Preferences and Design in Insurance and Pensions

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PhD Thesis

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Preface

This thesis has been prepared in fulfillment of the requirements for the PhD degree at the Department of Mathematical Sciences, Faculty of Science, University of Copenhagen, Denmark. The project was funded by SEB Pension Denmark and the Danish Agency for Sciences, Technology and Innovation under the Industrial PhD Program.

The work has been carried out under the supervision of Professor Mogens Steffensen, University of Copenhagen, and Frank Pedersen and Per Linnemann, SEB Pension Denmark. The work was carried out in the period from January 1, 2010 to March 31, 2013 (including 12 weeks of parental leave).

The main body of this thesis consists of an introduction to the overall work and five chapters on different but related topics. The five chapters are written as individual academic papers and are thus self-contained. They can be read individually with minor overlaps in the contents of them. There are minor notational discrepancy among the five chapters but it is unlikely to cause any confusion.

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Abstract

Life insurance and pension decisions are of the more important financial settlements to be decided in a household. In this thesis we investigate different aspects of relevance for decision making within a household, especially focusing on life insurance and pension decisions. The focus is on the relation between household preferences and the related optimal product design.

Optimal decisions of a household are considered in continuous-time stochastic control theory models. Within a standard expected utility framework we investigate the effects of differences between household members as well as tax-effects. The focus is on the consumption, investment and life insurance demands. In another modeling framework, we modify the utility measurement and propose a combination of forward and backward looking preferences. At last, a model with very explicit preferences for stability in consumption is investigated and we find that the optimal consumption pattern derived corresponds to the benefit pattern of a specific annuity product. This particular product is in a simulation study compared to Unit-Link annuities, which fit perfectly with the consumption patterns derived under the expected utility models.

Papers Included in the Thesis

- Kenneth Bruhn. Consumption, Investment and Life Insurance under Different Tax Regimes. Annals of Actuarial Science, available at `http://journals.cambridge.org/action/displayAbstract?fromPage=online&aid=8771898`.
- Kenneth Bruhn and Mogens Steffensen. Household Consumption, Investment and Life Insurance. Insurance: Mathematics and Economics 48 (2011) pp. 315-325.
- Kenneth Bruhn and Mogens Steffensen. Recursive Utility with Utility Driven Habit Formation.
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- Per Linnemann, Kenneth Bruhn and Mogens Steffensen.
  - English version: A Comparison of Modern Investment-Linked Pension Savings Products. Submitted.
  - Danish version: Sæt fokus på din udbetalingsprofil - en sammenligning af moderne pensionsprodukter med markedsrente. Finans/Invest 6 (2011) 3-11.

Summary

Life insurance and pensions decisions - go with your guts or think twice...

The aim and scope of this thesis is to shed light on the financial decision making of households. This is especially in relation to purchase of life insurance and saving for retirement in the younger years and decumulation of savings via purchase of annuities in retirement. To substantiate financial advice provided, (too) often based on simple guts feelings, we provide several models with different preference structures, providing you something to think twice about.

The first model that we investigate in this thesis builds on standard expected utility theory for an investor with standard preferences yielding a constant relative risk aversion. The model includes taxation to the problem of deciding optimal consumption (and saving), investment and purchase of life insurance (and life annuities). Analytical solutions are available in this model, allowing for simple comparison of model results with classical results for the related model without taxation. Results are summarized in Section 1.2 and the full model is presented in Chapter 2. The taxation significantly alters the optimal decisions derived in the classical model without taxation, maybe you should think twice.

In the second model, summarized in Section 1.3 and presented in Chapter 3, we focus on the interdependency between household members. The model flexibility allows for investigation of several interesting differences between household members. Diversity in salary, consumption needs in widowhood and expected remaining life time as widow(er) is investigated for a two-person household. The diversities translates into very different life insurance sums on the two lives in the household, reserving room for second thoughts.

The third model presented in this thesis is the first of two where we deviate from standard expected utility theory. Here, we analyze a particular mod-

ification that incorporate both forward and backward (in time) looking preferences. The forward looking feature of the utility functional dictates that the certainty equivalent of utility from future consumption is taken into consideration in measuring utility from present consumption. The backward looking feature incorporates previous utility from consumption in the measurement of utility from present consumption. In total, the analytical solution derived for the model is very similar to the solution to the classical standard expected utility model. The differences are shortly explained in the summary in Section 1.4 and further elaborated on in Chapter 4.

All of the three first models result in consumption patterns consistent with decumulation of savings through purchase of pure Unit-Link annuities. Due to the fact that annuities with smoother benefit streams are empirically more popular choices, the last model presented in this thesis investigates very explicit preferences for stability in consumption. The model and its solution is summarized in Section 1.5 and fully explored in Chapter 5. We find that the optimal consumption pattern that solves the model is in general replicated by the benefits of a particular annuity product, Tidspension. Tidspension is also analyzed in Chapter 5 and the similarities between optimal smooth consumption and benefits provided are explored.

A simulation study, comparing the benefits provided by two pure Unit-Link annuities and a Tidspension annuity, concludes the thesis. The focus in the comparison is on the year-to-year stability in benefits as well as the benefit level and expected trend in benefits. The simulation study is summarized in Section 1.6 and fully explored in Chapter 6. The focus in the study is not on the preferences that leads to purchase of the different annuities, these are investigated in the prior models: We only want to give room for second thoughts!

Summary, English-Danish Translation

Livs- og pensionsforsikringsbeslutninger - følg din mavefornemmelse eller tænk dig om ...

Formålet med denne afhandling er at belyse husholdningers finansielle beslutningstagen, specielt i forhold til køb af livsforsikring og opsparing til alderdommen i de yngre år samt nedsparring via køb af livrenter i pensionisttilværelsen. For at underbygge finansiel rådgivning, der (alt for) ofte baseres på simple mavefornemmelser, leverer vi flere modeller med forskellige præferencer strukturer, som giver dig noget at tænke over.

Den første model, som vi undersøger i denne afhandling, bygger på standard forventet nytte teori for en investor med standard præferencer, hvilket giver en konstant relativ risikoaversion. Modellen tilføjer beskatning til problemet med at beslutte det optimale forbrug (og opsparing), investeringer og køb af livsforsikring (og livrenter). Analytiske løsninger opnås for denne model, hvilket giver mulighed for en simpel sammenligning af modellens resultater med klassiske resultater for den tilhørende model uden beskatning. Resultaterne er opsummeret i afsnit 1.2 og den fulde model er præsenteret i kapitel 2. Beskatning ændrer væsentligt de optimale beslutninger udledt i den klassiske model uden beskatning, måske skulle du tænke dig om en ekstra gang.

I den anden model, som opsummeres i afsnit 1.3 og præsenteres i kapitel 3, fokuserer vi på den gensidige afhængighed mellem husstandsmedlemmer. Modellens fleksibilitet giver mulighed for undersøgelse af flere interessante forskelle mellem husstandsmedlemmer. Specielt undersøger vi i en to-personers husstand effekten af forskelle i løn, forbrugsbehov i enkestand og forventet resterende levetid som enke/enkemand. Forskellene resulterer i meget forskellige livsforsikringssummer for de to personer i husstanden, hvilket giver stof til eftertanke.

Den tredje model, der præsenteres i denne afhandling, er den første af to, hvor vi afviger fra standard forventet nytteteori. Her analyserer vi en særlig modifikation, som indbefatter præferencer både fremad og bagud (i tid). Det fremadskuende element i nyttefunktionen medfører, at sikkerhedsækvivalenten af nytte fra det fremtidige forbrug tages i betragtning ved bestemmelsen af nytte fra det nuværende forbrug. Det bagudskuende element medfører, at tidligere nytte fra forbrug tages i betragtning ved bestemmelse af nytte fra det nuværende forbrug. Alt I alt er den analytiske løsning afledt for modellen meget lig løsningen på den klassiske standard forventet nytte model. Forskellene er kort forklaret i resuméet i afsnit 1.4 og yderligere uddybet i kapitel 4.

De tre første modeller resulterer alle i forbrugsmønstre, der er i overensstemmelse med nedsparring gennem køb af rene Unit-Link livrenter. På grund af det faktum, at livrenter med mindre volatile (over tid) ydelsesstrømme empirisk er et mere populært valg, undersøger vi i den sidste model i denne afhandling meget eksplicitte præferencer for stabilitet i forbruget. Modellen og dens løsning er sammenfattet i afsnit 1.5 og detaljeret gennemgået i kapitel 5. Vi finder, at det optimale forbrugsmønster, der løser modellen, generelt opfyldes af ydelserne fra et bestemt livrente produkt, nemlig Tidspension. Tidspension er også analyseret i kapitel 5 og lighederne mellem det optimale forbrug, som løser modellen, og ydelserne fra Tidspension undersøges.

Afhandlingen afsluttes af et simuleringsstudie, som sammenligner ydelserne fra to rene Unit-Link ratepensioner og en Tidspension ratepension. Fokus i sammenligningen er på år-til-år stabiliteten i ydelserne samt ydelsesniveau og den forventede ydelsesudvikling. Simuleringsstudiet er sammenfattet i afsnit 1.6 og yderligere uddybet i kapitel 6. Fokus i undersøgelsen er ikke på hvilke præferencer, som fører til køb af de forskellige ratepensioner, disse er undersøgt i de tidligere modeller - vi ønsker kun at give stof til eftertanke!

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1. Introduction

The main purpose of this chapter is to provide an overview of the subsequent five chapters of the thesis. We establish a general connection between the topics of each chapter and position them within relevant related literature. Central aspects of the models in each chapter are commented on and results are interpreted across the five self-contained chapters.

1.1 Personal Finance

> *Personal finance refers to the financial decisions which an individual or a family unit is required to make to obtain, budget, save, and spend monetary resources over time, taking into account various financial risks and future life events.*[1]

Personal finance, or household finance, as a concept covers a big variety of interesting topics. The aim and scope of this thesis is an elaboration of models that considers a few of these topics, establishing new knowledge. Of particular interest to us is the area of life insurance and pension decisions. This includes a particular focus on preferences and taxation.

A central starting point in the field of continuous-time models for personal finance is the contributions Merton (1969, 1971). These first generation continuous-time models starts with an investor with the objective, to maximize the time-additive utility from consumption financed by the returns on his investments. The joint consumption and investment strategy that maximizes the investor's expected future utility is referred to as the optimal strategy.

Despite Merton (1969, 1971)'s rather narrow focus on only consumption and investment within the broader definition of personal finance, the fun-

[1] Definition by Wikipedia.com taken on June 2, 2013.

damental method of modeling is repeatedly used and the papers' citations counted in thousands. The methodology allows for straight forward extension to more control processes, including essentially whatever investment vehicle, insurance and housing decision etc. that one can think of.

The models in chapters 2-5 all build on the methodology of Merton (1969, 1971); deciding on optimal controls in a continuous-time setup by considering the expected future. Specifically, chapters 2 and 3 includes taxation and life insurance while chapters 4 and 5 considers intertemporal relations in preferences and consumption. In Chapter 6 we present an analysis of three annuity products that serves the preferences investigated in chapters 2-5.

1.2 Tax-Effects on Optimal Personal Behavior

Chapter 2 is the first of two that builds on the contribution of Richard (1975). In Richard (1975), the optimization problem in Merton (1969, 1971) is extended to include life insurance decisions along with consumption and investment. Here, the possibility of dying before all wealth is consumed is introduced and this way utility from bequest is included in the modeling. The uncertain life-time is modeled, in consistency with general continuous-time life insurance mathematics, by an age-dependent mortality intensity. Being subject to a possible early death and a connected bequest motive, investment in life insurance is introduced in order for the person to determine the optimal heritage to leave for his heirs. The model also includes a deterministic labor income stream to add to the motivation of life insurance purchase in order to protect the financial well being of heirs.

Richard (1975) take as given a set of preferences for the utility from consumption and bequest and a given distribution of remaining life time and derive optimal strategies for consumption, investment and life insurance. Formally, the optimization criteria in Richard (1975) reads

$$\sup \mathbb{E}\left(\int_0^T \mathbb{1}_{\{N_{s-}=0\}} \Big[u(s,c_s)ds + U(s,X_s+S_s)dN_s + \tilde{U}(s,X_s)d\varepsilon_T(s) \Big] \right),$$

where $I = 1 - N$ is the indicator for the person being alive and $\varepsilon_T(\cdot) = \mathbb{1}_{\{T \leq \cdot\}}$. The utility functions u, U and $\tilde{U}$ denotes utility from consumption, bequest and retirement wealth at time of retirement, T, and supremum is taken over consumption, investment and life insurance. The person's wealth follows the dynamics

$$\begin{aligned} dX_t &= rX_t dt + \pi_t(\alpha - r)X_t dt + \pi_t \sigma X_t dW_t \\ &\quad + a_t dt - c_t dt - \mu_t^* S_t dt + S_t dN_t, \\ X_0 &= x_0, \end{aligned}$$

where a is the rate of income, c is the rate of consumption and π is the proportion of wealth invested in the risky asset. The life insurance sum, S, is continuously adjustable and paid for by a natural premium intensity, μ^*.

In the model in Chapter 2 we elaborate on the model in Richard (1975). We include taxation in the financial model, especially focusing on tax-incentives for retirement saving and purchase of life insurance. The taxes considered in the model includes:

- Consumption tax (VAT and other consumption related taxes)
- Labor income tax
- Retirement benefits tax (connected to tax exempted contributions)
- Tax deduction on life insurance premium
- Tax paid on the life insurance sum upon death
- Tax on investment return

We investigate two specific tax-models for retirement savings, where especially one of the models serves a dual purpose. In the first model, retirement savings are tax exempt while benefits are taxed upon payout. When the tax on labor income, and thereby the value of tax exempt contributions, is larger than the tax on benefits, the model includes a financial motivation for retirement saving.

In the second tax-model considered, there is no possibility of tax exempting contributions to retirement savings and also no tax on retirement benefits. Instead, a proportional bonus is added on savings contributions during working life, thereby imposing an extraordinary motivation for saving for the third age. The special case where proportional bonus is set to zero serves as a base-line scenario in comparison of tax motivation of retirement saving, since it includes no extraordinary motivational effects.

Both models are solved in Chapter 2, leading to analytical solutions for the optimal controls. This allows for a simple detailed study of tax effects on optimal behavior under the considered tax-regimes. In general, the following rules are valid under either model:

- Consumption tax does not influence optimal behavior
- Increased tax on stock returns increases investment in stocks
- Increasing tax on the life insurance sum paid out upon death/decreasing tax deduction on the life insurance premium decreases net demand for life insurance

The tax incentives for motivation of retirement savings generally works but does not work equally well. To substantiate this result, a numerical analysis of optimal behavior under the different regimes is presented in Chapter 2. The numerical study is performed for two representative households, one inspired by American and one inspired by Danish tax regulations. Numerous comparative statics are included. A few highlights of the results are:

- Incenting retirement saving by lowering tax on investment return forces contributions to be relatively higher in early working life
- Adding a proportional bonus on retirement savings contributions adds a discontinuity point to optimal consumption at time of retirement
- Taxing retirement benefits (made by tax exempt contributions) lower than labor income adds a discontinuity point to optimal consumption at time of retirement

Based on the results, we conclude that the tax optimal retirement savings vehicle does not have a constant premium rate, nor does the premium rate coincide with the optimal rate derived for a model without taxation. This holds regardless of the proposed tax model. Summing up, the optimal product design depends on the underlying tax model.

As a last exercise in Chapter 2, the expected present value of tax cash flows is calculated for all tax regimes. From a government point of view, this quantity is relevant as a measure of future losses and gains while imposing either of the regimes. Not surprisingly, tax exempting investment returns and contributions to retirement savings is costly on short horizon but leads to higher tax revenues in retirement. Comparison of government value of tax regimes that people are indifferent between, is therefore highly dependent on the discount factor used in calculating the present value.

It is well known, that investment return tax revenues varies very much with financial markets performance and governments are left with a huge hedging exercise (if trying to keep tax-flows perfectly predictable, governments must monitor and react to every investment decision made by the population). An effect of closing down tax exemption of contributions is that the hedging amount shrinks. Also lowering investment return taxation moderates the amount to be hedged by governments. The analysis in Chapter 2 shows that there are reasonable combinations of tax-parameters that leaves populations indifferent between tax-regimes while lowering hedging amounts. For governments, it is a simple exercise to change tax motivation of retirement saving among the investigated regimes; government and population preferences can meet at sane tax levels.

In 2013, Danish tax regulations regarding retirement saving is changed (again!). This time, actually, the system is not worsened. The alteration dictates, that premiums to one of the three most popular savings products is no longer tax-exempt while benefits no longer are taxed. Since previous tax regulations dictated tax exemption worth roughly 37.5% of premiums (labor income tax in the lower tax bracket is roughly 37.5%), while benefits were taxed by 40%, the system is now slighly more attractive from the point of view of the population.

1.3 Interdependent Relationships within a Family

While Chapter 2 considers the interplay between tax rates and personal behavior regarding consumption, investment and life insurance, Chapter 3 has a more endogenous scope. As we saw above, tax regulations have an exogenous influence on preferences for consumption, investment and life insurance and altered both timing of saving/consumption and amounts spend. Chapter 3 develops and solves a model of optimal consumption, investment and life insurance for a multi-person household, thereby providing the opportunity to investigate interdependency among household members.

In the modeling in Chapter 3 we elaborate on the modeling in Richard (1975). We are inspired by Kraft and Steffensen (2008) who generalize the model of Richard (1975) to allow the person to insure himself against financial challenges connected to general changes in his life conditions, e.g. unemployment, disability and so on. State of life for the person is modeled by a continuous-time finite-state Markov chain with deterministic time-dependent intensities. Utility is connected to both consumption in the various life states and wealth and lump sum benefits paid upon transition between states. Furthermore, utility is scaled by different weights for consumption depending on time and life state of the person.

The model in Chapter 3 considers a household of multiple economically and probabilistically dependent person whose aim it is to maximize joint expected future utility from present consumption throughout their entire life time. The present state of the household (who is alive and who is not?) is modeled by a Markov process, Z, which determines the weight on utility from consumption as well as the mortality intensities of living household members. Considering a market where financial investing is exclusively in standard Black-Scholes vehicles (bond and stock) and life insurance is continuously paid for by a natural premium intensity, μ^*, the household faces the optimization criteria

$$\sup \mathbb{E}\Big(\int_0^\infty \sum_{j\in\{0,1\}^n} \mathbb{1}_{\{Z_s=j\}} u^j(s, c_s)ds \Big). \tag{1.1}$$

The optimization is subject to wealth dynamics

$$\begin{aligned} dX_t &= (r + \pi_t(\alpha - r))X_t dt + \pi_t \sigma X_t dW_t \\ &\qquad + a_t^{Z_t} dt - c_t dt + \sum_{j\in\mathcal{Z}_{t-}} S_t^j dM_t^{*Z_{t-}j}, \end{aligned} \tag{1.2}$$

$$X_0 = x_0, \tag{1.3}$$

where M^* is the compensated counting process related to Z and the intensity μ^*. Supremum is taken over life insurance sums on each individual, S, proportion of wealth invested in the stock, π, and consumption, c, and a is the time and state dependent income intensity of the household. We allow for purchase of negative life insurance and interprets this as purchase of life annuities.[2]

The optimal controls for the problem (1.1)-(1.3) is found in Chapter 3 for households consisting of n members, $n \in \mathbb{N}$. Due to the structure of the model we arrive at these controls by means of mathematical induction techniques. The model allows for analytical solutions of all controls, which permits simple comparison across individual household members. Effects of heterogeneity within a household is considered in Chapter 3 for the special case of a two-person household, inspired by the middle aged married couple with grown-up children. The following four cases are specifically investigated, focusing on the over time effects on optimal controls and the related wealth:

- Different labor income of household members until retirement
- Different needs for consumption in widowhood
- Different mortality intensities in widowhood
- No difference between household members

The result of the analysis in Chapter 3 is that any of the differences among spouses leads to difference in the optimal life insurance sums. For the case of difference in labor income, the life insurance sums, though, coincide in retirement, a natural result since life insurance is bought to protect the household against loss of income upon death of a household member. When needs for consumption in widowhood is different, life insurance sums differ since an unequal amount of capital is required upon death of the spouse in order to finance consumption in widowhood. The case of different mortality intensities in widowhood leads to different life insurance sums due to different life expectancy upon death of the spouse; if you expect to live longer, you expect a longer period of time for consumption and thereby require more capital.

In the search for the optimal pension product, the model in Chapter 3

[2]The model is more carefully explained in Chapter 3.

learns us, that dependency within a household plays a major role in the design. This holds in particular for the determination of the date from which on the purchase of life insurance is substituted by purchase of life annuities. A date, which most likely is different for the two household members.

For the work in chapters 2 and 3, a standard power utility function is used for the measurement of utility from consumption at any time. The focus in these chapters is on the internal preferences within a household and tax effects on these preferences and the optimal decisions made. The power utility function has only one parameter that covers both relative risk aversion and elasticity in intertemporal substitution and as we saw, using a time-additive optimization criteria leads to optimal investment and consumption that is proportional to wealth. Especially the proportional consumption is in contradiction with empirical studies, firstly pointed out in the consumption smoothing puzzle of Hansen and Singleton (1983). Chapters 4 and 5 presents optimization criteria that allow for more flexibility in the modeling of utility from consumption.

1.4 Forward and Backward Looking Preferences

The preference structure of Merton's original model for optimal investment and consumption is over the years generalized in certain ways. The papers Epstein and Zin (1989), Duffie and Epstein (1992a,b), Kraft and Seifried (2010) and Kraft et al. (2012) all consider over-time dependency in preferences in a continuous-time setup. They do so by use of recursive utility, i.e. by letting utility of present consumption depend on the certainty equivalent of expected future utility from consumption. The recursive utility modeling allows for separation of risk aversion and elasticity of intertemporal substitution in the utility measurement. This separation is also achieved in Steffensen (2011), where he considers consistency issues in a non-linear optimization criteria involving utility of future consumption.

While recursive utility is forward looking, the papers Sundaresan (1989), Abel (1990), Constantinides (1990) and Munk (2008) introduce backward looking preferences in the form of habit formation. The idea is that the utility you experience from present consumption depends on your own (or a reference groups) previous consumption.

Most literature on continuous-time utility optimization with habit formation models the present habit levels as accumulated (properly discounted) past consumption. This modeling has the undesired feature that a unit increase in present consumption raises the habit stock by one unit but only raises intertemporal utility by the value of felicity, i.e. there is no diminishing marginal in habit as there is in utility. For this reason, our habit modeling in Chapter 4 is inspired by the modeling in Toche (2005), as he models the present habit level as accumulated discounted past utility from consumption.

The model in Chapter 4 incorporates both forward and backward looking preferences, i.e. both recursive utility and habit formation. For the utility measurement we choose again a power utility function, where we, due to the recursive utility modeling, can separate relative risk aversion and elasticity of intertemporal substitution. The habit level at time t is calculated as

$$h_t = \left(\int_0^t \mathrm{e}^{-\int_s^t B d\tau} \left(A \frac{u(c_s, V_s)}{\theta V_s} ds + d\varepsilon_0(s) \right) \right)^{\frac{1}{1-\phi}},$$

where $\varepsilon_t(\cdot) = \mathbb{1}_{\{t \leq \cdot\}}$ and $u(c, V)$ is the forward looking recursive utility functional measuring utility from consumption. We think of the outer transformation, taking everything to the power $\frac{1}{1-\phi}$, as a certainty equivalent with respect to elasticity of intertemporal substitution. The habit level is incorporated in the utility functional in a multiplicative form, such that power utility is derived from the consumption to habit level ratio[3].

The model allows for analytical solutions regarding optimal consumption and investment. The solutions show remarkably conformity with the solutions of Merton (1969, 1971), allowing for easy identification of the effects of including the forward and backward looking preferences. Highlights of the solution's features include

- Investment in the risky asset and consumption is proportional to wealth as in Merton's original work

[3]Further details are provided in Chapter 4.

- The investment proportion is constant and equals the Merton's constant
- The consumption-to-wealth ratio is calculated based on an annuity that includes parameters connected to both the forward and backward looking part of the utility functional
- Optimal investment and consumption is both infinite variation processes in time, as in Merton's original work and the models in chapters 2 and 3.

The similarity with the classical results of Merton (1969, 1971) allows for simple comparison of results of results. The elasticity of intertemporal substitution influence consumption behavior as in Kraft et al. (2012) and Steffensen (2011), who consider the analog problem without habit formation. The inclusion of multiplicative habit driven by past utility adds a parameter of persistency to the optimal consumption, such that increased persistency in the habit formation decreases immediate consumption.

The infinite variation feature of the optimal control processes is classic in time-additive optimal investment and consumption problems. This characteristic arises since the consumer in his attempt to maximize time-additive utility, equals expected marginal utility at all future time-points. In order to do this, he continuously adjusts consumption as a reaction to any non-deterministic investment return. As noted above, the relative volatile consumption pattern that comes from the optimal consumption strategy contradicts empirical consumption data, see Hansen and Singleton (1983). The modeling in Chapter 5 targets this extreme consumption behavior and presents a model that allows for a smoother consumption patterns.

1.5 Further Consumption Smoothing

The model in Chapter 5 builds, as the model in Chapter 4, on the intertemporal relationship in life-time consumption. Where Chapter 4 follows the more traditional approach of incorporating the dependency in the preference structure, Chapter 5 deviates and models dependency directly in consumption dynamics.

The concern of the model is to explain annuity demands of retirees, where the majority purchases annuities of fixed of smooth benefits rather than pure Unit-Link annuities. The optimal consumption derived in either of the models in chapters 2-4 possesses short term variance in the sense that

$$dc_t^* = c_t^*(A(t)dt + B(t)\sigma dW_t),\ c_0^* = c_0,$$

where A and B are deterministic functions, σ is volatility of the risky investment and W the Brownian motion driving the risky investment. This result also holds for the models in Merton (1969, 1971) and Richard (1975). To replicate the optimal consumption stream by an annuity benefit, the solution is in all cases to purchase a continuous-time version of a pure Unit-Link annuity. This product has the feature that benefits are instantly determined as present capital invested divided by a deterministic time dependent function. Furthermore, the optimal investment strategies that we derive in chapters 2-4 are consistent with the more common life cycle investment strategies that normally comes with Unit-Link products.

In Chapter 5, we propose a model where consumers are only concerned with drift and variance in consumption and minimizes deviation in actual consumption away from a prespecified target. Short term variation in consumption, in the sense explained above, it not allowed in the model, meaning that we insist on

$$dc_t = a_t dt,\ c_0 > 0,$$

where c is the instant consumption rate and a is a feed back control process for the problem. The optimization criterion imposed in the model punishes quadratic variation of consumption away from a prespecified target as well as deviation in terminal wealth away from a fixed proportion of terminal consumption. The particular model is motivated by a retiree whose desire it is to consume his savings in a smooth manner before his life ends. We are especially focusing on the case where a positive trend in consumption is asked for, a trend that can not immediately be financed by present wealth and risk-free investment.

This particular modeling incorporates a habit formation directly in consumption dynamics, since the consumption of tomorrow is highly dependent on the consumption of today. The optimal adjustment of consumption, a, that we derive in Chapter 5 has the features, that

- it is linear in the difference between present wealth and the present value of preferred future consumption
- it balances the desire for meeting short term consumption needs and a terminal wealth target
- an attractive investment market attenuates undesired consumption adjustments

Since the model allows for an analytical solution, the above features are directly observable in the optimal control.

The optimal investment strategy derived for the model dictates a CPPI strategy in the sense that risky investments are linear in capital in excess of the present value of preferred future consumption. Especially, risky investment is zero when the present capital equals the present value of preferred future consumption. The proportionally constant is again Merton's constant and cause positive risky investments when present capital cannot finance preferred future consumption by investment in bonds alone.

In the remaining part of Chapter 5, the optimal consumption and investment strategies are compared to the benefit stream and investment profiles of a specific annuity product, Tidspension. Tidspension is a product in the class of formula based smoothed investment-linked annuities. The core concept of Tidspension is that investment returns are smoothed over time, such that volatile investment returns not immediately influence annuity benefits.

The comparison shows that the benefit structure of Tidspension fits the optimal consumption stream demanded in the optimization problem. For the case of no risky investments, we find that a constant factor in the original formulation of Tidspension must be replaced by a time-dependent deterministic function for a perfect fit. A numerical study of the actual product shows that the difference in benefits and demanded consumption is minor for reasonable preference parameters. Tidspension's benefits are not smoothed enough in the beginning but too much towards the end of the annuitization period.

When risky investments are included in the comparison, we find that one more technical modification of Tidspension is needed in order for benefits and optimal consumption to match. The structure of Tidspension is preserved but one more constant parameter must be replaced by a deterministic time-dependent function in order for the match of benefits and consumption to be perfect.

The models in chapters 2-4 lead to optimal consumption streams that possess the characteristics of pure Unit-Link annuities whereas the model in Chapter 5 leads to consumption dynamics in the form of Tidspension's. All models are in continuous-time leading to demand for continuous-time annuities, whereas the typical annuity is a discrete-time product where benefits are annually adjusted. Still, in discrete-time versions, the two types of annuities are very different products. Chapter 6 presents a comparison of two discrete-time pure Unit-Link annuities and a discrete-time Tidspension annuity.

1.6 Annuity Products and Stability in Benefits

An annuity is a financial product that provides a stream of benefits paid for at initiation of the contract by a lump sum. The variety of annuities range from fixed annuities to pure Unit-Link annuities; fixed annuities provide fixed benefit streams, pure Unit-Link annuities provide benefit streams perfectly linked to the investment returns of an underlying portfolio. In between these two products, with profits annuities, Unit-Link annuities with guarantees and Tidspension annuities are more moderate types of annuities.

In Chapter 6, we present a simulation study, where we simulate a financial model in order to compare two different pure Unit-Link annuities with a Tidspension annuity. The three annuities fundamentally differ in their exposure to investment risk and their benefit profiles (how much of the cash balance is paid out at different points in time). The Tidspension annuity also has a build-in mechanism that smoothens investment returns over time. The analysis presented in Chapter 6 answers, to some extent, the following questions:

- How does the expected benefit stream relate to the investment and benefit profiles?

- What is the effect on benefits of the Tidspension smoothing mechanism?
- How does the products perform in a financial crisis investment market?

The focus in Chapter 6 is in the first place on annual investment returns, changes in benefits and variability in these quantities. As expected, more risk taking in investment provides a larger expected return and higher standard deviation which passes on to the annual adjustment of benefits. The smoothing mechanism of Tidspension ensures that standard deviation in annual benefit adjustments is significantly lowered compared to the Unit-Link products, despite of more risk taking in investments. On the other hand, the risk taking in investment is equally paid for in all three products in the sense that accumulated benefits vary perfectly with risk exposure.

Also in a financial crisis situation, the smoothing mechanism of Tidspension stabilizes the year-to-year development in benefits compared to the Unit-Link products. Due to the monthly smoothing of returns, while only annual adjustment of benefits, the timing of financial losses proves important for Tidspension, whereas it is irrelevant for Unit-Link annuities. Having the benefits adjusted just after a major financial loss, the full blow is taken on Unit-Link annuity benefits, while as little as 1.84% of the loss affects the following benefit adjustment in Tidspension. The smoothing mechanism, though, ensures that the full loss effect is only smoothed over the coming years, not avoided.

The analysis in Chapter 6 does not consider the underlying preferences that customers could have for either of the products. As already pinned out, the modeling in chapters 2-4 lead to optimal consumption streams that possess the characteristics of pure Unit-Link annuities and the model in Chapter 5 leads to a consumption pattern best served for by a Tidspension-like annuity. Chapter 6 illustrates the characteristics of the two types of products as a demonstration of the consequence of going with one annuity over the other. We dare not say one product is better than the other, only that they serve different household demands.

Preferences are very diverse among households and a great variety of annuities serves different needs. Definitely, the more flexibility incorporated in the annuity, the more preferences are covered.

2. Consumption, Investment and Life Insurance under Different Tax Regimes

Kenneth Bruhn, Annals of Actuarial Science, available at `http://journals.cambridge.org/action/displayAbstract?fromPage=online&aid=8771898`.

Abstract We study the effects of introducing taxation in classical continuous-time optimization problems with utility from consumption, bequest and retirement savings. Inspired by actual tax favoured retirement savings programs, we formulate and solve the optimization problem for various tax regimes, and compare tax effects on consumption/savings contributions, investment and purchase of life insurance under the regimes. The optimization problems have analytical solutions, which allow for easy comparison of tax effects under the different regimes. To substantiate the results we also present a numerical analysis of the results based on realistic parameter values and regimes. Based on American and Danish tax regimes we estimate the values of existing retirement saving favouring to be $1-2$ percent of lifetime income.

Keywords *Personal finance; Stochastic control; Power utility; Linear taxation*

2.1 Introduction

Increasing human life time has put great pressure on public old age pension systems in many countries. Governments face a challenge of inspiring citizens to save for retirement, thereby reducing longevity issues for the national economies. As a consequence, economically attractive retirement savings programs have been introduced. The most characteristic features of these programs are (1) reduced investment return tax on retirement savings and (2) exempt labour income tax on contributions to retirement savings, where benefits are subsequently taxed at a lower tax rate. The latter feature is often criticized for favouring people with high labour income (due to progressive income taxation, see Gale et al., 2006), and for being excessively costly for governments (costs for various OECD countries are estimated in Antolín et al., 2004, Yoo and de Serres, 2004 and Caminada and Goudswaard, 2004). For these reasons, substituting tax exempt contributions with bonuses on contributions is proposed in Gale et al. (2006), and tax regimes with this feature are now introduced (e.g. the German Riester scheme has this feature, see Börsch-Supan et al., 2008 and Corneo et al., 2008).

Tax treatment of life insurance comes in several varieties. Even within a country there are different schemes allowing for different tax treatment of premiums and benefits (sum paid out upon death). In general, premiums are either paid by income-taxed money and benefits are tax free, or premiums are exempt from labour income tax and the benefits taxed. For the latter case, the benefits taxation is typically done by a tax rate that depends on the income of the inheritor, which potentially leads to tax favouring or even disfavouring of life insurance. Several studies of the relation between bequest motives, tax incentives and life insurance purchase has been carried out, this list is not complete: Sauter et al. (2010) provide a study of tax incentives and bequest motives on demand for life insurance, based on data from Germany. Sweeting (2009) investigates the tax treatment of pensions and saving incentives in the UK. Jappelli and Pistaferri (2003) analyze data on the tax treatment of life insurance and the introduction of incentives for life insurance in Italy. Finally, Bernheim (1991) presents empirical evidence that savings are motivated by a desire to leave bequests.

In this paper we consider a model for decision of optimal consumption, investment and life insurance purchase for an individual that is subject to different tax regimes. The model is based on Richard (1975), who considers the same problem without taxation[1]. For the numerical investigation

[1]As is done in Pliska and Ye (2007), we consider the case where the life time is

performed here, we parameterize our model in terms of a household model in the notion of Bruhn and Steffensen (2011). They develop a model for optimal consumption, investment and life insurance purchase for a general household consisting of multiple members. A similar model is developed in Huang and Milevsky (2008) for a two-person household with stochastic income, and Kwak et al. (2011) for parents with children, with separate risk preferences of parents and children.

Compared to Richard (1975), our model extensions addresses the introduction of relevant taxes and tax regimes. The taxes introduced are on consumption, investment returns, labour income, retirement benefits (with a related tax exemption of contributions) and life insurance (with a related tax deduction on premiums)[2]. We model taxation of investment returns as symmetric non-progressive mark-to-market. In general, non-progressive mark-to-market taxation of investment returns is the most common for retirement savings with reduced investment return taxation. For non-favoured savings, most countries have deferred capital gains taxation, but we omit this feature in our models[3]. The symmetric assumption on the tax on investment returns is not entirely realistic. In reality, most countries do not offer an immediate tax refund of capital losses. Instead they allow for building of a negative tax reserve that is later deductible from tax on capital gains. We make the simplifying assumption of symmetric taxation of investment returns in order to allow for tractable analytical solutions.

The contribution of this paper is the following: We formulate and solve the problem of optimal consumption, investment and purchase of life insurance under two different tax regimes, (1) immediate taxation of all labour income and bonus on savings contributions and (2) tax exempted contributions to retirement savings. When bonuses are set equal to zero, the first

random and unbounded, whereas Richard (1975) has a bounded distribution of life time.

[2]Any tax/tax exemption arising from housing costs and mortgage are omitted in the modeling in this paper, and we refer to Amromin et al. (2007) for comments on that.

[3]The papers of Dammon et al. (n.d.), Seifried (2010) and Kraft et al. (2010) among others investigate deferred capital gains taxation and Kraft et al. (2010) estimates the utility loss from assuming mark-to-market taxes instead of deferred taxes in investment decisions is at most 0.5% of present financial wealth and life time income.

regime also serves as a non-favoured regime. The optimization problem has explicit solutions for both regimes, which allows for explicit analysis of the tax effects on the optimal controls in both cases. For realistic parameterizations of the model (where taxes are inspired by the US and Danish tax rates), we find the tax rates which make an 'average' person/household indifferent between saving under the different regimes. We further investigate the expected optimal behaviour of the person over time under these indifference-regimes. Finally, we compute the expected present value of future tax incomes and expenditures for a government under the different regimes.

