntz–Krieger algebra and has intermediate cancellation. Then A is stably isomorphic over X to a tight Cuntz–Krieger algebra over X with intermediate cancellation.*

Proof. Let B be a C^*-algebra over X with intermediate cancellation that looks like a stabilized Cuntz–Krieger algebra. As in the previous proof, we see that $\mathbb{O}\mathrm{K}(B)$ has free quotients in odd degree and finite equal ranks. We turn the cosheaf $\mathbb{O}\mathrm{K}(B)$ into a pointed cosheaf by choosing an arbitrary element in $\mathrm{K}_0(B)$. By Theorem 7.7, there is a tight Cuntz–Krieger algebra $\mathcal{O}_A$ over X with intermediate cancellation such that $\mathbb{O}\mathrm{K}(B) \cong \mathbb{O}\mathrm{K}(\mathcal{O}_A)$. By Theorem 6.2, the algebras B and $\mathcal{O}_A$ are stably isomorphic over X. ◻

Theorem 8.4. *Let $I \rightarrowtail A \twoheadrightarrow B$ be an extension of C^*-algebras. Assume that A is unital. Then A is a Cuntz–Krieger algebra with intermediate cancellation if and only if*

- *the ideal I is stably isomorphic to a Cuntz–Krieger algebra with intermediate cancellation,*
- *the quotient B is a Cuntz–Krieger algebra with intermediate cancellation,*
- *the boundary map $\mathrm{K}_*(B) \to \mathrm{K}_{*+1}(I)$ vanishes.*

A similar assertion holds for extensions of unital purely infinite graph C^*-algebras with intermediate cancellation.

Proof. The crucial point is that the property of *looking like* a Cuntz–Krieger algebra behaves well with extensions (see Remark 3.12). So does intermediate cancellation when considered for separable purely infinite C^*-algebras by Corollary 3.11. We have that $A \in\in \mathfrak{KK}\big(\mathrm{Prim}(A)\big)$ looks like a Cuntz–Krieger algebra and has intermediate cancellation if and only if

- the stabilization $I \otimes \mathbb{K} \in\in \mathfrak{C}^*\mathfrak{alg}\big(\mathrm{Prim}(I)\big)$ of the ideal I looks like a stabilized Cuntz–Krieger algebra and has intermediate cancellation,
- the quotient $B \in\in \mathfrak{C}^*\mathfrak{alg}\big(\mathrm{Prim}(B)\big)$ looks like a Cuntz–Krieger algebra and has intermediate cancellation,
- the boundary map $\mathrm{K}_*(B) \to \mathrm{K}_{*+1}(I)$ vanishes.

Hence the result follows from Corollary 8.2 applied to A and B and from Corollary 8.3 applied to I. The assertion for unital graph C^*-algebras follows similarly from Theorem 7.6 and Corollary 6.7. ◻

As similar argument based on Theorems 7.5 and 6.2 leads to the following permanence result for stabilized purely infinite graph C^*-algebras.

Theorem 8.5. *Let $I \rightarrowtail A \twoheadrightarrow B$ be an extension of C^*-algebras. Assume that A has finite ideal lattice. Then A is stably isomorphic to a purely infinite graph C^*-algebra with intermediate cancellation if and only if*

- *the ideal I is stably isomorphic to a purely infinite graph C^*-algebra with intermediate cancellation,*
- *the quotient B is stably isomorphic to a purely infinite graph C^*-algebra with intermediate cancellation,*
- *the boundary map $\mathrm{K}_*(B) \to \mathrm{K}_{*+1}(I)$ vanishes.*

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ONE-PARAMETER CONTINUOUS FIELDS OF KIRCHBERG ALGEBRAS WITH RATIONAL K-THEORY

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Abstract. We show that separable continuous fields over the unit interval whose fibers are stable Kirchberg algebras that satisfy the universal coefficient theorem in KK-theory (UCT) and have rational K-theory groups are classified up to isomorphism by filtrated K-theory.

1. Introduction

The purpose of this paper is to investigate the classification problem for continuous fields of Kirchberg algebras over the unit interval by K-theory invariants. It is natural to associate to a $C[0,1]$-algebra A the family of all exact triangles of $\mathbb{Z}/2$-graded K-theory groups

$$\begin{tikzcd}
\mathrm{K}_*(A(U)) \arrow[rr] & & \mathrm{K}_*(A(Y)) \arrow[dl] \\
& \mathrm{K}_*(A(Y \setminus U)) \arrow[ul] &
\end{tikzcd}$$

where Y is a subinterval of $[0,1]$ and U is a relatively open subinterval of Y. The family of these exact triangles are assembled into an invariant $\mathrm{FK}(A)$ called the filtrated K-theory of A, see Definition 3.4.

In this article we exhibit several classes of separable continuous fields over the unit interval whose fibers are stable UCT Kirchberg algebras and for which filtrated K-theory is a complete invariant. In particular, we show that

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this is the case for fields which are stable under tensoring with the universal UHF-algebra. A C^*-algebra D has rational K-theory if $\mathrm{K}_*(D) \cong \mathrm{K}_*(D) \otimes \mathbb{Q}$.

Theorem 1.1. *Let A and B be separable continuous fields over the unit interval whose fibers are stable Kirchberg algebras that satisfy the UCT and have rational* K*-theory groups. Then any isomorphism of filtrated* K*-theory* $\mathrm{FK}(A) \cong \mathrm{FK}(B)$ *lifts to a $C[0,1]$-linear *-isomorphism $A \cong B$.*

The continuous fields classified by this theorem include fields that are nowhere locally trivial. It is for this reason that one needs to include infinitely many subintervals of $[0,1]$ in any complete invariant. However it suffices to consider intervals whose endpoints belong to a countable dense subset of $[0,1]$. The result does not extend to continuous fields of Kirchberg algebras if torsion is allowed, as we will explain shortly.

The main idea of our approach is to combine the following three crucial ingredients:

- Eberhard Kirchberg's isomorphism theorem for non-simple nuclear $\mathcal{O}_\infty$-absorbing C^*-algebras [15],
- the results from [10] which relate E-theory over a second countable space X with the corresponding version of KK-theory and with E-theory groups over finite approximating spaces of X,
- the universal coefficient theorem for accordion spaces from [2] (generalizing results from [4,18,20,23]) including a description of projective and injective objects in the target category of filtrated K-theory.

The relevance of accordion spaces in this framework is due to the fact that sufficiently many non-Hausdorff finite approximating spaces of the unit interval are accordion spaces.

A major difficulty in any attempt to use the result of [15] is the computation of the group $\mathrm{KK}(X; A, B)$ or at least a quotient of this group which allows to detect $\mathrm{KK}(X)$-equivalences. In [10], the second named author and Ralf Meyer proved a universal *multi*-coefficient theorem (abbreviated UMCT) for separable $\mathrm{C}(X)$-algebras over a totally disconnected compact metrizable space X. As a consequence, by Kirchberg's isomorphism theorem [15], separable stable continuous fields over such spaces whose fibres are UCT Kirchberg algebras are classified by an invariant the authors call *filtrated* K*-theory with coefficients*. This result is also implicit in [11].

The filtrated K-theory with coefficients of [10] comprises the K-theory with coefficients (the Λ-modules defined in [9], also called *total* K*-theory*) of all distinguished subquotients of the given field, along with the action of all natural maps between these groups. It is demonstrated in [10], generalising a result from [7], that coefficients are necessary for such a classification result over any infinite metrizable compact space. This means that filtrated K-theory (without coefficients) can only be a classifying invariant on subclasses of fields with special K-theoretical properties and this explains the need for additional assumptions in our results. For comparison let us recall that the classification result of [8] is restricted to fields whose fibers have torsion-free K_0-groups and vanishing K_1-groups or vice versa.

The construction of an effective filtrated K-theory with coefficients for C^*-algebras over the unit interval remains an open problem. In the final Section 5 we describe some of the technical difficulties that are encountered in potential constructions of such an invariant.

2. Preliminaries

In this section we summarize definitions and results by various authors which we shall use later. We make the convention $\mathbb{N} = \{1, 2, 3, \ldots\}$.

2.1. **C^*-algebras over topological spaces.** Let X be any topological space. Recall from [19]:

Definition 2.1. A *C^*-algebra over X* is a C^*-algebra A equipped with a continuous map $\mathrm{Prim}(A) \to X$.

Definition 2.2. Let A be a C^*-algebra over X. Let $U \subseteq X$ be an open subset. Taking the preimage under the map $\mathrm{Prim}(A) \to X$, we may naturally associate the *distinguished ideal* $A(U) \subseteq A$ to U. A morphism of C^*-algebras over X is a *-homomorphism preserving all distinguished ideals.

A subset $Y \subset X$ is called *locally closed* if it can be written as a difference $U \setminus V$ of two open subsets $V \subseteq U \subseteq X$. It can be shown that the *distinguished subquotient* $A(Y) := A(U)/A(V)$ is well-defined.

We assume that X is locally compact Hausdorff in the following two definitions.

Definition 2.3. A $C_0(X)$-algebra is a C^*-algebra A equipped with a nondegenerate *-homomorphism from $C_0(X)$ to the center of the multiplier algebra of A. A morphism of $C_0(X)$-algebras is a $C_0(X)$-linear *-homomorphism.

The category of C^*-algebras over X and the category of $C_0(X)$-algebras are isomorphic (see [19, Proposition 2.11]). We denote the category of separable C^*-algebras over X by $\mathfrak{C}^*\mathfrak{sep}(X)$.

Definition 2.4. For $x \in X$ and a $C_0(X)$-algebra A, we denote the quotient map $A \twoheadrightarrow A(x)$ onto the fiber by π_x. The algebra A is called *continuous* if the function $x \mapsto \|\pi_x(a)\|$ is a continuous function on X for every $a \in A$.

2.2. **Bivariant K-theory for C^*-algebras over topological spaces.** Let X be a second countable topological space. Let us recall that $\mathfrak{KK}(X)$ is the triangulated category that extends KK-theory to separable C^*-algebras over X, see [19]. In [10], the second named author and Meyer define a version of E-theory for separable C^*-algebras over X and establish its basic properties. This construction yields a triangulated category $\mathfrak{E}(X)$ and a functor $\mathfrak{C}^*\mathfrak{sep}(X) \to \mathfrak{E}(X)$ which is characterized by a universal property. We recall two results which are of particular importance for us.

