Centre for Symmetry and Deformation – Department of Mathematical Sciences
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Dissertation

The complex of looped diagrams and natural operations on Hochschild homology

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Abstract

In this thesis natural operations on the (higher) Hochschild complex of a given family of algebras are investigated. We give a description of all formal operations (in the sense of Wahl) for the class of commutative algebras using Loday's lambda operation, Connes' boundary operator and shuffle products. Furthermore, we introduce a dg-category of looped diagrams and show how to generate operations on the Hochschild complex of commutative Frobenius algebras out of these. This way we recover all operations known for symmetric Frobenius algebras (constructed via Sullivan diagrams), all the formal operations for commutative algebras (as computed in the first part of the thesis) and a shifted BV structure which has been investigated by Abbaspour earlier. We prove that this BV structure comes from a suspended Cacti operad sitting inside the complex of looped diagrams. Last, we generalize the setup of formal operations on Hochschild homology to higher Hochschild homology. We also generalize statements about the formal operations and give smaller models for the formal operations on higher Hochschild homology in certain cases.

Resumé

I denne afhandling undersøges naturlige operationer på det (højere) Hochschild kompleks af algebraer. Vi giver en beskrivelse af alle formelle operationer (som defineret af Wahl) for klassen af kommutative algebraer ved hjælp af Lodays lambda-operationer, Connes rand-operator og shuffle-produkter. Desuden introducerer vi en dg-kategori af diagrammer med sløjfer og viser hvordan man genererer operationer på Hochschild komplekset af kommutative Frobenius algebraer ud af disse. På denne måde kan vi få alle operationer kendt for symmetriske Frobenius algebraer (konstrueret via Sullivan diagrammer), alle de formelle operationer for kommutative algebraer (som blev udregnet i den første del af afhandlingen) samt en forskudt BV-struktur, der er blevet undersøgt af Abbaspour tidligere. Vi viser at denne BV-struktur kommer fra en suspenderet Kaktus operad som sidder inde i komplekset af diagrammer med sløjfe. Til sidst generaliserer vi teorien af formelle operationer på Hochschild homologi til højere Hochschild homologi. Vi generalisere resultater om de formelle operationer og giver mindre modeller for de formelle operationer på højere Hochschild homologi i visse tilfælde.

Contents

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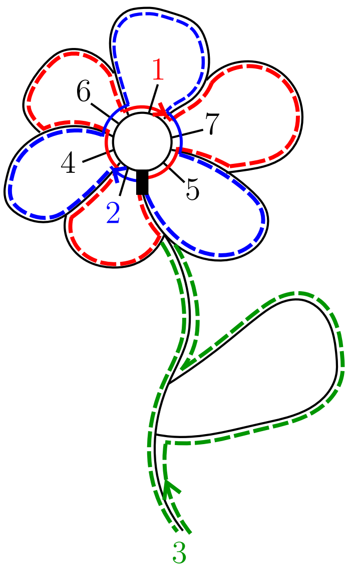
Angela Klamt
Copenhagen, October 2013



A $[{}^{1}_{(4,3)}]$–looped diagram of degree 11

Part I

Introduction and Summary

CHAPTER 1

Introduction

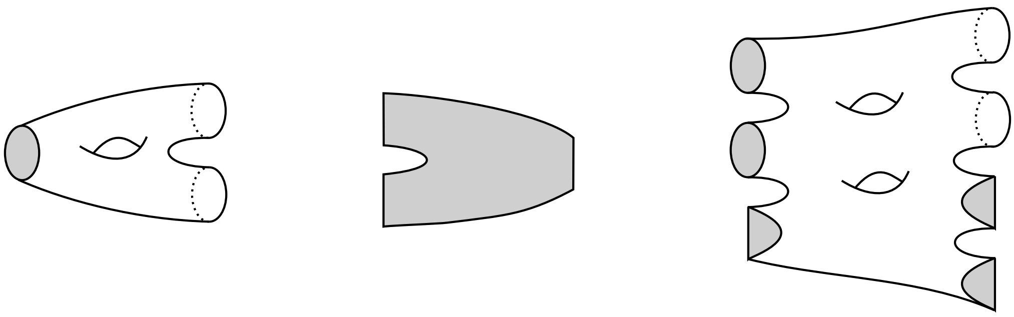
1. Motivation

1.1. String topology. A basic starting point for our interest in operations on Hochschild homology is string topology. String topology studies the structure on the homology of the free loop space LM of a manifold M, which is defined to be the unpointed mapping space from the circle S^1 to the manifold M. The subject started in 1999 when Chas and Sullivan gave a construction of a product $H_*(LM) \otimes H_*(LM) \to H_{*-d}(LM)$ for M a closed oriented manifold of dimension d (see [**CS99**]). This product is part of the structure of a Batalin-Vilkovisky algebra on $H_*(LM)$, an algebra with an operator Δ of degree one fulfilling a certain relation. The Δ operator is given by the action of the fundamental class of S^1 on $H_*(LM)$.

The construction of the Chas-Sullivan product is quite geometrically involved. However, these geometric ideas were used by many authors to generalize them to operations of the form $H_*(LM)^{\otimes n_1} \to H_*(LM)^{\otimes n_2}$ for two natural numbers n_1 and n_2. In [**God07**] Godin proved that the pair $(H_*(M), H_*(LM))$ has the structure of an open-closed homological conformal field theory, which means that we have operations

$$H_*(LM)^{\otimes n_1} \otimes H_*(M)^{\otimes m_1} \otimes HC(\left[\begin{smallmatrix} n_1 \\ m_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} n_2 \\ m_2 \end{smallmatrix}\right]) \to H_*(LM)^{\otimes n_2} \otimes H_*(M)^{\otimes m_2}$$

where $HC(\left[\begin{smallmatrix} n_1 \\ m_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} n_2 \\ m_2 \end{smallmatrix}\right])$ is the homology on the chains of the moduli space of open closed surfaces with n_1 incoming circles, m_1 incoming intervals, n_2 outgoing circles and m_2 outgoing intervals. Examples of open-closed cobordisms are given in Figure 1. We will come back to this kind of structure later in a more general context.



(a) A closed cobordism, $n_1 = 1$, $n_2 = 2$ and $m_1 = m_2 = 0$

(b) An open cobordism, $n_1 = n_2 = 0$, $m_1 = 2$ and $m_2 = 1$

(c) An open-closed cobordism $n_1 = 2$, $m_1 = 1$, $n_2 = 2$ and $m_2 = 2$

FIGURE 1. Examples of open-closed cobordisms

Another kind of structure, which is not part of Godin's construction and plays a motivating role in this thesis is the Goresky-Hingston coproduct (see [**GH09**]), a coproduct on relative homology, i.e. a map

$$H_*(LM, M) \to H_*(LM, M) \otimes H_*(LM, M).$$

However, we will not dwell on this relative setup for now.

Taking coefficients in a field and letting M be a 1-connected closed oriented manifold, Jones [**Jon87**] proved that there is an isomorphism

$$HH_*(C^{-*}(M), C^{-*}(M)) \cong H^{-*}(LM),$$

where $HH_*(A, A)$ denotes the Hochschild homology of an algebra A, introduced in more detail in the next section. Hence dual string topology operations

$$H^{-*}(LM)^{\otimes n_2} \otimes H^{-*}(M)^{\otimes m_2} \to H^{-*}(LM)^{\otimes n_1} \otimes H^{-*}(M)^{\otimes m_1}$$

are equivalent to operations

$$\begin{aligned} &HH_*(C^{-*}(M), C^{-*}(M))^{\otimes n_2} \otimes H^{-*}(M)^{\otimes m_2} \\ &\qquad\qquad\to HH_*(C^{-*}(M), C^{-*}(M))^{\otimes n_1} \otimes H^{-*}(M)^{\otimes m_1}. \end{aligned}$$

This motivates us to investigate operations on Hochschild homology more systematically.

1.2. Hochschild homology. We proceed with giving a definition of Hochschild homology. We restrict to associative algebras here, even though there is a more general setup for A_∞-algebras available (see for example [**KS09**, Section 7.24]). If not specified otherwise, we work over a commutative ring $\mathbb{K}$ and denote by $\mathrm{Ch}(\mathbb{K})$ the category of chain complexes over $\mathbb{K}$. First, we start with ungraded algebras. Let A be an associative algebra and M an A–bimodule. The Hochschild complex of A with coefficients in M denoted by $C_*(A, M)$ is the chain complex which in degree k is given by

$$C_k(A, M) = M \otimes A^{\otimes k}$$

with differentials $d : C_k(A, M) \to C_{k-1}(A, M)$ defined as the sum $d = \sum_{i=0}^{k} (-1)^i d_i$ and the d_i given by

$$\begin{aligned} d_0(m \otimes a_1 \otimes \cdots \otimes a_k) &= ma_0 \otimes a_1 \otimes \cdots \otimes a_k, \\ d_i(m \otimes a_1 \otimes \cdots \otimes a_k) &= m \otimes a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_k \quad \text{for } 1 \leq i \leq k-1 \text{ and} \\ d_k(m \otimes a_1 \otimes \cdots \otimes a_k) &= a_n m \otimes a_0 \otimes \cdots \otimes a_{k-1}. \end{aligned}$$

For a graded abelian group A_* we denote by $A_*[k]$ the shifted abelian group with $(A_*[k])_n = A_{n-k}$. Then, for a differential graded algebra A and a differential graded bimodule M, the Hochschild complex is generalized to

$$C_*(A, M) = \bigoplus_{k \geq 0} M \otimes A^{\otimes k}[k]$$

with differential $D + d$ where d is the differential from above (with a sign twist we do not deal with here) and D comes from the inner differentials on A and M, i.e.

$$D(m \otimes a_1 \otimes \cdots \otimes a_k) = d_M(m) \otimes a_1 \cdots \otimes a_k + \sum_{i=1}^{k} \pm m \otimes a_1 \otimes \cdots \otimes d_A(a_i) \otimes a_{i+1} \otimes \cdots \otimes a_k.$$

This means that $C_*(A, M)$ is the total complex of the double complex with horizontal grading the Hochschild grading and vertical grading the inner grading of A and M. In the thesis we only deal with the case $A = M$ and from now on restrict to it.

1.3. Operations on the Hochschild complex of algebras. In order to investigate string topology operations via Hochschild homology one is interested in finding operations

$$C_*(A,A)^{\otimes n_1} \otimes A^{\otimes m_1} \to C_*(A,A)^{\otimes n_2} \otimes A^{\otimes m_2}$$

which are natural in some class of algebras, for example the class of all associative algebras, commutative algebras or (symmetric/commutative) Frobenius algebras. In this section we are only interested in operations which descend to homology, i.e. operations which commute with the boundary. If $n_1 = n_2 = 0$ the question is more basic, since we actually look for operations $A^{\otimes m_1} \to A^{\otimes m_2}$. In particular all operations created by permutations and the structure of the algebra (e.g. the multiplication) are examples of such operations.

1.3.1. *The inclusion of the algebra into the complex.* The easiest operation involving the Hochschild complex is the inclusion of the algebra into it, i.e. the map $i : A \to \bigoplus_{k\geq 0} A^{\otimes k+1} = C_*(A,A)$ mapping A to the zeroth summand of $C_*(A,A)$.

1.3.2. *Connes' boundary operator.* Another example of an operation which is natural in all associative dg-algebras is Connes' boundary operator $B : C_*(A,A) \to C_{*+1}(A,A)$. For a Hochschild chain $a = a_0 \otimes \cdots \otimes a_k$ it is defined as

$$B(a) = \sum_{i=0}^{k} \pm 1 \otimes a_i \otimes \cdots \otimes a_k \otimes a_0 \otimes \cdots \otimes a_{i-1},$$

i.e. it puts a 1 in front of all cyclic permutations of the element a. The operator B commutes with the Hochschild boundary map and hence defines an operation on homology. If one works over reduced chains the operator squares to zero (and hence is a certain boundary itself).

1.4. Operations for commutative algebras. If we restrict ourselves to commutative instead of associative algebras, many more operations are known.

1.4.1. *The shuffle product.* The shuffle product allows us to multiply Hochschild chains together, i.e. it gives a degree preserving map

$$\mu : C_*(A,A) \otimes C_*(A,A) \to C_*(A,A)$$

which for two Hochschild chains $a = a_0 \otimes \cdots \otimes a_k$ and $b = b_0 \otimes \cdots \otimes b_l$ is defined as

$$\mu(a \otimes b) = \sum_{\substack{\sigma \in \Sigma_{k+l} \\ \text{a } (k,l)\text{–shuffle}}} \pm\, a_0 b_0 \otimes c_{\sigma(1)} \otimes \cdots \otimes c_{\sigma(k+l)}$$

with $c_i = a_i$ for $i \leq k$ and $c_i = b_{i-k}$ for $i > k$. A (k,l)–shuffle is a permutation which preserves the internal order of the first k and of the last l elements, i.e. $\sigma(1) < \cdots < \sigma(k)$ and $\sigma(k+1) < \cdots < \sigma(k+l)$. This means we multiply the first two elements and stick b into a in all possible manners. The shuffle product commutes with the boundaries, is associative and graded commutative.

1.4.2. *The projection from the Hochschild complex to the algebra.* We have already seen the map $A \to C_*(A,A)$ which includes the algebra into the Hochschild complex. For commutative algebras this map is split, i.e. the projection of the Hochschild complex onto the first summand is a chain map. This is equivalent to seeing that $C_*(A,A)$ splits into A and $\bigoplus_{k\geq 1} A^{\otimes k+1}$ as chain complexes, i.e. that the Hochschild differential $d : A \otimes A \to A$ is trivial. Recall that for $a_0 \otimes a_1 \in A \otimes A$ we have $d(a_0 \otimes a_1) = a_0 a_1 \pm a_1 a_0$ which by the commutativity (and since the signs actually fit) is zero. Thus we have an operation $C_*(A, A) \to A$ natural in all commutative algebras.

1.4.3. *Loday's lambda operations.* In [**Lod89**] Loday defined operations acting on the Hochschild complex of a commutative algebra with coefficients in a bimodule M, i.e. maps $\lambda^k : C_*(A, M) \to C_*(A, M)$. He defined them more generally for cyclic homology and McCarthy generalized them to an even broader setup in [**McC93**].

All these operations are given by a sum of permutations of the tensor factors in each Hochschild degree. To specify which permutations are used, the Euler decomposition of the symmetric group plays an important role. For a permutation $\sigma \in \Sigma_n$ a descent is a number i such that $\sigma(i) > \sigma(i+1)$. Then the Euler decomposition is the decomposition of Σ_n into the subsets $\Sigma_{n,k}$ defined as

$$\Sigma_{n,k} := \{\sigma \in \Sigma_n \mid \sigma \text{ has } k-1 \text{ descents}\}.$$

In [**Lod89**], up to a sign twist, the maps l_n^k acting on $M \otimes A^{\otimes n}$ (but not commuting with the boundaries) were defined as

$$l_n^k(m \otimes a_1 \otimes \cdots \otimes a_n) := \sum_{\sigma \in \Sigma_{n,k}} \pm m \otimes a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)}$$

for $n \geq 1$ and $1 \leq k \leq n$, $l_0^0(m) = m$ and $l_n^k = 0$ for all other n and k. Out of these the lambda operations were constructed as

$$\lambda_n^k = \sum_{i=0}^{k} \binom{n+k-i}{n} l_n^i.$$

The families λ^k commute with the boundary maps and hence give operations on homology.

This is not the only way one can build operations out of the l_n^k. We want to mention two further methods:

The shuffle operations $sh^k : C_*(A, M) \to C_*(A, M)$ are defined as

$$sh_n^k = \sum_{i=1}^{k} \binom{n-i}{k-i} l_n^i$$

for $n \geq 1$ and $1 \leq k \leq n$, $sh_0^0 = \mathrm{id}$ and $sh_0^k = sh_n^0 = 0$ for $k > 0$ and $n > 0$. For $n \geq 1$ we obtain $sh_n^1 = \mathrm{id}$. The shuffle operations lie in the linear span of the lambda operations and vice versa. One advantage of the shuffle operations is that sh^k acts trivially on all Hochschild degrees smaller than k, so the infinite sum of shuffle operations is still a well-defined operation on the Hochschild complex.

So far, we have been working with coefficients in any commutative ring $\mathbb{K}$. Taking coefficients in the rational numbers instead, the idempotents $e_n^{(i)}$ are defined as the solutions of the n equations

$$\lambda_n^k = \sum_{i=1}^{n} k^i e_n^{(i)}$$

for $1 \leq k \leq n$ and $e_n^{(i)} = 0$ if $n < i$. These have been studied earlier by Gerstenhaber and Schack in [**GS87**] as a generalization of Barr's idempotent (defined in [**Bar68**]) and were used to give a Hodge decomposition of Hochschild homology over the rationals (a decomposition into eigenspaces). In their work they also show that any natural operation which acts on each Hochschild degree separately can be written as a linear combination of these idempotents. Moreover, as the name suggests, they form a complete set of orthogonal idempotents. An explicit formula in terms of the l_n^k is given in [**Lod89**, Prop. 2.8].

1.5. Operations for symmetric Frobenius algebras. In this section we give an action of the complex of so called Sullivan diagrams on the Hochschild complex of symmetric Frobenius algebras. This appeared in Theorem 3.3 of [**TZ06**] and has been recovered in a more formal context (which we will come back to later) by Wahl and Westerland in [**WW11**, Theorem 6.7].

A *Frobenius dg-algebra* A is given by a chain complex equipped with the following data:

- a multiplication $m : A \otimes A \to A$ and a unit $\mathbb{1}_A : \mathbb{K} \to A$ such that m and $\mathbb{1}_A$ define a dg-algebra structure on A
- a comultiplication $\Delta : A \to A \otimes A$ and a counit $\eta : A \to \mathbb{K}$ such that they define a dg-coalgebra structure on A

satisfying the so called Frobenius relation

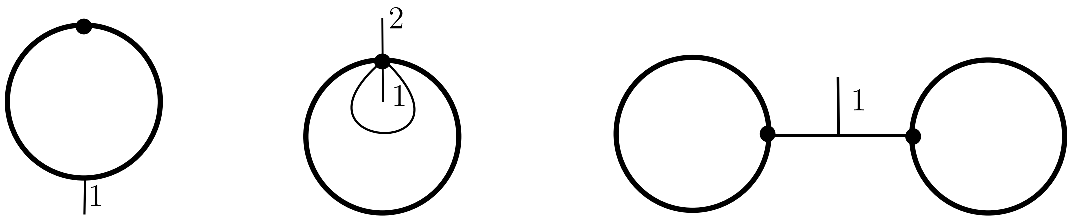
$$\Delta \circ m = (m \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta) = (\mathrm{id} \otimes m) \circ (\Delta \otimes \mathrm{id}).$$

We denote the twist map $A \otimes A \to A \otimes A$ by τ.

A Frobenius algebra is called *symmetric* if $\eta \circ m \circ \tau = \eta \circ m$ and it is *commutative* if $m \circ \tau = m$. A commutative Frobenius algebra is automatically cocommutative, i.e. $\tau \circ \Delta = \Delta$.

There is a graph complex of *Sullivan diagrams* $\mathcal{SD}([{n_1 \atop m_1}], [{n_2 \atop m_2}])$ which is a quotient of another graph complex $\mathcal{OC}([{n_1 \atop m_1}], [{n_2 \atop m_2}])$, whose homology is the homology of the moduli space of open-closed cobordisms with n_1 and n_2 incoming respective outgoing circles and m_1 and m_2 incoming respective outgoing intervals. A Sullivan diagram can be thought of as a graph with exceptional circles and a cyclic ordering of the edges incident at all vertices, up to some equivalence relation. Examples of Sullivan diagrams are shown in Figure 2. By [**TZ06**, Theorem 3.3] an element in $\mathcal{SD}([{n_1 \atop m_1}], [{n_2 \atop m_2}])$ gives an operation

$$C_*(A, A)^{\otimes n_1} \otimes A^{\otimes m_1} \to C_*(A, A)^{\otimes n_2} \otimes A^{\otimes m_2}.$$



(a) The BV-operator (b) The product (c) The coproduct

FIGURE 2. Examples of Sullivan diagrams

This means in particular, that we have a Batalin-Vilkovisky algebra and coalgebra structure given by the image of the BV-structure encoded in the homology of the moduli space of closed surfaces. The BV-operator is shown in Figure 2(a) and turns out to be Connes' boundary operator which we defined already in Section

1.3.2. In Figure 2(b) and Figure 2(c) the Sullivan diagrams giving multiplication and comultiplication, respectively, are shown. The formulas for these operations are given by (see [**WW11**, Section 6]):

(1) Multiplication:

$$(a_0 \otimes \cdots \otimes a_k) \otimes (b_0 \otimes \cdots \otimes b_l) \mapsto \begin{cases} 0 & \text{if } k > 0 \\ \sum \pm a_0'' a_0' b_0 \otimes b_1 \otimes \cdots \otimes b_l & \text{if } k = 0 \end{cases}$$

where we use the notation $\Delta(a_0) = \sum a_0' \otimes a_0''$ for the comultiplication of a_0. This product is zero on homology except for $HH_0(A, A) \otimes HH_0(A, A)$.

(2) Coproduct:

$$a_0 \otimes \cdots \otimes a_k \mapsto \sum_{i=1}^{k} \sum \pm(a_0'' \otimes a_1 \otimes \cdots \otimes a_i) \otimes (a_0' \otimes a_{i+1} \otimes \cdots \otimes a_k).$$

These constructions can be lifted to the homotopy associative case and give an action of the moduli space of open closed surfaces on a homotopy associative version of Frobenius algebras (which has been done in [**Cos07**] and [**KS09**]). We will restrict ourselves to the cases above, since these are the operations which are relevant in this thesis.

1.6. Commutative Frobenius algebras. Commutative Frobenius algebras lie in the intersection of all the classes of algebras we have looked at so far. However, in [**Abb13a**, Section 7] Abbaspour defined another product on the split complement of a commutative Frobenius algebra A in $C_*(A, A)$, i.e. a product which is zero on the image of the embedding $A \to C_*(A, A)$. He proves that this gives a shifted BV-structure on the homology of the chains of positive Hochschild degree (and thus on $HH_*(A, A)$ if we set the BV-operator zero on Hochschild degree zero) for A a commutative Frobenius algebra (or even a commutative cocommutative open Frobenius algebra, which means that we do not require a counit to exist). Again, we write $\Delta(a_0 b_0) = \sum (a_0 b_0)' \otimes (a_0 b_0)''$. Then the product of $a = a_0 \otimes \cdots a_k$ and $b = b_0 \cdots b_l$ is given by

$$a \cdot b = \begin{cases} 0 & \text{if } k = 0 \text{ or } l = 0 \\ \sum \pm (a_0 b_0)' \otimes a_1 \otimes \cdots \otimes a_k \otimes (a_0 b_0)'' \otimes b_1 \otimes \cdots \otimes b_l & \text{else.} \end{cases}$$

This product is commutative on Hochschild homology.

1.7. Identifying operations as string operations. Before we close this section we want to connect some of these operations back to string topology, which was our motivating example to start with. Hence we would like to show that some of the operations mentioned above are actually operations on $HH_*(C^{-*}(M), C^{-*}(M))$ for a compact oriented, simply connected manifold M. Working over the rationals, the complex $C^*(M)$ is quasi-isomorphic to the deRham complex of differential forms on M (which is a CDGA) and its homology is a strict Frobenius algebra. Following Lambrechts-Stanley [**LS07**] this implies that $C^*(M)$ is quasi-isomorphic as an algebra to a commutative Frobenius algebra A^{-*} and hence $HH_*(C^{-*}(M), C^{-*}(M)) \cong HH_*(A^{-*}, A^{-*})$. Thus we get an action of (a shifted version of) the complex $\mathcal{SD}$ on $HH_*(C^{-*}(M), C^{-*}(M))$. Using the dual statement, which was proved in [**FT08**], the authors of [**WW11**, Prop 6.10] prove that the (shifted) CoBV algebra structure on $HH_*(C^{-*}(M), C^{-*}(M))$ agrees with the one on $H^{-*}(LM)$ under the isomorphism $HH_*(C^{-*}(M), C^{-*}(M)) \cong H^{-*}(LM)$.

On the other hand, working over the rationals and denoting the deRham algebra by $\Omega^\bullet(M)$, we have a direct isomorphism

$$HH_*(\Omega^{-\bullet}(M), \Omega^{-\bullet}(M)) \cong H^{-*}(LM).$$

In [**BFG91**] it is shown that this isomorphism sends Loday's lambda operation λ^k to (some multiple) of the k–th power operation on LM, i.e. the map induced by precomposing elements in $LM = Map(S^1, M)$ with the k–fold covering map $S^1 \to S^1$.

2. The more formal setup

Our goal is to construct natural operations on Hochschild homology, i.e. operations which are natural in a certain class of algebras. Examples which have shown up so far are the class of associative algebras, commutative algebras or (symmetric/commutative) Frobenius algebras. More generally, these algebras are encoded as algebras over some specific PROP which in some cases comes from an operad. Both concepts are used in this thesis, so we start by recalling their definitions.

2.1. Operads. Operads were originally defined by Boardman-Vogt and May in their work on iterated loop spaces in [**BV73**] and [**May72**]. The data of an operad is given by operations with some number of inputs and one output together with a composition law that satisfies certain coherence axioms, generalizing the idea of an ordinary multiplication of an associative algebra. Operads are used to describe algebraic structures in symmetric monoidal categories, for example in chain complexes, simplicial sets or topological spaces. We denote by Σ_n the symmetric group on n elements. Then we can define a (unital) operad in a symmetric monoidal category $\mathscr{M}$ as follows:

Definition 2.1 (Operad). An operad $\mathscr{O}$ in a symmetric monoidal category $\mathscr{M}$ consists of a sequence of objects $\mathscr{O}(r) \in \mathscr{M}$ with $r \in \mathbb{N}$, where $\mathscr{O}(r)$ is equipped with an action of Σ_r together with the following data:

- A unit morphism $\mathbb{1} \to \mathscr{O}(1)$.
- A composition of morphisms

$$\mathscr{O}(n_1) \otimes \cdots \otimes \mathscr{O}(n_r) \otimes \mathscr{O}(r) \to \mathscr{O}(n_1 + \cdots + n_r)$$

  for any $r \geq 0$ and all $n_1, \cdots, n_r \geq 0$ fulfilling $\Sigma_{n_1} \times \cdots \times \Sigma_{n_r}$– and Σ_r–equivariance.

The data needs to satisfy a unit and an associativity axiom (for more details see for example [**Fre12**, Sec. 1.1]).

A non-unital operad is such a sequence defined for all $r > 0$.

It is possible to think of the operations as trees with r inputs and one output, so that the composition corresponds to gluing the output of the i–th operation in $\mathscr{O}(n_i)$ on the i–th input of the operation in $\mathscr{O}(r)$ (see Figure 3).

Before we give examples of operads, we define the notion of an algebra over an operad to connect the topic back to our interest of study:

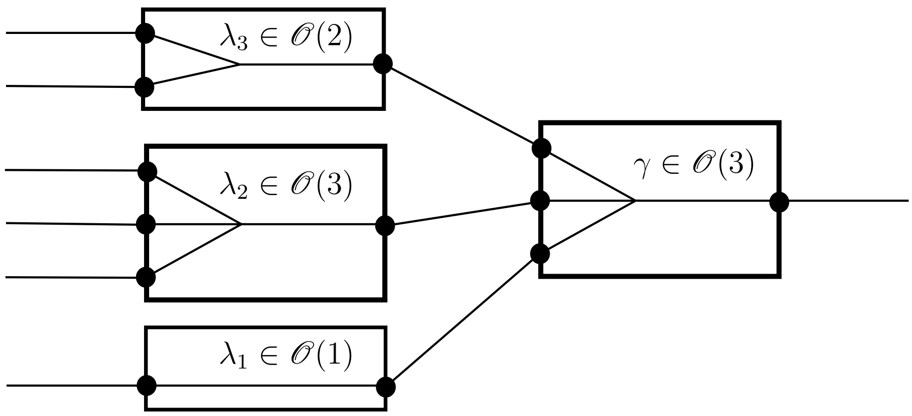


FIGURE 3. The composition $(\mathscr{O}(1) \otimes \mathscr{O}(3) \otimes \mathscr{O}(2)) \otimes \mathscr{O}(3) \to \mathscr{O}(6)$ visualized as gluing of trees

DEFINITION 2.2. An algebra over an operad $\mathscr{O}$ is an object $A \in \mathscr{M}$ together with morphisms

$$\lambda : A^{\otimes r} \otimes \mathscr{O}(r) \to A$$

for each $r \geq 0$ which satisfy equivariance, unit and associativity axioms.

An example is the unital associative operad in *Sets*. It is given by $Ass(r) = \Sigma_r$ with Σ_r acting through multiplication from the right. The linearization of this operad defines an operad Ass with $Ass(r) = \mathbb{K}[\Sigma_r]$ in $\mathbb{K}$-modules or chain complexes (with trivial differential in this case). Algebras over Ass in $\mathbb{K} - mod$ or $\mathrm{Ch}(\mathbb{K})$ are associative respectively differential graded associative algebras.

Similarly, we can define $\mathscr{C}om$ on the set-level to be the operad with one element in each degree, i.e. $\mathscr{C}om(r) = \{1\}$. Its linearization thus is given by $\mathscr{C}om(r) = \mathbb{K}$ in $\mathbb{K} - mod$ or $\mathrm{Ch}(\mathbb{K})$. For an algebra over $\mathscr{C}om$ there is precisely one way to multiply r elements, which makes the algebras over this operad to be commutative.

Most of the algebras we work with are algebras over an operad. Nevertheless, we sometimes need a more general setup. The language of PROPs generalizes the language of operads to a bigger class of algebras.

2.2. PROPs. In order to generalize operads, we want to view the operations as morphism spaces of categories. Then we can define:

DEFINITION 2.3. A PROP is a symmetric monoidal category with objects the natural numbers including zero (whose symmetric monoidal structure is given by $+$).

An algebra over a PROP $\mathcal{E}$ is a strong symmetric monoidal functor from $\mathcal{E}$ to *Sets*.

We can associate a PROP $\mathcal{E}_{\mathscr{O}}$ to an operad $\mathscr{O}$ in *Sets* by choosing it to be the symmetric monoidal category generated by $\mathscr{O}$ with $\mathscr{O}(n) \hookrightarrow \mathcal{E}_{\mathscr{O}}(n, 1)$. More precisely, this can be described as follows: The morphism spaces $\mathcal{E}_{\mathscr{O}}(n, m)$ are defined to be "all possible ways to multiply elements together after precomposing with a permutation", i.e. a morphism in the PROP is an equivalence class of elements obtained by first applying an element σ from Σ_n to the n inputs and then m operations γ_i each from a $\mathscr{O}(n_i)$ with $\sum n_i = n$ where given $\sigma' \in \Sigma_{n_1} \times \cdots \times \Sigma_{n_m}$, $\sigma \in \Sigma_n$ and $\gamma_i \in \mathscr{O}(n_i)$ we identify the elements given by the pair $(\sigma' \circ \sigma, \gamma_1 \times \cdots \times \gamma_m)$ and $(\sigma, (\gamma_1 \times \cdots \times \gamma_m) \circ \sigma')$.

An example of an operation built from an operad and a permutation is illustrated in Figure 4. Algebras over the PROP and over the operad agree.

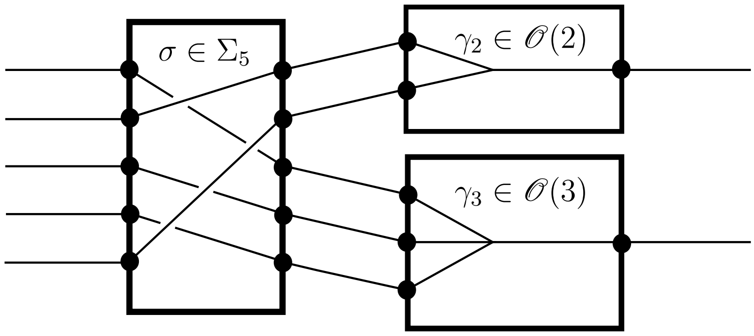


FIGURE 4. An element in $\mathcal{E}_{\mathscr{O}}(5,2)$ given by a permutation $\sigma \in \Sigma_5$ and two operations $\gamma_2 \in \mathscr{O}(2)$ and $\gamma_3 \in \mathscr{O}(3)$

A PROP $\mathcal{E}$ enriched over chain complexes has morphism chain complexes $\mathcal{E}(n,m)$ and all composition and structure maps are maps of chain complexes. Algebras are then precisely the enriched strong symmetric monoidal functors $\mathcal{E} \to \mathrm{Ch}(\mathbb{K})$. Thus for example the dg-PROP associated to the dg-operad Ass gives us differential graded associative algebras and algebras over the PROP associated to the operad $\mathscr{C}om$ are differential graded commutative algebras. In our notation we will not distinguish between an operad and its induced PROP.

However, the definition of a PROP does not only recover operads in a more formal way, it also encodes algebraic structures which cannot be encoded in an operad. These are all those operations which cannot be written as a permutation composed with a disjoint union of operations with a certain number of inputs and one output. An example is the comultiplication on a coalgebra. One might want to use the notation of cooperads instead - however, when we want to involve comultiplications and multiplications at the same time, the language of PROPs seems to be the appropriate one. An example of this are Frobenius algebras which we have defined in Section 1.5. Going back to the definition, one sees that all operations between $A^{\otimes n}$ and $A^{\otimes m}$ inducing the structure are generated by

- a multiplication $m : A \otimes A \to A$,
- the unit $\mathbb{1}_A : \mathbb{K} \to A$,
- the comultiplication $\Delta : A \to A \otimes A$,
- the counit $\eta : A \to \mathbb{K}$ and
- the twist map $\tau : A \otimes A \to A \otimes A$.

The PROP Fr of Frobenius algebras is the linearization of the unenriched PROP given by all operations one can build out of the above mentioned where we identify those giving associativity, unity, coassociativity, counity and the Frobenius relation. A strong symmetric monoidal functor from the linearization of this PROP to $\mathrm{Ch}(\mathbb{K})$ is then the same as a dg-Frobenius algebra.

The PROPs of symmetric and commutative Frobenius algebras sFr and cFr are defined analogously, dividing out the extra relations defining symmetry and commutativity, respectively. Following the work of Lauda and Pfeiffer (cf. [**LP08**, Cor. 4.5]), we give an alternative graphical way to view the PROP of symmetric Frobenius algebras as follows (see also Figure 5 for a general example of a morphism in $sFr(1,3)$):

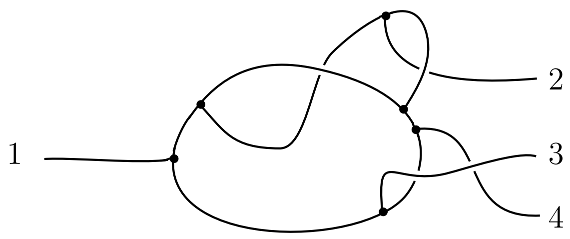


FIGURE 5. An element in $sFr(1,3)$

A morphism in $sFr(m,n)$ is a graph with $m+n$ (labeled) leaves, all vertices of valence 3 and a cyclic ordering of the edges at each vertex up to the relation jumping an edge over a vertex (a local picture of this relation is shown in Figure 6). Then splitting one edge into two corresponds to comultiplying the algebra element,

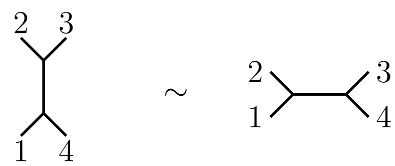
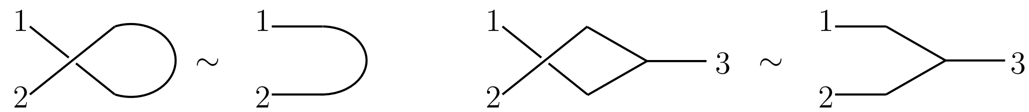


FIGURE 6. The equivalence relation in the PROP Fr

whereas unifying two edges is the multiplication. The relation in Figure 6 encodes associativity, coassociativity and the Frobenius relation depending on how it is read. Symmetry and the fact that doing the twist twice is the identity hold since we do not choose an embedding into the plane, i.e. symmetry can be unraveled as shown in Figure 7(a). The PROP cFr is the quotient of sFr by forgetting the cyclic ordering at the vertices which is the same as dividing out the relation shown in Figure 7(b).



(a) the symmetry relation in sFr (b) the commutativity relation in cFr

FIGURE 7. Symmetry and commutativity in terms of graphs

This leads us to another interesting example of a PROP: The PROP of open cobordisms, which is not concentrated in degree zero as all the other PROPs we worked with so far were.

EXAMPLE 2.4 (The PROP of open cobordisms). *The category of open cobordisms is the dg-PROP where we think of a number $n \in \mathbb{N}$ as a disjoint union of intervals and the morphisms $\mathcal{O}(m,n)$ are given by a graph complex whose homology is the same as the homology of the moduli spaces of 2–dimensional cobordisms (i.e. Riemann surfaces with boundary) between these intervals (for an example see Figure 1(b)). This turns out not to be an associative PROP, i.e. there is no functor $Ass \to \mathcal{O}$. However, it is an A_∞-PROP, i.e. there is a functor $A_\infty \to \mathcal{O}$.*

2.3. Back to Hochschild homology. In the previous section we have seen that differential graded associative algebras are equivalent to strong symmetric monoidal functors $Ass \to \mathrm{Ch}(\mathbb{K})$. It is a natural question to ask whether we can define a Hochschild complex for other functors $\Phi : Ass \to \mathrm{Ch}(\mathbb{K})$. For that, we follow [**WW11**, Section 2]. In order to see how the generalization arises, we first consider the case of a functor associated to an associative algebra A, i.e. we fix the strong symmetric monoidal functor Φ defined by $\Phi(n) = A^{\otimes n}$. The differential d_Φ on $\Phi(n)$ is given by the differential on the tensor product, i.e. the sum over all actions on one tensor factor (with a sign). The multiplication map $A \otimes A \to A$ then corresponds to the map induced by $m \in Ass(2,1)$ acting on $\Phi(2)$.

Hence the Hochschild complex as defined in the first section can be rewritten as

$$C_*(A,A) = \bigoplus_{k\geq 0} A^{\otimes k+1} = \bigoplus_{k\geq 0} \Phi(k+1).$$

Let $m_{j,k} \in Ass(n, n-1)$ be the element which multiplies the j–th and the k–th input. Then the differential on the summand $\Phi(k+1)$ of $C_*(A,A)$ can be rewritten as

$$d = d_\Phi + \sum_{i=0}^{k} \pm \Phi(m_{i+1,i+2})$$

where $m_{k+1,k+2} := m_{k+1,1}$. Stepping back for a second we notice that the definition of the Hochschild complex as stated above does not use the strong symmetric monoidality of Φ at all. Using the formulas written above, this defines the Hochschild complex $C(\Phi)$ for an arbitrary dg-functor $Ass \to \mathrm{Ch}(\mathbb{K})$. A big advantage as we will see is that this allows to apply the Hochschild complex to the representable functors $\mathcal{E}(m,-)$ for a PROP $\mathcal{E}$ with an associative multiplication, i.e. a PROP $\mathcal{E}$ together with a functor $Ass \to \mathcal{E}$ which is the identity on objects. We can add an extra functoriality into the definition and write

$$C(\Phi)(m) = \bigoplus_{k\geq 0} \Phi(k+1+m)$$

which then allows us to iterate the construction and obtain

$$C^n(\Phi)(m) = \bigoplus_{k_1,\dots,k_n} \Phi(k_1+1+\cdots+k_n+1+m).$$

For a strong symmetric monoidal functor Φ corresponding to an algebra A we get an isomorphism

$$C^n(\Phi)(m) \cong C(\Phi)(0)^{\otimes n} \otimes \Phi(1)^{\otimes m} \cong C_*(A,A)^{\otimes n} \otimes A^{\otimes m}.$$

2.4. Formal operations on the Hochschild complex. The main goal of this work is to understand the complex of operations

$$C_*(A,A)^{\otimes n_1} \otimes A^{\otimes m_1} \to C_*(A,A)^{\otimes n_2} \otimes A^{\otimes m_2}$$

which then for example might correspond to string topology operations. More concretely, we want to find such operations which are natural in some class of algebras A, that means natural in some associative algebras over a PROP $\mathcal{E}$, i.e. algebras over a PROP $\mathcal{E}$ such that there is a functor $Ass \to \mathcal{E}$ which is the identity on objects. We denote the complex of these operations by $Nat^{\otimes}([\begin{smallmatrix} n_1 \\ m_1 \end{smallmatrix}], [\begin{smallmatrix} n_2 \\ m_2 \end{smallmatrix}])$ and rewrite the definition in a closer form as the complex of morphisms

$$Nat^{\otimes}_{\mathcal{E}}([\begin{smallmatrix} n_1 \\ m_1 \end{smallmatrix}], [\begin{smallmatrix} n_2 \\ m_2 \end{smallmatrix}]) = \hom(C^{n_1}(\Phi)(m_1), C^{n_2}(\Phi)(m_2))$$

natural in all strong symmetric monoidal functors $\Phi : \mathcal{E} \to \mathrm{Ch}(\mathbb{K})$.

It turns out that this complex is hard to study. On the other hand, the obvious generalization we could do is to test on all functors and not only on strong symmetric monoidal functors. Thus, in [**Wah12**, Sec. 2] Wahl defines the *formal operations* as

$$Nat_{\mathcal{E}}([\begin{smallmatrix} n_1 \\ m_1 \end{smallmatrix}], [\begin{smallmatrix} n_2 \\ m_2 \end{smallmatrix}]) = \hom(C^{n_1}(\Phi)(m_1), C^{n_2}(\Phi)(m_2))$$

natural in **all** functors $\Phi : \mathcal{E} \to \mathrm{Ch}(\mathbb{K})$.

The big advantage of the second complex is that we can compute the formal operations more concretely. It was proven in [**Wah12**, Theorem 2.1] there is an isomorphism

$$Nat_{\mathcal{E}}([\begin{smallmatrix} n_1 \\ m_1 \end{smallmatrix}], [\begin{smallmatrix} n_2 \\ m_2 \end{smallmatrix}]) \cong \prod_{j_1,\dots,j_{n_1}} \bigoplus_{k_1,\dots,k_{n_2}} \mathcal{E}(j_1 + \cdots + j_{n_1} + m_1, k_1 + \cdots + k_{n_2} + m_2)$$

where the right hand side is equipped with a degree shift and a complicated differential coming from a coHochschild-Hochschild construction itself. It turns out that the homology of it can be computed explicitly in quite a few cases which we will come back to after saying a few more words about the relation between $\mathrm{Nat}^{\otimes}_{\mathcal{E}}$ and $\mathrm{Nat}_{\mathcal{E}}$. Every operation which is natural in all functors is in particular natural in all strong symmetric monoidal functors, so we have a restriction map $r : \mathrm{Nat}_{\mathcal{E}}([\begin{smallmatrix} n_1 \\ m_1 \end{smallmatrix}], [\begin{smallmatrix} n_2 \\ m_2 \end{smallmatrix}]) \to \mathrm{Nat}^{\otimes}_{\mathcal{E}}([\begin{smallmatrix} n_1 \\ m_1 \end{smallmatrix}], [\begin{smallmatrix} n_2 \\ m_2 \end{smallmatrix}])$. In general we define $\widehat{\mathcal{E}}$ to be the PROP with morphism spaces $\widehat{\mathcal{E}}(m,n)$ all the morphisms $A^{\otimes m} \to A^{\otimes n}$ natural in all $\mathcal{E}$–algebras A. Then r is injective/surjective/a quasi-isomorphism if and only if $\rho : \mathcal{E}(m,n) \to \widehat{\mathcal{E}}(m,n)$ is (cf. [**Wah12**, Theorem 2.9]). In particular, in the case of operads any two operations can be distinguished by their actions on some free algebra (i.e. ρ is injective), so in the case where $\mathcal{E}$ comes from an operad the map r is injective (see [**Wah12**, Example 2.11]).

2.5. Computations of $\mathrm{Nat}_{\mathcal{E}}$. Turning back to the operations on the Hochschild homology introduced in Section 1.3 we see that using the above definition these are operations in $\mathrm{Nat}^{\otimes}_{\mathcal{E}}([\begin{smallmatrix} n_1 \\ m_1 \end{smallmatrix}], [\begin{smallmatrix} n_2 \\ m_2 \end{smallmatrix}])$ for $\mathcal{E}$ the appropriate class of algebras. Thus, one might ask the question whether these generalize to operations in $\mathrm{Nat}_{\mathcal{E}}([\begin{smallmatrix} n_1 \\ m_1 \end{smallmatrix}], [\begin{smallmatrix} n_2 \\ m_2 \end{smallmatrix}])$. Going through the definitions in that section one checks that all operations were actually defined using only the structure maps like multiplication, comultiplication and the permutations, i.e. they are all induced by the action of the PROP on the Hochschild complex. Hence, rewriting them in terms of these generators and applying Φ, we get an action on the Hochschild complex of Φ for all $\Phi : \mathcal{E} \to \mathrm{Ch}(\mathbb{K})$. So we already know a bunch of formal operations for the PROPs Ass, $\mathscr{C}om$, sFr and cFr.