This paper proceeds as follows: In Section 2.2 we present and solve the classical optimization problem originally presented in Richard (1975). In Section 2.3 we present the two different tax regimes and solve the optimization problems related to them, and in Section 2.4 we present a numerical investigation of the results. The numerical investigation considers the values of different tax regimes (both for the person/household facing the optimization problem and for the tax authorities), and the expected behaviour of the person/household under these regimes. Section 2.5 concludes.

2.2 Classical Results

In this section we reproduce the classical results on optimal consumption, investment and purchase of life insurance. We need these results for comparison of tax effects in Section 2.3.

Classical continuous time utility optimization is formalized in Richard (1975) as the problem of optimizing expected future utility. The utility stems from consumption, bequest and a terminal utility from having wealth left at a specified future point in time. Here we think of this point in time as the time of the optimizer's retirement.

For a mathematical formulation of the problem we let

$$N = (N_t)_{t \geq 0},$$

be the indicator process for the person being alive. Thereby N takes values in $\{0,1\}$, such that $N_t = 0$ corresponds to the person being alive at time t and $N_t = 1$ corresponds to the person being dead.

The person has access to an investment market and a life insurance market. We model a Black-Scholes investment market that consists of a risk free asset, Z^1, and a risky asset, Z^2, with dynamics

$$\begin{aligned} dZ_t^1 &= rZ_t^1 dt,\ Z_0^1 = z^1 > 0, \\ dZ_t^2 &= \alpha Z_t^2 dt + \sigma Z_t^2 dW_t,\ Z_0^2 = z^2 > 0, \end{aligned}$$

where W is a standard Brownian motion. The results that we derive in this paper can be generalized to more advanced investment market models[4]. Since we are mainly concerned with the savings/consumption behaviour of the person, this simple market is sufficient for our analysis.

The processes N and W are assumed to be independent. We define them on the measurable space $(\Omega, \mathcal{F})$, where $\mathcal{F}$ is the natural filtration of (N, W). On $(\Omega, \mathcal{F})$ we define the equivalent probability measures $\mathbb{P}$ and $\mathbb{P}^*$. We refer to $\mathbb{P}$ as the objective measure and $\mathbb{P}^*$ as the pricing measure. The pricing measure is used for pricing both market risk (W) and life insurance risk (N) by the insurance company.

As in e.g. Richard (1975), Bruhn and Steffensen (2011) among others, we assume that N has intensity μ under $\mathbb{P}$ and μ^* under $\mathbb{P}^*$, and refer to them as the objective mortality intensity and the pricing intensity.

In the life insurance market the person buys life insurance with a sum insured at time t, S_t, and for that coverage he pays premium at the rate $\mu_t^* S_t$, where μ^* is the natural premium intensity decided by the life insurance company. Any premium loading that the company demands to cover general expenses for the contract is included in the pricing intensity.

Based on the introduced investment and life insurance market, the wealth

[4]In general, complete markets are needed in order to obtain analytical solutions to the optimization problems in this paper. Stochastic interest rate, stochastic excess-return on the stock etc. are straight forward generalizations of the model, see e.g. Munk et al. (2004)

process, X, follows the dynamics

$$\begin{aligned} dX_t = rX_t dt + \pi_t(\alpha - r)X_t dt + \pi_t \sigma X_t dW_t \\ + a_t dt - c_t dt - \mu_t^* S_t dt + S_t dN_t, \end{aligned} \tag{2.1}$$

$$X_0 = x_0, \tag{2.2}$$

where a is the rate of income, c is the rate of consumption and π is the proportion of wealth invested in the risky asset. The fraction of income that is not immediately consumed, $a - c$, is the savings premium of the person, which is paid into a savings account in a financial institution. The life insurance sum, S, is continuously adjustable and paid for by a natural premium intensity. The premium is paid out of the savings account, and we note that this savings vehicle replicates a Variable Universal Life Insurance[5]. This type of contract is widely sold in the US and in many European countries, though under different names.

Given the dynamics of the wealth process, X, the classical utility optimization problem is mathematically formulated as

$$\begin{aligned} &\sup_{c,\pi,S} \mathbb{E}_{0,x_0}\left(\int_0^T \mathbb{1}_{\{N_{s-}=0\}} \Big[u(s, c_s) ds + U(s, X_s + S_s) dN_s \right. \\ &\left. + \tilde{U}(s, X_s) d\varepsilon_T(s) \Big] \right) \\ = &\sup_{c,\pi,S} \mathbb{E}_{0,x_0}\left(\int_0^T e^{-\int_0^s \mu_\tau d\tau} \Big[\left(u(s, c_s) + \mu_s U(s, X_s + S_s) \right) ds \right. \\ &\left. + \tilde{U}(s, X_s) d\varepsilon_T(s) \Big] \right), \end{aligned}$$

where $\varepsilon_T(\cdot) = \mathbb{1}_{\{T \leq \cdot\}}$ and $\mathbb{E}_{t,x}$ is the conditional expectation under $\mathbb{P}$, given that the person is alive at time t and holds wealth $X_t = x$. The

[5]Instead of saying that the insurance premium is paid out of the savings account, we could say that it is paid out of the savings premium. Thereby $a_t - c_t - \mu_t^* S_t$ is the savings premium and $\mu_t^* S_t$ is the insurance premium paid out of the salary a_t. The wealth dynamics are identical under both interpretations, and we choose the first for simpler introduction of tax exempted premiums in Section 2.3.

functions u, U and $\tilde{U}$ denote utility of consumption, bequest and terminal wealth.

For the remainder of this paper we work with power utility with deterministic and time dependent weights. This type of utility is characterized by a constant relative risk aversion, which in our parametrization is $1-\gamma$, and constant elasticity of intertemporal substitution (EIS) which is $(1-\gamma)^{-1}$. We parameterize the utility functions as

$$\begin{aligned} u(t,c) &= \tfrac{1}{\gamma} w^{1-\gamma}(t) c^{\gamma}, \\ U(t,x) &= \tfrac{1}{\gamma} F^{1-\gamma}(t)(x+G(t))^{\gamma}, \\ \tilde{U}(t,x) &= \tfrac{1}{\gamma} \tilde{F}^{1-\gamma}(t)(x+\tilde{G}(t))^{\gamma}, \end{aligned}$$

with $\gamma \in (-\infty, 1] \backslash \{0\}$, $t \geq 0$ and w, F, $\tilde{F}$, G and $\tilde{G}$ being the deterministic time dependent weights. The case $\gamma = 0$ corresponds to logarithmic utility since $\lim_{\gamma \to 0}(c^{\gamma}-1)/\gamma = \ln(c)$. This particular case of unit relative risk aversion and EIS will not be dealt with explicitly in this paper[6].

The form of the utility functions regarding bequest and retirement savings, U and $\tilde{U}$, is highly inspired by the results of Bruhn and Steffensen (2011). For now we think of G and $\tilde{G}$ as measuring a financial value of future expected income of the inheritor and a financial value of public retirement payments for the person. The functions w, F and $\tilde{F}$ represent the individual's relative weights for the three different sources of utility (consumption, bequest and retirement savings). Note, that since F and G are deterministic functions, they can not capture a sudden change in the bequest motive at a future point in time, e.g. in case of death of the inheritor. We disregard this possibility in the models, as it is common in related literature. For a model with possible early death of the inheritor (spouse) see Bruhn and Steffensen (2011).

Based on the power utility functions, the optimal value function for the

[6]The optimal controls we derive in the following for $\gamma \in (-\infty, 1] \setminus \{0\}$ are in general also valid for $\gamma = 0$, even though the derivation for $\gamma = 0$ is different.

classical optimization problem is

$$
\begin{aligned}
V(t,x) = \sup_{c,\pi,S} \mathbb{E}_{t,x}\Bigg(\int_t^T & \mathrm{e}^{-\int_t^s \mu_\tau d\tau} \Big[\Big(\tfrac{1}{\gamma} w^{1-\gamma}(s) c_s^\gamma \\
& + \mu_s \tfrac{1}{\gamma} F^{1-\gamma}(s)(X_s + G(s) + S_s)^\gamma \Big) ds \\
& + \tfrac{1}{\gamma} \tilde{F}^{1-\gamma}(s)(X_s + \tilde{G}(s))^\gamma d\varepsilon_T(s) \Big] \Bigg),
\end{aligned} \tag{2.3}
$$

where X follows the dynamics (2.1)-(2.2). We solve this stochastic optimization problems via the Hamilton-Jacobi-Bellman (HJB) equation. For the problem described by (2.1)-(2.3), the HJB-equation is

$$
\begin{aligned}
& V_t + \sup_{c,\pi,S} \Big[\tfrac{1}{\gamma} w^{1-\gamma} c^\gamma + \mu(\tfrac{1}{\gamma} F^{1-\gamma}\,(x + G + S)^\gamma - V) \\
& + \big[rx + \pi(\alpha - r)x + a - c - \mu^* S \big] V_x + \tfrac{1}{2}\pi^2\sigma^2 x^2 V_{xx} \Big] = 0, \\
& V(T,x) = \tfrac{1}{\gamma} \tilde{F}^{1-\gamma}(T)(x + \tilde{G}(T))^\gamma.
\end{aligned}
$$

The problem is solved in e.g. Richard (1975), and in our parametrization the solution is

$$
V(t,x) = \tfrac{1}{\gamma} f^{1-\gamma}(t)(x + g(t))^\gamma,
$$

where

$$
\begin{aligned}
f(t) = & \int_t^T \mathrm{e}^{-\frac{1}{1-\gamma}\int_t^s \mu_\tau - \gamma(\mu_\tau^* + \varphi) d\tau} \\
& \quad + [(w(s) + \left(\frac{\mu_s}{\mu_s^{*\gamma}} \right)^{\frac{1}{1-\gamma}} F(s)) ds + \tilde{F}(T) d\varepsilon_T(s)], \\
g(t) = & \int_t^T \mathrm{e}^{-\int_t^s r + \mu_\tau^* d\tau} [(a_s + \mu_s^* G(s)) ds + \tilde{G}(T) d\varepsilon_T(s)],
\end{aligned}
$$

with

$$
\varphi = r + \frac{(\alpha - r)^2}{2\sigma^2(1-\gamma)}. \tag{2.4}
$$

As in Kraft and Steffensen (2008) we introduce the mortality intensity

$$\mu' = \left(\frac{\mu}{\mu^{*\gamma}}\right)^{\frac{1}{1-\gamma}},$$

and the adjusted interest rate

$$r' = -\tfrac{1}{1-\gamma}(\mu - \gamma(\mu^* + \varphi)) - \mu'.$$

Letting N have intensity μ' under the probability measure $\mathbb{P}'$, f and g have the following Feynman-Kač representations:

$$\begin{aligned} f(t) = \mathbb{E}'\Big(\int_t^T & \mathrm{e}^{-\int_t^s r'_\tau d\tau} \mathbb{1}_{\{\mathrm{N_{s-}}=0\}} \\ & + [w(s)ds + F(s)dN_s + \tilde{F}(s)d\varepsilon_T(s)] \Big| N_t = 0\Big), \\ g(t) = \mathbb{E}^*\Big(\int_t^T & \mathrm{e}^{-\int_t^s r_\tau d\tau} \mathbb{1}_{\{\mathrm{N_{s-}}=0\}} \\ & + [a(s)ds + G(s)dN_s + \tilde{G}(s)d\varepsilon_T(s)] \Big| N_t = 0\Big). \end{aligned}$$

We see that f has the interpretation of an expected present value of the future utility weights. Similarly, g is expected present value of future labour income and public pension for the person, where it also, through G, takes into account the human wealth of the inheritor. We refer to g as the human wealth of the person and $x + g$ as the total wealth of the person.

Optimal Controls

The optimal controls in this classical model without taxes (see also Richard, 1975, Kraft and Steffensen, 2008 among others) are

$$c_t^* = \frac{w(t)}{f(t)}(X_t + g(t)), \tag{2.5}$$

$$\pi_t^* = \frac{\alpha - r}{\sigma^2(1-\gamma)} \frac{X_t + g(t)}{X_t}, \tag{2.6}$$

$$S_t^* = \left(\frac{\mu_t}{\mu_t^*}\right)^{\frac{1}{1-\gamma}} \frac{F(t)}{f(t)}(X_t + g(t)) - (X_t + G(t)). \tag{2.7}$$

Based on the interpretations of f and g, the optimal controls have the following interpretations: The consumption is a fraction of the total wealth of the person, where the fraction is the weight for immediate consumption relative to expected present value of future utility weights. The investment strategy dictates that a constant fraction of total wealth relative to present wealth must be invested in the risky asset, and the optimal life insurance sum is found from weighting human wealth of the person relative to human wealth of the inheritor.

2.3 Models with Taxes

In this section we present the two models including tax, and solve the related optimization problems. Furthermore we comment on tax effects on the optimal controls.

Taxes

We restrict our modelling to constant relative taxes. Thus, the relation between gross (before tax) values and net (after tax) values are $\bar{a}(1-\tau) = a$, where τ is the proportional tax rate, $\bar{a}$ the gross value and a the net value. As a general rule, gross values are represented by barred variables, while the corresponding net values are without bars.

In general we introduce the following tax parameters:

- τ_C for the consumption tax (VAT and other consumption taxes)
- τ_L for the labour income tax
- τ_B for the retirement benefits tax (when contributions are tax exempted)
- τ_I for the tax deduction on life insurance premium
- τ_D for the tax paid on the life insurance sum upon death
- τ_1 for the tax on return from the risk free asset
- τ_2 for the tax on return from the risky asset

2.3.1 Immediate Labour Income Taxation with Bonus on Contributions

First we focus on the situation where all labour income is taxed immediately at pay-day, and a proportional bonus is added to the savings contributions. We assume that benefits from retirement savings are not subject to any tax, and that investment returns are taxed immediately upon realization, regardless of whether they are positive or negative.

In this case the wealth, X, follows the dynamics

$$\begin{aligned}
dX_t &= \bar{r}(1-\tau_1)X_t dt + \pi_t(\bar{\alpha}(1-\tau_2) - \bar{r}(1-\tau_1))X_t dt + \pi_t\bar{\sigma}(1-\tau_2)X_t dW_t \\
&\quad + \bar{a}_t(1-\tau_L)(1+\beta)dt - \bar{c}_t(1+\beta)dt - \mu_t^*\bar{S}_t(1-\tau_I)dt \\
&\quad + \bar{S}_t(1-\tau_D)dN_t \\
&= rX_t dt + \pi_t(\alpha - r)X_t dt + \pi_t\sigma X_t dW_t \\
&\quad + a_t(1+\beta)dt - \frac{c_t}{1-\tau_C}(1+\beta)dt - \mu_t^* S_t \tfrac{1-\tau_I}{1-\tau_D} dt + S_t dN_t, \qquad (2.8) \\
X_0 &= x_0. \qquad (2.9)
\end{aligned}$$

Here $\bar{a}$ denotes gross income, $\bar{c}$ is the gross consumption (before VAT and other consumption taxes), and $\bar{S}$ is the gross life insurance sum, while a, c and S are the corresponding net values. Analogously, $\bar{r}$, $\bar{\alpha}$ and $\bar{\sigma}$ are gross-return parameters of the investment market, and r, α and σ are the corresponding net values. The proportional bonus received on the savings contributions is given by β, and the special case $\beta = 0$ corresponds to non-favoured savings.

Under this regime where savings contributions are made after taxation of all labour income, we assume that no taxation of savings takes place upon death. The accumulated contributions bonus is in particular not paid back. Since we model utility from net consumption, c, and the net life insurance sum, S, the optimal value function for this problem with immediate taxation of all labour income is

$$
\begin{aligned}
V^{\lambda}(t,x) = \sup_{c,\pi,S} \mathbb{E}_{t,x}\Bigg(\int_t^T & \mathrm{e}^{-\int_t^s \mu_\tau d\tau} \Big[\Big(\tfrac{1}{\gamma} w^{1-\gamma}(s) c_s^{\gamma} \\
& + \mu_s \tfrac{1}{\gamma} F^{1-\gamma}(s)((1-\tau_C)(X_s + G(s) + S_s))^{\gamma}\Big) ds \qquad (2.10) \\
& + \tfrac{1}{\gamma} \tilde{F}^{1-\gamma}(s)((1-\tau_C)(X_s + \tilde{G}(s)))^{\gamma} d\varepsilon_T(s)\Big]\Bigg).
\end{aligned}
$$

Here G and $\tilde{G}$ measure the financial values of the *net* future income of the inheritor and *net* public pension for the person. Note that utility from bequest and from retirement savings is adjusted for consumption tax, since the amounts left for the inheritor and at the time of retirement are (sooner or later) used for consumption.

The HJB-equation for the problem described by (2.8)-(2.10) is

$$
\begin{aligned}
& V_t^{\lambda} + \sup_{c,\pi,S} \Big[\tfrac{1}{\gamma} w^{1-\gamma} c^{\gamma} + \mu(\tfrac{1}{\gamma} F^{1-\gamma}\, ((1-\tau_C)(x+G+S))^{\gamma} - V^{\lambda}) \\
& + \big[rx + \pi(\alpha - r)x + a(1+\beta) - \tfrac{c}{1-\tau_C}(1+\beta) - \mu^* S \tfrac{1-\tau_I}{1-\tau_D} \big] V_x^{\lambda} \\
& + \tfrac{1}{2}\pi^2\sigma^2 x^2 V_{xx}^{\lambda} \Big] = 0, \\
& V^{\lambda}(T,x) = \tfrac{1}{\gamma} \tilde{F}^{1-\gamma}(T)((1-\tau_C)(x + \tilde{G}(T)))^{\gamma},
\end{aligned}
$$

which is solved by

$$
V^{\lambda}(t,x) = \tfrac{1}{\gamma} f_{\lambda}^{1-\gamma}(t)((1-\tau_C)(x + g_{\lambda}(t)))^{\gamma}, \qquad (2.11)
$$

with

$$
\begin{aligned}
f_\lambda(t) = & \int_t^T \mathrm{e}^{-\frac{1}{1-\gamma}\int_t^s \mu_\tau - \gamma(\hat{\mu}_\tau+\varphi)d\tau}[(w(s)(1+\beta)^{-\frac{\gamma}{1-\gamma}} \\
& + \left(\frac{\mu}{\hat{\mu}^\gamma}\right)^{\frac{1}{1-\gamma}} F(s))ds + \tilde{F}(T)d\varepsilon_T(s)] \\
= & \,\mathbb{E}^\lambda\Big(\int_t^T \mathrm{e}^{-\int_t^s r_\tau^\lambda d\tau}\mathbb{1}_{\{\mathrm{N_{s-}=0}\}}[w(s)(1+\beta)^{-\frac{\gamma}{1-\gamma}}ds \\
& + F(s)dN_s + \tilde{F}(s)d\varepsilon_T(s)]\Big| N_t = 0\Big), \\
g_\lambda(t) = & \int_t^T \mathrm{e}^{-\int_t^s r+\hat{\mu}_\tau d\tau}[(a_s(1+\beta)+\hat{\mu}_s G(s))ds + \tilde{G}(T)d\varepsilon_T(s)] \\
= & \,\hat{\mathbb{E}}\Big(\int_t^T \mathrm{e}^{-\int_t^s r_\tau d\tau}\mathbb{1}_{\{\mathrm{N_{s-}=0}\}}[a_s(1+\beta)ds + G(s)dN_s \\
& + \tilde{G}(s)d\varepsilon_T(s)]\Big| N_t = 0\Big).
\end{aligned}
$$

Here, φ given by (2.4) and

$$
\hat{\mu} = \mu^* \tfrac{1-\tau_I}{1-\tau_D}, \tag{2.12}
$$

$$
\mu^\lambda = \left(\frac{\mu}{\hat{\mu}^\gamma}\right)^{\frac{1}{1-\gamma}},
$$

$$
r^\lambda = -\tfrac{1}{1-\gamma}(\mu - \gamma(\hat{\mu}+\varphi)) - \mu^\lambda. \tag{2.13}
$$

where for the Feynman-Kač representations, N has intensity μ^λ under $\mathbb{P}^\lambda$ and $\hat{\mu}$ under $\hat{\mathbb{P}}$.

The function f_λ is an expected value of the future utility weights. The function g_λ is expected present value of future net labour income including bonus and public pension for the person, where it also takes into account the human wealth of the inheritor. The expected values are calculated under different measures compared to the classical case, due to the fact that the pricing mortality intensity μ^* is tilted with the tax/tax deduction on life insurance/life insurance premium.

The bonus parameters influence the person's willingness to postpone consumption from the savings period to the retirement period, by changing the marginal utility of gross consumption until retirement. In f_λ, the bonus parameter therefore only affects the weight on the person's utility from consumption, w, relative to the weight on bequest and retirement savings. For a more risk averse person (low EIS), $\gamma < 0$, the weight on consumption is increasing in β, and vice versa for a less risk averse person (high EIS). The human capital, g_λ, is increasing in β and the contribution to human capital from income increases relative to the contribution from the two other sources (human capital of the inheritor and public pension).

Contributions to the savings account are made after labour income tax is paid, so that the present wealth, x, can be thought of as a net value. Since g_λ measures the net human capital (expected value of future net income including bonus), we refer to $x + g_\lambda$ as net total wealth.

Optimal Controls

The optimal controls are

$$
\begin{aligned}
c_t^* &= \frac{w(t)}{f_\lambda(t)} \frac{1-\tau_C}{(1+\beta)^{\frac{1}{1-\gamma}}} (X_t + g_\lambda(t)), \\
\pi_t^* &= \frac{\alpha - r}{\sigma^2(1-\gamma)} \frac{X_t + g_\lambda(t)}{X_t} = \frac{\bar{\alpha}(1-\tau_2) - \bar{r}(1-\tau_1)}{\bar{\sigma}^2(1-\tau_2)^2(1-\gamma)} \frac{X_t + g_\lambda(t)}{X_t}, \\
S_t^* &= \left(\frac{\mu_t}{\hat{\mu}_t}\right)^{\frac{1}{1-\gamma}} \frac{F(t)}{f_\lambda(t)} (X_t + g_\lambda(t)) - (X_t + G(t)).
\end{aligned}
$$

The optimal consumption, c^*, is the only control involving the consumption tax, τ_C, and it is linear in it. Especially we find that the optimal gross consumption

$$
\bar{c}_t^* = \frac{c_t^*}{1-\tau_C} = \frac{w(t)}{f_\lambda(t)} \frac{X_t + g_\lambda(t)}{(1+\beta)^{\frac{1}{1-\gamma}}},
$$

is independent of the consumption tax, which in particular means that the optimal savings ratio is independent of the consumption tax. Taxes relating to the investment and life insurance market affect the values of f_λ and g_λ and thus influence both gross and net consumption.

The optimal investment proportion and net life insurance sum have the same form as in the classical case, except that all investment and life insurance market parameters are tilted with their corresponding taxes. The investment proportion is calculated as a constant fraction of net total wealth relative to net present wealth, and the tax parameter for the returns on the risky asset, τ_2, is squared in the nominator of the fraction. The amount invested in the risky asset is therefore highly dependent on the investment return taxes, and especially on τ_2, such that higher investment return taxes lead to more risky investments. Since the human capital is increasing in β, a higher proportional bonus leads to more risky investments.

The net life insurance sum is found by weighting net human capital of the person against net human capital of the inheritor. The weighting explicitly takes the taxes and tax deductions related to life insurance into account.

2.3.2 Deferred Labour Income Taxation of Contributions

For the optimization problem under this second tax regime, we assume that contributions to retirement savings are exempt from immediate labour income taxation. Instead, benefits are subject to taxation upon withdrawal. Beside this change, the person is subject to the same taxes as under the previous regime.

Until retirement, the person's savings evolve according to the dynamics

$$\begin{aligned}
dX_t &= \bar{r}(1-\tau_1)X_t dt + \pi_t(\bar{\alpha}(1-\tau_2) - \bar{r}(1-\tau_1))X_t dt + \pi_t \bar{\sigma}(1-\tau_2)X_t dW_t \\
&\quad + \bar{a}_t dt - \tfrac{\bar{c}_t}{1-\tau_L} dt - \mu_t^* \bar{S}_t \tfrac{1-\tau_I}{1-\tau_L} dt + \bar{S}_t(1-\tau_D) dN_t \\
&= r X_t dt + \pi_t(\alpha - r)X_t dt + \pi_t \sigma X_t dW_t \\
&\quad + \bar{a}_t dt - \tfrac{c_t}{(1-\tau_L)(1-\tau_C)} dt - \mu_t^* \tfrac{S_t}{1-\tau_D} \tfrac{1-\tau_I}{1-\tau_L} dt + S_t dN_t, \qquad (2.14)\\
X_0 &= x_0. \qquad (2.15)
\end{aligned}$$

Note that the life insurance premium is paid out of the savings account, where contributions are exempt from labour income tax. Therefore the life insurance premium is subject to labour income tax before tax deduction by τ_I (an appealing special case is $\tau_I = \tau_L$).

The optimal value function for this problem concerning retirement saving with deferred labour income taxation of contributions is

$$
\begin{aligned}
V^\delta(t,x) = \sup_{c,\pi,S} \mathbb{E}_{t,x}\Bigg(\int_t^T & \mathrm{e}^{-\int_t^s \mu_\tau d\tau} \Big[\Big(\tfrac{1}{\gamma} w^{1-\gamma}(s) c_s^\gamma \\
& + \mu_s \tfrac{1}{\gamma} F^{1-\gamma}(s)((1-\tau_C)(X_s(1-\tau_D) + G(s) + S_s))^\gamma \Big) ds \\
& + \tfrac{1}{\gamma} \tilde{F}^{1-\gamma}(s) \left((1-\tau_C)(X_s(1-\tau_B) + \tilde{G}(s)) \right)^\gamma d\varepsilon_T(s) \Big] \Bigg).
\end{aligned}
\tag{2.16}
$$

As above G and $\tilde{G}$ measure the financial values of the inheritor's expected *net* lifetime income and the *net* public pension payments during retirement for the person. Compared to the optimization problem under the previous tax regime, the retirement savings are subject to taxation by τ_D upon death of the person or τ_B upon withdrawal at retirement. This feature is written directly in the utility from bequest and retirement savings in (2.16), and that enables us to use the same weight functions, w, F, G, $\tilde{F}$ and $\tilde{G}$, as under the previous regime.

The HJB-equation for the problem described by (2.14)-(2.16) is

$$
\begin{aligned}
& V_t^\delta + \sup_{c,\pi,S} \Big[\tfrac{1}{\gamma} w^{1-\gamma} c^\gamma + \mu(\tfrac{1}{\gamma} F^{1-\gamma} \left((1-\tau_C)((1-\tau_D)x + G + S)\right)^\gamma - V^\delta) \\
& + \left[rx + \pi(\alpha - r)x + \bar{a} - \tfrac{c}{(1-\tau_L)(1-\tau_C)} - \mu^* \tfrac{S}{1-\tau_D} \tfrac{1-\tau_I}{1-\tau_L} \right] V_x^\delta \\
& + \tfrac{1}{2} \pi^2 \sigma^2 x^2 V_{xx}^\delta \Big] = 0, \\
& V^\delta(T,x) = \tfrac{1}{\gamma} F^{1-\gamma}(T) \left((1-\tau_C)((1-\tau_B)x + \tilde{G}(T)) \right)^\gamma,
\end{aligned}
$$

and the solution to the equation is

$$
V^\delta(t,x) = \tfrac{1}{\gamma} f_\delta^{1-\gamma}(t)((1-\tau_C)(1-\tau_L)(x + g_\delta(t)))^\gamma, \tag{2.17}
$$

where

$$
\begin{aligned}
f_\delta(t) = & \int_t^T \mathrm{e}^{-\frac{1}{1-\gamma}\int_t^s \mu_\tau - \gamma(\check{\mu}_\tau + \varphi)d\tau}[(w(s) + \left(\frac{\mu_s}{\check{\mu}_s^\gamma}\right)^{\frac{1}{1-\gamma}} \left(\frac{1-\tau_D}{1-\tau_L}\right)^{\frac{\gamma}{1-\gamma}} F(s))ds \\
& + \left(\frac{1-\tau_B}{1-\tau_L}\right)^{\frac{\gamma}{1-\gamma}} \tilde{F}(T)d\varepsilon_T(s)] \\
= & \mathbb{E}^\delta\Big(\int_t^T \mathrm{e}^{-\int_t^s r_\tau^\delta d\tau}\mathbb{1}_{\{\mathrm{N_{s-}}=0\}}[w(s)ds + \left(\frac{1-\tau_D}{1-\tau_L}\right)^{\frac{\gamma}{1-\gamma}} F(s)dN_s \\
& + \left(\frac{1-\tau_B}{1-\tau_L}\right)^{\frac{\gamma}{1-\gamma}} \tilde{F}(s)d\varepsilon_T(s)]\Big| N_t = 0\Big), \\
g_\delta(t) = & \int_t^T \mathrm{e}^{-\int_t^s r + \check{\mu}_\tau d\tau}[(\bar{a}_s + \check{\mu}_s \frac{G(s)}{1-\tau_D})ds + \frac{\tilde{G}(T)}{1-\tau_B} d\varepsilon_T(s)] \\
= & \check{\mathbb{E}}\Big(\int_t^T \mathrm{e}^{-\int_t^s r d\tau}\mathbb{1}_{\{\mathrm{N_{s-}}=0\}}[\bar{a}(s)ds + \frac{G(s)}{1-\tau_D}dN_s \\
& + \frac{\tilde{G}(s)}{1-\tau_B}d\varepsilon_T(s)]\Big| N_t = 0\Big).
\end{aligned}
$$

Here, φ is given by (2.4) and

$$
\begin{aligned}
\check{\mu} &= \mu^* \frac{1-\tau_I}{1-\tau_L}, \\
\mu^\delta &= \left(\frac{\mu}{\check{\mu}^\gamma}\right)^{\frac{1}{1-\gamma}}, \\
r^\delta &= -\tfrac{1}{1-\gamma}(\mu - \gamma(\check{\mu} + \varphi)) - \mu^\delta,
\end{aligned}
$$

and N has intensity μ^δ under $\mathbb{P}^\delta$ and $\check{\mu}$ under $\check{\mathbb{P}}$.

The function f_δ has the interpretation of an expected present value of the future utility weights. Compared to the first tax regime, the weights F and $\tilde{F}$ are adjusted by tax quotients. For the weight on bequest, F, the adjustment is by a ratio of tax on the life insurance sum relative to labour income tax. For the weight on retirement savings, $\tilde{F}$, it is by the ratio of tax on retirement benefits relative to tax on labour income. Both adjustments also involve γ, such that the effect is opposite for high risk aversion/low EIS, $\gamma < 0$, and low risk aversion/high EIS, $\gamma > 0$.

We note that g_δ is a measure of gross future income, $\bar{a}$, and that the functions G and $\tilde{G}$ are 'grossified' to make them comparable in size to gross income. We refer to g_δ as gross human wealth and $x + g_\delta$ as gross total wealth of the person.

Optimal Controls

The optimal controls are

$$
\begin{aligned}
c_t^* &= \frac{w(t)}{f_\delta(t)}(1-\tau_L)(1-\tau_C)(X_t + g_\delta(t)), \\
\pi_t^* &= \frac{(\alpha - r)}{\sigma^2(1-\gamma)} \frac{X_t + g_\delta(t)}{X_t}, \\
S_t^* &= \left(\frac{\mu_t}{\hat{\mu}_t}\right)^{\frac{1}{1-\gamma}} \frac{F(t)}{f_\delta(t)}(1-\tau_L)(X_t + g_\delta(t)) - ((1-\tau_D)X_t + G(t)).
\end{aligned}
$$

The optimal controls under this regime are in general in the same form as under the regime with immediate taxation of all labour income. Since $x + g_\delta$ is gross total wealth, the optimal consumption, c^*, and the optimal life insurance sum, S^*, now directly involve the labour income tax, τ_L. Furthermore, the optimal proportion invested in the risky asset is a fraction of gross total wealth relative to gross wealth. The fraction is the same as under the first tax regime.

2.4 Numerical Analysis

In this section we perform a numerical analysis based on the results derived in Section 2.3. The parametrization of the models and the parameter values are presented in Section 2.4.1 and Section 2.4.2, and the results are presented in Section 2.4.3 and Section 2.4.4.

2.4.1 Utility Weights

Here we motivate and present the utility weights used in the numerical analysis. The utility weights for bequest and for pension savings at the time of retirement are highly inspired by Bruhn and Steffensen (2011), which takes the approach of deciding on the weights by solving the related optimization problems faced by the inheritor and the retired person.

Utility from Consumption

For the weight on consumption, w, we take the classical approach as to model a constant rate of impatience that puts more weight on present than expected future consumption. Since we also want to incorporate constant inflation into the numerical calculations, we end up with

$$w^{1-\gamma}(t) = \mathrm{e}^{-(\rho+\gamma i)t}, \ t \geq 0,$$

where ρ is the impatience factor and i is the inflation rate.

Utility from Bequest

One obvious way of deciding on utility from bequest is inspired by the utility that the inheritor experiences from consuming the heritage. If the inheritor faces a similar optimization problem as the one we are interested in, except for that the inheritor has no bequest motive (utility from leaving money upon death is zero), this leads to

$$\begin{aligned} F(t) &= \int_t^\infty \mathrm{e}^{-\frac{1}{1-\gamma}\int_t^s \mu_\tau - \gamma(\mu_\tau^* + \varphi)d\tau} W(s)ds, \\ G(t) &= \int_t^\infty \mathrm{e}^{-\int_t^s (r+\mu_\tau^*)d\tau} (\mathbb{1}_{\{s<T\}} A_s + \mathbb{1}_{\{s\geq T\}} \tilde{A}_s)ds. \end{aligned}$$

Here W is the weight that the optimizer puts on the inheritor's utility from consumption, A is the net income stream and $\tilde{A}$ the net public pension of the inheritor.

In accordance with the weight for the person's own consumption, w, we write

$$W^{1-\gamma}(t) = \bar{\theta} w^{1-\gamma}(t),\ t \geq 0,$$

where $\bar{\theta}$ is the weight that the person puts on the heir's consumption relative to his own. If the heir is the spouse of the person, θ may reflect aspects such as decreased costs and an expected different consumption pattern for the widow(er). Since the weight must change the marginal utility of consumption for the person and the heir, it depends on γ, and we reparametrize the model such that $\bar{\theta} = \frac{1}{2}\theta^{\gamma}$ (this parametrization is also used in Hong and Ríos-Rull, 2012 and Bruhn and Steffensen, 2011).

Utility upon Retirement

Utility from pension savings is based on an optimization problem regarding the retirement period. We propose these weights for utility at the time of retirement:

$$\begin{aligned}
\tilde{F}(T) &= \int_T^{\infty} \mathrm{e}^{-\frac{1}{1-\gamma}\int_T^s \mu_\tau - \gamma(\mu^*_\tau + \varphi) d\tau} [w(s) + \left(\frac{\mu}{\mu_s^{*\gamma}}\right)^{\frac{1}{1-\gamma}} F(s)] ds, \\
\tilde{G}(T) &= \int_T^{\infty} \mathrm{e}^{-\int_T^s r + \mu^*_\tau d\tau} (\tilde{a}_s + \mu^*_s G(s)) ds,
\end{aligned}$$

where $\tilde{a}$ is the net public pension rate. These weights correspond to the post retirement wealth dynamics

$$\begin{aligned}
dX_t &= \bar{r}(1-\tilde{\tau}_1)X_t dt + \pi_t(\bar{\alpha}(1-\tilde{\tau}_2) - \bar{r}(1-\tilde{\tau}_1))X_t dt + \pi_t \bar{\sigma}(1-\tilde{\tau}_2) X_t dW_t \\
&\quad + \tilde{a}_t dt - c_t dt - \mu^*_t S_t dt + S_t dN_t \\
&= r X_t dt + \pi_t(\alpha - r) X_t dt + \pi_t \sigma X_t dW_t \\
&\quad + \tilde{a}_t dt - c_t dt - \mu^*_t S_t dt + S_t dN_t, \\
X_0 &= x_0,
\end{aligned}$$

and the retiree solves the optimization problem given by these dynamics and the optimal value function

$$V(t,x) = \sup_{c,\pi,S} \mathbb{E}_{t,x}\left(\int_t^\infty \mathrm{e}^{-\int_t^s \mu_\tau d\tau} \Big[\tfrac{1}{\gamma} w^{1-\gamma}(s) c_s^\gamma \right.$$
$$\left. + \mu_s \tfrac{1}{\gamma} F^{1-\gamma}(s)(X_s + G(s) + S_s)^\gamma \Big] ds \right).$$

Note that the optimal controls related to this problem is given by (2.5)-(2.7) with f and g substituted by $\tilde{F}$ and $\tilde{G}$.