Let $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ be an ordered basis for the topology on X. Denote by X_n the finite topological space, which arises as the T_0-quotient of X equipped with the topology generated by the set $\{U_1, \ldots, U_n\}$. Observe that we have a projective system of spaces $\cdots \twoheadrightarrow X_2 \twoheadrightarrow X_1 \twoheadrightarrow X_0$ together with compatible maps $X \twoheadrightarrow X_n$. By functoriality in the space variable, we obtain a projective sequence of triangulated categories $\big(\mathfrak{E}(X_n)\big)_{n\in\mathbb{N}}$ together with compatible functors $\mathfrak{E}(X) \to \mathfrak{E}(X_n)$.

Proposition 2.5 ([10, Theorem 3.2])**.** *Let A and B be separable C^*-algebras over X. Then there is a natural short exact sequence of $\mathbb{Z}/2$-graded Abelian groups*

$$\varprojlim{}^1 \mathrm{E}_{*+1}(X_n; A, B) \rightarrowtail \mathrm{E}_*(X; A, B) \twoheadrightarrow \varprojlim \mathrm{E}_*(X_n; A, B).$$

Definition 2.6. The bootstrap class $\mathcal{B}_{\mathrm{E}}$ consists of all separable C^*-algebras that are equivalent in E-theory to a commutative C^*-algebra. The bootstrap class $\mathcal{B}_{\mathrm{E}}(X)$ consists of all separable C^*-algebras over X such that $A(U)$ belongs to $\mathcal{B}_{\mathrm{E}}$ for every open subset $U \subseteq X$.

Proposition 2.7 ([10, Theorem 4.6])**.** *Let A and B be separable C^*-algebras over X belonging to the bootstrap class $\mathcal{B}_{\mathrm{E}}(X)$. An element in $\mathrm{E}(X; A, B)$ is invertible if and only if the induced map $\mathrm{K}_*(A(U)) \to \mathrm{K}_*(B(U))$ is invertible for every open subset U of X.*

2.3. **Continuous fields of Kirchberg algebras.** In this subsection we assume that X is a finite-dimensional, compact, metrizable topological space.

Proposition 2.8. *Let A be a separable continuous $C(X)$-algebra whose fibers are stable Kirchberg algebras. Then A is stable, nuclear and $\mathcal{O}_\infty$-absorbing.*

Proof. Bauval shows in [1, Théorème 7.2] that A, being continuous and having nuclear fibers, is nuclear (in fact $C(X)$-nuclear). A combination of results by Blanchard, Kirchberg and Rørdam in [3, 16, 17, 22] implies that $A \otimes \mathcal{O}_\infty \otimes \mathbb{K} \cong A$, see [8, Theorem 7.4] and [14]. □

Corollary 2.9. *Let A and B be separable continuous $C(X)$-algebras whose fibers are stable Kirchberg algebras. Then every* $\mathrm{E}(X)$*-equivalence between A and B lifts to a $C(X)$-linear *-isomorphism.*

Proof. From [10, Theorem 5.4] we see that the given $\mathrm{E}(X)$-equivalence is induced by a $\mathrm{KK}(X)$-equivalence. By Proposition 2.8, we can apply Kirchberg's isomorphism theorem [15]. □

Proposition 2.10. *Let A be a separable nuclear continuous $C(X)$-algebra whose fibers satisfy the* UCT*. Then A belongs to the* $\mathrm{E}(X)$*-theoretic bootstrap class $\mathcal{B}_{\mathrm{E}}(X)$.*

Proof. This follows from [5, Theorem 1.4] applied to every open subset of X. □

2.4. **Filtrated K-theory over finite spaces.** In this subsection we assume that X is a finite T_0-space.

Definition 2.11. Let $\mathfrak{Ab}^{\mathbb{Z}/2}$ be the category of $\mathbb{Z}/2$-graded Abelian groups and $\mathbb{Z}/2$-graded homomorphisms. We denote the collection of non-empty, connected, locally closed subsets of X by $\mathbb{LC}(X)^*$. For $Y \in \mathbb{LC}(X)^*$, we have a functor $\mathrm{FK}_Y^X\colon \mathfrak{E}(X) \to \mathfrak{Ab}^{\mathbb{Z}/2}$ taking A to $\mathrm{K}_*(A(Y))$. Let $\mathcal{NT}^X$ be the $\mathbb{Z}/2$-graded pre-additive category whose object set is $\mathbb{LC}(X)^*$ and whose morphisms from Y to Z are the natural transformations from FK_Y^X to FK_Z^X regarded as functors from separable C^*-algebras over X with $\mathbb{Z}/2$-graded morphism groups $\mathrm{E}_*(X; \textvisiblespace, \textvisiblespace)$ to $\mathbb{Z}/2$-graded Abelian groups with arbitrary group homomorphisms. The collection $\big(\mathrm{FK}_Y^X(A)\big)_{Y \in \mathbb{LC}(X)^*}$ has a natural graded module structure over $\mathcal{NT}^X$. This module is denoted by $\mathrm{FK}^X(A)$. Hence we have a functor $\mathrm{FK}^X\colon \mathfrak{E}(X) \to \mathfrak{Mod}(\mathcal{NT}^X)^{\mathbb{Z}/2}$.

Remark 2.12. If the space X is not too complicated, it is possible to describe the category $\mathcal{NT}^X$ in explicit terms. Suppose for instance that X is an accordion space in the sense of [2]. Then $\mathcal{NT}^X$ is generated by six-term sequence transformations corresponding to inclusions of distinguished subquotients and an explicit generating list of relations can be given, see [2].

Proposition 2.13 ([2, Theorem 8.9])**.** *Let X be an accordion space. Let A and B be separable C^*-algebras over X. Assume that A belongs to the bootstrap class $\mathcal{B}_{\mathrm{E}}(X)$. Then there is a natural short exact sequence of $\mathbb{Z}/2$-graded Abelian groups*

$$\begin{aligned}\mathrm{Ext}^1_{\mathcal{NT}^X}(\mathrm{FK}^X(A), \mathrm{FK}^X(SB)) \rightarrowtail \mathrm{E}_*(X; A, B)&\\ \twoheadrightarrow \mathrm{Hom}_{\mathcal{NT}^X}(\mathrm{FK}^X(A), \mathrm{FK}^X(B)).&\end{aligned}$$

Here we have replaced $\mathrm{KK}_*(X; A, B)$ by $\mathrm{E}_*(X; A, B)$ in the original statement. This is possible as we explain in the following remark.

Remark 2.14. It was shown in [2] that $\mathrm{FK}^X(A)$ has a projective resolution of length 1 for every separable C^*-algebra A over X. Regarding FK^X as a functor from $\mathfrak{E}(X)$, it is the universal $\mathcal{I}$-exact stable homological functor, where $\mathcal{I}$ is now the ideal in $\mathfrak{E}(X)$ consisting of all elements inducing zero maps in FK^X. This is because $\mathrm{KK}(X; R, R) \cong \mathrm{E}(X; R, R)$ by [10, Theorem 5.5], where $R \in \mathcal{B}(X)$ is the representing object for FK^X. Now the result follows from the general UCT of [18]. (Here $\mathcal{B}(X)$ denotes the KK-theoretic bootstrap class of C^*-algebras over X defined by Meyer–Nest in [19] as the smallest class of C^*-algebras containing all one-dimensional C^*-algebras over X and closed under certain operations. If A is a nuclear C^*-algebra over X, then A belongs to $\mathcal{B}(X)$ if and only if it belongs to $\mathcal{B}_{\mathrm{E}}(X)$.)

Not every $\mathcal{NT}^X$-module belongs to the range of the invariant FK^X. In particular, $\mathrm{FK}^X(A)$ is an *exact* $\mathcal{NT}^X$-module for every C^*-algebra A over X as defined in [18, Definition 3.5].

Proposition 2.15. *Let X be an accordion space and M an $\mathcal{NT}^X$-module. Then M is projective/injective if and only if M is exact and the $\mathbb{Z}/2$-graded Abelian group $M(Y)$ is projective/injective for every $Y \in \mathbb{LC}(X)^*$.*

Proof. The statement about projective modules is proven in [2]. The claim about injective modules follows from a dual argument. ∎

3. Finite approximations of the unit interval

Let $I = [0,1]$ be the unit interval. Choose, once and for all, a dense sequence $(d_n)_{n\in\mathbb{N}}$ in I. For convenience, we may assume $d_m \neq d_n$ for $m \neq n$ and $d_n \notin \{0,1\}$ for all $n \in \mathbb{N}$. Consider the ordered subbasis $\mathcal{V} = (V_n)_{n\in\mathbb{N}}$ for the topology on I given by $V_{2n-1} = [0, d_n)$ and $V_{2n} = (d_n, 1]$; denote by I_n the T_0-quotient of I equipped with the topology generated by the set $\{V_1, \ldots, V_{2n}\}$.

Let A and B be separable C^*-algebras over I. Since the spaces I_n form a cofinal family in the projective sequence of approximations corresponding to the basis generated by the subbasis $\mathcal{V}$ above, Proposition 2.5 yields a short exact sequence

$$\varprojlim{}^1 \mathrm{E}_{*+1}(I_n; A, B) \rightarrowtail \mathrm{E}_*(I; A, B) \twoheadrightarrow \varprojlim \mathrm{E}_*(I_n; A, B). \tag{3.1}$$

We are therefore interested in the computation of the groups $\mathrm{E}_*(I_n; A, B)$.