In the original paper introducing formal operations [**Wah12**], Wahl gives three examples of PROPs $\mathcal{E}$ where the homology of $\mathrm{Nat}_{\mathcal{E}}$ can be identified with the homology of smaller and more combinatorial complexes. The first and most important example for us is $\mathcal{E} = sFr$. Recall that a Sullivan diagram in $\mathcal{SD}([\begin{smallmatrix} n_1 \\ m_1 \end{smallmatrix}], [\begin{smallmatrix} n_2 \\ m_2 \end{smallmatrix}])$ gives an operation

$$C_*(A,A)^{\otimes n_1} \otimes A^{\otimes m_1} \to C_*(A,A)^{\otimes n_2} \otimes A^{\otimes m_2}$$

and since all these operations are formal, it actually defines an operation

$$C^{n_1}(\Phi)(m_1) \to C^{n_2}(\Phi)(m_2)$$

natural in all $\Phi : sFr \to \mathrm{Ch}(\mathbb{K})$, i.e. we have dg-maps

$$J_{sFr} : \mathcal{SD}([\begin{smallmatrix} n_1 \\ m_1 \end{smallmatrix}], [\begin{smallmatrix} n_2 \\ m_2 \end{smallmatrix}]) \to \mathrm{Nat}_{sFr}([\begin{smallmatrix} n_1 \\ m_1 \end{smallmatrix}], [\begin{smallmatrix} n_2 \\ m_2 \end{smallmatrix}]),$$

which lift to dg-functors for $\mathcal{SD}$ and Nat_{sFr} categories with elements $\mathbb{N} \times \mathbb{N}$ and morphism spaces the above ones.

In [**Wah12**, Theorem 3.8] Wahl proves that J_{sFr} is a quasi-isomorphism of dg-categories and split injective on each morphism space, i.e. that the homology of the formal operations is completely described by the already known operations given by Sullivan diagrams.

The other two computations carried out in [**Wah12**] do not completely fit into our setup since the multiplication of the PROPs involved is only associative up to homotopy. As mentioned earlier, everything done so far can be carried out for A_∞-PROPs and thus $\mathrm{Nat}_{\mathcal{E}}$ makes sense for these PROPs, too. Recall from the beginning that $\mathcal{O}$ is the PROP of open cobordisms, i.e. $\mathcal{O}(m,n)$ is given by the chains on the moduli space of cobordisms from m to n intervals. Similarly, $\mathcal{OC}([{n_1 \atop m_1}],[{n_2 \atop m_2}])$ is given by the chains on the moduli space of cobordisms going from n_1 circles and m_1 intervals to n_2 circles and m_2 intervals. Then the chain version of the map J_{sFr} gives a dg-map

$$J_{\mathcal{O}} : \mathcal{OC}([{n_1 \atop m_1}],[{n_2 \atop m_2}]) \to \mathrm{Nat}_{\mathcal{O}}([{n_1 \atop m_1}],[{n_2 \atop m_2}]),$$

which by [**Wah12**, Theorem 3.1] is split injective and yields a quasi-isomorphism of categories.

Restricting $\mathcal{O}$ to the subPROP of unital A_∞-algebras A_∞^+ and $\mathcal{OC}([{n_1 \atop m_1}],[{n_2 \atop m_2}])$ to subspaces $Ann([{n_1 \atop m_1}],[{n_2 \atop m_2}])$ built out of disks and annuli (for a definition see [**WW11**, Prop. 6.12]) in [**Wah12**, Theorem 3.7] it is shown that the restriction of $J_{\mathcal{O}}$ to Ann factors through a quasi-isomorphism

$$J_{A_\infty^+} : Ann([{n_1 \atop m_1}],[{n_2 \atop m_2}]) \to \mathrm{Nat}_{A_\infty^+}([{n_1 \atop m_1}],[{n_2 \atop m_2}]).$$

All the quasi-isomorphisms mentioned in this section are actually split. These splittings seem to fail as soon as we implement commutativity, since a lot of structure on the PROP is lost.

3. Generalizations for higher dimensions

3.1. Higher dimensional string topology operations. Another way of generalizing Section 1 is to look at more general mapping spaces than $Map(S^1, M)$. For example higher string topology deals with operations on $Map(S^n, M)$. These have been investigated by various authors, see for example [**VPS76**], [**CHV06**, Chapter 6],[**Bar10**], [**GTZ10a**] and [**GTZ12**], where the last two use higher Hochschild homology to give an E_{n+1}-structure on $Map(S^n, M)$ if M is an n–connected Poincaré duality space (with little extra conditions).

To investigate operations on $Map(|X_\bullet|, |Y_\bullet|)$ for simplicial sets $X_\bullet$ and $Y_\bullet$ we can use the generalized Jones isomorphism

$$H^*(Map(|X_\bullet|, |Y_\bullet|)) \cong HH_{X_\bullet}(C^*(Y_\bullet), C^*(Y_\bullet)) \tag{3.1}$$

which holds if the dimension of $X_\bullet$ is smaller than or equal the connectivity of $Y_\bullet$ (cf. [**GTZ12**, Theorem 4.4]). Here, $HH_{X_\bullet}(C^*(Y_\bullet), C^*(Y_\bullet))$ is the higher Hochschild homology of $H^*(Y_\bullet)$ with respect to $X_\bullet$ which we define in the next section. Hence looking for operations on higher Hochschild homology is one way to investigate higher string topology.

3.2. Higher Hochschild homology and topological Chiral homology. Let A be a commutative algebra. Given two (ordered) finite sets S and T and a map $f : S \to T$ of sets this induces a map $f_* : A^{\otimes|S|} \to A^{\otimes|T|}$ such that

$$f_*(a_0 \otimes \cdots \otimes a_{|S|-1}) = \pm b_0 \otimes \cdots \otimes b_{|T|-1}$$

with $b_j = \prod_{i\in f^{-1}(j)} a_i$ (where we view $S = \{0, \cdots, |S| - 1\}$ and T similarly) and the sign comes from the permutations of the a_i. Drawing such a map in terms of trees we put the a_i on the inputs and multiply all those which are sent to the same output (cf. Figure 8).

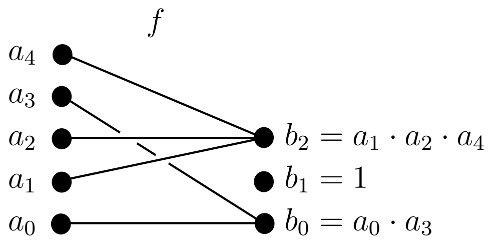


FIGURE 8. the map f_* induced by a map $f : \{0, \dots, 4\} \to \{0, 1, 2\}$

For a finite simplicial set $X_\bullet$ and a commutative algebra A the higher Hochschild chains of A with respect to $X_\bullet$ are defined as $CH_{X_\bullet}(A, A)_k = A^{\otimes|X_k|}$ with differential $\sum \pm d_{i*}$ induced by the $d_i : X_k \to X_{k-1}$ as explained above. For a commutative differential graded algebra we again take the total complex with respect to both differentials. Moreover, for a general simplicial (not necessarily finite) set $X_\bullet$ one takes the colimit over all finite subsets of $X_\bullet$. Higher Hochschild homology was originally defined in [**Pir00**] and further work was done in [**GTZ10a**] and [**GTZ12**]. In the original paper a version for arbitrary functors from the commutative PROP to vector spaces was also carried out.

Besides the isomorphism to the cohomology of the topological mapping spaces stated in Equation (3.1), we want to point out some further properties of higher Hochschild homology. First of all, there is a simplicial model of the circle given by the simplicial set $S^1_k = \{0, \cdots, k\}$ with boundaries

$$d_i(j) = \begin{cases} j & \text{for } i \leq j \\ j-1 & \text{for } i > j \end{cases} \quad \text{and } d_n(j) = \begin{cases} j & \text{for } j \neq n \\ 0 & \text{for } j = n \end{cases}$$

and degeneracies s_i the maps missing i, such that $H_*(S^1_\bullet) = H_*(S^1)$. Given a commutative algebra A, we get

$$C_{S^1_\bullet}(A, A) \cong C_*(A, A),$$

i.e. we recover the ordinary Hochschild chains as defined in Section 1.2.

Taking the simplicial set to be a point, one obtains $HH_{pt}(A, A) \cong A$. Moreover, $CH_{X_\bullet}(A, A)$ is always a commutative differential graded algebra itself using a higher analog of the shuffle product.

Given two simplicial sets $X_\bullet$ and $Y_\bullet$, one obtains

$$CH_{X_\bullet \amalg Y_\bullet}(A, A) \cong CH_{X_\bullet}(A, A) \otimes CH_{Y_\bullet}(A, A).$$

Finally, given another simplicial set $Z_\bullet$ with an injection $X_\bullet \hookrightarrow Z_\bullet$ and an arbitrary map $Y_\bullet \to Z_\bullet$ we put $W_\bullet = X_\bullet \underset{Z_\bullet}{\cup} Y_\bullet$. By [**GTZ10a**, Lemma 2.1.6], there is an isomorphism

$$CH_{X_\bullet}(A,A) \underset{CH_{Z_\bullet}(A,A)}{\otimes} CH_{Y_\bullet}(A,A) \to CH_{W_\bullet}(A,A).$$

In [**GTZ10b**] the authors work with an infinity version of higher Hochschild homology and prove that a lift of the above properties gives an axiomatic definition of higher Hochschild homology. More precisely, in [**GTZ10b**, Theorem 1 and Theorem 2] they prove that $CH : sSet_\infty \times CDGA_\infty \to CDGA_\infty$ is the unique bifunctor fulfilling the following axioms:

(1) **value on a point**: There is a natural equivalence of CDGAs $CH_{pt}(A) \cong A$.
(2) **monoidal**: There are natural equivalences of CDGAs

$$CH_{\amalg X_{i_\bullet}}(A) \cong \bigotimes CH_{X_{i_\bullet}}(A).$$

(3) **homotopy gluing/pushout**: CH sends homotopy pushout in $sSet_\infty$ to homotopy pushouts in $CDGA_\infty$.

Using this axiomatic definition, in [**GTZ10b**, Theorem 5] they furthermore show that for an m-dimensional framed manifold M the topological chiral homology (also called factorization homology) $\int_M(A)$ is equivalent to the higher Hochschild homology $CH_M(A)$.

Topological chiral homology is a homology theory for topological manifolds. It has been made precise in [**Lur12**] but arises from work of Beilinson and Drinfeld on factorization algebras in [**BD04**]. A good introduction into the topic can be found in [**Fra12**]. We finish with giving a definition for the infinity category of topological spaces:

Define the operad $Disk_m^{fr}(n)$ of framed embeddings of n disjoint copies of $\mathbb{R}^m$ to $\mathbb{R}^m$ and let $Disk_m^{fr}(n_1, n_2)$ be the PROP associated to the operad. Let M be a framed m–manifold and define $\mathbb{E}_M(n) = Emb^{fr}(\coprod_n \mathbb{R}^m, M)$. For $\Phi : Disk_m^{fr} \to Top$, the topological Chiral homology of M with coefficients in Φ is the functor

$$\int_M \Phi = \Phi \underset{Disk_m^{fr}}{\overset{\mathbb{L}}{\otimes}} \mathbb{E}_M.$$

The more general definition then goes via the derived coend in infinity categories.

CHAPTER 2

Summary of results

Paper A

The first paper is concerned with the complex of formal operations for the commutative PROP, more precisely it computes the homology of $\mathrm{Nat}_{\mathscr{C}om}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}],[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}])$ for natural numbers n_1, m_1, n_2, m_2. The main ingredient is the identification of the complex with the dual of tensor powers of the Hochschild homology of the cohomology of the circle.

To be more concrete, working over a field, we first prove that the homology of $\mathrm{Nat}_{\mathscr{C}om}([\begin{smallmatrix}1\\0\end{smallmatrix}],[\begin{smallmatrix}1\\0\end{smallmatrix}])$ is concentrated in degrees zero and one. A general element in the degree zero part is an infinite sum of scalar multiples of Loday's shuffle operations (see Section 1.4.3). This is a well-defined operation, since only finitely many shuffle operations are non-zero on each degree. A general element in degree one is the composition of such an element with Connes' boundary operator.

We furthermore prove that the homology of $\mathrm{Nat}_{\mathscr{C}om}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}],[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}])$ can be described in terms of operations built out of Loday's shuffle operations, Connes' boundary operator and the shuffle product. We specify a procedure to obtain a unique presentation of a homology class in $\mathrm{Nat}_{\mathscr{C}om}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}],[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}])$ in terms of the above building blocks.

Paper B

In the second paper we define a dg-category of looped diagrams $l\mathcal{D}$ inspired by the dg-category of Sullivan diagrams $\mathcal{SD}$. We define a dg-functor J_{cFr} from looped diagrams to the complex of formal operations for the PROP of commutative Frobenius algebras and hence obtain operations natural in all those algebras. We prove that we recover all operations known for symmetric Frobenius dg-algebras as well as the formal operations for differential graded commutative algebras (as identified in Paper A) which in particular include Loday's lambda and shuffle operations. Furthermore, we prove that there is a chain level version of a suspended cacti operad inside the complex of looped diagrams which recovers the shifted BV-structure on the Hochschild homology of commutative Frobenius algebras defined by Abbaspour (see Section 1.6).

A looped diagram consists of a commutative version of a Sullivan diagram together with a collection of loops inside it. We use an equivalent description of Sullivan diagrams as stated in [**WW11**, Section 2.10] in terms of black and white graphs. A looped diagram $(\Gamma, \gamma_1, \cdots, \gamma_n)$ then more precisely can be described as follows:

- The diagram Γ is a commutative Sullivan diagram with at least n labeled leaves, i.e. a graph with its vertices labeled black and white, all black vertices having valence 3, all white vertices having any valence ≥ 1, an ordering of the white vertices, a total ordering of the edges at each white vertex (i.e. a cyclic ordering at the vertex and a choice of start edge) and at least n labeled leaves.

- For each i we have a loop γ_i in Γ each starting at the labeled leaf i. If we view white vertices as circles, a loop in a commutative Sullivan diagram is a loop in Γ seen as a CW-complex up to the equivalence relation that it is irrelevant how the loop behaves on the parts away from the white vertices.

Examples of looped diagrams are given in Figure 9, where we dotted the part of the loop which does not contain data.

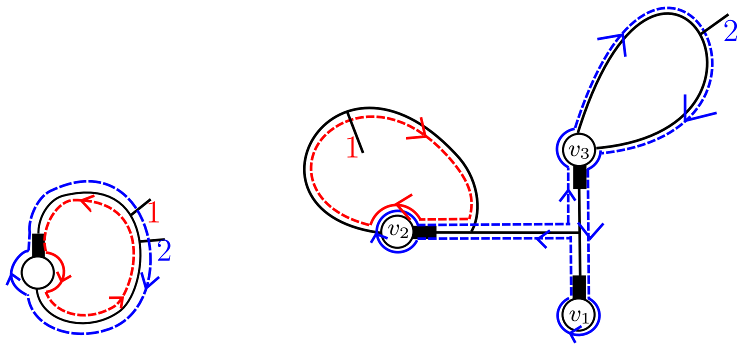


FIGURE 9. Two looped diagrams

We prove that out of the span of such diagrams we obtain a dg-category. There is a restriction functor from Sullivan diagrams to looped diagrams which forgets the cyclic ordering one has in Sullivan diagrams but puts a loop for each marked boundary cycle. The diagram

$$\begin{tikzcd}
\mathcal{SD} \arrow[r, "J_{sFr}"] \arrow[d] & \mathrm{Nat}_{sFr} \arrow[d] \\
l\mathcal{D} \arrow[r, "J_{cFr}"] & \mathrm{Nat}_{cFr}
\end{tikzcd}$$

commutes and hence all operations coming from symmetric Frobenius algebras are already encoded in the complex $l\mathcal{D}$.

Furthermore, by allowing certain products (i.e. infinite sums of operations) we can enlarge the complex $l\mathcal{D}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}],[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}])$ to $il\mathcal{D}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}],[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}])$ and still have a dg-map $J_{cFr} : il\mathcal{D}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}],[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}]) \to \mathrm{Nat}_{cFr}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}],[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}])$. Restricting to a subcomplex of tree-like diagrams, we define a complex $il\mathcal{D}_{\mathscr{C}om}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}],[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}])$. The image of the restriction of J_{cFr} to this subcomplex lies in $\mathrm{Nat}_{\mathscr{C}om}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}],[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}])$ and thus we have a dg-map $J_{\mathscr{C}om} : il\mathcal{D}_{\mathscr{C}om}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}],[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}]) \to \mathrm{Nat}_{\mathscr{C}om}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}],[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}])$. Then we can restate the results of Paper A in the form that the functor $J_{\mathscr{C}om}$ is a quasi-isomorphism. This gives a nicer combinatorial description of all formal operations for commutative algebras.

Last, we can recover the new shifted BV-structure on the Hochschild chains of commutative algebras introduced in [**Abb13a**]. The complex of looped diagrams in fact comes from a simplicial set. We prove that the geometric realization of a certain subcomplex is homotopy equivalent to a shifted version of the Cacti operad as defined by Voronov (cf. [**Vor05**] and for more details [**Kau05**]), whose homology is the BV-operad (by [**Vor05**], [**Kau05**] and [**Get94**]). This then proves that we have a shifted BV-structure on the Hochschild homology of commutative Frobenius dg-algebras coming from an action on the chains. More precisely, we prove the result for commutative, cocommutative open Frobenius dg-algebras.

Paper C

In this paper we generalize the methods of the Hochschild and coHochschild complex to the setup of higher Hochschild homology in the sense of Pirashvili ([**Pir00**]). Already in his original paper Pirashvili considered higher Hochschild homology of arbitrary functors $\mathscr{C}om \to \mathbb{K} - Vect$, which we generalize to graded functors.

Completely analogously to the way one generalizes Hochschild homology of algebras to the Hochschild homology of functors $Ass \to \mathrm{Ch}(\mathbb{K})$ as explained in Section 2.3, we can define the higher Hochschild homology of $\Phi : \mathscr{C}om \to \mathrm{Ch}(\mathbb{K})$ with respect to a simplicial (finite) set $X_\bullet$ as

$$C_{X_\bullet}(\Phi)(m) = \bigoplus \Phi(|X_\bullet| + m).$$

The differential is induced from the differential on Φ and the one coming from the simplicial abelian group structure on $\Phi(|X_\bullet| + m)$ with the boundary maps the ones induced from the boundary maps of the simplicial set d_i, i.e. the composition with the maps $d_i^* \in \mathscr{C}om(|X_k|, |X_{k-1}|)$. We prove that $C_{X_\bullet}(\Phi)(-)$ is again a dg-functor, preserves quasi-isomorphisms and is quasi-isomorphic to its reduced version. Moreover, following [**Wah12**, Section 2] we introduce the coHochschild construction $D_{X_\bullet}(\Psi)$ for functors $\Psi : \mathscr{C}om^{op} \to \mathrm{Ch}(\mathbb{K})$ and prove that all the above properties hold similarly.

For two simplicial sets $X_\bullet$, $Y_\bullet$ and $\mathcal{E}$ a commutative PROP we define the formal operations

$$\mathrm{Nat}_{\mathcal{E}}(X_\bullet, Y_\bullet) = \hom(C_{X_\bullet}(-), C_{Y_\bullet}(-))$$

natural in all functors $\Phi : \mathcal{E} \to \mathrm{Ch}(\mathbb{K})$.

Again, these can be computed explicitly, thus we get

$$\mathrm{Nat}_{\mathcal{E}}(X_\bullet, Y_\bullet) \cong \prod_l \bigoplus_k \mathcal{E}(|X_l|, |Y_k|) \cong D_{X_\bullet}(C_{Y_\bullet}(\mathcal{E}(-,-))) \tag{C.1}$$

which turns out to be true even in a more general context. Furthermore, using the above result we can give smaller models for the formal operations in certain cases:

First, working over a field $\mathbb{F}$, for $X_\bullet$ arbitrary and $Y_\bullet$ a simplicial finite set, a quasi-isomorphism of functors $C^*(Y_\bullet^{\times -}) \simeq A^{\otimes -} : \mathscr{C}om \to \mathrm{Ch}(\mathbb{K})$ induces a quasi-isomorphism

$$\mathrm{Nat}_{\mathscr{C}om}(X_\bullet, Y_\bullet) \simeq CH_{X_\bullet}(A)^*.$$

In particular if $\mathbb{Q} \subset \mathbb{F}$, the deRham algebra $\Omega^\bullet(Y_\bullet; \mathbb{F})$ fulfills this property and therefore

$$\mathrm{Nat}_{\mathscr{C}om}(X_\bullet, Y_\bullet) \simeq CH_{X_\bullet}(\Omega^\bullet(Y_\bullet; \mathbb{F}))^*.$$

Our second computation of $\mathrm{Nat}_{\mathscr{C}om}(X_\bullet, Y_\bullet)$ only holds when the dimension of the simplicial set $X_\bullet$ is smaller than the connectivity of $Y_\bullet$. Using Bousfield's spectral sequence (see [**Bou87**]), we get a quasi-isomorphism between $\mathrm{Nat}_{\mathscr{C}om}(X_\bullet, Y_\bullet)$ and the simplicial chains on the topological mapping space $\hom_{Top}(|X_\bullet|, |Y_\bullet|)$. Moreover, we show that this quasi-isomorphism preserves a certain structure close to a comultiplication on filtrations.

CHAPTER 3

Perspectives

The work described in the previous chapters leads to new questions for further research. We want to outline a few ideas and problems in this context.

The complex of looped diagrams and string operations:

Using the complex of looped diagrams we found a new model to generate string operations. By comparison to the results one has for ordinary Sullivan diagrams, some questions arise:

- What other (interesting) structures on the Hochschild complex of commutative Frobenius algebras can we detect using looped diagrams?
- Do we obtain all formal operations? I.e. is the map $il\mathcal{D} \to \mathrm{Nat}_{cFr}$ injective/surjective/an isomorphism on homology?
- Is there a direct geometric interpretation of the new string operations analog to the constructions by Godin in [**God07**] using fat graphs?
- Is there an ∞-version of looped diagrams acting on the Hochschild chains of C_∞–Frobenius algebras?

The first question could be answered by "a lot" since every looped diagram gives an operation. However, the existence of the new product and the shifted BV-structure leads to the question whether there is a whole shifted HCFT structure inside the operations we get from looped diagrams.

About the second question we can only say that taking all morphisms in $il\mathcal{D}$ will not lead to an injective map, thus we need to restrict to subcomplexes. One might want to restrict to $l\mathcal{D}$ instead of $il\mathcal{D}$ (i.e. not taking products into account). However, we have seen that we get a quasi-isomorphism between a subcomplex of $il\mathcal{D}([{n_1 \atop m_1}],[{n_2 \atop m_2}])$ and $\mathrm{Nat}_{\mathscr{C}om}([{n_1 \atop m_1}],[{n_2 \atop m_2}])$, i.e. we see that the product complexes seem to be necessary. Moreover, trying to imitate the proof of the weak equivalence $\mathcal{SD} \to \mathrm{Nat}_{sFr}$ directly seems to fail, since this proof heavily relies on the fact that there is an ordering at the black vertices.

The last two questions are connected. The work by Godin for fat graphs gives a geometric interpretation of the operations of the open-closed PROP, i.e. of an infinity version of Sullivan diagrams. We would like to find an infinity version of the looped diagrams, which then hopefully should allow a similar geometric construction. Since this structure is supposed to be much bigger than the one described by $\mathcal{OC}$, we expect there to be a connection to the recent work of Hepworth and Lahtinen in [**HL13**], where they introduce the richer structure of a so called HHGFT acting on the homology of the free loop space of BG for G a compact Lie group.

On the other hand, the construction of string operations relies on finding a wrong way map and afterward reading off along the boundary cycles. In ongoing work Ralph Cohen, Nancy Hingston and Nathalie Wahl try to achieve such a construction for Sullivan diagrams. It turns out, that in their approach the cyclic ordering at the black vertices is irrelevant and only the loops they read off along are important. Thus, looped diagrams seem to be the appropriate setup to work in. One first operation, which we hope to recover this way, is the Goresky-Hingston product on the relative cohomology $H^*(LM, M)$ (see [**GH09**]). We conjecture that the product in the shifted BV-structure on the Hochschild homology of commutative Frobenius algebras corresponds to the Goresky-Hingston product on $H^*(LM, M)$ under a relative version of the Jones isomorphism. This was conjectured simultaneously in [**Abb13b**].

Give operations on higher Hochschild homology:

A question arising from Paper C is whether the known operations on higher Hochschild homology (for example the E_n–structure in [**GTZ12**, Theorem 4.4] or the operations leading to the Hodge decomposition in [**Pir00**]) can be seen as formal operations on higher Hochschild homology and to give more explicit calculations of the formal operations for some simplicial set (similar to the computations done for $S^1_\bullet$ in Paper A).

Generalizing the formal operations to other monoidal categories:

At the end of Paper C we give an approach on how to view the Hochschild construction and formal operations in a broader setup summarizing the constructions used so far. The proofs given there do not work for all monoidal model categories yet. So some questions arising from that setup are:

- Can we generalize our constructions and proofs to the category of spectra? Does this give a generalization of the computations of formal operations done so far to topological Hochschild homology? What do we know about the relation of the formal operations and all natural operations in this setup?
- Can we give an infinity version of the proofs done in Section 5 of Paper C?

The first question seems to be approachable using the model structure of S–modules (cf. [**MM02**]) in which every object is fibrant. However, in the generalizations working we additionally need to have an enriched cofibrant replacement functor which is monoidal and forms a comonad. Even though a more general theory for cofibrant replacements via comonadic functors is known (see [**Gar09**, Theorem 3.3] and [**BR12**, Cor. 3.1]) we do not see a way to get the monoidality simultaneously. However, this might also be due to the author's little knowledge in the area of S–modules.

Formal operations for cyclic homology:

Restricting ourselves to Hochschild homology is a starting point for the area of cyclic homology. Many known operations (for example Loday's lambda operations) in fact have been constructed as acting on cyclic homology. One question to ask is how one can carry over the work on Hochschild homology and formal operations to the world of cyclic homology and how the formal operations of Hochschild homology and cyclic homology relate.

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Part II

Scientific papers

Paper A

THE COMPLEX OF FORMAL OPERATIONS ON THE HOCHSCHILD CHAINS OF COMMUTATIVE ALGEBRAS

ANGELA KLAMT

Abstract. We compute the homology of the complex of formal operations on the Hochschild complex of differential graded commutative algebras as defined by Wahl and prove that these can be built as infinite sums of operations obtained from Loday's shuffle operations, Connes' boundary operator and the shuffle product.

Introduction

Natural operations on the Hochschild homology of commutative algebras have been studied by several authors, see for example [Bar68], [GS87], [Lod89] and [McC93]. Recently, in [Wah12], Wahl defined a complex of so called formal operations for a given class of algebras, which comes with a dg-map to the complex of natural transformations. In the case of commutative algebras this map is an injection. In this paper we prove that the homology of the complex of formal operations in the commutative case can be built out of Loday's shuffle operations, Connes' boundary operator and shuffle products.

The commutative PROP $\mathscr{C}om$ is defined to be the symmetric monoidal category with objects the natural numbers (including zero) and morphism spaces $\mathbb{Z}[FinSet(-,-)]$. Denote by Ch chain complexes over $\mathbb{Z}$. Then a (unital) commutative differential graded algebra is a strong symmetric monoidal functor $\mathscr{C}om \to \mathrm{Ch}$. In [WW11], the Hochschild complex $C(\Phi)$ for general functors $\Phi : \mathscr{C}om \to \mathrm{Ch}$ is defined as $C(\Phi) = \bigoplus_k \Phi(k)[k-1]$, with differentials coming from the simplicial structure on $\Phi(k)$ which will be made explicit later. For a strong symmetric monoidal functor Φ (i.e. $\Phi(1)$ is a commutative algebra) this definition agrees with the classical definition of the Hochschild complex $C_*(\Phi(1), \Phi(1))$. Iterating the construction, one defines the iterated Hochschild complex $C^{(n,m)}(\Phi)$ (see Section 1.1 for a precise definition). In this paper we compute the homology of the formal transformations of the (iterated) Hochschild homology of commutative algebras $\mathrm{Nat}_{\mathscr{C}om}([\begin{smallmatrix} n_1 \\ m_1 \end{smallmatrix}], [\begin{smallmatrix} n_2 \\ m_2 \end{smallmatrix}])$ which in [Wah12] were defined as the complex of maps $C^{(n_1,m_1)}(\Phi) \to C^{(n_2,m_2)}(\Phi)$ natural in all functors $\Phi : \mathscr{C}om \to \mathrm{Ch}$.

In [Lod89], Loday defined the so called shuffle operations constructed from permutations $\{1, \cdots, n+1\} \to \{1, \cdots, n+1\}$ which keep the first entry fixed. These act on the n–th degree of the Hochschild complex of an algebra A by permuting the $(n+1)$ factors of A accordingly. Loday's lambda operations can be obtained by similar constructions. These correspond to the power operations on the homology of the free loop space of a manifold (as it is explained in [McC93]). Moreover, they have been used to give a Hodge decomposition of cyclic and Hochschild homology. Both, the lambda and the shuffle operations commute with the boundary maps and one can obtain the lambda operations as linear combination of the shuffle operations and vice versa. However, the shuffle operations fulfill one extra property which makes them suitable for our context: The k–th shuffle operation sh^k acts trivially on all Hochschild degrees smaller than k, i.e. $(sh^k)_l = 0$ if $l < k$. Hence the infinite sum of shuffle operations is still a well-defined operation on the Hochschild complex. Denoting Connes' boundary operator by B and defining operations $B^k = B \circ sh^k$, we can compute the homology of $\mathrm{Nat}_{\mathscr{C}om}([\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}], [\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}])$, i.e. the homology of the complex of operations $C(\Phi) \to C(\Phi)$ natural in all functors $\Phi : \mathscr{C}om \to \mathrm{Ch}$:

Theorem A (see Theorem 2.8)**.** The homology $H_*(\mathrm{Nat}([\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}], [\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}]))$ is concentrated in degrees 0 and 1. In these degrees an explicit description of the elements is given by the following:

(1) Every element in $H_0(\mathrm{Nat}([\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}], [\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}]))$ can be uniquely written as $\sum_{k=0}^{\infty} c_k[sh^k]$ with $c_k \in \mathbb{Z}$ and $[sh^k]$ the classes of the cycles sh^k in homology. In the i-th degree of the product this is given by $(\sum_{k=0}^{\infty} c_k[sh^k])_i = \sum_{k=0}^{i} c_k[(sh^k)_i]$, i.e. it is a finite sum in each component.

(2) Every element in $H_1(\mathrm{Nat}([\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}], [\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}]))$ can be uniquely written as $\sum_{k=0}^{\infty} c_k[B^k]$ with $c_k \in \mathbb{Z}$ and $[B^k]$ the classes of the cycles B^k in homology. In the i-th degree of the product this is given by $(\sum_{k=0}^{\infty} c_k[B^k])_i = \sum_{k=0}^{i} c_k[(B^k)_i]$.

The shuffle product generalizes to a degree zero map $C(\Phi) \otimes C(\Phi) \to C(\Phi)$. In the second half of the paper, we generalize the above theorem to the iterated Hochschild construction and see:

Theorem B (see Theorem 3.4)**.** The complex $\mathrm{Nat}_{\mathscr{C}om}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}],[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}])$ is quasi-isomorphic to the product

$$\prod_{k_1,\cdots,k_{n_1}} A_{k_1,\ldots,k_{n_1}}$$

where the complexes $A_{k_1,\ldots,k_{n_1}}$ have trivial differential and are spanned by objects build out of the B^k, sh^k and the shuffle product in a procedure described in Definition 3.3.

The complex $\prod_{k_1,\cdots,k_{n_1}} A_{k_1,\ldots,k_{n_1}}$ has also an alternative description in terms of graph complexes. In [Kla13b] we define a complex of looped diagrams and a subcomplex of special tree-like looped diagrams $\widetilde{iplD}_{\mathscr{C}om}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}],[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}])$ together with a dg-map $\widetilde{J}_{\mathscr{C}om}:\widetilde{iplD}_{\mathscr{C}om}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}],[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}])\to \mathrm{Nat}_{\mathscr{C}om}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}],[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}])$ such that the image $\widetilde{J}_{\mathscr{C}om}$ is exactly the complex $\prod_{k_1,\cdots,k_{n_1}} A_{k_1,\ldots,k_{n_1}}$. In terms of this data, Theorem B can be nicely rewritten as follows:

Theorem B'. The dg-map $\widetilde{J}_{\mathscr{C}om}:\widetilde{iplD}_{\mathscr{C}om}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}],[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}])\to \mathrm{Nat}_{\mathscr{C}om}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}],[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}])$ is a quasi-isomorphism.

Even though Theorem A is a special case of Theorem B, we give a separate proof of it in the first half of the paper. Parts of the arguments used in the proof of Theorem B are generalizations of those used in the proof of Theorem A.

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Contents

Conventions. Throughout the paper we work in the category Ch of chain complexes over the integers $\mathbb{Z}$. We use the usual sign convention on the tensor product, i.e. the differential $d_{V\otimes W}$ on $V\otimes W$ is defined as $d_{V\otimes W}(v\otimes w)=d_V(v)\otimes w+(-1)^{|v|}v\otimes d_W(w)$.

A *dg-category* $\mathcal{E}$ is a category enriched over chain complexes, i.e. the morphism sets are chain complexes. In this paper we use composition from the right, i.e. we require the composition maps $\mathcal{E}(m,n)\otimes\mathcal{E}(n,p)\to\mathcal{E}(m,p)$ to be chain maps. A *dg-functor* is an enriched functor $\Phi:\mathcal{E}\to \mathrm{Ch}$, so the structure maps $\Phi(m)\otimes\mathcal{E}(m,n)\to\Phi(n)$ are chain maps.

For a graded abelian group A we denote by $A[k]$ the shifted abelian group with $(A[k])_n=A_{n-k}$. Throughout the paper, the natural numbers are assumed to include zero.

1. Recollection of definitions and basic properties

We denote by $\mathscr{C}om$ the PROP of unital commutative algebras considered as a dg-PROP concentrated in degree zero. It is the dg-category with elements the natural numbers (including zero) and morphism spaces $\mathscr{C}om(m,n) = \mathbb{Z}[FinSet(m,n)]$ the linearization of the maps of finite sets, where m and n denote the finite sets with m and n elements, respectively. The PROP $\mathscr{C}om$ is an example of a PROP with A_∞-multiplication as used in [WW11] and [Wah12]. Moreover, it also fits in the context of [Kla13a] where we consider PROPs with commutative multiplication. In the first of the aforementioned papers a more general construction of Hochschild homology was defined. Denoting the Hochschild complex of a dg-algebra A by $C_*(A,A)$, this generalization allows us to define the complex of so-called formal operations which is a subcomplex of the operations

$$C_*(A,A)^{\otimes n_1} \otimes A^{m_1} \to C_*(A,A)^{\otimes n_2} \otimes A^{m_2}$$

natural in all commutative algebras A. This subcomplex is the complex we calculate in the paper. In this section, we recall the definition of the Hochschild complex for functors and the complex of formal operations.

1.1. **Hochschild and coHochschild complexes.** Recall that for a dg-algebra A its Hochschild complex $C_*(A,A)$ is defined as

$$C_*(A,A) \cong \bigoplus_k A^{\otimes k}[k-1]$$

with differential coming from the inner differential on A and the Hochschild differential which takes the sum over multiplying neighbors together (and an extra summand multiplying the last and first element). We start with generalizing this definition as it was done in [WW11, Section 5]:

For $1 \leq i < k$ Let $m^k_{i,i+1} \in FinSet(k,k-1)$ be the map which sends i and $i+1$ to i and is orderpreserving and injective on the other elements. For $\Phi : \mathscr{C}om \to \mathrm{Ch}$ a dg-functor the *Hochschild complex* of Φ is the functor $C(\Phi) : \mathscr{C}om \to \mathrm{Ch}$ defined by

$$C(\Phi)(n) = \bigoplus_{k\geq 1} \Phi(k+n)[k-1].$$

The sets $\Phi(k+1+n)$ for $k \geq 0$ form a simplicial abelian group with boundary maps $d_i = \Phi(m^{k+1}_{i+1,i+2} + \mathrm{id}_n)$ where we set $m^k_{k,k+1} = m^k_{1,k}$ and degeneracy maps induced by the map inserting a unit at the $i+1$–st position. Denoting the differential on Φ by d_Φ, we define the differential on $C(\Phi)(n)$ to be the differential coming from these boundary maps which explicitly is given by

$$d(x) = d_\Phi(x) + (-1)^{|x|} \sum_{i=1}^{k} (-1)^i \Phi(m^k_{i,i+1} + id_n)(x).$$

Note that we used the formula $d = \sum_{i=0}^k (-1)^{i+1} d_i$ for the differential on the chain complex associated to a simplicial set instead of the usual choice $d = \sum_{i=0}^k (-1)^i d_i$. We do so to make the signs fit with the original definition in [WW11, Section 5].

The *reduced Hochschild complex* $\overline{C}(\Phi)(n)$ is the reduced chain complex associated to this simplicial abelian group, i.e. it is given by

$$\overline{C}(\Phi)(n) = \bigoplus \Phi(k+n)/U(k)$$

with $U(k) = \sum_{1\leq i\leq k-1} im(u_i)$ where $u_i : \Phi(k-1+n) \to \Phi(k+n)$ is the map inserting a unit at the $(i+1)$–st position.

Iterating this construction, the complexes $C^{(n,m)}(\Phi)$ and $\overline{C}^{(n,m)}(\Phi)$ are given by

$$C^{(n,m)}(\Phi) := C^n(\Phi)(m) \qquad \text{and} \qquad \overline{C}^{(n,m)}(\Phi) := \overline{C}^n(\Phi)(m).$$

Working out the definitions explicitly we obtain

$$C^{(n,m)}(\Phi) \cong \bigoplus_{j_1\geq 1,\cdots,j_n\geq 1} \Phi(j_1+\cdots+j_n+m)[j_1+\cdots+j_n-n].$$

Before we move on to the coHochschild construction, we want to connect the above definition to the ordinary Hochschild complex of a commutative algebra:

Unital commutative dg-algebras correspond to strong symmetric monoidal functors $\Phi : \mathscr{C}om \to \mathrm{Ch}$ by sending an algebra A to the functor $\Phi(n) = A^{\otimes n}$ and vice versa. Then the Hochschild complex is given by

$$C_*(A^{\otimes -}) = \bigoplus_{k\geq 1} A^{\otimes k}[k-1] \cong C_*(A,A)$$

which is isomorphic to the ordinary Hochschild complex of an algebra. Using the strong monoidality again, we obtain

$$C^{(n,m)}(A^{\otimes -}) \cong C_*(A,A)^{\otimes n} \otimes A^{\otimes m}$$

and similarly for the reduced versions.

Dually, given a dg-functor $\Psi : \mathscr{C}om^{op} \to \mathrm{Ch}$ its *CoHochschild complex* is defined as

$$D(\Psi)(n) = \prod_{k\geq 1} \Psi(k+n)[1-k]$$

with the differential coming from the cosimplicial structure induced by the multiplications, so for $y \in \prod_{k\geq 1} \Psi(k+n)$ it is given by

$$d(y)_l = (-1)^{l+1}(d_\Psi(y_l) - \sum_{i=1}^{l+1} (-1)^i \Psi(m_{i,i+1}^{l+1} + id_n)(y_{k-1}))$$

(see [Wah12, Section 1]). As for the Hochschild construction, we twisted the differential coming from the cosimplicial structure maps by -1. For strong symmetric monoidal functors this construction is isomorphic to the classical coHochschild construction of a coalgebras as for example defined in [Yuk81, Sec. 3.1].

Again, we can take the reduced cochain complex $\overline{D}(\Psi)(n)$ which is the subcomplex

$$\overline{D}(\Psi)(n) = \prod_{k\geq 1} \bigcap_{i=2}^{k} ker(u_i).$$

By [Wah12, Prop. 1.7 + 1.8], the inclusion $\overline{D}(\Psi) \to D(\Psi)$ and the projection $C(\Phi) \to \overline{C}(\Phi)$ are quasi-isomorphisms.

Again, under the correspondence of counital cocommutative coalgebras and strong monoidal functors $\Psi : \mathscr{C}om^{op} \to \mathrm{Ch}$, the coHochschild construction defined above is isomorphic to the ordinary coHochschild construction of a coalgebra.

Furthermore, we can also spell out the iterated construction explicitly, i.e. for a functor $\Psi : \mathscr{C}om^{op} \to \mathrm{Ch}$ we get

$$D^n(\Psi)(m) \cong \prod_{j_1,\cdots,j_n} \Psi(j_1 + \cdots + j_n + m)[n - (j_1 + \cdots + j_n)].$$

1.2. **Formal operations.** The complex of formal operations $\mathrm{Nat}_{\mathscr{C}om}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}], [\begin{smallmatrix}n_2\\m_2\end{smallmatrix}])$ is defined as

$$\mathrm{Nat}_{\mathscr{C}om}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}], [\begin{smallmatrix}n_2\\m_2\end{smallmatrix}]) := \hom(C^{(n_1,m_1)}(\Phi), C^{(n_2,m_2)}(\Phi))$$

natural in all functors $\Phi : \mathscr{C}om \to \mathrm{Ch}$.

In [Wah12, Theorem 2.1] it was shown that

$$\mathrm{Nat}_{\mathscr{C}om}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}], [\begin{smallmatrix}n_2\\m_2\end{smallmatrix}]) \cong D^{n_1}C^{n_2}(\mathscr{C}om(-,-))(m_2)(m_1) \simeq \overline{D}^{n_1}\overline{C}^{n_2}(\mathscr{C}om(-,-))(m_2)(m_1).$$

Since every graded commutative algebra A defines a strong symmetric monoidal functor $A^{\otimes -} : \mathscr{C}om \to \mathrm{Ch}$, every element in $\mathrm{Nat}_{\mathscr{C}om}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}], [\begin{smallmatrix}n_2\\m_2\end{smallmatrix}])$ gives an operation

$$C^{n_1,m_1}(A^{\otimes -}) \cong C_*(A,A)^{\otimes n_1} \otimes A^{\otimes m_1} \to C_*(A,A)^{\otimes n_2} \otimes A^{\otimes m_2} \cong C^{n_2,m_2}(A^{\otimes -}).$$

More precisely, defining $\mathrm{Nat}^{\otimes}_{\mathscr{C}om}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}], [\begin{smallmatrix}n_2\\m_2\end{smallmatrix}])$ to consist of those transformations which are natural in all commutative dg-algebras A, we get a restriction functor $r : \mathrm{Nat}_{\mathscr{C}om}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}], [\begin{smallmatrix}n_2\\m_2\end{smallmatrix}]) \to \mathrm{Nat}^{\otimes}_{\mathscr{C}om}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}], [\begin{smallmatrix}n_2\\m_2\end{smallmatrix}])$. Since $\mathscr{C}om$ is the PROP coming from an operad, by [Wah12, Section 2.2] the map r is injective.

2. The homology of $\mathrm{Nat}_{\mathscr{C}om}$ for $n_1 = n_2 = 1$ and $m_1 = m_2 = 0$

In this section we recall Lodays's shuffle operations, use them to define cycles sh^k and B^k in $\mathrm{Nat}_{\mathscr{C}om}([\begin{smallmatrix}1\\0\end{smallmatrix}], [\begin{smallmatrix}1\\0\end{smallmatrix}])$ and move on to show that every element in the homology is built out of those. More precisely, we will explain that it is sensible to take infinite sums of the operations sh^k and B^k, denote the subcomplex generated by them by X and show that the inclusion $X \hookrightarrow \overline{DC}(\mathscr{C}om(-,-))$ is a quasi-isomorphism.

Our method of doing so is by considering the filtration on $\overline{DC}(\mathscr{C}om(-,-))$ arising from it being a total complex, together with the induced filtration on X. The inclusion yields a map of the associated spectral sequences which both come from exhaustive and complete. We show that the inclusion is an isomorphism on the E^1–page, hence gives (since X has trivial differential) an isomorphism from X to the homology of $\overline{DC}(\mathscr{C}om(-,-)) \simeq \mathrm{Nat}_{\mathscr{C}om}([\begin{smallmatrix}1\\0\end{smallmatrix}], [\begin{smallmatrix}1\\0\end{smallmatrix}])$.

2.1. **Loday's lambda and shuffle operations.** In this section we recall Loday's λ- respectively shuffle operations and give a short recap on their construction. Loday's operations, which can be defined over $\mathbb{Z}$, can be seen as a generalization of the Gerstenhaber-Schack idempotents e^n which can only be defined over $\mathbb{Q}$ (cf. [GS87]) and are a refinement of an operation defined by Barr in [Bar68]. These idempotents were used to define a Hodge decomposition of Hochschild and cyclic homology and any natural operation which acts on each Hochschild degree separately and which has trivial differential can be written as a linear combination of these operations. However, in [Lod89, Prop. 2.8] it was shown how to recover these idempotents from Loday's operations.

Recall from Section 1.2 that we have $\mathrm{Nat}([{}^1_0],[{}^1_0]) \cong DC(\mathscr{C}om(-,-))$. Explicitly, this means that in degree l we obtain

$$\mathrm{Nat}([{}^1_0],[{}^1_0])_l \cong DC(\mathscr{C}om(-,-))_l \cong \prod_{k\geq 0} \mathscr{C}om(k+1,k+l+1).$$

An element $f \in \prod_k \mathscr{C}om(k+1,k+l+1)$ acts on $\bigoplus_i \Phi(i+1)$ by applying f_k to $\Phi(k+1)$. We will use the same notation for the element in $DC(\mathscr{C}om(-,-))$ and $\mathrm{Nat}([{}^1_0],[{}^1_0])$.

We start with the definition of the Euler decomposition of the symmetric group Σ_n as given in [Lod89]. For a permutation $\sigma \in \Sigma_n$ a descent is a number i such that $\sigma(i) > \sigma(i+1)$. Then one defines

$$\Sigma_{n,k} := \{\sigma \in \Sigma_n \mid \sigma \text{ has } k-1 \text{ descents}\}.$$

To construct the operations, we notice that every element $\sigma \in \Sigma_n$ defines an element in $\mathscr{C}om(n+1,n+1) = \mathbb{Z}[FinSet(n+1,n+1)]$ by the embedding of Σ_n into Σ_{n+1} which sends σ to the permutation which leaves 1 fixed and applies the permutation σ to the elements $\{2,\cdots,n+1\}$. We denote the image of $\Sigma_{n,k}$ in Σ_{n+1} by $\Sigma^1_{n+1,k}$.