We have deliberately avoided tax parameters on public pension and life insurance after retirement, since these taxes are not considered in our numerical analysis. Furthermore the tax on consumption after retirement is taken care of in the utility functions in (2.10) and (2.16). In the case of tax exempt contributions to retirement savings, the taxation of benefits is also taken care of in the utility function in (2.16). Thereby this formulation of utility from retirement savings is meaningful under both tax regimes.

The parameters $\tilde{\tau}_1$ and $\tilde{\tau}_2$ are the tax rates on investment returns from risk free and risky investments. We equip the parameters with tildes to highlight that they might be different from those faced until retirement. This is especially the case with reduced investment return taxation until retirement and a lump sum benefit.

Household Finance Interpretation

All models so far are presented in terms of personal finance, taking into account the behaviour of one single person. The models, though, are easily generalized to cover an optimization problem for married couples that file their tax reports jointly.

In order for the optimization problems to be household finance related (a married couple), interpretations of some model elements are slightly changed. The wealth process is the total wealth of the household, the income stream, $\bar{a}$, is the total gross labour income of the married couple, and A the net income stream of the widow(er) after death of the spouse. Also $\tilde{a}$ is the net public pension of the couple and $\tilde{A}$ is the net public pension of the widow(er). Thereby the functions g_λ and g_δ quantify the household's human wealth.

The intensities μ and μ^* are the intensities by which one of the spouses dies, except for those used in F and G which represent the intensities by which the widow(er) dies. Therefore it seems reasonable to expect that the intensities used in the household optimization problem are two times the intensities used in F and G. All in all this means that the optimal life insurance sum is the amount paid out to the widow(er) upon death of the spouse, which means that identical amounts of life insurance is bought on both lives.

In general this way of introducing a household model denies us the possibility of imposing differences on the spouses. For a model with this possibility see Bruhn and Steffensen (2011).

2.4.2 Model Parameter Estimates

For a basic set of non-favoured tax parameters we rely on values from current tax codes in the US and Denmark. For the consumption tax we use the estimate from Trabandt and Uhlig (2009), since this includes both general VAT as well as consumption taxes on e.g. energy/fat/sugar/tobacco etc.[7] All tax parameters are shown in Table 2.1.

In the numerical investigation performed in this section, we restrict the analysis to the case of constant real income streams, such that nominal income grows by inflation. Non-constant income would in general blur the over-time effects of the tax regimes investigated, and constant real income is in line with the estimates of Cocco et al. (2005) (they find that life cycle income streams are in general not constant, but rather flat from ages around 30-65).

[7] Another way of getting robust estimates of tax parameters is by considering the macro-economic proposal presented in Mendoza et al. (1994). Since our analysis is performed on micro-level, we go with current tax codes and the estimates of Trabandt and Uhlig (2009).

Variable	Description	US	Denmark
τ_L	Labour tax	28%	47.5%
τ_I	Tax refund on life insurance	28%	47.5%
τ_D	Tax on life insurance sum	28%	47.5%
τ_1	Investment return tax (bonds)	15%	35%
τ_2	Investment return tax (shares)	15%	35%
τ_C	Consumption tax	4.8%	25.9%

Table 2.1: *Basic model tax parameter values for numerical results.*

We assume that the mortality intensities are the same under $\mathbb{P}$ and $\mathbb{P}^*$. This special case corresponds to zero market price of insurance risk, and is relevant due to a reference to diversification of risk in the insurance portfolio of the life insurance company.

The mortality intensities are of Gompertz-Makeham form such that

$$\mu_t = \mu_t^* = 2(M_1 + 10^{M_2 + M_3(z+t) - 10}), \tag{2.18}$$

where z is the age at time zero. Since the numerical analysis is performed in terms of a household (married couple) optimizing expected lifetime utility, z is the age of each of the spouses at time zero. Furthermore, the mortality intensities in the functions F and G are half these intensities (see Section 2.4.1).

In the following we present results for both an American and a Danish couple of initial age 30 with 35 years to retirement. The remaining parameter values for the studies are found in Table 2.2. Notice especially that the value $\gamma = -3$ corresponds to a risk aversion of 4 and that the value $\theta = 0.5$ implies that the couple's marginal utility from spending \$1.36 together equals the the marginal utility from spending \$1 for either of the widows. This level of shared costs is based on a study of American data performed in Hong and Ríos-Rull (2012).

Variable	Description	Value
z	Age of person and spouse at time of optimization	30
T	Time of retirement	35
x	Initial wealth at time of optimization	0
$\bar{a}$	Gross real labour income rate	\$150 000
$\tilde{a}$	Real net public pension rate	\$15 000
A	Real net labour income rate for American/Danish widow(er)	\$54 000/\$35 625
$\tilde{A}$	Real net public pension rate for American/Danish widow(er)	\$7 500/\$7 500
γ	Risk aversion/EIS parameter	-3
ρ	Impatience factor in w and W	3%
θ	Weight factor in W	0.5
i	Inflation factor	2%
$\bar{r}$	Constant drift of the risk free security	4%
$\bar{\alpha}$	Constant drift of the risky security	7%
$\bar{\sigma}$	Constant volatility of the risky security	20%
M_1	Parameter for mortality intensity	0.002353
M_2	Parameter for mortality intensity	5.102232
M_3	Parameter for mortality intensity	0.04550

Table 2.2: *Basic parameter values for numerical results. The mortality intensity parameters are estimated based on deaths of people over the age of 50 in America in 2006.*

2.4.3 Personal Preferences - Indifference Utility and Related Controls

We want to quantify the effect of introducing tax favoured pension savings accounts, and for that we compare the expected lifetime utility under different tax regimes.

Since indifference between two tax regimes does not imply identical behaviour under the two regimes, we also take a closer look at the related optimal controls and their expected development over time.

Tax Regimes of Interest

The most common tax favoured retirement saving vehicles in the US are IRAs and 401(k)s (and to some extent Universal Life Insurance and Deferred Life Annuities). All programs allow for tax exempt contributions (up to a certain amount per year), and tax free accumulation of savings. Taxation of benefits from the saving vehicles is progressive, which often leads to favourable benefit taxation compared to the exempt labour income taxation of contributions.

The Danish saving vehicles Kapitalpension, Ratepension and Livrente have the same properties as the American vehicles, except that investment returns on savings are not tax free but only favourably taxed. Taxation of benefits is linear for the benefits from a Kapitalpension (which must be paid out as a lump sum at the time of retirement), and progressive for the annuity benefits from the Ratepension and Livrente. The latter often leads to favourable benefit taxation as in the US. The annuity benefits are, though, accounted for when the citizens apply for public benefits as e.g. housing subsidies, medicine subsidies etc., and that makes the annuity benefit taxation less favourable.

Inspired by the American and Danish retirement savings regimes, we restrict our numerical analysis to the following two scenarios:

- The American couple face a tax regime where contributions to retirement savings are tax exempt, and benefits are taxed at same rate as labour income. Investment returns are tax free, and benefits are either paid out as a lump sum at the time of retirement or as an annuity.

- The Danish couple also face a tax regime where contributions to retirement savings are tax exempt and investment returns are favourable taxed (by 15% tax). Benefits (paid out as a lump sum at the time of retirement or as an annuity) are taxed at a lower rate than labour income. We set the rate to be 5%-points lower than the labour income tax.

Indifference Measures

In order to compare the two tax regimes, we calculate the values of certain parameters for an alternative tax regime, which makes the American and Danish couples indifferent between saving under the regimes.

One benchmark is given by a regime without any favouring of retirement savings. In order to be indifferent between the two regimes, the couples each demand an indifference sum, ψ, that solves

$$V^{\lambda}(t = 0, \psi) = V^{\delta}(t = 0, 0),$$

where V^{λ} and V^{δ} are given by (2.11) and (2.17), V^{λ} is calculated with $\beta = 0$ and V^{δ} is calculated based on the favoured tax values. In line with common practice we report the indifference sums in terms of percentage of total wealth, $x + g_{\lambda}$.

Similarly we define the indifference bonus as the bonus that the households demand on contributions to retirement savings, in order to be indifferent between the two retirement saving regimes. Mathematically we define the indifference bonus as the β that solves

$$V^{\lambda}(t = 0, 0) = V^{\delta}(t = 0, 0),$$

where V^{λ} and V^{δ} are given by (2.11) and (2.17), and V^{δ} is calculated based on the favoured tax values.

Finally, since retirement savings for the Danish couple are favoured both due to favourable benefit taxation and investment returns taxation, we define a third indifference measure for the Danes. This is the investment return taxation, τ, that makes the couple indifferent between the favoured tax regime and a tax regime where contributions are not tax exempt, but investment returns (irrespective of origin) are subject to taxation by τ. With abuse of notation, τ solves

$$V^{\lambda}(t = 0, 0, \tau) = V^{\delta}(t = 0, 0).$$

where again V^λ and V^δ are given by (2.11) and (2.17), V^λ is calculated with $\beta = 0$ and V^δ is calculated based on the favoured tax values.

No investment Return Taxation - 'The American Dream'

To investigate the effects of the absence of tax on investment returns for retirement savings, we turn to the married American couple introduced above. They are given the opportunity to save in a tax deferred savings account where there is no investment return taxation, and benefits are paid out as either a lump sum at the time of retirement or an annuity. When paid out as a lump sum, investment return taxation during retirement is 15% ($\tilde{\tau}_1 = \tilde{\tau}_2 = 15\%$), while it is 0% when benefits are paid out as an annuity[8].

Table 2.3 shows the indifference sum/bonus demanded by the couple in order to be indifferent between saving under the different regimes. For a robustness-check of the results, further indifference sums/bonuses are shown for other values of parameters than those in Table 2.1 and Table 2.2. Table 2.4 shows the related initial saving ratios for the couple.

For a lump sum benefit, the indifference sum for the American couple of 1.14% (1.54% for annuity benefits) of human wealth corresponds to a net sum of just over \$36 000 (\$49 000). The value of annuity benefits in this setup is thereby 35% higher than that of lump sum benefits, since the couple have more time to exploit the favourable investment return taxation.

The indifference bonus for a lump sum benefit of nearly 7% (9.39% for annuity benefits) corresponds to a first year bonus of around \$1 000 (\$1 250). Due to preferences and a stochastic investment market, consumption and saving contributions vary over time, and these amounts are not constant over the savings period.

The life insurance premium is paid out of the savings, and is not reported

[8]Both products are sold in reality, and for that reason we investigate both the lump sum and the annuity payment. In the setup presented in this paper, the annuity payments are obviously more valuable to the couple, but in reality there could be numerous reasons for the couple to choose the lump sum payment.

Lump Sum Benefit:

Setting	Indif. Sum	Indif. Bonus
Basic	1.14%	6.84%
z +/− 5	0.81%/1.46%	5.69%/7.62%
T +/− 5	0.82%/1.39%	6.21%/6.80%
γ +/− 2.5	0.47%/1.34%	5.87%/6.92%
ρ +/− 3%-point	0.43%/1.87%	3.38%/8.92%
Public pension +/− 100%	0.33%/2.11%	3.35%/8.55%
i +/− 1%-point	0.87%/1.27%	5.25%/8.09%
τ_D +/− 10%-points	0.73%/1.46%	4.31%/8.85%
$\tau^\dagger$ +/− 10%-points	1.03%/1.23%	7.02%/6.72%
$\tau^\ddagger$ +/− 15%-points	2.27%/0%	13.69%/0%

Annuity Benefit:

Setting	Indif. Sum	Indif. Bonus
Basic	1.54%	9.39%
z +/− 5	1.01%/2.13%	7.15%/11.34%
T +/− 5	1.02%/2.11%	7.76%/10.46%
γ +/− 2.5	0.74%/1.79%	9.11%/9.37%
ρ +/− 3%-point	0.68%/2.46%	5.35%/11.92%
Public pension +/− 100%	0.39%/2.94%	3.91%/12.13%
i +/− 1%-point	1.21%/1.68%	7.36%/10.82%
τ_D +/− 10%-points	1.12%/1.87%	6.72%/11.52%
$\tau^\dagger$ +/− 10%-points	1.38%/1.68%	9.52%/9.30%
$\tau^\ddagger$ +/− 15%-points	3.10%/0%	19.15%/0%

Table 2.3: *Indifference sums and bonuses for American couple. Top panel is the case of benefits paid out as a lump sum at the time of retirement (equal investment return taxes for all 3 regimes during retirement), bottom panel for annuity payments during retirement (no investment return taxation under favoured regime during retirement).* † *Change of* τ_L, τ_B, τ_I *and* τ_D *by +/− 10%-points.* ‡ *Change of non-favoured* τ_1, $\tilde{\tau}_1$, τ_2 *and* $\tilde{\tau}_2$ *by +/− 15%-points.*

Lump Sum Benefit:

Setting	Tax favoured	Indif. Sum	Indif. Bonus
Basic	14.3%	12.0%	12.4%
z +/− 5	11.4%/17.3%	9.1%/14.9%	9.5%/15.5%
T +/− 5	9.1%/20.4%	6.5%/18.4%	6.9%/18.9%
γ +/− 2.5	−2.7%/19.7%	−9.9%/18.3%	−8.9%/18.6%
ρ +/− 3%-point	−0.8%/28.3%	−3.4%/26.2%	−3.2%/26.8%
Public pension +/− 100%	7.1%/21.6%	4.5%/19.4%	4.8%/20.0%
i +/− 1%-point	11.9%/15.7%	9.3%/13.5%	9.7%/14.0%
τ_D +/− 10%-points	14.8%/13.9%	13.0%/11.1%	13.3%/11.7%
$\tau^\dagger$ +/− 10%-points	11.7%/16.6%	9.8%/13.6%	10.3%/14.1%
$\tau^\ddagger$ +/− 15%-points	14.5%/14.2%	9.6%/14.3%	10.6%/14.3%

Annuity Benefit:

Setting	Tax favoured	Indif. Sum	Indif. Bonus
Basic	14.2%	11.6%	12.3%
z +/− 5	11.3%/17.0%	8.9%/14.4%	9.6%/15.2%
T +/− 5	9.1%/20.2%	6.4%/17.8%	6.8%/18.6%
γ +/− 2.5	−2.5%/19.4%	−10.2%/17.9%	−8.7%/18.3%
ρ +/− 3%-point	−0.1%/28.1%	−3.7%/25.8%	−3.3%/26.5%
Public pension +/− 100%	7.2%/21.1%	4.5%/18.8%	4.8%/19.5%
i +/− 1%-point	11.8%/15.5%	9.0%/13.1%	9.6%/13.8%
τ_D +/− 10%-points	14.6%/13.8%	12.7%/10.8%	13.1%/11.5%
$\tau^\dagger$ +/− 10%-points	11.5%/16.4%	9.5%/13.2%	10.2%/13.9%
$\tau^\ddagger$ +/− 15%-points	14.2%/14.2%	8.9%/14.2%	10.2%/14.2%

Table 2.4: *Initial savings ratio for American couple. Top panel is the case of benefits paid out as a lump sum at the time of retirement (equal investment return taxes for all 3 regimes during retirement), bottom panel for annuity payments during retirement (no investment return taxation under favoured regime during retirement).* † *Change of* τ_L, τ_B, τ_I *and* τ_D *by +/− 10%-points.* ‡ *Change of non-favoured* τ_1, $\tilde{\tau}_1$, τ_2 *and* $\tilde{\tau}_2$ *by +/− 15%-points.*

in the tables. The net life insurance sum under the different regimes are very similar and initially the premium paid is around 5%-points of the savings-ratios.

Robustness Values

The assumed investment return tax of 15% is based on temporary tax rules that are to expire in 2011, and holds only for specific tax brackets. The robustness-check in Table 2.3 shows that the value of the retirement savings regime without investment return taxation doubles when investment return taxation is doubled.

Increasing the mortality intensities of the persons ($z + 5$), decreases the indifference sum and bonus since the couple have a smaller probability of staying alive until retirement, and expect less years of retirement. The values on the other hand increase when time to retirement is increased ($T + 5$), since there are now more years to take advantage of the low tax.

Increasing γ has two major effects. It decreases the risk aversion and the couple invest more in the risky asset, which in general has a positive effect on the indifference sum/bonus. The consumption also increases with γ and the savings ratio gets low, such that the couple's retirement savings are mainly generated by the investment return (no short-selling constraint allows for generating savings by shorting bonds and investing in the risky asset). This way the couple miss the tax exemption of contributions (since they are low), but pay taxes on the benefits, which in total decrease the value of the favoured regime.

Changing the life insurance tax, τ_D, has more effect on the favoured regime than the other two, since wealth upon death is also taxed by τ_D in the favoured regime. The initial savings ratios are not very influenced by a change in τ_D, while the initial purchase of life insurance changes by $15 - 20\%$ when changing τ_D by 10%-points (values not shown in tables).

The remaining robustness values also show expected effects on the indifference sums and bonuses as well as on the initial savings ratios.

Behaviuor over Time

As we have already seen on the initial savings ratios in Table 2.4, indifference between saving under the different regimes does not mean acting identically under them. In Figure 2.1 we show, for all three tax regimes, the expected development of the optimal controls and the wealth of the household, given that both persons are alive[9]. We only illustrate the case of a lump sum benefit at the time of retirement, and values are in real terms (corrected for inflation).

The regime with the initial indifference sum as starting wealth has no tax regulations encouraging more or less retirement saving at any time until retirement. This regime is therefore referred to as the baseline regime. Under this regime we find that consumption (in real terms) decreases over time, mainly due to the values of the impatience factor, ρ, the risk aversion/EIS parameter, γ, and the expected rate of net returns on investments.

The value of not paying investment return tax is higher, if savings are made while young rather than old. Therefore consumption under the favoured regime starts relatively lower than under the baseline regime. The consumption is increasing over time, again mainly due to the value of the impatience factor, the risk aversion/EIS parameter and the expected rate of net returns on investments (lower taxation of investment returns than in baseline regime).

Adding bonus on the savings premium impose an incentive for retirement saving for the household that remains stable over the savings period (since the bonus-percentage is constant). The effect of this is that the savings ratio is higher (consumption until retirement lower) than under the baseline regime. At the time of retirement the net consumption jumps to a higher level. The size of the jump is $(1+\beta)^{1/(1-\gamma)} - 1$, which is the change in the marginal utility of consumption when the retirement savings motive disappears.

The net life insurance sum for the household saving under the bonus regime is higher than under the baseline regime, since the bonus regime reduces motive for consumption during the savings period relatively to the bequest

[9] In this notion, expected refers to that we have inserted the expected stock value for all future time points, i.e. the stock gives an annual return corresponding to the drift of $\alpha = 7\%$.

motive. Remember that the household under the baseline regime starts out with an initial wealth of just over \$36 000, and that explains some of the difference in the first years.

Favourable Benefit and Investment Return Taxation - 'The Danish Double Advantage'

The Danish couple's retirement savings are tax favoured in two ways, by tax exempt contributions with a favourable benefit taxation and by reduced investment return taxation. For the numerical results presented here, we let the benefit taxation, τ_B, be 42.5%, and the favoured investment return taxation be 15%. In Table 2.5 we show the indifference sum, bonus and investment return tax (equal for risk free and risky investments), that the couple demands in order to be indifferent between the tax favoured regime, and the three alternative regimes. We show the results both for lump sum and annuity benefits (for the favoured regime, lump sum benefits means $\tilde{\tau}_1 = \tilde{\tau}_2 = 35\%$ and annuity benefits means $\tilde{\tau}_1 = \tilde{\tau}_2 = 15\%$), along with robustness checks for several parameters. The related initial savings ratios are shown in Table 2.6.

For the Danish couple, the value of the favoured regime is an indifference sum of 1.60% of human wealth (1.95% for annuity benefits), which is a net sum of roughly \$44 000 (\$53 500). The indifference bonus of 12.10% (14.95%) corresponds to a first year bonus of \$600 (\$750), which is substantially lower than for the Americans, despite a higher bonus-percentage. Higher labour income taxation and a lower optimal savings ratio for the Danes accounts for the difference.

The indifference tax on investment returns is 3.0% (6.0%) or a reduction of taxation by 80% (60%) compared to the favoured regime. When benefits are paid out as an annuity, the indifference tax is then assumed paid on investment returns both before and after retirement. That gives the couple a longer expected time to take advantage of the favourable tax, and that accounts for the higher value.

The initial savings ratios are very different for the four regimes. The regime with the indifference sum has the lowest initial savings ratio, since nothing

Lump Sum Benefit:

Setting	Indif. Sum	Indif. Bonus	Indif. Inv. Tax
Basic	1.60%	12.10%	3.0%
z +/− 5	0.85%/2.35%	7.69%/15.14%	4.1%/2.6%
T +/− 5	0.94%/2.23%	8.60%/14.14%	4.4%/1.0%
γ +/− 2.5	−0.05%/2.09%	−1.11%/13.28%	33.5%/4.2%
ρ +/− 3%-point	0.34%/2.92%	3.57%/16.91%	−23.6%/5.8%
Public pension +/− 100%	−0.71%/4.69%	−19.64%/18.76%	7.5%/3.8%
i +/− 1%-point	0.80%/2.04%	6.88%/15.18%	−0.8%/4.7%
τ_D +/− 10%-points	1.19%/1.88%	8.73%/14.61%	11.5%/ − 2.8%
$\tau^\dagger$ +/− 10%-points	1.17%/1.87%	11.88%/11.90%	0.5%/4.9%
$\tau^\ddagger$ +/− 20%-points	2.39%/0.64%	18.65%/4.71%	3.0%/3.0%
τ_B +/− 5%-points	0.96%/2.18%	7.09%/16.84%	15.0%/10.4%

Annuity Benefit:

Setting	Indif. Sum	Indif. Bonus	Indif. Inv. Tax
Basic	1.95%	14.95%	6.0%
z +/− 5	0.96%/3.04%	8.78%/19.98%	5.8%/6.3%
T +/− 5	1.10%/2.86%	10.15%/18.50%	6.4%/5.4%
γ +/− 2.5	0.05%/2.53%	1.14%/16.30%	36.3%/6.8%
ρ +/− 3%-point	0.46%/3.55%	4.88%/20.87%	−9.7%/8.0%
Public pension +/− 100%	−1.01%/5.93%	−29.39%/24.34%	11.7%/7.0%
i +/− 1%-point	0.94%/2.47%	8.11%/18.75%	2.7%/7.1%
τ_D +/− 10%-points	1.52%/2.24%	11.34%/17.69%	12.3%/1.6%
$\tau^\dagger$ +/− 10%-points	1.37%/2.34%	14.00%/15.10%	3.4%/7.5%
$\tau^\ddagger$ +/− 20%-points	3.09%/0.64%	24.68%/4.71%	6.0%/6.0%
τ_B +/− 5%-points	1.30%/2.53%	9.78%/19.82%	15.0%/ − 1.9%

Table 2.5: *Indifference sums, bonuses and investment return taxes for Danish couple. Top panel is the case of benefits paid out as a lump sum at the time of retirement (equal investment return taxes for all 3 regimes during retirement), bottom panel for annuity payments during retirement (15% investment return taxation under favoured regime during retirement).* † *Change of τ_L, τ_B, τ_I and τ_D by +/− 10%-points.* ‡ *Change of non-favoured τ_1, $\tilde{\tau}_1$, τ_2 and $\tilde{\tau}_2$ by +/− 20%-points.*

Lump Sum Benefit:

Setting	Tax favoured	Indif. Sum	Indif. Bonus	Indif. Inv. Tax
Basic	8.8%	4.9%	5.8%	10.4%
z +/− 5	6.0%/11.9%	2.2%/7.9%	2.8%/9.1%	7.5%/13.6%
T +/− 5	4.1%/14.4%	0.0%/10.8%	0.6%/11.9%	5.8%/16.0%
γ +/− 2.5	−12.7%/15.0%	−24.8%/12.7%	−25.0%/14.3%	−23.9%/16.8%
ρ +/− 3%-point	−7.5%/23.9%	−11.8%/20.3%	−11.5%/21.5%	−1.2%/24.9%
Public pension +/− 100%	−2.1%/19.8%	−6.5%/16.4%	−8.7%/17.5%	−1.1%/21.1%
i +/− 1%-point	4.0%/11.7%	−0.4%/8.2%	0.2%/9.1%	6.5%/12.9%
τ_D +/− 10%-points	9.5%/8.3%	6.5%/3.7%	7.2%/4.7%	10.5%/10.3%
$\tau^\dagger$ +/− 10%-points	4.8%/12.1%	0.8%/7.7%	1.7%/8.6%	7.0%/12.8%
$\tau^\ddagger$ +/− 20%-points	8.9%/8.8%	1.0%/8.4%	2.5%/8.8%	10.5%/10.4%
τ_B +/− 5%-points	9.0%/8.6%	5.5%/4.4%	9.0%/9.6%	9.0%/9.6%

Annuity Benefit:

Setting	Tax favoured	Indif. Sum	Indif. Bonus	Indif. Inv. Tax
Basic	8.9%	4.6%	5.7%	10.0%
z +/− 5	6.0%/11.7%	2.1%/7.3%	2.7%/8.8%	7.3%/12.9%
T +/− 5	4.1%/14.3%	−0.1%/10.2%	0.5%/11.7%	5.5%/15.5%
γ +/− 2.5	−12.1%/14.8%	−24.9%/12.3%	−24.7%/13.0%	−25.7%/15.4%
ρ +/− 3%-point	−7.4%/23.7%	−11.9%/19.8%	−11.6%/21.2%	−3.1%/24.4%
Public pension +/− 100%	−1.5%/19.1%	−6.2%/15.4%	−9.7%/16.8%	−1.3%/20.1%
i +/− 1%-point	4.2%/11.5%	−0.6%/7.8%	0.2%/8.9%	6.3%/12.5%
τ_D +/− 10%-points	9.4%/8.3%	6.2%3.3%	7.0%/4.6%	10.4%/9.7%
$\tau^\dagger$ +/− 10%-points	4.8%/11.9%	0.6%/7.3%	1.7%/8.4%	6.7%/12.4%
$\tau^\ddagger$ +/− 20%-points	8.9%/8.9%	0.3%/8.4%	2.2%/8.9%	10.0%/10.0%
τ_B +/− 5%-points	9.0%/8.6%	5.2%/4.0%	5.9%/5.4%	9.0%/10.9%

Table 2.6: *Initial savings ratio for Danish couple. Top panel is the case of benefits paid out as a lump sum at the time of retirement (equal investment return taxes for all 3 regimes during retirement), bottom panel for annuity payments during retirement (15% investment return taxation under favoured regime during retirement).* † *Change of* τ_L, τ_B, τ_I *and* τ_D *by +/− 10%-points.* ‡ *Change of non-favoured* τ_1, $\tilde{\tau}_1$, τ_2 *and* $\tilde{\tau}_2$ *by +/− 20%-points.*

motivates extraordinary savings compared to the three other regimes (and the indifference sum is already added to the savings). The bonus regime has a savings ratio that is lower than that of the regimes with reduced investment return taxation, since bonus does not motivate early savings. The highest savings ratio comes with the lowest investment return taxation.

Robustness Values

The initial savings ratios are almost equal for the favoured and the bonus regime when investment return taxation is even (they are different in the fourth decimal, not shown in Table 2.6). This occurs since bonus and favourable benefit taxation motivates retirement savings in the same manner. The small difference in the savings ratios occurs since bonus is not paid back if one spouse dies before retirement.

The indifference measures are very sensitive to changes in the public pension, and in fact doubling the public pension leads to a negative indifference sum and bonus. This may seem counter-intuitive, but arises since the high level of public pension leads to negative savings contributions over most of the savings period for the favoured regime. The retirement savings are thereby mainly generated by shorting bonds and investing in stocks. The fact that savings benefits in the favoured regime are subject to taxation by τ_B at retirement thereby has a negative value that exceeds the value of the low investment return taxation. The same explanation holds for the negative indifference sum and bonus in the case of an increase in γ.

Behaviour over Time

In Figure 2.2, we present the expected development over time for the wealth and the controls for all four regimes, given that both persons stay alive.

As expected, the regime with the lowest investment return taxation has the lowest initial consumption and the steepest increase until retirement. The jumps of the optimal consumption at retirement for the regimes with bonus or favourable benefit taxation are due to the change in marginal utility of consumption, when the savings period ends.

The optimal investment and life insurance sums evolve as expected over time.

2.4.4 Non-Indifference from a Government's Point of View

Despite the fact that we can formulate several tax regimes that the American and Danish couples are indifferent between, the couples behave differently under the regimes. One consequence of that is different tax cash flows experienced by the tax authorities.

In order to investigate the preferred regime for the tax authorities, we compute the expected present value of tax income and expenditures for a government. We only take into account the taxes introduced in this paper. For the regime with immediate labour income taxation and no retirement savings favouring, we include the taxes τ_L (labour income), τ_C (consumption), τ_1, τ_2 (investment return) and τ_I and τ_D (life insurance). An indifference sum paid out at time zero is a lump sum expenditure for the tax authorities, and bonus paid during the savings period is a continuous expenditure. For the regime with tax exempt contributions to retirement savings, only the part of the salary that is spent on immediate consumption generates a labour tax income during the saving period. Instead, a lump sum tax income at the time of retirement is generated by τ_B. We do not take the public pension into account, since its value is the same for all regimes.

In Figure 2.3 we show the expected present values of the tax streams generated by the different investigated regimes. Retirement savings are assumed paid out as a lump sum in the favoured regimes.

For low values of the discount factor, the favoured regimes (tax exempt contributions) are most favourable for the tax authorities. This is due to the retirement savings tax, τ_B, that generates a large revenue if both persons are alive upon retirement. If one of the persons dies before the retirement age, and contributions are tax exempt, both the life insurance sum and the retirement savings are taxed by the life insurance tax, τ_D. That also adds to the value of the favoured regimes.

Tax exempt contributions to retirement savings on the other hand generates less labour income tax for the tax authorities. Therefore the regimes with indifference sums or bonuses are preferable for tax authorities when the discounting factor is high. This is, though, also due to the higher investment return taxation in the regimes of indifference sums and bonuses. In addition we see that the Danish regime with immediate labour income taxation and very low investment return tax (3.0%) is not preferable even at high values of the discount factor.

2.5 Conclusion

In this paper we investigated the problem of optimizing lifetime utility with bequest motive under two different taxation regimes. We quantified the tax impact under the different regimes and found that this was not identical, some taxes matter more in one regime than another. The regimes each motivated retirement savings in different ways, and the numerical analysis showed that this led to savings of different size. Moreover, the contributions to savings were made at different times during the savings period.

The indifference values calculated indicates that governments wanting to shift retirement savings incites from tax exempt contributions to bonus on contributions can do that for reasonable values of the proportional bonus. The effect of that is generally a larger tax revenue during the savings period, and for sufficiently high values of the discounting-factor, also larger expected present value of the future tax flows.

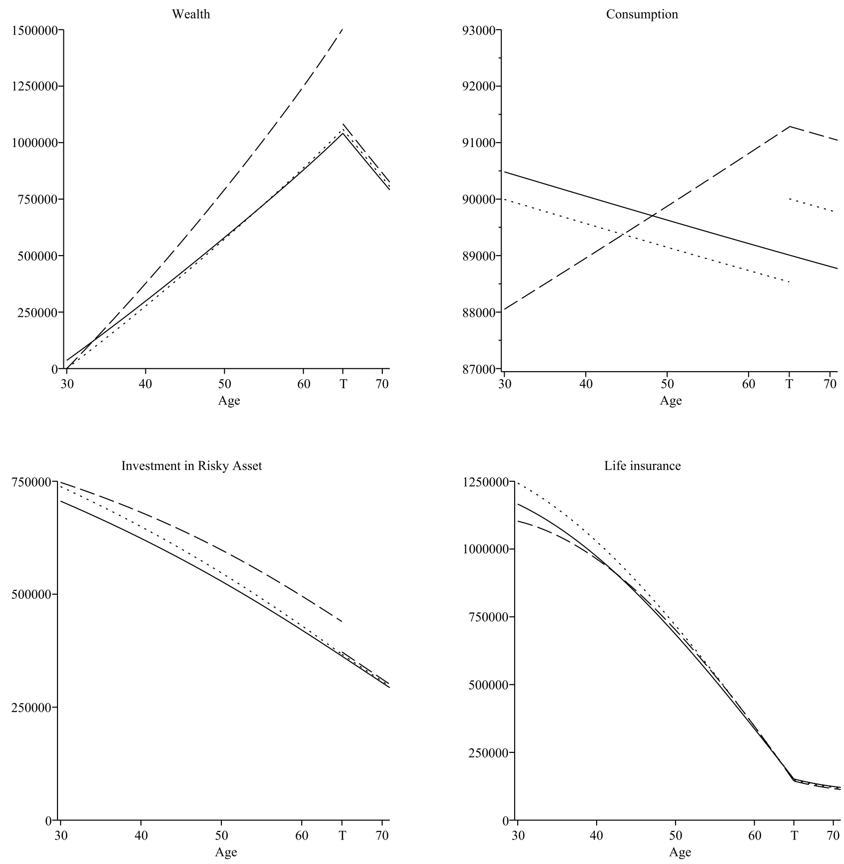

Figure 2.1: *Expected development of real values of wealth, consumption, investment and life insurance for American couple. Dashed line is the favoured regime (no investment return tax during savings period), the full line is the regime with indifference sum and the dotted line the regime with bonus on contributions.*

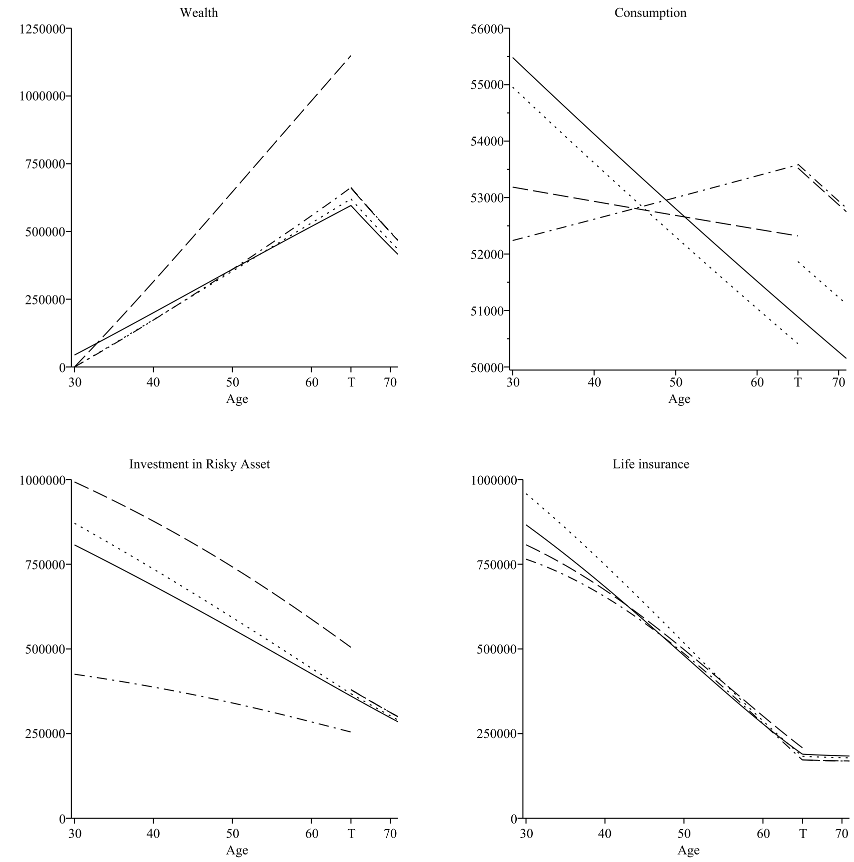

Figure 2.2: *Expected development of real values of wealth, consumption, investment and life insurance for Danish couple. Dashed line is the favoured regime (low investment return tax during savings and favoured tax of lump sum benefit), the dash-dotted line is regime with low investment return tax during savings, the full line is the regime with indifference sum and the dotted line the regime with bonus on contributions.*

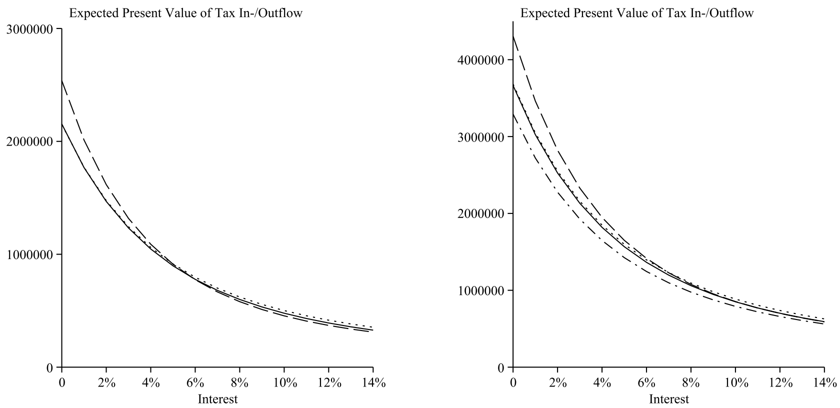

Figure 2.3: *Expected value of future income and expenditure for tax authorities for different rates of discounting. Lifetime of the household members is calculated by the objective mortality in* (2.18)*. American data to the left, Danish to the right. Dashed line is the favoured regime (tax exempt savings contributions and low investment return tax), the dash-dotted line (in Danish data) is regime with low investment return tax during savings, the full line is the regime with indifference sum and the dotted line the regime with bonus on contributions.*

3. Household Consumption, Investment and Life Insurance

Kenneth Bruhn and Mogens Steffensen, Insurance: Mathematics and Economics 48 (2011) 315-325.