Lemma 3.2. *The spaces I_n are accordion spaces.*

Proof. For a given natural number $n \in \mathbb{N}$, we order the set $\{d_1, \ldots, d_n\}$ by writing $\{d_1, \ldots, d_n\} = \{e_1, \ldots, e_n\}$ where $e_k < e_{k+1}$ for $1 \leq k < n$. Then we have

$$I_n = \{[0, e_1), \{e_1\}, (e_1, e_2), \{e_2\}, (e_2, e_3), \ldots, \{e_n\}, (e_n, 1]\}.$$

Denoting $u_0 = [0, e_1)$, $u_k = (e_k, e_{k+1})$ for $1 \leq k < n$, $u_n = (e_n, 1]$ and $c_k = \{e_k\}$ for $1 \leq k \leq n$, a basis for the topology on I_n given by the family of open subsets

$$\{\{u_k\} \mid 0 \leq k \leq n\} \cup \{\{u_k, c_k, u_{k+1}\} \mid 0 \leq k < n\}.$$

Hence I_n is an accordion space of a specific form, the Hasse diagram of the specialization order of which is indicated in the diagram below.

$$\bullet \to \bullet \leftarrow \bullet \to \bullet \leftarrow \bullet \to \cdots \leftarrow \bullet \to \bullet \leftarrow \bullet \to \bullet \leftarrow \bullet \qquad \square$$

For $n \in \mathbb{N}$, we briefly write $\mathcal{NT}_n$ for $\mathcal{NT}^{I_n}$ and $\mathrm{FK}_n(A)$ for $\mathrm{FK}^{I_n}(A)$.

Assume that A belongs to the bootstrap class $\mathcal{B}_{\mathrm{E}}(I)$. By Proposition 2.13, for every $n \in \mathbb{N}$, we have a short exact sequence

$$\begin{aligned} \mathrm{Ext}^1_{\mathcal{NT}_n}(\mathrm{FK}_n(A), \mathrm{FK}_n(SB)) &\rightarrowtail \mathrm{E}_*(I_n; A, B) \\ &\twoheadrightarrow \mathrm{Hom}_{\mathcal{NT}_n}(\mathrm{FK}_n(A), \mathrm{FK}_n(B)). \end{aligned} \tag{3.3}$$

Definition 3.4. Let A be a C^*-algebra over $[0,1]$. The *filtrated* K*-theory of* A consists of the $\mathbb{Z}/2$-graded Abelian groups $\mathrm{K}_*(A(Y))$ for all locally closed subintervals $Y \subseteq I$ together with the graded group homomorphisms in the six-term exact sequence $\mathrm{K}_*(A(U)) \to \mathrm{K}_*(A(Y)) \to \mathrm{K}_*(A(Y \setminus U)) \to \mathrm{K}_{*+1}(A(U))$ for every relatively open subinterval U of a locally closed interval $Y \subseteq I$ with the property that the set $Y \setminus U$ is connected. A *homomorphism* from $\mathrm{FK}(A)$ to $\mathrm{FK}(B)$ is a family of graded group homomorphisms

$$\{\varphi_Y \colon \mathrm{K}_*(A(Y)) \to \mathrm{K}_*(B(Y))\}_Y$$

such that for all pairs $U \subset Y$ as above, all squares in the diagram

$$\begin{array}{ccccccc}
\mathrm{K}_*(A(U)) & \longrightarrow & \mathrm{K}_*(A(Y)) & \longrightarrow & \mathrm{K}_*(A(Y\setminus U)) & \longrightarrow & \mathrm{K}_{*+1}(A(U)) \\
\downarrow{\scriptstyle \varphi_U} & & \downarrow{\scriptstyle \varphi_Y} & & \downarrow{\scriptstyle \varphi_{Y\setminus U}} & & \downarrow{\scriptstyle \varphi_U} \\
\mathrm{K}_*(B(U)) & \longrightarrow & \mathrm{K}_*(B(Y)) & \longrightarrow & \mathrm{K}_*(B(Y\setminus U)) & \longrightarrow & \mathrm{K}_{*+1}(B(U))
\end{array}$$

commute.

The $\mathbb{Z}/2$-graded Abelian group of homomorphisms from $\mathrm{FK}(A)$ to $\mathrm{FK}(B)$ is denoted by $\mathrm{Hom}_{\mathcal{NT}}(\mathrm{FK}(A), \mathrm{FK}(B))$.

We note that one may consider a variation $\mathrm{FK}'(A)$ of $\mathrm{FK}(A)$ where only intervals with endpoints from the sequence $(d_n)_{n\in\mathbb{N}}$ and $0, 1$ are used. It is not hard to show that the restriction map $\mathrm{Hom}_{\mathcal{NT}}(\mathrm{FK}(A), \mathrm{FK}(B)) \to \mathrm{Hom}_{\mathcal{NT}}(\mathrm{FK}'(A), \mathrm{FK}'(B))$ is bijective. It follows that

$$\mathrm{Hom}_{\mathcal{NT}}(\mathrm{FK}(A), \mathrm{FK}(B)) = \varprojlim \mathrm{Hom}_{\mathcal{NT}_n}(\mathrm{FK}_n(A), \mathrm{FK}_n(B)).$$

Remark 3.5. We can regard $\mathrm{FK}(A)$ as a $\mathbb{Z}/2$-graded module over a $\mathbb{Z}/2$-graded pre-additive category $\mathcal{NT}$ with objects the locally closed subintervals of I and morphisms generated by elements i_U^Y, $r_Y^{Y\setminus U}$, $\delta_{Y\setminus U}^U$ for every relatively open subinterval U of a locally closed interval $Y \subseteq I$ such that $Y \setminus U$ is connected. Regardless of the relations among these generators, homomorphisms from $\mathrm{FK}(A)$ to $\mathrm{FK}(B)$ would then simply be graded module homomorphisms. This justifies the notation $\mathrm{Hom}_{\mathcal{NT}}(\mathrm{FK}(A), \mathrm{FK}(B))$.

4. Classification results

We are now ready to put together the facts from the previous sections to derive classification results.

Applying inverse limits to the UCT-sequences (3.3), and using that $\varprojlim^1$ is a derived functor of $\varprojlim$, we obtain the exact sequence

$$
\begin{aligned}
(4.1) \quad 0 &\to \varprojlim \mathrm{Ext}^1_{\mathcal{NT}_n}(\mathrm{FK}_n(A), \mathrm{FK}_n(SB)) \to \varprojlim \mathrm{E}_*(I_n; A, B) \\
&\to \mathrm{Hom}_{\mathcal{NT}}(\mathrm{FK}(A), \mathrm{FK}(B)) \xrightarrow{d} \varprojlim^1 \mathrm{Ext}^1_{\mathcal{NT}_n}(\mathrm{FK}_n(A), \mathrm{FK}_n(SB)).
\end{aligned}
$$

Definition 4.2. Let $\mathcal{K}ir(I)$ denote the class of separable continuous $C(I)$-algebras whose fibers are stable Kirchberg algebras satisfying the UCT.

Theorem 4.3. *Let $\mathscr{C}$ be a subclass of $\mathcal{K}ir(I)$ such that for all A and B in $\mathscr{C}$, the map d in* (4.1) *vanishes. Then, for all A and B in $\mathscr{C}$, the map* $\mathrm{E}_*(I; A, B) \to \mathrm{Hom}_{\mathcal{NT}}(\mathrm{FK}(A), \mathrm{FK}(B))$ *is surjective and every isomorphism* $\mathrm{FK}(A) \cong \mathrm{FK}(B)$ *lifts to a $C(I)$-linear *-isomorphism.*

Proof. Let $\alpha \in \mathrm{Hom}_{\mathcal{NT}}(\mathrm{FK}(A), \mathrm{FK}(B))$. If $d(\alpha) = 0$, we can use the exact sequences (4.1) and (3.1) to lift α to an element $\tilde{\alpha} \in \mathrm{E}(I; A, B)$. If α was an isomorphism, then $\tilde{\alpha}$ is an $\mathrm{E}(I)$-equivalence by Proposition 2.7. We conclude the proof by applying Corollary 2.9. ☐

Remark 4.4. Theorem 4.3 does not hold for the whole class $\mathscr{C} = \mathcal{K}ir(I)$ as shown by Example 6.5 from [10]. If the fibers of A and B have torsion in K-theory, then the map $\mathrm{E}_*(I; A, B) \to \ker(d) \subset \mathrm{Hom}_{\mathcal{NT}}(\mathrm{FK}(A), \mathrm{FK}(B))$ is typically not surjective.

We will now verify the hypotheses of Theorem 4.3 for certain classes of C^*-algebras over $[0, 1]$. Our first example yields (in particular) a proof of Theorem 1.1.

Example 4.5. (*Proof of Theorem* 1.1.) By Proposition 2.15, the conclusion of Theorem 4.3 holds for the class $\mathscr{C}$ of $C(I)$-algebras A in $\mathcal{K}ir(I)$ for which $\mathrm{K}_*(A(Y))$ is a divisible Abelian group for every locally closed interval $Y \subseteq I$. By the Künneth formula for tensor products, the class $\mathscr{C}$ contains all objects in $\mathcal{K}ir(I)$ which are stable under tensoring with the universal UHF-algebra $M_{\mathbb{Q}}$. Let A be as in Theorem 1.1. Since $\mathrm{K}_*(A(x)) \cong \mathrm{K}_*(A(x)) \otimes \mathbb{Q}$ it follows that $A(x) \cong A(x) \otimes M_{\mathbb{Q}}$, for all $x \in I$, by the Kirchberg–Phillips classification theorem. We conclude the argument by noting that if each fiber of a $C(I)$-algebra A is stable under tensoring with the universal UHF-algebra $M_{\mathbb{Q}}$, then so is A itself by [14].

Example 4.6. Again by Proposition 2.15, the conclusion of Theorem 4.3 holds for the class $\mathscr{C}$ of $C(I)$-algebras A in $\mathcal{K}ir(I)$ for which $\mathrm{K}_*(A(Y))$ is a free Abelian group for every locally closed interval $Y \subseteq I$ because the Ext^1-terms in (4.1) vanish.

Example 4.7. Fix $i \in \{0,1\}$. Consider the class $\mathscr{C}$ of $C(I)$-algebras A in $\mathcal{K}ir(I)$ which satisfy $\mathrm{K}_i(A(Y)) = 0$ for every locally closed interval $Y \subseteq I$. For parity reasons, the Ext^1-terms in (4.1) vanish. Hence the class $\mathscr{C}$ satisfies the condition of Theorem 4.3.

Remark 4.8. Fix $i \in \{0,1\}$. It follows from the main result in [8] that the condition of Theorem 4.3 is also satisfied for the class $\mathscr{C}$ of $C(I)$-algebras A in $\mathcal{K}ir(I)$ whose fibers have vanishing K_d-groups and torsion-free K_{d+1}-groups. However, we have not been able to reprove this by an independent, purely K-theoretical argument.