In [Lod89], up to a sign twist, the operations l^k_n were defined as

$$l^k_n := \sum_{\sigma\in\Sigma^1_{n+1,k}} sgn(\sigma)\sigma$$

for $n \geq 1$ and $1 \leq k \leq n$, $l^0_0 = 1$ and $l^k_n = 0$ else. Out of these, two families of operations were constructed, the λ- and shuffle operations:

$$\lambda^k_n = \sum_{i=0}^{k} \binom{n+k-i}{n} l^i_n$$

for all n, k and

$$sh^k_n = \sum_{i=1}^{k} \binom{n-i}{k-i} l^i_n$$

for $n \geq 1$ and $1 \leq k \leq n$, $sh^0_0 = \mathrm{id}$ and $sh^k_0 = sh^0_n = 0$ for $k > 0$ and $n > 0$. For $n \geq 1$ we obtain $sh^1_n = \mathrm{id}$. For $n \geq 1$ and $k \geq 2$ the shuffles can be seen via another combinatorial description: For each k consider all $(p_1,\cdots,p_k)$-shuffles in Σ_n with $p_1 + \cdots p_k = n$ and all $p_j \geq 1$. As above, we can embed them into Σ_{n+1} by applying the permutation to $\{2,\cdots,n+1\}$ and leaving 1 fixed. Taking the sum over all the images (with sign), we obtain sh^k_n. We write $\lambda^k = \prod_n \lambda^k_n$ and $sh^k = \prod_n sh^k_n$ for the products in $\prod_n \mathscr{C}om(n+1,n+1)$. In particular, both define families of formal operations in $\mathrm{Nat}([{}^1_0],[{}^1_0])_0$.

The elements λ^k lie in the span of the sh^k, more precisely

$$\lambda^k = \sum_{m=0}^{k} \binom{k}{m} sh^m. \tag{2.1}$$

The shuffle operations can also be expressed in terms of the lambda operations as

$$sh^k = \sum_{m=0}^{k} (-1)^{k-m} \binom{k}{m} \lambda^m. \tag{2.2}$$

Remark 2.1. A nice property of the λ^k is their multiplicative behavior. It is shown in [Lod89, Theorem 1.7] that

$$\lambda^k \cdot \lambda^{k'} = \lambda^{kk'}.$$

Together with equations (2.2) and (2.1) we can extract a formula for the multiplication of the shuffle elements and get

$$sh^k \cdot sh^{k'} = \sum_{i=0}^{k}\sum_{i'=0}^{k'}\sum_{j=0}^{ii'} (-1)^{k+k'-(i+i')} \binom{k}{i}\binom{k'}{i'}\binom{ii'}{j} sh^j.$$

A special property of the sh^k is that $sh^k_n = 0$ for $n < k$. This allows us to take infinite sums $\sum_{k=0}^{\infty} c_k sh^k$ and still obtain a well-defined element in $\prod_n \mathscr{C}om(n+1, n+1) \cong \mathrm{Nat}([^1_0], [^1_0])_0$, since in each degree only finitely many terms are nonzero. This is not possible for the λ^k which is the reason why we need to work with the sh^k.

As a next step we see that the operations are actually cycles in $\mathrm{Nat}([^1_0], [^1_0])$. The j–th part of the differential is given by $d(x)_j = (-1)^j(d_h(x)_j - d^{co}(x)_j)$ with $d_h(x) = \sum (-1)^{i+1} d_i(x)$ and $d^{co}(x) = \sum (-1)^{i+1} d^i(x)$.

Proposition 2.2 ([Lod89, Proposition 2.3., Cor. 2.5.]). *The following holds:*

$$d_h(l^k_n) = d^{co}(l^k_{n-1} - l^{k-1}_{n-1})$$

and thus $d(\lambda^k) = 0$ *and* $d(sh^k) = 0$.

We move on to the definition of a second family of elements B^k which will be used to build the degree one part of the homology of $\mathrm{Nat}([^1_0], [^1_0])$. We start with the definition of the BV-operator $B \in \prod_l \mathscr{C}om(l, l+1)$, which as an operation on Hochschild chains corresponds to the well-known Connes' boundary operator. Precomposing this element with the already constructed elements sh^k we obtain the elements we are looking for.

Definition 2.3. The element $B \in \prod_l \mathscr{C}om(l, l+1)$ has as its l-th component $B_l \in \mathscr{C}om(l, l+1)$, defined as

$$B_l = \sum_{i=1}^{l} (-1)^{i(l+1)} g_i$$

with

$$g_i(t) = \begin{cases} t+i+1 & \text{if } t+i+1 \le l+1 \\ t+i-l & \text{else,} \end{cases}$$

i.e. $g_i^{-1}(1) = \emptyset$ and we sum over all cyclic permutations of the set $\{2, \cdots, k+1\}$.

Finally we define B^k as the composition $B \circ sh^k$.

By the usual computations, one sees that B is a cycle. Thus, since the composition of cycles is a cycle, the elements B^k are cycles, too. Analogously to above, we can consider infinite sums $\sum_{i=0}^{\infty} c_k B^k$ since only finitely many B^k are non-trivial in each degree of the image. They can be described explicitly similarly to the elements sh^k:

We consider the n embeddings of $\Sigma_n \to \mathscr{C}om(n, n+1)$ given by composition of maps $\Sigma_n \to \mathscr{C}om(n, n)$ with the embedding of $\mathscr{C}om(n, n)$ into $\mathscr{C}om(n, n+1)$ not hitting the first element, where the l–th map from Σ_n to $\mathscr{C}om(n, n)$ is given by adding l (modulo n) to the image of the permutations. We denote the union of the images of these embeddings of $\Sigma^1_{n,k}$ in $\mathscr{C}om(n, n+1)$ by $\Sigma^+_{n,k}$.

Then we can define

$$R^l_n := \sum_{g \in \Sigma^+_{n+1,l}} sgn(g) g$$

and obtain $(B^k)_n = \sum_{l=1}^{k} \binom{n-l}{k-l} R^l_n$ for $k > 0$, $(B^0)_0 = R^1_0$ and $(B^0)_i = 0$ for $i \neq 0$.

Now we are able to define the subcomplex X which will be shown to be isomorphic to the homology of $Nat_{\mathscr{C}om}([^1_0], [^1_0])$.

Definition 2.4. Let X be the graded abelian group which is defined as

$$X_i = \begin{cases} \{\sum_{k=0}^{\infty} c_k sh^k \mid c_k \in \mathbb{Z}\} & \text{for } i = 0 \\ \{\sum_{k=0}^{\infty} c_k B^k \mid c_k \in \mathbb{Z}\} & \text{for } i = 1 \\ 0 & \text{else.} \end{cases}$$

All the elements in X are elements in $Nat_{\mathscr{C}om}([^1_0], [^1_0]) \cong DC(\mathscr{C}om(-,-))$. Actually, we can take their equivalence classes with respect to the reduced Hochschild construction and check that they lie in the reduced coHochschild construction. Thus we can view X as a subcomplex of $\overline{DC}(\mathscr{C}om(-,-))$ with differential zero.

2.2. **The spectral sequence of $\overline{DC}(\mathscr{C}om(-,-))$.** In order to compute the homology of $\overline{D}(\overline{C}(\mathscr{C}om(-,-)))$ it will be practical to give a different description of it in terms of the homology of products of S^1. that by the definition of the commutative PROP we have $\mathscr{C}om(k,l) = \mathbb{Z}[FinSet(k,l)] \cong \mathbb{Z}[(FinSet(1,l)^{\times k}]$. Thus as abelian groups, we have an isomorphism $\mathscr{C}om(k,l) \cong \mathscr{C}om(1,l)^{\otimes k}$. Viewing the Hochschild construction $C(\mathscr{C}om(k,-))$ as a simplicial abelian group, the l–th level is given by $\mathscr{C}om(k, l+1) \cong \mathscr{C}om(1, l+1)^{\otimes k}$ and the boundary maps are given by post composition with the multiplications of neighbors in $\mathscr{C}om(l+1, l)$, which acts diagonally on the space $\mathscr{C}om(1, l+1)^{\otimes k}$. On the other hand, $FinSet(1, l+1)$ is the standard model for the simplicial circle with one non-degenerate zero- and one non-degenerate one-cell, i.e. $H_*(C_*(FinSet(1,l))) \cong H_*(S^1)$. We denote $S^1_\bullet = FinSet(1, \bullet + 1)$.

Applying the coHochschild construction, out of $\overline{C}(\mathscr{C}om(-,-))$ we form a cosimplicial abelian group whose coboundary maps are given by precomposition with multiplication. Under the above isomorphism $\overline{C}(\mathscr{C}om(k,-)) \cong \overline{C}_*(\mathscr{C}om(1,-)^{\times k})$ this corresponds to doubling the information of the i–th input, i.e. it is given by the i–th diagonal map. Recall that the reduced complex $\overline{D}(\overline{C}_*((S^1_\bullet)^{\times k}))$ is given as

$$\prod_{k\geq 1}\bigcap_{i=2}^{k} ker(u_i)$$

where u_i forgets the i–th factor in $\overline{C}_*((S^1_\bullet)^{\times k+1}))$. Note that because neither the Alexander-Whitney nor the Eilenberg-Zilber map preserve both the multiplication and the twist map, we cannot go further and replace this cosimplicial simplicial abelian group with the one $\overline{C}(S^1_\bullet)^{\otimes k+1}$.

We are interested in computing the homology of the product total complex of the double complex underlying $\overline{D}(\overline{C}(\mathscr{C}om(-,-))$. Recall from [Wei95, Sec. 5.6] that there is an associated second quadrant spectral sequence (obtained by filtering by columns) which comes from an exhaustive and complete filtration of the total complex $\overline{D}(\overline{C}(\mathscr{C}om(-,-))$. The E^1–page is given by taking the homology with respect to the vertical differential, and since the reduced complex is a direct summand and thus permutes with taking homology, we have

$$E^1_{p,q} \cong \bigcap_{i=2}^{p+1} ker(u_i : H_q({S^1_\bullet}^{\times p+1}) \to H_q({S^1_\bullet}^{\times p})).$$

The Alexander-Whitney map (cf. [Wei95, Section 8.5.4]) gives an isomorphism $H_q({S^1_\bullet}^{\times p}) \cong (H_*(S^1_\bullet)^{\otimes \times p})_q$. We denote the generator of $H_0(S^1_\bullet)$ by 1 and the generator of $H_1(S^1_\bullet)$ by y. Then we can rewrite the maps u_i as

$$u_i(x_1 \otimes \cdots \otimes x_{i-1} \otimes x_i \otimes x_{i+1} \otimes \cdots \otimes x_p) = \begin{cases} x_1 \otimes \cdots \otimes x_{i-1} \otimes x_{i+1} \otimes \cdots \otimes x_p & \text{if } x_i = 1 \\ 0 & \text{if } x_i = y. \end{cases}$$

In particular, we see that the only elements which lie in the kernel of all the u_i with $i \geq 2$ are elements of the form $x \otimes y \otimes \cdots \otimes y$ with x arbitrary. Therefore under the Alexander-Whitney map we can rewrite

$$E^1_{p,q} \cong \begin{cases} \langle 1 \otimes \underbrace{y \otimes \cdots \otimes y}_{p} \rangle & \text{if } p = q \\ \langle y \otimes \underbrace{y \otimes \cdots \otimes y}_{p} \rangle & \text{if } q = p+1 \\ 0 & \text{else.} \end{cases}$$

2.3. **The complex X under the filtration.** Since X is a subcomplex of $\overline{DC}(\mathscr{C}om(-,-))$, we also obtain an induced filtration F_pX of X. Since there is no differential, the corresponding spectral sequence collapses on the E^0–page already. Concretely, the filtration is given by

$$F_pX_0 \cong \left\{ \sum_{k\geq p} c_k sh^k \;\middle|\; c_k \in \mathbb{Z} \right\}$$

and similar in degree 1 using the B^k. Since every element in X possesses a unique representation as an infinite sum, we get that $\lim X/F_pX \cong X$ and hence the filtration is complete. Since it starts with X itself, it is exhaustive.

The E^0–page of the corresponding spectral sequence can be described as

$$E^0_{p,q} = F_pX_{q-p}/F_{p+1}X_{q-p} \cong \begin{cases} \langle sh^k \rangle & \text{if } p = q \\ \langle B^k \rangle & \text{if } p+1 = q \\ 0 & \text{else.} \end{cases}$$

Furthermore, since X has no differentials we have $E^r_{p,q} = E^0_{p,q}$ for all r.

2.4. **The isomorphism on E^1–pages.** Next we show that the inclusion of X into $\overline{DC}(\mathscr{C}om(-,-))$ induces an isomorphism on E^1–pages. Therefore, recall that the E^0–page of $\overline{DC}(\mathscr{C}om(-,-))$ is isomorphic to the double complex itself since the p–th column of the E^0–page is the quotient of the total complex of all the columns indexed greater than or equal to p by the columns indexed greater than p. The map from the filtration of X into there thus sends $\langle sh^p \rangle$ to $sh^p_p \in \mathscr{C}om(p+1, p+1)$ since it forgets all parts of sh^p living in columns greater than p. Since these elements have trivial vertical differential, their image on the E^1–page of $\overline{DC}(\mathscr{C}om(-,-))$ is the same and we only need to apply the Alexander-Whitney map to sh^p_p and see that this agrees with the generator $1 \otimes \underbrace{y \otimes \cdots \otimes y}_{p}$ and similar for the B^k.

Lemma 2.5. *Let σ be a permutation in Σ_{n+1} with $\sigma(1)=1$. We consider its image in $\overline{C}_n((S^1)^{\times n+1})$ under the projection from $C_n(S^{1\times n+1})$. Then $AW(\sigma)=0$ if $\sigma\neq\mathrm{id}$ and $AW(\mathrm{id}_{n+1})=1\otimes\underbrace{y\otimes\cdots\otimes y}_{n}$.*

Proof. We first describe how a permutation σ looks as an element of $((S^1)^{\times n+1}_n)$ and explain what the boundary maps are. The n-simplices of $S^1_\bullet$ are given by $\{1,\dots,n+1\}$. For $i<n$ the boundary map d_i maps both $i+1$ and $i+2$ to $i+1$ and is injective and monotone otherwise. The last boundary map d_n maps both $n+1$ and 1 to 1. For a permutation σ with $\sigma(1)=1$ we have $d_i(\sigma)(1)=1$ for any i. In general, for an element $j\in S^1_n$ with $j\neq n+1$ we have

$$d_i(j)=\begin{cases} j\in S^1_{n-1} & \text{if } i+1\geq j\\ j-1\in S^1_{n-1} & \text{if } i+1<j\end{cases}\tag{2.3}$$

and for $n+1\in S^1_n$

$$d_i(n+1)=\begin{cases} n\in S^1_{n-1} & \text{if } i+1\leq n\\ 1\in S^1_{n-1} & \text{if } i=n.\end{cases}\tag{2.4}$$

Denote by $1\in\overline{C}_0(S^1_\bullet)$ the projection of the element $1\in S^1_0$ and by $y\in\overline{C}_1(S^1_\bullet)$ the image of the element $2\in S^1_1$. All other elements in S^1_k are degenerate in $\overline{C}_*(S^1_\bullet)$, i.e. zero after passing to the reduced complex.

We consider the reduced Alexander-Whitney map

$$AW:\overline{C}_n((S^1)^{\times n+1})\to((\overline{C}(S^1))^{\otimes n+1})_n=\bigoplus_{\substack{k_1,\cdots k_{n+1}\\ \sum k_i=n}}\overline{C}_{k_1}(S^1)\otimes\cdots\otimes\overline{C}_{k_n+1}(S^1).$$

Since $\overline{C}_k(S^1)=0$ if $k\neq 0,1$ we have

$$\bigoplus_{\substack{k_1,\cdots k_{n+1}\\ \sum k_i=n}}\overline{C}_{k_1}(S^1)\otimes\cdots\otimes\overline{C}_{k_{n+1}}(S^1)\cong\bigoplus_{\substack{1\leq i\leq n+1\\ k_i=0,k_j=1\text{ for }j\neq i}}\overline{C}_{k_1}(S^1)\otimes\cdots\otimes\overline{C}_{k_{n+1}}(S^1).$$

By [Wei95, Section 8.5.4] the Alexander Whitney can be described as

$$AW(x)=\sum_{\substack{k_1,\cdots k_{n+1}\\ \sum k_i=n}}\overline{D}^1_{k_1,\cdots,k_{n+1}}(x)\otimes\cdots\otimes\overline{D}^{n+1}_{k_1,\cdots,k_{n+1}}(x)$$

with $D^j_{k_1,\cdots,k_{n+1}}=\underbrace{d^j_0\cdots d^j_0}_{k_1+\cdots+k_{j-1}}d^j_{k_1+\cdots+k_j+1}\cdots d^j_n\circ pr_j$ and $\overline{D}^j_{k_1,\cdots,k_{n+1}}$ the map after projecting to the reduced complex. Here, $pr_j:(S^1_\bullet)^{\times n+1}\to S^1_\bullet$ is the projection onto the j–th factor.

We fix $\sigma\in\Sigma_{n+1}$ with $\sigma(1)=1$ and compute $AW(\sigma)$:

Assume $k_1=1$. We show that the map to the summand $\overline{C}_{k_1}(S^1)\otimes\cdots\otimes\overline{C}_{k_{n+1}}(S^1)$ is zero. To do so, we show that $\overline{D}^1_{k_1,\cdots,k_{n+1}}(\sigma)$ is zero. We have

$$D^1_{k_1,\cdots,k_{n+1}}(\sigma)=d^1_2\cdots d^1_n(\sigma)pr_1\in C_1(S^1).$$

Since $pr_1(\sigma)=1$, using the description of the boundary maps above we see that $D^1_{k_1,\cdots,k_{n+1}}(\sigma)=1\in S^1_1$ which is degenerate in $\overline{C}_1(S^1_\bullet)$, so after projecting to the reduced complex the element becomes zero.

Therefore, the only possible non-zero part of the map $AW(\sigma)$ to the reduced complex is the one corresponding to $k_1=0$ and $k_i=1$ for $1<i\leq n+1$. Hence we are left to show that

$$\overline{D}^j_{0,1,\cdots,1}(\mathrm{id})=\begin{cases}1 & \text{if } j=1\\ y & \text{if } j>1\end{cases}$$

and that for $\sigma\neq\mathrm{id}$ there exists a j with $1<j\leq n+1$ such that $\overline{D}^j_{0,1,\cdots,1}(\sigma)=0$.

For the first part, we take $\sigma=\mathrm{id}$, i.e. $\sigma(j)=j$. We want to show that for the element $j\in S^1_n$, $\underbrace{d^j_0\cdots d^j_0}_{j-2}d^j_j\cdots d^j_n(j)$ is $2\in S^1_2$, i.e. its image in $C_*(S^1_\bullet)$ is given by y. Iterating equation (2.3) for $1<j<n+1$ we obtain

$$d^j_j\cdots d^j_n(j)=j\in S^1_{n-((n+1)-j)}=S^1_{j-1}$$

and hence after applying the second case of Equation (2.3) $(j-2)$ times, we obtain

$$\underbrace{d^j_0\cdots d^j_0}_{j-2}(j)=j-(j-2)=2\in S^1_1,$$

so we have shown the claim for all $j \neq n+1$. For $j = n+1$, equation (2.4) implies

$$\underbrace{d_0^j \cdots d_0^j}_{n-1}(n+1) = 2 \in S_1^1.$$

Therefore, $AW(\mathrm{id}) = 1 \otimes y \otimes \cdots \otimes y$.

Now assume that $\sigma \neq \mathrm{id}$. Then there is a j such that $\sigma(j) < j$. Again

$$d_j^j \cdots d_n^j(\sigma(j)) = \sigma(j) \in S_{j-1}^1$$

but in this case we reach the element 1 by applying d_0^j only $\sigma(j) - 1$ times, i.e.

$$\underbrace{d_0^j \cdots d_0^j}_{\sigma(j)-1}(\sigma(j)) = 1 \in S_{j-\sigma(j)}^1.$$

Applying d_0^j more often keeps the result as 1, i.e.

$$\underbrace{d_0^j \cdots d_0^j}_{j-2} d_j^j \cdots d_n^j(\sigma(j)) = 1 \in S_1^1.$$

This element is degenerate, i.e. zero after projecting to $\overline{C}_*(S^1)$ and hence $AW(\sigma) = 0$. □

Similarly to compute the Alexander-Whitney map on the elements B^k one can show

Lemma 2.6. *Let g be a bijection $\{1, \dots, n\} \to \{2, \dots, n+1\}$, viewed as an element in $\mathscr{C}om(n, n+1)$. Then $AW(g) = 0$ if $g \neq \widetilde{id_n}$, with $\widetilde{id_n} : \{1, \dots, n\} \to \{2, \dots, n+1\}$ the map defined by $\widetilde{id_n}(j) = j+1$. Moreover, $AW(\widetilde{\mathrm{id}_n}) = \underbrace{y \otimes \cdots \otimes y}_{n}$.*

Proof. Similar to above, one shows that

$$\overline{D}_{1,1,\cdots,1}^j(\widetilde{\mathrm{id}}) = y$$

and that for $g \neq \widetilde{\mathrm{id}}$ there exists a j with $1 \leq j \leq n$ such that $\overline{D}_{1,1,\cdots,1}^j(g) = 0$. The arguments are completely analog to the ones in the previous proof. □

Since $\overline{C}_*(S^1) \cong H_*(S^1)$ we have actually computed the image of permutations in homology. In particular, we can apply this to $(sh^k)_k$ and $(B^k)_k$. Recall that $(sh^k)_k = \sum_{i=1}^k l_k^i = \sum_{\sigma \in \Sigma_n^1} sgn(\sigma)\sigma$, so it is a sum over all permutations leaving the first entry fixed and thus contains the summand of the identity exactly once. Similarly, $(B^k)_k$ contains the summand $\widetilde{\mathrm{id}_{k+1}}$ once. Now we can conclude:

Corollary 2.7. *The inclusion of X into $\overline{DC}(\mathscr{C}om(-,-))$ induces an isomorphism on the E^1–pages of the corresponding spectral sequences.*

Proof. By the above Lemma we get $AW((sh_k)^k) = AW(\mathrm{id}_{k+1}) = 1 \otimes \underbrace{y \otimes \cdots \otimes y}_{k}$ and $AW((B^k)_k) = AW(\widetilde{\mathrm{id}_{k+1}}) = y \otimes \underbrace{y \otimes \cdots \otimes y}_{k}$. Thus after applying the Alexander-Whitney map (which is an isomorphism) the inclusion from the E^1–page of X to the E^1–page of $\overline{DC}(\mathscr{C}om(-,-))$ is an isomorphism. □

2.5. **Result.** We are ready to state our main theorem of this section:

Theorem 2.8. *The homology $H_*(\mathrm{Nat}([{}^1_0], [{}^1_0]))$ is concentrated in degrees 0 and 1. In these degrees an explicit description of the elements is given by the following:*

(1) *Every element in $H_0(\mathrm{Nat}([{}^1_0], [{}^1_0]))$ can be uniquely written as $\sum_{k=0}^\infty c_k[sh^k]$ with $c_k \in \mathbb{Z}$ and $[sh^k]$ the classes of the cycles sh^k in homology. Hence, in the i-th degree of the product this is given by $(\sum_{k=0}^\infty c_k[sh^k])_i = \sum_{k=0}^i c_k[(sh^k)_i]$, i.e. it is a finite sum in each component.*
(2) *Every element in $H_1(\mathrm{Nat}([{}^1_0], [{}^1_0]))$ can be uniquely written as $\sum_{k=0}^\infty c_k[B^k]$ with $c_k \in \mathbb{Z}$ and $[B^k]$ the classes of the cycles B^k in homology. In the i-th degree of the product this is given by $(\sum_{k=0}^\infty c_k[B^k])_i = \sum_{k=0}^i c_k[(B^k)_i]$.*

Proof. Both filtrations of the complexes $\overline{DC}(\mathscr{C}om(-,-))$ and X are complete and exhaustive. By the Eilenberg-Moore comparison theorem (cf. [Wei95, Theorem 5.5.1]) the inclusion of X into $\overline{DC}(\mathscr{C}om(-,-))$, which is an isomorphism on E^1–pages induces an isomorphism on the homology. Since the differential on X is trivial, we get isomorphisms

$$X \cong H_*(\overline{DC}(\mathscr{C}om(-,-))) \cong H_*(\mathrm{Nat}([{}^1_0], [{}^1_0]))$$

which proves the theorem. □

Remark 2.9. (1) We have $\mathrm{id} = sh^0 + sh^1$ and the BV-operator $B = B^0 + B^1$. If the reader prefers to have these two as part of the generating family, we can replace sh^0 by id and B^0 by B.
(2) By equation (2.1) the lambda operations lie in the span of the sh^k. Even though each sh^k also lies in the finite span of the λ^i for $i \leq k$, we cannot replace all sh^k by λ^k since then the infinite sums taken above would not anymore be degree-wise finite.

3. Iterated Hochschild homology

In this section we generalize our previous computations and describe the elements in the homology of the complex $\mathrm{Nat}_{\mathscr{C}om}([\begin{smallmatrix} n_1 \\ m_1 \end{smallmatrix}], [\begin{smallmatrix} n_2 \\ m_2 \end{smallmatrix}])$. We start with stating the theorem, give an example of an operation and then give an outline of the proof.

3.1. **Definition of extra generators and the main theorem.** To state the main theorem we need to define a few more elementary operations which are the building blocks for general operations:

Definition 3.1. (1) Let $p \in \mathrm{Nat}_{\mathscr{C}om}([\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}], [\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}])$ be the map $\oplus_{k\geq 0}\Phi(k+1) \to \Phi(1)$ given by the projection onto the first summand.
(2) Define sh^0 and B^0 in $\mathrm{Nat}_{\mathscr{C}om}([\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}], [\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}])$ the restriction of the corresponding elements in $\mathrm{Nat}_{\mathscr{C}om}([\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}], [\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}])$ to the summand $\Phi(1)$. Hence, $sh^0 : \Phi(1) \to C_*(\Phi)$ is the inclusion of Φ into the Hochschild complex and B^0 is this inclusion composed with Connes boundary operator.
(3) The shuffle product $m^{1,2} \in \mathrm{Nat}_{\mathscr{C}om}([\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}], [\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}])$ is defined as

$$(m^{1,2})_{j_1,j_2} = \sum_{\substack{\sigma\in\Sigma_{j_1+j_2} \\ (j_1,j_2)\text{–shuffle}}} sgn(\sigma)F(\sigma) \in \mathscr{C}om(j_1+1+j_2+1, j_1+j_2+1)$$

where the sum runs over all (j_1, j_2)–shuffles $\sigma \in \Sigma_{j_1+j_2}$ and the map $F(\sigma)$ sends 1 and $j_1 + 2$ to 1, i to $\sigma(i) + 1$ if $1 < i \leq j_1$ and to $\sigma(i) + 2$ if $j_1 + 2 < i \leq j_2 + 2$. Note that we use the identification $\prod_{j_1,j_2} \mathscr{C}om(j_1 + 1 + j_2 + 1, j_1 + j_2 + 1) \cong (D^2C(\mathscr{C}om(-,-)))_0 \cong \mathrm{Nat}_{\mathscr{C}om}([\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}], [\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}])_0$ as described in Section 1. An illustration of $(m^{1,2})_{2,1}$ is given in Figure 2. The shuffle product is associative and commutative.
Define $m^{1,\cdots,r} \in \mathrm{Nat}_{\mathscr{C}om}([\begin{smallmatrix} r \\ 0 \end{smallmatrix}], [\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}])$ to be the iterated shuffle product. If we write m^M for some subset M of $\{1, \ldots, r\}$, we mean the element only applying the shuffle product to this subset.
(4) Define $\overline{m}^{1,\cdots,r} \in \mathrm{Nat}_{\mathscr{C}om}([\begin{smallmatrix} 0 \\ r \end{smallmatrix}], [\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}])$ the morphism multiplying all elements together. Again, if we label by a subset, we mean the operation only multiplying the elements of the subset.

Now one can check:

Lemma 3.2. *All the operations defined above are cycles.*

Using all these operations, we can define subcomplexes $A_{k_1,\cdots,k_{n_1}}$ of $\mathrm{Nat}_{\mathscr{C}om}([\begin{smallmatrix} n_1 \\ m_1 \end{smallmatrix}], [\begin{smallmatrix} n_2 \\ m_2 \end{smallmatrix}])$ which we then take products of to get all formal operations. Elements of $A_{k_1,\cdots,k_{n_1}}$ are the composition of first applying the operations sh^k and B^k from before to each factor (and precomposing with the inclusion from the algebra into the Hochschild complex if needed), projecting some of the resulting terms onto the algebra and then composing with a tensor product of shuffle products and ordinary products in the algebra. We write this formally as follows:

Definition 3.3. For $k_i \geq 0$ let $A_{k_1,\ldots,k_{n_1}} = \bigoplus_{f,s}\langle x_{f,s}\rangle \subset \mathrm{Nat}([\begin{smallmatrix} n_1 \\ m_1 \end{smallmatrix}], [\begin{smallmatrix} n_2 \\ m_2 \end{smallmatrix}])$ where f and s are functions

- with $f : \{1, \cdots, n_1 + m_1\} \to \{1, \cdots, n_2 + m_2\}$ such that $f(i) \leq n_2$ if $k_i > 0$
- $s : f^{-1}(\{1, \cdots, n_2\}) \to \{0, 1\}$

and $x_{f,s}$ is the composition $x_{f,s} = x_2 \circ x_1$ where for $c := |f^{-1}(\{1, \cdots, n_2\})|$ we define the elements $x_1 \in \mathrm{Nat}_{\mathscr{C}om}([\begin{smallmatrix} n_1 \\ m_1 \end{smallmatrix}], [\begin{smallmatrix} c \\ m_1+n_1-c \end{smallmatrix}])$ and $x_2 \in \mathrm{Nat}_{\mathscr{C}om}([\begin{smallmatrix} c \\ m_1+n_1-c \end{smallmatrix}], [\begin{smallmatrix} n_2 \\ m_2 \end{smallmatrix}])$ as follows:

- The element $x_1 = z_1 \otimes \ldots \otimes z_{n_1+m_1}$ is the tensor product of operations z_i. The operations z_i are defined as follows:
  - If $1 \leq i \leq n_1$ and
    - if $f(i) \leq n_2$ then $z_i \in \mathrm{Nat}_{\mathscr{C}om}([\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}], [\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}])$ is given by

$$z_i = \begin{cases} sh^{k_i} & \text{if } s(i) = 0 \\ B^{k_i} & \text{if } s(i) = 1, \end{cases}$$

    - if $f(i) > n_2$ and thus $k_i = 0$ then $z_i = p \in \mathrm{Nat}_{\mathscr{C}om}([\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}], [\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}])$.
  - If $n_1 + 1 \leq i \leq n_1 + m_1$ and

   * if $f(i) \leq n_2$ then $z_i \in \mathrm{Nat}_{\mathscr{C}om}(\left[{0 \atop 1}\right], \left[{1 \atop 0}\right])$ given by

$$z_i = \begin{cases} sh^0 & \text{if } s(i) = 0 \\ B^0 & \text{if } s(i) = 1, \end{cases}$$

   * if $f(i) > n_2$ then $z_i = \mathrm{id} \in \mathrm{Nat}_{\mathscr{C}om}(\left[{0 \atop 1}\right], \left[{0 \atop 1}\right])$.
- The element $x_2 \in \mathrm{Nat}(\left[{c \atop m_1+n_1-c}\right], \left[{n_2 \atop m_2}\right])$ takes the shuffle product of all elements with same value j under f (and this is the output j). More precisely,

$$x_2 = m^{\{f^{-1}(1)\}} \otimes \ldots \otimes m^{\{f^{-1}(n_2)\}} \otimes \overline{m}^{\{f^{-1}(n_2+1)\}} \otimes \ldots \otimes \overline{m}^{\{f^{-1}(n_2+m_2)\}}.$$

Since both x_1 and x_2 got constructed out of cycles the element x is a cycle again. Hence, all complexes $A_{k_1,\cdots,k_{n_1}}$ have trivial differential.

Now we are able to state the main theorem of the paper:

Theorem 3.4. *The complex* $\mathrm{Nat}_{\mathscr{C}om}(\left[{n_1 \atop m_1}\right], \left[{n_2 \atop m_2}\right])$ *is quasi-isomorphic to the product*

$$\prod_{k_1,\cdots,k_{n_1}} A_{k_1,\ldots,k_{n_1}}$$

and hence a general element in $H_*(\mathrm{Nat}_{\mathscr{C}om}(\left[{n_1 \atop m_1}\right], \left[{n_2 \atop m_2}\right]))$ *is a unique infinite sum of scalar multiples of the elements described in Definition 3.3, which are tensor products of the basic operations* sh^k *and* B^k *composed with tensor products of shuffle and ordinary products.*

Remark 3.5. In [Kla13b] we define a complex of looped diagrams and a subcomplex of tree-like looped diagrams $\widetilde{ipl\mathcal{D}_{\mathscr{C}om}}(\left[{n_1 \atop m_1}\right], \left[{n_2 \atop m_2}\right])$ together with a dg-map $J_{\mathscr{C}om} : ipl\mathcal{D}_{\mathscr{C}om}(\left[{n_1 \atop m_1}\right], \left[{n_2 \atop m_2}\right]) \to \mathrm{Nat}_{\mathscr{C}om}(\left[{n_1 \atop m_1}\right], \left[{n_2 \atop m_2}\right])$. There is a subcomplex $\widetilde{ipl\mathcal{D}}_{\mathscr{C}om}(\left[{n_1 \atop m_1}\right], \left[{n_2 \atop m_2}\right]) \subset ipl\mathcal{D}_{\mathscr{C}om}(\left[{n_1 \atop m_1}\right], \left[{n_2 \atop m_2}\right])$ such that the image of this complex is given by $\prod_{k_1,\cdots,k_{n_1}} A_{k_1,\ldots,k_{n_1}}$. Denoting the restriction of $J_{\mathscr{C}om}$ to $\widetilde{ipl\mathcal{D}}_{\mathscr{C}om}(\left[{n_1 \atop m_1}\right], \left[{n_2 \atop m_2}\right])$ by $\widetilde{J}_{\mathscr{C}om}$, the above theorem can be restated as saying that $\widetilde{J}_{\mathscr{C}om}$ is a quasi-isomorphism (see [Kla13b, Section 3]).

Before we deal with the proof of the theorem, we want to give an example of an operation:

Example 3.6. We give an example of an element in $\mathrm{Nat}(\left[{2 \atop 2}\right], \left[{2 \atop 1}\right])$ belonging to the factor $A_{0,2}$ as defined in Definition 3.3. So we fixed $n_1 = 2$, $m_1 = 2$, $n_2 = 2$ and $m_2 = 1$. Moreover, we choose $k_1 = 0$ and $k_2 = 2$.

To give a generator in $A_{0,2}$, we first need a function $f : \{1, \ldots, 2+2\} \to \{1, \ldots, 2+1\}$ such that $f(2) \leq 2$. We choose

$$1 \mapsto 3 \qquad 2 \mapsto 2 \qquad 3 \mapsto 3 \qquad 4 \mapsto 2.$$

So $\{i \mid f(i) > n_2 = 2\} = \{1, 3\}$ and therefore we need a function $s : \{1, \ldots, 4\} \backslash \{1, 3\} \to \{0, 1\}$. We take $s(2) = 0$ and $s(4) = 1$.

We first describe the x_1 part in Definition 3.3. We have $x_1 = z_1 \otimes z_2 \otimes z_3 \otimes z_4$ with $z_1 = p$ (since $f(1) > 2$), $z_2 = sh^2$ (since $f(2) \leq 2$ and $s(2) = 0$), $z_3 = \mathrm{id}$ (since $f(3) > 2$) and $z_4 = B^0$ (since $f(4) \leq 2$ and $s(4) = 1$).

We know that p acts trivially on all degrees greater zero, so x_1 can only act non-trivial on degrees $(0, l)$ for some positive l. The degree $(0, 2)$ part of x_1 is illustrated in Figure 1.

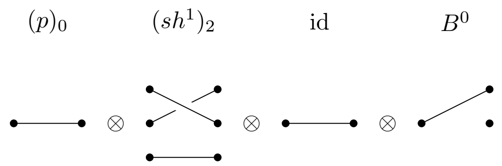

Figure 1. The operation $(x_1)_{0,2}$

Next we need to illustrate the composition with x_2. The element x_2 was defined to take the shuffle products of the outputs which agree on a $f(i) \leq n_2$ and the ordinary product for those outputs which agree on a $f(i) > n_2$. We have $f(3) = f(1) = 3$. This means, that the single outputs of segment 3 and 1 are multiplied and give the output of segment 3. Moreover $f(4) = f(2) = 2$. Here we have to apply the shuffle product. The fourth segment has 2 outputs, the second has 3, so the second output segment will have $2 + 3 - 1 = 4$ outputs. The first output of both segments is multiplied together and we take the shuffles of the rest. We first illustrate what happens on outputs in general (i.e. illustrate $m^{2,4}$) and then plug in our elements. In Figure 2 on the left are the old outputs of the two elements (i.e. 3 and 2 outputs) and on the right their merged outputs. Now we can take everything together, i.e. plug in our elements from before to compute $x_{f,s} = x_2 \circ x_1$. The degree $(0, 2)$ part of $x_{f,s}$ is illustrated in Figure 3.

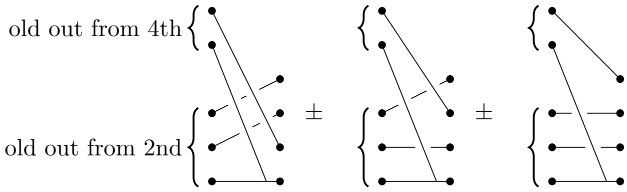

FIGURE 2. merging of outputs

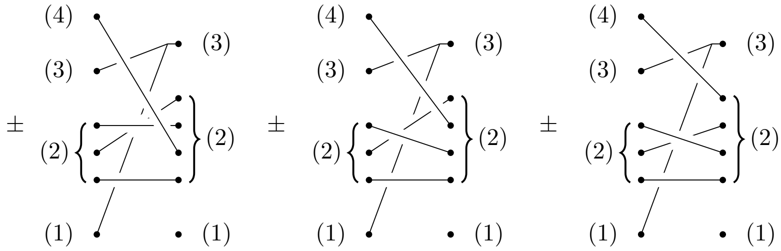

FIGURE 3. The operation $(x_{f,s})_{0,2}$

We also describe how the natural transformation associated to this acts on the Hochschild homology of a commutative algebra A. The element $x_{f,s} = x_2 \circ x_1$ corresponds to a map

$$C_*(A,A) \otimes C_*(A,A) \otimes A \otimes A \to C_*(A,A) \otimes C_*(A,A) \otimes A.$$

In the pictures we have given a description (up to sign) of what happens to an element with the first tensor factor being of length 1 and the second tensor factor of length 3, i.e. for

$$(q_0) \otimes (r_0 \otimes r_1 \otimes r_2) \otimes (g) \otimes (h) \in C_0(A,A) \otimes C_2(A,A) \otimes A \otimes A$$

the pictures in Figure 3 describe the operation by plugging in q_0 at (1), $(r_0 \otimes r_1 \otimes r_2)$ at (2), g at (3) and h at (4). If there is no input mapping to an output, a unit is inserted at that part of the output. Doing this in the same order as the pictures are given in Figure 3, we get

$$\begin{aligned}(q_0) \otimes (r_0 \otimes r_1 \otimes r_2) \otimes g \otimes h \mapsto\ & \pm(1) \otimes (r_0 \otimes h \otimes r_2 \otimes r_1) \otimes (q_0 \cdot g) \\ & \pm (1) \otimes (r_0 \otimes r_2 \otimes h \otimes r_1) \otimes (q_0 \cdot g) \pm (1) \otimes (r_0 \otimes r_2 \otimes r_1 \otimes h) \otimes (q_0 \cdot g) \\ & \in C_0(A,A) \otimes C_3(A,A) \otimes A.\end{aligned}$$

3.2. **Outline of the proof of Theorem 3.4.** The proof of Theorem 3.4 is structured as follows:

- In Section 3.3 we construct a subcomplex D of the formal operations such that $D \hookrightarrow \mathrm{Nat}([{}^{n_1}_{m_1}], [{}^{n_2}_{m_2}])$ is a weak equivalence.
- In Section 3.4 we split $D \cong \widetilde{D} \oplus \widetilde{D}'$ and prove that the homology of $\widetilde{D}'$ vanishes.
- In Section 3.5 we define another complex of operations $\widehat{D}$ and show that the Eilenberg-Zilber quasi-isomorphism defines a map $\widehat{D} \xrightarrow{EZ} \widetilde{D}$ which on each component on the level of elements corresponds to "multiplication with the elements x_2" as defined in Definition 3.3.
- Last, in Section 3.6 similarly to the proof of Theorem 2.8 we show that $\widehat{D'}$ is spanned by elements x_1 as defined in Definition 3.3.

Before we can start with our actual computations, we recall a few results about total complexes of multi-chain complexes and the order of totalization. The two propositions follow from the fact that the spectral sequences of the half plane double complexes are conditionally convergent together with work of Boardman [Boa99, Theorem 7.2].

Proposition 3.7 ([Kla13a, Cor. B.12])**.** *Let $f : C_{p,q} \to D_{p,q}$ be a map of left (respectively right) halfplane double complexes. If f is a quasi-isomorphism with respect to the vertical differential (i.e. an isomorphism after taking homology in the vertical direction), f induces a quasi-isomorphism $f : \prod_{p,q} C_{p,q} \to \prod_{p,q} D_{p,q}$ respectively $f : \bigoplus_{p,q} C_{p,q} \to \bigoplus_{p,q} D_{p,q}$.*

Proposition 3.8 ([Kla13a, Cor. B.14])**.** *Let $f : C_{p,q} \to D_{p,q}$ be a map of left (respectively right) halfplane double complexes. If f is a chain homotopy equivalence with respect to the horizontal differential (i.e. there* *exist g and h s.t. $d_{hor}\circ h+h\circ d_{hor}=g\circ f-id$ and h' such that $d_{hor}\circ h'+h'\circ d_{hor}=f\circ g-id$), f induces a quasi-isomorphism $f:\prod_{p,q}C_{p,q}\to\prod_{p,q}D_{p,q}$ respectively $f:\bigoplus_{p,q}C_{p,q}\to\bigoplus_{p,q}D_{p,q}$.*

We will use the two propositions frequently throughout the computations.

3.3. **A smaller subcomplex of the operations.** Recall from the definition of the iterated Hochschild construction in Section 1.2 that

$$\begin{aligned} N :=& \mathrm{Nat}_{\mathscr{C}om}([{}^{n_1}_{m_1}],[{}^{n_2}_{m_2}]) \\ \cong& D^{n_1}C^{n_2}(\mathscr{C}om(-,-))(m_2))(m_1) \\ \cong& \prod_{h_1,\dots,h_{n_1}}\bigoplus_{j_1,\dots,j_{n_2}} C^{h_1}\cdots C^{h_{n_1}}C_{j_1}\cdots C_{j_{n_2}}(\mathscr{C}om(-+m_1,-+m_2)),\end{aligned}$$

the chain complex of a multi cosimplicial-simplicial abelian group which for fixed h_i's and j_l's is given by $\mathscr{C}om(h_1+1+\cdots+h_{n_1}+1+m_1,j_1+1+\cdots+j_{n_2}+1+m_2)$. To treat m_1 and m_2 equal to the other inputs, we can view them as extra directions of the multicomplex which we only use in degree zero, i.e.

$$N\cong\prod_{h_1,\dots,h_{n_1}}\bigoplus_{j_1,\dots,j_{n_2}} C^{h_1}\cdots C^{h_{n_1}}C^0\cdots C^0C_{j_1}\cdots C_{j_{n_2}}C_0\cdots C_0(\mathscr{C}om(-,-)).$$

To rewrite the complex even further, we need some notation:

Let A be a d–multisimplicial abelian group with indexing set $\{1,\cdots,d\}$. For a set $M=\{m_1,\cdots,m_n\}\subseteq\{1,\cdots,d\}$ define $\mathrm{diag}^M A$ as the $d-(n-1)$–multisimplicial abelian group where we have taken the diagonal in $m_1,\cdots,m_n$, i.e. these indices now agree. Write $d_{\underline{h}}=h_1+1+\cdots+h_{n_1}+1+m_1$.