Abstract This paper develops a continuous-time Markov model for utility optimization for households. The household optimizes expected future utility from consumption by controlling consumption, investments and purchase of life insurance for each person in the household. The optimal controls are investigated in the special case of a two-person household, and we present graphics illustrating how differences between the two persons affect the controls.

Keywords *Personal Finance; Household Finance; Multi-State Model; Stochastic Control; Power Utility*

3.1 Introduction

Original consumption-investment problems are formulated in terms of optimizing utility of consumption and a terminal utility over a fixed time horizon for a single person, see Merton (1969) and Merton (1971). Richard (1975) included the problem of finding an optimal life insurance strategy, and formulated the problem of optimizing expected utility over an uncertain life time, where utility now arose from consumption and from leaving a positive amount of money upon death. Apart from introducing life insurance, Richard (1975) also modeled a continuous life time income, and found that the expected life time income had a positive effect on the demand for life insurance. Actually, the inclusion of an insurance decision in the personal finance optimization problem was first formulated in a discrete-time setting by Yaari (1965).

Since the path-breaking article of Hoem (1969), the continuous-time finite state Markov chain has played a prominent role in the theory of life insurance, and Kraft and Steffensen (2008) applied the continuous-time finite state Markov chain to the ideas established by Richard (1975). Kraft and Steffensen (2008) motivated the set-up by a personal finance model which allowed the customer to insure himself against disability, unemployment and similar personal risks.

Inspired by Kraft and Steffensen (2008) we use the Markov chain set-up for modeling household finance in the sense of optimizing expected future utility for a household consisting of economically and probabilistically dependent persons. The modeling is flexible enough to capture several interesting differences between the members of the household, and leads to closed form solutions for the optimal controls of investments, consumption of the household and purchase of life insurance for each of its members.

The paper is organized as follows: In Section 3.2 we present the general Markov model including the dynamics of the wealth of the household. Furthermore, we describe the assumptions concerning utility, and the general optimal value function for the problem. Section 3.3 presents the problem and the solution in the case of a one-person household, thereby setting the foundation for the multiple-person models. In Section 3.4 we solve the problem for a two-person household. We comment on the optimal control processes regarding consumption, investment and life insurance purchase, and in Section 3.5 we show numerical examples of these based on expectations to the investment market. In Section 3.6 we explain the mathematical induction technique used for solving the multiple-person problem and write up the optimal controls in this case. Finally, in Section 3.7 we present ideas for further development of the model.

3.2 The General Optimization Problem

We let the state of the household be represented by a finite state multi-dimensional Markov chain, Z, and the state of the economy be represented by a standard Brownian motion W. These processes are assumed to be independent and defined on the measurable space $(\Omega, \mathcal{F})$, where $\mathcal{F}$ is the natural filtration of (Z, W).

We let $\mathbb{P}$ and $\mathbb{P}^*$ be equivalent probability measures on the measurable space $(\Omega, \mathcal{F})$ and refer to $\mathbb{P}$ as the objective measure and $\mathbb{P}^*$ as the pricing measure, used for pricing both market risk (W) and life insurance risk (Z) by the insurance company. We hereby take the modern approach and consider life insurance policies as standard tradeable financial contracts, as is done in e.g. Richard (1975) and Kraft and Steffensen (2008). Illiquidity issues could be dealt with on the top of that, e.g. by introducing an illiquidity risk premium, but this is beyond the scope of this article.

It is essential for our studies below that the pricing measure exists such that pricing is unique and linear. Whereas this is conventional for e.g. equity risk, the assumption is a less conventional restriction for insurance risk. The pricing measure with respect to insurance risk may be equal to the objective measure with reference to diversification. Our results are not restricted to that case of zero market price on insurance risk but, as it can be seen below, the results become particularly simple in that case.

When modeling a household consisting of n persons, the state process Z takes values in $\{0,1\}^n$, and by convention it starts in $\{0, 0, \ldots, 0\}$ at time 0. The n marginal processes of Z indicate, for each person, whether or not that person is dead, and thereby Z is given by

$$(Z_t)_{t \geq 0} = \left(Z_t^1, Z_t^2, \ldots, Z_t^n\right)_{t \geq 0},$$

where $Z^k = (Z_t^k)_{t \geq 0}$ counts the number of deaths for person k, $k \in \{1, 2, \ldots, n\}$.

The state process Z has jump-intensities, $\hat{\mu}^{ij}$ under $\mathbb{P}$ and $\hat{\mu}^{*ij}$ under $\mathbb{P}^*$, and we denote the set of states to which Z can jump at time t by $\mathcal{Z}_t$. As we do not allow for multiple deaths in a small time interval or for resurrection, the number of states in $\mathcal{Z}_t$ equals the number of persons being alive at time t.

For any given $i = (i_1, i_2, \ldots, i_n)$ and $j = (j_1, j_2, \ldots, j_n)$, we write the transition rate functions

$$\hat{\mu}_t^{ij} = \prod_{l=1}^{n} (1 - Z_t^l)^{1-i_l} (Z_t^l)^{i_l} \mu_t^{ij}, \qquad \hat{\mu}_t^{*ij} = \prod_{l=1}^{n} (1 - Z_t^l)^{1-i_l} (Z_t^l)^{i_l} \mu_t^{*ij},$$

for some deterministic continuous transition rate functions μ_t^{ij} and μ_t^{*ij}. These functions are non-null only for i and j such that the transition $i \to j$ is possible, i.e. $i_k = 0$ and $j_k = 1$ for exactly one k and $i_l = j_l$ for $l \neq k$. In order to have well-defined problems we assume that $\mu_t^{ij} \to \infty$ and $\mu_t^{*ij} \to \infty$ for those pairs of states (i,j) for which the transition $i \to j$ is possible. That implies in particular that

$$\lim_{t\to\infty} \mathbb{P}(Z_t = \{1, 1, \ldots, 1\}) = \lim_{t\to\infty} \mathbb{P}^*(Z_t = \{1, 1, \ldots, 1\}) = 1.$$

The compensated jumping process is a martingale under the respective measures, meaning that $M = Z - \int \hat{\mu}$ is a martingale under $\mathbb{P}$ and $M^* = Z - \int \hat{\mu}^*$ is a martingale under $\mathbb{P}^*$. In particular, we will use the marginal processes, and for $j \in \mathcal{Z}_t$ write

$$dM_t^{*Z_t j} = dZ_t^{\psi(Z_t, j)} - \mu_t^{*Z_t j} dt$$

for the dynamics at time t of the marginal martingale given Z_t, where $\psi(i,j)$ gives the coordinate of Z that changes from 0 to 1 upon a jump of Z from state i to state j. Note that $\hat{\mu}^{*Z_t j} = \mu^{*Z_t j}$ since $j \in \mathcal{Z}_t$.

Wealth Dynamics

The household decides on an optimal allocation of wealth in a risky asset and a risk free asset at all times. The household has access to an investment market consisting of a bond (B) and a stock (S) with Black-Scholes dynamics:

$$\begin{aligned} dB_t &= rB_t dt, \\ B_0 &= b_0 > 0, \\ dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\ S_0 &= s_0 > 0, \end{aligned}$$

where r, α and σ are constants and W is a standard Brownian motion. The proportion of household wealth invested in the stock is described by the process π. This simple investment market model is chosen since the focus here is on the life insurance decisions, but the results can be generalized to more advanced investment market models.

Allowing the household to purchase life insurance for each person at all times, the household wealth, X, follows the dynamics

$$\begin{aligned} dX_t &= (r + \pi_t(\alpha - r))X_t dt + \pi_t \sigma X_t dW_t \\ &\qquad + a_t^{Z_t} dt - c_t dt + \sum_{j \in \mathcal{Z}_{t-}} S_t^j dM_t^{*Z_{t-}j}, \\ X_0 &= x_0, \end{aligned}$$

where a^j is the deterministic income process corresponding to state j, $j \in \{0,1\}^n$, and c is the total consumption process of the household. The processes S^j are the sums insured such that S^j is the amount payed out upon a jump of Z from state Z_{t-} to state j, and $\mu_t^{*Z_{t-}j} S_t^j$ is the natural premium intensity that the household pays at time t for that life insurance. The linearity of the premium as a function of the sum insured is a consequence of assuming existence of a pricing measure. This linearity is essential for our studies and the application of our results below is restricted to that situation. The special case of zero market price of insurance risk corresponds to setting $\mu^* = \mu$ and represents a relevant and particularly simple special case.

In practical, building of reserves in insurance companies is needed for trading life annuities. That could be dealt with by formulating our optimization problem with two types of wealth, personal wealth and institutional wealth, as it is done by Kraft and Steffensen (2008). They find, however, that if we impose no constraints on these wealth processes and allow utility to depend only on the sum of them, then we need not model two separate wealth processes (the optimization problem with two wealth processes results in the same optimal controls as the problem with only one). Since we allow for utility of consumption only, in our case it is sufficient to model one wealth process of the household.

A life annuitant leaves his institutional wealth to the insurance company upon death. When we do not distinguish between personal and institutional wealth, this is just seen as a sum paid out of the wealth of the individual to the insurance company. Therefore, we speak of a negative sum insured as a life annuity payment. Thus, not restricting the sum insured to be positive is essentially equivalent to allowing for purchase of life annuities.

The Optimization Problem

We consider the problem of maximizing expected utility for the household, where the utility is assumed to come from consumption only. In particular, we assume that there is no utility from leaving a positive amount of money at the time where the last person in the household dies (as is done by e.g. Richard (1975) and Kraft and Steffensen (2008)). Writing $u^j(t,c)$ for the utility of consuming c at time t, given that $Z_t = j$, the optimization problem is

$$\sup_{q \in \mathcal{Q}_{[0,\infty)}} \mathbb{E}_{0,x,0}\Big(\int_0^\infty \sum_{j \in \{0,1\}^n} \mathbb{1}_{\{Z_s = j\}} u^j(s, c_s) ds \Big),$$

where q is a control process and $\mathcal{Q}_{[0,\infty)}$ is the set of controls for the time after time 0, which are admissible at time 0 and $\mathbb{E}_{t,x,z}$ denotes the conditional expectation given that $X_t = x$ and $Z_t = z$. For this problem we introduce the optimal value function

$$V^z(t,x) = \sup_{q \in \mathcal{Q}_{[t,\infty)}} \mathbb{E}_{t,x,z}\Big(\int_t^\infty \sum_{j \in \{0,1\}^n} \mathbb{1}_{\{Z_s = j\}} u^j(s, c_s) ds \Big).$$

Note that for a fixed time $T > t$ the optimal value function has the recursive definition

$$V^z(t,x) = \sup_{q \in \mathcal{Q}_{[t,T)}} \mathbb{E}_{t,x,z}\Big(\int_t^T \sum_{j \in \{0,1\}^n} \mathbb{1}_{\{Z_s = j\}} u^j(s, c_s) ds + \tilde{V}^{Z_T}(T, X_T) \Big),$$

with

$$\tilde{V}^z(T,x) = \sup_{q \in \mathcal{Q}_{[T,\infty)}} \mathbb{E}_{T,x,z}\Big(\int_T^\infty \sum_{j \in \{0,1\}^n} \mathbb{1}_{\{Z_s = j\}} u^j(s, c_s) ds \Big),$$

which is convenient when investigating optimization problems where the controls are constrained in some time periods. In this paper, this is used for investigating a problem where the household consumes according to a fixed consumption strategy until time T.

Using the dynamic programming principle, and the fact that the jump intensities of Z do not depend on the wealth process X, the optimal value function can be rewritten as

$$\begin{aligned}V^z(t,x) = \sup_{q\in\mathcal{Q}_{[t,\infty)}} \mathbb{E}_{t,x}\Big(\int_t^s \mathrm{e}^{-\int_t^r \mu_\tau^{z\cdot} d\tau}\big(u^z(r,c_r) \\ + \sum_{j\in\mathcal{Z}_t} \mu_r^{zj} V^j(r, X_r + S_r^j)\big)dr + \mathrm{e}^{-\int_t^s \mu_\tau^{z\cdot} d\tau} V^z(s, X_s)\Big),\end{aligned}$$

where $\mu_t^{z\cdot} = \sum_{j\in\mathcal{Z}_t} \mu_t^{zj}$ is the total intensity of a jump out of state z at time t.

We allow the utility of consumption to depend on time and on who is alive. In the rest of this article we work with power utility, i.e. the utility functions are given by

$$\begin{aligned}u^j(t,c) &= w_j^{1-\gamma}(t)\tilde{u}(c), \\ \tilde{u}(c) &= \begin{cases} \frac{1}{\gamma}c^\gamma, & c > 0, \\ -\infty, & c \le 0, \end{cases}\end{aligned}$$

for $j \in \{0,1\}^n \backslash \{1,1,\ldots,1\}$, $t > 0$ and $\gamma \in (-\infty, 1)\backslash\{0\}$, while $u^j(t,c) = 0$ for $j = \{1,1,\ldots,1\}$ (which means that there is no utility from consumption after all members of the household are death). The separability of the state and time dependence in the deterministic weight function, w, and the consumption in the power utility function, u, allows for closed form solutions of the optimization problems.

The case $\gamma = 0$, which corresponds to logarithmic utility since $\lim_{\gamma\to 0}(c^\gamma - 1)/\gamma = \ln(c)$, will not be dealt with.

3.3 One-Person Household

This section deals with the problem of optimizing expected future utility of consumption for one person. This problem was solved by Richard (1975), but since the result is fundamental for solving the problems concerning multiple-person households we reproduce the results here. Apart from solving the problem of optimizing expected future utility in the situation where the person can control consumption, investment and life insurance purchase until his death, we consider the special case where he is not allowed to control consumption until a fixed point in time. This point in time can be thought of as the time of retirement.

For the one-person model, the state process Z takes values in $\{0, 1\}$. We write μ_t for the intensity under $\mathbb{P}$ of a jump from state 0 to state 1 at time t, and μ_t^* for the corresponding intensity under $\mathbb{P}^*$. This two-state model, often referred to as a survival model, is illustrated in Figure 3.1.

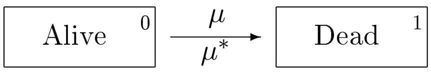

Figure 3.1: *The survival model.*

In this case the wealth of the one-person household follows the dynamics

$$\begin{aligned} dX_t &= (r + \pi_t(\alpha - r))X_t dt + \pi_t \sigma X_t dW_t + a_t dt - c_t dt + S_t dM_t^*, \\ X_0 &= x_0, \end{aligned}$$

where we assume that $a_t = 0$ for $t \geq T$ (time of retirement). Because we allow for a negative sum insured S (purchase of life annuity) and since the person obtains no utility from leaving money upon death, it is easy to convince ourselves that the optimal sum insured at all times shall be as small as possible (why not cash in the risk premium that the insurance company is willing to pay you to be your only inheritor!). As the life insurance company will not take over your debt upon your death, we arrive at the sum insured $S_t = -X_{t-}$ for all $t > 0$. Mathematically, this intuitive result is shown by introducing utility from leaving money upon death and solve this optimization problem, thereafter letting the utility upon death go to zero, see e.g. Richard (1975) and Kraft and Steffensen (2008).

Capitalizing on the above intuition leads to the following dynamics of the wealth process while the person is alive:

$$dX_t = (r + \pi_t(\alpha - r))X_t dt + \pi_t \sigma X_t dW_t + a_t dt - c_t dt + X_t \mu_t^* dt, \quad (3.1)$$
$$X_0 = x_0. \quad (3.2)$$

Now the problem of optimizing expected future utility can be described by the wealth dynamics (3.1)-(3.2) and the optimal value function

$$V(t,x) = \sup_{q \in \mathcal{Q}_{[t,T)}} \mathbb{E}_{t,x}\Big(\int_t^T \mathrm{e}^{-\int_t^s \mu_\tau d\tau} \tfrac{1}{\gamma} w^{1-\gamma}(s) c_s^\gamma ds + \mathrm{e}^{-\int_t^T \mu_\tau d\tau} \tilde{V}(T, X_T) \Big),$$

where

$$\tilde{V}(T,x) = \sup_{q \in \mathcal{Q}_{[T,\infty)}} \mathbb{E}_{T,x}\Big(\int_T^\infty \mathrm{e}^{-\int_T^s \mu_\tau d\tau} \tfrac{1}{\gamma} w^{1-\gamma}(s) c_s^\gamma ds \Big).$$

We now solve this problem in two cases; the case where the person controls investment, life insurance purchase and consumption both before and after the retirement time T, and the case where the person's consumption until time T is described by a deterministic process c. The latter case is motivated by a pension saving scheme dictating a fixed amount or a fixed percentage of salary going into a pension savings account during the savings period, leaving the residual salary as fixed consumption.

We start by solving the optimization problem at the time of retirement and thereafter use the solution obtained as the boundary condition in the problem of optimizing before retirement. The Hamilton-Jacobi-Bellman equation for the problem after the time of retirement is

$$\begin{aligned} &\tilde{V}_t(t,x) - \mu_t \tilde{V}(t,x) + \sup_{c,\pi} \Big[\tfrac{1}{\gamma} w^{1-\gamma}(t) c^\gamma \\ &+ \big[(r + \pi(\alpha - r))x - c + \mu_t^* x\big] \tilde{V}_x(t,x) + \tfrac{1}{2}\pi^2 \sigma^2 x^2 \tilde{V}_{xx}(t,x) \Big] = 0, \\ &\lim_{t \to \infty} \tilde{V}(t,x) = 0, \end{aligned}$$

where the boundary condition stems from the assumption $\mu_t \to \infty$ for $t \to \infty$.

Using the technique of Kraft and Steffensen (2008) we find the solution

$$\tilde{V}(t,x) = \tfrac{1}{\gamma}\tilde{f}^{1-\gamma}(t)x^{\gamma}, \tag{3.3}$$

with

$$\tilde{f}(t) = \int_t^{\infty} \mathrm{e}^{-\frac{1}{1-\gamma}\int_t^s (\mu_\tau - \gamma(\mu^*_\tau + \varphi))d\tau} w(s)ds,$$

where

$$\varphi = r + \frac{(\alpha - r)^2}{2\sigma^2(1-\gamma)}.$$

The optimal investment and consumption strategies are in this case given by

$$\begin{aligned} c^*_t &= \frac{w(t)}{f(t)} X_t, \\ \pi^*_t X_t &= \frac{\alpha - r}{\sigma^2(1-\gamma)} X_t, \end{aligned}$$

and we recognize them from Richard (1975) in the special case where utility upon death equals zero.

Now inserting the obtained solution as boundary condition at time T, we get the following Hamilton-Jacobi-Bellman equation for the problem of finding optimal investment and consumption strategies until retirement:

$$\begin{aligned} &V_t(t,x) - \mu_t V(t,x) + \sup_{c,\pi}\Big[\tfrac{1}{\gamma}w^{1-\gamma}(t)c^{\gamma} \\ &+ \big[(r + \pi(\alpha - r))x + a - c + \mu^*_t x\big]V_x(t,x) + \tfrac{1}{2}\pi^2\sigma^2x^2V_{xx}(t,x)\Big] = 0, \\ &V(T,x) = \tfrac{1}{\gamma}\tilde{f}^{1-\gamma}(T)x^{\gamma}. \end{aligned}$$

Solving this problem leads to

$$V(t,x) = \tfrac{1}{\gamma}f^{1-\gamma}(t)(x + g(t))^{\gamma}, \tag{3.4}$$

with

$$f(t) = \int_t^\infty \mathrm{e}^{-\frac{1}{1-\gamma}\int_t^s (\mu_\tau - \gamma(\mu_\tau^* + \varphi))d\tau} w(s)ds, \tag{3.5}$$

$$g(t) = \int_t^T \mathrm{e}^{-\int_t^s (r+\mu_\tau^*)d\tau} a_s ds, \tag{3.6}$$

and optimal controls

$$\begin{aligned} c_t^* &= \frac{w(t)}{f(t)}(X_t + g(t)), \\ \pi_t^* X_t &= \frac{\alpha - r}{\sigma^2(1-\gamma)}(X_t + g(t)). \end{aligned}$$

Inspired by the ideas of Kraft and Steffensen (2008) the functions f and g have the following interpretations:

The function f measures the expected value of the future utility weight w with

$$\bar{\mu} = \frac{1}{1-\gamma}\mu - \frac{\gamma}{1-\gamma}\mu^*$$

as a utility adjusted mortality intensity and

$$\bar{r} = \frac{\gamma}{1-\gamma}\varphi$$

as utility adjusted interest rate, taking into account that the person invests money in the risky asset. Note, that since Kraft and Steffensen (2008) have no risky asset in their optimization problem, they get $\varphi = r$. Letting the counting process Z have intensity $\bar{\mu}$ under $\bar{\mathbb{P}}$, f has the following Feynman-Kač representation:

$$f(t) = \bar{\mathbb{E}}\Big(\int_t^\infty \mathrm{e}^{-\int_t^s \bar{r}_\tau d\tau} \mathbb{1}_{\{Z_s = 0\}} w(s)ds \Big| Z_t = 0 \Big),$$

where the measure $\bar{\mathbb{P}}$ is created with the sole purpose of representing f as a conditional expectation, and is connected to neither the objective nor the pricing measure (note, though, that the intensity $\bar{\mu}$ is a weighted average of the intensities under the objective and the pricing measure, μ and μ^*).

The function g measures the expected value of discounted future income, where the risk-free rate is used as the discounting rate, and the expectation is calculated under the pricing measure $\mathbb{P}^*$. This gives the following Feynman-Kač representation of g:

$$g(t) = \mathbb{E}^*\Big(\int_t^T \mathrm{e}^{-\int_t^s r d\tau} \mathbb{1}_{\{Z_s=0\}} a_s ds \Big| Z_t = 0 \Big).$$

This value is often referred to as human wealth since it represents the financial value of future income. We speak of $x+g$ as the total wealth.

With these interpretations of f and g we find that the optimal consumption at any time is a fraction of total wealth, where the fraction expresses the demand for immediate consumption relative to the demand for future consumption. The optimal amount invested in the risky asset, $\pi^* X$, is a constant proportion of total wealth. In particular, the proportion of wealth invested in the risky asset is constant after retirement.

Fixed Consumption Until Retirement

Assuming that the person can not control consumption until retirement but instead consumes according to a deterministic process c, leads to the optimal value function

$$\begin{aligned} V(t,x) &= \sup_{\pi \in \mathcal{Q}_{[t,T)}} \mathbb{E}_{t,x}\Big(\int_t^T \mathrm{e}^{-\int_t^s \mu_\tau d\tau} \frac{1}{\gamma} w^{1-\gamma}(s) c_s^\gamma ds + \mathrm{e}^{-\int_t^T \mu_s ds} \tilde{V}(T, X_T) \Big) \\ &= \int_t^T \mathrm{e}^{-\int_t^s \mu_\tau d\tau} \frac{1}{\gamma} w^{1-\gamma}(s) c_s^\gamma ds + \bar{V}(t,x), \end{aligned}$$

hereby defining

$$\bar{V}(t,x) \equiv \sup_{\pi \in \mathcal{Q}_{[t,T)}} \mathbb{E}_{t,x}\Big(\mathrm{e}^{-\int_t^T \mu_s ds} \tilde{V}(T, X_T) \Big).$$

Here again X follows the dynamics (3.1)-(3.2). In order for this problem to be well defined, we need to require that the prespecified deterministic consumption process, c, fulfills the condition

$$\int_0^T \mathrm{e}^{-\int_0^s (r+\mu^*_\tau)d\tau}(a_s - c_s)ds > -x_0,$$

which in words means that the person should be able to avoid bankruptcy with probability one.

Inserting $\tilde{V}$ from (3.3), the HJB-equation for the problem formulated in terms of $\bar{V}$ is

$$\begin{aligned}
&\bar{V}_t(t,x) - \mu_t \bar{V}(t,x) \\
&+ \sup_\pi \Big[\big[(r + \pi(\alpha - r))x + a_t - c_t + \mu^*_t x \big] \bar{V}_x(t,x) + \tfrac{1}{2}\pi^2\sigma^2 x^2 \bar{V}_{xx}(t,x) \Big] = 0, \\
&\bar{V}(T,x) = \tfrac{1}{\gamma} \tilde{f}^{1-\gamma}(T) x^\gamma.
\end{aligned}$$

Solving this problem leads to the optimal value function

$$V(t,x) = h(t) + \tfrac{1}{\gamma} f^{1-\gamma}(t)(x + g(t))^\gamma, \tag{3.7}$$

where the functions h, f and g are given by

$$h(t) = \int_t^T \mathrm{e}^{-\int_t^s \mu_\tau d\tau} \tfrac{1}{\gamma} w^{1-\gamma}(s) c_s^\gamma ds, \tag{3.8}$$

$$f(t) = \int_T^\infty \mathrm{e}^{-\frac{1}{1-\gamma}\int_t^s (\mu_\tau - \gamma(\mu^*_\tau + \varphi))d\tau} w(s) ds, \tag{3.9}$$

$$g(t) = \int_t^T \mathrm{e}^{-\int_t^s (r+\mu^*_\tau)d\tau}(a_s - c_s)ds, \tag{3.10}$$

and the optimal amount invested in the risky asset is given by

$$\pi^*_t X_t = \frac{\alpha - r}{\sigma^2(1-\gamma)}(X_t + g(t)).$$

Notice that the Feynman-Kač representation of g is now

$$g(t) = \mathbb{E}^* \Big(\int_t^T \mathrm{e}^{-\int_t^s r d\tau} \mathbb{1}_{\{Z_s = 0\}}(a_s - c_s)ds \Big| Z_t = 0 \Big),$$

which is the present value of expected future income less consumption. As we motivated this problem by saying that the amount $a_t - c_t$ was the prespecified savings premium rate paid to the insurance company at time t, we find that g now can be interpreted as the financial value of future premiums. The optimal investment strategy in the case of fixed consumption until retirement dictates that a fixed proportion of this new total wealth be invested in the risky asset. For the insurance company this strategy seems more appealing as it is only based on the savings, X, and the financial value of future premiums. Still, though, for small values of X_t, we could have that π_t^* is greater than one, which is normally not allowed on savings contracts in practice.

3.4 Two-Person Household

Optimization problems regarding a two-person household are motivated by married couples that either never had children or where the children have left home and are no longer economically dependent of their parents.

In this two-person model the state process Z takes values in $\{0,1\}^2$. For a more compact notation we renumber the states such that state $\{0,0\}$, corresponding to both persons being alive, is state 0, state $\{0,1\}$, corresponding to person A being alive and person B being dead, is state 1, state $\{1,0\}$, corresponding to the opposite, is state 2 and state $\{1,1\}$, corresponding to both persons being dead, is state 3. The states and jump intensities are shown in Figure 3.2.

Given that $Z_t = 0$ (both persons are alive), the wealth of the household follows the dynamics

$$\begin{aligned} dX_t &= (r + \pi_t(\alpha - r))X_t dt + \pi_t \sigma^2 X_t dW_t \\ &\quad + a_t^0 dt - c_t dt + S_t^1 dM_t^{*01} + S_t^2 dM_t^{*02}, \end{aligned} \tag{3.11}$$

$$X_0 = x_0, \tag{3.12}$$

while the wealth process, given that $Z_t = i$, $i = 1, 2$ (one person is alive), follows these dynamics:

$$dX_t = (r + \pi_t(\alpha - r))X_t dt + \pi_t \sigma^2 X_t dW_t + a_t^i dt - c_t dt + S_t^3 dM_t^{*i3}, \tag{3.13}$$

$$X_{\tau_i} = X_{\tau_i -} + S_{\tau_i -}^i. \tag{3.14}$$

Here τ_i is the time of a jump of Z into state i and $S_t^3 = -X_{t-}$ as it is known from Section 3.3.

Written in terms of the optimal value function, the problem of maximizing the expected utility from future consumption looks as follows:

$$\begin{aligned} V^0(t,x) = \sup_{q \in \mathcal{Q}^0_{[t,T)}} \mathbb{E}_{t,x}\Big(\int_t^T & \mathrm{e}^{-\int_t^s (\mu_\tau^{01} + \mu_\tau^{02}) d\tau} \big(\tfrac{1}{\gamma} w_0^{1-\gamma}(s) c_s^\gamma \\ & + \mu_s^{01} V^1(s, X_s + S_s^1) + \mu_s^{02} V^2(s, X_s + S_s^2)\big) ds \\ & + \mathrm{e}^{-\int_t^T (\mu_\tau^{01} + \mu_\tau^{02}) d\tau} \tilde{V}^0(T, X_T)\Big), \end{aligned} \tag{3.15}$$

where X follows the dynamics from (3.11)-(3.14). The supremum in (3.15) is taken over controls regarding state 0, highlighted by writing $\mathcal{Q}^0$, as we

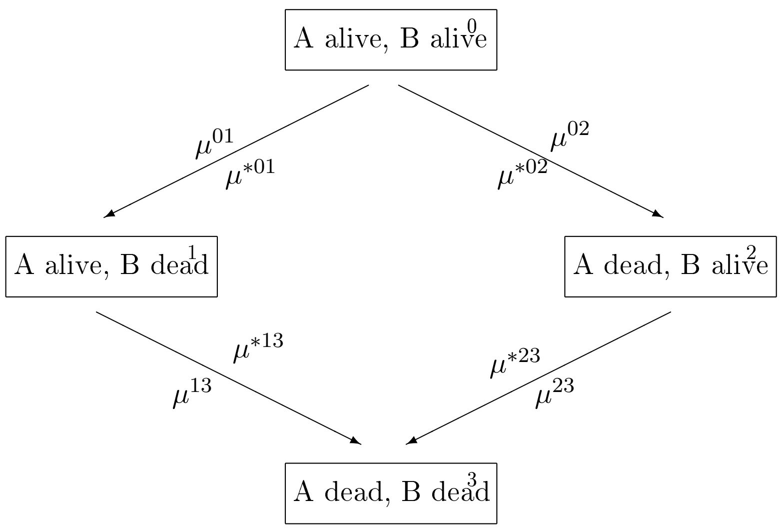

Figure 3.2: *State space and intensities for the two-person model.*

have used the Markov property of Z and X, which allows us to insert the solution from the one-person problem once one of the members of the household is dead. The functions $V^1(t,x)$ and $V^2(t,x)$ are in the form given by (3.4) or (3.7), but with different parameters. The exact form depends on which optimization problem we solve, i.e. whether consumption until time T is controllable or not.

Like in the one-person model we see that the problem of optimizing expected future utility before retirement requires that the problem of optimizing upon retirement be solved in advance. We will not go into details with this particular problem, but just assume that the solution is known.

In order to find the optimal investment, consumption and life insurance processes prior to retirement when both persons are alive, we specify the HJB-equation

$$\begin{aligned}
&V_t^0(t,x) + \sup_{c,\pi,S^1,S^2} \Big[\tfrac{1}{\gamma} w_0^{1-\gamma}(t) c^\gamma + \mu_t^{01}\big(V^1(t,x+S^1) - V^0(t,x)\big) \\
&+ \mu_t^{02}\big(V^2(t,x+S^2) - V^0(t,x)\big) + \big[(r+\pi(\alpha-r))x + a_t^0 - c \\
&- \mu_t^{*01} S^1 - \mu_t^{*02} S^2\big] V_x^0(t,x) + \tfrac{1}{2}\pi^2\sigma^2 x^2 V_{xx}^0(t,x)\Big] = 0, \\
&V^0(T,x) = \tilde{V}^0(T,x).
\end{aligned}$$

Capitalizing on the fact that V^1 and V^2 in this case are in the form (3.4), we arrive at the solution

$$V^0(t,x) = \tfrac{1}{\gamma} f_0^{1-\gamma}(t)(x + g_0(t))^\gamma,$$

with f_0 and g_0 given by

$$\begin{aligned}
f_0(t) &= \int_t^\infty \mathrm{e}^{-\frac{1}{1-\gamma}\int_t^s (\mu_\tau^{01} + \mu_\tau^{02} - \gamma(\mu_\tau^{*01} + \mu_\tau^{*02} + \varphi))d\tau} \big(w_0(s) \\
&\quad + \left(\frac{\mu_s^{01}}{\mu_s^{*01\gamma}}\right)^{\frac{1}{1-\gamma}} f_1(s) + \left(\frac{\mu_s^{02}}{\mu_s^{*02\gamma}}\right)^{\frac{1}{1-\gamma}} f_2(s)\big) ds, \\
g_0(t) &= \int_t^T \mathrm{e}^{-\int_t^s (r + \mu_\tau^{*01} + \mu_\tau^{*02}) d\tau} (a_s^0 + \mu_s^{*01} g_1(s) + \mu_s^{*02} g_2(s)) ds.
\end{aligned}$$

As in the one-person model, introducing the mortality intensities

$$
\begin{aligned}
\bar{\mu}^{0i} &= \left(\frac{\mu^{0i}}{\mu^{*0i\gamma}}\right)^{\frac{1}{1-\gamma}}, \\
\bar{\mu}^{i3} &= \frac{1}{1-\gamma}\mu^{i3} - \frac{\gamma}{1-\gamma}\mu_t^{*i3},
\end{aligned}
$$

and artificial interest rates

$$
\begin{aligned}
\bar{r}_t^0 &= -\frac{1}{1-\gamma}(\mu_t^{01} + \mu_t^{02} - \gamma(\mu_t^{*01} + \mu_t^{*02} + \varphi)) - \bar{\mu}_t^{01} - \bar{\mu}_t^{02}, \\
\bar{r}_t^i &= \frac{\gamma}{1-\gamma}\varphi,
\end{aligned}
$$

for $i = 1, 2$, we get these Feynman-Kač representations for f_0 and g_0:

$$
\begin{aligned}
f_0(t) = \bar{\mathbb{E}}\Big(\int_t^\infty & \mathrm{e}^{-\int_t^s \bar{r}_\tau^{Z_\tau} d\tau} (\mathbb{1}_{\{Z_s=0\}} w_0(s) \\
&+ \mathbb{1}_{\{Z_s=1\}} w_1(s) + \mathbb{1}_{\{Z_s=2\}} w_2(s)) ds \Big| Z_t = 0 \Big), \\
g_0(t) = \mathbb{E}^*\Big(\int_t^T & \mathrm{e}^{-\int_t^s r d\tau} (\mathbb{1}_{\{Z_s=0\}} a_s^0 + \mathbb{1}_{\{Z_s=1\}} a_s^1 + \mathbb{1}_{\{Z_s=2\}} a_s^2) ds \Big| Z_t = 0 \Big).
\end{aligned}
$$

Like in the one-person model we find that f_0 measures the expected value of the future utility weights, and that g_0 is the human wealth of the household.

The optimal controls, given that $Z_t = 0$, are

$$
\begin{aligned}
c_t^* &= \frac{w_0(t)}{f_0(t)}(X_t + g_0(t)), \\
\pi_t^* X_t &= \frac{\alpha - r}{\sigma^2(1-\gamma)}(X_t + g_0(t)), \\
S_t^{i*} &= \left(\frac{\mu_t^{0i}}{\mu_t^{*0i}}\right)^{\frac{1}{1-\gamma}} \frac{f_i(t)}{f_0(t)}(X_t + g_0(t)) - (X_t + g_i(t)), i = 1, 2.
\end{aligned}
$$

Again, as in the one-person model, the optimal consumption rate at any time is a fraction of total wealth, balancing the demands for immediate and future consumption. We notice that the demand for future consumption takes into account also the demand for consumption after the death of one of the persons.

For the optimal investment strategy we see that the household finds it optimal to invest a constant proportion of the total wealth of the household. The optimal proportion in the two-person model is exactly the same as in the one-person model. This means that after the death of one of the persons in the household, the surviving person invests the same proportion of the new total wealth in the risky security as before the spouse's death. Mathematically, this effect arises because we assume that the risk aversion parameter, γ, used in the utility functions, is the same in all states.