5. A remark on coefficients

In order to get a classification result without any K-theoretical assumptions, one expects, as indicated in the introduction, to need some version of filtrated K-theory with coefficients for C^*-algebras over the unit interval. This requires, to begin with, the correct definition of filtrated K-theory with coefficients for C^*-algebras over accordion spaces. It was observed in [13] that, already over the two-point Sierpiński space S, the naïve candidate for such a definition—using the corresponding six-term sequence of Λ-modules—produces an invariant which lacks desired properties such as a UMCT.

We argue that, in order to give a fully satisfactory definition of filtrated K-theory with coefficients for C^*-algebras over S, one has to allow all finitely generated, indecomposable exact six-term sequences of Abelian groups as coefficients—just as all finitely generated, indecomposable Abelian groups as coefficients are needed in the UMCT of [9]. It is easy to see that there is a countable number of isomorphism classes of such six-term sequences. However, unlike in the case of Abelian groups, it follows from the main result in [21] that their classification is controlled $\mathbb{Z}/p$-wild for every prime p. This wildness phenomenon seems to make filtrated K-theory with (generalized) coefficients as sketched above very hard to compute explicitly, limiting its rôle in the theory to a rather theoretical one.

We conclude by remarking that recent results of Eilers, Restorff and Ruiz in [12] indicate that additional K-theoretical assumptions allow the usage of a smaller, more concrete invariant.

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CLASSIFICATION OF CERTAIN CONTINUOUS FIELDS OF KIRCHBERG ALGEBRAS

RASMUS BENTMANN

Abstract. We show that the K-theory cosheaf is a complete invariant for separable continuous fields with vanishing boundary maps over a finite-dimensional compact metrizable topological space whose fibers are stable Kirchberg algebras with rational K-theory groups satisfying the universal coefficient theorem. We provide a range result for fields in this class with finitely generated K-theory. There are versions of both results for unital continuous fields.

1. Introduction

The present article is part of a programme aimed at deciding when two C^*-algebras over a (second countable) topological space X are equivalent in ideal-related KK-theory. In consequence of a fundamental result due to Eberhard Kirchberg [19, Folgerung 4.3], this is a central question in the classification theory of non-simple purely infinite C^*-algebras.

We briefly review the existing results in the literature. These are divided into two classes, namely finite (non-Hausdorff) spaces on the one hand and finite-dimensional compact metrizable spaces on the other hand. Universal coefficient theorems (UCT), which compute the KK(X)-groups in terms of K-theoretic invariants and which imply a solution to the given classification problem, have been established for certain classes of finite T_0-spaces in [27, 7, 25, 21, 3, 5]. A solution for finite unique path spaces using a more complicated kind of invariant is provided in [6]. For arbitrary finite spaces, the problem remains unsolved and seems rather unfeasible because certain wildness phenomena occur; see [2]. In the context of finite-dimensional compact metrizable spaces the strongest results are available in the totally disconnected case [14, 13] and in the case of the unit interval [11, 12, 4].

As these examples illustrate, the feasibility of the classification problem under consideration depends critically on the space X. However, it is possible to obtain solutions for more general base spaces under additional K-theoretical assumptions. For instance, Kirchberg's isomorphism theorem [19, Folgerung 2.18] states that two separable nuclear stable $\mathcal{O}_2$-absorbing C^*-algebras are isomorphic once their primitive ideal spaces are homeomorphic ($\mathcal{O}_2$-absorption entails in particular the vanishing of all K-theoretic data). In [1], a UCT for C^*-algebras with *vanishing boundary maps* (as we shall define in §3) over an arbitrary finite T_0-space is proven.

The main result of the present article is the following; it is based on the UCT in [1] and Dadarlat–Meyer's approximation of ideal-related E-theory over an infinite space X by ideal-related E-theory over finite quotient spaces of X from [13], together with Kirchberg's theorem [19].

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Theorem 1.1. *Let A and B be separable continuous fields over a finite-dimensional compact metrizable topological space X whose fibers are stable Kirchberg algebras that satisfy the UCT and have rational* K*-theory groups. Assume that A and B have vanishing boundary maps. Then any isomorphism of* K*-theory cosheaves $\mathbb{O}\mathrm{K}(A) \cong \mathbb{O}\mathrm{K}(B)$ lifts to a $C(X)$-linear *-isomorphism $A \cong B$.*

The K-theory cosheaf $\mathbb{O}\mathrm{K}$ is a rather simple (but large) K-theoretic invariant which we shall define in §3; it comprises the K-theory groups of all (distinguished) ideals of the algebra, together with the maps induced by all inclusions of such ideals.

The proof of this theorem is concluded in §4. We provide a version of the theorem for unital algebras in Theorem 4.2. An Abelian group G is *rational* if it is isomorphic to its tensor product with the field of rational numbers $\mathbb{Q}$; this is equivalent to G being torsion-free and divisible and to G being a vector space over $\mathbb{Q}$.

Our method of proof is largely parallel to the one in [4], where the UCT from [5] for C^*-algebras over finite accordion spaces was used, based on the observation that the unit inverval has sufficiently many finite quotients of accordion type. The main result of the present article is instead valid for an arbitrary finite-dimensional compact metrizable base space, but this comes at the expense of the assumption of vanishing boundary maps.

Using a result from [17], Kirchberg's isomorphism theorem for $\mathcal{O}_2$-absorbing C^*-algebras mentioned above implies in particular that a separable continuous C^*-bundle over a finite-dimensional compact metrizable space X whose fibers are stable UCT Kirchberg algebras with trivial K-theory groups is isomorphic to the trivial C^*-bundle $C(X, \mathcal{O}_2 \otimes \mathbb{K})$; see also [10]. Our classification result may be considered as an extension of this automatic triviality theorem for continuous $\mathcal{O}_2 \otimes \mathbb{K}$-bundles: instead of asking the C^*-bundles to have entirely trivial K-theory, we only require the collection of the K-theory groups of all ideals in the algebra to form a flabby cosheaf of $\mathbb{Z}/2$-graded $\mathbb{Q}$-vector spaces (this terminology is explained in §3).

In Section 5, we determine the range of the invariant in the classification result above, but under the requirement of finitely generated K-theory. More precisely, we show:

Theorem 1.2. *Let M be a flabby cosheaf of $\mathbb{Z}/2$-graded $\mathbb{Q}$-vector spaces on X such that $M(X)$ is finite-dimensional. Then M is a direct sum of a finite number of skyscraper cosheaves and $M \cong \mathbb{O}\mathrm{K}(A)$ for a continuous field A as in Theorem* 1.1.

This theorem also has an analogue for unital continuous fields. The range question in the general case where $M(X)$ may be countably infinite-dimensional remains open.

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2. Preparations

Throughout this article, we let X denote a finite-dimensional compact metrizable topological space (arbitrary topological spaces will be denoted by Y). The topology of X (its lattice of open subsets) is denoted by $\mathbb{O}(X)$. We choose an ordered basis $(U_n)_{n\in\mathbb{N}}$ for $\mathbb{O}(X)$ and consider the (finite) T_0-quotient X_n of X equipped with the topology $\mathbb{O}(X_n)$ generated by the family $\{U_1, \ldots, U_n\}$ (see [13, §3]).

Our reference for continuous fields of C^*-algebras (or, synonymically, C^*-bundles) is [15]. For basic definitions, facts and notation concerning C^*-algebras over (possibly non-Hausdorff) topological spaces, we refer to [22]. Versions of KK-theory and E-theory for separable C^*-algebras over second countable topological spaces have been constructed in [22] and [13], respectively. By [13, Theorem 3.2], there is a natural short exact sequence of $\mathbb{Z}/2$-graded Abelian groups

$$\varprojlim{}^1 \mathrm{E}_{*+1}(X_n; A, B) \rightarrowtail \mathrm{E}_*(X; A, B) \twoheadrightarrow \varprojlim \mathrm{E}_*(X_n; A, B) \tag{2.1}$$

for every pair A, B of separable C^*-algebras over X.

Recall from [22, §3.2] that there is an exterior product for $\mathrm{KK}(X)$-theory. In particular, we can form the (minimal) tensor product of a C^*-algebra A over X with a C^*-algebra D and obtain a C^*-algebra $A \otimes D$ over X. We let $M_{\mathbb{Q}}$ denote the universal UHF-algebra. Hence $\mathrm{K}_0(M_{\mathbb{Q}}) \cong \mathbb{Q}$ and $\mathrm{K}_1(M_{\mathbb{Q}}) = 0$. A C^*-algebra B (over X) is called *$M_{\mathbb{Q}}$-absorbing* if $B \cong B \otimes M_{\mathbb{Q}}$. If A and B are C^*-algebras over X and B is $M_{\mathbb{Q}}$-absorbing, then the exterior product

$$\mathrm{KK}_*(X; A, B) \otimes \mathrm{KK}_*(\mathbb{C}, M_{\mathbb{Q}}) \to \mathrm{KK}_*(X; A \otimes \mathbb{C}, B \otimes M_{\mathbb{Q}}) \cong \mathrm{KK}_*(X; A, B) \tag{2.2}$$

turns $\mathrm{KK}_*(X; A, B)$ into a rational vector space.

3. Vanishing boundary maps and flabby cosheaves

A C^*-algebra A over an arbitrary topological space Y is said to have *vanishing boundary maps* if the natural map $i_U^V \colon \mathrm{K}_*\big(A(U)\big) \to \mathrm{K}_*\big(A(V)\big)$ is injective for all open subsets $U \subseteq V \subseteq Y$ (it suffices to consider the case $V = Y$); this is equivalent to the condition in [1, Definition 3.2] because of the six-term exact sequence.

If A has vanishing boundary maps, one can deduce from the Mayer–Vietoris sequence and continuity of K-theory that, for every covering $(V_i)_{i\in I}$ of an open subset $V \subseteq Y$ by open subsets $V_i \subseteq V$, one has an exact sequence

$$\bigoplus_{j,k\in I} \mathrm{K}_*\big(A(V_j \cap V_k)\big) \xrightarrow{(i^{V_j}_{V_j\cap U_k} - i^{V_k}_{V_j\cap V_k})} \bigoplus_{i\in I} \mathrm{K}_*\big(A(V_i)\big) \xrightarrow{(i^U_{V_i})} \mathrm{K}_*\big(A(V)\big) \longrightarrow 0; \tag{3.1}$$

see [8, Proposition 1.3].