Since $\mathscr{C}om(h,l)\cong\mathscr{C}om(1,l)^{\otimes h}$, we can rewrite

$$\begin{aligned} &C^{h_1}\cdots C^{h_{n_1}}C^0\cdots C^0C_{j_1}\cdots C_{j_{n_2}}C_0\cdots C_0(\mathscr{C}om(-,-)) \\ \cong& C^{h_1}\cdots C^{h_{n_1}}C^0\cdots C^0C_{j_1}\cdots C_{j_{n_2}}C_0\cdots C_0(\mathrm{diag}^{all}_\bullet(\underbrace{\mathscr{C}om(1,-)\otimes\cdots\otimes\mathscr{C}om(1,-)}_{d_{\underline{h}}}))\end{aligned}$$

where *all* indicates the set of all indices. The cosimplicial boundary maps are given by doubling a factor $\mathscr{C}om(1,-)$. Let $\pi\in\mathscr{C}om(j_1+1+\cdots+j_{n_2}+1+m_2,n_2+m_2)$ be the projection to the intervals, i.e. $\pi(i)=k$ for $\sum_{i<k}(j_i+1)\le i<\sum_{i\le k}(j_i+1)$ (setting $j_l=0$ for $l>n_2$). Then given a generator $g\in\mathscr{C}om(d_{\underline{h}},j_1+1+\cdots+j_{n_2}+1+m_2)$ (i.e. g is a map of finite sets) we define $f=\pi\circ g:\{1,\cdots,d_{\underline{h}}\}\to\{1,\cdots,n_2+m_2\}$. The map f is invariant under applying simplicial boundary maps to g and hence for each fixed tuple $\{h_1,\dots,h_{n_1}\}$ we can split

$$\bigoplus_{j_1,\dots,j_{n_2}} C_{j_1}\cdots C_{j_{n_2}}C_0\cdots C_0(\mathrm{diag}^{all}_\bullet(\underbrace{\mathscr{C}om(1,-)\otimes\cdots\otimes\mathscr{C}om(1,-)}_{d_{\underline{h}}}))$$

into subcomplexes indexed by maps $f:\{1,\cdots,d_{\underline{h}}\}\to\{1,\cdots,n_2+m_2\}$ such that the map in the i–th tensor factor maps 1 to the $f(i)$–th interval. We write $\mathscr{C}om_{f(j)}(1,-)$ to indicate in which interval 1 is mapped.

Fix such a map f. Now we focus on a single j_i for a moment. For simplicity of notation we choose $i=1$. The complex

$$C_*(\mathrm{diag}^{all}_\bullet(\underbrace{\mathscr{C}om_{f(1)}(1,-+j_2+1+\cdots+m_2)\otimes\cdots\otimes\mathscr{C}om_{f(d_{\underline{h}})}(1,-+j_2+1+\cdots+m_2)}_{d_{\underline{h}}}))$$

by the Eilenberg-Zilber Theorem (cf. [Wei95, Sec. 8.5]) is quasi-isomorphic to the totalization of the complex

$$\underbrace{C_*(\mathscr{C}om_{f(1)}(1,-+j_2+1+\cdots+m_2))\otimes\cdots\otimes C_*(\mathscr{C}om_{f(d_{\underline{h}})}(1,-+j_2+1+\cdots+m_2))}_{d_{\underline{h}}}.$$

Hence we can look at each $C_*(\mathscr{C}om_{f(j)}(1,-+j_2+1+\cdots+m_2))$ separately. Assume that this is the j–th factor, i.e. if $f(j)>1$ we do not hit the interval belonging $*$. Then the differential on $C_*(\mathscr{C}om_{f(j)}(1,-+j_2+1+\cdots+m_2))$ is alternate zero and the identity and we have $C_*(\mathscr{C}om_{f(j)}(1,-+j_2+1+\cdots+m_2))\simeq C_0(\mathscr{C}om_{f(j)}(1,-+j_2+1+\cdots+m_2))\cong\mathscr{C}om_{f(j)}(1,j_2+1+\cdots+m_2)$ where the last isomorphism follows from the fact, that it does not matter, whether we include a point, we never map to, or not. Iterating this argument for all $1\le i\le j_{n_2}$ (and using Proposition 3.7), we can contract

$$\bigoplus_{j_1,\dots,j_{n_2}} C_{j_1}\cdots C_{j_{n_2}}C_0\cdots C_0(\mathrm{diag}^{all}_\bullet(\underbrace{\mathscr{C}om_{f(1)}(1,-)\otimes\cdots\otimes\mathscr{C}om_{f(d_{\underline{h}})}(1,-)}_{d_{\underline{h}}}))$$

to

$$\bigoplus_{j_1,\dots,j_{n_2}} C_{j_{f(1)}}(\mathscr{C}om(1,-+1))\otimes\cdots\otimes C_{f(d_{\underline{h}})}(\mathscr{C}om(1,-+1))$$

with $j_l := 0$ if $n_2 < l \leq n_2 + m_2$. The differential in the i–th direction comes from the simplicial boundaries acting diagonally on all factors j with $f(j) = i$. Concluding, we can rewrite this complex as

$$\bigoplus_{j_1,\cdots,j_{n_2}} \bigoplus_{\substack{f:\{1,\cdots,d_{\underline{h}}\}\\ \to\{1,\cdots,n_2+m_2\}}} C_{j_1}\cdots C_{j_{n_2}} C_0 \cdots C_0(\\ \operatorname{diag}^{\{i|f(i)=1\}} \cdots \operatorname{diag}^{\{i|f(i)=n_2+m_2\}} (\underbrace{\mathscr{C}om(1,-)\otimes\cdots\otimes\mathscr{C}om(1,-)}_{d_{\underline{h}}})$$

and (after using Proposition 3.7 and Proposition 3.8) get a quasi-isomorphism

$$N \simeq \prod_{h_1,\cdots,h_{n_1}} \bigoplus_{\substack{f:\{1,\cdots,d_{\underline{h}}\}\\ \to\{1,\cdots,n_2+m_2\}}} \bigoplus_{j_1,\cdots,j_{n_2}} C^{h_1}\cdots C^{h_{n_1}} C^0 \cdots C^0 C_{j_1}\cdots C_{j_{n_2}} C_0\cdots C_0(\\ \operatorname{diag}^{\{i|f(i)=n_2+m_2\}} \cdots \operatorname{diag}^{\{i|f(i)=1\}} (\underbrace{\mathscr{C}om(1,-)\otimes\cdots\otimes\mathscr{C}om(1,-)}_{d_{\underline{h}}}).$$

We denote the last complex by D.

3.4. **Splitting off an acyclic subcomplex.** As a next step we split off an acyclic subcomplex from D.

Recall $d_{\underline{h}} = h_1 + 1 + \cdots + h_{n_1} + 1 + m_1$ and define the subset $F \subset \left\{f : \{1,\cdots,d_{\underline{h}}\} \to \{1,\cdots,n_2+m_2\}\right\}$ by

$$F = \left\{ f : \{1,\cdots,d_{\underline{h}}\} \to \{1,\cdots,n_2+m_2\} |\text{ s.t. } f|_{\{\sum_{j<i}(h_j+1)+1,\cdots,\sum_{j\leq i}(h_j+1)\}} \text{ is constant for all } i \leq n_2 \right\}$$

i.e. all the values belonging to one h_i are equal.

The complement of this set is given by

$$F' = \left\{ f : \{1,\cdots,d_{\underline{h}}\} \to \{1,\cdots,n_2+m_2\} |\text{ s.t. } \exists\, i \text{ s.t. } f|_{\{\sum_{j<i}(h_j+1)+1,\cdots,\sum_{j\leq i}(h_j+1)\}} \text{ is not constant} \right\}.$$

Define

$$\widetilde{D} := \prod_{h_1,\cdots,h_{n_1}} \bigoplus_{F} \bigoplus_{j_1,\cdots,j_{n_2}} C^{h_1}\cdots C^{h_{n_1}} C^0 \cdots C^0 C_{j_1}\cdots C_{j_{n_2}} C_0\cdots C_0(\\ \operatorname{diag}^{\{i|f(i)=n_2+m_2\}} \cdots \operatorname{diag}^{\{i|f(i)=1\}} (\underbrace{\mathscr{C}om(-,-)\otimes\cdots\otimes\mathscr{C}om(-,-)}_{d_{\underline{h}}}))$$

and

$$\widetilde{D}' := \prod_{h_1,\cdots,h_{n_1}} \bigoplus_{F'} \bigoplus_{j_1,\cdots,j_{n_2}} C^{h_1}\cdots C^{h_{n_1}} C^0 \cdots C^0 C_{j_1}\cdots C_{j_{n_2}} C_0\cdots C_0(\\ \operatorname{diag}^{\{i|f(i)=n_2+m_2\}} \cdots \operatorname{diag}^{\{i|f(i)=1\}} (\underbrace{\mathscr{C}om(-,-)\otimes\cdots\otimes\mathscr{C}om(-,-)}_{d_{\underline{h}}})).$$

Then we can show:

Lemma 3.9. *Both $\widetilde{D}$ and $\widetilde{D}'$ are subcomplexes of D and we have a splitting*

$$D \cong \widetilde{D} \oplus \widetilde{D}'.$$

Proof. We need to show that the coboundary maps preserve the decomposition. Recall that the j–th coboundary map belonging to a h_i doubles the information of the $h_1 + 1 + \cdots + h_{i-1} + 1 + j$–th factor and hence adds in the same value for f. So if f was constant on $\{\sum_{j<i} h_j + i, \cdots, \sum_{j\leq i} h_j + i\}$ before, it will stay constant on $\{\sum_{j<i} h_j + i, \cdots, \sum_{j\leq i} h_j + i + 1\}$ (and on all other intervals, since they did not get touched). Similarly, if f was not constant on one of the intervals, it cannot become constant that way. This proves that both complexes are actual subcomplexes and hence the lemma is proven. ☐

Lemma 3.10. *The complex $\widetilde{D}'$ has trivial homology.*

Proof. Recall that

$$\widetilde{D}' = \prod_{h_1,\cdots,h_{n_1}} \bigoplus_{F'} \bigoplus_{j_1,\cdots,j_{n_2}} C^{h_1}\cdots C^{h_{n_1}} C^0 \cdots C^0 C_{j_1}\cdots C_{j_{n_2}} C_0\cdots C_0(\\ \operatorname{diag}^{\{i|f(i)=n_2+m_2\}} \cdots \operatorname{diag}^{\{i|f(i)=1\}} (\underbrace{\mathscr{C}om(-,-)\otimes\cdots\otimes\mathscr{C}om(-,-)}_{d_{\underline{h}}})).$$

We give a decomposition of F' into disjoint sets which are preserved by the boundary and coboundary maps. This gives a decomposition of $\widetilde{D}'$ into a direct sum of chain complexes.

Set $F'_1 = \left\{f \in F', f|_{\{1,\cdots,h_1+1\}} \text{ not constant}\right\}$ and in general

$$F'_t = \left\{f \in F', f|_{\{\sum_{j<i}(h_j+1)+1,\cdots,\sum_{j\leq i}(h_j+1)\}} \text{ const. for all } i<t, f|_{\{\sum_{j<t}(h_j+1)+1,\cdots,\sum_{j\leq t}(h_j+1)\}} \text{ not const.}\right\},$$

i.e. F'_t consists of those functions which are constant on the first $t-1$ intervals and non-constant on the t–th one.

The coboundary maps send an element in F'_t to an element in the same F'_t since they preserve the set of values of f on an interval. So $F' = \coprod_t F'_t$ and

$$\widetilde{D}' = \bigoplus_t \widetilde{D}'_t$$

with

$$\begin{aligned}\widetilde{D}'_t = \prod_{h_1,\cdots,h_{n_1}} \bigoplus_{f\in F'_t} \bigoplus_{j_1,\cdots,j_{n_2}} & C^{h_1}\cdots C^{h_{n_1}} C^0\cdots C^0 C_{j_1}\cdots C_{j_{n_2}} C_0 \cdots C_0(\\ & \operatorname{diag}^{\{i|f(i)=n_2+m_2\}}\cdots \operatorname{diag}^{\{i|f(i)=1\}}(\underbrace{\mathscr{C}om(-,-)\otimes\cdots\otimes\mathscr{C}om(-,-)}_{d_{\underline{h}}})).\end{aligned}$$

Define the multi-cosimplicial chain complex

$$\begin{aligned}& A^{\bullet,\cdots,\bullet} \\ =& \bigoplus_{f\in F'_t} \bigoplus_{j_1,\cdots,j_{n_2}} C_{j_1}\cdots C_{j_{n_2}} C_0\cdots C_0(\operatorname{diag}^{\{i|f(i)=n_2+m_2\}}\cdots \operatorname{diag}^{\{i|f(i)=1\}}(\mathscr{C}om(-,-)\otimes\cdots\otimes\mathscr{C}om(-,-)),\end{aligned}$$

thus

$$\widetilde{D}'_t = \prod_{h_1,\cdots,h_{n_1}} C^{h_1}\cdots C^{h_{n_1}} C^0\cdots C^0 A^{\bullet,\cdots,\bullet}.$$

Since changing the order in the product total complex is an isomorphism, we can totalize first in the t–th direction and get

$$\widetilde{D}'_t \cong \prod_{h_1,\cdots,h_{n_1}} C^{h_1}\cdots C^{h_{n_1}} C^0\cdots C^0 C^{h_t} A^{\bullet,\cdots,\bullet}.$$

We view this as a double chain complex with the first differential the totalization of all h_i besides h_t and the second the totalization of the t–th direction and the chain differential of $A^{\bullet,\cdots,\bullet}$. We want to give a retraction of the total complex of this double complex. Giving a retraction of the double complex is a chain homotopy equivalence between the double complex and zero. By Proposition 3.7 this yields a quasi-isomorphism $\widetilde{D}'_i$ to 0.

Let $A_t^{\bullet,\cdots,\bullet} = C^*(A^{\bullet,\cdots,\bullet})$ be the multi-simplicial cochain chain complex where we applied the Moore functor in the t–th direction. A contraction of this cochain complex for each multi-simplicial degree compatible with all the other coboundary maps gives a prove that the associated chain complex $\widetilde{D}'_t$ is acyclic.

We need to define a map $s : \widetilde{D}'_t \to \widetilde{D}'_t$ such that $d_t \circ s + s \circ d_t = \mathrm{id}$. We will actually give a map $s : \widetilde{D}'_t \to D$ fulfilling this property. Since we have split D into direct summands, the projection of this map to $\widetilde{D}'_t$ gives the contraction we asked for.

Fix $f \in F'_t$ and let $x \in A_t^{\bullet,\cdots,\bullet}$ be in the summand belonging to f. Denote by s_t^i the codegeneracies of the t–th cosimplicial set.

Let $u(x)$ be minimal such that $f(\sum_{j<t} h_j + i + u(x)) \neq f(\sum_{j<t} h_j + i + u(x) + 1)$ ($u(x)$ exists since f was not constant on that interval). Define

$$s(x)_{h_1,\cdots,h_n} = (-1)^{h_t+u(x)+1} s_t^{u(x)}(x)_{h_1,\cdots,h_n} = (-1)^{h_t+u(x)+1} s_t^{u(x)}(x_{h_1,\cdots,h_t+1,\cdots,h_n}).$$

This map is a retraction, which can be checked by using the simplicial identities several times (and is omitted here). $\square$

3.5. **Applying Eilenberg-Zilber to unify outputs.** Since we requested the functions in F defining $\widetilde{D}$ to be constant on the intervals belonging to the h_i, a lot of data is redundant. Hence, we rewrite

$$\widetilde{D} \cong \bigoplus_{\substack{f:\{1,\cdots,n_1+m_1\}\\ \to\{1,\cdots,n_2+m_2\}}} \prod_{h_1,\ldots,h_{n_1}} \bigoplus_{j_1,\cdots,j_{n_2}} C^{h_1}\cdots C^{h_{n_1}} C^0 \cdots C^0 C_{j_1}\cdots C_{j_{n_2}} C_0 \cdots C_0(\\ \operatorname{diag}^{\{i|f(i)=1\}}\cdots \operatorname{diag}^{\{i|f(i)=n_2+m_2\}}(\underbrace{\mathscr{C}om(-,-)\otimes\cdots\otimes\mathscr{C}om(-,-)}_{n_1+m_1})).$$

So far we have shown $\widetilde{D} \simeq N$ where the map is the embedding. Next, we show that $\widetilde{D}$ is quasi-isomorphic to yet another complex $\widehat{D}$, which we then are able to describe explicitly. Furthermore, the quasi-isomorphism is given by the Eilenberg-Zilber map and corresponds to the composition with x_2 in the elements of Definition 3.3.

Define

$$\widehat{D} := \bigoplus_{\substack{f:\{1,\cdots,n_1+m_1\}\\ \to\{1,\cdots,n_2+m_2\}}} \prod_{h_1,\ldots,h_{n_1}} \bigoplus_{\substack{l_1,\ldots,l_{n_1+m_1}\\ l_i=0 \text{ if } f(i)>n_2}} C^{h_1}\overline{C}_{l_1}(\mathscr{C}om(-,-))\otimes\cdots\otimes C^{h_{n_1}}\overline{C}_{l_{n_1}}(\mathscr{C}om(-,-))) \\ \otimes \overline{C}_{l_{n_1+1}}(\mathscr{C}om(1,-))\otimes\cdots\otimes\overline{C}_{l_{n_1+m_1}}(\mathscr{C}om(1,-))$$

which is a subcomplex of

$$\bigoplus_f \operatorname{Nat}\left(\left[\begin{smallmatrix} n_1 \\ m_1\end{smallmatrix}\right], \left[\begin{smallmatrix} |f^{-1}(1,\cdots,n_2)| \\ |f^{-1}(n_2+1,\cdots n_2+m_2)|\end{smallmatrix}\right]\right).$$

Instead of summing over all l_i with the condition that $l_i = 0$ if $f(i) > 0$, we can introduce a new summation by first summing over natural numbers j_i for $1 \le i \le n_2$ (and setting $j_{n_2+1}\ldots, j_{n_2+m_2}$ equal to zero) and then summing over all l_i such that $\sum_{f(i)=t} l_i = j_t$, so

$$\widehat{D} \cong \bigoplus_{\substack{f:\{1,\cdots,n_1+m_1\}\\ \to\{1,\cdots,n_2+m_2\}}} \prod_{h_1,\ldots,h_{n_1}} \bigoplus_{j_1,\cdots,j_{n_2}} \bigoplus_{\substack{l_1,\ldots,l_{n_1+m_1}\\ \sum_{f(i)=t} l_i=j_t}} \\ C^{k_1}\cdots C^{h_{n_1}}\overline{C}_{l_1}\cdots\overline{C}_{l_{n_1+m_1}}(\mathscr{C}om(-,-)\otimes\cdots\otimes\mathscr{C}om(-,-)\otimes\mathscr{C}om(1,-)\otimes\cdots\otimes\mathscr{C}om(1,-))$$

where C^{h_i} and $\overline{C}_{l_i}$ correspond to the i–th factor in the tensor product.

We now reorder the way we totalize the l_i's: We can first totalize in all the directions of each subset of l_i's with $f(i) = j$ for all j and then totalize all those together. For an ordered set $M = \{m_1, \cdots, m_n\}$ we write $\overline{C}_M = \overline{C}_{m_n}\cdots\overline{C}_{m_1}$. Given a multisimplicial set A and a fixed t applying Eilenberg-Zilber in all directions with $f(i) = t$

$$\bigoplus_{\substack{l_i\\ f(i)=t\\ \sum_{f(i)=t} l_i=j_t}} C_{\{l_i\}}A \simeq C_{j_t}\operatorname{diag}^{\{i|f(i)=t\}} A$$

and hence after applying Proposition 3.7 and Proposition 3.8 we get a quasi-isomorphism

$$EZ: \widehat{D} \to \widetilde{D} \cong \bigoplus_{\substack{f:\{1,\cdots,n_1+m_1\}\\ \to\{1,\cdots,n_2+m_2\}}} \prod_{h_1,\ldots,h_{n_1}} \bigoplus_{j_1,\cdots,j_{n_2}} C^{h_1}\cdots C^{h_{n_1}} C^0 \cdots C^0 C_{j_1}\cdots C_{j_{n_2}} C_0 \cdots C_0(\\ \operatorname{diag}^{\{i|f(i)=1\}}\cdots \operatorname{diag}^{\{i|f(i)=n_2+m_2\}}(\underbrace{\mathscr{C}om(-,-)\otimes\cdots\otimes\mathscr{C}om(-,-)}_{n_1+m_1}))$$

which on elements applies the Eilenberg-Zilber morphism to the outputs which have equal value under f. By the definition of the Eilenberg-Zilber map (cf. [Wei95, Sec. 8.5]) this corresponds to taking the shuffle product and hence is multiplication with the element x_2 given in Definition 3.3 (which was independent of the choice of the h_i's).

3.6. **Describing a subcomplex in terms of operations.** As a last step we need to show that $\widehat{D}$ is generated by infinite sums of linear combinations of elements of the form x_1 as described in Definition 3.3. We split $\widehat{D}$ into summands $\widehat{D}_f$ for $f : \{1, \cdots, n_1 + m_1\} \to \{1, \cdots, n_2 + m_2\}$. Then $\widehat{D}_f$ is given by

$$\widehat{D}_f := \prod_{h_1,\dots,h_{n_1}} \bigoplus_{\substack{l_1,\dots,l_{n_1+m_1}\\ l_i=0 \text{ if } f(i)>n_2}} C^{h_1}\overline{C}_{l_1}(\mathscr{C}om(-,-)) \otimes \cdots \otimes C^{h_{n_1}}\overline{C}_{l_{n_1}}(\mathscr{C}om(-,-)) \otimes \overline{C}_{l_{n_1+1}}(\mathscr{C}om(1,-)) \otimes \cdots \otimes \overline{C}_{l_{n_1+m_1}}(\mathscr{C}om(1,-))$$

Furthermore, $C^*C_0(\mathscr{C}om(-,-)) = C^*(\mathscr{C}om(-,1))$ is the cochain complex $\mathscr{C}om(h,1) \cong *$ in each degree $h-1$ and has differentials 0 and the identity, alternately. The inclusion of the cochain complex with only one nonzero entry $\mathbb{Z} = C^0C_0(\mathscr{C}om(-,-))$ in degree 0 is a homotopy equivalence and hence induces a quasi-isomorphism on total complexes. So after reordering, we get a quasi-isomorphism

$$\begin{aligned}\widehat{D}_f \simeq \prod_{\substack{h_1,\dots,h_{n_1}\\ f(i)\le n_2}} \bigoplus_{\substack{l_1,\dots,l_{n_1+m_1}\\ f(i)\le n_2}} & C^{h_1}\overline{C}_{l_1}(\mathscr{C}om(-,-)) \otimes \cdots \otimes C^{h_{n_1}}\overline{C}_{l_{n_1}}(\mathscr{C}om(-,-)) \\ & \otimes \overline{C}_{l_{n_1+1}}(\mathscr{C}om(1,-)) \otimes \cdots \otimes \overline{C}_{l_{n_1+m_1}}(\mathscr{C}om(1,-)) \\ \otimes C^0C_0(\mathscr{C}om(-,-)) \otimes \cdots \otimes C^0C_0(\mathscr{C}om(-,-)) & \otimes C_0(\mathscr{C}om(1,-)) \otimes \cdots \otimes C_0(\mathscr{C}om(1,-))\end{aligned}$$

The terms $C^0C_0(\mathscr{C}om(-,-))$ correspond to those i with $i \le n_1$ and $f(i) > n_2$, whereas the terms of the form $C_0(\mathscr{C}om(1,-))$ give those i with $i > n_1$ and $f(i) > n_2$. The first ones are spanned by the element $p \in \mathrm{Nat}([{}^1_0], [{}^0_1])$ and the second ones by $\mathrm{id} \in \mathrm{Nat}([{}^0_1], [{}^0_1])$.

Now we denote $c = |f^{-1}(\{1,\cdots,n_2\})|$ and $c' = |f^{-1}(\{1,\cdots,n_2\}) \cap \{1,\cdots,n_1\}|$ and after relabeling the h_i and l_i are left to compute

$$\prod_{h_1,\dots,h_{c'}} \bigoplus_{l_1,\cdots,l_c} C^{h_1}\overline{C}_{l_1}(\mathscr{C}om(-,-)) \otimes \cdots \otimes C^{h_{c'}}\overline{C}_{l_{c'}}(\mathscr{C}om(-,-)) \otimes \overline{C}_{l_{c'+1}}(\mathscr{C}om(1,-)) \otimes \cdots \otimes \overline{C}_{l_c}(\mathscr{C}om(1,-))$$

which is congruent to

$$\prod_{h_1,\dots,h_{c'}} \bigoplus_{l_1,\cdots,l_{c'}} \overline{C}^{h_1}\overline{C}_{l_1}(\mathscr{C}om(-,-)) \otimes \cdots \otimes C^{h_{c'}}\overline{C}_{l_{c'}}(\mathscr{C}om(-,-)) \otimes \bigoplus_{l_{c'+1},\dots,l_c} \overline{C}_{l_{c'+1}}(\mathscr{C}om(1,-)) \otimes \cdots \otimes \overline{C}_{l_c}(\mathscr{C}om(1,-)).$$

We know that

$$\overline{C}_l(\mathscr{C}om(1,-)) \cong \overline{C}_l(S^1) = \begin{cases} 1 & \text{if } l = 0 \\ y & \text{if } l = 1 \\ 0 & \text{else.} \end{cases}$$

In terms of operations, 1 corresponds to sh^0 and y to B^0 and we thus conclude that the complex

$$\bigoplus_{l_{c'+1},\dots,l_c} \overline{C}_{l_{c'+1}}(\mathscr{C}om(1,-)) \otimes \cdots \otimes \overline{C}_{l_c}(\mathscr{C}om(1,-))$$

is homotopy equivalent to the complex spanned by tensor products of sh^0 and B^0.

Now we can deal with the last part of the complex: Using the elements sh^k and B^k which we constructed combinatorially earlier on, we can describe a general element in the homology of the above product. This implies that the subcomplex generated by these cycles is quasi-isomorphic to the complex we were computing so far.

For $j = 0,1$ let $\tilde{c}^{j,h_i} \in C^{h_i}\overline{C}_{h_i+j}(\mathscr{C}om(-,-))$ be defined via $\tilde{c}^{0,h_i} = sh^{h_i}$ and $\tilde{c}^{1,h_i} = B^{h_i}$. Completely analogous to the proof of Theorem 2.8, we can show:

Proposition 3.11. *The complex*

$$\prod_{h_i} \bigoplus_{l_i} \overline{C}^{h_1}\overline{C}_{l_1}(\mathscr{C}om(-,-)) \otimes \cdots \otimes \overline{C}^{h_{c'}}\overline{C}_{l_{c'}}(\mathscr{C}om(-,-))$$

is quasi-isomorphic to the subcomplex which in degree n is given by elements of the form

$$x = \sum_{\substack{s:\{1,\cdots c'\}\to\{0,1\}\\ \sum s(i)=n}} \sum_{h_1=0}^{\infty} \cdots \sum_{h_{c'}=0}^{\infty} r^s_{h_1,\cdots,h_{c'}} \tilde{c}^{s(1),h_1} \otimes \cdots \otimes \tilde{c}^{s(c'),h_{c'}}$$

with $r^s_{h_1,\cdots,h_{c'}} \in \mathbb{Z}$.

Again, in each degree this is a finite sum, since all the $(\tilde{c}_{h_i})_j = 0$ *for* $j < h_i$. *More explicitly,*

$$(x)_{j_1,\cdots,j_{c'}} = \sum_{\substack{s:\{1,\cdots c'\}\to\{0,1\}\\ \sum s(i)=n}} \sum_{h_1=0}^{j_{1'}} \cdots \sum_{h_{c'}=0}^{j_{c'}} r_{h_1,\cdots,h_{c'}} (\tilde{c}^{s(1),h_1})_{j_1} \otimes \cdots \otimes (\tilde{c}^{s(c'),h_{c'}})_{j_{c'}}.$$

Proof. The proof is an iteration of the argument in Section 2.

Denote by X the subcomplex described in the theorem, i.e.

$$X_n = \left\{ \sum_{\substack{s:\{1,\cdots c'\}\to\{0,1\}\\ \sum s(i)=n}} \sum_{h_1=0}^{\infty} \cdots \sum_{h_{c'}=0}^{\infty} r^s_{h_1,\cdots,h_{c'}} \tilde{c}^{s(1),h_1} \otimes \cdots \otimes \tilde{c}^{s(c'),h_{c'}} \,\middle|\, r^s_{h_1,\cdots,h_{c'}} \in \mathbb{Z} \right\}$$

with trivial differential. We are going to multifilter the total complex and the complex X.

(1) For fixed numbers $h_1, \ldots, h_{c'} \in \mathbb{N}$ we define

$$X_n^{h_1,\ldots,h_{c'}} = \left\{ \sum_{\substack{s:\{1,\cdots c'\}\to\{0,1\}\\ \sum s(i)=n}} r^s \tilde{c}^{s(1),h_1} \otimes \cdots \otimes \tilde{c}^{s(c'),h_{c'}} \,\middle|\, r^s \in \mathbb{Z} \right\}$$

The map $G_{c'}^{h_1,\ldots,h_{c'}} : X^{h_1,\ldots,h_{c'}} \to \left(\overline{C}^{h_1}\overline{C}_*(\mathscr{C}om(-,-)) \otimes \cdots \otimes \overline{C}^{h_{c'}}\overline{C}_*(\mathscr{C}om(-,-))\right)[-\sum h_i]$, i.e. to the shifted tensor product of the chain complexes $\overline{C}^{h_1}\overline{C}_*(\mathscr{C}om(-,-))$ defined by sending $\tilde{c}^{s(1),h_1} \otimes \cdots \otimes \tilde{c}^{s(c'),h_{c'}}$ to $(\tilde{c}^{s(1),h_1})_{h_1} \otimes \cdots \otimes (\tilde{c}^{s(c'),h_{c'}})_{h_{c'}}$ is a quasi-isomorphism. Since all the involved chains are free, the tensor product commutes with homology. Then this map is the c'–fold tensor product of the map in Corollary 2.7 and hence an isomorphism on homology.

(2) Fix $0 \le i \le c'$ and $h_1, \cdots, h_i \in \mathbb{N}$. Now define

$$X_n^{h_1,\ldots,h_i} = \left\{ \sum_{\substack{s:\{1,\cdots c'\}\to\{0,1\}\\ \sum s(k)=n}} \sum_{h_{i+1}=0}^{\infty} \cdots \sum_{h_{c'}=0}^{\infty} r^s_{h_{i+1},\cdots,h_{c'}} \tilde{c}^{s(1),h_1} \otimes \cdots \otimes \tilde{c}^{s(c'),h_{c'}} \,\middle|\, r^s_{h_{i+1},\cdots,h_{c'}} \in \mathbb{Z} \right\}$$

i.e. the complex where the values of the first i indices h_j is fixed. We next prove that the map

$$G_i^{h_1,\cdots,h_i} : X^{h_1,\ldots,h_i} \to \prod_{h_{i+1},\ldots,h_{c'}} \bigoplus_{l_1,\ldots,l_{c'}} \overline{C}^{h_1}\overline{C}_{l_1}(\mathscr{C}om(-,-)) \otimes \cdots \otimes \overline{C}^{h_{c'}}\overline{C}_{l_{c'}}(\mathscr{C}om(-,-))$$

defined on generators by

$$\tilde{c}^{s(1),h_1} \otimes \cdots \otimes \tilde{c}^{s(c'),h_{c'}} \mapsto (\tilde{c}^{s(1),h_1})_{h_1} \otimes \cdots \otimes (\tilde{c}^{s(i),h_i})_{h_i} \otimes \tilde{c}^{s(i+1),h_{i+1}} \otimes \cdots \otimes \tilde{c}^{s(c'),h_{c'}}$$

is a quasi-isomorphism. For $i = 0$ this gives the desired result.

The proof goes by decreasing induction on i. The case $i = c'$ this was shown in the previous step. Now choose $0 \le i < c'$ and assume the statement holds for $i + 1$. We view the complex as the total complex of the double complex where we first totalized all but the h_{i+1} direction and then totalize the last direction. Thus we can filter the complex by h_{i+1}, i.e. the p-th term in the filtration is given by

$$F_p = \prod_{h_{i+1}\ge p, h_{i+2},\ldots,h_{c'}} \bigoplus_{l_1,\ldots,l_{c'}} \overline{C}^{h_1}\overline{C}_{l_1}(\mathscr{C}om(-,-)) \otimes \cdots \otimes \overline{C}^{h_{c'}}\overline{C}_{l_{c'}}(\mathscr{C}om(-,-)).$$

By [Wei95, Sec. 5.6] this filtration is exhaustive and complete. Again, its E_1-page is given by taking homology in the vertical direction.

On the other hand we can define a similar filtration of $X^{h_1,\ldots,h_i}$:

$$F_p(X_n^{h_1,\ldots,h_i}) = \left\{ \sum_{\substack{s:\{1,\cdots c'\}\to\{0,1\}\\ \sum s(k)=n}} \sum_{h_{i+1}\ge p}^{\infty} \cdots \sum_{h_{c'}=0}^{\infty} r^s_{h_{i+1},\cdots,h_{c'}} \tilde{c}^{s(1),h_1} \otimes \cdots \otimes \tilde{c}^{s(c'),h_{c'}} \,\middle|\, r^s_{h_{i+1},\cdots,h_{c'}} \in \mathbb{Z} \right\}$$

Since $\tilde{c}_k^{s(i),h_i} = 0$ for $k < h_i$ and any $s(i)$, the map $G_i^{h_1,\cdots,h_i}$ respects the filtration.

Furthermore, this filtration is exhaustive (since it starts with the whole complex) and complete (as we take infinite sums, i.e. the product over h_i). It collapses on the E^0–page which then as the quotient $F_p(X_n^{h_1,\ldots,h_i})/F_{p+1}(X_n^{h_1,\ldots,h_i})$ consists of all those terms where $h_{i+1} = p$. Hence $E^0_{p,q} = E^1_{p,q} \cong X_q^{h_1,\ldots,p}$. The map $G_i^{h_1,\cdots,h_i}$ on the p-th quotient now is precisely $G_{i+1}^{h_1,\cdots,h_i,p}$ and thus by induction it induces an isomorphism on the E^1–pages of two complete, exhaustive spectral sequences. Hence by the Eilenberg-Moore comparison theorem (see for example [Wei95, Theorem 5.1]) it is an isomorphism on homology which proves the claim.

The proposition now follows from the case $i = 0$, since $G_0^{\emptyset}$ is the inclusion of X into the total complex.

□

Now we are finally able to put everything together and prove the main theorem:

Proof of Theorem 3.4. We have seen that the composition

$$\widehat{D} \xrightarrow{EZ} \widetilde{D} \hookrightarrow \mathrm{Nat}([{n_1 \atop m_1}], [{n_2 \atop m_2}])$$

is a quasi-isomorphism.

The first map actually splits into quasi-isomorphisms

$$\widehat{D}_f \xrightarrow{EZ} \widetilde{D}_f$$

given by multiplication with the element x_2 described in Definition 3.3. Moreover, taking the results of the last section together, we have seen that $\widehat{D}_f \subset \mathrm{Nat}([{n_1 \atop m_1}], [{c \atop n_1+m_1-c}])$ is spanned by infinite linear combinations of elements as described above. The only difference of these elements to the elements described in Definition 3.3 is, that we first chose f and then the k_i's. However, this commutes (it is equivalent to pulling out the direct sum over the functions f out of the product over the k_i's). Hence the result follows. □

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Paper B

NATURAL OPERATIONS ON THE HOCHSCHILD COMPLEX OF COMMUTATIVE FROBENIUS ALGEBRAS VIA THE COMPLEX OF LOOPED DIAGRAMS

ANGELA KLAMT

Abstract. We define a dg-category of looped diagrams which we use to construct operations on the Hochschild complex of commutative Frobenius dg-algebras. We show that we recover the operations known for symmetric Frobenius dg-algebras constructed using Sullivan chord diagrams as well as all formal operations for commutative algebras (including Loday's lambda operations) and prove that there is a chain level version of a suspended Cacti operad inside the complex of looped diagrams. This recovers the suspended BV algebra structure on the Hochschild homology of commutative Frobenius algebras defined by Abbaspour and proves that it comes from an action on the Hochschild chains.

Introduction

To a dg-algebra A one associates the Hochschild chain complex $C_*(A, A)$. Operations of the form

$$C_*(A,A)^{\otimes n_1} \otimes A^{\otimes m_1} \to C_*(A,A)^{\otimes n_2} \otimes A^{\otimes m_2}$$

have been investigated by many authors. We are interested in those operations which exist for all algebras of a certain class, more concretely all algebras over a given operad or PROP. In [Wah12], the complex of so-called formal operations is introduced, a more computable complex approximating the complex of all natural operations for a given class of (A_∞–)algebras. In this paper we build a combinatorial complex mapping to the complex of formal operations for the case of commutative Frobenius dg-algebras, defining in particular a large family of operations for commutative Frobenius dg-algebras.

Before describing the complex of operations, we recall some of the operations known so far, which we want to be covered by our new complex.

One of the main motivations for investigating operations on Hochschild homology is given by string topology. String topology started in 1999 when Chas and Sullivan in [CS99] gave a construction of a product $H_*(LM) \otimes H_*(LM) \to H_{*-d}(LM)$ for M a closed oriented manifold of dimension d and LM the free loop space on M, that makes $H_*(LM)$ into a BV-algebra. Afterward, more operations were discovered and in [God07] the structure of an open-closed HCFT was exhibited on the pair $(H_*(M), H_*(LM))$, yielding a whole family of operations

$$H_*(LM)^{\otimes n_2} \otimes H_*(M)^{\otimes m_2} \to H_*(LM)^{\otimes n_1} \otimes H_*(M)^{\otimes m_1}$$

parametrized by the moduli space of Riemann surfaces.

Taking coefficients in a field and M to be a 1-connected closed oriented manifold, Jones [Jon87] proved that there is an isomorphism

$$HH_*(C^{-*}(M), C^{-*}(M)) \cong H^{-*}(LM).$$

Moreover, if M is a formal manifold, i.e. if we have a weak equivalence $C^*(M) \simeq H^*(M)$ preserving the multiplication, we obtain an isomorphism $HH_*(C^{-*}(M), C^{-*}(M)) \cong HH_*(H^{-*}(M), H^{-*}(M))$. Thus, under these conditions, operations

$$HH_*(H^{-*}(M), H^{-*}(M))^{\otimes n_1} \otimes H^{-*}(M)^{\otimes m_1} \to HH_*(H^{-*}(M), H^{-*}(M))^{\otimes n_2} \otimes H^{-*}(M)^{\otimes m_2}$$

are equivalent to operations

$$H^{-*}(LM)^{\otimes n_1} \otimes H^{-*}(M)^{\otimes m_1} \to H^{-*}(LM)^{\otimes n_2} \otimes H^{-*}(M)^{\otimes m_2},$$

which are dual to the operations we ask for in string topology. On the other hand, $H^{-*}(M)$ is a commutative Frobenius algebra, thus constructing operations on the Hochschild homology of commutative Frobenius algebras gives us (dual) string operations. This correspondence can be applied even more generally. Working with a field of characteristic zero and taking the deRham complex $\Omega^\bullet(M)$ instead of singular cochains, in [LS07] Lambrechts and Stanley prove that there is a commutative differential graded Poincaré duality algebra A weakly equivalent to $\Omega^\bullet(M)$. A Poincaré duality algebra is a graded version of a commutative Frobenius algebra. Hence, the Hochschild complex is isomorphic to $HH_*(A^{-*}, A^{-*})$ and string operations on $H^{-*}(LM)$ correspond to operations on the Hochschild homology of $HH_*(A^{-*}, A^{-*})$.

In [GH09] Goresky and Hingston investigate a product on the relative cohomology $H^*(LM, M)$ that is an operation which is not part of the HCFT mentioned above. We define a product on the Hochschild homology of commutative Frobenius dg-algebras (or more precisely on the split summand of positive Hochschild degree) and show that it is part of a shifted BV-structure. Such a product also occurs in [Abb13a, Section 7] and [Abb13b, Section 6]. Simultaneously with the aforementioned paper we conjecture that this product is the operation corresponding to the Goresky-Hingston product under the above isomorphism (see Conjecture 2.14).

Since commutative Frobenius algebras are in particular symmetric, we want our complex to recover all operations known for symmetric Frobenius algebras. Following the work of Kontsevich and Soibelman for the action of the chains of the moduli space of open-closed surfaces on the Hochschild chains of A_∞-algebras (cf. [KS09]), in [TZ06] Tradler and Zeinalian show that a certain chain complex of Sullivan chord diagrams acts on the Hochschild cochain complex of a symmetric Frobenius algebra (a dual construction on the Hochschild chains was done by Wahl and Westerland in [WW11]). In [Wah12, Theorem 3.8] this complex is shown to give all formal operations for symmetric Frobenius algebras up to a split quasi-isomorphism.

On the other hand, every commutative Frobenius dg-algebra is of course a differential graded commutative algebra. In [Kla13] we give a description of the homology of all formal operations for differential graded commutative algebras in terms of Loday's shuffle operations (defined in [Lod89]) and the Connes' boundary operator. Well-known operations which are covered in this complex are Loday's λ–operations and the shuffle product $C_*(A, A) \otimes C_*(A, A) \to C_*(A, A)$.

The chain complex of operations on commutative Frobenius dg-algebras constructed in this paper recovers all the operations just mentioned: The shifted BV-structure, the operations coming from Sullivan diagrams and the more classical operations on the Hochschild chains of commutative algebras. In addition, this complex provides a large class of other non-trivial operations and can be used to compute relations between the previously known operations.

We now present our results in more detail. The main new object introduced in this paper is what we call a *looped diagram*. A looped diagram of type $[{}_{(m_1+m_2,n_1)}^{\quad n_2}]$ is a pair $(\Gamma, \mathscr{C})$ where Γ can be described as an equivalence class of one-dimensional cell complexes (a "commutative Sullivan diagram") built from n_2 circles by attaching chords with $n_1 + m_1 + m_2$ marked points on the circles and $\mathscr{C}$ is a collection of n_1 loops on Γ starting at the marked points labeled 1 to n_1. An example of a $[{}_{(2,2)}^{\ 1}]$–looped diagram is given in Figure 1.

The set of looped diagrams forms a multi-simplicial set with the boundary maps given by identifying neighbored marked points on the circles and taking the induced loops. The corresponding reduced chain complex defines $l\mathcal{D}([{}^{n_1}_{m_1}], [{}^{n_2}_{m_2}])$, the chain complex of looped diagrams. Inside $l\mathcal{D}([{}^{n_1}_{m_1}], [{}^{n_2}_{m_2}])$, we have a subcomplex $l\mathcal{D}_+([{}^{n_1}_{m_1}], [{}^{n_2}_{m_2}])$ of diagrams with an

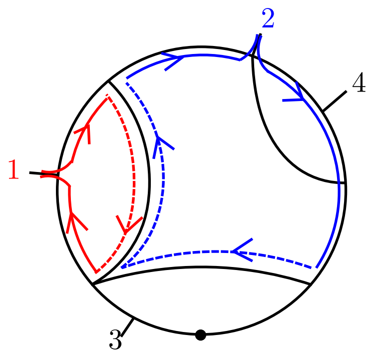

FIGURE 1. A $[^{1}_{(2,2)}]$–looped diagram

open boundary condition which we use to construct operations on commutative cocommutative open Frobenius dg-algebras (commutative Frobenius algebras without a counit). In both complexes we can compose elements, i.e. we in fact construct dg-categories $l\mathcal{D}$ and $l\mathcal{D}_+$ with objects $\mathbb{N}\times\mathbb{N}$ and morphism spaces $l\mathcal{D}([^{n_1}_{m_1}],[^{n_2}_{m_2}])$ and $l\mathcal{D}_+([^{n_1}_{m_1}],[^{n_2}_{m_2}])$ and show:

Theorem A (see Theorem 2.3)**.** For any commutative Frobenius dg-algebra A there is a map of chain complexes

$$C_*(A,A)^{\otimes n_1}\otimes A^{\otimes m_1}\otimes l\mathcal{D}([^{n_1}_{m_1}],[^{n_2}_{m_2}])\to C_*(A,A)^{\otimes n_2}\otimes A^{\otimes m_2}$$

natural in A and commuting with the composition in $l\mathcal{D}$.

For any commutative, cocommutative open Frobenius dg-algebra A, we have a chain map

$$C_*(A,A)^{\otimes n_1}\otimes A^{\otimes m_1}\otimes l\mathcal{D}_+([^{n_1}_{m_1}],[^{n_2}_{m_2}])\to C_*(A,A)^{\otimes n_2}\otimes A^{\otimes m_2}$$

natural in A and preserving the composition of $l\mathcal{D}$. Moreover, all these operations are formal operations in the sense of [Wah12, Section 2].

Moving on, we enlarge $l\mathcal{D}([^{n_1}_{m_1}],[^{n_2}_{m_2}])$ to a complex $il\mathcal{D}([^{n_1}_{m_1}],[^{n_2}_{m_2}])$, where we take products over specific types of diagrams. In this complex not all elements are composable, but we can show:

Theorem B (see Theorem 2.4)**.** For A a commutative Frobenius dg-algebra there is a map of chain complexes

$$C_*(A,A)^{\otimes n_1}\otimes A^{\otimes m_1}\otimes il\mathcal{D}([^{n_1}_{m_1}],[^{n_2}_{m_2}])\to C_*(A,A)^{\otimes n_2}\otimes A^{\otimes m_2}$$

natural in A and commuting with the composition of composable elements in $il\mathcal{D}$. Again, for A a commutative, cocommutative open Frobenius dg-algebra, we have a chain map

$$C_*(A,A)^{\otimes n_1}\otimes A^{\otimes m_1}\otimes il\mathcal{D}_+([^{n_1}_{m_1}],[^{n_2}_{m_2}])\to C_*(A,A)^{\otimes n_2}\otimes A^{\otimes m_2}$$

natural in A.