Instead of commenting on the optimal sum insured we focus on the optimal wealth of the surviving person in case the other person dies at time t. In case of a jump into state i, $i = 1, 2$, this amount is given by

$$X_t + g_i(t) + S_t^{i*} = \left(\frac{\mu_t^{0i}}{\mu_t^{*0i}} \right)^{\frac{1}{1-\gamma}} \frac{f_i(t)}{f_0(t)} (X_t + g_0(t)).$$

The amount is calculated as a fraction of total wealth of the household. The first term,

$$\left(\frac{\mu_t^{0i}}{\mu_t^{*0i}} \right)^{\frac{1}{1-\gamma}},$$

measures the market price of the life insurance risk relative to the household's expectation, where the term also involves the risk aversion parameter, γ.

The second term,

$$\frac{f_i(t)}{f_0(t)},$$

measures the demand for future consumption after the person's death relative to the demand for future consumption before the person's death.

Note that with the convention $g_i(t) = 0$ for $t \geq T$ and $i = 0, 1, 2$, the above optimal value function, controls and the interpretation of them hold also for the time after retirement.

Fixed Consumption Until Retirement

Like in the one-person model, inspired by fixed contribution saving schemes, we introduce the problem of finding optimal investment and life insurance processes for the time until retirement, given that consumption in state i follows a deterministic process c^i, $i = 0, 1, 2$. The optimal value function for this problem is given by (3.15), where X follows the dynamics given by (3.11)-(3.14) and $q \in \mathcal{Q}^0_{[t,T)}$ reflects that the consumption-process until retirement is fixed.

Solving this problem is similar to solving the original problem and we therefore immediately write up the solution to the problem. Given that the prespecified fixed consumption process, c, fulfills the bankruptcy criterion given by

$$\int_0^T \mathrm{e}^{-\int_t^s (r+\mu^{*01}_\tau+\mu^{*02}_\tau)d\tau}(a^0_s - c^0_s + \mu^{*01}_s g_1(s) + \mu^{*02}_s g_2(s))ds > -x_0,$$

with

$$g_i(t) = \int_t^T \mathrm{e}^{-\int_t^s (r+\mu^{*i3}_\tau)d\tau}(a^i_s - c^i_s)ds, \ i = 1, 2,$$

the optimal value function is given by

$$V^0(t,x) = h_0(t) + \tfrac{1}{\gamma} f_0^{1-\gamma}(t)(x + g_0(t))^\gamma,$$

with h_0, f_0 and g_0 given by

$$\begin{aligned}
h_0(t) = & \int_t^T \mathrm{e}^{-\int_t^s (\mu^{01}_\tau+\mu^{02}_\tau)d\tau}(\tfrac{1}{\gamma} w_0^{1-\gamma}(s)(c^0_s)^\gamma + \mu^{01}_s h_1(s) + \mu^{02}_s h_2(s))ds, \\
f_0(t) = & \int_t^T \mathrm{e}^{-\frac{1}{1-\gamma}\int_t^s (\mu^{01}_\tau+\mu^{02}_\tau-\gamma(\mu^{*01}_\tau+\mu^{*02}_\tau+\varphi))d\tau}\Big(\left(\frac{\mu^{01}_s}{\mu^{*01\gamma}_s}\right)^{\frac{1}{1-\gamma}} f_1(s) \\
& + \left(\frac{\mu^{02}_s}{\mu^{*02\gamma}_s}\right)^{\frac{1}{1-\gamma}} f_2(s)\Big)ds + \int_T^\infty \mathrm{e}^{-\frac{1}{1-\gamma}\int_t^s (\mu^{01}_\tau+\mu^{02}_\tau-\gamma(\mu^{*01}_\tau+\mu^{*02}_\tau+\varphi))d\tau} \\
& \Big(w_0(s) + \left(\frac{\mu^{01}_s}{\mu^{*01\gamma}_s}\right)^{\frac{1}{1-\gamma}} f_1(s) + \left(\frac{\mu^{02}_s}{\mu^{*02\gamma}_s}\right)^{\frac{1}{1-\gamma}} f_2(s)\Big)ds, \\
g_0(t) = & \int_t^T \mathrm{e}^{-\int_t^s (r+\mu^{*01}_\tau+\mu^{*02}_\tau)d\tau}(a^0_s - c^0_s + \mu^{*01}_s g_1(s) + \mu^{*02}_s g_2(s))ds.
\end{aligned}$$

The functions h_i, f_i and g_i, $i = 1, 2$ are in the forms (3.8)-(3.10). We find optimal controls that are in the same form as in the original problem in the two-person model, namely

$$\pi_t^* X_t = \frac{\alpha - r}{\sigma^2(1-\gamma)}(X_t + g_0(t)),$$
$$S_t^{i*} = \left(\frac{\mu_t^{0i}}{\mu_t^{*0i}}\right)^{\frac{1}{1-\gamma}} \frac{f_i(t)}{f_0(t)}(X_t + g_0(t)) - (X_t + g_i(t)), \; i = 1, 2.$$

As the f and g functions are changed compared to the original problem, we find that the optimal amount of capital invested risky in this case is smaller than in the original problem due to the fact that the function g_0 is smaller in this case. Like in the one-person model this effect arises as the function takes into account that the household consumes according to a fixed consumption strategy until retirement.

Also the optimal sums insured is different from in the original case, but whether they are smaller or larger depends on the fixed consumption rate, c.

3.5 Numerical Examples for Two-Person Households

In this section a few interesting and realistic examples illustrate what this type of model can capture. For each different parametrization of the model we show the expected development over time of the wealth process and the control processes in the case where the household controls consumption, investment and life insurance purchase both before and after retirement.

The utility of consumption in the household should somehow reflect that some costs are shared. E.g., a couple should not spend $2 together to be as happy as they would be by spending $1 each separately (they might be happier separated, but we are not trying to model that!).

Hong and Ríos-Rull (2012) study a discrete model, where the connection between the utility functions is in the form

$$u_0(t, \tfrac{c}{\theta_0}) = \xi u_1(t, \tfrac{c}{\theta_1}) + (1-\xi)u_2(t, \tfrac{c}{\theta_2}),$$

with

$$u_i(t,c) = \mathrm{e}^{-\rho t}\tfrac{1}{\gamma}c^{\gamma},\ i = 0,1,2,$$

for $\gamma = -2$. They estimate parameters based on American data on life insurance holdings of households consisting of persons with ages varying between 15-85. Under the hypothesis of no difference between the two persons in the household (which they formalize as $\xi = \frac{1}{2}$ and $\theta_1 = \theta_2 = 1$) they estimate $\theta_0 = 1.08$. Under this hypothesis, and with the risk aversion factor $\gamma = -2$, we find

$$u_0(t, 1.53c) = u_1(t,c) + u_2(t,c), \tag{3.16}$$

which means that the marginal utility from spending \$1.53 together equals the marginal utility from spending \$1 separately. Note, that Hong and Ríos-Rull (2012) reject the hypothesis of no differences between man and wife at the 0.1 percent level, but for comparison of the graphics based on differences between man and wife, we still choose to present graphics based on this hypothesis.

Assuming that all time dependence is captured by the impatience factor, ρ, we arrive at these utility weights for the three states:

$$w_0^{1-\gamma}(t) = 1.53^{-\gamma}\mathrm{e}^{-\rho t}, \tag{3.17}$$
$$w_i^{1-\gamma}(t) = \mathrm{e}^{-\rho t},\ i = 1,2. \tag{3.18}$$

The jump intensities of Z are assumed to be equal across possible jumps, across the measures $\mathbb{P}$ and $\mathbb{P}^*$, and in the form

$$\mu_t = \mu_t^* = \tilde{A} + 10^{\tilde{B}+\tilde{C}(z^i+t)-10},\ i = A, B, \tag{3.19}$$

where z is the age at time 0 for the person exposed to death.

Based on the parameters in Table 3.1 we get the expected optimal controls illustrated in Figure 3.3. Since we have not introduced any differences between the two persons, the optimal controls connected to the states where only one person is alive are the same no matter who it is. The expected consumption is decreasing, which is basically due to the fact that the impatience factor is greater than the expected drift of the investment portfolio. Note that we have the relationship $c_t^{i*} = \frac{w_i(t)}{w_0(t)} c_t^{0*}$, $t \geq 0$, $i = 1, 2$, which only holds in this case where $\mu = \mu^*$.

Now, we turn to the case of investigating how differences between the two persons in the household influence the optimal controls. We consider differences in the salary processes, and differences in the utility functions and mortality intensities after the death of a spouse.

Assuming that $a^0 = 90\,000$, $a^1 = 30\,000$ and $a^2 = 60\,000$, corresponding

	Description	Value
z^A, z^B	Age of person A and B at time of optimization	50
T	Time of retirement	15
x	Initial wealth in USD at time of optimization	400 000
a^0	Constant income rate in USD until retirement in state 0	90 000
a^1, a^2	Constant income rates in USD until retirement in state 1 and 2	45 000
ρ	Impatience factor for all states of Z	0.05
γ	Risk aversion parameter needed in utility functions	-2
r	The constant drift of the risk free security	0.02
α	The constant drift of the risky security	0.05
σ	The constant volatility of the risky security	0.2
A	Parameter for mortality intensity	0.002353
B	Parameter for mortality intensity	5.102232
C	Parameter for mortality intensity	0.04550

Table 3.1: *Parameters needed for numerical results in Section 3.5. Note that r, α and σ are thought of as corrected for inflation, and that the mortality intensity parameters are estimated based on deaths of people over the age of 50 in America in 2006.*

to person A's income rate equaling half person B's, the expected optimal controls are as shown in Figure 3.4. As we have only changed the salary intensities, the household demands the same consumption after the death of each spouse. In order to finance that, the demand for life insurance on each life is different until the time of retirement.

The opposite scenario occurs if we instead introduce differences between the two persons in their demand for consumption. Hong and Ríos-Rull (2012) estimate $\theta_0 = 1.33$, $\theta_1 = 4.76$, $\theta_2 = 1$ and $\xi = 0.07$ in their full model, where the widow is alive in state 1 and the widower is alive in state 2. Based on these estimates, and with $\gamma = -2$, we find

$$u_0(t, 1.38c) = u_1(t, 1.30c) + u_2(t, c),$$

which shows that marriage forms economic habit for the widow but not for the widower.

With person A being the wife and person B being the husband we find these weight functions

$$w_0^{1-\gamma}(t) = 1.38^{-\gamma} \mathrm{e}^{-\rho t}, \tag{3.20}$$

$$w_1^{1-\gamma}(t) = 1.30^{-\gamma} \mathrm{e}^{-\rho t}, \tag{3.21}$$

$$w_2^{1-\gamma}(t) = \mathrm{e}^{-\rho t}. \tag{3.22}$$

Expected optimal controls based on these utility weights are presented in Figure 3.5. Now, the household decides on different consumption strategies for the widow and widower, and in order to finance those strategies they choose different sums insured on their lives while they are both alive. Note, that we still have $c_t^{i*} = \frac{w_i(t)}{w_0(t)} c_t^{0*}$, $t \geq 0$, $i = 1, 2$.

The last example is a change in the mortality intensities upon death of the spouse. Numerous empirical studies have shown an increase in the mortality intensity of a widower upon death of his wife (see e.g. Liu (2009) and Martikainen and Valkonen (1996)). Most studies also suggest increasing mortality intensity of a widow upon death of her husband, but a few (see e.g. Nagata et al. (2003) and de Leon et al. (1993), who find it only in small scale and only for women over the age of 75) suggest that the effect could be opposite (the mourning widow sometimes looks surprisingly refreshed under the veil). In order to fully see the effects on the controls that the household decides upon, when introducing an upward jump in the mortality intensity for the husband and a downward jump for the wife, upon death of their spouse, we introduce these very dramatic intensities:

$$\mu^{01} = \mu^{02} = \mu^{*}, \tag{3.23}$$

$$\mu^{13} = \tfrac{1}{4}\mu^{*}, \tag{3.24}$$

$$\mu^{23} = 4\mu^{*}. \tag{3.25}$$

Based on these intensities we find the expected controls as is seen in Figure 3.6. Notice that even though the sums insured are different, the initial consumption upon death of a spouse is the same for the widow and the widower, since

$$\begin{aligned}
\hat{c}_t^{i*} &= \frac{w_i(t)}{f_i(t)}(X_t + S_t^{i*} + g_i(t)) \\
&= \frac{w_i(t)}{f_0(t)}\left(\frac{\mu_t^{0i}}{\mu_t^{0i*}}\right)^{\frac{1}{1-\gamma}}(X_t + g_0(t)),
\end{aligned}$$

$i = 1, 2$, where $\hat{c}_t^{i*}$ is the initial consumption in state i upon a jump into the state at time t.

The expected consumption after the death of the spouse evolves in different directions for the widow and the widower. Since $\mu^{13} < \mu^{*13} = \mu^{*23} < \mu^{23}$, we find that $f_1 < f_2$, which again means that the widower will consume a larger fraction of his total wealth than the widow will. Furthermore, as the initial wealth of the widower upon death of his spouse is smaller than that of the widow, we find that his consumption is decreasing in time relative to the consumption when they are both alive, while her consumption is increasing.

3.6 Multiple-Person Household

In this section we give a brief description of how to formulate and solve the optimization problem for a household consisting of multiple persons. This problem is motivated by married couples with economically dependent children or economically dependent parents/grand parents. We only deal with the problem of optimizing prior to retirement, assuming that the problem of optimizing at time of retirement is solved in advance.

When modeling an n-person household, $n \in \mathbb{N}$, the state process Z takes values in $\{0,1\}^n$, where state $\{0,0,\dots,0\}$ is the state where all persons are alive, while state $\{1,1,\dots,1\}$ is the state where all persons are dead. For a more compact notation we rename state $\{0,0,\dots,0\}$ to state 0.

The utility connected to consumption in state i at time t is given by

$$\tfrac{1}{\gamma} w_i^{1-\gamma}(t) c^\gamma, \;\; i \in \{0,1\}^n \setminus \{1,1,\dots,1\}$$

and the wealth process of the household follows these dynamics

$$\begin{aligned} dX_t &= (r + \pi_t(\alpha - r))X_t dt + \pi_t \sigma^2 X_t dW_t \\ &\quad + a_t^{Z_t} dt - c_t dt + \sum_{i \in \mathcal{Z}_{t-}} S_t^i dM_t^{*Z_{t-} i}, \qquad (3.26) \\ X_0 &= x_0. \qquad (3.27) \end{aligned}$$

As in the previous models concerning one- and two-person households, we assume that $a_t^i = 0$ for all $t \geq T$ and $i \in \{0,1\}^n \setminus \{1,1,\dots,1\}$.

Now the general optimization problem for the n-person household is specified through the optimal value function

$$\begin{aligned} V^0(t,x) = \sup_{q \in \mathcal{Q}_{[t,T)}} \mathbb{E}_{t,x} \Big(\int_t^T & \mathrm{e}^{-\int_t^s \mu_\tau^{0\cdot} d\tau} \big(\tfrac{1}{\gamma} w_0^{\gamma-1}(s) c_s^\gamma \\ & + \sum_{i \in \mathcal{Z}_t} \mu_s^{0i} V^i(s, X_s + S_s^i)\big) ds + \mathrm{e}^{-\int_t^T \mu_\tau^{0\cdot} d\tau} \tilde{V}^0(T, X_T) \Big), \end{aligned} \qquad (3.28)$$

where X follows the dynamics from (3.26)-(3.27).

As we want to solve this problem by mathematical induction we now proceed as follows:

- State the solution to the problem.
- Show that the solution is correct for $n = 1$.

- For $n \geq 2$: Assume that the solution is correct for $n-1$ and show that the solution is then also correct for n.

Inspired by the work in Sections 3.3 and 3.4 we now come up with the following solution to the problem described by (3.26)-(3.28), where consumption is controlled at all times, stated in terms of the optimal value function and optimal controls:

$$V^0(t,x) = \tfrac{1}{\gamma} f_0^{1-\gamma}(t)(x+g_0(t))^\gamma, \tag{3.29}$$

where f_0 and g_0 are given by

$$\begin{aligned} f_0(t) &= \int_t^\infty \mathrm{e}^{-\frac{1}{1-\gamma}\int_t^s (\mu_\tau^{0\cdot} - \gamma(\mu_\tau^{*0\cdot}+\varphi))d\tau} (w_0(s) + \sum_{i\in\mathcal{Z}_t} \left(\frac{\mu_s^{0i}}{\mu_s^{*0i\gamma}}\right)^{\frac{1}{1-\gamma}} f_i(s))ds, \\ g_0(t) &= \int_t^T \mathrm{e}^{-\int_t^s (r+\mu_\tau^{*0\cdot})d\tau} (a_s^0 + \sum_{i\in\mathcal{Z}_t} \mu_s^{*0i} g_i(s))ds. \end{aligned}$$

The optimal controls in state 0 are

$$\begin{aligned} c_t^* &= \frac{w_0(t)}{f_0(t)}(X_t + g_0(t)), \qquad (3.30) \\ \pi_t^* X_t &= \frac{\alpha - r}{\sigma^2(1-\gamma)}(X_t + g_0(t)), \qquad (3.31) \\ S_t^{i*} &= \left(\frac{\mu_t^{0i}}{\mu_t^{*0i}}\right)^{\frac{1}{1-\gamma}} \frac{f_i(t)}{f_0(t)}(X_t + g_0(t)) - (X_t + g_i(t)),\ i \in \mathcal{Z}_t. \qquad (3.32) \end{aligned}$$

In Section 3.3 we showed that (3.29) is correct for $n = 1$. For $n \geq 2$ we assume that (3.29) is correct for $n-1$. Considering the optimization problem for an n-person household, the optimal value function (3.28) becomes

$$\begin{aligned} V^0(t,x) = \sup_{q\in\mathcal{Q}^0_{[t,T)}} \mathbb{E}_{t,x}\Big(\int_t^T & \mathrm{e}^{-\int_t^s \mu_\tau^{0\cdot} d\tau} (\tfrac{1}{\gamma} w_0^{\gamma-1}(s) c_s^\gamma \\ & + \sum_{i\in\mathcal{Z}_t} \mu_s^{0i} \tfrac{1}{\gamma} f_i^{1-\gamma}(s)(X_s + S_s^i + g_i(s))^\gamma)ds + \mathrm{e}^{-\int_t^T \mu_\tau^{0\cdot} d\tau} \tilde{V}^0(T, X_T)\Big). \end{aligned}$$

Here X follows the dynamics from (3.26)-(3.27), and $\tilde{V}^0(T,x) = \frac{1}{\gamma} f_0^{1-\gamma}(s)(x + g_0(T))^\gamma$ is assumed to be known from solving the optimization problem at the time of retirement. Note, that the assumption that the solution (3.29) holds for $n-1$ means, in particular, that all functions V^i in (3.28) are in the form (3.29).

The HJB-equation for this problem is

$$
\begin{aligned}
&V_t^0(t,x) + \sup_{c,\pi,S} \Big[\tfrac{1}{\gamma} w_0^{1-\gamma}(t) c^\gamma - \sum_{i \in \mathcal{Z}_t} \mu_t^{0i} \big(V^0(t,x) - \tfrac{1}{\gamma} f_i^{1-\gamma}(t)(x + S^i + g_i(t))^\gamma \big) \\
&+ \big[(r + \pi(\alpha - r))x + a_t^0 - c - \sum_{i \in \mathcal{Z}_t} \mu_t^{*0i} S^i \big] V_x^0(t,x) + \tfrac{1}{2}\pi^2\sigma^2 x^2 V_{xx}^0(t,x) \Big] = 0, \\
&V^0(T,x) = \tfrac{1}{\gamma} f_0^{1-\gamma}(T)(x + g_0(T))^\gamma,
\end{aligned}
$$

where $S = \{S^i\}_{i \in \mathcal{Z}_0}$. This problem is solved by the optimal value function (3.29) and the optimal controls in (3.30)-(3.32).

Fixed Consumption Until Retirement

As in the one- and two-person models, we can also formulate this problem in the n-person model. The bankruptcy condition is now

$$
\int_0^T \mathrm{e}^{-\int_0^s (r + \mu_\tau^{*0\cdot}) d\tau} (a_s^0 - c_s^0 + \sum_{i \in \mathcal{Z}_0} \mu_s^{*0i} g_i(s)) ds > -x_0.
$$

This problem can also be solved by mathematical induction, which leads to the optimal value function

$$
V^0(t,x) = h_0(t) + \tfrac{1}{\gamma} f_0^{1-\gamma}(t)(x + g_0(t))^\gamma,
$$

with h_0, f_0 and g_0 given by

$$\begin{aligned}
h_0(t) &= \int_t^T \mathrm{e}^{-\int_t^s \mu_\tau^{0\cdot} d\tau} \left(\tfrac{1}{\gamma} w_0^{1-\gamma}(s)(c_s^0)^\gamma + \sum_{i\in\mathcal{Z}_t} \mu_s^{0i} h_i(s)\right) ds, \\
f_0(t) &= \int_t^T \mathrm{e}^{-\frac{1}{1-\gamma}\int_t^s (\mu_\tau^{0\cdot} - \gamma(\mu_\tau^{*0\cdot}+\varphi)) d\tau} \sum_{i\in\mathcal{Z}_t} \left(\frac{\mu_s^{0i}}{\mu_s^{*0i\gamma}}\right)^{\frac{1}{1-\gamma}} f_i(s) ds \\
&\quad + \int_T^\infty \mathrm{e}^{-\frac{1}{1-\gamma}\int_t^s (\mu_\tau^{0\cdot} - \gamma(\mu_\tau^{*0\cdot}+\varphi)) d\tau} \left(w_0(s) + \sum_{i\in\mathcal{Z}_t} \left(\frac{\mu_s^{0i}}{\mu_s^{*0i\gamma}}\right)^{\frac{1}{1-\gamma}} f_i(s)\right) ds, \\
g_0(t) &= \int_t^T \mathrm{e}^{-\int_t^s (r+\mu_\tau^{*0\cdot}) d\tau} \left(a_s^0 - c_s^0 + \sum_{i\in\mathcal{Z}_t} \mu_s^{*0i} g_i(s)\right) ds.
\end{aligned}$$

The optimal controls for state 0 are

$$\begin{aligned}
\pi_t^* X_t &= \frac{\alpha - r}{\sigma^2(1-\gamma)} (X_t + g_0(t)), \\
S_t^{i*} &= \left(\frac{\mu_t^{0i}}{\mu_t^{*0i}}\right)^{\frac{1}{1-\gamma}} \frac{f_i(t)}{f_0(t)} (X_t + g_0(t)) - (X_t + g_i(t)),\ i \in \mathcal{Z}_t.
\end{aligned}$$

3.7 Final Remarks

The model framework introduced in this paper can be extended in several ways. This could happen through a more sophisticated investment market, more decisions to make for the household regarding insurance against loss of income upon onset of disability or loss of employment, but also housing finance, e.g. mortgage financing of houses, could be introduced to the model. Of particular interest we find the introduction of more realism in the modeling of utility, though. This could be through the introduction of time and/or state dependent risk aversion, recursive utility or habit formation arising from consumption in the past.

Acknowledgment

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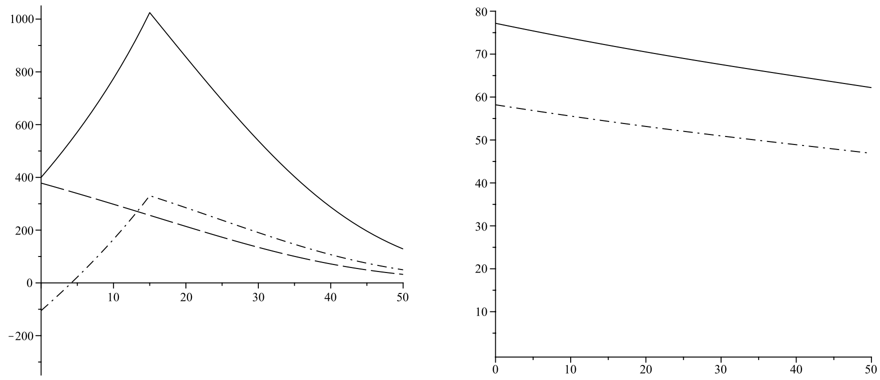

Figure 3.3: *Numerical results in USD 1 000, based on the parameters in Table 3.5, the utility weights in* (3.17)-(3.18) *and the mortality intensities in* (3.19). *Left: Wealth in state 0 (full line), optimal amount spent on life annuities in state 0 (that is $-S^1$ and $-S^2$) for each person (dash-dotted line) and optimal investment in the risky security in state 0 (long-dashed line). Right: Optimal consumption in state 0 (full-drawn line) and optimal consumption after death of one of the persons (dash-dotted line).*

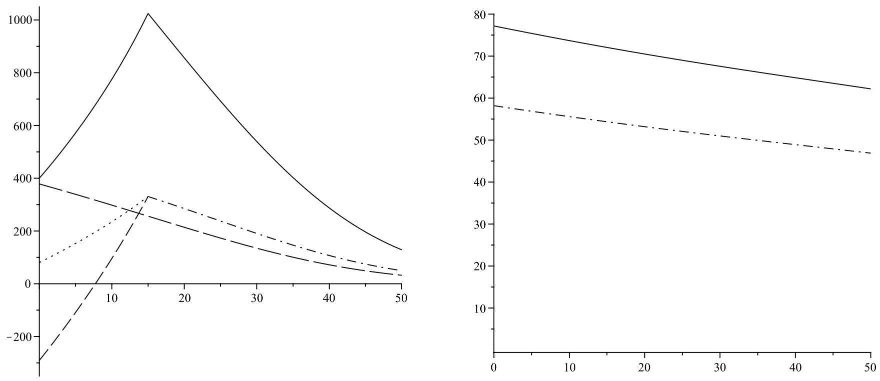

Figure 3.4: *Numerical results in USD 1 000, based on the parameters in Table 3.5, the utility weights in* (3.17)-(3.18) *and the mortality intensities in* (3.19)*, except for that a^0=90 000, a^1=30 000 and a^2=60 000. Left: Wealth in state 0 (full line), optimal amount spent on life annuity in state 0 on person A's life ($-S^1$ =dashed line), optimal amount spent on life annuity on person B's life ($-S^2$=dotted line) and optimal investment in the risky security in state 0 (long-dashed line). Right: Optimal consumption in state 0 (full-drawn line) and optimal consumption after death of one of the persons (dash-dotted line).*

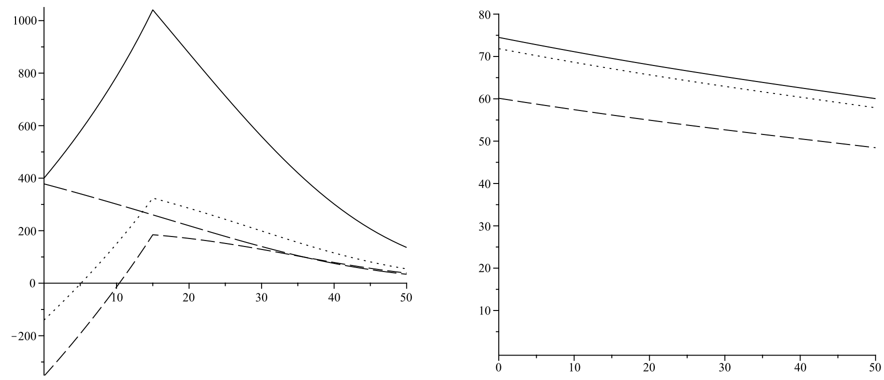

Figure 3.5: *Numerical results in USD 1 000, based on the parameters in Table 3.5, the utility weights in* (3.20)-(3.22) *and the mortality intensities in* (3.19)*. Left: Wealth in state 0 (full line), optimal amount spend on life annuity in state 0 on person A's life ($-S^1$=dashed line), optimal amount spend on life annuity on person B's life ($-S^2$=dotted line) and optimal investment in the risky security in state 0 (long-dashed line). Right: Optimal consumption in state 0 (full-drawn line), optimal consumption after death of person A (dotted line) and optimal consumption after death of person B (dashed line).*

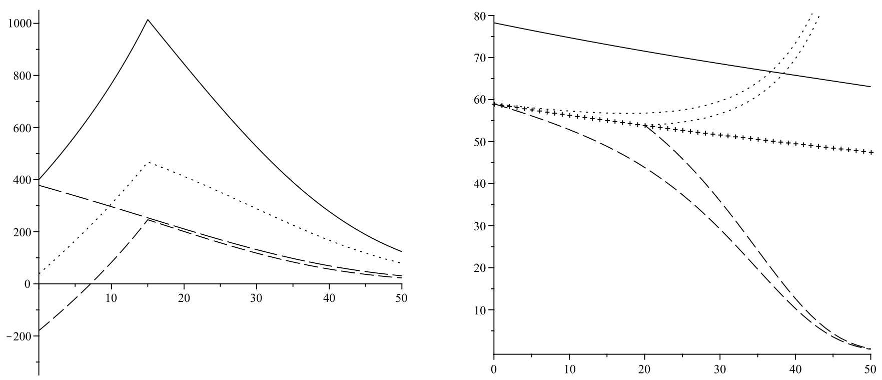

Figure 3.6: *Numerical results in USD 1 000, based on the parameters in Table 3.5, the utility weights in* (3.17)-(3.18) *and mortality intensities from* (3.23)-(3.25). *Left: Total wealth in state 0 (full line), optimal amount spend on life annuity in state 0 on person A's life ($-S^1$=dashed line), optimal amount spend on life annuity in state 0 on person B's life ($-S^2$=dotted line) and optimal investment in the risky security in state 0 (long-dashed line). Right: Optimal consumption in state 0 (full line), optimal initial consumption for the surviving person upon death of the other (marked with '+'), optimal consumption after death of person A at time 0 and time 20 (dotted line) and after death of person B at time 0 and time 20 (dashed line).*

4. Recursive Utility with Utility Driven Habit Formation

Kenneth Bruhn and Mogens Steffensen.

Abstract Literature on optimal investment and consumption includes forward looking utility functionals (recursive utility) as well as backward looking functionals (habit formation). In either case, present utility relates to consumption at other points in time, in general leading to more flexible modeling of preferences allowing for a better fit with observable behavior. Whereas recursive utility relates present utility to expected future utility, habit formation traditionally relates utility to past consumption.

The contribution of this paper is a continuous-time model that combines forward and backward looking preferences. We choose to model internal habit formation stemming from past utility of consumption. The model allows for analytical solutions regarding optimal consumption and investment in a Black-Scholes investment market.

Keywords *Stochastic Control; Recursive Utility; Multiplicative Habit Formation; Utility Driven Habit Formation*

4.1 Introduction

Samuelson (1952): *"The amount of wine I drank yesterday and will drink tomorrow can be expected to have effects upon my today's indifference slope between wine and milk"*.

Based on the novel idea that present utility depends on past or expected future utility, we develop a model for preferences that combines the two.

The foundation for each element of this modeling is established in great detail by references to previous related literature.

The strand of literature on habit formation specification starts with Duesenberry (1949) who proposes that utility from current consumption can be affected by the level of past consumption. The idea was first formally developed by Pollak (1970) and Ryder and Heal (1973). More recent analysis has shown that habit formation can help explain e.g. the equity-premium puzzle (see. Abel (1990), Campbell and Cochrane (1999), Constantinides (1990)) as well as extraordinary saving in high growth scenarios, see Carroll et al. (2000).

When we refer to continuous-time recursive utility, we have in mind stochastic differential utility developed by Duffie and Epstein (1992a,b), which is a continuous-time limit of the type of recursive utility studied in Kreps and Porteus (1978) and Epstein and Zin (1989) among others. The modeling allows for disentangling of relative risk aversion (RRA) and elasticity of intertemporal substitution (EIS) in the utility functional and has been used for addressing various asset pricing puzzles, e.g. see the list in Kraft et al. (2012).

Recently, Steffensen (2011) developed a new approach that allows for disentangling RRA and EIS. The idea is based on time-consistency in the solution to a non-linear optimization problem involving the certainty equivalent of future consumption rates. The modeling in Steffensen (2011) introduces a different method for disentangling of RRA and EIS in continuous-time. The method allows us a different understanding of the preference structure in optimal consumption and investment problems. The methodology is in this paper used for motivating the structure of habit formation arising from past utility of consumption.

Combining recursive utility and habit formation in preferences is briefly discussed in Duffie and Epstein (1992b), who establish foundations for the existents of a solution to the problem. We build on the existence in the formulation of a model allowing for the combination of recursive utility and habit formation in a simple Black-Scholes investment market. The analytical solution to the optimization problem related to the model allows for simple comparison with existing related problems and increase flexibility in calibration to empirical consumption behavior.

In this paper, we reserve a great amount of space for introduction of the modeling in existing literature mentioned above. Our motivation for this is, that the foundation on which we build our model must be well established. Part of this paper can be considered a small survey on habit formation and recursive utility literature but is entirely meant for setting the stage for integrating the two.

The paper proceeds as follows: Section 4.2 presents the investment market model and Section 4.3 contains the mini-survey on the relevant related optimization problems briefly presented in the previous paragraphs. In Section 4.4 we present the model and solves the optimization problem and Section 4.5 concludes.

4.2 Investment Market and Wealth Dynamics

Throughout the paper we work with a complete Black-Scholes investment market with *bond* dynamics

$$dM_t = rM_t dt, \ M_0 = m,$$

and *stock* dynamics

$$dS_t = S_t[(r+\lambda)dt + \sigma dW_t], \ S_0 = s.$$

The interest rate, r, the excess return, λ, and the volatility, σ, are assumed constant for simplicity. This particular simple investment market model is chosen for its simplicity since we are not interested in the interaction between preferences and investment market structure. For the case of recursive utility with a more complex investment market structure we refer to Kraft et al. (2012) and Steffensen (2011). Utility maximization with standard power utility and more complex investment markets is investigated in numerous papers, see e.g. Munk et al. (2004) and the references therein.

We consider a price-taking investor who takes decisions concerning consumption and investment over a fixed time horizon. The *wealth* of our investor follows the dynamics:

$$dX_t = X_t[(r + \pi_t\lambda)dt + \pi_t\sigma dW_t] - c_t dt,\ X_0 = x_0, \tag{4.1}$$

where π_t is the proportion of wealth invested in the stock, and c_t is the total consumption rate at time t.

4.3 Existing Relevant Optimization Problems

In this section we present existing optimization models based on the theory of time-additive utility maximization, recursive utility and habit formation. Based on these foundations we propose a new model in Section 4.4.

4.3.1 Merton's Problem

Before we turn to models allowing for recursive utility and habit formation in preferences, we present the classical/original time-additive continuous-time utility optimization problem, formulated and solved in Merton (1969, 1971). The optimization problems discussed in the following are all generalizations of this particular model. Merton (1969, 1971) consider a utility maximizer with time-additive preferences for consumption, considering the optimization criterion

$$\sup_{c,\pi} \mathbb{E}\left[\int_0^n \mathrm{e}^{-\delta t}\tfrac{1}{1-\gamma}c_t^{1-\gamma}dt\right], \tag{4.2}$$

subject to wealth dynamics (4.1), where $\delta \geq 0$ is a subjective utility discount rate. The problem comes from considering utility from consumption based on a power utility function in the form

$$u(c) = \tfrac{1}{1-\gamma}c^{1-\gamma},$$

where γ is relative risk aversion (RRA) and γ^{-1} is elasticity of intertemporal substitution (EIS).

In order to solve the problem (4.1)-(4.2), it is embedded in a value function given by

$$V(t,x) = \sup_{c,\pi} \mathbb{E}_{t,x}\left[\int_t^n \mathrm{e}^{-\delta(s-t)} \tfrac{1}{1-\gamma} c_s^{1-\gamma} ds\right], \tag{4.3}$$

where $\mathbb{E}_{t,x}$ is the conditional expectation given that $X_t = x$. We see that the value function gives us the indirect utility from wealth at any time $t \in [0, n]$.

The problem is solved by the indirect utility function

$$V(t,x) = \tfrac{1}{1-\gamma} g(t)^{\gamma} x^{1-\gamma},$$

with

$$g(t) = \int_t^n \mathrm{e}^{-\frac{1}{\gamma}\int_t^s \left[\delta-(1-\gamma)\left(r+\frac{\lambda^2}{2\sigma^2\gamma}\right)\right] d\tau} ds,$$

and the optimal controls

$$c_t^* = \frac{X_t}{g(t)},$$
$$\pi_t^* = \frac{\lambda}{\sigma^2\gamma}.$$

The optimal controls dictate that consumption at time t is proportional to the wealth at time t. The function g that determine the proportion consumed has the interpretation of an annuity weighting preferences for immediate relative to future consumption. The optimal exposure of wealth to investment risk dictates that a fixed constant proportion of wealth is invested in the risky asset. The constant is often referred to as Merton's constant.

4.3.2 Disentangling RRA and EIS

Now, two different approaches has been dealing with the separation of RRA and EIS in continuous-time optimization problems with time-global objectives as the one given by (4.2).