Definition 3.2. The K-*theory cosheaf* $\mathbb{O}\mathrm{K}^Y(A)$ of a C^*-algebra A over Y with vanishing boundary maps consists of the collection of $\mathbb{Z}/2$-graded Abelian groups $\big(\mathrm{K}_*\big(A(U)\big) \mid U \in \mathbb{O}(Y)\big)$ together with the collection $\big(i_U^V \mid V \in \mathbb{O}(Y), U \in \mathbb{O}(V)\big)$ of graded group homomorphisms.

In Theorem 1.1 we briefly wrote $\mathbb{O}\mathrm{K}(A)$ for $\mathbb{O}\mathrm{K}^X(A)$. For an arbitrary C^*-algebra A over Y, $\mathbb{O}\mathrm{K}^Y(A)$ would only define a precosheaf, that is, a covariant functor on $\mathbb{O}(X)$. However, if A has vanishing boundary maps, then by (3.1), $\mathbb{O}\mathrm{K}^Y(A)$ is indeed a *flabby cosheaf* of $\mathbb{Z}/2$-graded Abelian groups in the technical sense of [8, §1]. We reproduce the definition below for the reader's convinience. The partially ordered set $\mathbb{O}(Y)$ is considered as a category with morphisms given by inclusions.

Definition 3.3. A *precosheaf* on Y is a covariant functor M from $\mathbb{O}(Y)$ to the category of modules over some ring. For open subsets $U \subseteq V \subseteq Y$, the induced map $M(U) \to M(V)$ is denoted by i_U^V. A precosheaf M is a *cosheaf* if the sequence

$$\bigoplus_{j,k\in I} M(V_j \cap V_k) \xrightarrow{(i^{V_j}_{V_j\cap U_k} - i^{V_k}_{V_j\cap V_k})} \bigoplus_{i\in I} M(V_i) \xrightarrow{(i^U_{V_i})} M(V) \longrightarrow 0 \tag{3.4}$$

is exact for every open covering $(V_i)_{i\in I}$ of an open subset $V \subseteq Y$. It is *flabby* if the map $i_U^V \colon M(U) \to M(V)$ is injective for all open subset $U \subseteq V \subseteq Y$. A morphism of cosheaves is a natural transformation of the corresponding functors.

Remark 3.5. The invariant $\mathbb{O}\mathrm{K}$ is necessarily large (in a non-technical sense), as its purpose is to classify certain C^*-bundles that need not be locally trivial. However, we may minimize its size without losing essential information by restricting to a fixed countable basis of $\mathbb{O}(X)$. (This is a standard fact about (co)sheaves.)

Consider now one of the approximating spaces X_n (this may have the homeomorphism type of any finite T_0-space). The crucial step towards the UCT in [1] was to show that $\mathbb{O}\mathrm{K}^{X_n}(A)$ has a projective resolution of length one (combine Lemmas 4.3 and 4.6 from [1]). When we work over the rational numbers, it follows by exactly the same argument that $\mathbb{O}\mathrm{K}^{X_n}(A) \otimes \mathbb{Q}$ is projective as a cosheaf of $\mathbb{Z}/2$-graded $\mathbb{Q}$-vector spaces, and the homological algebra in the $\mathbb{Q}$-linear triangulated category $\mathcal{B}_{\mathrm{E}}(X_n) \otimes \mathbb{Q}$ for the homological functor $\mathbb{O}\mathrm{K}^{X_n} \otimes \mathbb{Q}$ (see [23, 18]) provides an isomorphism

$$\mathrm{E}_*(X_n; A, B) \otimes \mathbb{Q} \cong \mathrm{Hom}_*\big(\mathbb{O}\mathrm{K}^{X_n}(A) \otimes \mathbb{Q}, \mathbb{O}\mathrm{K}^{X_n}(B) \otimes \mathbb{Q}\big), \tag{3.6}$$

where we write $\mathrm{Hom}_*(M, N)$ for $\mathrm{Hom}(M, N) \oplus \mathrm{Hom}(M, N[1])$.

4. Proof of Theorem 1.1

We are now prepared to prove our main result. As in Theorem 1.1, we abreviate $\mathbb{O}\mathrm{K}^X$ by $\mathbb{O}\mathrm{K}$. Assume that A and B as in Theorem 1.1 are given. By [4, Propositions 2.8 and 2.10], A and B are stable, nuclear and $\mathcal{O}_\infty$-absorbing, and belong to the $\mathrm{E}(X)$-theoretic bootstrap class $\mathcal{B}_{\mathrm{E}}(X)$ defined in [13, Definition 4.1]. Hence, by the comparison theorem [13, Theorem 5.4], Kirchberg's classification theorem [19, Folgerung 4.3] and the invertibility criterion [13, Theorem 4.6], it suffices to show that a given homomorphism $\mathbb{O}\mathrm{K}(A) \to \mathbb{O}\mathrm{K}(B)$ lifts to an element in $\mathrm{E}_0(X; A, B)$.

Since the fibers of A and B have rational K-theory groups, they are $M_{\mathbb{Q}}$-absorbing by the Kirchberg–Phillips classification theorem [26, §8.4]. By [17], the algebras A and B themselves are $M_{\mathbb{Q}}$-absorbing. Hence, by (2.2), (3.6), the comparison theorem [13, Theorem 5.4] and the Künneth formula for tensor products,

$$\mathrm{E}_*(X_n; A, B) \cong \mathrm{Hom}_*\big(\mathbb{O}\mathrm{K}^{X_n}(A), \mathbb{O}\mathrm{K}^{X_n}(B)\big) \tag{4.1}$$

because $\mathbb{Q} \otimes \mathbb{Q} \cong \mathbb{Q}$. This implies $\varprojlim \mathrm{E}_*(X_n; A, B) \cong \mathrm{Hom}_*\big(\mathbb{O}\mathrm{K}(A), \mathbb{O}\mathrm{K}(B)\big)$ (by Remark 3.5 applied to the chosen basis $(U_n)_{n\in\mathbb{N}}$). The claim now follows from (2.1).

4.1. Classification of unital C^*-bundles.

For a unital C^*-bundle over X, we may equip $\mathbb{O}\mathrm{K}(A)$ with the unit class $[1_A] \in \mathrm{K}_0(A)$. This pair is denoted by $\mathbb{O}\mathrm{K}^+(A)$; it is a *pointed* cosheaf, that is, a cosheaf M with a distinguished element in the degree-zero part of $M(X)$. Morphisms of such pointed cosheaves are of course required to preserve the distinguished element. By [16, Theorem 3.3], we immediately obtain the following version of our main result for unital algebras.

Theorem 4.2. *Let A and B be separable unital continuous fields over X whose fibers are UCT Kirchberg algebras with rational* K*-theory groups. Assume that A and B have vanishing boundary maps. Then any isomorphism $\mathbb{O}\mathrm{K}^+(A) \cong \mathbb{O}\mathrm{K}^+(B)$ lifts to a $C(X)$-linear *-isomorphism $A \cong B$.*

5. Range results

We investigate the question of the range of the invariant in Theorem 4.2 (the same considerations apply mutatis mutandis and without keeping track of the unit class in the stable case and yield a proof for Theorem 1.2). For the results in this section, it would suffice to assume that X is a compact Hausdorff space.

If a unital C^*-bundle A of the form classified by our result is locally trivial on an open subset U of X, then $A(U)$ must be isomorphic to $C_0(U, \mathcal{O}_2)$. Hence interesting examples cannot be locally trivial (around every point in X).

Example 5.1. We will now describe some basic non-trivial examples of C^*-bundles satisfying the conditions in our classification theorem. Let $D_1, \ldots, D_n$ be unital UCT Kirchberg algebras. By the Exact Embedding Theorem [20, Theorem 2.8], we may find unital *-monomorphisms $\gamma_i \colon D_i \to \mathcal{O}_2$. For points $x_1, \ldots, x_n$ in X, we define

$$A = \{f \in C(X, \mathcal{O}_2) \mid f(x_i) \in \gamma_i(D_i) \text{ for } i = 1, \ldots, n\}. \tag{5.2}$$

This is clearly a continuous field of Kirchberg algebras, with fiber D_i at x_i and fiber $\mathcal{O}_2$ at all other points. A simple computation using excision shows that

$$\mathrm{K}_*\big(A(U)\big) \cong \bigoplus_{i\colon x_i \in U} \mathrm{K}_*(D_i).$$

Hence $\mathbb{O}\mathrm{K}(A)$ is the direct sum of so-called skyscraper cosheaves $i_{x_i}\big(\mathrm{K}_*(D_i)\big)$ based at x_i with coefficient group $\mathrm{K}_*(D_i)$. Here $i_x(G)$ is defined by

$$i_x(G)(U) = \begin{cases} G & \text{if } x \in U, \\ 0 & \text{else.} \end{cases}$$

These cosheaves are indeed flabby. It follows that the continuous field A has vanishing boundary maps. So, if the algebras D_i have rational K-theory groups, then A satisfies the conditions of Theorem 4.2. Under the identification $\mathrm{K}_0(A) \cong \bigoplus_{i=1}^n \mathrm{K}_0(D_i)$, we have $[1_A] = \sum_{i=1}^n [1_{D_i}]$. Using the range result for K-theory on unital UCT Kirchberg algebras [26, §4.3], it follows that an arbitrarily pointed finite direct sum of skyscraper cosheaves whose coefficient groups are countable $\mathbb{Z}/2$-graded Abelian groups can be realized as the pointed K-theory cosheaf of a unital continuous field as in (5.2).

The following proposition shows that, if A is a unital continuous field as in Theorem 4.2 and the $\mathbb{Q}$-vector space $\mathrm{K}_*(A)$ is finite-dimensional, then A must be of the form (5.2).