We first explain how we recover the operations known from symmetric Frobenius algebras. We denote the complexes of Sullivan diagrams by $\mathcal{SD}([^{n_1}_{m_1}],[^{n_2}_{m_2}])$ (see Section 1.1 for a definition). As already mentioned, Sullivan diagrams define natural operations for symmetric Frobenius algebras, i.e. for A a symmetric Frobenius dg-algebra there is an action

$$C_*(A,A)^{\otimes n_1}\otimes A^{\otimes m_1}\otimes \mathcal{SD}([^{n_1}_{m_1}],[^{n_2}_{m_2}])\to C_*(A,A)^{\otimes n_2}\otimes A^{\otimes m_2}$$

natural in A. There is a canonical way to build a looped diagram out of a Sullivan diagram Γ and thus there is a dg-map $K:\mathcal{SD}([^{n_1}_{m_1}],[^{n_2}_{m_2}])\to l\mathcal{D}([^{n_1}_{m_1}],[^{n_2}_{m_2}])$ commuting with the composition of diagrams. In Proposition 2.7 we show that both actions (the one of $\mathcal{SD}$ and the one of $l\mathcal{D}$) are compatible, i.e. that given a commutative Frobenius dg-algebra A the diagram

$$\begin{tikzcd} C_*(A,A)^{\otimes n_1} \otimes A^{\otimes m_1} \otimes \mathcal{SD}([{n_1 \atop m_1}],[{n_2 \atop m_2}]) \arrow[d, "\mathrm{id}^{\otimes n_1}\otimes \mathrm{id}^{\otimes m_1}\otimes K"'] \arrow[dr] & \\ C_*(A,A)^{\otimes n_1} \otimes A^{\otimes m_1} \otimes l\mathcal{D}([{n_1 \atop m_1}],[{n_2 \atop m_2}]) \arrow[r] & C_*(A,A)^{\otimes n_2} \otimes A^{\otimes m_2} \end{tikzcd}$$

commutes.

Second, restricting to those diagrams which only have leaves and no chords glued in, we define a subcomplex $ipl\mathcal{D}_{\mathscr{C}om}([{n_1 \atop m_1}],[{n_2 \atop m_2}])$ of $il\mathcal{D}_+([{n_1 \atop m_1}],[{n_2 \atop m_2}])$. This subcomplex defines operations for commutative algebras and contains Loday's lambda operations. More precisely, by [Kla13, Theorem 3.4] it gives all formal operations of differential graded commutative algebras up to quasi-isomorphism. So in particular the complex of looped diagrams includes (up to quasi-isomorphism) all formal operations on the Hochschild chains of differential graded commutative algebras.

Last, we define another subcomplex $pl\mathcal{D}^{>0}_{cact}(n_1,n_2) \subset l\mathcal{D}_+([{n_1 \atop 0}],[{n_2 \atop 0}])$ with $pl\mathcal{D}^{>0}_{cact}(n_1,n_2)$ the PROP coming from an operad $pl\mathcal{D}^{>0}_{cact}(n,1)$. We show that a topological version of this operad is homeomorphic to a topologically desuspended Cacti operad and hence deduce:

Theorem C (see Theorem 4.7)**.** The complex $pl\mathcal{D}^{>0}_{cact}(n,1)$ is a chain model for the twisted operadic desuspended BV-operad, i.e.

$$H_*(pl\mathcal{D}^{>0}_{cact}(-,1)) \cong \widetilde{s}^{-1}BV$$

as graded operads.

Here $\widetilde{s}^{-1}$ denotes a desuspension with twisted sign. The sign twist comes from the fact that we actually work with topological operads and suspend topologically by smashing with the sphere operad (see Definition 4.3). As a corollary of the above theorem we can deduce:

Corollary D (see Corollary 4.8)**.** There is a desuspended BV-algebra structure on the Hochschild homology of a commutative cocommutative open Frobenius dg-algebra (in particular on the Hochschild homology of a commutative Frobenius dg-algebra) which comes from an action of a chain model of the suspended Cacti operad on the Hochschild chains. The BV-operator is the ordinary BV-operator on positive Hochschild degrees and trivial on Hochschild degree zero.

The paper is organized as follows: The combinatorics used in the paper are given in Section 1. We start with recalling the definitions of black and white graphs and Sullivan diagrams in Section 1.1. In Section 1.2 we define looped diagrams and show that $l\mathcal{D}$ and $l\mathcal{D}_+$ are well-defined dg-categories. The two following sections 1.3 and 1.4 are very technical and not needed for the actual construction of operations (they will be used to construct the commutative operations and the action of the Cacti operad). For getting an overview over the results, we suggest the reader to skip them and come back later, if needed. More precisely, in Section 1.3 we prove that the subcomplex of diagrams with a constant loop is a split subcomplex and investigate how the composition looks like on the split complement of non-constant diagrams. In Section 1.4 we give a finer subdivision of $l\mathcal{D}^{>0}$ on the level of abelian groups and afterward take the product over all the subgroups to get the complexes $il\mathcal{D}([{n_1 \atop m_1}],[{n_2 \atop m_2}])$ and $il\mathcal{D}_+([{n_1 \atop m_1}],[{n_2 \atop m_2}])$. Composition is not well-defined on these complexes, but we give some subcomplexes for which composition with every other element is well-defined. Section 2 deals with the formal operations on the Hochschild chains of commutative Frobenius dg-algebras. We start by recalling the definition of Frobenius algebras in Section 2.1 and the definition and results on formal operations in Section 2.2. In Section 2.3 we explain how to build formal operations out of looped diagrams, i.e. prove Theorem A and Theorem B. In Section 2.4 we investigate the connection to the operations on symmetric Frobenius algebras stated above. Finally, in Section 2.5 we show how the shuffle product, the Chas-Sullivan coproduct, the BV-operator and the shifted commutative product defined in [Abb13a, Section 7] and [Abb13b, Section 6] look like in terms of looped diagrams and use the techniques of looped diagrams to prove a relation between the new product and the BV-operator. In Section 3 we define the subcomplexes of graphs giving the operations of commutative algebras and recall [Kla13, Theorem 3.4] in terms of these diagrams. In Section 4 we define the complex $pl\mathcal{D}^{>0}_{cact}(n_1, n_2)$, prove Theorem C and thus obtain the action of a desuspended cacti operad on the Hochschild chains of a commutative Frobenius dg-algebra (see Corollary D).

In Appendix A we have listed all complexes defined in the paper together with a short explanation and reference. We hope that this is helpful to keep track of the definitions throughout the paper.

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Contents

Conventions. If not specified otherwise we work in the category Ch of chain complexes over $\mathbb{Z}$. We use the usual sign convention on the tensor product, i.e. the differential $d_{V\otimes W}$ on $V \otimes W$ is given by $d_{V\otimes W}(v \otimes w) = d_V(v) \otimes w + (-1)^{|v|} v \otimes d_W(w)$.

A *dg-category* $\mathcal{E}$ is a category enriched over chain complexes, i.e. the morphism sets are chain complexes. We use composition from the right, i.e. we require the composition maps $\mathcal{E}(m, n) \otimes \mathcal{E}(n, p) \to \mathcal{E}(m, p)$ to be chain maps. A *dg-functor* is an enriched functor $\Phi : \mathcal{E} \to \mathrm{Ch}$, so the structure maps $\Phi(m) \otimes \mathcal{E}(m, n) \to \Phi(n)$ are chain maps.

Given a graded abelian group A we denote by $A[k]$ the shifted abelian group with $(A[k])_n = A_{n-k}$. Throughout the paper the natural numbers are always assumed to include zero.

1. Definitions of graph complexes

In this section we define the chain complex of looped diagrams and its subcomplex of positive diagrams. The complexes are an extension of a quotient of the chain complex of Sullivan diagrams which we first recall. We mainly follow [WW11, Section 2].

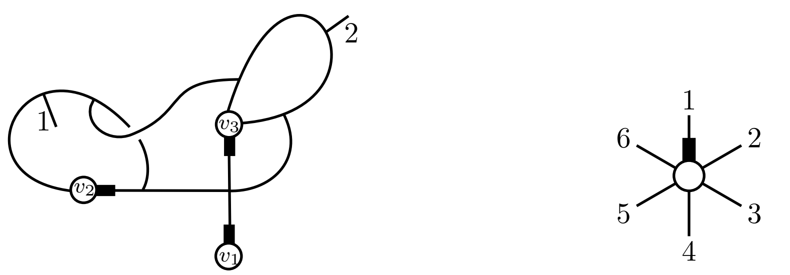
1.1. **Graphs.** A *graph* is a tuple (V, H, s, i) with V the vertices, H the half-edges, $s : H \to V$ the source map and $i : H \to H$ an involution. A half-edge is a *leaf* if it is a fixed point under i. A *fat graph* is a graph with a cyclic ordering of the half-edges at the vertices. The cyclic orderings define *boundary cycles* on the graph which correspond to the boundary cycles of the surface one gets by thickening the graph (for more details see [WW11, Section 2.1]).

In the graphical representation the half-edges are glued to the vertices using s and to each other using i.

An *orientation* of a graph is a unit vector in $\det(\mathbb{R}(V \amalg H))$. Note that any odd-valent fat graph has a canonical orientation and that an orientation of a fat graph with even-valent vertices is given by an ordering of the (even-valent) vertices together with a choice of a start half-edge h_1^i for each (even-valent) vertex (changing the position of the odd-valent vertices in the ordering or changing the choice of their start half-edge does not change the orientation). Denoting the half-edges belonging to a vertex v_i by h_j^i starting from the start half-edge and following the cyclic ordering, the canonical orientation is given by

$$v_1 \wedge h_1^1 \wedge \cdots h_{n_1}^1 \wedge v_2 \wedge \ldots \wedge h_{n_r}^r.$$

A *black and white graph* is an oriented fat graph where we label the vertices black or white and allow the white vertices to have any positive valence, whereas the black vertices are requested to have valence at least three. The white vertices are ordered and each white vertex is equipped with a choice of a start half-edge. A $\genfrac{[}{]}{0pt}{}{p}{m}$–*graph* is a black and white graph with p white vertices and m labeled leaves, quotiening out the equivalence relation of forgetting unlabeled leaves which are not the start half-edge of a white vertex. In the graphical representation we mark the start half-edges by black blocks. An example of a $\genfrac{[}{]}{0pt}{}{3}{2}$–graph is given in Figure 2(a). A special example of a $\genfrac{[}{]}{0pt}{}{1}{n}$–graph is the graph l_n which will play a crucial role in defining operations. This graph is given by attaching n leaves to the white vertex and labeling them starting from the start half-edge (for an example see Figure 2(b)). In general we omit the label v_1 in the pictures if there is only one white vertex.



(a) A general black and white graph of degree 4

(b) The graph l_6

Figure 2. A $\genfrac{[}{]}{0pt}{}{3}{2}$–graph and the $\genfrac{[}{]}{0pt}{}{1}{6}$–graph l_6

The *degree* of a black vertex of valence v_b is given by $v_b - 3$ whereas the degree of a white vertex v_w is defined as $v_w - 1$. The degree of a black and white graph is the sum of the degrees over all its vertices.

The differential of a black and white graph is given by the sum of all graphs obtained by blowing up all vertices of degree at least 1 in all possible ways, i.e. splitting the set of half-edges attached to the vertex into two subsets (respecting the cyclic ordering and with at least 2 elements in each if the vertex was black) and adding an edge in between these. For more details on the differential and examples see [WW11, Section 2.5].

The chain complex of $\left[\begin{smallmatrix}p\\n\end{smallmatrix}\right]$–*Sullivan diagrams* is defined as a quotient of the above complex of $\left[\begin{smallmatrix}p\\n\end{smallmatrix}\right]$–graphs by the subcomplex spanned by the graphs with at least one black vertex of valence at least 4 and the boundaries of these graphs. Hence an element in this complex is an equivalence class of graphs with all its black vertices of valence exactly 3 and the equivalence relation is generated by the relation shown in Figure 3.

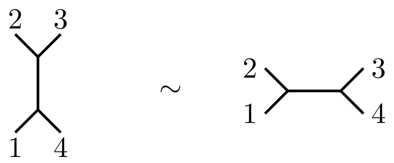


FIGURE 3. The equivalence relation on Sullivan diagrams

In [WW11, Theorem 2.7] it was shown that this complex is isomorphic to the complex of *Cyclic Sullivan chord diagrams* defined in [TZ06, Def. 2.1].

The dg-category $\mathcal{SD}$ is defined to have pairs of natural numbers $\left[\begin{smallmatrix}n\\m\end{smallmatrix}\right]$ as objects and morphism complexes $\mathcal{SD}(\left[\begin{smallmatrix}n_1\\m_1\end{smallmatrix}\right], \left[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}\right]) \subset \left[\begin{smallmatrix}n_1\\n_1+m_1+m_2\end{smallmatrix}\right]$–Sullivan diagrams the subcomplex of the graphs with the first n_1 leaves being sole labeled leaves in their boundary cycle. The composition is defined in [WW11, Section 2.8].

We want to define looped diagrams as an enlargement of a quotient of these.

1.2. Commutative Sullivan diagrams and looped diagrams.

Definition 1.1. A $\left[\begin{smallmatrix}p\\m\end{smallmatrix}\right]$–*commutative Sullivan diagram* is an equivalence class of $\left[\begin{smallmatrix}p\\m\end{smallmatrix}\right]$–Sullivan diagrams by forgetting the ordering at the black vertices, i.e. the chain complex $\left[\begin{smallmatrix}p\\m\end{smallmatrix}\right] - CSD$ of $\left[\begin{smallmatrix}p\\m\end{smallmatrix}\right]$–commutative Sullivan diagrams is the quotient of $\left[\begin{smallmatrix}p\\m\end{smallmatrix}\right]$–graphs by

- graphs with black vertices of valence at least 4,
- the boundaries of such graphs and
- reordering of the half-edges at black vertices.

An example of two equivalent $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$–commutative Sullivan diagrams is given in Figure 4.

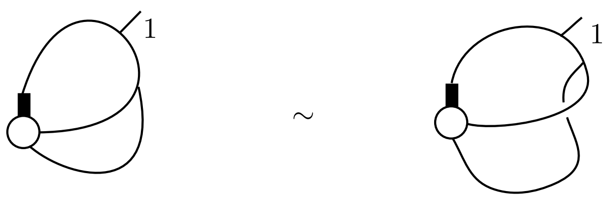
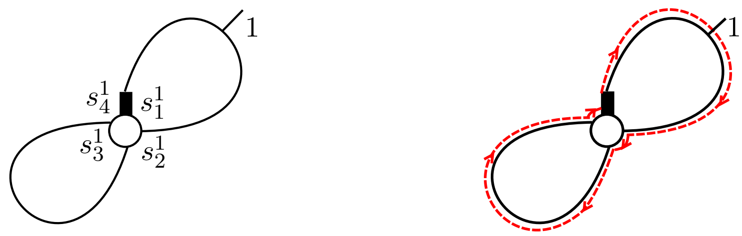


FIGURE 4. Two equivalent commutative Sullivan diagrams

Remark 1.2. Black and white graphs were defined with an orientation. As mentioned above, any trivalent graph has a canonical orientation and so does every white vertex (starting from the start half edge). Thus every Sullivan diagram has a canonical orientation. The relation we divide out is the commutativity relation together with the canonical orientation of the two graphs. From now on, we always work with the canonical orientation. However, the orientation was also used to define the sign of the differential and the sign of the composition. We will spell out these sign explicitly, too.

In order to be able to define a composition, we need an additional structure analogous to the one we lost going from Sullivan diagrams to commutative Sullivan diagrams. We define a larger category of looped diagrams which includes all Sullivan diagrams without free boundary as a subcategory.

Given a white vertex in a black and white graph with k half-edges attached to it, the half-edges cut the circle around the white vertex into k parts. Given a labeling of the white vertices $v_1, \cdots, v_p$ we label the segments at the vertex v_i by $s^i_1, \cdots, s^i_{k_i}$ following the ordering of the half-edges at v_i. These segments inherit a canonical orientation from the ordering at the white vertex and hence we can talk about their start and end. By $-s^i_j$, we mean the segment with the opposite orientation, i.e. start and end got interchanged. An example of the labeling of segments for one white vertex is given in Figure 5(a). An *arc component* of a black and white graph is a set of half-edges and black vertices which is path-connected with the paths in the graph not passing through a white vertex (i.e. in pictures, a connected component of the graph after "deleting" the white vertices). For example, the commutative Sullivan diagram in Figure 4 has one arc component, whereas the underlying diagram of Figure 5(a) has two arc components (the one with the labeled leaf and the one without).



(a) The segments around one white vertex (b) The loop $\gamma = \{s^1_2, s^1_4\}$

FIGURE 5. A commutative Sullivan diagram with segments and a loop starting from leaf 1

Definition 1.3. A *loop in a $[{p \atop m}]$–commutative Sullivan diagram* from an arc component a to itself is an ordered set of oriented segments of the boundary circles of the white vertices $\{\varepsilon_1 s^{i_1}_{t_1}, \cdots, \varepsilon_r s^{i_r}_{t_r}\}$ for $r \geq 0$, $1 \leq i_1, \cdots, i_r \leq p$ and $\varepsilon_i \in \{-1, 1\}$ such that the following conditions hold:

(1) $\varepsilon_1 s^{i_1}_{t_1}$ starts at the arc component a.
(2) $\varepsilon_r s^{i_r}_{t_r}$ ends at the arc component a.
(3) $\varepsilon_{w+1} s^{i_{w+1}}_{t_{w+1}}$ starts at the arc component at which $\varepsilon_w s^{i_w}_{t_w}$ ends (which can be at a different white vertex!).
(4) $\varepsilon_w s^{i_w}_{t_w} \neq -\varepsilon_{w+1} s^{i_{w+1}}_{t_{w+1}}$.

The loop is called *constant* if the set of segments is empty, i.e. if $r = 0$.

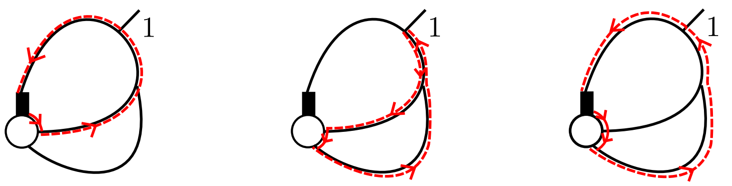
A loop from a leaf k to itself is a loop starting at the arc component the leaf belongs to.

The composition $\gamma_1 * \gamma_2$ of loops γ_1 and γ_2 both starting at the same leaf k is the concatenation of these two loops (see Figure 6).

A loop γ starting at the leaf k is called *irreducible* if it cannot be written as the composition of two non-trivial loops (i.e. the loop does not return to the arc component of k before it finishes).

A *positively oriented* loop in a $[{p \atop m}]$–commutative Sullivan diagram is a loop such that all orientations of the boundary segments are positive (i.e. $\varepsilon_w = 1$ for all w).

We draw a loop from a leaf by starting at the leaf and marking the segments of the white vertex (with orientation). To keep track of their ordering, we also draw the loop through arc components (dotted) even though this is not part of the data (i.e. changing the way we walk through an arc component does not change the loop). This way, a loop in a diagram is really represented by a loop in the picture. An example is given in Figure 5(b).



(a) $\gamma_1 = (s_1^1)$ (b) $\gamma_2 = (s_2^1)$ (c) $\gamma_1 * \gamma_2 = (s_1^1, s_2^1)$

FIGURE 6. Juxtaposition of loops

Definition 1.4. A $[{p \atop (m,n)}]$*–looped diagram* is a $[{p \atop n+m}]$–commutative Sullivan diagram with n loops such that the i-th loop starts at the i-th labeled leaf.

An example of a $[{1 \atop (0,2)}]$–looped diagram and an example of a $[{3 \atop (0,2)}]$–looped diagram are given in Figure 7. To indicate which path starts at which leaf we color the label of the leaf with the same color as the corresponding loop.

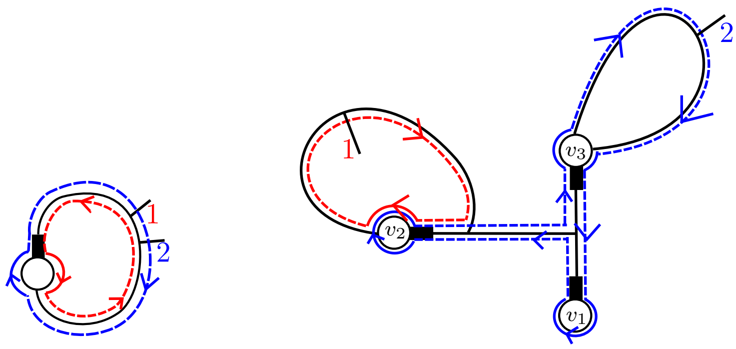


FIGURE 7. Two looped diagrams

The decomposition of the underlying commutative Sullivan diagram into connected components gives a decomposition of the looped diagram into connected components, since every loop has to stay in a connected component.

We want to make a complex out of these diagrams and thus need to define a differential. The differential is defined just as for Sullivan diagrams, where we blow up every possible pair of neighbored vertices at the white vertex with alternating sign (this is equivalent to the sign given by orientations, cf. [WW11, Section 2.10]).

For a $[{}^{\,p}_{(m,n)}]$–looped diagram $(\Gamma, \gamma_1, \cdots, \gamma_n)$ the i-th blow up at the white vertex v_k is the $[{}^{\,p}_{(m,n)}]$–looped diagram $(\Gamma', \gamma_1', \cdots, \gamma_n')$ where Γ' is the blow up of Γ (as explained in the end of Section 1.1) and the γ_l' are given by the γ_l after forgetting the segment s_i^k whenever it was part of the loop and relabeling all s_j^k for $j > i$. We define the differential d to be the sum of all blow ups with the sign at each vertex v_i alternating and starting with $(-1)^{1+\sum_{j<i}|v_j|}$. Examples are given in Figure 8 and Figure 13.

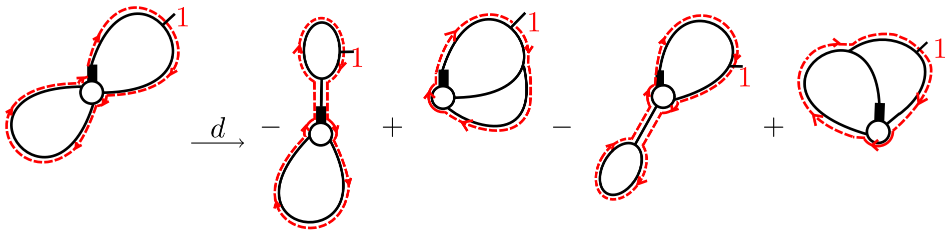


FIGURE 8. The differential of a looped diagram (with one loop)

Lemma 1.5. *The map d defines a differential on $[{}^{\,p}_{(m,n)}]$–looped diagrams. The chain complex of these diagrams is called $[{}^{\,p}_{(m,n)}]$–lD.*

Proof. One checks that the complex is the reduced chain complex of a p–multisimplicial set obtained by allowing unlabeled leaves at arbitrary positions. At each white vertex v_k we have a simplicial set with boundary maps d_i blowing up the $(i-1)$-st and i-th incoming leaves (and doing the induced procedure to the loops) and degeneracies adding in a unit leaf in between the i-th and $(i+1)$-st vertex and replacing s_i^k by the two pieces (and relabeling the other segments at that vertex). The simplicial identities hold. $\square$

Remark 1.6. In the previous proof we added a leaf to a vertex, replaced the corresponding boundary segment by the two new pieces and relabeled the others. Morally, we did nothing to our loop (cf. Figure 9), but being precise, given a looped diagram (Γ, γ) after adding a leaf to the white vertex v_k in between the old edges j and $j+1$ we get the diagram (Γ', γ') as follows: The commutative Sullivan diagram Γ' is the diagram Γ with the extra half-edge. The new loop γ' is obtained from γ by replacing all s_i^k for $i > j$ by s_{i+1}^k and if s_j^k was in γ with positive orientation, it is replaced by s_j^k followed by s_{j+1}^k and if it appeared with negative orientation by $-s_{j+1}^k$ and $-s_j^k$.

FIGURE 9. Adding a leaf

Remark 1.7. The set of $[{}^{\,p}_{(m,n)}]$–looped diagrams with all loops positively oriented is a subcomplex of the whole complex. We denote this chain complex by $[{}^{\,p}_{(m,n)}]$–plD.

We now construct a dg-category out of $[{}^{\,p}_{(m,n)}]$–lD, extending the category of Sullivan diagrams.

Definition 1.8. Let $l\mathcal{D}$ be the dg-category of looped diagrams with objects pairs of natural numbers $\left[{n \atop m}\right]$ and morphism $l\mathcal{D}(\left[{n_1 \atop m_1}\right], \left[{n_2 \atop m_2}\right])$ given by the chain complex $\left[{n_2 \atop (m_1+m_2, n_1)}\right]$–lD and composition defined by taking elements $x = (\Gamma, \gamma_1, \cdots, \gamma_{n_1}) \in l\mathcal{D}(\left[{n_1 \atop m_1}\right], \left[{n_2 \atop m_2}\right])$ and

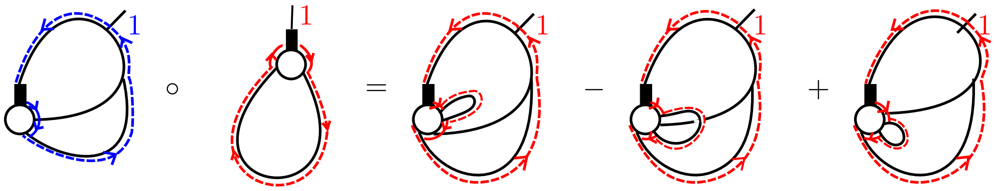


FIGURE 10. Composition $\circ$

$y = (\Gamma', \gamma'_1, \cdots, \gamma'_{n_2}) \in l\mathcal{D}(\left[{n_2 \atop m_2}\right], \left[{n_3 \atop m_3}\right])$ to the element $y \circ x = (\widetilde{\Gamma}, \widetilde{\gamma}_1, \cdots, \widetilde{\gamma}_{n_1}) \in l\mathcal{D}(\left[{n_1 \atop m_1}\right], \left[{n_3 \atop m_3}\right])$ which is zero if there is an i such that the loop γ'_i is constant and the vertex v_i in Γ has more than one half-edge attached to it and else is computed as the sum of all possible $([G], g_1, \cdots, g_{n_1})$ (with a sign) obtained as follows

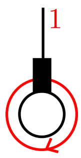
(1) The commutative Sullivan diagrams $[G]$ are the equivalence classes of black and white graphs obtained from x and y by
  (a) removing the n_2 white vertices from Γ,
  (b) For each $i = 1, \cdots, n_2$ identifying the start half-edge of the i–th white vertex v_i of Γ with the i–th labeled leaf of Γ',
  (c) starting with $i = 1$ and continuing inductively
    (i) attaching the remaining half-edges from the vertex v_i to the white vertices of Γ' along the loop γ'_i following their cyclic ordering (in general we here have several possibilities),
    (ii) replacing the γ'_j by the induced ones as describe in Remark 1.6 (which morally does not do anything),
  (d) attaching the last m_2 labeled leaves of Γ to the leaves of Γ' labeled $n_2 + 1, \cdots n_2 + m_2$ respecting the order.

(2) The gluing defines a map from the boundary segments around the vertex v_i in Γ to ordered subsets of the loop γ'_i which are sets of boundary segments in $[G]$. All the subsets are disjoint and putting them together following the order of the boundary segments of v_i reproduces the loop γ'_i. Since we have such a map for each i, we get a map from {boundary segments in Γ} to {boundary segments in $[G]$}. We define g_j to be the image of γ_j under this map.

The fact that the g_j are again loops follows directly from the construction.

The orientation (and thus the sign) is obtained by juxtaposition of the orientations of Γ and Γ' as explained in [WW11, Section 2.8]. However, we give a more explicit (but equivalent) way to compute the sign. It is computed by putting all non-start half-edges of Γ to the right of the start half-edge of the first white vertex in Γ' in their cyclic order and the order of the white vertices in Γ (i.e. the first half-edge of the first white vertex of Γ is next to the start half-edge of Γ') and then computing the parity of the number of half-edges they have to pass to move to their final position. If they are glued left of the start half-edge of a white vertex then they can move to the right of the start half-edge of the next white vertex in Γ' which does not change the sign.

The identity element $\mathrm{id}_{\left[{n_1 \atop m_1}\right]}$ in $l\mathcal{D}(\left[{n_1 \atop m_1}\right], \left[{n_1 \atop m_1}\right])$ is given by the element $\mathrm{id}_{\left[{1 \atop 0}\right]} \amalg \cdots \amalg \mathrm{id}_{\left[{1 \atop 0}\right]} \amalg \mathrm{id} \amalg \cdots \amalg \mathrm{id}$ with $\mathrm{id}_{\left[{1 \atop 0}\right]} \in l\mathcal{D}(\left[{1 \atop 0}\right], \left[{1 \atop 0}\right])$ the $\left[{1 \atop (0,1)}\right]$–looped diagram shown in Figure 11(a) and



(a) The first building block of the identity

(b) The second building block of the identity

FIGURE 11. The element $\mathrm{id}_{[\substack{1\\0}]} \in l\mathcal{D}_+^{>0}([\substack{1\\0}],[\substack{1\\0}])$ and the element $\mathrm{id} \in l\mathcal{D}([\substack{0\\1}],[\substack{0\\1}])$

$\mathrm{id} \in l\mathcal{D}([\substack{0\\1}],[\substack{0\\1}])$ the $[\substack{0\\(2,0)}]$–looped diagram (without white vertices and loops) shown in Figure 11(b).

The composition defined above is associative. Examples are given in Figure 10 and Figure 12.

Definition 1.9. Let $l\mathcal{D}_+$ be the dg-category of *looped diagrams with positive boundary condition* with the same objects as $l\mathcal{D}$ and morphisms $l\mathcal{D}_+([\substack{n_1\\m_1}],[\substack{n_2\\m_2}])$ those looped diagrams in $[\substack{n_2\\(m_1+m_2,n_1)}]$–lD where every connected component contains at least one white vertex or one of the m_2 last labeled leaves, i.e. a leaf labeled by a number in $\{n_1+m_1+1,\ldots,n_1+m_1+m_2\}$.

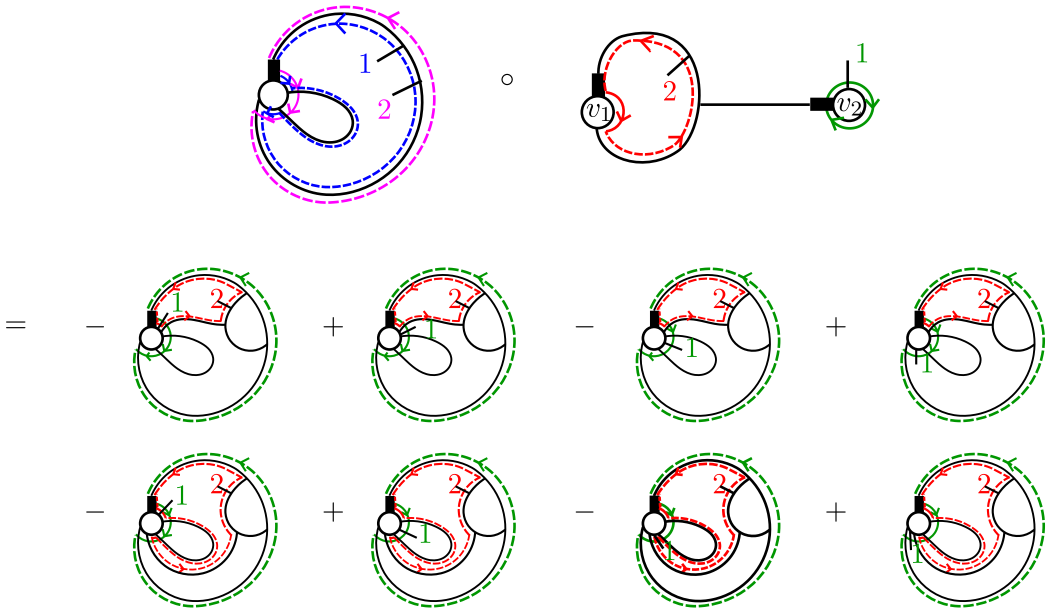


FIGURE 12. Composition $\circ$

Definition 1.10. The dg-category $pl\mathcal{D}$ (and $pl\mathcal{D}_+$) of *positively oriented looped diagrams (with positive boundary condition)* has the same objects as $l\mathcal{D}$ and morphism $pl\mathcal{D}([\substack{n_1\\m_1}],[\substack{n_2\\m_2}])$ given by the chain complex $[\substack{n_2\\(m_1+m_2,n_1)}]$–plD (lying in $l\mathcal{D}_+([\substack{n_1\\m_1}],[\substack{n_2\\m_2}])$, respectively).

Similarly, the dg-category $pl\mathcal{D}_+$ of *positively oriented looped diagrams with positive boundary condition* has the same objects as $l\mathcal{D}$ and and morphism $pl\mathcal{D}_+([\substack{n_1\\m_1}],[\substack{n_2\\m_2}])$ those looped diagrams lying in the intersection of $l\mathcal{D}_+([\substack{n_1\\m_1}],[\substack{n_2\\m_2}])$ and $pl\mathcal{D}([\substack{n_1\\m_1}],[\substack{n_2\\m_2}])$.

Proposition 1.11. *The composition $\circ$ of looped diagrams defined above is a chain map*

$$l\mathcal{D}([\substack{n_1\\m_1}],[\substack{n_2\\m_2}]) \otimes l\mathcal{D}([\substack{n_2\\m_2}],[\substack{n_3\\m_3}]) \to l\mathcal{D}([\substack{n_1\\m_1}],[\substack{n_3\\m_3}]).$$

Proof. Take $x = (\Gamma, \gamma_1, \cdots, \gamma_{n_1}) \in l\mathcal{D}([{n_1 \atop m_1}], [{n_2 \atop m_2}])$ and $y = (\Gamma', \gamma_1', \cdots, \gamma_{n_2}') \in l\mathcal{D}([{n_2 \atop m_2}], [{n_3 \atop m_3}])$ We need to show that $(-1)^{|x|} dy \circ x + y \circ dx = d(y \circ x)$.

We refer to edges at the white vertex of the composition as coming from Γ if they where attached in the gluing process and as coming from Γ' if they were belonging to Γ' before. The differential in $y \circ x$ comes from four different kinds of boundaries, which is illustrated on the example in Figure 13 where we compute the differential of the second to last summand of the composition shown in Figure 12. The four kinds of boundaries

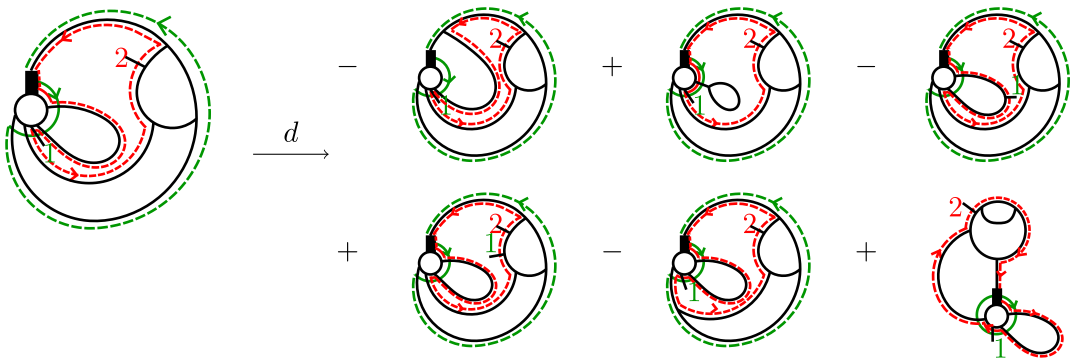


FIGURE 13. The differential of the second to last term of the composition in Figure 12

are described as follows:

1. The boundaries coming from the multiplication of edges belonging to Γ' together (cf. first, second and last summand in Figure 13).
2. The boundaries coming from the multiplication of edges originally belonging to the same white vertex v_j in Γ or from two special kinds of boundaries: Those arising from contracting the first segment of a loop g_i if this segment starts at the arc component which the old start half-edge of v_j was attached to and ends at the old second half-edge of v_j. Similarly, we additionally take the terms obtained from contracting the last segment of a loop g_i if this segment starts at the old last half-edge of v_j and ends at the arc component which the old start half-edge of v_j was attached to (cf. second last summand in Figure 13).
3. The boundaries arising from the multiplication of edges of Γ and Γ' together (except for the two cases mentioned in the step before) (cf. third summand in Figure 13).
4. The boundaries obtained from the multiplication of edges of Γ coming from two different white vertices (cf. fourth summand in Figure 13).

We now show that $dy \circ x$ gives exactly all summands appearing in 1., $y \circ dx$ all those appearing in 2. and that the sums of the diagrams in 3. and in 4. are zero (i.e. that every diagram shows up twice with opposite sign).

1. In dy we have two kinds of boundaries, the ones coming from contracting boundary segments which are not contained in any loop and those contracting segments appearing in loops.

   In the first case the two neighbored edges will also be neighbored in $y \circ x$ and multiplying them first and then composing with x or first composing and then multiplying is the same.

   In the second case we again have to distinguish two cases. First, if we consider contracting a segment s_j^k which appears in a loop but none of the loops only consists of this segment (i.e. $\gamma_i' \neq \{s_j^k\}$). Then in the composition there are summands where we did not glue any edges into this segment. Contracting the segment in these summands of $y \circ x$ agrees with the composition of x with those boundaries of y where we contracted that segment in Γ'. Second, if we have a loop $\gamma'_i = \{s^k_j\}$ but s^k_j is not the whole boundary segment (in which case there is no contraction on either of the sides) then the contraction of this segment is a constant loop, so it only gives a term in the composition if there is only one edge attached to v_i in Γ and the composition then makes the part of the loop going around v_i (if there was one) constant. But in $y \circ x$ the according segment can only be empty in exactly this case and thus the terms agree.

So up to sign we have shown that $dy \circ x$ agrees with the terms described in 1.

For the sign we divide the edges glued onto Γ' into two pairs, those left of the pair of edges of Γ' we multiply to get the considered element and those right (if we have several white vertices in Γ' and the edges we multiply are attached to v_i, edges attached to a white vertex v_j with $j < i$ count as being left). Denote the numbers by e_{left} and e_{right}. We have $|x| = e_{left} + e_{right}$. The sign of the boundary that multiplies the two vertices in $d(y \circ x)$ changed by $(-1)^{e_{left}}$ against the sign of the boundary of multiplying these edges in y. On the other hand, if we first multiply the two edges in Γ' and then glue Γ on, all the edges right of this boundary have to move over one edge less to get to their position, so the sign of the composition changes by $(-1)^{e_{right}}$. Thus the total difference in sign is $(-1)^{e_{left}+e_{right}} = (-1)^{|x|}$, so the terms of $(-1)^{|x|} dy \circ x$ show up with the same sign as those in $d(y \circ x)$.

(2) It is not hard to see that first multiplying neighbored edges around a vertex in x (and contracting the loop accordingly) and then gluing the result onto y or taking those summands of $d(y \circ x)$ were we multiplied neighbored vertices coming from x (and contracted the piece of the loop) agrees. The special cases described in 2 come from multiplying the start edge with the second or last edge and then gluing it onto Γ'. Similar considerations as in the first case show that the two terms show up with the same sign.

(3) Write $\gamma'_i = (s_1, \ldots, s_l)$ with the s_j boundary segments in Γ'. If in $x \circ y$ an edge e of Γ was glued as the last edge in a segment s_j for $j < l$, then there is another summand in $x \circ y$ where all edges of Γ different to e was glued to the same segment as before, but e was glued as the first edge to s_{j+1}. The element obtained from the first element by multiplying the edge e onto the arc component following s_j agrees with the one obtained from the second by multiplying e onto the arc component before s_{j+1} which is the same as the arc component following s_j (by the definition of a loop). An example is given by the red edge in the first and fifth summand in the composition in Figure 12. The boundary multiplying it onto the little loop (in the first case from the left in the second from the right) is the same.

Hence, all terms in the differential where we multiplied an edge of Γ onto one of Γ' show up twice by the above argumentation.

Assume that the first of these two elements form a term in $y \circ x$ with sign ω and its differential had sign σ, so in $d(y \circ x)$ the element has sign $\omega \cdot \sigma$. To get the second the specified edge of Γ is moved by r steps (to the right of the boundary component). This means that it shows up with sign $(-1)^r \omega$ in $y \circ x$. Moving over the neighbored edge does not change the sign of the differential and moving over more edges changes it by -1 each time. Thus the boundary shows up with sign $(-1)^{r-1}\sigma$, so in $d(y \circ x)$ the element has sign $(-1)^r \omega \cdot (-1)^{r-1}\sigma = -\omega\sigma$, i.e. both terms have opposite signs and cancel.

(4) We are left to show that terms where we multiplied two edges of Γ belonging to two different white vertices together show up twice, too. However, since we glued the edges inductively, the argumentation from the previous step works if we view the edges of the first vertex as fixed and glue the edges of the second vertex onto that graph.

$\square$

It is not hard to check that $l\mathcal{D}_+$, $pl\mathcal{D}$ and $pl\mathcal{D}_+$ are subcategories.

The way we defined our category $l\mathcal{D}$ we obtain a functor $K : \mathcal{SD} \to l\mathcal{D}$ which takes a Sullivan diagram $\Gamma \in \mathcal{SD}([{n_1 \atop m_1}], [{n_2 \atop m_2}])$ and sends it to the looped diagram $([\Gamma], \gamma_1, \cdots, \gamma_{n_1})$ with γ_i the loop starting from the i–th labeled leaf and following the boundary cycle the leaf was in before (see Figure 14 for an example).

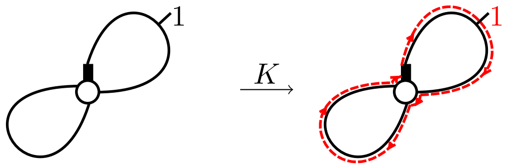


FIGURE 14. The functor K applied to a diagram in $\mathcal{SD}([{1 \atop 0}], [{1 \atop 0}])$

Before moving on we want to imitate one more construction done to Sullivan diagrams: To make them fit for string topology, in [WW11, Section 6.3+6.5] a shifted version was considered. A Sullivan diagram S was shifted by $-d \cdot \chi(S, \partial_{out})$, where $\chi(S, \partial_{out})$ is the Euler characteristic of a representative of S as a CW-complex relative to its outgoing boundary (the n_2 white vertices and the m_2 labeled outgoing leaves). Similarly, we define:

Definition 1.12. Let $l\mathcal{D}_d$ to be the shifted version of $l\mathcal{D}$ where a looped diagram $(\Gamma, \gamma_1, \cdots, \gamma_{n_1})$ gets shifted by $-d \cdot \chi(\Gamma, \partial_{out})$.

In particular, the functor K also gives a functor $K : \mathcal{SD}_d \to l\mathcal{D}_d$. For more details on this construction we refer to [WW11, Section 6.5].

1.3. **The split subcomplex of non-constant diagrams.** For later purpose we want to split off those diagrams which have constant loops. They clearly form a subcomplex and as we will see below this subcomplex is split. In some situations it will be more natural to work with the non-constant diagrams only.

Definition 1.13. A $[{p \atop (m,n)}]$–looped diagram $(\Gamma, \gamma_1, \cdots, \gamma_n)$ is called *partly constant* if one of the γ_j is a constant loop. The subcomplex of $l\mathcal{D}([{n_1 \atop m_1}], [{n_2 \atop m_2}])$ spanned by these diagrams is denoted by $l\mathcal{D}^{cst}([{n_1 \atop m_1}], [{n_2 \atop m_2}])$.

We define the map $p_j : l\mathcal{D}([{n_1 \atop m_1}], [{n_2 \atop m_2}]) \to l\mathcal{D}^{cst}([{n_1 \atop m_1}], [{n_2 \atop m_2}]) \subseteq l\mathcal{D}([{n_1 \atop m_1}], [{n_2 \atop m_2}])$ by

$$p_j((\Gamma, \gamma_1, \ldots, \gamma_{n_1})) = (\Gamma, \gamma_1, \ldots, \gamma_{j-1}, cst, \gamma_{j+1}, \ldots, \gamma_{n_1})$$

where cst is the constant loop starting at the leaf j.

Lemma 1.14. *The map p_j is a chain map.*

Proof. Contracting a boundary segment in Γ which was part of the loop γ_j and then forgetting the rest of the loop commutes with first forgetting the whole loop and then contracting the boundary segment, thus p_j commutes with the differential. $\square$

For a set $T = \{t_1, \cdots, t_k\} \subseteq \{1, \cdots, n_1\}$ we define $p_T = p_{t_k} \circ \cdots \circ p_{t_1}$, i.e. the map making the loops corresponding to T constant.

Moreover, we define $p_{cst} : l\mathcal{D}([{n_1 \atop m_1}], [{n_2 \atop m_2}]) \to l\mathcal{D}^{cst}([{n_1 \atop m_1}], [{n_2 \atop m_2}])$ by

$$p_{cst} = \sum_{k=1}^{n_1} \sum_{\substack{T \subseteq \{1, \cdots, n_1\} \\ |T|=k}} (-1)^{k+1} p_T$$

Proposition 1.15. *The map p_{cst} is a splitting of the inclusion of the subcomplex $i : l\mathcal{D}^{cst}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}],[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}]) \hookrightarrow l\mathcal{D}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}],[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}])$, i.e.*

$$p_{cst} \circ i : l\mathcal{D}^{cst}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}],[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}]) \to l\mathcal{D}^{cst}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}],[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}])$$

is the identity.

Proof. Let $x = (\Gamma, \gamma_1, \ldots, \gamma_{n_1}) \in l\mathcal{D}^{cst}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}],[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}])$. For simplicity of notation we assume that γ_1 is constant. For a set T with $1 \notin T$ we have that $p_T = p_{\{1\}\cup T}$. Since $|\{1\} \cup T| = |T| + 1$ these terms show up with opposite signs and thus cancel. The family of all non-empty subsets T with $1 \notin T$ together with $\{1\}\cup T$ are all non-empty subsets of $\{1, \cdots, n_1\}$ except for the set $\{1\}$. Thus the only non-trivial term in $p_{cst}(x)$ is $p_{\{1\}}(x)$ which is x since the first loop was already constant. □

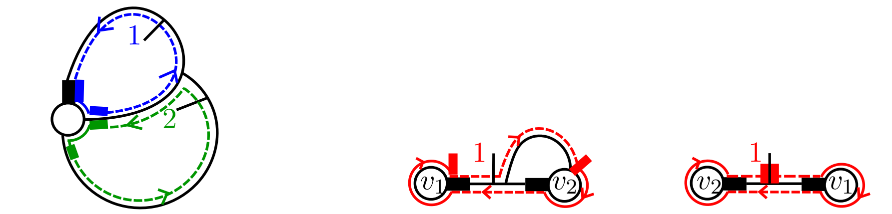
Definition 1.16. For a looped diagram $(\Gamma, \gamma_1, \ldots, \gamma_{n_1}) \in l\mathcal{D}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}],[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}])$ we define

$$(\Gamma, \langle\gamma_1\rangle, \ldots, \langle\gamma_{n_1}\rangle) = (\Gamma, \gamma_1, \ldots, \gamma_{n_1}) - p_{cst}((\Gamma, \gamma_1, \ldots, \gamma_{n_1})).$$

The complex spanned by these diagrams is denoted by $l\mathcal{D}^{>0}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}],[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}])$.