First Approach

The first approach is to consider the continuous-time limit of the local discrete-time objective

$$V(t,x) = W\left(\Delta, c_t, u^{-1}(\mathbb{E}_{t,x}[V(t+\Delta, X_{t+\Delta})])\right). \tag{4.4}$$

Here W is an aggregator, aggregating the utility of time t consumption and the time t certainty equivalent of having $X_{t+\Delta}$ for consumption from time $t+\Delta$ and onwards. An important case of this is Epstein-Zin preferences (introduced in Epstein and Zin (1989) and developed to continuous-time in Duffie and Epstein (1992a,b). Here, up to a constant not influencing the optimal controls solving the problem,

$$W(\Delta, c, v) \triangleq \bar{u}(\Delta u(c) + (1-\delta\Delta)u(\bar{u}^{-1}(v))),$$

with $u(z) = \frac{1}{1-\phi}z^{1-\phi}$ and $\bar{u}(z) = \frac{1}{1-\gamma}z^{1-\gamma}$, where again γ measures RRA and $\psi = \phi^{-1}$ is EIS (also referred to as CES for Constant Elasticity of intertemporal Substitution). The continuous-time version now stems from letting $\Delta \to 0$ and this results in the continuous-time aggregator

$$f(c,v) = \theta v \left[\left(\frac{c}{((1-\gamma)v)^{-\frac{1}{1-\gamma}}} \right)^{1-\phi} - \delta \right],$$

where $\theta = \frac{1-\gamma}{1-\phi}$. Kraft and Seifried (2010) provide an alternative notion of differentiability compared to Duffie and Epstein (1992a,b) in order to make the notion of stochastic differential utility more general and robust to e.g. inclusion of non-Brownian markets.

For the problem of Epstein-Zin preferences, an admissible pair of a consumption strategy $c = \{c_t\}_{t\in[0,n]}$ and an investment strategy $\pi = \{\pi_t\}_{t\in[0,n]}$ delivers utility V_0^c over $[0,n]$, with

$$V_t^c = \mathbb{E}_{t,x}\left[\int_t^n f(c_s, V_s^c)ds\right],$$

where $V(t, X_t^*) = V_t^*$ when optimal consumption and portfolio strategies exists and $X^* = \{X_t^*\}_{t\in[0,n]}$ and $V^* = \{V_t^*\}_{t\in[0,n]}$. This particular formulation of the problem and its solution provides a certain motivation in the modeling in Section 4.4.

The Hamilton-Jacobi-Bellman equation for problem of Epstein-Zin preferences reads

$$
\begin{aligned}
0 &= \sup_{c,\pi} \Big[V_t + f(c,V) + [(r+\pi\lambda)x - c]V_x + \tfrac{1}{2}\pi^2\sigma^2x^2V_{xx} \Big], \\
V(T,x) &= 0,
\end{aligned}
$$

and we provide its solution later in this section.

Second Approach

Recently, Steffensen (2011) formalized a somewhat equivalent optimization problem allowing for separation of RRA and EIS. This problem is based on the certainty equivalence of expected future consumption rather than that of expected future wealth used in the first approach above.

Steffensen (2011) suggests to consider the problem

$$
\sup_{c,\pi} \left(\int_0^n \delta \mathrm{e}^{-\delta t} \left(\mathbb{E}\left[\tfrac{1}{1-\gamma} c_t^{1-\gamma} \right] \right)^{\frac{1}{\theta}} dt \right)^{\theta}, \tag{4.5}
$$

which we embed in the value function

$$
V(t,x) = \sup_{c,\pi} \left(\int_t^n \delta \mathrm{e}^{-\delta (s-t)} \left(\mathbb{E}_{t,x}\left[\tfrac{1}{1-\gamma} c_s^{1-\gamma} \right] \right)^{\frac{1}{\theta}} ds \right)^{\theta}. \tag{4.6}
$$

The outer power θ as well as the multiplication with δ inside the integral is taken for mathematical tractability since they do not alter the optimal controls stemming from the problem. Due to the non-linearity of the optimization objective, dynamic programming does not work in this case. This means in particular that a solution to (4.6) is most likely inconsistent with the problem in (4.5). Steffensen (2011) therefore suggest that we only search for a solution to (4.6) among the time-consistent strategies. He formalizes that as the control processes that solves a problem with the similar structure at all points in time, i.e. the optimal control should also for all $s > t$ realize the supremum in

$$
V(s,X_s) = \sup_{c,\pi} \left(\int_s^n \delta \mathrm{e}^{-\delta (u-s)} \left(\mathbb{E}_{s,X_s}\left[\tfrac{1}{1-\gamma} c_u^{1-\gamma} \right] \right)^{\frac{1}{\theta}} du \right)^{\theta}.
$$

Steffensen (2011) provide what he call a 'pseudo-Bellman equation' that characterizes the value function (4.6) and the optimal controls. In the case of a Black-Scholes investment market as we work with here, the result of Steffensen (2011) reads:

Pseudo-Bellman: *Define*

$$V(t,x) = \sup_{c,\pi} \left(\int_t^n \delta e^{-\delta(s-t)} \left(\mathbb{E}_{t,x} \left[\tfrac{1}{1-\gamma} c_s^{1-\gamma} \right] \right)^{\frac{1}{\theta}} ds \right)^{\theta}.$$

Assume that there exists a regular U such that $U(t,x)$ solves the pseudo-Bellman equation

$$0 = \sup_{c,\pi} \left[U_t + f(c,U) + [(r+\pi\lambda)x - c]U_x + \tfrac{1}{2}\pi^2\sigma^2x^2U_{xx} \right], \quad (4.7)$$

$$U(T,x) = 0, \quad (4.8)$$

where

$$f(c,U) = \delta\theta U \left[\left(\frac{c}{((1-\gamma)U)^{-\frac{1}{1-\gamma}}} \right)^{1-\phi} - 1 \right].$$

Then

$$V(t,x) = U(t,x),$$

and the optimal control is given by (c^*,π^*) realizing the supremum in (4.7).

It is not hard to see that the two approaches for a problem with disentangled RRA and EIS have the same optimal controls in this case of a Black-Scholes investment market. Namely the solution to either of the problems are

$$c_t^* = \frac{X_t}{g(t)},$$

$$\pi_t^* = \frac{\lambda}{\sigma^2\gamma},$$

with

$$g(t) = \int_t^n \mathrm{e}^{-\frac{1}{\phi}\int_t^s \left[\delta-(1-\phi)\left(r+\frac{\lambda^2}{2\sigma^2\gamma}\right)\right]d\tau} ds.$$

Up to a constant, the indirect utility from wealth under both approaches is

$$V(t,x) = \frac{1}{1-\gamma} g(t)^{\phi\theta} x^{1-\gamma}.$$

Especially, the structure of the optimal solution is preserved compared to classic model of Merton, which in particular illustrates the effect of the disentanglement of RRA and EIS. We note that the investment proportion is again given by Merton's constant. Also consumption is again proportional to wealth. The consumption-to-wealth ratio differs from that found by Merton, but it is seen that they are based on annuities calculated with different interest rate levels, though.

4.3.3 Allowing for Habit Formation in Preferences

The modeling of habit formation in preferences varies across existing literature in several ways. In particular, existing literature deviate in how the habit level is generated and how the present habit level affects utility of present consumption. In the following we sketch models of relevance for this work and comment on the strengths of the different models.

A general distinction in the literature is whether the habit level is internally or externally generated (or in equivalent formulation, endogenously or exogenously generated). The model of externally generated habit, since Abel (1990)/Gali (1994) also referred to as catching/keeping up with the Joneses, is motivated by the idea that the utility you get from a specific consumption good dependens on the good consumed by a reference group (the utility experienced from driving a Mercedes depends on whether your neighbor drives a Skoda or a Ferrari). This class of models is also referred to as reference-utility models. Despite it's motivational tractability, we omit the external generation of habitual preferences and relate our modeling to internally generated habit formation.

The internally generated habit level relates utility of present consumption to a functional of past consumption. The idea behind this modeling is that consumers get used to a certain standard of living and measure utility from present consumption by relating it to what they have previously consumed. In continuous-time models, the internal habit dynamics are most often in the form

$$\frac{dh_t}{dt} = a_1 c_t - a_2 h_t, \; h_0 > 0, \tag{4.9}$$

a_1, $a_2 \geq 0$. These habit dynamics dates back to Ryder and Heal (1973). With these dynamics, the habit level is a discounted average of past consumption rates so that the more recent consumption rates are given higher weights. The constant a_1 is interpreted as a scaling parameter, while a_2 is a persistence parameter, see Munk (2008).

Previous literature proposes several ways of incorporating the habit level in the modeling of utility from present consumption. The more common incorporation of habit formation in preferences is in additive form, so that utility is measured by a utility function given by

$$u(c, h) = \bar{u}(c - h), \tag{4.10}$$

where c and h represent consumption and habit level and $\bar{u}$ is a utility function taking values in $\mathbb{R}$. The papers Sundaresan (1989) and Constantinides (1990) include this modeling for a power utility investor and manage to solve the Equity-premium puzzle. The main reason for their success is that the inclusion of additive habit formation in a Constant Relative Risk Aversion (CRRA) function changes it to a Decreasing Relative Risk Aversion (DRRA) function.

Sundaresan (1989) also investigates a model of habit formation in the form of (4.10) with exponential utility. He shows that the model of exponential utility with additive habit formation can help explain the smooth consumption puzzle of Hansen and Singleton (1983), i.e. his model provides more stability over time in consumption.

The literature on multiplicative habit formation is far from as extensive as is the case with additive habit formation. Abel (1990), Gali (1994) and Carroll et al. (2000) work under the hypothesis that the inclusion of habit formation in the utility measurement from consumption is in the form

$$u(c,h) = \bar{u}(\frac{c}{h^{a_3}}),$$

where $\bar{u}$ is again a power utility function and $a_3 \in [0,1]$ is the weight on habit. The habit level, h, is driven by past consumption and the modeling allows for a different approach to solving the equity premium puzzle. Also Carroll et al. (2000) provide evidence that this modeling can lead to more moderate consumption behavior in growth economies, which is consistent with empirical observations.

In Alpanda and Woglom (2009), a model of multiplicative habit formation and exponential utility is considered. The strength of modeling exponential utility is well-known to be that it resolves the smooth consumption and the risk-premium puzzle. The combined model of Alpanda and Woglom (2009) allows for a resolution of the risk-free rate puzzle in addition. However, they do not otherwise substantiate their proposed habit dynamics, namely

$$\frac{dh_t}{dt} = a_4 \frac{h_t}{X_t}(c_t - h_t), \; h_0 > 0. \tag{4.11}$$

The dynamics (4.11) allows for an analytical solution to an optimal consumption problem with exponential utility and habit formation, and that fact serves as their main argument for considering the habit dynamics in (4.11).

One particular interesting paper modeling internal habit dynamics in a different form than (4.9) is Toche (2005). The habit dynamics proposed in that paper is in the form

$$\frac{dh_t}{dt} = a_5(U(c_t,h_t) - h_t), \; h_0 \in \mathbb{R}, \tag{4.12}$$

where $U(c,h)$ is instant utility from consuming c given that the habit level is h. The sign for h_0 in (4.12) depends on the sign of utility function U. The habit dynamics in (4.12) results in a habit level driven by past utility of consumption rather than past consumption as in (4.9). The motivation for considering the habit dynamics in (4.12) is the flattering idea that people do not get used to a certain level of consumption but to a certain level of utility. In the words of Toche (2005): "*According to this alternative definition* (the habit dynamics in (4.12)), *developing a consumption habit means that past experienced felicity levels are 'remembered' (consciously or not) as felicity levels, rather than as consumption levels*".

The traditional habit dynamics in (4.9) specifies that a unit increase in present consumption raises the habit stock by one unit but raises only intertemporal utility by the value of felicity. In particular, the one millionth unit of consumption raises habit stock as much as the first. Toche (2005) proposes the habit dynamics (4.12) to allow for diminishing marginal return with respect to increases in past consumption.

The utility function included in the model in Toche (2005) has the form

$$U(c,h) = u(c)(-h)^{-a_6},$$

where u is a power utility function and $a_6 \geq 0$ is the weight on the habit level. Toche (2005) works with a power utility investor of moderate risk aversion such that $u(c) < 0$ (i.e. RRA> 1) and includes the minus in front of h to ensure the right curvature of the utility function when the habit stock is negative[1]. The parameter a_6 influences the infinite-horizon EIS considered in the growth model in Toche (2005). The combined model of Toche (2005) allows for a further weakened relation between saving and growth compared to Carroll et al. (2000).

We elaborate further on the idea underlying the habit dynamics in (4.12) and the correspondence between habit formation and EIS in the model proposed in the following section.

4.4 The Optimization Criteria and the Model

In this section we present the optimization criteria of our investor and the related optimization problem. The criteria combines multiplicative habit of past utility from consumption with recursive utility. The optimization

[1]The utility function U satisfy that $U_c > 0$, $U_{cc} < 0$, $U_h \leq 0$, $U_{hh} \leq 0$, $U_{ch} \geq 0$, and $U_{cc}U_{hh} - (U_{ch})^2 \geq 0$.

criteria is presented in Section 4.4.1 and the problem is presented and solved in Section 4.4.2.

4.4.1 Optimization criteria with Habit Formation and Recursive Utility

The motivation for the habit dynamics proposed below is somewhat obvious from models with disentangled RRA and EIS that was presented in Section 4.3.2. For the problem of Epstein-Zin preferences we saw that an admissible pair of a consumption and investment strategies delivered utility V_0^c over $[0, n]$, with

$$V_t^c = \mathbb{E}_{t,x}\left[\int_t^n f(c_s, V_s^c)ds\right]. \tag{4.13}$$

In the Bellman equations in Section 4.3.2, the functional f 'takes the place of' the utility function in the classical Merton-problem, and therefore we can think of f as a utility function that takes expected future utility from consumption into account.

We are tempted to say, that we maximize time-additive expected future utility where utility is measured in terms of the functional f (using the deterministic time discounting factor $\delta\theta$). This is very much in line with Bjork and Murgoci (2010), saying that every time-inconsistent problem (with a solution) has a time-consistent equivalent with a 'modified' objective functional. In this case the inconsistent problem is (4.6) and the consistent equivalent is (4.13).

Let us for now separate f so that

$$\begin{aligned} f(c, v) &= \theta v\left[\left(\frac{c}{((1-\gamma)v)^{-\frac{1}{1-\gamma}}}\right)^{1-\phi} - \delta\right] \\ &= u(c, v) - \delta\theta v, \end{aligned}$$

with

$$u(c,v) = \theta v \left(\frac{c}{((1-\gamma)v)^{-\frac{1}{1-\gamma}}} \right)^{1-\phi}.$$

In words we would say that the utility function u measures power utility of consumption relative to the certainty equivalent of utility from future consumption, and then multiplied with the indirect utility from future consumption. Incorporating multiplicative habit now seems natural to do by considering maximizing utility from $u(\frac{c}{h}, v)$, where h is the present habit level.

Preference driven habit formation means that h is a measure of past utility rather than past consumption, as proposed by Toche (2005). Inspired by the form of u we suggest that h follows the dynamics

$$\frac{dh_t}{dt} = \tfrac{1}{1-\phi} h_t A \left(\frac{c_t}{h_t((1-\gamma)V_t)^{-\frac{1}{1-\gamma}}} \right)^{1-\phi} - \tfrac{1}{1-\phi} B h_t, \; h_0 = 1, \tag{4.14}$$

meaning that h has the representation

$$\begin{aligned} h_t &= \left(\int_0^t \mathrm{e}^{-\int_s^t B d\tau} \left(A((1-\gamma)V_s)^{-\theta^{-1}} c_s^{1-\phi} ds + d\varepsilon_0(s) \right) \right)^{\frac{1}{1-\phi}} \\ &= \left(\int_0^t \mathrm{e}^{-\int_s^t B d\tau} \left(A \frac{u(c_s, V_s)}{\theta V_s} ds + d\varepsilon_0(s) \right) \right)^{\frac{1}{1-\phi}}. \end{aligned}$$

In line with Munk (2008) we interpret the constant $A \in (0,\infty)/\{1\}$ as a scaling parameter, while $B \geq 0$ is a persistence parameter. The outer transformation, taking everything to the power of $\frac{1}{1-\phi}$, brings the habit level back to a monetary unit so that the fraction c/h is 'money'/'money'. We think of the transformation as a certainty equivalent with respect to EIS rather than RRA, which somehow seems meaningful when h is a measure of past utility (the past contains no risk given the present).

In total we arrive at the continuous-time aggregator for the Epstein-Zin preferences including multiplicative habit of past utility

$$
\begin{aligned}
f(c,h,v) &= \theta v \left[\left(\frac{c}{h((1-\gamma)v)^{-\frac{1}{1-\gamma}}} \right)^{1-\phi} - \delta \right] \\
&= u(\tfrac{c}{h}, v) - \delta\theta v.
\end{aligned}
$$

When $\gamma = \phi$, the functional f collapses into well known time-additive expected utility with multiplicative habit formation.

4.4.2 Model and Solution

The (pseudo-)Hamilton-Jacobi-Bellman equation for the problem of finding optimal consumption and investment strategies is found by straight forward derivation in line with the calculations in Steffensen (2011). For the Black-Scholes investment market that we work with here, see Section 4.2, the ultimate solution that we derive coincides for the approaches in Steffensen (2011) and in Duffie and Epstein (1992a,b) and Kraft and Seifried (2010). We therefore go straight at the Hamilton-Jacobi-Bellman equation for the investor's indirect utility, assuming that there exist optimal portfolio and consumption strategies π^* and c^* for the problem.

$$
\begin{aligned}
0 = \sup_{c,\pi} \Big[& V_t + f(c,h,V) + [(r+\pi\lambda)x - c]V_x + \tfrac{1}{2}\pi^2\sigma^2x^2V_{xx} \\
& + \left(\tfrac{1}{1-\phi} hA \left(\frac{c}{h((1-\gamma)V)^{-\frac{1}{1-\gamma}}} \right)^{1-\phi} - \tfrac{1}{1-\phi} Bh \right) V_h \Big],
\end{aligned} \tag{4.15}
$$

$$
V(T,h,x) = 0. \tag{4.16}
$$

The first order conditions for the supremum in (4.15) are solved by

$$
\begin{aligned}
c^* =& \left[V_x^{-1}((1-\gamma)V)^{1-\theta^{-1}}\left(h^{-(1-\phi)}+Ah^{\phi}((1-\gamma)V)^{-1}V_h\right)\right]^{\frac{1}{\phi}},\\
\pi^* =& -\frac{\lambda V_x}{\sigma^2 x V_{xx}}.
\end{aligned}
$$

We conjecture that a solution to the Bellman equation (4.15)-(4.16) is in the form

$$
V(t,h,x)=\frac{1}{1-\gamma}g(t)^{\phi\theta}\left(\frac{x}{h}\right)^{1-\gamma}.
$$

Inserting everything into the HJB-equation gives:

$$
\begin{aligned}
0 =& V\phi\theta g_t g^{-1}+\theta V\left((1-\gamma)\tfrac{1}{1-\gamma}g^{\phi\theta}\left(\frac{x}{h}\right)^{1-\gamma}\right)^{-\theta^{-1}}h^{-(1-\phi)}\\
&\times\left[g^{-\frac{\phi\theta}{\phi\theta}}x(1-A)^{\frac{1}{\phi}}\right]^{1-\phi}-\delta\theta V\\
&+x\left(r+\frac{\lambda^2}{\sigma^2\gamma}\right)V(1-\gamma)x^{-1}-(1-A)^{\frac{1}{\phi}}g^{-\frac{\phi\theta}{\phi\theta}}xV(1-\gamma)x^{-1}\\
&+\tfrac{1}{2}\left(\frac{\lambda}{\sigma^2\gamma}\right)^2\sigma^2x^2V(1-\gamma)(-\gamma)x^{-2}\\
&+\left(\frac{A}{1-\phi}h^{\phi}\left((1-\gamma)\tfrac{1}{1-\gamma}g^{\phi\theta}\left(\frac{x}{h}\right)^{1-\gamma}\right)^{-\theta^{-1}}\left((1-A)^{\frac{1}{\phi}}g^{-\frac{\phi\theta}{\phi\theta}}x\right)^{1-\phi}\right.\\
&\left.-\frac{B}{1-\phi}h\right)(-V(1-\gamma)h^{-1})\\
\Rightarrow&\\
0 =& \phi\theta g_t g^{-1}+\theta(1-A)^{\frac{1}{\phi}-1}g^{-1}-\delta\theta\\
&+r(1-\gamma)+\tfrac{1}{2}\frac{\lambda^2}{\sigma^2}\frac{1-\gamma}{\gamma}-(1-\gamma)(1-A)^{\frac{1}{\phi}}g^{-1}\\
&-\theta A(1-A)^{\frac{1}{\phi}-1}+\theta B,\\
g(T) =& 0.
\end{aligned}
$$

Since we can seperate of state and time variables for the conjectured solution, we find that the ODE for g is:

$$\begin{aligned} g_t &= \frac{1}{\phi}\left[\delta - B - (1-\phi)\left(r + \frac{\lambda^2}{2\sigma^2\gamma}\right)\right] g - (1-A)^{\frac{1}{\phi}}, \\ g(T) &= 0, \end{aligned}$$

with solution

$$g(t) = (1-A)^{\frac{1}{\phi}} \tilde{g}(t)$$

where

$$\tilde{g}(t) = \int_t^T \mathrm{e}^{-\frac{1}{\phi}\int_t^s \left[\delta - B - (1-\phi)\left(r + \frac{\lambda^2}{2\sigma^2\gamma}\right)\right] d\tau} ds.$$

The optimal controls solving our proposed model are

$$\begin{aligned} c_t^* &= \frac{X_t}{\tilde{g}(t)}, \\ \pi_t^* &= \frac{\lambda}{\sigma^2\gamma}. \end{aligned}$$

The controls are similar to the controls from the recursive utility model in Section 4.3.2, except for the habit persistence parameter B. We see that present consumption increases in B, which is only natural since increasing B decreases persistence. The consumption-to-wealth ratio is again calculated based on an annuity but with a different interest rate level.

Remarkably we find that none of the controls involve the habit level h. We see that there is somehow an offset in the modeling! The offset is even more evident from the fact that the parameter A (scaling the weight of habit in utility from present consumption) from the habit dynamics is not present in the controls.

4.5 Conclusion

In this paper we have proposed a model that includes forward and backward looking preferences. In either direction, we modeled that present utility from consumption took into account a certainty equivalent of past or present utility from consumption; the forward looking certainty equivalent was calculated including the risk aversion coefficient, the backward looking was calculated with the coefficient for elasticity of intertemporal substitution. Remarkably we found that forward utility functional proposed merely offset the effect of modeling backward looking preferences.

As is the case in classical time-additive utility optimization as well as recursive utility optimization, the consumption-to-wealth ratio is here calculated based on an annuity. The similar structure makes comparison simple and effects of different model elements are easy recognizable.

5. Optimal Smooth Consumption and Annuity Design

Kenneth Bruhn and Mogens Steffensen, to appear in Journal of Banking and Finance.

Abstract We propose an optimization criterion that yields extraordinary consumption smoothing compared to the well known results of the life-cycle model. Under this criterion we solve the related consumption and investment optimization problem faced by individuals with preferences for intertemporal stability in consumption. We find that the consumption and investment patterns demanded under the optimization criterion is in general offered as annuity benefits from products in the class of 'Formula Based Smoothed Investment-Linked Annuities'.

Keywords *Stochastic Control; Quadratic optimization; Linear Regulation; Consumption Smoothing; Formula Based Smoothed Investment-Linked Annuities*

5.1 Introduction

This paper illustrates that preferences for smooth consumption streams can be explained by constrained marginal consumption and an associated quadratic optimization criterion. The optimal consumption stream derived under the preferences just mentioned, shows remarkably conformity with the benefit stream of a particular annuity product from the product class 'Formula Based Smoothed Investment-linked Annuities'.

We were puzzled: Why are smooth-benefit and fixed annuities so much more popular than Unit-Link annuities with unsmoothed benefits?[1]

A wage earner's life can be divided in to two phases; an accumulation and a decumulation phase. During the accumulation phase, the person consumes part of his earnings while saving the remainder for retirement. After the person has left the labor market, consumption is financed by decumulation of savings, typically through an annuity. The annuity is either bought in the accumulation period as a deferred annuity or at the date of retirement as an immediate annuity. Several types of annuities exists, e.g. fixed, with-profit, participating life, formula based smoothed investment-linked, Variable, Unit-Link annuity etc. They differ in the stability over time in the benefits provided, ranging from a fixed annuity with a constant benefit stream to a pure Unit-link annuity where benefits vary perfectly with investment returns. In this paper we address the task of matching the preference structure of an individual to the consumption structure stemming from realized benefits of an annuity. Especially, we are interested in which preferences relate to the annuities providing the more smooth benefits.

The widely accepted "Life-Cycle Hypothesis" (LCH) of Modigliani & Broomberg and "Permanent Income Hypothesis" (PIH) of Friedman suggest that a person's consumption is proportional to his/her total wealth (the sum of financial and human wealth). In post-retirement, total wealth consists mainly of financial wealth (human wealth is zero unless we account for public pension as part of human wealth). Consumption proportional to income thereby means that consumption in post-retirement varies perfectly with financial wealth, yielding that decumulation of savings is preferably done via a pure Unit-link annuity.

The fact that people do not decumulate via Unit-Link annuities is partly formulated in "The Consumption Smoothing Puzzle", which dates back to Hansen and Singleton (1983). They present evidence that observed

[1]Insurance Information Institute (www.iii.org) reports \$7.6 billions of individual immediate fixed annuities sales and \$0.1 billions of variable annuities in 2010 in the US. The Association of British Insurers Research Department reports that new sales volumes in UK in 2007 consists of 89% conventional annuities (level/fixed and escalating) and 4% Unit Linked/With profit annuities, the rest being Enhanced/Impaired Life (see Gunawardena et al. (2008)).

consumption is much smoother than predicted by the life cycle models. Various solutions to the puzzle have been proposed, most of these modifying the market assumption underlying the model in Hansen and Singleton (1983), see e.g. Zeldes (1989) for stochastic income, Black (1990) for mean reverting returns or Fleming and Hernández-Hernández (2003) for stochastic volatility of returns.

Also preference modifications have been proposed in order to solve the puzzle. One preference modification that is of special interest here, is the introduction of endogenous habit formation, see e.g. Sundaresan (1989), Constantinides (1990) and Abel (1990). Furthermore, Munk (2008) solves the optimal investment and consumption problem with stochastic variations in investment opportunities and habit formation, thereby modifying both the underlying market and preferences. The key concept in habit forming preferences is intertemporal dependence in preferences in the sense that utility of present consumption depends on past consumption. Under these preferences, in comparison with the LCH and PIH, consumption is somewhat smoothed even for low or no human wealth.

This paper contributes to the understanding of the dependence structure in preferences that implies increased consumption smoothing. The optimization criterion proposed in this paper, though, does not include intertemporally dependent preferences but instead explicitly intertemporally dependent consumption. This is modeled by allowing for only limited control of consumption, in the sense that only the rate of change in the consumption rate is controllable.

In the object function of our proposed model we punish quadratic distance between the consumption rate derivative and a prespecified target and between terminal wealth and a target consumption ratio. The quadratic criterion is a classic in pension fund control where numerous papers examine the connection between the classical linear regulator and optimal pension funding. We refer the reader to the review article by Cairns (2000) and references therein. A main difference to our paper is that our criterion concerns the consumption rate derivative rather than the consumption rate itself. This adds a technical dimension through an additional state variable and changes the interpretation of the problem formulation and its solution. Also, we find that it makes sense to interpret the problem, not only as that of a pension fund controlling portfolio level contributions, but more as that of an individual decision maker, who expresses his preferences through a consumption growth rate target.

We find that the consumption patterns solving our proposed problem formulation shows remarkably conformity with the characteristics of annuity products from a particular class of products, namely "Formula Based Smoothed Investment-Linked Annuities". The conformity is established by direct comparison with one particular product from that product-class[2].

The structure of the remainder of the paper is: Section 5.2 contains the classical results on optimal consumption and investment problems. In Section 5.3 we present and motivate our problem of optimal smooth consumption and we solve it in Section 5.4. Section 5.5 contains a formalization of the product "Tidspension" from the product-class "Formula Based Smoothed Investment-Linked Annuities". In Section 5.6 we compare the problem solution with the product characteristics of Tidspension.

5.2 Classical Results on Consumption and Investment

In this section we present the classical power utility continuous-time optimal investment and consumption problem and the related problem with habit persistence in preferences.

Academic literature on dynamic consumption and investment decisions in continuous-time starts with Merton (1969) and Samuelson (1969). The investment market in which the investor can invest consists of a bond with constant interest rate r and a stock with constant excess-return λ and volatility σ. Thereby, the investor faces the wealth dynamics

$$dX_t = (r + \pi_t \lambda)X_t dt + \pi_t X_t \sigma dW_t - c_t dt, \;\; X_0 > 0, \tag{5.1}$$

where W is a standard Brownian motion, π is the proportion of wealth invested in the stock and c is the consumption rate. The optimal strategies are derived for a time-additive power utility maximizer with constant

[2]For a description of the product-class see Jørgensen and Linnemann (2012).

relative risk aversion γ and time-preference coefficient ρ, facing the problem

$$\sup_{c,\pi} \mathbb{E}\Big(\int_0^T \mathrm{e}^{-\rho s} \left[\tfrac{1}{1-\gamma} c_s^{1-\gamma} ds + \tfrac{1}{1-\gamma} X_s^{1-\gamma} d\varepsilon_T(s) \right] \Big), \tag{5.2}$$

where $\varepsilon_T(\cdot) = \mathbb{1}_{\{T \leq \cdot\}}$.

The optimal solution for consumption (c^*) and investment proportion (π^*) is

$$c_t^* = \frac{X_t}{f_1(t)},$$
$$\pi_t^* = \frac{\lambda}{\sigma^2 \gamma},$$

for a deterministic function f_1, see Merton (1969, 1971). The optimal consumption is proportional to savings which is very much in line with the LCH and the PIH. The investment proportion is constant and non-zero and we easily find that

$$dc_t^* = c_t^*(A_1(t)dt + \frac{\lambda}{\sigma^2\gamma}\sigma dW_t), \; c_0^* = X_0/f(0),$$

for a deterministic function A_1. The point here is that consumption possess short term volatility in the sense that stock market fluctuations (through σdW) has an immediate effect on present consumption.

The related problem with additive endogenous habit formation in preferences consists of solving

$$\sup_{c,\pi} \mathbb{E}\Big(\int_0^T \mathrm{e}^{-\rho s} \left[\tfrac{1}{1-\gamma} (c_s - h_s)^{1-\gamma} ds + \tfrac{1}{1-\gamma} (X_s - \xi h_s)^{1-\gamma} d\varepsilon_T(s) \right] \Big),$$

subject to the wealth dynamics (5.1) and habit dynamics

$$dh_t = (\varphi_1 c_t - \varphi_2 h_t)dt, \; h_0 > 0, \tag{5.3}$$

for $\xi \geq 0$ and $\varphi_1, \varphi_2 > 0$, see e.g. Munk (2008). With additive habit formation in preferences, utility from consuming c_t at time point $t \in [0, T]$ comes from the part of the consumption that exceeds a minimum consumption requirement given by the habit level h_t.

The optimal consumption and investment proportion for this problem is

$$c_t^* = h_t + (1 + \varphi_1 f_3(t))^{-\frac{1}{\gamma}} \frac{X_t - f_3(t)h_t}{f_2(t)},$$
$$\pi_t^* = \frac{\lambda}{\sigma^2\gamma} \frac{X_t - f_3(t)h_t}{X_t},$$

for deterministic functions f_2 and f_3, with

$$f_3(t) = \int_t^T \mathrm{e}^{-(r+\varphi_2-\varphi_1)(s-t)} ds.$$

We note that optimal consumption at time t is given by the sum of the minimum consumption, h_t, and a proportion of the present capital in excess of the capital value of future minimum consumption, $X_t - f_3(t)h_t$[3]. The consumption is affine in capital, leading to a smoother consumption pattern than under time-additive power utility without habit formation. We now find

$$d(c_t^* - h_t^*) = (c_t^* - h_t^*)(A_2(t)dt + \frac{\lambda}{\sigma^2\gamma}\sigma dW_t),$$
$$c_0^* - h_0^* = (1 + \varphi_1 f_3(0))^{-\frac{1}{\gamma}} \frac{x_0 - f_3(0)h_0}{f_2(0)},$$

for a deterministic function A_2. The point of the above dynamics is, that consumption under additive habit formation still possesses short term volatility. Compared to the case without habit formation, the short term volatility is dampened since only consumption in excess of present habit level is volatile. We note that the proportionality constant of the market fluctuations effect on the consumption dynamics, $\frac{\lambda}{\sigma^2\gamma}$, is the same in both cases with and without habit formation.

[3]Note, that if $c_s = h_s$ for all $s \geq t$, then the future habit levels will be given by $h_s = \mathrm{e}^{-(\varphi_2-\varphi_1)(s-t)}h_t$, and we can interpret $f_3(t)h_t$ as the capital value of future minimum consumption (see Munk (2008)).

The short term volatility of consumption under time-additive power utility with and without additive habit formation comes from the fact that stock market fluctuations immediately and diffusively influence consumption. A similar characterization holds for a pure Unit-link annuity, where investment returns are immediately converted into benefits, leading to short term volatility of benefits. On the other hand, the smooth-benefit annuity is characterized by investment returns that are smoothed over time, so that benefits do not reveal short term volatility. Put differently, if an investor has preferences as assumed in one of the formulations above, he would decumulate savings via a pure Unit-link annuity rather than a smooth-benefit annuity.

As noted in Section 6, the short term volatility in the optimal consumption streams under power utility somehow contradicts observed annuity demands of retirees. We therefore now propose another model for determining optimal consumption in retirement.

5.3 Consumption Smoothing and the Quadratic Criterion

In this section we present the model of smooth consumption. The model is based on an alternative formulation of habit formation and a quadratic optimization criterion.

The Alternative Habit Formation

Instead of allowing consumption to be as rapidly adjusted as in the optimization problems studied in Section 5.2, we propose that the consumption rate follows the dynamics

$$dc_t = adt, c_0 > 0. \tag{5.4}$$

In Longstaff (2001), this idea was implemented for the number of shares held by the investor facing a Merton-like optimization problem like the one given by (5.1)-(5.2). Whereas Longstaff (2001) motivates the constrained investment dynamics by an illiquid financial market, we motivate the constrained consumption modeled by (5.4) with friction in consumption adjustment. Being used to a certain consumption level, e.g. largely driven by housing costs, it is hard to adjust it significantly in short time.

In line with the modeling of habit formation in preferences given by (5.3), the constrained consumption dynamics in (5.4) ties current consumption to previous consumption. The difference is, that habit in preferences states that you are used to a certain standard of living and therefore only derives utility from the part of consumption that exceeds your habit level. Habit in consumption, on the other hand, states that your current consumption can only be marginally changed in a short time interval, simply because you are committed to a stable consumption pattern. In the following, we let a be a control variable for the optimization problem, so that the household can choose the rate of change in the consumption rate. Thus, we are controlling not consumption but the acceleration of consumption over time.

The Quadratic Optimization Criterion

Given that the household is subject to the wealth dynamics in (5.1) and consumption dynamics in (5.4), we must decide on a optimization criterion for our model. Choosing the Merton-criterion given by (5.2) would lead to a bang-bang strategy for the rate of change in the consumption rate. Derivation of this result is straightforward using the Hamilton-Jacobi-Bellman methodology (see e.g. Longstaff (2001) who arrives at a bang-bang strategy for the number of shares under a Merton-criterion). The reasoning for this result is that you control a but measure utility from c such that a is not directly present in your optimization objective. This way, the control process a appears as a factor on the optimal value function's derivative in c only in the related Hamilton-Jacobi-Bellman equation. If we solve this problem under the condition that a is to be chosen from the interval $[a^-, a^+]$, we easily find that the first order condition for a is

$$a = \begin{cases} a^+, & V_c^{\text{Longstaff}} > 0, \\ a^-, & V_c^{\text{Longstaff}} < 0, \end{cases}$$

where $V_c^{\text{Longstaff}}$ is the derivative in c of the related optimal value function.

The result is intuitively obvious; maximizing utility of consumption in a time-additive manner is essentially all about equaling marginal utility over time. Any instantaneous unexpected investment return is therefore used to adjust the marginal utility at all future time-points and that forces the person to rapidly adjust current consumption (consumption possesses short term volatility). Since the purpose of this paper is to discuss preferences for smooth consumption and the annuity products related to these preferences, we do not elaborate further on this. Though, an annuity product inspired by the sketched bang-bang strategy can be designed within the class of Unit-link products with financial guarantees. To the knowledge of the authors, such a product does not (yet) exist in the market.

Instead of focusing on the actual level of consumption, we propose that the optimization criterion directly incorporates the rate of change in the consumption rate. The motivation is, that people are very focused on the increase in welfare, regardless of their actual present level of consumption (the pauper and the millionaire are both interested in the same thing, namely that the next jacket/TV/car etc. is better than the one they bought last time). We especially bear in mind the situation where the household is eager for an instantly marginal increase in consumption, which they cannot finance by investing their present capital in a riskless asset.