Proposition 5.3. *Let $\mathbb{F}$ be a field and let Y be an arbitrary topological space. Let M be a flabby cosheaf of $\mathbb{F}$-vector spaces over Y. If $M(Y)$ is finite-dimensional, then M is a direct sum of a finite number of skyscraper cosheaves.*

Proof. We proceed by induction on the dimension of $M(Y)$. If the dimension is zero, then there is nothing to prove. Otherwise, by [9, V. Proposition 1.5], there exists $y \in Y$ such that $M(Y \setminus \overline{\{y\}})$ is a proper subspace of $M(Y)$. By assumption, the subcosheaf N of M defined by

$$N(U) = M(U \setminus \overline{\{y\}})$$

for $U \in \mathbb{O}(Y)$ is a direct sum of skyscraper cosheaves. Since the quotient $Q = M/N$ vanishes on $Y \setminus \overline{\{y\}}$, it follows from the exact sequence (3.4) that Q is a skyscraper cosheaf of the form $Q = i_y(V)$ for some $\mathbb{F}$-vector space V. It remains to show that the extension $N \rightarrowtail M \twoheadrightarrow Q$ splits. We have $\mathrm{Hom}(i_y(V), N) \cong \mathrm{Hom}\big(V, \varprojlim_{U \ni x} N(U)\big)$ and thus

$$\mathrm{Ext}^1(Q, N) \cong \mathrm{Hom}\big(V, {\varprojlim_{U \ni x}}^1 N(U)\big) = 0$$

by the Mittag–Leffler condition using that $N(U)$ is a finite-dimensional vector space for every U. $\square$

The considerations above are summarized in the following version of Theorem 1.2 for unital continuous fields:

Theorem 5.4. *Let (M, m) be a pointed flabby cosheaf of $\mathbb{Z}/2$-graded $\mathbb{Q}$-vector spaces on X such that $M(X)$ is finite-dimensional. Then M is a direct sum of a finite number of skyscraper cosheaves and $(M, m) \cong \mathbb{O}\mathrm{K}^+(A)$ for a continuous field A as in Theorem* 4.2.

Combining the range result above with our classification results, we obtain an explicit description of the isomorphism classes of the classified continuous fields A whose K-theory $\mathrm{K}_*(A)$ is finite-dimensional over $\mathbb{Q}$. In the case that $\mathrm{K}_*(A)$ is an arbitrary (countable) $\mathbb{Q}$-vector space the situation is unclear: it remains open whether a countable direct sum of skyscraper cosheaves whose coefficient groups are countable $\mathbb{Z}/2$-graded $\mathbb{Q}$-vector spaces can be realized as the K-theory cosheaf $\mathbb{O}\mathrm{K}(A)$ of a continuous field A as in Theorem 1.1; it also remains open whether any flabby cosheaf of countable $\mathbb{Q}$-vector spaces over X is necessarily a direct sum of skyscraper cosheaves.

6. Further remarks

6.1. **Real rank zero.** We briefly comment on the relationship of the assumptions in our classification theorem to real rank zero, a property that is often useful for classification purposes. It was shown in [24, Theorem 4.2] that a separable purely infinite C^*-algebra A has real rank zero if and only if the primitive ideal space of A has a basis consisting of compact open subsets and A is K_0-liftable (meaning, in our terminology, that A has "vanishing exponential maps"). While a C^*-bundle (with non-vanishing fibers) over a compact metrizable space of positive dimension cannot satisfy the first condition, the second condition of K_0-liftability is built into our assumptions (we also assume that A has "vanishing index maps"). As Theorem 1.2 shows, at least in the case of finitely generated K-theory, the K-theory cosheaf of a separable continuous field with vanishing boundary maps has a very zero-dimensional flavour.

6.2. **Cosheaves versus sheaves.** The following explanations clarify the relationship (in the setting of fields with vanishing boundary maps) between our K-theory cosheaf and the K-theory sheaf defined in [11] for C^*-bundles over the unit interval. In [9, Propositions V.1.6 and V.1.8], Glen Bredon provides a structure result for flabby cosheaves: the compact sections functor provides a one-to-one correspondence between soft sheaves and flabby cosheaves on $\mathbb{O}(X)$. A sheaf is *soft* if sections over closed subsets can be extended to global sections. If $\widehat{\mathbb{O}\mathrm{K}(A)}$ denotes the soft sheaf corresponding to the flabby cosheaf $\mathbb{O}\mathrm{K}(A)$, then we have $\widehat{\mathbb{O}\mathrm{K}(A)}(Z) \cong \mathrm{K}_*\bigl(A(Z)\bigr)$ for every *closed* subset $Z \subseteq X$. Regarding the range question considered in §5, we remark that [12, Theorem 5.8] provides a range result for unital C^*-bundles over the unit interval, but it is not clear when the constructed algebras have vanishing boundary maps.

6.3. **Another classification result.** We conclude the note by stating one more result which follows in essentially the same way as our main result. We comment below on the required modifications in the proof. Again, a version for unital algebras can be obtained from [16, Theorem 3.3].

Theorem 6.1. *Fix $i \in \{0, 1\}$. Let A and B be separable continuous fields of stable UCT Kirchberg algebras over a finite-dimensional compact metrizable topological space X. Assume that $\mathrm{K}_i\bigl(A(Z)\bigr) = 0$ for all locally closed subsets $Z \subseteq X$. Then any isomorphism $\mathbb{O}\mathrm{K}(A) \cong \mathbb{O}\mathrm{K}(B)$ lifts to a $C(X)$-linear *-isomorphism $A \cong B$.*

Notice that we do not assume that the fibers of A and B have rational K-theory groups. The K-theoretical assumption in the theorem implies that A and B have vanishing boundary maps. Hence the universal coefficient theorem [1, Theorem 5.2] applies (we may write $\mathbb{O}\mathrm{K}^{X_n}$ instead of $\mathrm{X}_n\mathrm{K}$ by [1, Lemma 4.3]) and simplifies to an isomorphism because the relevant Ext^1-term vanishes for parity reasons. The remainder of the proof is analogous.

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Homotopy-theoretic E-theory and n-order

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Abstract The bootstrap category in E-theory for C^*-algebras over a finite space is embedded into the homotopy category of certain diagrams of $\mathbf{K}$-module spectra. Therefore it has infinite n-order for every $n \in \mathbb{N}$. The same holds for the bootstrap category in G-equivariant E-theory for a compact group G and for the Spanier–Whitehead category in connective E-theory.

Keywords n-order · Triangulated categories · E-theory · Ring spectra

Mathematics Subject Classification (2010) 18E30 · 19K35 · 46L80 · 55P43

1 Introduction

Triangulated categories arise in various contexts such as algebraic geometry, representation theory and algebraic topology. This motivates their distinction into *algebraic*, *topological* (and non-algebraic), and *exotic* (that is, non-topological) triangulated categories; see [17]. Every algebraic triangulated category is topological. The converse is false; topological triangulated categories may exhibit certain torsion phenomena which cannot occur in algebraic triangulated categories. The most well-known such

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phenomenon is the fact that the endomorphism ring of the mod-2 Moore spectrum is not annihilated by multiplication by 2.

In [16, 17], Schwede introduced the notion of *n-order* for triangulated categories (a non-negative integer or infinity for every $n \in \mathbb{N}$). This is an invariant (up to triangulated equivalence) that can often be used to distinguish non-algebraic triangulated categories from algebraic ones by measuring the occurrence of the afore-mentioned torsion phenomena: Schwede shows that the n-order of every algebraic triangulated category is infinite for every $n \in \mathbb{N}$; on the other hand, he proves that, if p is a prime number, the Spanier–Whitehead category in stable homotopy theory has p-order equal to $p - 1$.

One aim of this note is to determine the n-order of certain triangulated categories arising in C^*-algebra theory. More specifically, we are interested in the bivariant homology theories

- connective E-theory for separable C^*-algebras, denoted by bu, as defined by Thom [18],
- G-equivariant E-theory for separable C^*-algebras with a continuous action of a compact group G by *-automorphisms, denoted by E^G, as defined by Guentner, Higson and Trout [5],
- ideal-related E-theory for separable C^*-algebras over a finite space X, denoted by $\mathrm{E}(X)$, as defined by Dadarlat and Meyer [3].

These give rise to triangulated categories denoted by $\mathfrak{bu}$, $\mathfrak{E}^G$ and $\mathfrak{E}(X)$, respectively. The *Spanier–Whitehead category* $\mathcal{SW}_{\mathrm{bu}} \subset \mathfrak{bu}$ in connective E-theory is the thick triangulated subcategory of $\mathfrak{bu}$ generated by the C^*-algebra $\mathbb{C}$ of complex numbers. The *bootstrap categories* $\mathcal{B}_{\mathrm{E}}^G \subset \mathfrak{E}^G$ and $\mathcal{B}_{\mathrm{E}}(X) \subset \mathfrak{E}(X)$ are the $\aleph_0$-localizing subcategories generated by the objects with one-dimensional underlying C^*-algebra, respectively. (While there are no non-trivial G-actions by *-automorphisms on $\mathbb{C}$, there are as many mutually non-isomorphic ways to turn $\mathbb{C}$ into a C^*-algebra over X as there are points in the space X).

Our computational result is the following; it may be regarded as a generalization of Schochet's observation in [12, Proposition 2.4], stating that $\mathbf{K}$-theory with coefficients in $\mathbb{Z}/n$ is annihilated by multiplication by n

Theorem 1.1 *The triangulated categories $\mathcal{SW}_{\mathrm{bu}}$, $\mathcal{B}_{\mathrm{E}}^G$ and $\mathcal{B}_{\mathrm{E}}(X)$ have infinite n-order for every $n \in \mathbb{N}$.*

The theorem is an application of the following result from [17, Example 2.9], which is based on results due to Tyler Lawson and to Vigleik Angeltveit [1].

Theorem 1.2 *Let $\mathbf{R}$ be a commutative symmetric ring spectrum such that $\pi_*\mathbf{R}$ is torsion-free and concentrated in even dimensions. Let $\mathbf{A}$ be an $\mathbf{R}$-algebra spectrum. Then the derived category of $\mathbf{A}$-module spectra has infinite n-order for every $n \in \mathbb{N}$.*

In order to apply this theorem, we need to embed our bootstrap categories into appropriate derived categories of module spectra. The theorem then follows because the n-order can only increase when we pass to a triangulated subcategory. For connective E-theory and G-equivariant E-theory we get the desired embeddings essentially for free from the literature. More specifically, we use a result from Andreas Thom's thesis in the case of connective E-theory and a construction of Dell'Ambrogio–Emerson–Kandelaki–Meyer in the G-equivariant case. In both cases, Theorem [1] may be applied with $\mathbf{A} = \mathbf{R}$. This is not surprising because the categories $\mathfrak{bu}$ and $\mathfrak{E}^G$ are monoidal.