Since $p_\emptyset = \mathrm{id}$ we can rewrite

$$(\Gamma, \langle\gamma_1\rangle, \ldots, \langle\gamma_{n_1}\rangle) = \sum_{T\subseteq\{1,\cdots,n_1\}} (-1)^{|T|} p_T(\Gamma, \gamma_1, \ldots, \gamma_{n_1}).$$



(a) The diagram $(\Gamma, \langle\gamma_1\rangle, \langle\gamma_2\rangle)$ with $\gamma_1 = \{s_1^1\}$ and $\gamma_2 = \{s_2^1\}$

(b) The diagram $(\Gamma, \langle\gamma\rangle)$ with $\gamma = \{s_2^2, s_1^1\}$

(c) The diagram $(\Gamma, \langle\gamma\rangle)$ with $\gamma = \{s_1^1, s_1^2\}$

FIGURE 15. Three non-constant diagrams in the new notation

In pictures we mark the loops $\langle\gamma\rangle$ by a bar at the start of the first boundary segment contained in the loop and a bar at the end of the last boundary segment (so it is the picture of $(\Gamma, \gamma_1, \ldots, \gamma_{n_1})$ with extra bars in there). Examples are given in Figure 15.

Corollary 1.17. *We have a splitting*

$$l\mathcal{D}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}],[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}]) \cong l\mathcal{D}^{cst}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}],[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}]) \oplus l\mathcal{D}^{>0}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}],[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}]).$$

The part of the differential on $l\mathcal{D}^{>0}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}],[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}])$ coming from a boundary map contracting a loop to a constant loop is trivial.

The above splitting induces an isomorphism of chain complexes $l\mathcal{D}^{>0}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}],[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}]) \cong l\mathcal{D}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}],[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}])/l\mathcal{D}^{cst}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}],[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}])$ under which the class of $(\Gamma, \langle\gamma_1\rangle, \ldots, \langle\gamma_{n_1}\rangle)$ is equivalent to the class of $(\Gamma, \gamma_1, \ldots, \gamma_{n_1})$. One might want to use the second one to compute the composition of two elements in $l\mathcal{D}^{>0}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}],[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}])$. Unfortunately, the partly constant terms which get subtracted in the definition of $(\Gamma, \langle\gamma_1\rangle, \ldots, \langle\gamma_{n_1}\rangle)$ can contribute non-trivially to the composition. In the next part of the section we provide conditions under which this phenomenon cannot occur.

To do so, we introduce a bit of notation:

We call a white vertex v_i in a commutative Sullivan diagram Γ *singular* if it is of degree zero, i.e. there is only one boundary segment.

For a looped diagram $x = (\Gamma, \gamma_1, \cdots, \gamma_{n_1})$ and a singular vertex v_i we write $x \backslash s_1^i := (\Gamma, \gamma_1 \backslash s_1^i, \cdots, \gamma_{n_1} \backslash s_1^i)$, where $\gamma_j \backslash s_j^i$ is the loop without the boundary segment s_j^i. An example is given in Figure 16.

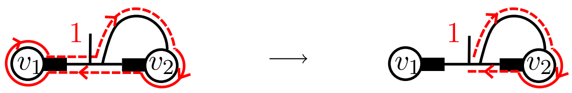


FIGURE 16. v_1 singular and $v_1\backslash\{s_1^1\}$

For $x = (\Gamma, \gamma_1, \ldots, \gamma_{n_1}) \in l\mathcal{D}([{}^{n_1}_{m_1}], [{}^{n_2}_{m_2}])$ denote the set of singular vertices of Γ by $S_x \subseteq \{1, \cdots, n_2\}$ and define $s_T(x) = (x\backslash T)$ for $T \subseteq S_x$. One checks that for $T \subseteq S_x$ we have $p_{cst}(s_T(x)) = s_T(p_{cst}(x))$ and thus $s_T(x - p_{cst}(x)) = (\mathrm{id} - p_{cst})(s_T(x))$ hence $s_T : l\mathcal{D}^{>0}([{}^{n_1}_{m_1}], [{}^{n_2}_{m_2}]) \to l\mathcal{D}^{>0}([{}^{n_1}_{m_1}], [{}^{n_2}_{m_2}])$.

For a looped diagram $x = (\Gamma, \langle\gamma_1\rangle, \ldots, \langle\gamma_n\rangle) \in l\mathcal{D}^{>0}([{}^{n_1}_{m_1}], [{}^{n_2}_{m_2}])$ we call a subset T of the singular vertices S_x *loop-covering* if there is at least one loop γ_i that only consists of boundary segments belonging to the white vertices in T (equivalently, $\gamma\backslash T = cst$). A singular white vertex v_i is called loop-covering, if $T = \{v_i\}$ is-loop covering. Note that if all vertices are loop-covering, also all subsets $T \subseteq S_x$ are loop-covering. The white vertex in Figure 15(a) is not singular. In Figure 15(b) the vertex v_1 is singular, but not loop-covering. In Figure 15(c) both vertices v_1 and v_2 are singular, but none of them is loop-covering. However, the set $T = \{v_1, v_2\}$ is loop-covering.

We define

$$\overline{s}(x) = \sum_{\substack{T \subseteq S_x \\ T \text{ not loop-covering}}} (-1)^{|T|} s_T(x).$$

Note that if for $x \in l\mathcal{D}^{>0}([{}^{n_1}_{m_1}], [{}^{n_2}_{m_2}])$ all singular vertices are loop-covering, then $\overline{s}(x) = x$.

For elements $x = (\Gamma, \langle\gamma_1\rangle, \ldots, \langle\gamma_{n_1}\rangle) \in l\mathcal{D}^{>0}([{}^{n_1}_{m_1}], [{}^{n_2}_{m_2}])$ and $y = (\Gamma', \langle\gamma'_1\rangle, \cdots, \langle\gamma'_{n_2}\rangle) \in l\mathcal{D}^{>0}([{}^{n_2}_{m_2}], [{}^{n_3}_{m_3}])$ we denote their lifts by $\widehat{x} = (\Gamma, \gamma_1, \cdots, \gamma_{n_1}) \in l\mathcal{D}([{}^{n_1}_{m_1}], [{}^{n_2}_{m_2}])$ and $\widehat{y} = (\Gamma', \gamma'_1, \cdots, \gamma'_{n_2}) \in l\mathcal{D}([{}^{n_2}_{m_2}], [{}^{n_3}_{m_3}])$. Suppose their composition in $l\mathcal{D}$ is given by $\widehat{y} \circ \widehat{x} = \sum([G], g_1, \cdots, g_{n_1})$. Then we define

$$y \,\tilde{\circ}\, x := \widehat{y} \circ \widehat{x} - p_{cst}(\widehat{y} \circ \widehat{x}) = \sum([G], \langle g_1\rangle, \cdots, \langle g_{n_1}\rangle).$$

This is not a chain map, but it is not far from being one as in most cases it agrees with the actual composition $y \circ x$. More precisely, we get:

Proposition 1.18. *For elements $x = (\Gamma, \langle\gamma_1\rangle, \ldots, \langle\gamma_{n_1}\rangle) \in l\mathcal{D}^{>0}([{}^{n_1}_{m_1}], [{}^{n_2}_{m_2}])$ and $y = (\Gamma', \langle\gamma'_1\rangle, \cdots, \langle\gamma'_{n_2}\rangle) \in l\mathcal{D}^{>0}([{}^{n_2}_{m_2}], [{}^{n_3}_{m_3}])$ their composition in terms of the above notation is given by*

$$y \circ x = y \,\tilde{\circ}\, \overline{s}(x).$$

In particular, $y \circ x \in l\mathcal{D}^{>0}([{}^{n_1}_{m_1}], [{}^{n_3}_{m_3}])$.

Proof. Using the definition of x and y, we need to show that

$$(\widehat{y} - p_{cst}(\widehat{y})) \circ (\widehat{x} - p_{cst}(\widehat{x})) = \widehat{y} \circ \widehat{\overline{s}(x)} - p_{cst}(\widehat{y} \circ \widehat{\overline{s}(x)})$$

Let $T \subseteq S_x$ be a subset that is not loop-covering. We claim that $\widehat{s_T(x)} = s_T(\widehat{x})$. To see so, recall that x takes a representation of $\widehat{x}$ as a linear combination of looped diagrams and throws away the partly constant ones. Moreover, $x = \widehat{x} - p_{cst}(\widehat{x})$. We need to see that the partly constant terms of $x\backslash T = \widehat{x}\backslash T - p_{cst}(\widehat{x})\backslash T$ are exactly given by $p_{cst}(\widehat{x})\backslash T$. It is clear that all terms in $p_{cst}(\widehat{x})\backslash T$ are still partly constant. Moreover, by the assumption that no loop of x (and thus no loop of $\widehat{x}$) is completely covered by T, the diagram $\widehat{x}\backslash T$ cannot be partly constant, which shows the claim.

Now, the above formula is equivalent to showing that

$$\sum_{T \subseteq \{1,\cdots,n_2\}} \sum_{U \subseteq \{1,\cdots,n_1\}} p_T(\widehat{y}) \circ p_U(\widehat{x}) = \sum_{\substack{T \subseteq S_x \\ T \text{ not loop-covering}}} \sum_{U \subseteq \{1,\cdots,n_1\}} p_U(\widehat{y} \circ s_T(\widehat{x})).$$

The proposition follows via the following steps which hold for general elements $a = (\Lambda, \lambda_1, \cdots, \lambda_{n_1}) \in l\mathcal{D}([{n_1 \atop m_1}], [{n_2 \atop m_2}])$ and $b = (\Lambda', \lambda'_1, \cdots, \lambda'_{n_2}) \in l\mathcal{D}([{n_2 \atop m_2}], [{n_3 \atop m_3}])$:

(1) For $U \subseteq \{1, \cdots, n_1\}$ we have $p_U(a \circ b) = a \circ p_U(b)$.
(2) The singular vertices of a and $p_T(a)$ agree. For any sets $T \subseteq S_a$ and $U \subseteq \{1, \cdots, n_1\}$ the equality $s_T(p_U(a)) = p_U(s_T(a))$ holds, since removing first part of a loop via a singular vertex and then the whole loop commutes with first removing the whole loop and then everything else at the white vertex.
(3) We have $p_T(b) \circ a = 0$ for $T \not\subseteq S_a$, since then we have a vertex with more than one edge glued to a constant loop.
(4) We have $b \circ s_T(a) = p_T(b) \circ a$ for $T \subseteq S_a$ since in $p_T(b) \circ a$ all the loops around the singular vertices in T become constant (because we removed them in b) and this is the same as first removing them and then gluing them onto b.
(5) For $T \subseteq S_a$ such that T is loop-covering, we have $\sum_{U \subseteq \{1, \cdots, n_1\}} (-1)^{|U|} s_T(p_U(a)) = 0$. To see so, assume that T covers the loop γ_j of a, i.e. γ_j only consists of boundary segments of white vertices belonging to T. For $U \subseteq \{1, \cdots, n_1\}$ with $j \notin U$, we get $s_T(p_U(x)) = s_T(p_{U \cup \{j\}}(x))$, which implies the above claim.

Plugging this in, we obtain

$$\begin{aligned}
\sum_{T \subseteq \{1, \cdots, n_2\}} \sum_{U \subseteq \{1, \cdots, n_1\}} p_T(\widehat{y}) \circ p_U(\widehat{x}) &\overset{(3)}{=} \sum_{T \subseteq S_x} \sum_{U \subseteq \{1, \cdots, n_1\}} p_T(\widehat{y}) \circ p_U(\widehat{x}) \\
&\overset{(4)}{=} \sum_{T \subseteq S_x} \sum_{U \subseteq \{1, \cdots, n_1\}} \widehat{y} \circ s_T(p_U(\widehat{x})) \\
&\overset{(5)}{=} \sum_{\substack{T \subseteq S_x \\ T \text{ not loop-covering}}} \sum_{U \subseteq \{1, \cdots, n_1\}} \widehat{y} \circ s_T(p_U(\widehat{x})) \\
&\overset{(2)}{=} \sum_{\substack{T \subseteq S_x \\ T \text{ not loop-covering}}} \sum_{U \subseteq \{1, \cdots, n_1\}} \widehat{y} \circ p_U(s_T(\widehat{x})) \\
&\overset{(1)}{=} \sum_{\substack{T \subseteq S_x \\ T \text{ not loop-covering}}} \sum_{U \subseteq \{1, \cdots, n_1\}} p_U(\widehat{y} \circ s_T(\widehat{x}))
\end{aligned}$$

and thus the proposition is proven.

□

Corollary 1.19. *Let $x \in l\mathcal{D}^{>0}([{n_1 \atop m_1}], [{n_2 \atop m_2}])$ and $y \in l\mathcal{D}^{>0}([{n_2 \atop m_2}], [{n_3 \atop m_3}])$ and assume that all singular vertices in x are loop-covering. Then their composition is given by*

$$y \circ x = y \,\tilde{\circ}\, x.$$

This holds in particular if x has no singular vertices.

1.4. **The type of a diagram and products of diagrams.** In this section we provide a finer decomposition of $l\mathcal{D}([{n_1 \atop m_1}], [{n_2 \atop m_2}])$ on the level of abelian groups. This part is particularly technical and we invite the reader to skip it and come back later if needed.

As already described above, we can concatenate subloops in a commutative Sullivan diagram which start at the same labeled leaf. We denote the concatenation of two loops γ and γ' by $\gamma * \gamma'$. For a commutative Sullivan diagram Γ with t_j loops γ_j^i for $1 \leq i \leq t_j$ starting at the j-th labeled leaf for $1 \leq j \leq n$, we define the element $(\Gamma, \langle \gamma_1^1, \ldots, \gamma_1^{t_1} \rangle, \ldots, \langle \gamma_n^1, \ldots, \gamma_n^{t_n} \rangle) \in l\mathcal{D}([{n_1 \atop m_1}], [{n_2 \atop m_2}])$ as

$$(\Gamma, \langle \gamma_1^1, \ldots, \gamma_1^{t_1} \rangle, \ldots, \langle \gamma_n^1, \ldots, \gamma_n^{t_n} \rangle) := \sum_{U_1 \subseteq \{1, \ldots, t_1\}} \cdots \sum_{U_n \subseteq \{1, \ldots, t_n\}} (-1)^{\sum_j (t_j - |U_j|)} (\Gamma, \gamma_1^{*U_1}, \ldots, \gamma_n^{*U_n})$$

where we define $\gamma_j^{*U_j} = \gamma_j^{u_j^1} * \cdots * \gamma_j^{u_j^{|U_j|}}$ if $U_j = \{u_j^1, \cdots, u_j^{|U_j|}\}$ is non-empty and $\gamma_j^{*\emptyset} = cst$ the constant loop if U_j is empty.

Note that we can assume that all the γ_j's are not constant since otherwise the element $(\Gamma, \langle\gamma_1^1, \ldots, \gamma_1^{t_1}\rangle, \ldots, \langle\gamma_n^1, \ldots, \gamma_n^{t_n}\rangle)$ is zero. Moreover if all t_j were 1 we recover the definition of the non-constant diagrams $l\mathcal{D}^{>0}$ used in the previous section. Furthermore, if at least one of the t_i is zero, then $(\Gamma, \langle\gamma_1^1, \ldots, \gamma_1^{t_1}\rangle, \ldots, \langle\gamma_n^1, \ldots, \gamma_n^{t_n}\rangle)$ is a partially constant diagram.

In the pictures we draw the diagram $(\Gamma, \langle\gamma_1^1, \ldots, \gamma_1^{t_1}\rangle, \ldots, \langle\gamma_n^1, \ldots, \gamma_n^{t_n}\rangle)$ like the diagram $(\Gamma, \gamma_1^1 * \cdots * \gamma_1^{t_1}, \cdots, \gamma_n^1 * \cdots * \gamma_n^{t_n})$ but adding bars at the end of each of the loops γ_i^j (for an example see Figure 17).

(a) The diagram $(\Gamma, \langle\gamma^1, \gamma^2\rangle)$ with $\gamma^1 = \{s_1^1, s_2^1\}$ and $\gamma^2 = \{s_3^1\}$

(b) The diagram $(\Gamma, \langle\lambda^1, \lambda^2, \lambda^3\rangle)$ with $\lambda^1 = \{s_1^1\}$, $\lambda^2 = \{s_2^1\}$ and $\lambda^3 = \{s_3^1\}$

FIGURE 17. Two elements of $lD(\left[{1 \atop 0}\right], \left[{1 \atop 0}\right])$ depicted in the new notation

Now we restrict to those diagrams where all the subloops γ_i^j are irreducible, i.e. cannot be written as concatenation of non-constant loops.

Definition 1.20. An *irreducible* $\left[{p \atop (m,n)}\right]$*-looped diagram* is given by an element of the form $(\Gamma, \langle\gamma_1^1, \ldots, \gamma_1^{t_1}\rangle, \ldots, \langle\gamma_n^1, \ldots, \gamma_n^{t_n}\rangle)$ such that all the loops γ_i^j are irreducible. The *type* of an irreducible looped diagram $(\Gamma, \langle\gamma_1^1, \ldots, \gamma_1^{t_1}\rangle, \ldots, \langle\gamma_n^1, \ldots, \gamma_n^{t_n}\rangle)$ is defined to be the tuple $(t_1, \cdots, t_n)$.

The abelian group spanned by irreducible looped diagrams of type $(t_1, \cdots, t_n)$ is denoted by $l\mathcal{D}^{t_1,\ldots,t_{n_1}}(\left[{n_1 \atop m_1}\right], \left[{n_2 \atop m_2}\right])$.

The type of a looped diagram is not preserved by the differential, hence the groups $l\mathcal{D}^{t_1,\ldots,t_{n_1}}(\left[{n_1 \atop m_1}\right], \left[{n_2 \atop m_2}\right])$ are in general not chain complexes.

One checks that every $\left[{p \atop (m,n)}\right]$-looped diagram can be written as the linear combination of irreducible $\left[{p \atop (m,n)}\right]$-looped diagrams and vice versa. Hence we can rewrite the complex of looped diagrams as

$$l\mathcal{D}(\left[{n_1 \atop m_1}\right], \left[{n_2 \atop m_2}\right]) \cong \bigoplus_{t_1,\cdots,t_{n_1}} l\mathcal{D}^{t_1,\ldots,t_{n_1}}(\left[{n_1 \atop m_1}\right], \left[{n_2 \atop m_2}\right])$$

with

$$l\mathcal{D}^{cst}(\left[{n_1 \atop m_1}\right], \left[{n_2 \atop m_2}\right]) \cong \bigoplus_{\substack{t_1,\cdots,t_{n_1} \\ \exists\, j \text{ s.t. } t_j=0}} l\mathcal{D}^{t_1,\ldots,t_{n_1}}(\left[{n_1 \atop m_1}\right], \left[{n_2 \atop m_2}\right])$$

and

$$l\mathcal{D}^{>0}(\left[{n_1 \atop m_1}\right], \left[{n_2 \atop m_2}\right]) \cong \bigoplus_{\substack{t_1,\cdots,t_{n_1} \\ t_j>0 \text{ for all } j}} l\mathcal{D}^{t_1,\ldots,t_{n_1}}(\left[{n_1 \atop m_1}\right], \left[{n_2 \atop m_2}\right]).$$

Similarly to working out the composition for positive diagrams explicitly, we want to say a few words about the composition of irreducible looped diagrams. For x and y two irreducible looped diagrams, in $y \circ x$ all old loops of y are not allowed to be empty. This means that we either have to glue an edge in there or they afterward have to be covered by a loop again. Moreover, if they were covered but no edge was glued, the diagram where this subloop is omitted has to be subtracted. For an example see Figure 18. Instead of taking the direct sum over all types of irreducible looped diagrams (which as explained just is the complex of looped diagrams) we want to take the product. Unfortunately, in the product complex over all types composition is not always well-defined. Nevertheless, we will use this complex and later on deal with composition.

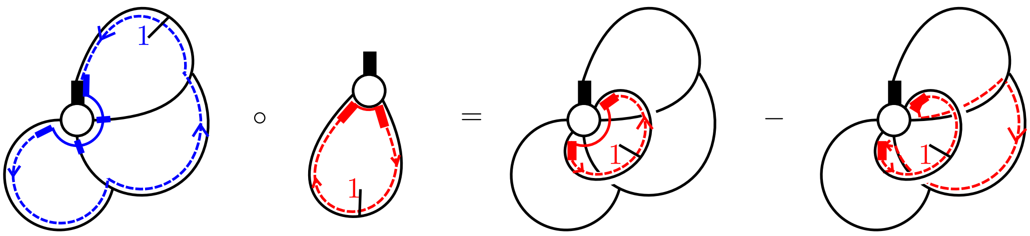


FIGURE 18. composition of irreducible looped diagrams

Definition 1.21. For $n_1 > 0$ the chain complex of *products of irreducible looped diagrams* $il\mathcal{D}([\substack{n_1 \\ m_1}], [\substack{n_2 \\ m_2}])$ is defined as

$$il\mathcal{D}([\substack{n_1 \\ m_1}], [\substack{n_2 \\ m_2}]) = \prod_{t_1, \cdots, t_{n_1}} l\mathcal{D}^{t_1, \ldots, t_{n_1}}([\substack{n_1 \\ m_1}], [\substack{n_2 \\ m_2}])$$

and

$$il\mathcal{D}^{>0}([\substack{n_1 \\ m_1}], [\substack{n_2 \\ m_2}]) = \prod_{t_i > 0} l\mathcal{D}^{t_1, \ldots, t_{n_1}}([\substack{n_1 \\ m_1}], [\substack{n_2 \\ m_2}])$$

Similarly, we define $il\mathcal{D}_+([\substack{n_1 \\ m_1}], [\substack{n_2 \\ m_2}])$, $ipl\mathcal{D}([\substack{n_1 \\ m_1}], [\substack{n_2 \\ m_2}])$ and $ipl\mathcal{D}_+([\substack{n_1 \\ m_1}], [\substack{n_2 \\ m_2}])$ as the products over all types restricted to these subcomplexes.

We see that

$$il\mathcal{D}([\substack{n_1 \\ m_1}], [\substack{n_2 \\ m_2}]) \cong l\mathcal{D}^{cst}([\substack{n_1 \\ m_1}], [\substack{n_2 \\ m_2}]) \oplus il\mathcal{D}^{>0}([\substack{n_1 \\ m_1}], [\substack{n_2 \\ m_2}]).$$

If $n_1 = 0$, we obtain

$$il\mathcal{D}([\substack{0 \\ m_1}], [\substack{n_2 \\ m_2}]) = ipl\mathcal{D}([\substack{0 \\ m_1}], [\substack{n_2 \\ m_2}]) = l\mathcal{D}([\substack{0 \\ m_1}], [\substack{n_2 \\ m_2}]).$$

In order to see that the differential on $il\mathcal{D}$ and the other product complexes is well-defined, we need to check that for a fixed type $(t_1, \ldots, t_n)$ there can only be finitely many types $(t'_1, \ldots, t'_n)$ such that the differential has non-trivial elements of type $(t_1, \ldots, t_n)$. However, one checks similarly to the computations done for non-constant diagrams that the differential is zero if an irreducible loop gets contracted and hence in the resulting summands of the differential there are at least as many irreducible loops as before. Therefore, the differential of a looped diagram of type $(t'_1, \ldots, t'_n)$ has summands of type $(t_1, \ldots, t_n)$ only if $t'_i \leq t_i$. Thus, on the product over all types, the differential is still welldefined.

Definition 1.22. For two elements $a = \sum_{t_1=1}^{\infty} \cdots \sum_{t_{n_1}=1}^{\infty} a_{t_1,\dots,t_{n_1}} \in il\mathcal{D}([{n_1 \atop m_1}],[{n_2 \atop m_2}])$ and $b = \sum_{t_1=1}^{\infty} \cdots \sum_{t_{n_2}=1}^{\infty} b_{t_1,\dots,t_{n_2}} \in il\mathcal{D}([{n_2 \atop m_2}],[{n_3 \atop m_3}])$ the pair (a,b) is called *composable* if

$$\sum_{t_1=1}^{\infty} \cdots \sum_{t_{n_1}=1}^{\infty} \sum_{t'_1=1}^{\infty} \cdots \sum_{t'_{n_2}=1}^{\infty} b_{t'_1,\dots,t'_{n_2}} \circ a_{t_1,\dots,t_{n_1}}$$

only contains finitely many summands of type $(u_1,\dots,u_{n_1})$ for arbitrary $u_i \in \mathbb{N}$.

Definition 1.23. Let $pl\mathcal{D}_{start}$ consist of those graphs in $pl\mathcal{D}$, where all loops consist of exactly one boundary segment of a white vertex which is the first boundary segment of that white vertex.

Proposition 1.24. *Let $a \in il\mathcal{D}([{n_1 \atop m_1}],[{n_2 \atop m_2}])$ and $b \in il\mathcal{D}([{n_2 \atop m_2}],[{n_3 \atop m_3}])$. If one of the following conditions holds, the pair (a,b) is composable:*

1. *$n_1 = 0$, i.e. $a \in l\mathcal{D}([{0 \atop m_1}],[{n_2 \atop m_2}])$ and b arbitrary,*
2. *b lies in the direct sum complex, i.e. $b \in l\mathcal{D}([{n_2 \atop m_2}],[{n_3 \atop m_3}])$ and a is arbitrary,*
3. *we have $a \in ipl\mathcal{D}_{start}([{n_1 \atop m_1}],[{n_2 \atop m_2}])$ and b arbitrary.*

Note that for $a \in l\mathcal{D}([{n_1 \atop m_1}],[{n_2 \atop m_2}])$ and $b \in l\mathcal{D}([{n_2 \atop m_2}],[{n_3 \atop m_3}])$ their composition is just the composition $b \circ a$ by definition.

Proof. If $n_1 = 0$ by definition a is contained in $l\mathcal{D}([{0 \atop m_1}],[{n_2 \atop m_2}])$ and thus it is a finite sum of diagrams in this complex. It is sufficient to show that for x a looped diagram in $l\mathcal{D}([{0 \atop m_1}],[{n_2 \atop m_2}])$ and $y \in l\mathcal{D}([{n_2 \atop m_2}],[{n_3 \atop m_3}])$ there are only finitely many types $(t'_1,\dots,t'_{n_2})$ the diagram y can have such that the composition $y \circ x$ is non-trivial. We have $(\sum_{i=1}^{n_2} t'_i)$ irreducible loops in y. By the observations we made earlier, we know, that we either have to glue an edge into each of these loops or cover it with a loop of x. Since x does not have any loops, we have to glue at least one edge of x in into each irreducible loop for the composition to be non-trivial. There are only $|x|$ edges which get glued to y, thus the composition is trivial whenever $(\sum_{i=1}^{n_2} t'_i) > |x|$.

If $n_1 \neq 0$ we show that for a given type $(t_1,\dots,t_{n_1})$ there are only finitely many types $(t_1,\dots,t_{n_1})$ and $(t'_1,\dots,t'_{n_2})$ such that the compositions $(b_{t'_1,\dots,t'_{n_2}} \circ a_{t_1,\dots,t_{n_1}})$ have summands of type $(u_1,\dots,u_{n_1})$. In the second case by assumption only finitely many types occur in b, i.e. we only need to show that for an arbitrary looped diagram y there are only finitely many types a looped diagram x can have, such that $y \circ x$ has type $(u_1,\dots,u_{n_1})$. However, for a looped diagram x of type $(t_1,\dots,t_{n_1})$ the type of the composition $y \circ x$ is bounded below by $(t_1,\dots,t_n)$. Therefore, in the $(u_1,\dots,u_{n_1})$ component of the composition, we can only have elements resulting from the composition of $a_{t_1,\dots,t_{n_1}}$ with $t_i \leq u_i$.

The proof of the last case is particularly technical and difficult to explain. Since the fact is not used later on, we omit the proof. $\square$

The later proposition and the associativity of $\circ$ imply that the union of morphism spaces $\coprod_{n_1,m_1} il\mathcal{D}([{n_1 \atop m_1}],[{n_2 \atop m_2}])$ is a left $ipl\mathcal{D}_{start}$-module and $\coprod_{n_2,m_2} il\mathcal{D}([{n_1 \atop m_1}],[{n_2 \atop m_2}])$ a right $l\mathcal{D}$-module.

2. The natural operations for commutative Frobenius algebras

2.1. The category of commutative Frobenius algebras.

This paper deals with commutative Frobenius algebras, so we start with recalling definitions on the subject.

Definition 2.1. A *Frobenius algebra A* is given by an abelian group equipped with the following data:

- a multiplication $m : A \otimes A \to A$ and a unit $\mathbb{1}_A : \mathbb{Z} \to A$ such that m and $\mathbb{1}_A$ define an algebra structure on A

- a comultiplication $\Delta : A \to A \otimes A$ and a counit $\eta : A \to \mathbb{Z}$ such that they define a coalgebra structure on A

satisfying the so called Frobenius relation $\Delta \circ m = (m \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta) = (\mathrm{id} \otimes m) \circ (\Delta \otimes \mathrm{id})$. If A is a chain complex, we obtain *Frobenius dg-algebras.*

We denote the twist map $A \otimes A \to A \otimes A$ by τ.

A Frobenius algebra is called *symmetric* if $\eta \circ m \circ \tau = \eta \circ m$ and it is *commutative* if $m \circ \tau = m$. A commutative Frobenius algebra is cocommutative, i.e. $\tau \circ \Delta = \Delta$.

An *open Frobenius algebra* is a Frobenius algebra without a counit. It is cocommutative if $\tau \circ \Delta = \Delta$. In this case commutativity does not imply cocommutativity (but we will only work with commutative cocommutative open Frobenius algebras).

The open cobordism category $\mathscr{O}$ was defined in [WW11, section 2.6] to be the dg-category with objects the natural numbers and morphism $\mathscr{O}(n,m)$ the chain complex of oriented fat graphs with $n+m$ labeled leaves (for the definition see Section 1.1), i.e. $\mathscr{O}(n,m) = [{}^{\,0}_{m+n}]$–Graphs. The category $sFr = H_0(\mathscr{O})$ is the category with the same objects but with cobordisms $sFr(n,m) = H_0(\mathscr{O})(n,m) := H_0(\mathscr{O}(n,m))$. This chain complex consists of trivalent graphs module the sliding relation (cf. Figure 3), i.e. $sFr(n,m) = [{}^{\,0}_{m+n}]$–Sullivan diagrams, so these are Sullivan diagrams without white vertices. A split symmetric monoidal functor $\Phi : H_0(\mathscr{O}) \to \mathrm{Ch}$, i.e. an sFr-algebra is an open TQFT and by [LP08, Cor 4.5] these algebras are precisely the symmetric Frobenius dg-algebras. Usually one would pass to the closed cobordism category to deal with commutative Frobenius algebras, but we instead want to continue working with graphs (as it is for example done in [Koc04, Chapter 3]). Adding the commutativity relation is equivalent to forgetting the ordering of the edges at the vertices. Thus the PROP cFr of commutative Frobenius algebras can be defined to have objects the natural numbers and morphisms $cFr(m_1,m_2) = l\mathcal{D}([{}^{\,0}_{m_1}],[{}^{\,0}_{m_2}]) = H_0(\mathscr{O})(m_1,m_2)/\sim = [{}^{\;\;0}_{m_1+m_2}]$–commutative Sullivan diagrams. So a commutative Frobenius dg-algebra is a strong symmetric monoidal functor from cFr to chain complexes. Forgetting the counit is equivalent to restricting to diagrams with the positive boundary condition (i.e. forcing every connected component to have an output). Thus we can define the PROP cFr_+ of commutative cocommutative open Frobenius algebras to have morphism spaces $cFr_+(m_1,m_2) = l\mathcal{D}_+([{}^{\,0}_{m_1}],[{}^{\,0}_{m_2}])$. Hence a commutative cocommutative open dg-Frobenius algebra is a strong symmetric monoidal functor $cFr_+ \to \mathrm{Ch}$.

Moreover, we also have a graded version of commutative Frobenius algebras where the comultiplication has degree d and the counit has degree $-d$. Analogously to [WW11, Section 6.3] for symmetric Frobenius algebras, the shifted PROP cFr_d then agrees with the PROP where we shifted the commutative Sullivan diagrams Γ by $-d \cdot \chi(\Gamma, \partial_{out})$ (as defined in the end of Section 1.2), i.e. $cFr_d(m_1,m_2) = l\mathcal{D}_d([{}^{\,0}_{m_1}],[{}^{\,0}_{m_2}])$.

2.2. **Formal operations.**

2.2.1. *Definitions of the Hochschild complex and formal operations.* Let $\mathcal{E}$ be a PROP with a multiplication, i.e. a dg-PROP with a functor $Ass \to \mathcal{E}$ which is the identity on objects. We define $m^k_{i,j} \in \mathcal{E}(k,k-1)$ to be the image of the map in $Ass(k,k-1)$ which multiplies the i–th and j–th input and is the identity on all other elements.

We recall the definitions of the Hochschild and coHochschild constructions of functors from [Wah12, Section 1].

For $\Phi : \mathcal{E} \to \mathrm{Ch}$ a dg-functor the *Hochschild complex* of Φ is the functor $C(\Phi) : \mathcal{E} \to \mathrm{Ch}$ defined by

$$C(\Phi)(n) = \bigoplus_{k \geq 1} \Phi(k+n)[k-1].$$

The differential is the total differential of the differential on Φ and the differential coming from the simplicial abelian group structure with boundary maps $d_i = \Phi(m^k_{i+1,i+2} + \mathrm{id}_n)$

where we set $m^k_{k,k+1} = m^k_{k,1}$ and degeneracy maps induced by the map inserting a unit at the $i + 1$–st position.

The *reduced Hochschild complex* $\overline{C}(\Phi)(n)$ is the reduced chain complex associated to this simplicial abelian group, i.e. the quotient by the image of the degeneracies.

Iterating this construction, the functors $C^{(n,m)}(\Phi)$ and $\overline{C}^{(n,m)}(\Phi)$ are given by

$$C^{(n,m)}(\Phi) := C^n(\Phi)(m) \qquad \text{and} \qquad \overline{C}^{(n,m)}(\Phi) := \overline{C}^n(\Phi)(m).$$

Working out the definitions explicitly, one sees that

$$C^{(n,m)}(\Phi) \cong \bigoplus_{j_1 \geq 1, \cdots, j_n \geq 1} \Phi(j_1 + \cdots + j_n + m)[j_1 + \cdots + j_n - n].$$

The category of $\mathcal{E}$–algebras is equivalent to strong symmetric monoidal functors $\Phi : \mathcal{E} \to \mathrm{Ch}$, sending an algebra A to the functor $A^{\otimes -}$. For an algebra A, the Hochschild complex $C(A^{\otimes -})(0)$ is the ordinary Hochschild complex $C_*(A, A)$ (and similarly for the reduced complexes). Furthermore, we have an isomorphism

$$C^{(n,m)}(A^{\otimes -}) \cong C_*(A, A)^{\otimes n} \otimes A^{\otimes m} \tag{2.1}$$

natural in all $\mathcal{E}$–algebras A.

Dually, given a dg-functor $\Psi : \mathcal{E}^{op} \to \mathrm{Ch}$ its *CoHochschild complex* is defined as

$$D(\Psi(n)) = \prod_{k \geq 1} \Psi(k + n)[1 - k]$$

with the differential coming from the cosimplicial structure induced by the multiplications and the inner differential on Ψ. Again, we can take the reduced cochain complex $\overline{D}(\Psi)(n)$. By [Wah12, Prop. 1.7 + 1.8], the inclusion $\overline{D}(\Psi) \to D(\Psi)$ and the projection $C(\Phi) \to \overline{C}(\Psi)$ are quasi-isomorphisms.

We can also spell out the iterated construction explicitly, i.e. for a functor $\Psi : \mathcal{E}^{op} \to \mathrm{Ch}$ we get

$$D^n(\Psi)(m) \cong \prod_{j_1, \cdots, j_n} \Psi(j_1 + \cdots + j_n + m)[n - (j_1 + \cdots + j_n)].$$

The complex of formal operations $\mathrm{Nat}_{\mathcal{E}}([\substack{n_1 \\ m_1}], [\substack{n_2 \\ m_2}])$ is defined as all maps

$$C^{(n_1,m_1)}(\Phi) \to C^{(n_2,m_2)}(\Phi)$$

natural in all functors $\Phi : \mathcal{E} \to \mathrm{Ch}$.

In [Wah12, Theorem 2.1] it is shown that

$$\mathrm{Nat}_{\mathcal{E}}([\substack{n_1 \\ m_1}], [\substack{n_2 \\ m_2}]) \cong D^{n_1} C^{n_2}(\mathcal{E}(-, -))(m_2)(m_1),$$

which is used to compute the complex of formal operations explicitly.

Instead of testing on all functors $\Phi : \mathcal{E} \to \mathrm{Ch}$ we could test on strong symmetric monoidal functors only and denote the operations obtained this way by $\mathrm{Nat}^{\otimes}_{\mathcal{E}}([\substack{n_1 \\ m_1}], [\substack{n_2 \\ m_2}])$. Via the isomorphism in equation (2.1) a transformation in $\mathrm{Nat}^{\otimes}_{\mathcal{E}}([\substack{n_1 \\ m_1}], [\substack{n_2 \\ m_2}])$ corresponds to an operation

$$C_*(A, A)^{\otimes n_1} \otimes A^{\otimes m_1} \to C_*(A, A)^{\otimes n_2} \otimes A^{\otimes m_2}$$

natural in all $\mathcal{E}$–algebras A, so in other words it is a natural transformation of the Hochschild complex. Since every transformation in $\mathrm{Nat}_{\mathcal{E}}([\substack{n_1 \\ m_1}], [\substack{n_2 \\ m_2}])$ is in particular natural in all strong symmetric monoidal functors, we have a restriction map $\rho : \mathrm{Nat}_{\mathcal{E}}([\substack{n_1 \\ m_1}], [\substack{n_2 \\ m_2}]) \to \mathrm{Nat}^{\otimes}_{\mathcal{E}}([\substack{n_1 \\ m_1}], [\substack{n_2 \\ m_2}])$, so every formal operation gives us a natural operations of the Hochschild complex of $\mathcal{E}$–algebras. In general we do not know whether this map is injective or surjective (for more details on this matter see [Wah12, Section 2.2]).

2.2.2. *Formal operations for commutative Frobenius algebras.* We now focus on the case where $\mathcal{E} = cFr$. Using the definitions of the previous section, we can describe the complexes $\overline{C}^n(cFr(m_1,-))(m_2)$ and $\overline{C}^n(cFr_+(m_1,-))(m_2)$ as follows:

Lemma 2.2. *There are isomorphisms*

$$\overline{C}^n(cFr(m_1,-))(m_2) \cong l\mathcal{D}([{}^{0}_{m_1}],[{}^{n}_{m_2}]) = [{}_{m_1+m_2}^{n}] - cSD$$

and

$$\overline{C}^n(cFr_+(m_1,-))(m_2) \cong l\mathcal{D}_+([{}^{0}_{m_1}],[{}^{n}_{m_2}]).$$

This is a direct analog of [WW11, Lemma 6.1] in the commutative setting and the proof works completely similar.

Applying the coHochschild construction n_1 times, we can describe the formal operations for cFr and cFr_+ via

$$\begin{aligned}\overline{\mathrm{Nat}}_{cFr}([{}^{n_1}_{m_1}],[{}^{n_2}_{m_2}]) &\simeq D^{n_1}(\overline{C}^{n_2}(cFr(-,-)(m_2))(m_1)\\ &\cong \prod_{j_1,\cdots,j_{n_1}} l\mathcal{D}([{}_{j_1+\cdots+j_{n_1}+m_1}^{0}],[{}^{n_2}_{m_2}])[n_1-\Sigma j_i]\end{aligned}$$

and

$$\overline{\mathrm{Nat}}_{cFr_+}([{}^{n_1}_{m_1}],[{}^{n_2}_{m_2}]) \simeq \prod_{j_1,\cdots,j_{n_1}} l\mathcal{D}_+([{}_{j_1+\cdots+j_{n_1}+m_1}^{0}],[{}^{n_2}_{m_2}])[n_1-\Sigma j_i].$$

Since every commutative Frobenius algebra is in particular a commutative cocommutative open Frobenius algebra, we have an induced inclusion $\overline{\mathrm{Nat}}_{cFr_+}([{}^{n_1}_{m_1}],[{}^{n_2}_{m_2}]) \hookrightarrow \overline{\mathrm{Nat}}_{cFr}([{}^{n_1}_{m_1}],[{}^{n_2}_{m_2}])$. Under the above equivalences, this inclusion corresponds to the inclusions of the subcomplexes $l\mathcal{D}_+([{}_{j_1+\cdots+j_{n_1}+m_1}^{0}],[{}^{n_2}_{m_2}]) \hookrightarrow l\mathcal{D}([{}_{j_1+\cdots+j_{n_1}+m_1}^{0}],[{}^{n_2}_{m_2}])$.

The composition in $\overline{\mathrm{Nat}}_{cFr}$ (and thus also in $\overline{\mathrm{Nat}}_{cFr_+}$) is described in terms of the right hand side as follows:

For $\Gamma \in \prod_{j_1,\cdots,j_{n_1}} [{}_{j_1+\cdots+j_{n_1}+m_1+m_2}^{n_2}] - cSD$ and $\Gamma' \in \prod_{j_1,\cdots,j_{n_2}} [{}_{j_1+\cdots+j_{n_2}+m_2+m_3}^{n_3}] - cSD$ we get $(\Gamma'\circ\Gamma)_{j_1,\dots,j_{n_1}}$ by attaching a summand G in $(\Gamma)_{j_1,\dots,j_{n_1}}$ which has n_2 white vertices with each $k_1,\dots,k_{n_2}$ half-edges, to the element $(\Gamma')_{k_1,\dots,k_{n_2}}$. This is done by taking away the white vertices from G and gluing the $k_1+\cdots+k_{n_2}$ half-edges onto the according labeled leaves of $(\Gamma')_{k_1,\dots,k_{n_2}}$.

Before we move on, we want to say a few words about how to view an element in $x \in \prod_{j_1,\cdots,j_{n_1}} l\mathcal{D}([{}_{j_1+\cdots+j_{n_1}+m_1}^{0}],[{}^{n_2}_{m_2}])$ as an operation on commutative Frobenius dg-algebras, i.e. how to extract an operation

$$CC_*(A,A)^{\otimes n_1} \otimes A^{m_1} \to CC_*(A,A)^{\otimes n_2} \otimes A^{m_2}.$$

We fix a tuple $(j_1,\cdots,j_{n_1})$ and an element

$$(a_0^1\otimes\cdots\otimes a_{j_1}^1)\otimes\cdots\otimes(a_0^{n_1}\otimes\cdots\otimes a_{j_{n_1}}^{n_1})\otimes b_1\otimes\cdots\otimes b_{m_1}\in CC_{j_1}(A,A)\otimes\cdots\otimes CC_{j_{n_1}}(A,A)\otimes A^{m_1}.$$

To get the resulting element in $CC_*(A,A)^{\otimes n_2}\otimes A^{m_2}$, we need to consider $x_{j_1+1,\cdots,j_{n_1}+1} \in l\mathcal{D}([{}_{j_1+1+\cdots+j_{n_1}+1+m_1}^{0}],[{}^{n_2}_{m_2}])$ which is a linear combination of $[{}_{j_1+1+\cdots+j_{n_1}+1+m_1+m_2}^{n_2}]$–looped diagrams. We start with writing $a_0^1,\cdots,a_{j_1}^1$ on the first j_1+1–leaves and continue by putting the other a_i^k following their order. Then we write the b_k on the leaves labeled $j_1+1+\cdots+j_{n_1}+1+1$ to $j_1+1+\cdots+j_{n_1}+1+m_1$. We put units on all unlabeled leaves. Now we have m_2 labeled leaves where we have not written an element of A on. We view these and the half-edges attached to the white vertices as ends of the graph for a moment. Starting from the leaves labeled with elements of A, we read the black vertices of the diagram as multiplications and comultiplications, i.e. whenever two edges meet the assigned elements are multiplied together and if an edge is split up into two the assigned element is comultiplied. Iterating this until we hit the ends of the graph (the remaining m_2 leaves and all half-edges attached to the white vertices) we obtain a linear combination of the diagram where we assigned values in A to all ends of the graph.

We read off our element in $CC_*(A,A)^{\otimes n_2} \otimes A^{m_2}$ by starting with the white vertices: Each white vertex corresponds to one copy of $CC_*(A,A)$. In the procedure described above, an element of A is assigned to each half-edge attached to a white vertex. If the white vertex has degree k, i.e. $k+1$ edges attached to with labels $c_0, \cdots c_k$, the resulting element lies in $CC_k(A,A)$ and is given by $c_0 \otimes c_k$. The elements assigned to the m_2 leaves give the resulting elements in $A^{\otimes m_2}$.