Instead of working with the Merton-criterion for optimization in this model, we propose a time-additive quadratic criterion. We formalize it for our model as

$$\inf_{a,\pi} \mathbb{E}\left(\int_0^T \left[\ell(c_s, a_s)ds + \mathcal{L}(c_s, X_s)d\varepsilon_T(s)\right]\right), \tag{5.5}$$

with loss-functions

$$\begin{aligned} \ell(c,a) &= \tfrac{1}{2}(a - \bar{a}c)^2, \\ \mathcal{L}(c,x) &= \tfrac{B}{2}(x - \xi c)^2, \end{aligned}$$

where $\bar{a}$, $B \in \mathbb{R}$ and $\xi \geq 0$ are constants. The controls a and π are the rate of change in the consumption rate and the proportion of wealth invested in risky assets, respectively.

Based on the instantaneous loss-function, ℓ, we say that the household is only concerned with changes in their consumption rate, a. Their objective is to minimize the instantaneous quadratic distance to the specified target $\bar{a}$ ($\bar{a}$ is thus reflecting time-preferences in consumption and since a is a dollar amount, $\bar{a}$ has to be in percentage in order for a and $\bar{a}c$ to be on the same scale in ℓ). The structure of ℓ dictates that the household aims at exponential growth (decay) in consumption when $\bar{a}$ is positive (negative).

The terminal loss-function, $\mathcal{L}$, is reflecting the bequest motive of the household. The value of ξ sets the target proportion of the final consumption rate that the household wants to have left at time T, and the quadratic distance to this target is punished. The punishment is weighted by B in order to allow for different weights on the two loss-functions in the criterion. A special example is $\xi = 0$, where the household has no bequest motive and aims at spending all wealth before time T.

The parameters $\bar{a}$ and ξ specify preferences for growth in consumption and a bequest motive and therefore also indirectly the risk attitude of the household. When the preferred future consumption pattern and bequest amount cannot be financed with riskless investments, more risky positions are taken as a consequence (as we will see below). In a sense, the parameters proxy for risk aversion in our problem formulation, but the specification is indirect and not as straightforward as known from e.g. classical utility optimization.

There is obviously less control of future consumption under the problem formulation in this section than in the classical models in Section 5.2. Since the problem still allows for a consumption pattern consistent with benefits from smoothed annuities, we do not see this as a drawback for the model.

5.4 The Solution to the Smooth Consumption Problem

In this section we solve our proposed smooth consumption problem. We then analyze the derived optimal consumption dynamics and the optimal investment strategy.

We solve the optimization problem (5.5) by dynamic programming with wealth and consumption dynamics given in (5.1) and (5.4). The optimization problem is therefore embedded in the optimal value function

$$V(t,c,x) = \inf_{a,\pi} \mathbb{E}_{t,c,x}\left(\int_t^T [\ell(c_s,a_s)ds + \mathcal{L}(c_s,X_s)d\varepsilon_T(s)]\right), \tag{5.6}$$

where $\mathbb{E}_{t,c,x}$ is the conditional expectation given that the person consumes at rate $c_t = c$ and holds wealth $X_t = x$ at time t. The Hamilton-Jacobi-Bellman (HJB) equation for solving this problem is

$$V_t + \inf_{a,\pi}\left[\tfrac{1}{2}(a-\bar{a}c)^2 + [(r+\pi\lambda)x - c]V_x + \tfrac{1}{2}\pi^2\sigma^2x^2V_{xx} + aV_c\right] = 0,$$
$$V(T,c,x) = \tfrac{B}{2}(x-\xi c)^2,$$

and the first order conditions for a and π are solved by

$$a^* = \bar{a}c - V_c,$$
$$\pi^* = -\frac{\lambda}{\sigma^2}\frac{V_x}{xV_{xx}}.$$

Plugging these controls into the HJB equation leads to the PDE

$$V_t - \tfrac{1}{2}V_c^2 + [rx-c]V_x - \frac{\lambda^2}{2\sigma^2}\frac{V_x^2}{V_{xx}} + \bar{a}cV_c = 0, \tag{5.7}$$
$$V(T,c,x) = \tfrac{B}{2}(x-\xi c)^2, \tag{5.8}$$

and we make an ansatz for a solution in the form

$$V(t,c,x) = \frac{1}{2}\frac{(x-g(t)c)^2}{f(t)}.$$

Now, (5.7)-(5.8) becomes

$$\begin{aligned}\left(\frac{x-g(t)c}{f(t)}\right)\Big[-\tfrac{1}{2}f'(t)-\tfrac{1}{2}g(t)^2+(r-\tfrac{\lambda^2}{2\sigma^2})f(t)\Big]&\\ +c\big[-g'(t)+(r-\bar{a})g(t)-1\big]&=0,\\ \frac{1}{2}\frac{(x-g(T)c)^2}{f(T)}&=\tfrac{B}{2}(x-\xi c)^2,\end{aligned}$$

which leads to the following ODEs for f and g

$$\begin{aligned}f'(t)&=\bar{r}f(t)-g(t)^2, \quad f(T)=\tfrac{1}{B},\\ g'(t)&=(r-\bar{a})g(t)-1, \quad g(T)=\xi,\end{aligned} \tag{5.9}$$

with

$$\bar{r}=2r-\tfrac{\lambda^2}{\sigma^2}.$$

These ODEs are solved by

$$\begin{aligned}f(t)&=\int_t^T \mathrm{e}^{-\bar{r}(s-t)}[g(s)^2ds+\tfrac{1}{B}d\varepsilon_T(s)],\\ g(t)&=\int_t^T \mathrm{e}^{-(r-\bar{a})(s-t)}[ds+\xi d\varepsilon_T(s)].\end{aligned} \tag{5.10}$$

In order to verify that the optimal controls are indeed solving the infimum, note that the second order derivative with respect to consumption gives a 1, which is indeed positive for all values of (t,c,x) in $[0,T]\times\mathbb{R}\times\mathbb{R}$. The second order derivative with respect to the investment amount πx is $\sigma^2 f(t)^{-1}$ which is also positive such that also the optimal investment is solving the infimum[4]. In total we conclude that the optimal controls (a^*,π^*) solves the infimum for all values of (t,c,x) in $[0,T]\times\mathbb{R}\times\mathbb{R}$.

We see that given the present level of consumption, c_t, $g(t)c_t$ is the time t present value of the *preferred* consumption in the time period $[t,T]$ and the

[4]Since the proportion of wealth invested in the risky asset is not well-defined when wealth is 0, the second order condition for the optimal proportion invested is not meaningful to consider for $x=0$. Specifying the whole optimization problem in terms of an optimal proportion or amount invested in the risky asset is equivalent, and we conclude that the controls solve the infimum.

preferred bequest amount. The present value is calculated with a discount rate equal to the risk free interest rate, r. Preferred means that the rate of change in the consumption rate equals exactly the target set by the parameter $\bar{a}$, and therefore also that the targeted bequest amount is met. As a consequence of this, $x = cg$ minimizes (5.6).

The function f is the present value of a payment stream of g^2 from time t to T and a terminal sum at time T of $1/B$, discounted with an interest rate of $\bar{r}$. We provide no specific interpretation of this quantity which, of course, balances the preference for consumption growth and bequest.

5.4.1 The optimal control of a

The optimal rate of change in the consumption rate, a^*, is

$$a_t^* = \bar{a}c_t + \frac{g(t)}{f(t)}(X_t - g(t)c_t). \tag{5.11}$$

The first term is exactly the preferred change in the consumption rate at time t, whereas the second term adjust if present consumption is not set on target ($X_t \neq c_t g(t)$). In the second term, we say that $X_t - g(t)c_t$ is the consumption buffer and $g(t)/f(t)$ is the adjustment speed. If the buffer is negative, the present capital is lower than the present value of preferred future consumption and bequest and consequently the rate of change in the consumption rate is decreased. The decrease is proportional to the buffer and the adjustment speed determines the immediate influence on the consumption rate.

We note that the optimal consumption dynamics are a linear combination of present consumption and the consumption buffer.

Illustration

In Figure 5.1 we illustrate the evolution of savings, consumption and optimal adjustment over a 10 year period. We illustrate here the case where there is no risky investment ($\lambda = 0 \Rightarrow \pi^* = 0$), just not to blur the picture with investment market fluctuations.

In the case presented in Figure 5.1, the initial consumption in combination with the preferred rate of change in consumption is higher than what can be financed from riskless investment of the capital. The consumption is underfinanced by 10% corresponding to the negative buffer. We note, that due to the inertia in consumption, it is optimal to adjust consumption more in the beginning of than late in the period. This results in numerically decreasing adjustments over time.

Comparative statics for a^*

The optimal change in the consumption rate consists of three parts; the preferred change, $\bar{a}c$, the consumption buffer, $x - gc$, and the adjustment speed, g/f. For the first two terms, comparative statics are simple, whereas the fact that f includes the function g complicates analysis of the third part. As a result of the latter, analytical results are not informative for all variables and we present a numerical study.

The weight on bequest, B, and the squared sharpe ratio of the risky asset, $S = \frac{\lambda^2}{\sigma^2}$, only influences the adjustment speed through the function f. We find that the adjustment speed is increasing in B and decreasing in S, so that we have

$$\begin{aligned}
\text{sign}\{\frac{\partial a^*}{\partial B}\} &= \text{sign}\{X^* - g(t)c_t^*\}, \\
\text{sign}\{\frac{\partial a^*}{\partial S}\} &= -\text{sign}\{X^* - g(t)c_t^*\}.
\end{aligned}$$

Since B weights the importance of bequest relative to stable consumption, it is only natural that increasing B increases the speed of adjustment. For the squared sharpe ratio, an increase herein means that the investment market becomes a more effective tool for adjusting the future wealth of the household. As a consequence, less extraordinary adjustment of consumption away from $\bar{a}$ is needed (note that $a^* \to \bar{a}$ monotonically when $S \to \infty$).

The interest rate, r, and the preferred adjustment speed, $\bar{a}$, influence the value of g and therefore also f. Analytical comparative statics are therefore not informative, since the net effect on the adjustment speed is inconclusive. For the preferred change in the consumption rate and the consumption buffer we easily find that

$$\frac{\partial}{\partial r}(\bar{a}c_t^*) = 0, \ \frac{\partial}{\partial r}(X_t^* - g(t)c_t^*) > 0,$$
$$\frac{\partial}{\partial \bar{a}}(\bar{a}c_t^*) > 0, \ \frac{\partial}{\partial \bar{a}}(X_t^* - g(t)c_t^*) < 0,$$

whereas numerical studies for the adjustment speed and the overall effect on the optimal change in the consumption rate is presented in Figure 5.2.

An increase in the interest rate, r, increases the consumption buffer since the present value of the preferred future consumption stream decreases when discounting gets harder. As is seen in Figure 5.2, for the parameter values used here, the adjustment speed also increases in r. Here, an increased market return encourage a stronger motive for numerically lowering the consumption buffer, i.e. keeping the actual consumption closer to the preferred one.

The initial consumption is 10% too high with the parameter values used for the numerical analysis. Therefore, an increase in $\bar{a}$ brings the initial consumption even more off target. The effect on the present change in the consumption rate from increasing $\bar{a}$ is negative, which may seem counterintuitive, since $\bar{a}c_0$ increases. However, increasing $\bar{a}$ also decreases the consumption buffer (preferred future consumption increases and that increases g) and only modestly decreases the adjustment speed, such that the overall effect is negative for the parameter values used here.

5.4.2 The optimal control of π

The optimal proportion of capital invested in the risky asset is

$$\pi_t^* = \frac{\lambda}{\sigma^2}\frac{g(t)c_t - X_t}{X_t}. \tag{5.12}$$

We note that there is some similarity with the optimal investment strategies in the utility maximization problems presented in Section 5.2. The optimal amount invested in the risky asset is in all cases proportional to a functional of wealth, where all proportions are calculated as the sharpe ratio divided by the volatility of the stock. In the utility maximization problems, the proportions also involve the coefficient of risk aversion, γ.

In the utility maximization problems in Section 5.2, the investment in risky assets is either proportional to capital or capital in excess of the capital value of future minimum consumption (in the case of habit in preferences). In this case, where we model habit formation directly in the consumption dynamics, the risky investments are proportional to minus the consumption buffer. This result in positive risky investments when the buffer is negative and preferred future consumption cannot be financed without risk-taking in investments.

Another consequence of the optimal investment strategy in (5.12) is that risky investments tends to zero when the consumption buffer tends to zero. In particular we have, that given the initial optimal risky investment (at time 0) is positive, the optimal risky investment stay positive ($a.s.$) for the entire time period of interest, $[0, T]$.

5.4.3 From Optimal Consumption to Annuity Design

The optimal consumption pattern derived for the control problem in this section possesses no short term volatility, as is the case for the classical problems in Section 5.2. We already know that the optimal consumption patterns in the classical problems can be mimicked by benefits of Unit-link annuities. Now, we wish to characterize and find, in the market, the annuity product that provides benefits consistent with the consumption pattern derived here in Section 5.4. The consumption dynamics and the investment profile are both mathematically well-defined and it is therefore only natural to search within the product class “Formula Based Smoothed Investment-Linked Annuities”.

5.5 Tidspension - a Formula Based Smoothed Investment-Linked Annuity

Tidspension is a Danish pension product from the class “Formula Based Smoothed Investment-Linked Annuities”. The product was introduced in the Danish pension market in 2002 as an alternative to the existing traditional with-profits/participating life products and the ongoing introduction of Unit Link-inspired products. A two-account construction with a Personal Benefit Account (PBA) and an Individual Smoothing Account (ISA) provides the foundation for the product design. The product is inspired by the Unit Link product in its fully transparent transition of individual investment returns to the policyholders' savings, whereas the with-profit products' smoothing of returns inspired the ISA of Tidspension. The product is described and analyzed in great detail in Guillen et al. (2006) and Jørgensen and Linnemann (2012), as well as in a good many Danish papers (see the list in Linnemann et al. (2011)).

Annuities in Tidspension

In the following we only illustrate Tidspension for an immediate annuity in the hypothetical case where benefits are adjusted continuously[5]. As for the optimization problems considered above, an underlying Black-Scholes investment market is assumed. Parameters are again r, λ and σ for the interest rate, excess return on the stock and volatility of stock return.

In this setting, the PBA earns a deterministic return equal to the riskless rate, r. Benefits are continuously paid out of the account and smoothed investment returns are accrued from the ISA. Now, the PBA, P, follows the dynamics

$$dP_t = rP_t dt + \alpha U_t dt - b_t dt,\ P_0 = p_0,$$

where U is the ISA and $\alpha > 0$ is the smoothing constant (determining the speed at which actual investment returns are transferred from the ISA to the PBA).

[5] The actual sold annuity product has yearly adjustments of benefits.

The benefits are continuously determined such that the net present value of a constant future benefit stream equals the PBA. The net present value (often referred to as the annuity factor) is based on the interest rate r^*. This construction ensures that if the total return on the PBA for the remainder of the annuitization period corresponds to r^*, then the PBA can finance exactly the benefit level determined by the interest rate, r^*. In mathematical terms, the benefit payment, b, is continuously determined as P/q, where q is the annuity factor given by

$$q(t) = \int_t^T \mathrm{e}^{-\int_t^s r^* d\tau} ds. \tag{5.13}$$

The investment return in Tidspension goes into the ISA, from where it is transfered to the PBA (smoothed over time). Thereby, the ISA follows the dynamics

$$\begin{aligned} dU_t = {} & rU_t dt + \pi\lambda(P_t + U_t)dt + \pi\sigma(P_t + U_t)dW_t \\ & - \alpha U_t dt - q(t)^{-1} U_t dt, \end{aligned} \tag{5.14}$$

$$U_0 = u_0. \tag{5.15}$$

The last term in the dynamics in (5.14) specifies the risk sharing of the contract. The company numerically reduces the ISA by the proportion $b/P = q^{-1}$ when benefits are paid out from the PBA. The reduction is made regardless of whether the ISA is positive or negative and is financially fair in the sense that no arbitrage is possible.

In (5.14), π denotes the allocation of the underlying investments to stocks. Due to the risk sharing and the smoothing of investment returns, the actual investment risk borne by the annuitant is actually less than specified by π. Furthermore, it systematically decreases towards termination of the contract. This is related to the ownership of the ISA which is shared by the annuitant and the company, since the smoothing and risk sharing implies that not all of the ISA will be transferred to the PBA (and paid out as benefits) before time T.

In total, we note that the sum of the PBA, P, and the ISA, U, follows the dynamics

$$
\begin{aligned}
d(P_t + U_t) &= (P_t + U_t)((r + \pi\lambda - q(t)^{-1})dt + \pi\sigma dW_t), \\
P_0 + U_0 &= p_0 + u_0.
\end{aligned}
$$

Market Value of Annuity

We take the modern valuation approach, and define the market value of the contract as the arbitrage-free price of discounted expected future cash-flow, see e.g. Steffensen (2006). With this approach, we easily find that the market value of the annuity is $P_t + \kappa(t)U_t$, with

$$
\kappa(t) = \alpha \int_t^T \mathrm{e}^{-\int_t^s \alpha - q(\tau)^{-1} d\tau} ds. \tag{5.16}
$$

The function κ is a measure of the proportion of the ISA that will be transferred to the PBA before termination of the contract. We refer to κU as the part of the ISA owned by the customer.

Benefit Dynamics

Since we assume that benefits are continuously adjusted, we easily arrive at the benefit dynamics

$$
db_t = d\left(\frac{P_t}{q(t)}\right) = (r - r^*)b_t dt + \frac{\alpha}{q(t)\kappa(t)}\kappa(t)U_t dt.
$$

We note that the dynamics has linear terms in the present benefit level, b, and the part of the ISA owned by the customer, κU.

In Tidspension, benefits are extraordinarily changed if the value of the ISA is different from zero. This is the case if there are investment returns that have not been transferred to the PBA. Relating the benefit dynamics of Tidspension to the optimal consumption dynamics of the optimization problem, we say that $(r - r^*)b$ is the preferred change of the consumption rate, $\alpha/(\kappa q)$ is the speed of adjustment and κU is the consumption buffer.

5.6 Comparison of Optimization Solution and Tidspension

In this section we perform an analysis of similarities and differences between the solution to the optimization problem and the annuity in Tidspension. We focus on the optimal consumption stream of the optimization problem and the benefit dynamics of Tidspension. We only consider the case of $\xi = 0$ for the optimization problem, in words this means that there is no bequest motive and the household aim at leaving no money at time T.

The main question we want to answer is:

"Under which conditions are the consumption stream in the optimization problem equal to the benefit stream in Tidspension?"

Given the optimal consumption dynamics derived in Section 5.4 and the benefit dynamics for Tidspension found in Section 5.5, we see that a more natural question is:

"Given that the initial consumption in Section 5.4 equals the initial benefit in Section 5.5, under which conditions are the consumption dynamics equal to the benefit dynamics in Tidspension?"

In order to answer this question, we remind ourselves that the optimal consumption follows the dynamics

$$\frac{dc_t^*}{dt} = a_t^* = \bar{a}c_t + \frac{g(t)}{f(t)}(X_t - g(t)c_t), \ c_0^* = c_0, \tag{5.17}$$

whereas the benefits in Tidspension follows the dynamics

$$\frac{db_t}{dt} = (r - r^*)b_t + \frac{\alpha}{q(t)\kappa(t)}\kappa(t)U_t, \ b_0 = b_0. \tag{5.18}$$

Barring these dynamics in mind, we take as given an investment market and set of preferences $(r, \bar{a})$ leading to the above optimal consumption dynamics and the optimal investment strategy given in (5.12). For the given $(r, \bar{a})$ we search for a parameterization (r^*, α, π) in Tidspension so that the benefit dynamics in this product equals the optimal consumption dynamics.

We start with an analysis of the case without investment risk.

5.6.1 The Case Without Investment Risk

For the optimization problem in Section 5.4, no investment risk follows from assuming $\lambda = 0$, whereas for Tidspension we just put $\pi = 0$. Now, relaxing the constraint that α is a constant in Tidspension, we find that if $\bar{a} = r - r^*$ and

$$\alpha(t) = \frac{g(t)q(t)\kappa(t)}{f(t)}, \tag{5.19}$$

the dynamics in (5.17) and (5.18) equals if $X_t - g(t)c_t = \kappa(t)U_t$. The latter is exactly the case under the assumption $\bar{a} = r - r^*$ if risky investments are zero, see 5.A.

The interpretation of the above is the following: $\bar{a} = r - r^*$ means that the optimal change in the consumption rate, $\bar{a}$, equals the benefit drift rate of Tidspension, $r - r^*$. When α solves (5.19), the adjustment speeds of the optimization problem and Tidspension equals, and $X_t - g(t)c_t = \kappa(t)U_t$ ensures that the consumption buffers of the optimization problem and Tidspension coincide. Note also that $\bar{a} = r - r^*$ ensures that $g = q$ (when $\xi = 0$).

For the actual sold product Tidspension, α is a constant factor. The consequence of this, in accordance with the above analysis, is that neither the adjustment speed nor the consumption buffer is identical in the optimization problem and Tidspension. The latter is a consequence of the first.

Figure 5.3 contains a plot of the adjustment speed from the optimization problem, g/f, and Tidspension, $\alpha/(\kappa g_T)$, for the case $\bar{a} = r - r^*$. The value of α corresponds to an annual smoothing of 20% of the ISA, which is used in the actual sold product. A result of this value is that the adjustment speed of Tidspension is relative higher in the beginning and smaller towards the end. An initial relative higher adjustment speed means that the consumption buffer numerically decreases faster in the beginning. Note, that there is a crossover around $t = 4$, though. Therefore this particular choice of parameters results in essentially equivalent buffers towards the end of the time interval.

5.6.2 The Case With Investment Risk

In Section 5.6.1 we saw that the consumption dynamics and benefit dynamics are not equal when $\bar{a} = r - r^*$ and $\pi \neq 0$. In order for the consumption and benefit dynamics to coincide, we must therefore also relax the assumption for Tidspension that r^* is constant and furthermore also determine the investment strategy of Tidspension.

As in the case without investment risk, we let α be given by (5.19) so that the adjustment speeds in the consumption dynamics and benefit dynamics coincide. Furthermore, we let r^* be a deterministic function of time that solves

$$r - \bar{a} = \alpha(t) + r^*(t). \tag{5.20}$$

With this particular choice of r^*, we have that $g(t) = (1 - \kappa(t))q(t)$, see 5.A. Now, the consumption dynamics (5.17) can be rewritten as

$$\begin{aligned}\frac{dc_t^*}{dt} &= (r - \alpha(t) - r^*(t))c_t + \frac{g(t)}{f(t)}(X_t - (1 - \kappa(t))q(t)c_t) \\ &= (r - r^*(t))c_t + \frac{g(t)}{f(t)}(X_t - q(t)c_t), \ c_0^* = c_0.\end{aligned}$$

Comparing this with the benefit dynamics in (5.18), we immediately see that drift rate and adjustment speed coincides. The consumption buffer in the optimization problem is now written in terms of q. Again, the benefit dynamics in Tidspension equals the consumption dynamics in the optimization problem if the consumption buffers equals. This equality is shown in 5.A when the investment strategy in Tidspension is specified by

$$\pi = -\frac{\lambda}{\sigma^2}.$$

Under this investment strategy we find that the exposure to investment risk coincides for the optimization problem and Tidspension:

$$\begin{aligned}
\pi_t^* X_t &= \frac{\lambda}{\sigma^2}(g(t)c_t - X_t) \\
&= -\pi((1-\kappa(t))q(t)c_t - q(t)c_t - \kappa(t)U_t) \\
&= -\pi(-\kappa(t)q(t)c_t - \kappa(t)U_t) \\
&= \pi\kappa(t)(P_t + U_t).
\end{aligned}$$

5.6.3 Reverse Engineering

In sections 5.6.1 and 5.6.2 we have been concerned with the question of finding an annuity product that suited the preferences for stability in consumption that we found in Section 5.4. A similar task would be to take as given the annuity in Tidspension and search for a set of market parameters and preferences that would suit this particular product. Based on the above findings, it seems intuitively clear that we can rig the investment market and preferences in such a way that a set $(r, \bar{a})$ lead to optimal consumption dynamics that coincide with benefit dynamics of Tidspension. We leave this task to the interested reader.

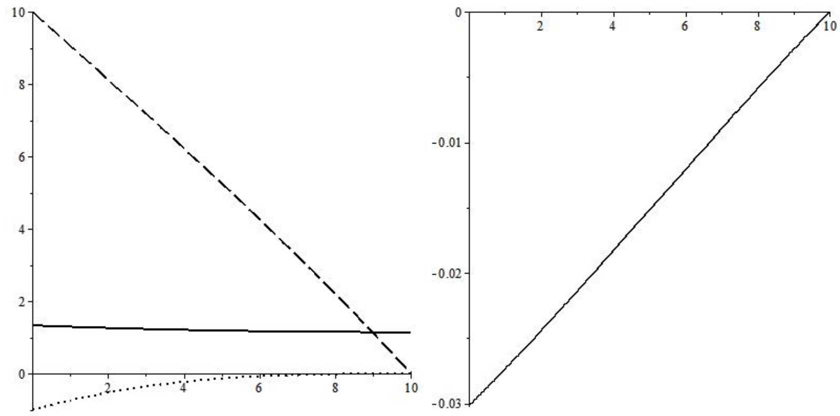

Figure 5.1: Left graph: Capital X (dashed line), consumption c (full line) and buffer $X - gc$ (dotted line). Right graph: Relative change in the optimal consumption rate a^*/c. Parameters are $X_0 = 10$, $c_0 = 1.36$, $r = 4\%$, $\lambda = 0\%$, $\bar{a} = 0\%$, $T = 10$, $B = 5$ and $\xi = 0$.

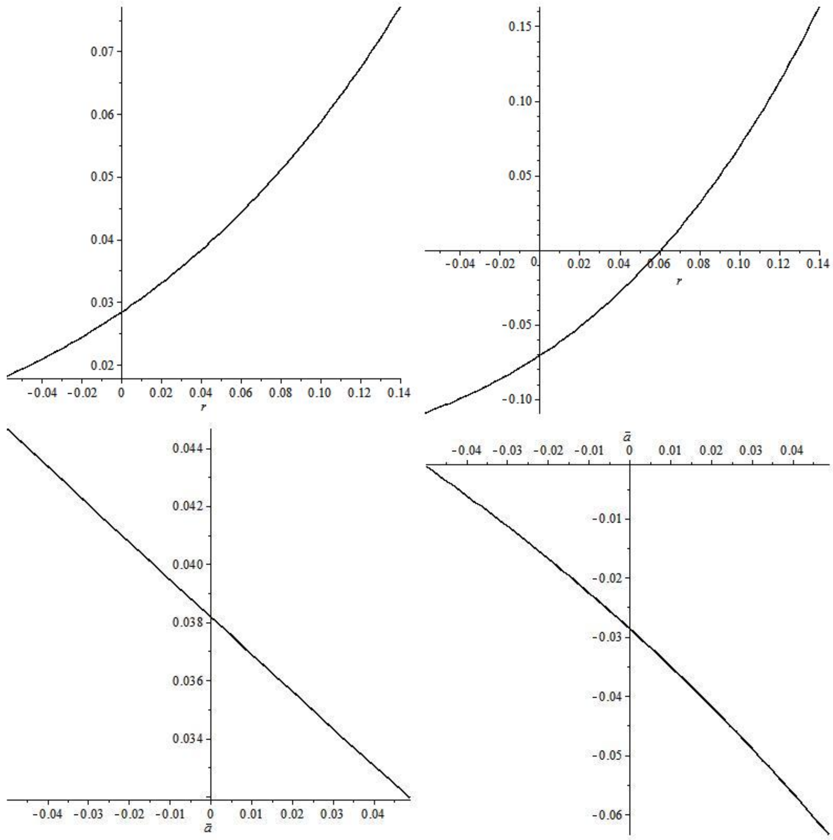

Figure 5.2: Top left graph: Optimal initial adjustment speed for various values of r. Top right graph: Optimal initial relative change in the optimal consumption rate for various values of r. Bottom left graph: Optimal initial adjustment speed for various values of $\bar{a}$. Bottom right graph: Optimal initial relative change in the optimal consumption rate for various values of $\bar{a}$. Parameters are $x_0 = 10$, $c_0 = 1.36$, $r = 4\%$, $\lambda = 3\%$, $\sigma = 20\%$, $\bar{a} = 0\%$, $T = 10$, $B = 5$ and $\xi = 0$.

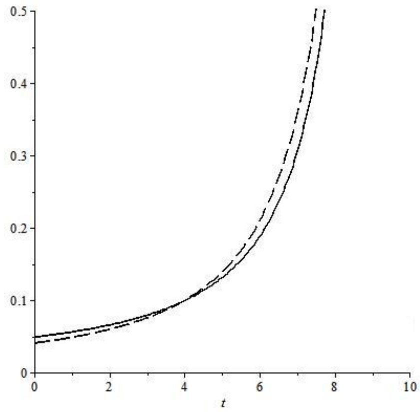

Figure 5.3: Adjustment speed for the optimization problem and for Tidspension. Parameters are $r = 4\%$, $\lambda = 0\%$, $\bar{a} = 0\%$, $T = 10$, $B = 5$, $\xi = 0$, $r^* = 4\%$ and $\alpha = -\ln(1 - 20\%)$. Full line is $\alpha/(\kappa q)$ and dashed line is g/f.

5.A Appendix

Dynamics of customers part of ISA (consumption buffer in Tidspension):

$$
\begin{aligned}
d(\kappa(t)U_t) &= U_t d\kappa(t) + \kappa(t) dU_t \\
&= U_t(\kappa(t)(\alpha + q(t)^{-1}) - \alpha)dt + \kappa(t)(rU_t dt + \pi\lambda(P_t + U_t)dt \\
&\quad + \pi\sigma(P_t + U_t)dW_t - \alpha U_t dt - q(t)^{-1}U_t dt) \\
&= r\kappa(t)U_t dt + \pi\lambda\kappa(t)(P_t + U_t)dt + \pi\sigma\kappa(t)(P_t + U_t)dW_t - \alpha U_t dt, \\
\kappa(0)U_0 &= \kappa(0)u_0.
\end{aligned}
$$

Dynamics of consumption buffer in terms of g:

$$
\begin{aligned}
d(X_t^* - g(t)c_t^*) &= dX_t^* - g(t)dc_t^* - c_t^* dg(t) \\
&= rX_t^* dt - \frac{\lambda^2}{\sigma^2}(X_t^* - g(t)c_t^*)dt - \frac{\lambda}{\sigma}(X_t^* - g(t)c_t^*)dW_t - c_t^* dt \\
&\quad - g(t)(\bar{a}c_t^* + \frac{g(t)}{f(t)}(X_t^* - g(t)c_t^*))dt - c_t^*((r - \bar{a})g(t) - 1)dt \\
&= r(X_t^* - g(t)c_t^*)dt - \frac{\lambda^2}{\sigma^2}(X_t^* - g(t)c_t^*)dt \\
&\quad - \frac{\lambda}{\sigma}(X_t^* - g(t)c_t^*)dW_t - \frac{g(t)^2}{f(t)}(X_t^* - g(t)c_t^*)dt, \\
X_0^* - g(0)c_0^* &= X_0 - g(0)c_0.
\end{aligned}
$$

Dynamics of consumption buffer in terms of q:

$$
\begin{aligned}
d(X_t^* - q(t)c_t^*) &= dX_t^* - q(t)dc_t^* - c_t^* dq(t) \\
&= rX_t^* dt - \frac{\lambda^2}{\sigma^2}(X_t^* - g(t)c_t^*)dt - \frac{\lambda}{\sigma}(X_t^* - g(t)c_t^*)dW_t - c_t^* dt \\
&\quad - q(t)((r - r^*(t))c_t + \frac{g(t)}{f(t)}(X_t^* - q(t)c_t))dt - c_t^*(r^* g(t) - 1)dt \\
&= r(X_t^* - q(t)c_t^*)dt + \frac{\lambda^2}{\sigma^2}(q(t)c_t^* + X_t^* - \kappa(t)q(t)c_t^*)dt \\
&\quad + \frac{\lambda}{\sigma}(q(t)c_t^* + X_t^* - \kappa(t)q(t)c_t^*)dW_t - \frac{g(t)q(t)}{f(t)}(X_t^* - q(t)c_t^*)dt, \\
X_0^* - q(0)c_0^* &= X_0 - q(0)c_0.
\end{aligned}
$$

Proof that $g(t) = (1 - \kappa(t))q(t)$, when $\xi = 0$ and r^* solves (5.20):

If we write q from (5.13) and κ from (5.16) in differential form, we have

$$\begin{aligned}
\frac{d}{dt}q(t) &= r^*(t)q(t) - 1,\ q(T) = 0,\\
\frac{d}{dt}\kappa(t) &= (\alpha(t) - q(t)^{-1})\kappa(t) - 1,\ \kappa(T) = 0.
\end{aligned}$$

Now, straight forward differentiation shows that

$$\begin{aligned}
\frac{d}{dt}(1 - \kappa(t))q(t) = &- ((\alpha(t) - q(t)^{-1})\kappa(t) - 1)q(t) + (1 - \kappa(t))(r^*(t)q(t) - 1)\\
= &\ (r^*(t) + \alpha(t))(1 - \kappa(t))q(t) - 1,\\
(1 - \kappa(T))q(T) = &\ 0,
\end{aligned}$$

and we see that when $\xi = 0$ and r^* solves (5.20), this is exactly the dynamics of g given in (5.9).

6. A Comparison of Modern Investment-Linked Pension Savings Products

Per Linnemann, Kenneth Bruhn and Mogens Steffensen, submitted.

Abstract This paper contributes with insight and transparency in the pension area. We analyse and compare three modern pension products; two different life-cycle products and one product falling within a new product category. By means of simulation of the investment market, we explore the determinants of annuity benefits within the three products. The results show that not only investment profiles define the stability of annuity benefits provided over time. Also more fundamental elements of the product design are important. Especially, a mathematical defined return-smoothing mechanism provides substantial stability in benefits, even for an aggressive investment profile. The perspective on product design and development is Danish, but two of the compared products are generic life-cycle products that exist in equivalent forms in most countries. Similarly, the third product with a return-smoothing mechanism is also an alternative product design in international perspective.

Keywords *Comparison of modern retirement products; Stochastic financial analysis; Life-cycle; Formula based smoothed investment-linked; Payout phase; Stabilisation of retirement income payments*

Introduction

Many pension savers know little or nothing about the type of old-age pension that is going to provide their regular income in retirement. At this stage of life, your financial conditions typically depend substantially on how your pension scheme works during the decumulation period. It is therefore important to focus on the following factors: What income profile can you expect in retirement? Will your pension scheme provide a stable or fluctuating retirement income from one year to the next? How will your retirement income be affected in case of sharp price falls in the financial markets?

We address these questions in relation to two different life-cycle products in comparison to the product TimePension, which belongs to the new product class we call "smoothed investment-linked annuities".

It is demonstrated in this paper that TimePension has particularly attractive return-risk properties, also seen in relation to life-cycle products. This is because of the unique product design of TimePension, which allows risky investment throughout the decumulation period with high expected returns and high expected pension benefits, along with great stability in retirement income payments and the underlying returns accrued in the pension benefit account.

A life-cycle product also provides an opportunity for risky investment and the achievement of high expected returns, but this results in substantial fluctuations in pension benefits. Conversely, these fluctuations can be reduced by less riskier investments. Such a step would also imply a reduction in expected returns and expected pension benefits. Our findings show that a life-cycle product investing exclusively in short-term (2.5-year) bonds throughout the decumulation period is able to offer stability in retirement income payments in line with the stability achieved with TimePension. On the other hand, the expected returns generated by such a life-cycle product are significantly lower than the returns achieved with TimePension.

Up until now, it has been widely accepted that the life-cycle products are going to succeed the traditional with profits/participating pension products. Contrary to this belief, our research shows that products of the type smoothed investment-linked annuities are serious alternatives that can deliver the next generation of pension products. Danish design may give many retirees - also on the international scene - an opportunity to combine "the best of two worlds".[1]

[1]We thank Per Klitgård and Frank Pedersen of SEB Pension for inspiring discussions

Traditional with profits/participating and investment-linked pension products

We start by providing an overview of developments in pension products in Denmark. This shows a transition from traditional with profits pension products to investment-linked products, including life-cycle products and the new product class, smoothed investment-linked annuities.

Traditional with profits schemes typically have a relatively conservative investment strategy with a relatively limited allocation to equities in the portfolio. It is necessary to build up and maintain collective bonus reserves as a buffer against price fluctuations. Sharp price falls in the financial markets result, everything else being equal, in a reduction in bonus reserves and, consequently, in a limited scope of investment opportunities. Reducing the technical interest rate partly solves this problem but may not be what the pension saver wants either.