In the case of ideal-related E-theory, we have to work a little harder to obtain the desired embedding. We apply the proposition with $\mathbf{R}$ equal to the Dell'Ambrogio–Emerson–Kandelaki–Meyer spectrum $\mathbf{K} = \mathbf{K}(\mathbb{C})$ for the trivial group and $\mathbf{A}$ equal to a certain $\mathbf{K}$-algebra spectrum $\mathbf{K}X$ which may be called the *incidence algebra* over $\mathbf{K}$ of the partially ordered set X (a finite T_0-space is essentially the same as a partially ordered set). A construction of this form in the special case of upper-triangular 3×3-matrices is indicated by Schwede in [15, Section 4.5].

The category of $\mathbf{K}X$-module spectra is Quillen equivalent to the category of diagrams of $\mathbf{K}$-module spectra indexed by X. The obtained embedding

$$\mathcal{B}_{\mathrm{E}}(X) \hookrightarrow \mathrm{Der}(\mathbf{K}X) \cong \mathrm{Ho}\big(\mathfrak{Mod}(\mathbf{K})^X\big)$$

is interesting in its own right: it sets the stage for Morita theory, allowing us to construct equivalences $\mathcal{B}_{\mathrm{E}}(X) \cong \mathcal{B}_{\mathrm{E}}(Y)$ for many pairs of finite spaces (X, Y). This will enable us to treat many spaces X at once when answering questions such as: is there a manageable homology theory on $\mathcal{B}_{\mathrm{E}}(X)$ computing the $\mathrm{E}(X)$-groups via a universal coefficient theorem? These consequences will be pursued elsewhere.

Some preliminaries We refer to [6] as a general reference on model categories and to [13,14] for the theory of symmetric spectra. Recall that if $\mathcal{M}$ is a stable model category, then its homotopy category $\mathrm{Ho}(\mathcal{M})$, defined as the localization of $\mathcal{M}$ at its weak equivalences, is naturally triangulated. The stable model category of module spectra over a ring spectrum $\mathbf{R}$ is denoted by $\mathfrak{Mod}(\mathbf{R})$; its homotopy category is called the *derived category* of $\mathbf{R}$-module spectra and denoted by $\mathrm{Der}(\mathbf{R})$. We will use several times that an $\mathbf{R}$-module map is a weak equivalence if it induces isomorphisms on stable homotopy groups.

We write $C \in\in \mathcal{C}$ to denote that C is an object in a category $\mathcal{C}$. The C^*-algebra of complex numbers is denoted by $\mathbb{C}$.

2 Connective E-theory

In his thesis [18], Andreas Thom defines *connective E-theory* for separable C^*-algebras. This is the universal triangulated homology theory on separable C^*-algebras satisfying matrix stability (and full excision); it is denoted by bu. The category of separable C^*-algebras with bu-groups as morphisms is denoted by $\mathfrak{bu}$; it carries a triangulation structure inherited from the (C^*-algebra) stable homotopy category; see [18, Section 3.3 and Section 4.2].

Definition 2.1 The *Spanier–Whitehead category* $\mathcal{SW}_{\mathrm{bu}} \subset \mathfrak{bu}$ in connective E-theory is the thick triangulated subcategory generated by the C^*-algebra $\mathbb{C}$.

It is shown in [18, Theorem 5.1.2] that there is a triangulated functor

$$\mathbf{K}^H : \mathfrak{bu} \to \mathrm{Der}(\mathbf{bu})$$

inducing a graded ring isomorphism $\mathrm{bu}_*(\mathbb{C}, \mathbb{C}) \cong \mathrm{Der}(\mathbf{bu})(\mathbf{bu}, \mathbf{bu})_*$, where $\mathbf{bu} := \mathbf{K}^H(\mathbb{C})$ is a commutative symmetric ring spectrum equivalent to the connective K-theory spectrum (see [18, Propositions 5.1.1 and D.1.1]).

Proposition 2.2 *The functor* $\mathbf{K}^H : \mathfrak{bu} \to \mathrm{Der}(\mathbf{bu})$ *is fully faithful on the Spanier–Whitehead category* $\mathcal{SW}_{\mathrm{bu}}$.

Proof This is a standard argument; compare for instance [15, Proposition 3.10]. Consider the full subcategory of $\mathfrak{bu}$ consisting of the objects B such that the map $\mathrm{bu}_*(\mathbb{C}, B) \to \mathrm{Der}(\mathbf{bu})\big(\mathbf{bu}, \mathbf{K}^H(B)\big)_*$ is an isomorphism. This subcategory contains $\mathbb{C}$ and is closed under suspension. It is also closed under exact triangles because $\mathbf{K}^H$ is triangulated. Hence it contains $\mathcal{SW}_{\mathrm{bu}}$. A similar argument shows that, for fixed $B \in\in \mathcal{SW}_{\mathrm{bu}}$, the map $\mathrm{bu}_*(A, B) \to \mathrm{Der}(\mathbf{bu})\big(\mathbf{K}^H(A), \mathbf{K}^H(B)\big)_*$ is bijective for all $A \in\in \mathcal{SW}_{\mathrm{bu}}$. Hence $\mathbf{K}^H$ is fully faithful on $\mathcal{SW}_{\mathrm{bu}}$. □

Proposition 2.3 *The essential image of the restriction of the functor* $\mathbf{K}^H$ *to* $\mathcal{SW}_{\mathrm{bu}}$ *is a triangulated subcategory of Der*$(\mathbf{bu})$.

Proof It suffices to prove that every morphism in the image of the restriction of $\mathbf{K}^H$ to $\mathcal{SW}_{\mathrm{bu}}$ has a cone in the image of the restriction of $\mathbf{K}^H$. Such a cone can be obtained as the image of a cone of the lifting of f to $\mathfrak{bu}$. □

Recall from [18, Theorem 5.1.2] that

$$\pi_*\mathbf{bu} \cong \mathrm{Der}(\mathbf{bu})(\mathbf{bu}, \mathbf{bu})_* \cong \mathrm{bu}_*(\mathbb{C}, \mathbb{C}) \cong \mathbb{Z}[u],$$

where u is of degree two. Together with the previous propositions this shows that Theorem 1.2 may be applied to prove Theorem 1.1 for the category $\mathcal{SW}_{\mathrm{bu}}$.

Remark 2.4 We have restricted ourselves to the Spanier–Whitehead subcategory of $\mathfrak{bu}$ because we do not expect $\mathfrak{bu}$ to possess all countable coproducts.

3 G-equivariant E-theory

Let G be a compact group. A general reference for G-equivariant E-theory is [5].

Lemma 3.1 *If a functor from* $\mathfrak{C}^*\mathfrak{sep}^G$ *maps all* E^G*-equivalences to isomorphisms, then it factors through the canonical functor* $\mathfrak{C}^*\mathfrak{sep}^G \to \mathfrak{E}^G$.

Proof First, we observe that every element in $\mathrm{E}_0^G(A, B)$ can canonically be written as the composition of a G-equivariant *-homomorphism and the inverse of another (it is straight-forward to check that the construction in [2, Section 25.6] goes through in the G-equivariant case). To construct the factorization, we can thus proceed as in the proof of the universal property of E-theory; see [2, Proof of 25.6.1] for details in the non-equivariant but analogous case. □

Now we consider the functor $\mathbf{K}^G : \mathfrak{C}^*\mathfrak{sep}^G \to \mathfrak{Mod}\big(\mathbf{K}^G(\mathbb{C})\big)$ constructed in [4, Section 3.3]. The construction in the non-equivariant case appeared earlier in [7].

Proposition 3.2 *The composition*

$$\mathfrak{C}^*\mathfrak{sep}^G \xrightarrow{\mathbf{K}^G} \mathfrak{Mod}\big(\mathbf{K}^G(\mathbb{C})\big) \to \mathrm{Der}\big(\mathbf{K}^G(\mathbb{C})\big)$$

descends to a triangulated functor

$$\mathbf{K}^G\colon \mathfrak{E}^G \to \mathrm{Der}\big(\mathbf{K}^G(\mathbb{C})\big).$$

Proof By the previous lemma, it suffices to check that the functor $\mathbf{K}^G\colon \mathfrak{C}^*\mathfrak{sep}^G \to \mathfrak{Mod}(\mathbf{K}^G(\mathbb{C}))$ maps E^G-equivalences to weak equivalences. This is a consequence of the natural isomorphism

$$\mathrm{Der}\big(\mathbf{K}^G(\mathbb{C})\big)\big(\mathbf{K}^G(\mathbb{C}), \mathbf{K}^G(B)\big)_* \cong \mathrm{E}^G_*(\mathbb{C}, B)$$

following from (3.6) and (3.7) in [4] and the identification $\mathrm{E}^G_*(\mathbb{C}, B) \cong \mathrm{KK}^G_*(\mathbb{C}, B)$. The fact that $\mathbf{K}^G$ is triangulated follows from [4, Remark 3.6] as in the proof of [4, Theorem 3.8]. □

Definition 3.3 The *bootstrap category* (*of the tensor unit*) $\mathcal{B}^G_{\mathrm{E}} \subset \mathfrak{E}^G$ in G-equivariant E-theory is the $\aleph_0$-localizing subcategory generated by the C^*-algebra $\mathbb{C}$ with the trivial G-action.

Remark 3.4 Results in [8] indicate that, if G is a (higher-dimensional) torus, then the class $\mathcal{B}^G$ in fact provides the correct domain for a potential universal coefficient theorem.

Proposition 3.5 *The functor* $\mathbf{K}^G\colon \mathfrak{E}^G \to \mathrm{Der}\big(\mathbf{K}^G(\mathbb{C})\big)$ *is fully faithful on the bootstrap category* $\mathcal{B}^G_{\mathrm{E}}$.