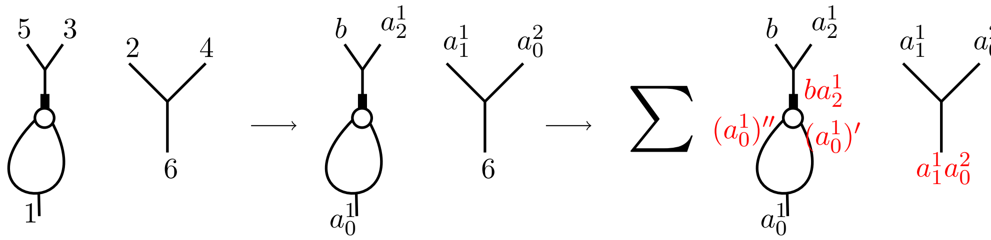


FIGURE 19. How to read off operations

In Figure 19 we have illustrated how to evaluate an element of $l\mathcal{D}([{}^{\;\,0}_{3+1+1}],[{}^1_1])$, as part of an operation in $\mathrm{Nat}_{cFr}([{}^2_1],[{}^1_1])$, on an element $(a_0^1 \otimes a_1^1 \otimes a_2^1) \otimes (a_0^2) \otimes b \in CC_2(A,A) \otimes CC_0(A,A) \otimes A$. The sum in the last picture is the sum coming from the comultiplication of a_0^1 using Sweedler's notation $\Delta(a_0^1) = \sum (a_0^1)' \otimes (a_0^1)''$. The element we read off is $(\sum ba_2^1 \otimes (a_0^1)' \otimes (a_0^1)'') \otimes a_1^1 a_0^2 \in CC_2(A,A) \otimes A$.

2.2.3. *Splitting off zero chains.* It is well-known that for a commutative algebra A the Hochschild chains $C_*(A,A)$ split as $C_0(A,A) \oplus C_{>0}(A,A)$. This relies on the fact that $d(a_0 \otimes a_1) = a_0 \cdot a_1 - (-1)^{|a_0||a_1|} a_1 \cdot a_0 = 0$ by the commutativity of A. It generalizes to the Hochschild complex of functors, since we already get $d_0 = -d_1 \in \mathscr{C}om(2,1)$ and thus $d = 0 \in \mathscr{C}om(2,1)$. Hence for $\Phi : \mathscr{C}om \to \mathrm{Ch}$ the differential on degree one of the Hochschild complex $C_*(\Phi)$ is trivial and we get a splitting $C_*(\Phi) \cong C_0(\Phi) \oplus C_{>0}(\Phi)$. This generalizes to the iterated complex $C^{(n,m)}(\Phi) \cong \bigoplus_{j_1 \geq 1, \cdots, j_n \geq 1} \Phi(j_1 + \cdots + j_n + m)$ which we therefore can rewrite as:

$$C^{(n,m)}(\Phi) \cong \bigoplus_{S \subseteq \{1,\cdots,n\}} \bigoplus_{j_i \geq 2, i \notin S} \Phi(j_1 + \cdots + j_n + m)$$

with $j_i = 1$ if $i \in S$, where the first direct sum is a direct sum of chain complexes.

For each $S \subseteq \{1, \cdots, n\}$ with $|S| = k$ the sum $\bigoplus_{j_i \geq 2, i \notin S} \Phi(j_1 + \cdots + j_n + m)$ with $j_i = 1$ for $i \in S$ is isomorphic to $\bigoplus_{j_{r_i} \geq 2} \Phi(j_{r_1} + \cdots + j_{r_{n-k}} + k + m)$ given by relabeling those j_i with $i \notin S$ to j_{r_l} and moving the j_i with $i = 1$ to the end (with a sign involved). Defining $C_{\mathcal{E}}^{>0,(n,m)} := \bigoplus_{j_1 \geq 2, \cdots, j_n \geq 2} \Phi(j_1 + \cdots + j_n + m)$, we see that

$$\bigoplus_{j_{r_i} \geq 2} \Phi(j_{r_1} + \cdots + j_{r_{n-k}} + k + m) \cong C^{>0,(n-k,m+k)}(\Phi)$$

and hence we get can rewrite it as a direct sum of chain complexes as

$$C^{(n,m)}(\Phi) \cong \bigoplus_{S \subseteq \{1,\cdots,n\}} C^{>0,(n-|S|,m+|S|)}(\Phi).$$

Defining $\mathrm{Nat}_{\mathcal{E}}^{>0}([{}^{n_1}_{m_1}],[{}^{n_2}_{m_2}]) := \hom_{\mathcal{E}}(C^{>0,(n_1,m_1)}(-), C^{(n_2,m_2)}(-))$ we conclude

$$\begin{aligned}
\mathrm{Nat}_{\mathcal{E}}([{}^{n_1}_{m_1}],[{}^{n_2}_{m_2}]) &= \hom_{\mathcal{E}}(C^{(n_1,m_1)}(-), C^{(n_2,m_2)}(-)) \\
&= \bigoplus_{S \subseteq \{1,\cdots,n_1\}} \hom_{\mathcal{E}}(C^{>0,(n_1-|S|,m_1+|S|)}(-), C^{(n_2,m_2)}(-)) \\
&= \bigoplus_{S \subseteq \{1,\cdots,n_1\}} \mathrm{Nat}_{\mathcal{E}}^{>0}([{}^{n_1-|S|}_{m_1+|S|}],[{}^{n_2}_{m_2}]).
\end{aligned}$$

The same works for the reduced Hochschild construction and reduced natural transformations, i.e.

$$\overline{\mathrm{Nat}}_{\mathcal{E}}([{}^{n_1}_{m_1}],[{}^{n_2}_{m_2}]) = \bigoplus_{S\subseteq\{1,\cdots,n_1\}} \overline{\mathrm{Nat}}^{>0}_{\mathcal{E}}([{}^{n_1-|S|}_{m_1+|S|}],[{}^{n_2}_{m_2}]),$$

which again is a splitting on the level of chain complexes. Similarly, for a functor $\Psi : \mathcal{E}^{op} \to \mathrm{Ch}$ we define $D^{>0,n}(\Psi)(m) := \prod_{j_i>2}\Psi(j_1+\cdots+j_n+m)$. Going through the proof of [Wah12, Theorem 2.1] one sees that $\overline{\mathrm{Nat}}^{>0}_{\mathcal{E}}([{}^{n_1}_{m_1}],[{}^{n_2}_{m_2}]) \cong \overline{D}^{>0,n_1}(\overline{C}^{n_2}(\mathcal{E}(-,-)(m_2))(m_1) \simeq D^{>0,n_1}(\overline{C}^{n_2}(\mathcal{E}(-,-)(m_2))(m_1)$.

Using the positive coHochschild construction, we can identify the subcomplex $\mathrm{Nat}^{>0}_{cFr}$ via the following:

$$\overline{\mathrm{Nat}}^{>0}_{cFr}([{}^{n_1}_{m_1}],[{}^{n_2}_{m_2}]) \simeq \prod_{j_1>1,\cdots,j_{n_1}>1} l\mathcal{D}([{}^{0}_{j_1+\cdots+j_{n_1}+m_1}],[{}^{n_2}_{m_2}])[n_1-\Sigma j_i].$$

2.3. **Building operations out of looped diagrams.** We describe a dg-functor from $l\mathcal{D}$ to $\overline{\mathrm{Nat}}_{cFr}$ which is the identity on objects. Thus we assign an operation on commutative Frobenius dg-algebras to every looped diagram. In the two sections afterward we will show that this actually covers interesting operations.

Recall the graph $l_j \in l\mathcal{D}([{}^0_j],[{}^1_0])$ defined in Section 1.1 given by a single white vertex with j leaves attached to it and let $\mathrm{id}_{m_1} \in l\mathcal{D}([{}^{0}_{m_1}],[{}^{0}_{m_1}])$ be the identity element, so we have $(l_{j_1}\amalg\cdots\amalg l_{j_{n_1}}\amalg \mathrm{id}_{m_1})\} \in l\mathcal{D}([{}^{0}_{j_1+\cdots+j_{n_1}+m_1}],[{}^{n_1}_{m_1}])$ of degree $\sum j_i - n_1$.

Theorem 2.3. *There is a functor of dg-categories*

$$J : l\mathcal{D} \to \overline{\mathrm{Nat}}_{cFr}$$

which is the identity on objects and sends a looped diagram $G \in l\mathcal{D}([{}^{n_1}_{m_1}],[{}^{n_2}_{m_2}])$ to $(G\circ(l_{j_1}\amalg\cdots\amalg l_{j_{n_1}}\amalg\mathrm{id}_{m_1}))_{j_1,\cdots,j_{n_1}} \in \prod_{j_1,\cdots,j_{n_1}} l\mathcal{D}([{}^{0}_{j_1+\cdots+j_{n_1}+m_1}],[{}^{n_2}_{m_2}])[n_1-\Sigma j_i]$. The functor restricts to functors

$$l\mathcal{D}_+ \to \overline{\mathrm{Nat}}_{cFr_+}, \qquad l\mathcal{D}^{>0} \to \overline{\mathrm{Nat}}^{>0}_{cFr} \qquad \text{and} \qquad l\mathcal{D}^{>0}_+ \to \overline{\mathrm{Nat}}^{>0}_{cFr_+}.$$

Proof. After assembling everything together, Lemma 2.2 says that $l\mathcal{D}$ is an extension of cFr in the sense of [WW11]. Then the theorem follows by [WW11, Lemma 5.12]. However, one can also directly check that the functor preserves the identity and composition. □

The above theorem provides us with a big family of operations. Unfortunately, these do not cover all operations we know, in particular not all operations coming from operations on commutative algebras (which will be investigated in Section 3). The next theorem provides a bigger set of operations, but we have to be more careful with composition, since as seen earlier, $il\mathcal{D}$ is not a category anymore.

Theorem 2.4. *We have dg-maps*

$$J_{cFr} : il\mathcal{D}([{}^{n_1}_{m_1}],[{}^{n_2}_{m_2}]) \to \overline{\mathrm{Nat}}_{cFr}([{}^{n_1}_{m_1}],[{}^{n_2}_{m_2}])$$

and

$$J_{cFr_+} : il\mathcal{D}_+([{}^{n_1}_{m_1}],[{}^{n_2}_{m_2}]) \to \overline{\mathrm{Nat}}_{cFr_+}([{}^{n_1}_{m_1}],[{}^{n_2}_{m_2}])$$

preserving the composition of composable objects.

Proof. We need to show that the map is well-defined, i.e. that an infinite sum $a = \sum_{t_1=1}^{\infty}\cdots\sum_{t_{n_1}=1}^{\infty} a_{t_1,\dots,t_{n_1}} \in il\mathcal{D}^{>0}([{}^{n_1}_{m_1}],[{}^{n_2}_{m_2}])$ is taken to a well-defined element in the complex $\prod_{j_1,\cdots,j_{n_1}} l\mathcal{D}([{}^{0}_{j_1+\cdots+j_{n_1}+m_1}],[{}^{n_2}_{m_2}])$. So we show that for a fixed tuple $(j_1,\cdots,j_{n_1})$ only finitely many of the $J(a_{t_1,\dots,t_{n_1}})_{(j_1,\cdots,j_{n_1})}$ are non-zero. In the composition with a diagram of type $(t_1,\cdots,t_{n_1})$ none of the $\sum t_i$ loops can be empty and by composing with l_i's only leaves are glued on. Hence for the composition to be non-zero we need that $j_i > t_i$. Thus the claim is shown. We only need to see that composition of composable elements is preserved. In order to do so we use that all operations in Nat_{cFr} are composable, so if two elements $a = \sum_{t_1=1}^{\infty} \cdots \sum_{t_{n_1}=1}^{\infty} a_{t_1,\dots,t_{n_1}} \in il\mathcal{D}([{n_1 \atop m_1}],[{n_2 \atop m_2}])$ and $b = \sum_{t'_1=1}^{\infty} \cdots \sum_{t'_{n_2}=1}^{\infty} b_{t'_1,\dots,t'_{n_2}} \in il\mathcal{D}([{n_2 \atop m_2}],[{n_3 \atop m_3}])$ are composable then

$$\begin{aligned} J(b \circ a) &= J\left(\sum_{(t_1,\cdots,t_{n_1})} \sum_{(t'_1,\cdots,t'_{n_2})} b_{t'_1,\dots,t'_{n_2}} \circ a_{t_1,\dots,t_{n_1}} \right) \\ &= \sum_{(t_1,\cdots,t_{n_1})} \sum_{(t'_1,\cdots,t'_{n_2})} J\left(b_{t'_1,\dots,t'_{n_2}} \circ a_{t_1,\dots,t_{n_1}} \right) \\ &= \sum_{(t_1,\cdots,t_{n_1})} \sum_{(t'_1,\cdots,t'_{n_2})} J\left(b_{t'_1,\dots,t'_{n_2}} \right) \circ J\left(a_{t_1,\dots,t_{n_1}} \right) \end{aligned}$$

so the composition agrees. □

Remark 2.5 (Operations of type $(t_1,\cdots,t_n)$)**.** At this point we want to explain how the map J_{cFr} actually acts on an element of type $(t_1,\cdots,t_{n_1})$. There is an easy way to read off the operation of such an element without going back to the original definition of the type. For a general element $x = (\Gamma, \langle \gamma_1^1,\dots,\gamma_1^{t_1}\rangle,\dots,\langle \gamma_{n_1}^1,\dots,\gamma_{n_1}^{t_{n_1}}\rangle) \in il\mathcal{D}([{n_1 \atop m_1}],[{n_2 \atop m_2}])$ of type $(t_1,\cdots,t_{n_1})$ the composition with $(l_{j_1} \amalg \cdots \amalg l_{j_{n_1}} \amalg \mathrm{id}_{m_1})$ is trivial if there is a $j_i < t_i$ and is given by all possible ways of (for each i) gluing the j_i labeled leaves along the γ_i^r (respecting the order of the leaves and the loops) such that we glued at least one leaf to each γ_i^r (and gluing the m_1 extra leaves as usual). In particular the image $J_{cFr}(x) \in \mathrm{Nat}([{n_1 \atop m_1}],[{n_2 \atop m_2}])$ acts trivial on all Hochschild degrees $(j_1,\dots,j_{n_1})$ with $j_i < t_i$ for some i.

2.4. **Connection to non-commutative operations.** The analog of the functor J has been defined in the context of symmetric Frobenius algebras in [Wah12, Section 3]. There, it was even shown to be a split quasi-isomorphism of complexes:

Theorem 2.6 ([Wah12, Theorem 3.8])**.** *The functor*

$$J_{H_0} : \mathcal{SD} \to \mathrm{Nat}_{sFr}$$

defined by sending a graph G to $(G \circ (l_{j_1} \amalg \cdots \amalg l_{j_{n_1}} \amalg \mathrm{id}_{m_1}))_{j_1,\cdots,j_{n_1}}$ is a split quasi-isomorphism.

Using the functor $sFr \to cFr$ which induces a functor $\mathrm{Nat}_{sFr} \to \mathrm{Nat}_{cFr}$ and recalling the functor $K : \mathcal{SD} \to l\mathcal{D}$ defined at the end of Section 1.2 (cf. Figure 14), one checks that the definitions were made to make the following proposition true:

Proposition 2.7. *The diagram*

$$\begin{tikzcd} \mathcal{SD} \arrow[r, "J_{H_0}"] \arrow[d, "K"] & \mathrm{Nat}_{sFr} \arrow[d] \\ l\mathcal{D} \arrow[r, "J"] & \mathrm{Nat}_{cFr} \end{tikzcd}$$

commutes.

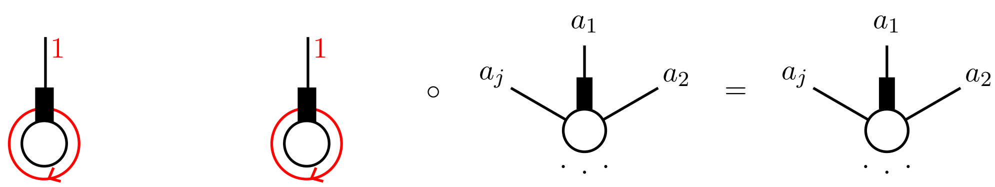
Remark 2.8. Everything done so far works also in the shifted setup, in particular we get a functor

$$J : l\mathcal{D}_d \to \mathrm{Nat}_{cFr_d}$$

and a commutative diagram

$$\begin{tikzcd} \mathcal{SD}_d \arrow[r, "J_{H_0}"] \arrow[d, "K"] & \mathrm{Nat}_{sFr_d} \arrow[d] \\ l\mathcal{D}_d \arrow[r, "J"] & \mathrm{Nat}_{cFr_d} . \end{tikzcd}$$

2.5. **First examples of operations and relations.** Before we investigate some subcomplexes of $l\mathcal{D}$ more systematically, we want to represent some of the known operations on the Hochschild homology of commutative Frobenius algebras by looped diagrams and apply the new tools to easily prove a relation between some of them.

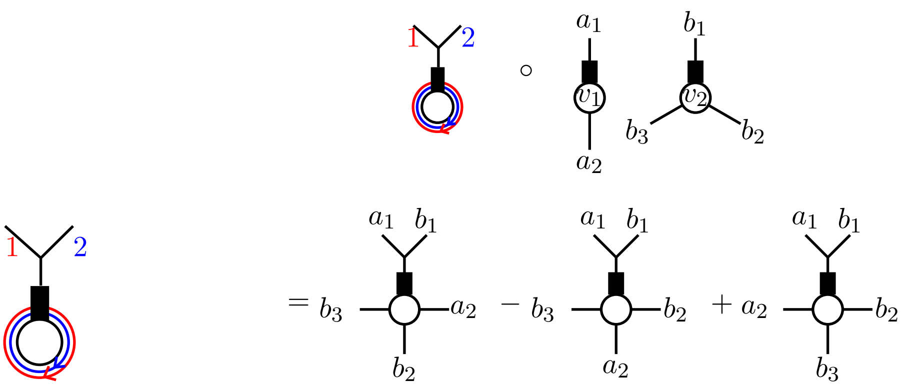


(a) The identity

(b) The image of the identity under J

FIGURE 20. The element $\mathrm{id}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} \in l\mathcal{D}_+^{>0}(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right])$ and its image in $\mathrm{Nat}_{cFr_+}(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right])$

Example 2.9 (Identity)**.** As seen before, the diagram $\mathrm{id}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} \in l\mathcal{D}_+^{>0}(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right])$ shown in Figure 20(a) corresponds to the identity in $\mathrm{Nat}_{cFr_+}(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right])$. In order to see this in pictures, we spell out the map $J : l\mathcal{D}_+^{>0}(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]) \to \mathrm{Nat}_{cFr_+}(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right])$ explicitly. By definition in degree j we have $J(\mathrm{id}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]})_j = \mathrm{id}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} \circ l_j$. There is only one way to glue the edges of l_j onto $\mathrm{id}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}$, so the resulting diagram in $l\mathcal{D}_+(\left[\begin{smallmatrix}0\\j\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right])$ is shown in Figure 20(b), where we labeled the leaves of l_j by $a_1, \cdots, a_j$. Given a Hochschild chain $a_1 \otimes \cdots \otimes a_j$ the image of the operation is now given by reading off around the white vertex. Thus we get $a_0 \otimes \cdots \otimes a_j$ back.

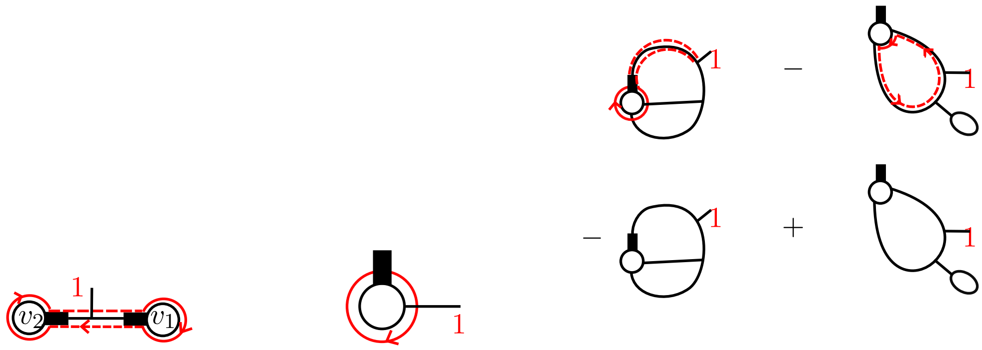


(a) The shuffle product pr

(b) $J(pr)_{2,3} = pr \circ (l_2 \amalg l_3)$

FIGURE 21. The element $pr \in l\mathcal{D}_+^{>0}(\left[\begin{smallmatrix}2\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right])$ and the degree $(2, 3)$ part of its image in $\mathrm{Nat}_{cFr_+}(\left[\begin{smallmatrix}2\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right])$

Example 2.10 (Shuffle product)**.** The operation $pr \in l\mathcal{D}_+^{>0}(\left[\begin{smallmatrix}2\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right])$ shown in Figure 21(a) is the shuffle product on the Hochschild homology. The composition $pr \circ (l_{j_1} \amalg l_{j_2})$ glues the first labeled leaves of l_{j_1} and l_{j_2} onto the start half-edge of pr and all other edges around the white vertex keeping the cyclic ordering of the edges coming from l_{j_1} and the cyclic ordering of the edges coming from l_{j_2}. Thus it produces all shuffles of these edges and hence corresponds to the shuffle product on the Hochschild chains. The example is illustrated in Figure 21(b) for $j_1 = 2$ and $j_2 = 3$, where we again already labeled the leaves by a_i and b_i to give a clearer understanding of the final operation.



(a) The comultiplication Δ (b) The BV operator (c) The element D

FIGURE 22. The comultiplication, the BV-operator and the boundary element D

Recall from the introduction that for a 1–connected closed oriented manifold, we have an isomorphism $HH_*(C^{-*}(M), C^{-*}(M)) \cong H^{-*}(LM)$. On $H^{-*}(LM)$ we have a coproduct (the dual of the Chas-Sullivan product) and a BV-operator. On the other hand, working with coefficients in $\mathbb{Q}$, by [LS07] there is a commutative Frobenius algebra A of degree d for $d = \dim M$ such that we have a weak equivalence $C^*(M) \simeq A$ and hence $HH_*(C^{-*}(M), C^{-*}(M)) \cong HH_*(A^{-*}, A^{-*})$. Since A^{-*} is a commutative Frobenius algebra of degree $-d$, we have an action of $l\mathcal{D}_{-d}$ on $HH_*(A^{-*}, A^{-*})$ and can show:

Proposition 2.11 (BV-structure on $H^*(LM, \mathbb{Q})$)**.** *Working with coefficients in $\mathbb{Q}$, the coBV structure on $HH_*(C^{-*}(M), C^{-*}(M)) \cong HH_*(A^{-*}, A^{-*})$, induced via the above isomorphism by the dual of the Chas-Sullivan product and the BV-operator on $H^{-*}(LM, \mathbb{Q})$, is generated by the operations $J(\Delta) \in \mathrm{Nat}_{cFr_{-d}}(\left[{1 \atop 0}\right], \left[{2 \atop 0}\right])$ and $J(B) \in \mathrm{Nat}_{cFr_{-d}}(\left[{1 \atop 0}\right], \left[{1 \atop 0}\right])$ with $\Delta \in l\mathcal{D}_{-d}(\left[{1 \atop 0}\right], \left[{2 \atop 0}\right])$ and $B \in l\mathcal{D}_{-d}(\left[{1 \atop 0}\right], \left[{1 \atop 0}\right])$ the shifted versions of the diagrams illustrated in Figure 22(a) and Figure 22(b).*

Proof. The diagrams Δ and B are the images of the diagrams illustrated in [WW11, Figure 13] under the functor $K : \mathcal{SD} \to l\mathcal{D}^{>0}$. Thus, the result is a direct consequence of [WW11, Prop. 6.10]. $\square$

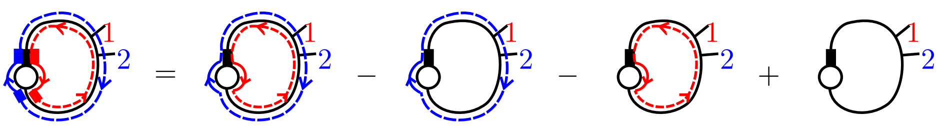


FIGURE 23. The product μ

Example 2.12 (Product of suspended BV-structure on HH_*)**.** The image of the diagram $\mu \in l\mathcal{D}_+^{>0}(\left[{2 \atop 0}\right], \left[{1 \atop 0}\right])$ shown in Figure 23 (in both ways of visualizing it as an element in $l\mathcal{D}^{>0}$ and $l\mathcal{D}$) defines a product on the Hochschild chains of commutative cocommutative open Frobenius dg-algebras. This product was introduced in [Abb13a, Section 7] and [Abb13b, Section 6], where it was also shown that together with the BV-operator it induces a BV-structure on the Hochschild homology. In Section 4 we will show that it is part of a desuspended Cacti operad.

The product μ shows similar behavior as the Goresky-Hingston product on $H^*(LM, M)$ (cf. [GH09]). For example, the composition of the Goresky-Hingston coproduct with the Chas-Sullivan product is zero. We can show a similar observation for $\mu \circ \Delta$, namely:

Proposition 2.13. *The composition $\mu \circ \Delta$ is a boundary in $l\mathcal{D}([\begin{smallmatrix}1\\0\end{smallmatrix}], [\begin{smallmatrix}1\\0\end{smallmatrix}])$, i.e. it gives a trivial operation on Hochschild homology of commutative cocommutative open Frobenius dg-algebras.*

Proof. In Figure 24 we have computed $\mu \circ \Delta$. This is equal to the boundary of the element D defined in Figure 22(c). $\square$

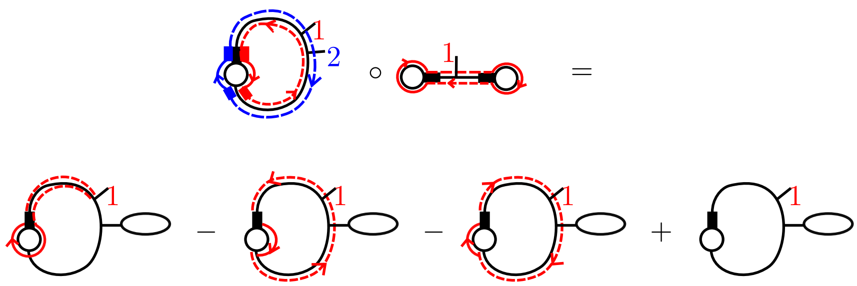


FIGURE 24. The composition $\mu \circ \Delta = d(D)$

Hence simultaneously with [Abb13b] we conjecture:

Conjecture 2.14. *Under the isomorphism $H^{-*}(LM, \mathbb{Q}) \cong HH_*(C^{-*}(M), C^{-*}(M)) \cong HH_*(A^{-*}, A^{-*})$ the the Goresky-Hingston product corresponds to the (shifted) operation induced by $\mu \in l\mathcal{D}^{>0}_{-d}([\begin{smallmatrix}2\\0\end{smallmatrix}], [\begin{smallmatrix}1\\0\end{smallmatrix}])$ shown in Figure 23.*

3. THE OPERATIONS COMING FROM COMMUTATIVE ALGEBRAS

We have a map of PROPs $\mathscr{C}om \to cFr$ which is the identity on objects and an inclusion on morphism spaces (since the structure of commutative Frobenius algebras includes the structure of commutative algebras). Therefore, we get an inclusion $\mathrm{Nat}_{\mathscr{C}om} \to \mathrm{Nat}_{cFr}$. This inclusion is split and factors through Nat_{cFr_+}.

In [Kla13] we recalled the shuffle operations defined in [Lod89] and computed the homology of $\mathrm{Nat}_{\mathscr{C}om}$ in terms of infinite sums of shuffles of these. In this section we define a split subcomplex of $ipl\mathcal{D}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}], [\begin{smallmatrix}n_2\\m_2\end{smallmatrix}])$ whose image under the map $J_{cFr} : ipl\mathcal{D}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}], [\begin{smallmatrix}n_2\\m_2\end{smallmatrix}]) \to \mathrm{Nat}_{cFr_+}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}], [\begin{smallmatrix}n_2\\m_2\end{smallmatrix}])$ defined in Theorem 2.4 is quasi-isomorphic to $\mathrm{Nat}_{\mathscr{C}om}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}], [\begin{smallmatrix}n_2\\m_2\end{smallmatrix}])$. On the level of complexes we give an even smaller subcomplex of $ipl\mathcal{D}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}], [\begin{smallmatrix}n_2\\m_2\end{smallmatrix}])$ which has trivial differential such that the map to $\mathrm{Nat}_{\mathscr{C}om}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}], [\begin{smallmatrix}n_2\\m_2\end{smallmatrix}])$ is a quasi-isomorphism, too.

Definition 3.1. The subcomplex $pl\mathcal{D}_{\mathscr{C}om}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}], [\begin{smallmatrix}n_2\\m_2\end{smallmatrix}])$ of $pl\mathcal{D}_+([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}], [\begin{smallmatrix}n_2\\m_2\end{smallmatrix}])$ is spanned by all looped diagrams $(\Gamma, \gamma_1, \cdots, \gamma_{n_1})$ such that Γ is a disjoint union of n_2 white vertices with trees attached to it (i.e. leaves multiplied together and attached to the white vertex and thus each irreducible loop goes once around the whole vertex) and m_2 labeled outgoing leaves with trees attached to them. The complex $ipl\mathcal{D}_{\mathscr{C}om}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}], [\begin{smallmatrix}n_2\\m_2\end{smallmatrix}])$ is given as above by taking products over the type of diagrams as defined in Section 1.4.

Note that an element of type $(t_1, \cdots, t_{n_1})$ corresponds to the i–th loop given by $\gamma_i = \underbrace{\langle \sigma, \cdots, \sigma \rangle}_{t_i}$ with σ the loop going once around the vertex starting at the leaf i. An example

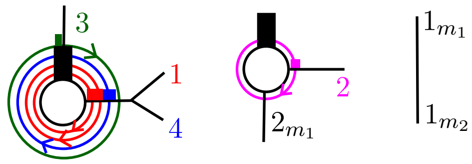


FIGURE 25. An element in $pl\mathcal{D}_{\mathscr{C}om}([{4 \atop 2}],[{2 \atop 1}])$ of type $(2,1,1,1)$

of an element in $pl\mathcal{D}_{\mathscr{C}om}([{4 \atop 2}],[{2 \atop 1}])$ of type $(2,1,1,1)$ is given in Figure 25 (where we label the leaves corresponding to m_1 and m_2 with subindices).

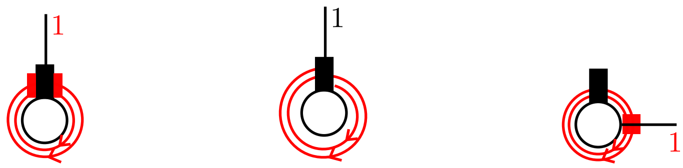
Example 3.2. We can describe the generators of $pl\mathcal{D}_{\mathscr{C}om}([{1 \atop 0}],[{1 \atop 0}])$ explicitly:

In degree zero it is generated by a family of elements sh^n. The element sh^n of type n is defined as $(\Gamma, \langle\sigma,\cdots,\sigma\rangle)$ where Γ is the diagram with one white vertex and the leaf attached to the start half-edge and σ the loop going once around the vertex (i.e. we have n irreducible loops around the vertex). For $n=0$ we only have the underlying diagram, representing the inclusion of the algebra. For an example see Figure 26(a).

The above elements have been defined using the type decomposition from Section 1.4 which we need to use if we want to take products over the elements. However, we saw that we can rewrite them by ordinary elements in $pl\mathcal{D}([{1 \atop 0}],[{1 \atop 0}])$. Thus for $n>0$, we define the element λ^n to be the diagram in $pl\mathcal{D}([{1 \atop 0}],[{1 \atop 0}])$ looping around the white vertex n times (i.e. the loop σ^{*n} which is not irreducible) and thus $\lambda^0 = sh^0$. These elements give us another generating family of $pl\mathcal{D}^{>0}([{1 \atop 0}],[{1 \atop 0}])$ and we will come back to them when explaining the operations corresponding to these diagrams. An example is given in Figure 26(b).

For the degree one part of $pl\mathcal{D}_{\mathscr{C}om}([{1 \atop 0}],[{1 \atop 0}])$ we only give the generating family in terms of the type decomposition: Recall from Figure 22(b) that the BV-operator is the looped diagram with a unit at the start half-edge, one labeled incoming leaf attached to the white vertex and a loop going once around from that leaf.

The elements B^n of type n are defined as $B \circ sh^n$, so they are given by the same graphs as the BV-operator but have n irreducible loops going around (cf. Figure 26(c)).



(a) The element sh^2 (b) The element λ^2 (c) The element B^2

FIGURE 26. General elements in $pl\mathcal{D}([{1 \atop 0}],[{1 \atop 0}])$

Lemma 3.3. *All morphisms in $ipl\mathcal{D}_{\mathscr{C}om}$ are composable and thus $ipl\mathcal{D}_{\mathscr{C}om}$ is a dg-category.*

Proof. It is enough to check the claim on generators. Given $a \in pl\mathcal{D}_{\mathscr{C}om}([{n_1 \atop m_1}],[{n_2 \atop m_2}])$ of type $(t_1,\dots,t_{n_1})$ and $b \in pl\mathcal{D}_{\mathscr{C}om}([{n_2 \atop m_2}],[{n_3 \atop m_3}])$ of type $(t'_1,\dots,t'_{n_2})$ we give conditions on the type of the composition being non-zero. First, every irreducible loop of a becomes an irreducible loop in the composition, i.e. the type of non-trivial elements in $a \circ b$ is bounded below by $(t_1,\dots,t_{n_1})$.

In $pl\mathcal{D}_{\mathscr{C}om}([{n_1 \atop m_1}],[{n_2 \atop m_2}])$ we can rewrite every diagram as a diagram without loops together with its type. In particular, in $a\circ b$ the underlying diagrams are of the form $a \circ \Gamma'$ for Γ' the underlying diagram of b. Assuming that the minimal non-trivial type of the composition $a \circ b$ is $(u_1,\dots,u_{n_1})$, we obtain that composition with $(l_{u_1+1} \amalg \cdots \amalg l_{u_{n_1}+1} \amalg \mathrm{id}_{m_2})$ is non-trivial (different underlying diagrams of the same type have different images under this composition). Thus $J(b\circ a)_{(u_1+1,\cdots,u_{n_1}+1)}$ is non-zero. However, we know that in the composition $(J(b)\circ J(a))_{l_1,\cdots,l_{n_1}}$ one glues the summands of $J(a)_{l_1,\cdots,l_{n_1}}$ onto terms $J(b)_{j_1,\cdots,j_{n_2}}$ with $\sum j_i=\deg(a)+\sum(l_i-1)$. For this to be non-zero, we need $J(b)_{j_1,\cdots,j_{n_2}}$ to be non-zero, which is only true if $j_i>t'_i$ for all i. Thus we conclude that $\sum t'_i<\sum j_i=deg(a)+\sum u_i$ and hence for any type $(u_1\cdots,u_{n_1})$ occurring in $b\circ a$ we have $\sum t'_i-\deg(a)<\sum u_i$ and $t_i\le u_i$ for all i. This proves the lemma. $\square$

There is an analog of Lemma 2.2 for commutative algebras:

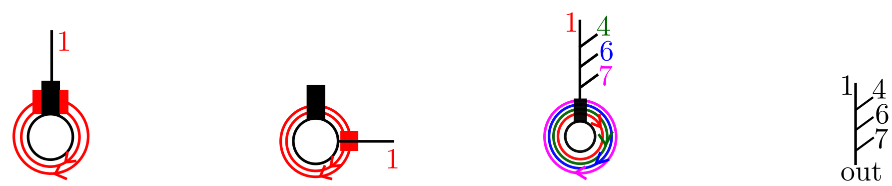
Lemma 3.4. *We have an isomorphism*
$$\overline{C}^n(\mathscr{C}om(m_1,-))(m_2)\cong pl\mathcal{D}_{\mathscr{C}om}([{}^{0}_{m_1}],[{}^{n}_{m_2}])$$
and hence a weak equivalence
$$\prod_{j_1,\cdots,j_{n_1}} pl\mathcal{D}_{\mathscr{C}om}([{}^{0}_{j_1+\cdots+j_{n_1}+m_1}],[{}^{n_2}_{m_2}])[n_1-\Sigma j_i]\to\overline{\mathrm{Nat}}_{\mathscr{C}om}([{}^{n_1}_{m_1}],[{}^{n_2}_{m_2}]).$$

It follows that the diagram
$$\begin{tikzcd}
\prod_{j_1,\cdots,j_{n_1}} pl\mathcal{D}_{\mathscr{C}om}([{}^{0}_{j_1+\cdots+j_{n_1}+m_1}],[{}^{n_2}_{m_2}])[n_1-\Sigma j_i] \arrow[r, "\simeq"] \arrow[d, hook] & \overline{\mathrm{Nat}}_{\mathscr{C}om}([{}^{n_1}_{m_1}],[{}^{n_2}_{m_2}]) \arrow[d, hook] \\
\prod_{j_1,\cdots,j_{n_1}} pl\mathcal{D}([{}^{0}_{j_1+\cdots+j_{n_1}+m_1}],[{}^{n_2}_{m_2}])[n_1-\Sigma j_i] \arrow[r, "\simeq"] & \overline{\mathrm{Nat}}_{cFr}([{}^{n_1}_{m_1}],[{}^{n_2}_{m_2}])
\end{tikzcd}$$
commutes.

Hence the dg-map $J_{cFr}:ipl\mathcal{D}([{}^{n_1}_{m_1}],[{}^{n_2}_{m_2}])\to\overline{\mathrm{Nat}}_{cFr}([{}^{n_1}_{m_1}],[{}^{n_2}_{m_2}])$ restricts to a dg-map $J_{\mathscr{C}om}:ipl\mathcal{D}_{\mathscr{C}om}([{}^{n_1}_{m_1}],[{}^{n_2}_{m_2}])\to\overline{\mathrm{Nat}}_{\mathscr{C}om}([{}^{n_1}_{m_1}],[{}^{n_2}_{m_2}])$.

By Lemma 3.3 all morphisms in $ipl\mathcal{D}_{\mathscr{C}om}$ are composable and since J_{cFr} and hence also $J_{\mathscr{C}om}$ preserve the composition of composable objects, the map $J_{\mathscr{C}om}$ is a natural transformation of categories $J_{\mathscr{C}om}:ipl\mathcal{D}_{\mathscr{C}om}\to\mathrm{Nat}_{\mathscr{C}om}$.

Let $m_{r_1,\cdots,r_n}$ be the $[{}^{1}_{(0,n)}]$–looped diagram with the tree with n leaves labeled r_1 to r_n attached to the start half-edge, no other half-edges attached to the white vertex and a loop going around the white vertex for each i. For $n=0$ the diagram $m_\emptyset$ only has a unit at the start half-edge. Denote by $\overline{m}_{r_1,\cdots,r_n}$ the $[{}^{0}_{(n+1,0)}]$–looped diagram consisting of the tree with incoming leaves labeled by $r_1,\cdots r_n$ and one outgoing leaf. For $m=0$ this diagram only has one unlabeled half-edge which is the outgoing leaf (i.e. it is a unit).



(a) The element sh^2 (b) The element B^2 (c) The element $m_{1,4,6,7}$ (d) The element $\overline{m}_{1,4,6,7}$

FIGURE 27. Building blocks of $\widetilde{pl\mathcal{D}}_{\mathscr{C}om}$

Definition 3.5. The subcomplex $\widetilde{pl\mathcal{D}}_{\mathscr{C}om}([{}^{n_1}_{m_1}],[{}^{n_2}_{m_2}])$ of $pl\mathcal{D}_{\mathscr{C}om}([{}^{n_1}_{m_1}],[{}^{n_2}_{m_2}])$ spanned by elements x obtained as follows: Given

- a function $f:\{1,\cdots,n_1+m_1\}\to\{1,\cdots,n_2+m_2\}$;
- a tuple of integers $(t_1,\cdots,t_{n_1})$ (the type), with $t_i=0$ if $f(i)>n_2$,
- a function $s:f^{-1}(\{1,\cdots,n_2\})\to\{0,1\}$

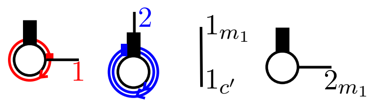
we define $x=x_2\circ x_1$ as the composition of two looped diagrams x_1 and x_2 with $x_1\in pl\mathcal{D}_{\mathscr{C}om}([{}^{n_1}_{m_1}],[{}^{c}_{m_1+n_1-c}])$ and $x_2\in pl\mathcal{D}_{\mathscr{C}om}([{}^{c}_{m_1+n_1-c}],[{}^{n_2}_{m_2}])$ for $c:=|f^{-1}(\{1,\cdots,n_2\})|$.

- The element $x_1 = x_1^1 \amalg \ldots \amalg x_1^{n_1+m_1}$ is defined as the disjoint union of n_1+m_1 elements with each one incoming leaf (such that we relabeled the leaf of the ith element by i for $i \leq n_1$ and j_{m_1} for $j = i - n_2$ for $n_1 \leq i \leq m_1 + n_1$ after taking disjoint union). These elements are defined as follows:
  - If $1 \leq i \leq n_1$ and
    * if $f(i) \leq n_2$ then $x_1^i \in pl\mathcal{D}_{\mathscr{C}om}([\begin{smallmatrix}1\\0\end{smallmatrix}],[\begin{smallmatrix}1\\0\end{smallmatrix}])$ is given by
    $$x_1^i = \begin{cases} sh^{t_i} & \text{if } s(i) = 0 \\ B^{t_i} & \text{if } s(i) = 1. \end{cases}$$
    * if $f(i) > n_2$ and thus $t_i = 0$, then $x_1^i = \mathrm{id} \in pl\mathcal{D}^{>0}_{\mathscr{C}om}([\begin{smallmatrix}1\\0\end{smallmatrix}],[\begin{smallmatrix}0\\1\end{smallmatrix}])$, the constant diagram with the incoming leaf glued to the outgoing leaf.
  - For $n_1 + 1 \leq i \leq n_1 + m_1$ we have
    * If $f(i) \leq n_2$ then $x_1^i \in pl\mathcal{D}_{\mathscr{C}om}([\begin{smallmatrix}0\\1\end{smallmatrix}],[\begin{smallmatrix}1\\0\end{smallmatrix}])$ given by
    $$x_1^i = \begin{cases} sh^{0} & \text{if } s(i) = 0 \\ B^{0} & \text{if } s(i) = 1. \end{cases}$$
    * If $f(i) > n_2$ then $x_1^i = \mathrm{id} \in pl\mathcal{D}^{>0}_{\mathscr{C}om}([\begin{smallmatrix}0\\1\end{smallmatrix}],[\begin{smallmatrix}0\\1\end{smallmatrix}])$.

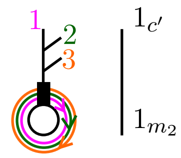
  Thus $x_1 = x_1^1 \amalg \ldots \amalg x_1^{n_1+m_1} \in pl\mathcal{D}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}],[\begin{smallmatrix}c\\m_1+n_1-c\end{smallmatrix}])$.
- The element $x_2 \in pl\mathcal{D}([\begin{smallmatrix}c\\m_1+n_1-c\end{smallmatrix}],[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}])$ multiplies the i–th incoming vertex onto the outgoing vertex $f(i)$ (and the incoming leaf onto $f(i)$ for $i > n_2$). More precisely,
  $$x_2 = m_{\{f^{-1}(1)\}} \amalg \ldots \amalg m_{\{f^{-1}(n_2)\}} \amalg \overline{m}_{\{f^{-1}(n_2+1)\}} \amalg \ldots \amalg \overline{m}_{\{f^{-1}(n_2+m_2)\}}.$$

We define $\widetilde{ipl\mathcal{D}}_{\mathscr{C}om}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}],[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}])$ analogous to before by taking products over the type of these elements.

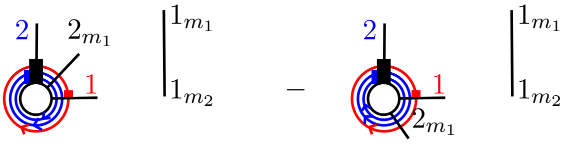
An example of such an element in $\widetilde{pl\mathcal{D}}_{\mathscr{C}om}([\begin{smallmatrix}2\\2\end{smallmatrix}],[\begin{smallmatrix}1\\1\end{smallmatrix}])$ with $t_1 = 1$, $t_2 = 2$, $f(1) = f(2) = f(4) = 1$ and $f(3) = 2$, $s(1) = s(3) = 1$ and $s(2) = 0$ is given in Figure 28. We have $x_1 \in pl\mathcal{D}_{\mathscr{C}om}([\begin{smallmatrix}2\\2\end{smallmatrix}],[\begin{smallmatrix}3\\1\end{smallmatrix}])$ and $x_2 \in pl\mathcal{D}_{\mathscr{C}om}([\begin{smallmatrix}3\\1\end{smallmatrix}],[\begin{smallmatrix}1\\1\end{smallmatrix}])$.



(a) The element x_1



(b) The element x_2



(c) The element $x = x_2 \circ x_1$

FIGURE 28. An element in $\widetilde{pl\mathcal{D}}_{\mathscr{C}om}([\begin{smallmatrix}2\\2\end{smallmatrix}],[\begin{smallmatrix}1\\1\end{smallmatrix}])$

Since both elements x_1 and x_2 have trivial differential, the differential on the complexes $\widetilde{pl\mathcal{D}}_{\mathscr{C}om}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}],[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}])$ and $\widetilde{ipl\mathcal{D}}_{\mathscr{C}om}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}],[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}])$ is also trivial. Note moreover, that $ipl\mathcal{D}_{\mathscr{C}om}([\begin{smallmatrix}1\\0\end{smallmatrix}],[\begin{smallmatrix}1\\0\end{smallmatrix}]) \cong \widetilde{ipl\mathcal{D}}_{\mathscr{C}om}([\begin{smallmatrix}1\\0\end{smallmatrix}],[\begin{smallmatrix}1\\0\end{smallmatrix}])$.