The media are conveying the impression that "Traditional with profits pension savings have no future", see Andersen (2010). Henrik Ramlau-Hansen, former CEO of Danica Pension, has described it as follows: "If you want a higher interest rate, you have to switch to investment-linked products. To be quite honest, I think that bonus reserves will never come back to their previous level", see Dengsøe (2009a). In the longer term, it could therefore be difficult to achieve satisfactory returns for pension savers in the traditional with profits product segment.

The traditional with profits pension product is also found to be difficult to understand, see for instance (Grosen, 2005a, p. 333-334): "Contracts involving bonus entitlement are complex, not to say non-transparent". This also appears from the report entitled "Det fordelingsmæssige Kontributionsprincip. Om fordeling og omfordeling" (The principle of contribution for distribution purposes. On distribution and redistribution.), see Danish Society of Actuaries (2008).

By contrast, the typical investment-linked products are characterised by a higher level of transparency. Here the individual customers achieve an-

in connection with the paper.

nual returns on their savings that correspond to the market returns on the underlying investments. There are no undistributed reserves or non-transparent redistributions between customers, and funds are distributed on the individual customers' savings.

Moreover, investment-linked products present an opportunity to formulate an investment strategy for the purpose of offering higher expected returns than with traditional with profits pension products. This is due to the possibility of selecting investment profiles with larger proportions of savings invested in equities and other so-called high-risk assets. This may provide a platform for achieving higher retirement incomes, since experience shows that equities have generated higher returns in the long run, see for instance (Møller and Nielsen, 2009, p. 172-173).

There is also "a well-founded theory supporting the assumption that prices of equities and bonds will develop in such a way that equities must be assumed to generate higher returns than bonds" in the long term, see (Møller and Nielsen, 2009, p. 168). They add that "Equity prices must develop in such a way that the expected additional returns on equities compared with bonds completely outweigh the higher risk".

Equities are associated with higher risk and wider fluctuations in investment returns. In a typical investment-linked product, the size of retirement income payments is determined on the basis of the market value of pension savings. This means that the annual adjustment of or change in pension funds becomes directly dependent on annual market returns.

As market returns fluctuate from one year to the next, the benefits paid under an investment-linked pension product fluctuate as well. This trend is illustrated in Jørgensen and Linnemann (2012). In case of sharp price falls in the financial markets, pension payments may be reduced significantly. Such circumstances make it difficult to forecast your opportunities of consumption in retirement.

Why life-cycle products?

Life-cycle products are investment-linked products adjusting investments to suit the customer's specific age and expected time of retirement. Savings are invested more cautiously as the customer approaches the retirement age. The lower investment risk reduces fluctuations in returns and, therefore, in retirement income payments.

In this paper, we compare two different life-cycle products and look at the impact of different investment profiles on stability in retirement income payments.

The fundamental idea behind the life-cycle products is for instance mentioned in Grosen and Nielsen (2006). Young people's aggregate net assets are primarily made up of their future labour income - known as human capital. For most savers, this capital resembles more an investment in bonds than in equities. "Conventional portfolio theory is therefore in favour of a high allocation to equities in the young years and a lower allocation later on when the remaining labour income is moderate". In addition, "By working more hours per year or postponing retirement, the investor can insure against poor returns in for instance the equity market." see (Engsted et al., 2011, p. 27). These conditions may therefore support investment in a high equity allocation in the young years and a lower allocation later on when the remaining labour income is moderate.

Life-cycle products have gained ground over the last decade in Denmark. This development was seen in many countries, with some variation in the timing of and demand for life-cycle product development and with some variation in the design of the products introduced. Grosen (2005b) provides an overview of the introduction of life-cycle products in Denmark. It appears that the first life-cycle product, "Markedspension", was introduced in Denmark by SEB Pension (formerly Codan Pension), see Nielsen and Nielsen (2004). Next in line were ATP with "atpValg", see Preisel et al. (2005), Nordea Liv og Pension with "Vækstpension", see Endersen and Bork (2005) and Danica Pension with "Danica Balance", see Jørgensen (2005). See also Faurdal (2006a,b) for a description of product development within investment-linked products in Denmark.

As late as 2010, Nykredit introduced a life-cycle product called "Livslang Pension", see Linder (2010). Other life-cycle products are available in the market, but here we refer exclusively to products described in published papers.

Svendsen (2010) states that about 50% of all new pension customers choose to invest their savings in life-cycle products and that "One of the product-related disadvantages of life-cycle products is that senior citizens and old-age pensioners have difficulty maintaining both a high expected return and a reasonable investment risk". Svendsen adds that SEB Pension has addressed this challenge with the product TimePension.

It is therefore also interesting to include a product type like TimePension in a comparison with life-cycle products. Also, in an international context, TimePension is a new product type, which we describe in more detail below.

Smoothed investment-linked annuities

Life-cycle products were introduced in the United States as early as at the beginning of the 1990s. The first life-cycle product was, as already mentioned, launched in Denmark in 2004 by SEB Pension. On the other hand, the new product class, smoothed investment-linked annuities, was invented in Denmark. This took place as early as 2002, where the pension product TimePension was launched.

It is noteworthy that TimePension has been analysed by researchers in several published papers, see Nielsen and Jørgensen (2002), Grosen and Jørgensen (2002), Jakobsen (2003), Guillen et al. (2006), Jœrgensen (2007), Steffensen and Waldstrøm (2009) and Jørgensen and Linnemann (2012). TimePension is a product comprising a high allocation to equities and equity-like assets - also after retirement. This engenders expectations of high returns and high expected pension benefits. The new and noteworthy aspect is that the product design ensures the concurrent achievement of great stability in retirement income payments, see Jørgensen and Linnemann (2012).

The fundamental idea behind TimePension and the product class smoothed investment-linked annuities is to combine the best properties of traditional with profits pension schemes involving bonus entitlement with those of modern investment-linked products. This is achieved through:

1. The smoothing of investment returns and stability in retirement income payments from the traditional with profits product class (avoiding, however, a conservative investment strategy, inter-customer redistribution and the lack of transparency usually associated with traditional with profits pension products).

2. The possibility of maintaining a high allocation to equities - also in the decumulation phase - thereby making best use of the investment-linked product potential for higher returns (avoiding, however, the fluctuating and volatile retirement income payments that, everything else being equal, are associated with such products).

As mentioned in Nielsen and Jørgensen (2002), it further applies "that the path of returns, from they are generated by market investments until they are irrevocably paid out to the customer as a pension benefit, is controlled by an accurately specified mathematical mechanism".

The fundamental mechanism behind TimePension is unique and simple. The product works on the basis of two accounts:

1. The individual pension benefit account that is used for calculating the smoothed income payments. The pension benefit account balance does not fluctuate with realized investment returns.

2. The individual smoothing account that serves as an investment buffer to smoothen out investment returns. The account balance fluctuates with realized financial returns and can be negative.

In addition, the company bears some of the investment risk that can be hedged away, though. In the decumulation period, mathematically specified risk sharing is applied between the individual retiree and the company, see Nielsen and Jørgensen (2002).[2]

[2]The company is assumed to be able to finance its own risk sharing. This means that we disregard the risk of bankruptcy.

Market investment returns on the underlying assets are added to the individual smoothing account on a monthly basis. Smoothed investment returns are paid into the pension benefit account and financed from the individual smoothing account also on a monthly basis.

In this manner, the pension benefit account grows with the addition of returns where, however, investment returns are smoothed and a certain loss-restraining effect is incorporated. As mentioned in Jørgensen and Linnemann (2012), movements in the pension benefit account are stabilised and smoothed by the structure of the TimePension product. The individual smoothing account makes up the "buffer" between the market value of investments and the balance of the pension benefit account. Movements in the two above-mentioned accounts and the market value of the underlying cash flow are illustrated in Figure 1c in Jørgensen and Linnemann (2012).

It is the stability in the TimePension pension benefit account that generates the high level of stability in retirement income payments. In life-cycle and other investment-linked products, retirement income payments fluctuate with annual market returns, and the customer bears the full investment risk.[3]

In other words, there is a difference between life-cycle products and TimePension, and the question is, therefore, how various life-cycle products and TimePension perform against each other. We analyse this issue below.

Presentation of the three pension products

We aim at providing insight and transparency in the pension area. We therefore compare the different pension products as seen from the consumer's point of view. In this connection, it is vital to take a look at the benefits in the form of the retirement income that the customer can expect to receive under the different pension schemes.

[3]Investment risk can be reduced by means of various accumulation and benefit guarantees.

We compare pension conditions for "senior citizens" who have another 10 years to reach the expected retirement age of 65 and where pension benefits are payable over a 20 year period. For the sake of simplicity, this paper focuses exclusively on annuity pension schemes that provide periodic benefits for the specified period and where calculations include no mortality assumptions.

It is assumed that the investment profiles in the three selected pension products can be represented by two main classes of assets; equities/equity-like assets and bonds. In the following, we refer to these investments as "equities" and "bonds", respectively.

The investment profiles for the two life-cycle products and TimePension is shown in Figure 6. For the two life-cycle products, the investment profiles are inspired by profiles of actually sold Danish products. Especially, for LifeCycle1, the investment profile is determined by a method taking into account the size of the human capital. According to Jørgensen (2005), this is set as the net present value of future premium contributions. The method "is based on the fact that most pension schemes involve periodic premium contributions, which, as far as most of the schemes are concerned, are associated with relatively small fluctuations." Note that the two life-cycle products reflect two different views on what is the 'natural' shape of the investment profile. For LifeCycle1 the profile is concave whereas for LifeCycle2 the profile is convex as a function of age. This is a key property that seperates these two products.

In TimePension, we model a constant equity allocation of 60% (and 40% bonds) during both the accumulation and decumulation periods as shown in Figure 6. It should be emphasised that when we talk about the equity allocation here and below in connection with TimePension, it means the allocation to equities in the underlying investments (the sum of the pension benefit and individual smoothing accounts). Due to the smoothing mechanism, this allocation reflects only partially the customer's real exposure to equity risk. This is contrary to the life-cycle products where real exposure is precisely determined by the equity allocation of invested funds.

In the following, we assume that investments correspond to the investment profiles in Figure 6 for the three products. Thus, we assume that the paid

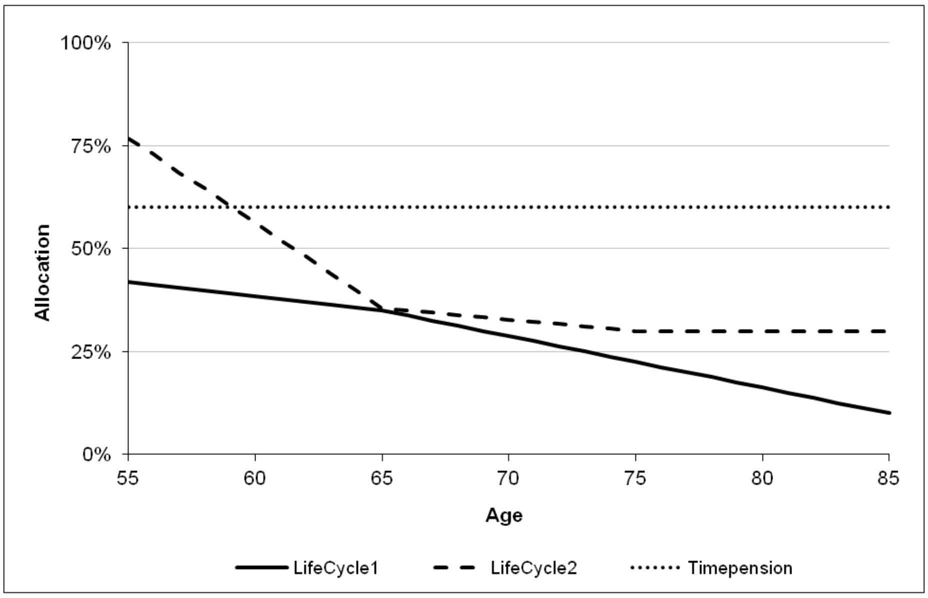

Figure 6.1: Investment profiles - allocation to equities for the three products.

premiums are continuously invested in equities according to the allocations specified in Figure 6, while the rest is invested in bonds. Furthermore, we model that the investment portfolio is rebalanced at the beginning of each quarter of the year during both the accumulation and decumulation periods.

Determining the amount of retirement income

The retirement income is determined annually under the assumption that the net present value of future retirement income payments of the same size equals the pension savings at the time of calculation. The net present value is calculated by means of an annuity factor (equalling the capital value of a pension benefit of DKK 1 per annum paid during the remainder of the decumulation period).

The annuity factor is calculated under the assumption that future yearly returns, equals what is known as the assumed interest rate. This means that when future returns on pension savings precisely correspond to the amount derived from the assumed interest rate, pension savings can finance exactly the size of the retirement income, determined on the basis of the assumed interest rate, for the remainder of the decumulation period.

As far as the two life-cycle products are concerned, the value of pension savings included in the determination of retirement income payments equals the market value of savings. When it comes to TimePension, it is the size of the pension benefit account that is used for calculating the amount of retirement income.

The financial model

In this paper, we address the questions mentioned in the introduction in connection with a comparison of the three pension products. We do this by simulating the underlying investments over the accumulation and decumulation periods 50,000 times for the three pension schemes. This gives an opportunity to compare the return and risk profiles of the products.

For the investment market simulation, we employ the model used in Jørgensen and Linnemann (2012). The model is based on Wachter (2002), Munk and Vinther (2004) and Vasicek (1977). Moreover, parameters for the model have been identified with inspiration from Jørgensen and Linnemann (2012) and in a manner ensuring compliance with the economic assumptions for pension projections for 2011, prepared by the Danish Bankers Association and the Danish Insurance Association, see e.g. Danish Insurance Association (2010). Parameter values are specified in the Technical Appendix.

Basis of comparison

As previously mentioned, we want to compare the three annuity pension products for "senior citizens" who have another 10 years left until retirement at age 65 and where pension benefits are payable over a period of 20 years. Persons who are 55 years old typically have a certain level of pension savings. In our calculations, we include these savings as a lump-sum payment (contribution) to the pension scheme. In addition, we expect periodic contributions to be made throughout the accumulation period. Hence, at age 55 it is assumed that a person has savings of DKK 2,500 (approximately 330 Euro). The amounts can be thought of in thousands in order to obtain realistic figures. The annual contribution during the accumulation period has been set at DKK 100 (approximately 15 Euro) in the first accumulation year[4] and is subsequently adjusted for inflation by 2% per annum. The amounts can be thought of in thousands in order to obtain realistic figures. The inflation rate is in compliance with the economic assumptions for pension projections, see Danish Insurance Association (2010).

We expect both the periodic contributions and the retirement income decumulations to be made on a monthly basis and to be prepaid (i.e. at the beginning of each month). The amount of retirement income is adjusted (i.e. modified) once a year. This is in contrast to the discretization of investments that are rebalanced quartely, as explained above. Pension investment returns are subject to 15% tax, corresponding to the effective tax rate under the Danish Taxation of Pension Investment Returns Act.[5] We have not taken costs and expenses into account. In other words, we analyse the basic structure of the products.

[4]We could multiply the two amounts by a given factor. This would merely mean that the size of the calculated pension benefits would be scaled by the same given factor. The result of our comparisons would still apply.

[5]In compliance with the current Danish Taxation of Pension Investment Returns Act, negative pension investment returns tax is offset against positive returns tax for a calendar year, whereas negative returns tax that is not eligible for offsetting in the statement for a given calendar year is carried forward to the following years (positive returns are offset against negative returns generated over the preceding five years).

Account returns

Initially, we compare the account returns achieved in connection with the three pension products. This is an easy and simple way of gaining fundamental insight into both the return performance and risk levels of the respective products. As mentioned above, account returns are thought of as the interest or returns credited to the account that is used for determining annual retirement income payments. The level of account returns then, essentially, reflects how much the annually determined retirement income is adjusted.

For the two life-cycle products, the account returns are given as the actual market returns achieved on the underlying investments at the age in question. For TimePension, the account returns are given as the returns accrued in the pension benefit account. As in Jørgensen and Linnemann (2012), we assume that account returns in TimePension are determined on the basis of a five-year zero-coupon rate in the financial model.[6] In addition, the monthly smoothing amount is taken into account, corresponding to 20% per annum of the individual smoothing account balance, whether positive or negative.

For each pension product, we have carried out 50,000 simulations of the full 10-year accumulation and 20-year decumulation phases. Against this background, we have calculated average (i.e. expected) full-year account returns and standard deviations in these for the respective ages. Here the standard deviation is the usual (risk) measure for the size of the dispersion of or variation in the 50,000 cases of simulated account returns. The more the account returns vary, the greater is the standard deviation.

In Table 6.1 we show average account returns and standard deviations, for a few selected ages in the decumulation period for the three products. As far as the two life-cycle products are concerned, it is the allocation to equities (see the investment profiles in Figure 6) that solely determine the percentages in Table 6.1. A higher allocation to equities generates higher

[6] In practice, the effective interest rate in a specified bond index (EFFAS index with a duration of 5-7 years) is used. The approximation we carry out is of no relevance to subsequent results.

Age	LifeCycle1		LifeCycle2		Tidspension	
	Avg.	Std.dev.	Avg.	Std.dev.	Avg.	Std.dev.
65	5.0%	5.9%	5.1%	5.9%	5.2%	2.8%
69	4.9%	5.4%	5.0%	5.7%	5.4%	2.9%
74	4.7%	4.8%	5.0%	5.4%	5.5%	2.9%
79	4.5%	4.4%	4.9%	5.4%	5.5%	2.9%

Table 6.1: Average (i.e. expected) pre-tax annual account returns (before pension investment returns tax) and standard deviations for the distribution of annual account returns for the ages specified below in the three pension products.

average account returns (i.e. investment returns) and standard deviations (i.e. risk) in the distribution of annual account returns.

It further appears from Table 6.1 that average account returns for TimePension exceed the corresponding account returns for the two life-cycle products. It is noteworthy that this result is achieved while the standard deviation in the distribution of account returns is smaller in TimePension than in the two life-cycle products. This is the consequence of the unique product design in the new product category, smoothed investment-linked annuities, of which TimePension forms part.

Note, that a life-cycle product investing exclusively in 2.5-year zero-coupon bonds throughout the decumulation period achieves a standard deviation in the distribution of annual account returns that equals 2.9%. This offers stability on level with TimePension, but expected returns are on the other hand substantially lower, at a mere 3.6% per annum, compared to the 5.2-5.5% per annum achieved with TimePension.

Below we analyze the extent to which the three pension products are able to provide year-over-year stability in retirement income payments. Before carrying out this analysis, however, we clarify the importance of the three pension products' different assumed interest rates for determining retirement income payments. This proves to be of vital importance for the associated retirement income profiles.

Expected retirement income profiles

The assumed interest rates are not identical in the different products. For example, we use an assumed interest rate equal to 1.5% for LifeCycle1, whereas the assumed interest rate for LifeCycle2 is equal to 0%. Both rates are taken from the actual sold products inspiring the lifecycle products analysed here. TimePension currently offers an assumed interest rate equal to 3.5%. The question is which retirement income profiles we can expect with the three products, given the difference in their assumed interest rates.

Figure 6 shows - for each of the products - the average (expected) adjusted retirement income for each year of the decumulation period based on the 50,000 simulations of the financial market. The retirement income is specified in percent of the amount of first-year expected retirement income for TimePension.

It appears from the figure that the LifeCycle1 product provides higher average retirement income than the LifeCycle2 product up to and including age 71. TimePension provides higher average retirement income up to and including age 73 compared with the LifeCycle2 product and up to and including age 76 compared with the LifeCycle1 product.

For many retirees, the need is typically greatest in the first years after retirement. This is the stage where the retirement income profile of TimePension offers the highest expected pension benefits.

If we sum up (without adding interest on) average retirement income payments for the individual product over the years of the decumulation period, it turns out that TimePension provides higher accumulated expected pension benefits than the LifeCycle2 product up to and including age 81, whereas TimePension outperforms the LifeCycle1 product at all ages. Finally, the LifeCycle1 product outperforms the LifeCycle2 product up to and including age 77.[7]

[7] We have found inspiration for these calculations in the publication from the British Financial Services Authority (FSA), "Just the facts about your retirement options", published in November 2007 as part of the "Money Made Clear" guides.

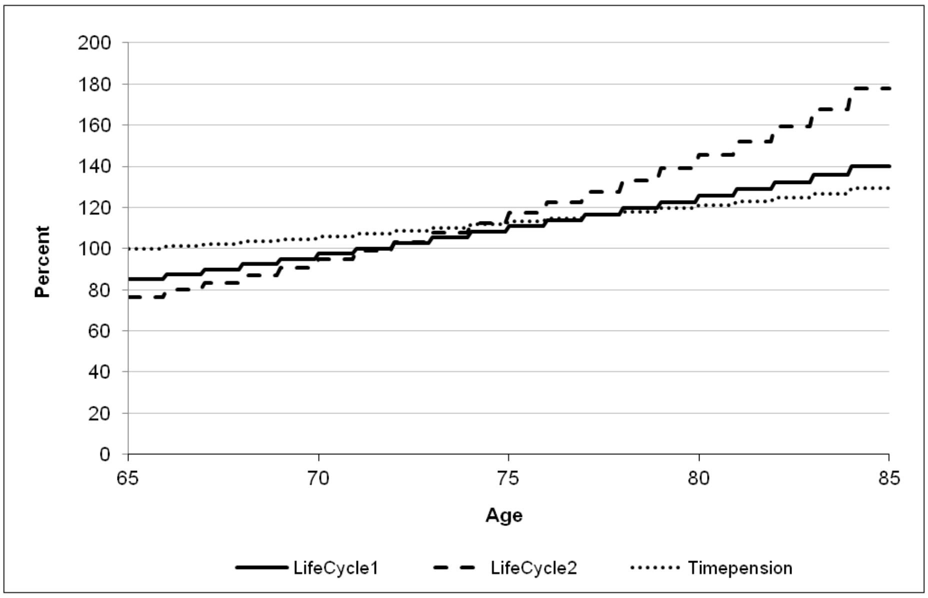

Figure 6.2: Average (i.e. expected) annually adjusted monthly retirement income payments for the three products, specified in percent of the amount of first-year expected retirement income for TimePension.

It also appears from Figure 6 that a product based on a lower assumed interest rate provides lower expected initial retirement income and a higher rate of increase in the following expected pension benefits than does a product based on a higher assumed interest rate. Generally, a higher assumed interest rate will, all else being equal, result in higher expected retirement income payments during the the beginning of the decumulation phase but lower rate of increase for expected pension benefits.

The results described above demonstrate the importance of paying attention to the retirement income profile associated with your pension scheme. Another important factor is whether and to what extent retirement income payments fluctuate from one year to the next. This issue is addressed in the next section.

Age	LifeCycle1		LifeCycle2		Tidspension	
	Avg.	Std.dev.	Avg.	Std.dev.	Avg.	Std.dev.
65	2.8%	5.3%	4.4%	5.4%	0.9%	2.4%
69	2.7%	4.9%	4.4%	5.2%	1.1%	2.5%
74	2.6%	4.4%	4.4%	5.0%	1.2%	2.5%
79	2.5%	4.2%	4.5%	5.2%	1.2%	2.6%

Table 6.2: Average (expected) percentage change and standard deviation in the annually adjustment of retirement income and standard deviation in the distribution of the percentage change in annually adjusted retirement income at the end of the specified ages for the three pension products.

Are retirement income payments stable or variable?

As mentioned earlier, the idea underlying the life-cycle products is to reduce the investment risk associated with the financial portfolio, as a function of age, by decreasing the allocation to equities, see Figure 1. The question is to what extent this provides stability in retirement income payments.

We therefore analyze how retirement income varies from one year to the next within the two different life-cycle products, LifeCycle1 and LifeCycle2. Moreover, we compare the results with the corresponding results for TimePension, which has a different investment profile and product design.

To illustrate the retirement income variation, for each of the 50,000 simulated accumulation and decumulation phases we calculate the percentage change in retirement income for each year in the decumulation period, i.e. from one given age to the next. This presents an opportunity to calculate the expectation and the standard deviation in the distribution for the percentage change in retirement income at the end of each year in the decumulation period. Table 6.2 shows the results for selected ages for the three products.

It comes as no surprise that the average change in annually adjusted retirement income is smaller for the LifeCycle1 product (at an assumed interest rate of 1.5%) than for the LifeCycle2 product (at an assumed interest rate of 0%) and that TimePension (at an assumed interest rate of 3.5%) has the smallest expected percentage changes in annually adjusted retirement income. This matches the results in Figure 6.

There is a link between account returns and percentage changes in annually adjusted retirement income payments (paid out monthly). This is due to the fact that the percentage change in annually adjusted retirement income is practically equal to account returns after tax less applied interest. As account returns are credited monthly whereas retirement income is adjusted annually, the above-mentioned link is not exact.

The variation in account returns is therefore significant to how much retirement income varies from one year to the next. A smaller standard deviation in the distribution of annual account returns is associated with a smaller variation and, hence, greater stability in retirement income payments from one year to the next. The standard deviations in the distribution of annual account returns in Table 1 correspond to the respective standard deviations for the percentage changes in annually adjusted retirement income payments in Table 6.2. The difference in values primarily emerges because account returns are calculated before pension investment returns tax whereas a minor difference also emerges because account returns are credited monthly and retirement income annually.[8]

It appears from Table 6.2 that TimePension has the smallest standard deviation in the distribution for the percentage change in annually adjusted retirement income. This means that TimePension offers the greatest stability in actual retirement income payments, which is achieved in spite of the fact that the equity allocation is substantially higher in TimePension throughout the decumulation phase than it is in the two other products.

The smoothing mechanism of TimePension is designed to offer stable re-

[8]It should be noted that the level of the assumed interest rate has only a marginal effect on the respective standard deviations in the distribution of the percentage changes in annually adjusted retirement income payments in Table 6.2.

turns on the pension benefit account and, accordingly, stability in retirement income payments. On the other hand, life-cycle products transfer the full variation in annual returns to variation in annual pension benefits.

We have also illustrated this in Figure 6 by showing the annual pension benefit for a simulation of one and the same financial scenario for each of the three products. The pension benefits are specified in percent of the amount of first-year retirement income for TimePension. It is evident that the variability (over time) in retirement income payments is greater for the two life-cycle products than for TimePension. It appears that TimePension provides great stability in retirement income throughout the decumulation period. This corresponds to the analysis in Jørgensen and Linnemann (2012), where the conclusion is that TimePension provides stability in retirement income payments in line with traditional withprofits pension products.

It is also interesting to compare the overall performance of the three products. Here it is necessary to decide how to measure performance over time on products characterised by widely different withdrawal profiles. We have chosen to compare the amounts of aggregate retirement income payments for the three products. To increase comparability, we have set an assumed interest rate of 3.5% for all three products. Furthermore, pension benefits are assumed to generate no returns after they have been paid out. This method is also employed by Jørgensen and Linnemann (2012).

For each of the 50,000 simulated accumulation and decumulation phases, we have thus calculated the amount of aggregate retirement income that is realised over the 20 years of decumulation. When considering aggregate retirement income payments, the TimePension and LifeCycle2 products are at the same level in terms of expected value and standard deviation, whereas LifeCycle1 is lower on both counts. This means that the choice between the different products, when considering aggregate retirement income payments, corresponds to a classic balancing of return and risk. In addition, however, the customer may have preferences for stability in the paid-out pension benefits, and this is achieved only with TimePension.

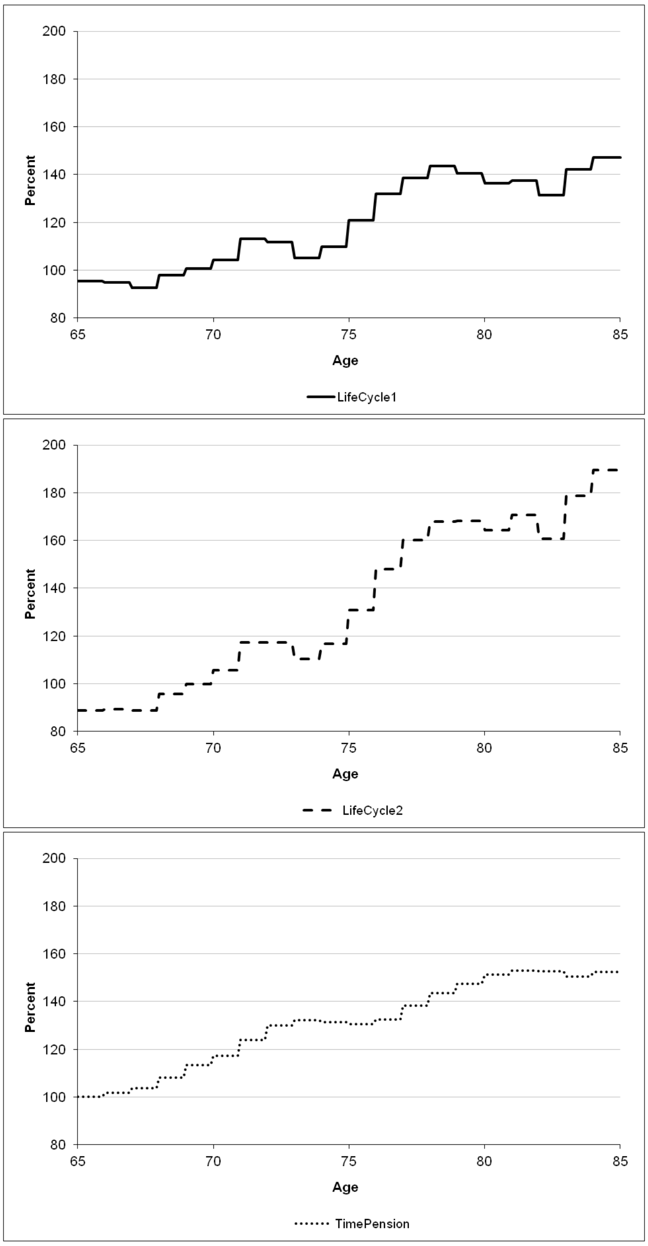

Figure 6.3: Annually adjusted monthly retirement income payments from the three products, specified in percent of the amount of first-year retirement income for TimePension.

Age	LifeCycle1	LifeCycle2	Tidspension
65	-15.5%	-17.7%	-1.0%
66	-15.6%	-15.8%	-1.0%
75	-10.4%	-13.6%	-1.0%
80	-7.6%	-13.4%	-1.0%

Table 6.3: Percentage change in average (i.e. expected) annual retirement income (paid monthly) at the beginning of the specified ages in case of a 45% fall in equity prices (relative to expected retirement income with no price fall). The price fall is assumed to occur in the last quarter of the year before the person attains the specified age.

Major price falls in the financial markets

Experience from recent years and from the beginning of the millennium has demonstrated the importance of paying attention to how pension products function in the event of financial crises, see also the chapter "Pensions and financial crises" in Dengsøe (2009b) and Mowbray (2010).

We thus compare how the average retirement income payment is affected by a major price fall of 45% for equities[9] and 10% for bonds[10], respectively. We calculate the percentage change in annual retirement income (paid monthly) in the first year (i.e. initial retirement income), second year, eleventh year and sixteenth year, respectively, on the assumption that the above-mentioned price falls occur in the last quarter of the year before pension payments for the year in question start. Table 6.3 presents the numbers for the 45% fall in equity prices and Table 6.4 for the 10% fall in bond prices.

When comparing the two life-cycle products, the drops in expected annual retirement income in Table 6.3 and Table 6.4 correspond to the allocation invested in equities and bonds, respectively, see Figure 6. TimePension, in contrast, has a loss-restraining property in connection with price falls in the

[9]In connection with the financial crisis in late 2008 and early 2009, we have seen price falls of this magnitude.

[10]This level has been set with inspiration from the Solvency II stress scenarios.

Age	LifeCycle1	LifeCycle2	Tidspension
65	-6.4%	-5.9%	0.0%
66	-6.4%	-6.4%	0.0%
75	-7.7%	-6.9%	0.0%
80	-8.4%	-7.1%	0.0%

Table 6.4: Percentage change in average (i.e. expected) annual retirement income (paid monthly) at the beginning of the specified ages in case of a 10% fall in bond prices. The price fall is assumed to occur in the last quarter of the year before the person attains the specified age.

financial markets. Although the equity allocation is 60% in TimePension, the percentage change in expected annual retirement income is only -1% in case of a 45% fall in equity prices. The change in expected annual retirement income is limited by the smoothing mechanism. As mentioned above, for the results in Table 6.3, we assume that the price fall happens during the last quarter of the year before the pension benefit is determined at the beginning of the following year. Returns are smoothed on a monthly basis at 1.84% (corresponding to 20% per annum) of the individual smoothing account balance.[11]

Note, that in case of a stock market rebound, TimePension smooth out the effect of these downward and upward movements in the financial markets. Contrary to this, retirement income payments vary with the annual market returns actually realised in the life-cycle products. Hence, we conclude that TimePension, also in relation to sharp price changes in the financial markets, offers stability in the determination of retirement income pay-

[11] For the sake of simplicity, we also look at the case where the equity price fall of -45% occurs in early December with the effect that smoothing for only one month is included in the calculations. It is assumed that the invested funds have a 60% equity allocation. The impact of the equity price fall is therefore, seen in isolation, 60%x(-45%) = -27%. After smoothing, the impact on pension benefit account returns is 1.84%x(-27%) = -0.4968%-point. This is almost equal to -0.5%, which is the percentage change in expected retirement income payments because of the equity price fall relative to expected retirement income with no price fall (calculated by means of 50,000 simulations).

ments.

Conclusion

Schopenhauer invoked the idea that innovations tend to pass through three distinct phases: "In the first stage, the new idea is ridiculed; in the second stage, it is severely opposed; and in the third stage, having become accepted, it is considered as being self-evident".

Jakobsen (2003) concludes that "TimePension represents a group of pension products which will undoubtedly be capable of capturing a large market share in the years ahead". Indeed, the Danish life and pension insurance company selling the product experienced that 20% of its new business in 2010 was accounted for by the TimePension products.[12] With time, it is expected that it becomes commonly known - also outside the pensions industry and academic circles - what qualities and advantages retirees and pension savers can achieve with the products in the product class smoothed investment-linked annuities. This is because of the unique opportunity for obtaining high expected retirement income concurrently with great stability in benefits provided in the decumulation phase. Added to this, there is an accurately specified mathematical mechanism that connects returns on investments with adjustments of retirement income payments. TimePension is in this manner a transparent product.

We also mention, that TimePension, as the only modern Danish pension product, has been analysed by researchers in several papers published in international scientific journals. This contributes to insight and transparency in the pensions area.

Many indications are that the type smoothed investment-linked annuities is a serious alternative to traditional life-cycle products - also on the international scene. The product class was internationally recognised in 2009 when the business magazine Life and Pensions awarded the "Innovation of the Year" prize to the product "TimePension with guaranty", see Carver

[12] TimePension, launched in 2002, and TimePension med Garanti, launched in 2009.

(2009). Earlier years' winners of this innovation prize had been the international life and pension insurance groups etc. in the form of AXA with over 200,000 employees and Aegon with more than 30,000 employees.

It is astounding to note that, while we in Denmark have been inspired by the United States to introduce the life-cycle products, it will come as no surprise to us if the Americans find inspiration for introducing products that are similar to TimePension and the new product class, smoothed investment-linked annuities. In other words, a Danish product design is a serious challenger to traditional life-cycle products in future product development.

Technical Appendix

Parameters for the model have been identified with inspiration from Jørgensen and Linnemann (2012) and in a manner ensuring compliance with the economic assumptions for pension projections for 2011, prepared by the Danish Bankers Association and the Danish Insurance Association, see e.g. Danish Insurance Association (2010).[13] The stock, the risk premium process and the interest rate follows the dynamics

$$
\begin{aligned}
dS_t &= (x_t + r_t)S_t dt + \sigma_s S_t dW^1(t), S_0 = s_0, \\
dx_t &= \alpha(\bar{x} - x_t)dt - \sigma_x dW^1(t), x_0 = x_0, \\
dr_t &= \kappa(\theta - r_t)dt + \sigma_r dW^2(t), r_0 = r_0.
\end{aligned}
$$

We have applied an equity volatility, σ_S, of 14% and an equilibrium risk premium for the equity relative to the short-term interest rate, $\bar{x}$, of 3.91%. The other parameters in the risk premium process, x_0, α and σ_x, are set at 3.91%, 10% and 0.5%. For the interest rate process, the parameters r_0, θ, κ and σ_r have been set at 2.86%, 2.86%, 25% and 1.5%, respectively. Moreover, the market price for interest rate risk is -25%, which is employed to determine the price of the bond portfolio, and the correlation between interest rate changes and equity returns, ρ, has been set at 0%.

[13] It is assumed that return on equities is 7% per annum and that return on bonds is 4% per annum.

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