Proof The proof again proceeds along the lines of [15, Proposition 3.10]. We have to check that the category $\mathfrak{E}^G$ has countable coproducts and that the functor $\mathbf{K}^G$ preserves them. The former is shown in [5, Proposition 7.1]. To see the latter, we must show that the canonical map

$$\varinjlim_{n\in\mathbb{N}} \mathbf{K}^G\left(\bigoplus_{k=1}^{n} A_k\right) \to \mathbf{K}^G\left(\bigoplus_{k=1}^{\infty} A_k\right)$$

is a weak equivalence for every sequence of objects $A_k \in\in \mathfrak{E}^G$. Since G-equivariant K-theory preserves countable direct sums, this map induces isomorphisms on stable homotopy groups and is thus a weak equivalence. □

As in the previous section, the essential image of the restriction of $\mathbf{K}^G$ to $\mathcal{B}^G_{\mathrm{E}}$ is a triangulated subcategory of $\mathrm{Der}\big(\mathbf{K}^G(\mathbb{C})\big)$.

The spectrum $\mathbf{K}^G(\mathbb{C})$ is by construction a commutative symmetric ring spectrum. In order to apply Theorem 1.2 to prove Theorem 1.1 in the case of $\mathcal{B}^G_{\mathrm{E}}$, we need to check that the stable homotopy groups $\pi_*\mathbf{K}^G(\mathbb{C})$ are torsion-free and concentrated in even degrees. We may identify

$$\pi_*\mathbf{K}^G(\mathbb{C}) \cong \mathrm{Der}\big(\mathbf{K}^G(\mathbb{C})\big)\big(\mathbf{K}^G(\mathbb{C}), \mathbf{K}^G(\mathbb{C})\big)_* \cong \mathrm{E}^G_*(\mathbb{C}, \mathbb{C}) \cong \mathrm{R}(G) \otimes \mathbb{Z}[\beta, \beta^{-1}].$$

Here $\mathrm{R}(G)$ denotes the representation ring of the group G concentrated in degree zero and β is an invertible element of degree 2 (see [2, Proposition 20.4.4]). Recall that the underlying Abelian group of the representation ring of G is freely generated by the isomorphism classes of simple G-modules (see for instance [11]). In particular, $\mathrm{R}(G)$ is torsion-free.

4 Ideal-related E-theory

Let X be a finite T_0-space and let $\mathfrak{C}^*\mathfrak{sep}(X)$ denote the category of separable C^*-algebras over X as defined in [9]. In particular, a C^*-algebra over X is a pair (A, ψ) consisting of a C^*-algebra A and a continuous map from the primitive ideal space of A to X. Every open subset U of X naturally gives rise to an ideal $A(U)$ in A. Let $\mathfrak{E}(X)$ denote the version of E-theory for C^*-algebras over X defined by Dadarlat and Meyer [3]. We refer to $\mathfrak{E}(X)$ as *ideal-related E-theory*.

Lemma 4.1 *If a functor from $\mathfrak{C}^*\mathfrak{sep}(X)$ maps all* E(X)*-equivalences to isomorphisms, then it factors through the canonical functor $\mathfrak{C}^*\mathfrak{sep}(X) \to \mathfrak{E}(X)$.*

Proof By [3, Lemma 2.26] (and its proof) every element in $\mathrm{E}_0(X; A, B)$ can canonically be written as the composition of a $*$-homomorphism over X and the inverse of another. To construct the factorization, we can then proceed again as in [2, Proof of 25.6.1]. □

We denote the smallest open neighbourhood of a point x in X by U_x. We consider X as a partially ordered set by setting $x \leq y$ if and only if $U_x \supseteq U_y$. In order to make sense of diagrams indexed by X, we regard X as a category with a unique morphism from x to y if and only if $x \geq y$. For a category $\mathcal{C}$, the diagram category $\mathcal{C}^X$ consists of all functors from X to $\mathcal{C}$.

Definition 4.2 Let $D\colon \mathfrak{C}^*\mathfrak{sep}(X) \to \mathfrak{C}^*\mathfrak{sep}^X$ be the functor taking a C^*-algebra A over X to the diagram $D(A)$ in $\mathfrak{C}^*\mathfrak{sep}$ given by $D(A)(x) = A(U_x)$ and the ideal inclusions $D(A)(x \to y) = \big(A(U_x) \hookrightarrow A(U_y)\big)$. Let $\mathbf{K}\colon \mathfrak{C}^*\mathfrak{sep} \to \mathfrak{Mod}(\mathbf{K}(\mathbb{C}))$ denote the functor of Dell'Ambrogio–Emerson–Kandelaki–Meyer with trivial group G (see [4, Section 3.3]). In the following we abbreviate $\mathbf{K} := \mathbf{K}(\mathbb{C})$. Let

$$\mathbf{K}^X : \mathfrak{C}^*\mathfrak{sep}(X) \to \mathfrak{Mod}(\mathbf{K})^X$$

be the composition of D with pointwise application of $\mathbf{K}$.

We equip the category $\mathfrak{Mod}(\mathbf{K})^X$ with the stable model structure described in [6, Theorem 5.1.3]; the weak equivalences and fibrations are defined pointwise.

Proposition 4.3 *The composition*

$$\mathfrak{C}^*\mathfrak{sep}(X) \xrightarrow{\mathbf{K}^X} \mathfrak{Mod}(\mathbf{K})^X \to \mathrm{Ho}\left(\mathfrak{Mod}(\mathbf{K})^X\right)$$

descends to a triangulated functor

$$\mathbf{K}^X\colon \mathfrak{E}(X) \to \mathrm{Ho}\left(\mathfrak{Mod}(\mathbf{K})^X\right).$$

Proof Every $\mathrm{E}(X)$-equivalence is in particular a pointwise E-equivalence. Hence it is taken to a pointwise weak equivalence in $\mathfrak{Mod}(\mathbf{K})^X$. By definition, every pointwise weak equivalence in $\mathfrak{Mod}(\mathbf{K})^X$ is a weak equivalence and thus becomes invertible in $\mathrm{Ho}\left(\mathfrak{Mod}(\mathbf{K})^X\right)$. The existence of the factorization now follows from Lemma 4.1. The fact that the induced functor $\mathbf{K}^X$ is triangulated is a consequence of the observations in [4, Remark 3.6]. □

For $x \in X$, we let $i_x\mathbb{C}$ denote the C^*-algebra of complex numbers together with the map taking its unique primitive ideal to the point $x \in X$. This is an object in $\mathfrak{C}^*\mathfrak{sep}(X)$. We set $\mathcal{R} = \bigoplus_{x\in X} i_x\mathbb{C}$.

Let $\mathbf{K}X$ denote the endomorphism ring spectrum of a stably fibrant approximation of $\mathbf{K}^X(\mathcal{R})$. We call this symmetric (non-commutative) $\mathbf{K}$-algebra spectrum the *incidence algebra* of X over $\mathbf{K}$. This construction is motivated by [15, Example 4.5(2)]. By [15, Theorem 4.16], there is a Quillen equivalence between $\mathfrak{Mod}(\mathbf{K})^X$ and $\mathfrak{Mod}(\mathbf{K}X)$. In particular, we will henceforth identify $\mathrm{Ho}\left(\mathfrak{Mod}(\mathbf{K})^X\right)$ with $\mathrm{Der}(\mathbf{K}X)$. We have

$$\mathbf{K}^X(i_x\mathbb{C})(y) = \begin{cases} \mathbf{K} & \text{for } y \le x \\ * & \text{else} \end{cases}$$

with all maps between non-trivial entries being identities. This yields a natural identification $\mathrm{Hom}\left(\mathbf{K}^X(i_x\mathbb{C}), M\right) \cong M(x)$ for every $x \in X$ and $M \in\in \mathfrak{Mod}(\mathbf{K})^X$. The corresponding relation in E-theory is the adjunction $\mathrm{E}(X; i_x\mathbb{C}, B) \cong \mathrm{E}\left(\mathbb{C}, B(U_x)\right)$ from [3, (4.3)]. It follows that the graded ring homomorphism from $\mathrm{E}_*(X, \mathcal{R}, \mathcal{R})$ to $\mathrm{Der}(\mathbf{K}X)\left(\mathbf{K}^X(\mathcal{R}), \mathbf{K}^X(\mathcal{R})\right)_*$ induced by the functor $\mathbf{K}^X$ is an isomorphism and that both graded rings are isomorphic to the tensor product of the (ungraded) integral incidence algebra $\mathbb{Z}X$ with the ring of Laurent polynomials $\mathbb{Z}[\beta, \beta^{-1}]$, where β has degree 2. The incidence algebra $\mathbb{Z}X$ is the category ring of the universal pre-additive category generated by the category X.

Definition 4.4 The bootstrap category $\mathcal{B}_{\mathrm{E}}(X) \subset \mathfrak{E}(X)$ is the $\aleph_0$-localizing subcategory generated by the object $\mathcal{R}$.

Proposition 4.5 *The functor* $\mathbf{K}^X\colon \mathfrak{E}(X) \to$ *Der*$(\mathbf{K}X)$ *is fully faithful on the bootstrap category* $\mathcal{B}_{\mathrm{E}}(X)$.

Proof The proof is essentially analogous to the one of Proposition 3.5. We use the fact that ideal-related K-theory preserves countable inductive limits. □

As before, the essential image of the restriction of $\mathbf{K}^X$ to $\mathcal{B}_{\mathrm{E}}(X)$ is a triangulated subcategory of $\mathrm{Der}(\mathbf{K}X)$ and the computation

$$\pi_*\mathbf{K}X \cong \mathrm{Der}(\mathbf{K}X)\big(\mathbf{K}^X(\mathcal{R}), \mathbf{K}^X(\mathcal{R})\big)_* \cong \mathrm{E}_*(X; \mathcal{R}, \mathcal{R}) \cong \mathbb{Z}X \otimes \mathbb{Z}[\beta, \beta^{-1}]$$

with $\mathbb{Z}X$ concentrated in degree zero and β of degree 2 shows that we can apply Theorem 1.2 to prove Theorem 1.1 in the case of $\mathcal{B}_{\mathrm{E}}(X)$.

5 Conclusion

We have shown that certain triangulated categories related to C^*-algebras have infinite n-order for every $n \in \mathbb{N}$ by relating them with certain ring spectra. This means that they share many structural properties of algebraic triangulated categories, but it is not clear whether they are actually algebraic. To conclude, we mention that in the specific case of the derived category of $\mathbf{K}$-module spectra there is an equivalence to an algebraic triangulated category, but it is not known whether this equivalence is triangulated; see [10, Theorem 3.1.4.(ii)].

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