We denote the further restriction of $J_{\mathscr{C}om}$ to $\widetilde{ipl\mathcal{D}}_{\mathscr{C}om}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}],[\begin{smallmatrix}n_2\\m_2\end{smallmatrix}])$ by $\widetilde{J}_{\mathscr{C}om}$. In [Kla13, Section 2.3] we recalled Loday's lambda and shuffle operations λ^k and sh^k (cf. [Lod89]) and defined operations B^k as the composition of the shuffle operations with Connes' boundary operator and used the families sh^k and B^k to build general operations in

$\mathrm{Nat}_{\mathscr{C}om}([{n_1 \atop m_1}],[{n_2 \atop m_2}])$. Up to sign, the shuffle operations act on the Hochschild degree n by taking all $(p_1,\cdots,p_k)$-shuffles in Σ_n with $p_1+\cdots p_k=n$ and all $p_j\geq 1$ and view these as elements in $\mathscr{C}om(n+1,n+1)$ leaving the first element fixed. Using this combinatorial description and recalling from Remark 2.5 how to read off operations of type k, it is not hard to see that $\widetilde{J}_{\mathscr{C}om}$ sends the looped diagram sh^k to the operation $sh^k\in pl\mathcal{D}([{1 \atop 0}],[{1 \atop 0}])$. Both, as diagrams and as operations we can rewrite the family λ^k in terms of the sh^k with the same coefficients occurring and hence see that $\widetilde{J}_{\mathscr{C}om}$ also sends the diagrams λ^k to the corresponding operations $\lambda^k\in pl\mathcal{D}([{1 \atop 0}],[{1 \atop 0}])$. Hence this way we recover Loday's lambda operations. Since we have already seen that Connes' boundary operator is send to B under $\widetilde{J}_{\mathscr{C}om}$, the same follows for the B^k. Moreover, the looped diagrams x_1 and x_2 constructed in Definition 3.5 are send to the operations x_1 and x_2 in Definition [Kla13, Def. 3.3]. Therefore the complex spanned by the diagrams of type $(t_1,\ldots,t_{n_1})$ in $\widetilde{ipl\mathcal{D}}_{\mathscr{C}om}([{n_1 \atop m_1}],[{n_2 \atop m_2}])$ is mapped to the complex $A_{t_1,\cdots,t_{n_1}}$ as defined in [Kla13, Def. 3.3] and thus $\widetilde{J}_{\mathscr{C}om}(\widetilde{ipl\mathcal{D}}_{\mathscr{C}om}([{n_1 \atop m_1}],[{n_2 \atop m_2}]))\cong\prod A_{t_1,\ldots,t_{n_1}}$. In [Kla13, Theorem 3.4] we prove that the inclusion of this complex into $\mathrm{Nat}_{\mathscr{C}om}([{n_1 \atop m_1}],[{n_2 \atop m_2}])$ is a quasi-isomorphism, thus in terms of looped diagrams the theorem can be restated as:

Theorem 3.6 ([Kla13, Theorem 3.4])**.** *The map $\widetilde{J}_{\mathscr{C}om}$ is a quasi-isomorphism, i.e.*

$$\widetilde{J}_{\mathscr{C}om}:\widetilde{ipl\mathcal{D}}_{\mathscr{C}om}([{n_1 \atop m_1}],[{n_2 \atop m_2}])\xrightarrow{\simeq}\mathrm{Nat}_{\mathscr{C}om}([{n_1 \atop m_1}],[{n_2 \atop m_2}])$$

and the left complex has trivial differential, thus on homology

$$H_*(\widetilde{J}_{\mathscr{C}om}):\widetilde{ipl\mathcal{D}}_{\mathscr{C}om}([{n_1 \atop m_1}],[{n_2 \atop m_2}])\xrightarrow{\cong}H_*(\mathrm{Nat}_{\mathscr{C}om}([{n_1 \atop m_1}],[{n_2 \atop m_2}])).$$

Corollary 3.7. *The inclusion $\widetilde{ipl\mathcal{D}}_{\mathscr{C}om}([{n_1 \atop m_1}],[{n_2 \atop m_2}])\to ipl\mathcal{D}_{\mathscr{C}om}([{n_1 \atop m_1}],[{n_2 \atop m_2}])$ is a quasi-isomorphism.*

Proof. Since the differential does not contract a loop going around a whole vertex and all loops are irreducible and of this kind and there is an isomorphism of chain complexes $pl\mathcal{D}^{t_1,\cdots,t_{n_1}}_{\mathscr{C}om}([{n_1 \atop m_1}],[{n_2 \atop m_2}])\cong pl\mathcal{D}_{\mathscr{C}om}([{0 \atop m_1+n_1}],[{n_2 \atop m_2}])$ and similarly by the restriction of this isomorphism we have $\widetilde{pl\mathcal{D}}^{t_1,\cdots,t_{n_1}}_{\mathscr{C}om}([{n_1 \atop m_1}],[{n_2 \atop m_2}])\cong\widetilde{pl\mathcal{D}}_{\mathscr{C}om}([{0 \atop m_1+n_1}],[{n_2 \atop m_2}])$. Moreover, by Lemma 3.4 and the following constructions $\mathrm{Nat}_{\mathscr{C}om}([{0 \atop m_1+n_1}],[{n_2 \atop m_2}])\cong pl\mathcal{D}_{\mathscr{C}om}([{0 \atop m_1+n_1}],[{n_2 \atop m_2}])$ and the map $\widetilde{J}_{\mathscr{C}om}$ in this case is given by the embedding of $\widetilde{pl\mathcal{D}}_{\mathscr{C}om}([{0 \atop m_1+n_1}],[{n_2 \atop m_2}])$ into this complex. Since $\widetilde{J}_{\mathscr{C}om}$ is a quasi-isomorphism, the embedding

$$\widetilde{pl\mathcal{D}}_{\mathscr{C}om}([{0 \atop m_1+n_1}],[{n_2 \atop m_2}])\to pl\mathcal{D}_{\mathscr{C}om}([{0 \atop m_1+n_1}],[{n_2 \atop m_2}])$$

is a quasi-isomorphism and thus by the isomorphism of complexes stated above, so is

$$\widetilde{pl\mathcal{D}}^{t_1,\cdots,t_{n_1}}_{\mathscr{C}om}([{n_1 \atop m_1}],[{n_2 \atop m_2}])\to pl\mathcal{D}^{t_1,\cdots,t_{n_1}}_{\mathscr{C}om}([{n_1 \atop m_1}],[{n_2 \atop m_2}]).$$

Since homology commutes with products, the map

$$\prod_{(t_1,\cdots,t_{n_1})}\widetilde{pl\mathcal{D}}^{t_1,\cdots,t_{n_1}}_{\mathscr{C}om}([{n_1 \atop m_1}],[{n_2 \atop m_2}])\to\prod_{(t_1,\cdots,t_{n_1})}pl\mathcal{D}^{t_1,\cdots,t_{n_1}}_{\mathscr{C}om}([{n_1 \atop m_1}],[{n_2 \atop m_2}])$$

is an isomorphism on homology and thus the corollary is proven. □

Corollary 3.8. *The map $J_{\mathscr{C}om}:ipl\mathcal{D}_{\mathscr{C}om}\to\mathrm{Nat}_{\mathscr{C}om}$ is a quasi-isomorphism of dg-categories.*

Proof. We have seen earlier that $J_{\mathscr{C}om}$ is a dg-functor. Thus we only need to show that it is a quasi-isomorphism. By Theorem 3.6 and Corollary 3.7 the maps $\widetilde{J}_{\mathscr{C}om}$ and $i_{\mathscr{C}om}$ are quasi-isomorphisms. Furthermore, $J_{\mathscr{C}om}\circ i_{\mathscr{C}om}=\widetilde{J}_{\mathscr{C}om}$ and hence it follows that $J_{\mathscr{C}om}$ is a quasi-isomorphism for all morphism spaces. □

4. The suspended cacti operad and its action

In this section we define a subcategory $pl\mathcal{D}^{>0}_{cact}(n_1,n_2)$ of $pl\mathcal{D}^{>0}_{+}([\begin{smallmatrix}n_1\\0\end{smallmatrix}],[\begin{smallmatrix}n_2\\0\end{smallmatrix}])$ and show that it is quasi-isomorphic to a suspension of the cacti quasi-operad. We start with operadic constructions and definitions.

4.1. **Operadic constructions.** First we give some operadic tools. We only consider non-unital operads, i.e. operads indexed on the positive integral numbers. Furthermore, sometimes we will have to work with quasi-operads. A quasi-operad fulfills the same axioms as an operad beside associativity (for more details see [Kau05, Section 1]).

We work with (quasi-)operads in chain complexes, topological spaces and pointed topological spaces.

To switch between these, we need the following constructions:

Proposition 4.1.
- *For $\mathcal{P}$ an operad in topological spaces with all structure maps proper, the level-wise one-point compactification $\mathcal{P}^c$ is an operad in pointed spaces (cf.* [AK13, Prop. 4.1]*).*
- *Let $\mathcal{P}$ be an operad in topological spaces and I an operadic ideal. Define $\mathcal{P}/I$ in level n to be the pointed space $\mathcal{P}(n)/I(n)$ with basepoint $I(n)$. Then $\mathcal{P}/I$ is an operad in pointed spaces.*

Next we define some operads used later on:

Definition 4.2. The open simplex operad $\mathcal{D}$ is the topological operad with $\mathcal{D}(n) = \mathring{\Delta}^{n-1} = \{(s_1,\cdots,s_n) \mid 0 < s_i < 1, \sum s_i = 1\}$ the open standard $n-1$–simplex, with Σ_n–action given by permuting the coordinates and composition defined by

$$\circ_i : \mathring{\Delta}^{k-1} \times \mathring{\Delta}^{n-1} \to \mathring{\Delta}^{n+k-2}$$
$$(t_1,\cdots t_k) \circ_i (s_1,\cdots,s_n) = (s_1,\cdots,s_{i-1},s_i\cdot t_1,\cdots,s_i\cdot t_k,s_{i+1},\cdots,s_n).$$

The sphere operad $\mathcal{S}ph$ is the one-point compactification of $\mathcal{D}$, i.e. $\mathcal{S}ph(n) = \mathcal{D}(n)^c$.

Note that the operad $\mathcal{D}$ is a retract of the scaling operad $\mathcal{R}_{>0}$ defined in [Kau05, 5.1.1] and given by $\mathcal{R}_{>0}(n) = \mathbb{R}^n_{>0}$. The retraction from $\mathcal{R}_{>0}$ to $\mathcal{D}$ sends a tuple $(r_1,\ldots,r_n) \in \mathbb{R}^n_{>0}$ with $R = \sum r_i$ to $(\frac{r_1}{R},\ldots,\frac{r_n}{R}) \in \mathring{\Delta}^{n-1}$.

The operads $\mathcal{D}$ and $\mathcal{S}ph$ have also been defined in [AK13] where they are denoted by Δ_1^{n-1} and S, respectively. There, it is also mentioned that one can use the sphere operad to define operadic suspension. To make this more precise, we recall:

Definition 4.3 ([LV12, Section 7.2.2])**.** The desuspension operad $\mathcal{S}^{-1}$ is defined as the endomorphism operad of $s^{-1}\mathbb{K}$ with s^{-1} the shift operator, thus

$$\mathcal{S}^{-1}(n) = \hom((s^{-1}\mathbb{K})^{\otimes n}, s^{-1}\mathbb{K}).$$

For a graded operad $\mathcal{O}$ its operadic desuspension is defined as

$$s^{-1}\mathcal{O}(n) = \mathcal{S}^{-1}(n) \otimes \mathcal{O}(n),$$

as the Hadamard tensor product of operads (cf. [LV12, Section 5.3.3]). We define the twisted desuspension operad $\widetilde{\mathcal{S}}^{-1} = \widetilde{H}_*(\mathcal{S}ph)$ the reduced homology of the sphere operad. This operad equals $\mathcal{S}^{-1}(n)$ as a graded Σ_n–module, but the signs in the composition differ. We define the twisted operadic desuspension by

$$\widetilde{s}^{-1}\mathcal{O}(n) = \widetilde{\mathcal{S}}^{-1}(n) \otimes \mathcal{O}(n).$$

Defining the topological operadic desuspension of an operad $\mathcal{P}$ in pointed spaces as $\mathcal{S}ph \wedge \mathcal{P}$, on reduced homology one obtains $\widetilde{H}_*(\mathcal{S}ph \wedge \mathcal{P}) \cong \widetilde{s}^{-1}\widetilde{H}_*(\mathcal{P})$.

4.2. **The cacti-like diagrams.** In this section we define the different kinds of looped and Sullivan diagrams used later on.

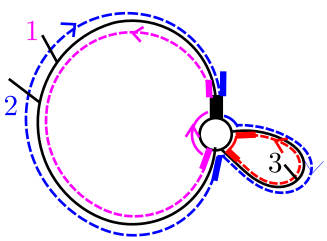
Definition 4.4. An $\left[{n_2 \atop (0,n_1)}\right]$–looped diagram $(\Gamma, \gamma_1, \cdots, \gamma_{n_1})$ belongs to $pl\mathcal{D}_{cact}(n_1, n_2)$ if it fulfills the following properties:

- The white vertices in Γ are not connected, i.e. Γ is the disjoint union of n_2 commutative diagrams.
- The underlying commutative Sullivan diagram has a representation embeddable into the plane.
- Every boundary segment of any white vertex in Γ is part of exactly one loop γ_i and all these loops γ_i are irreducible and positively oriented.
- If an arc component (i.e. the connected components after removing the white vertex) has genus g (as a graph) there are exactly g constant loops attached to it.

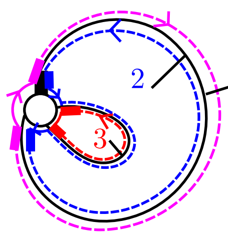
This defines a complex, since by the irreducibility of the loops around the white vertex, genus in the graph can only be created if a loop is contracted completely and thus there is a constant loop belonging to the new genus.

Denote by $pl\mathcal{D}^{cst}_{cact}(n_1, n_2)$ the subcomplex of partly constant diagrams (i.e. at least one of the loops is constant) which in particular contains all diagrams with genus in the arc components by the last condition in the definition. Let $pl\mathcal{D}^{>0}_{cact}(n_1, n_2)$ be the subcategory of $pl\mathcal{D}^{>0}(\left[{n_1 \atop 0}\right], \left[{n_2 \atop 0}\right])$ given by the non-constant diagrams as introduced in Section 1.3. Since the white vertices are not connected, all singular vertices must be loop covering and hence by Corollary 1.19 we have an isomorphism of dg-categories

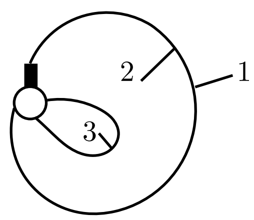
$$pl\mathcal{D}^{>0}_{cact}(n_1, n_2) \cong pl\mathcal{D}_{cact}(n_1, n_2)/pl\mathcal{D}^{cst}_{cact}(n_1, n_2).$$



(a) $\Gamma \in pl\mathcal{D}^{>0}_{cact}(3,1)$



(b) A different embedding of Γ



(c) The corresponding cacti-like Sullivan diagram $F(\Gamma)$

FIGURE 29. A looped diagram in $pl\mathcal{D}^{>0}_{cact}(3,1)$, the embedding used to obtain a Sullivan diagram and the corresponding cacti-like Sullivan diagram

By the description every diagram is a disjoint union of looped diagrams with at least one incoming leaf and hence

$$pl\mathcal{D}_{cact}(n_1, n_2) = \coprod_{\substack{f:\{1,\ldots,n_1\}\to\{1,\cdots,n_2\} \\ f \text{ surj}, \ t_i := |f^{-1}(i)|}} pl\mathcal{D}_{cact}(t_1, 1) \times \cdots \times pl\mathcal{D}_{cact}(t_{n_2}, 1)$$

and similarly for $pl\mathcal{D}^{cst}_{cact}(n_1, n_2)$ and $pl\mathcal{D}^{>0}_{cact}(n_1, n_2)$. Furthermore, the chain complexes $pl\mathcal{D}_{cact}(n, 1)$, $pl\mathcal{D}^{cst}_{cact}(n, 1)$ and $pl\mathcal{D}^{>0}_{cact}(n, 1)$ define non-unital dg-operads. An example of an element in $pl\mathcal{D}^{>0}_{cact}(3, 1)$ is given in Figure 29(a).

Before we give an actual definition of the cacti operad, we define the complex of cacti-like $\left[{1 \atop n}\right]$–Sullivan diagrams:

Definition 4.5. A Sullivan diagram $\Gamma \in \left[{1 \atop n}\right]$ − Sullivan diagrams is *cacti-like* if

- it is embeddable into the plane,

- the arc components have genus zero as graphs (i.e. viewed as CW-complexes they are contractible),
- the diagram has exactly n boundary cycles, each labeled by a leaf.

In a first step, we show that on the level of diagrams we have a bijection:

Lemma 4.6. *We have a bijection*

$$K : \{\textit{Cacti-like } [\tfrac{1}{n}] - \textit{Sullivan diagrams}\} \longrightarrow \left\{ \begin{array}{c} \textit{non-constant looped diagrams} \\ (\Gamma, \gamma_1, \cdots, \gamma_n) \in pl\mathcal{D}_{cact}(n,1) \end{array} \right\}$$

with K the map described at the end of Section 1.2 forgetting about the cyclic ordering at the black vertices though remembering a loop for each boundary cycle.

Proof. The map K has an inverse F which can be described as follows: After forgetting the labeled leaves, for any looped diagram $(\Gamma, \gamma_1, \cdots, \gamma_n) \in pl\mathcal{D}_{cact}(n,1)$ with no constant loops the commutative Sullivan diagram Γ can be uniquely (up to the equivalence relation on Sullivan diagrams) embedded into the plane such that the last segment of the white vertex is on the outside of the graph. To see so, one verifies that the data of a commutative Sullivan diagram of degree d without genus in the arc components is equivalent to dividing $\{0, \cdots, d\}$ into a disjoint union of subsets and connecting all elements of one subset by an arc component. This becomes unique if the diagram was embeddable into the plane before and if we glue onto an interval instead of a circle (which we do by the condition that the last boundary segment lies outside the graph). To define the inverse map F, for a looped diagram $(\Gamma, \gamma_1, \cdots, \gamma_n) \in pl\mathcal{D}_{cact}(n,1)$ we then choose the labeled leaves to be inside the loop which starts at this leaf (this is possible since the loops do not overlap). It is not hard to see that after forgetting the loops this embedding into the plane gives a cacti-like $[\frac{1}{n}]$–Sullivan diagram. An example of the chosen embedding of the diagram in Figure 29(a) and the corresponding cacti-like Sullivan diagram are shown in Figure 29. By definition, the composition $F \circ K$ is the identity. In order to see that $K \circ F$ is the identity too, one checks that every cacti-like $[\frac{1}{n}]$–Sullivan diagram can be embedded into the plane such that the last segment of the white vertex is on the outside of the diagram. Choosing this embedding and using the fact that F is well-defined, i.e. independent of the embedding, it follows directly that $K \circ F$ is the identity. $\square$

4.3. **The cacti operad.** In this section we recall the definition of the (normalized) Cacti operad with spines. The original definition goes back to Voronov in [Vor05]. For an overview over different definitions of cacti see [Kau05]. We use the definition given in [CV05, Section 2.2].

An element in $Cacti(n)$ is given by a treelike configuration of n labeled circles with positive circumferences c_i such that $\sum c_i = 1$ (usually one uses the radii, but for our setup working with the circumferences immediately is easier) together with the following data: (1) A cyclic ordering at each intersection point, (2) the choice of a marked point on each circle and (3) the choice of a global marked point on the whole configuration together with a choice of a circle this point lies on. Treelike means that the dual graph of this configuration, whose vertices correspond to the lobes and which has an edge whenever two circles intersect, is a tree. In the normalized cacti $Cacti^1(n)$ all circles have the same radius/circumference. This is only a quasi-operad, since associativity fails (cf. [Kau05, Remark 2.9.19]).

In [LUX08] it is shown that $Cact$ can be equivalently defined using the space of chord diagrams. To avoid even more notation, we apply the equivalence of Sullivan chord diagrams and Sullivan diagrams as defined in Section 1.1 and give a description of $Cacti^1$ and $Cacti$ in these terms: $Cact^1(n)$ is a finite CW complex with cells $\Delta^{f_\Gamma(1)-1} \times \cdots \times \Delta^{f_\Gamma(n)-1}$ for each Γ a cacti like $[\frac{1}{n}]$–Sullivan diagram whose boundary cycle belonging to the i–th labeled incoming leaf consists of $f_\Gamma(i)$ pieces. Thus the $t^j_1, \ldots, t^j_{f_\Gamma(j)}$ give the

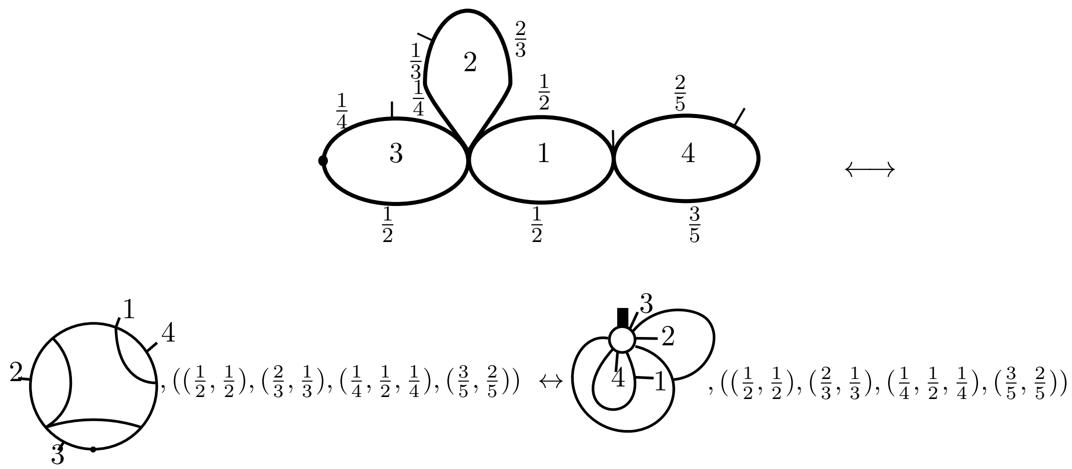


FIGURE 30. Three different representations of the same element in $Cacti^1(4)$

lengths of the pieces of a lobe starting at the spine (marked point) of the lobe. Hence we have an isomorphism

$$Cacti^1(n) \cong \bigcup_{\Gamma\in[\frac{1}{n}]-\text{Sullivan diagrams, cacti- like}} \Delta^{f_\Gamma(1)-1}\times\cdots\times\Delta^{f_\Gamma(n)-1}$$

where Δ^m is the standard simplex. The correspondence is shown in Figure 30, where we have drawn an element in $Cacti^1(4)$ first as an ordinary cactus, then as a Sullivan chord diagram with the corresponding arc lengths in $\Delta^1\times\Delta^1\times\Delta^2\times\Delta^1$ (i.e. the description corresponding to [LUX08]) and then in terms of the definition given in the lines above (Sullivan diagram with arc lengths in $\Delta^1\times\Delta^1\times\Delta^2\times\Delta^1$).

The attaching maps identify an element $(\Gamma,((t_1^1,\ldots,t^1_{f_\Gamma(1)}),\ldots,(t_1^n,\ldots,t^n_{f_\Gamma(n)})))$ with one of the $t_i^j=0$ with the pair $(\Gamma',((t_1^1,\ldots,t^1_{f_\Gamma(1)}),\ldots,t^j_{i-1},t^j_{i+1},\ldots,t^n_{f_\Gamma(n)})))$ where t_i^j is omitted and Γ' is the Sullivan diagram where the i–th boundary segment belonging to the boundary cycle labeled by the j–th leaf is contracted. In Figure 31 we have given an example, where in the left representative the length t_3^3 (so the last length belonging to the boundary cycle labeled by the leaf 3) equals zero and in the right representative the last segment of the boundary cycle of the diagram (which in this case is the first boundary segment belonging to white vertex) got contracted.

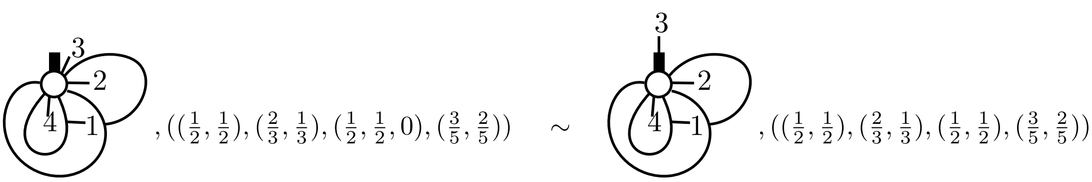


FIGURE 31. The gluing relation in $Cacti^1(4)$

As a space, we have $Cacti(n)\cong Cacti^1(n)\times\mathring{\Delta}^{n-1}$ where the extra parameters specify the lengths of the loops. So we get

$$Cacti(n) \cong \bigcup_{\Gamma\in[\frac{1}{n}]-\text{Sullivan diagrams, cacti- like}} \Delta^{f_\Gamma(1)-1}\times\cdots\times\Delta^{f_\Gamma(n)-1}\times\mathring{\Delta}^{n-1}$$

and the equivalence relation coming from the one in $Cacti^1(n)$.

By [Kau05, Cor. 5.2.3] the homology of the quasi-operad $Cacti^1$ is equivalent to the homology of $Cacti$ as a graded operad. As announced in [Vor05, Theorem 2.3] (see

[Kau05, Theorem 5.3.6] for a complete proof), the cacti operad is homotopy equivalent to the framed little disc operad, and by [Get94] the singular homology of the framed little disc operad is isomorphic as an algebraic operad to the BV-operad. Thus,

$$H_*(Cacti^1) \cong BV$$

as an equivalence of graded operads.

4.4. **Result and proof.** Now we have given enough definitions to state the main theorem of the section:

Theorem 4.7. *The complex $pl\mathcal{D}^{>0}_{cact}(n,1)$ is a chain model for the twisted operadic desuspended BV-operad, i.e.*

$$H_*(pl\mathcal{D}^{>0}_{cact}(-,1)) \cong \widetilde{s}^{-1}BV$$

as graded operads.

Before we prove the theorem, we first want to point out the consequence for the operations on Hochschild homology:

Corollary 4.8. *There is a non-trivial desuspended BV-algebra structure on the Hochschild homology of a commutative cocommutative open Frobenius dg-algebra (in particular on the Hochschild homology of a commutative Frobenius dg-algebra) which comes from an action of a chain model of the suspended Cacti operad on the Hochschild chains.*

More precisely, the product is the product given by the action of the looped diagram shown in Figure 23 and the BV-operator is trivial on Hochschild degree zero and the ordinary BV-operator on all other degrees. Spelling out the product explicitly, one sees that this BV-structure agrees with the BV-structure on the Hochschild homology of positive Hochschild degree given in [Abb13a, Section 7] and [Abb13b, Section 6].

To avoid introducing more notation, we write $pl\mathcal{D}_{cact}(n,1)_\bullet$ and $pl\mathcal{D}^{cst}_{cact}(n,1)_\bullet$ for the associated simplicial sets of the chain complexes (where we allow unlabeled incoming leaves at any point of the white vertex, cf. the proof of Lemma 1.5) and take their geometric realization. We define an operad structure on this space and show that on homology we have an isomorphism of operads $H_*(pl\mathcal{D}_{cact}(-,1)/pl\mathcal{D}^{cst}_{cact}(-,1)) \cong H_*(|pl\mathcal{D}_{cact}(-,1)_\bullet|, |pl\mathcal{D}^{cst}_{cact}(-,1))_\bullet|)$.

Write $X_\bullet = pl\mathcal{D}_{cact}(-,1)_\bullet$. We first explain why we do not need to care about degenerate simplices (i.e. diagrams with unlabeled leaves attached to the white vertex at other positions than the start half-edge). Denote the non-degenerate part by $X_\bullet^{non-deg}$. Since the boundary of a non-degenerate element in $X_\bullet$ is non-degenerate, the simplicial realization is the realization of $X_\bullet^{non-deg}$ as a semi-simplicial set, i.e.

$$|X_\bullet| \cong \coprod_k (X_k^{non-deg} \times \Delta^k)/\sim$$

with $(x, \delta^i y) \sim (d_i x, y)$.

So an element in the geometric realization is an equivalence class of a looped diagram together with an assignment of lengths to each piece of the white vertex. Thus a point in the geometric realization is a diagram where we consider the white vertex as a circle of length one up to the equivalence relation of rescaling and sliding the arc components. For composing y onto the i–th loop of x we assume that the boundary segment belonging to the i–th loop of x have lengths $(s_1, \dots, s_r)$. Then composition onto the i–th loop rescales the white vertex of y to lengths $\sum s_i$ and glues the diagram into these pieces gluing the start half-edge onto the labeled leaf and considering the arc components to have lengths zero. This is similar to the gluing in the arc complex and fat graph considered in [Pen87], [KLP03] and others. An example of such a composition is shown in Figure 32, where we glue an element in $|pl\mathcal{D}_{cact}(2,1)_\bullet|$ onto the second loop of an element in $|pl\mathcal{D}_{cact}(3,1)_\bullet|$ and thus obtain an element in $|pl\mathcal{D}_{cact}(4,1)_\bullet|$. We have written the lengths of the boundary segments of the white vertices next to the vertex in the picture. To stay with the notation we use in operads we write $y \circ_i x$ if we glue y onto the i–th loop of x (i.e. we switch the order against the ordinary gluing in the chain complex of looped diagrams).

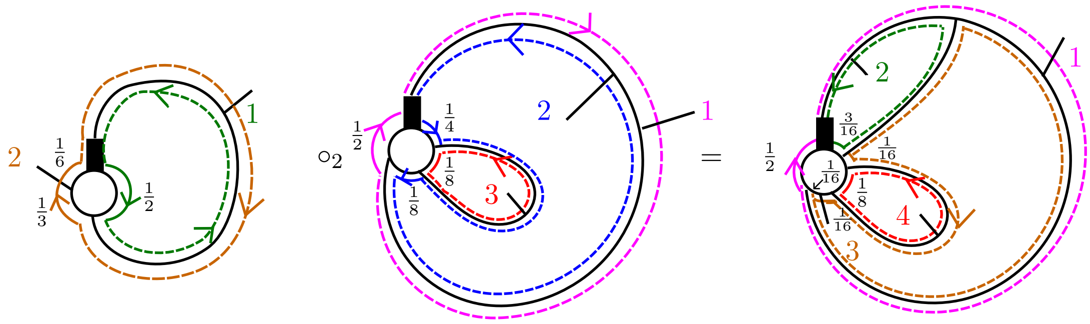


FIGURE 32. An example of the composition in $|pl\mathcal{D}_{cact}(-,1)_\bullet|$, gluing the left element onto the second (blue) loop of the right one

Lemma 4.9. *With the above operad structure on $pl\mathcal{D}_{cact}(-,1)$, we obtain an isomorphism of operads*

$$H_*(pl\mathcal{D}_{cact}(-,1)/pl\mathcal{D}^{cst}_{cact}(-,1)) \cong H_*(|pl\mathcal{D}_{cact}(-,1)_\bullet|\,,|pl\mathcal{D}^{cst}_{cact}(-,1))_\bullet|).$$

Proof. Given a simplicial set $X_\bullet$, we have an isomorphism of chain complexes $\overline{C}_*(X_\bullet) \cong C^{CW}_*(|X_\bullet|)$ where $\overline{C}_*$ denotes the normalized chains whereas C^{CW}_* stands for the cellular chains of a complex. In our situation this induces isomorphisms $pl\mathcal{D}_{cact}(n,1) \cong C^{CW}_*(|pl\mathcal{D}_{cact}(n,1)_\bullet|)$ and $pl\mathcal{D}^{cst}_{cact}(n,1) \cong C^{CW}_*(|pl\mathcal{D}^{cst}_{cact}(n,1)_\bullet|)$. Since the operadic composition on the geometric realization is a cellular map we get an induced composition on the cellular complexes. Hence we only have to check that on homology the composition commutes with the isomorphisms.

A diagram $x \in pl\mathcal{D}_{cact}(n,1)$ of degree k is mapped to the cell given by the $x \times \Delta^k$. Given a cell $x \times \Delta^k$ and a cell $y \times \Delta^l$ with the i–th loop of y non-constant, gluing $x \times \Delta^k$ onto the i–th loop of $y \times \Delta^l$ we obtain the union of the Δ^{k+l} cells which one gets by gluing the diagram y onto the diagram x in all possible ways. The signs come from the orientation of the cells. By the definition the complex $x \in pl\mathcal{D}^{cst}_{cact}(n,1)$ gets mapped to the cells belonging to the operad $|pl\mathcal{D}^{cst}_{cact}(-,1))|$. Since both are operadic ideals, we get the wished isomorphism on homology. $\square$

Lemma 4.10. *We have a homeomorphism of operads in pointed spaces*

$$|pl\mathcal{D}_{cact}(-,1)_\bullet|\,/\,|pl\mathcal{D}^{cst}_{cact}(-,1)_\bullet| \cong Cacti^c$$

where $Cacti^c$ is the one-point compactification of the cacti operad.

We do not see an easy proof of the fact that the structure maps in $Cacti$ are proper and hence that the structure maps in $Cacti^c$ are continuous. However, proving the homeomorphism in the above statement and checking that it preserves the obvious structure maps then implies the continuity of these maps.

Proof. Note that $|pl\mathcal{D}^{cst}_{cact}(n,1)_\bullet|$ is a closed subspace of the compact space $|pl\mathcal{D}_{cact}(n,1)_\bullet|$ (which is the realization of a finite simplicial set). It is a point set topological exercise that when given a compact space X with a closed subspace X', the one-point compactification of $X \backslash X'$ (the complement of X' in X) is homeomorphic to X/X'. Hence proving a homeomorphism

$$|pl\mathcal{D}_{cact}(n,1)_\bullet| \setminus |pl\mathcal{D}^{cst}_{cact}(n,1)_\bullet| \cong Cacti$$

induces a homeomorphism

$$|pl\mathcal{D}_{cact}(-,1)_\bullet|\,/\,|pl\mathcal{D}^{cst}_{cact}(-,1)_\bullet| \cong Cacti^c$$

which sends $\left|pl\mathcal{D}^{cst}_{cact}(-,1)_\bullet\right|$ to the point at infinity of $Cacti^c$.

Take (x,y) with $x=(\Gamma,\gamma_1,\cdots,\gamma_n)\in pl\mathcal{D}_{cact}(-,1)_k$ and $y=(s_0,\cdots,s_k)\in\Delta^n$ such that $[(x,y)]\in|pl\mathcal{D}_{cact}(n,1)_\bullet|\setminus\left|pl\mathcal{D}^{cst}_{cact}(n,1)_\bullet\right|$. This is equivalent to assuming that x is non-constant and that in y not all of the s_i belonging to the same loop in x are zero. More precisely, for $F(x)$ the corresponding cacti-like Sullivan diagram defined in Lemma 4.6 and f_Γ the map which counts how many of the boundary components of the white vertex belong to each boundary cycle, we reorder and relabel $(s_0,\cdots,s_k)$ to $(t^1_1,\ldots,t^1_{f_{F(\Gamma)}(1)},t^2_1,\ldots,t^n_{f_{F(\Gamma)}(n)})$. Then the assumption on y is equivalent to $R_i=\sum_j t^i_j\neq 0$ for all i.

We send the pair (x,y) to

$$((\frac{t^1_1}{R_1},\ldots,\frac{t^1_{f_{F(\Gamma)}(1)}}{R_1}),\ldots,(\frac{t^n_1}{R_n},\ldots,\frac{t^n_{f_{F(\Gamma)}(n)}}{R_n}),(R_1,\ldots,R_n))\in\Delta^{f_\Gamma(1)-1}\times\cdots\times\Delta^{f_\Gamma(n)-1}\times\mathring{\Delta}^{n-1}$$

lying in $Cacti(n)$ in the "cell" belonging to $F(\Gamma)$. Away from the boundary the coordinates of the simplex (i.e. the s_i) just give us the lengths of the pieces of the arc of a cacti.

The map is well-defined, since the equivalence relation given by the geometric realization (contracting a piece of the boundary is equivalent to setting the corresponding t^j_i zero) is the same equivalence relation as we have on cacti.

We next construct an inverse. Given

$$((t^1_1,\cdots,t^1_{f_\Gamma(1)},t^2_1,\ldots,t^n_{f_\Gamma(n)}),(R_1,\ldots,R_n))\in\Delta^{f_\Gamma(1)-1}\times\cdots\times\Delta^{f_\Gamma(n)-1}\times\mathring{\Delta}^{n-1}$$

in the "cell" corresponding to a cacti-like Sullivan diagram Γ, it is mapped to

$$[(K(\Gamma),(R_1\cdot t^1_1,\cdots,R_1\cdot t^1_{f_\Gamma(1)},R_2\cdot t^2_1,\ldots,R_n\cdot t^n_{f_\Gamma(n)}))]\in|pl\mathcal{D}_{cact}(n,1)_\bullet|\,.$$

It is not hard to check that this map is an inverse to the first one. Moreover, both maps are continuous, hence we have constructed the asked homeomorphism.

Because of the way we defined the maps, it is also clear that composition is preserved on $|pl\mathcal{D}_{cact}(n,1)_\bullet|\setminus\left|pl\mathcal{D}^{cst}_{cact}(n,1)_\bullet\right|$. Composition on $|pl\mathcal{D}_{cact}(n,1)_\bullet|\,/\left|pl\mathcal{D}^{cst}_{cact}(n,1)_\bullet\right|$ sends everything containing the basepoint to the basepoint and hence agrees with the composition of the one-point compactification.

□

The next step of the proof goes along the lines of [Kau05, Section 5]. We first need to define two further operads, one given by a semi-direct product and the analog given by the semi-direct smash product.

Recall the simplex operad $\mathcal{D}$ with $\mathcal{D}(n)=\mathring{\Delta}^{n-1}$. Let $\mathcal{D}\ltimes Cacti^1$ be the operad with $(\mathcal{D}\ltimes Cacti^1)(n)=\mathcal{D}(n)\times Cacti^1(n)$ with diagonal Σ_n–action and the composition which for $(d,c)\in\mathcal{D}(n)\times Cacti^1(n)$ and $(d',c')\in\mathcal{D}(k)\times Cacti^1(k)$ is defined by

$$(d,c)\circ_i(d',c')=(d\circ_i d',c\circ^{d'}_i c')$$

with $c\circ^{d'}_i c'$ computed via the following procedure: Write $d'=(d_1,\cdots,d_k)$ and rescale c' by d', i.e. scale the j–th lobe by d_j. Then we glue c' into the i–th lobe of c and rescale all lobes back to length 1. A more general theory of the semi-direct product of operads can be found in [Kau05, Section 1.3].

Similarly, we define the operad $\mathcal{S}ph\overset{\ltimes}{\wedge}Cacti^1_+$ with pointed spaces $(\mathcal{S}ph\overset{\ltimes}{\wedge}Cacti^1_+)(n)=\mathcal{S}ph(n)\wedge Cacti^1_+(n)$, where $Cacti^1_+(n)$ is the space $Cacti^1(n)$ with added disjoint basepoint. The composition is defined by

$$(d\wedge c)\circ_i(d'\wedge c')=(d\circ_i d')\wedge(c\;\widetilde{\circ^{d'}_i}\;c')$$

with

$$c\;\widetilde{\circ^{d'}_i}\;c'=\begin{cases}* & \text{if any of } c,c' \text{ or } d \text{ is the base point}\\ c\circ^{d'}_i c' & \text{else.}\end{cases}$$

On the level of spaces we have

$$(\mathcal{D}(n) \times Cacti^1(n))^c \cong \mathcal{D}(n)^c \wedge Cacti^1(n)^c \cong \mathcal{S}ph(n) \wedge Cacti^1_+(n)$$

since $Cacti^1(n)$ is a finite CW-complex and hence compact, i.e. its one-point compactification just adds a disjoint basepoint. We defined the operad structures exactly such that

$$(\mathcal{D} \ltimes Cacti^1)^c \cong \mathcal{S}ph \overset{\ltimes}{\wedge} Cacti^1_+.$$

Rewriting the first part of [Kau05, Theorem 5.2.2] in terms of the simplex operad, we have:

Lemma 4.11. *There is a homeomorphism of operads*

$$Cacti \cong \mathcal{D} \ltimes Cacti^1$$

and hence

$$Cacti^c \cong (\mathcal{D} \ltimes Cacti^1)^c \cong \mathcal{S}ph \overset{\ltimes}{\wedge} Cacti^1_+.$$

As the last step, we need the analog of the second part [Kau05, Theorem 5.2.2], which is the analog of the comparison between loops and Moore loops:

Lemma 4.12. *There is a homotopy equivalence of pointed quasi-operads*

$$\mathcal{S}ph \overset{\ltimes}{\wedge} Cacti^1_+ \simeq \mathcal{S}ph \wedge Cacti^1_+.$$

Proof. Completely similar to [Kau05], the perturbed and unperturbed multiplications are homotopic. Choosing a line segment from $d' \in \Delta^{k-1}$ to the midpoint of the simplex $(1/k, \ldots, 1/k)$ and denoting the corresponding path d'_t, the homotopy for the pointed quasi-operad composition is defined by

$$(d, c) \circ_i (d', c') = (d \circ_i d', c \mathbin{\widetilde{\circ_i^{d'_t}}} c')$$

where we rescale c' by d'_t. ☐

Now we are able to prove the main proposition of the section:

Proof of Theorem 4.7. Collecting all the homeomorphisms and homotopy equivalences of (quasi-)operads shown in this section, we get the following isomorphism of operads,

$$\begin{aligned}
H_*(pl\mathcal{D}^{>0}_{cact}(-,1)) &\cong H_*(pl\mathcal{D}_{cact}(-,1)/pl\mathcal{D}^{cst}_{cact}(-,1)) \\
&\cong H_*(|pl\mathcal{D}_{cact}(-,1)_\bullet|, |pl\mathcal{D}^{cst}_{cact}(-,1))_\bullet|) \\
&\cong \widetilde{H}_*(|pl\mathcal{D}_{cact}(-,1)_\bullet| / |pl\mathcal{D}^{cst}_{cact}(-,1))_\bullet|) \\
&\cong \widetilde{H}_*(Cacti^c) \\
&\cong \widetilde{H}_*(\mathcal{S}ph \overset{\ltimes}{\wedge} Cacti^1_+) \\
&\cong \widetilde{H}_*(\mathcal{S}ph \wedge Cacti^1_+) \\
&\cong \widetilde{\mathcal{S}}^{-1} \otimes H_*(Cacti)
\end{aligned}$$

where the last step is the Kuenneth morphism $\widetilde{H}_*(X) \otimes \widetilde{H}_*(Y) \to \widetilde{H}_*(X \wedge Y)$ which is an isomorphism since $\widetilde{H}_*(X)$ is free in the case considered here. Moreover, the Kuenneth morphism is a symmetric monoidal functor and thus preserves the operad structure.

Since $H_*(Cacti)$ is the BV operad, the isomorphism of operads

$$H_*(pl\mathcal{D}^{>0}_{cact}(-,1)) \cong \widetilde{\mathcal{S}}^{-1} \otimes BV$$

follows. ☐

Appendix A. An overview over the complexes of looped diagrams

In the two tables below we give an overview over all the complexes defined in the paper. Table 1 gives the basic definitions of the complexes and Table 2 applies constructions to complexes $\mathscr{C}$ from the first table.

$\mathscr{C}$	Name (if given)	Description of generators $x = (\Gamma, \gamma_1, \cdots, \gamma_{n_1}) \in \mathscr{C}$	Place defined in the paper
$l\mathcal{D}([{n_1 \atop m_1}], [{n_2 \atop m_2}])$	*looped diagrams*	Γ a commutative Sullivan diagram with $n_1 + m_1 + m_2$ labeled leaves and n_1 loops starting at the first n_1–labeled leaves	Def. 1.8
$l\mathcal{D}_+([{n_1 \atop m_1}], [{n_2 \atop m_2}])$	*looped diagrams with positive boundary condition*	$x \in l\mathcal{D}([{n_1 \atop m_1}], [{n_2 \atop m_2}])$ such that every connected component contains at least one white vertex or one of the m_2 last labeled leaves	Def. 1.9
$pl\mathcal{D}([{n_1 \atop m_1}], [{n_2 \atop m_2}])$, $pl\mathcal{D}_+([{n_1 \atop m_1}], [{n_2 \atop m_2}])$	*positively oriented looped diagrams (with positive boundary condition)*	$x \in l\mathcal{D}([{n_1 \atop m_1}], [{n_2 \atop m_2}])$ (or $l\mathcal{D}_+([{n_1 \atop m_1}], [{n_2 \atop m_2}])$, resp.) such that all loops are positively oriented	Def. 1.10
$pl\mathcal{D}_{start}([{n_1 \atop m_1}], [{n_2 \atop m_2}])$		$x \in pl\mathcal{D}([{n_1 \atop m_1}], [{n_2 \atop m_2}])$ such that each loop γ_i consists of exactly one boundary segment of a white vertex which is the first boundary segment of that white vertex	Def. 1.23
$pl\mathcal{D}_{\mathscr{C}om}([{n_1 \atop m_1}], [{n_2 \atop m_2}])$		$x \in pl\mathcal{D}_+([{n_1 \atop m_1}], [{n_2 \atop m_2}])$ such that Γ is a disjoint union of n_2 white vertices with trees attached to it and m_2 labeled outgoing leaves with trees attached to them	Def. 3.1
$\widetilde{pl\mathcal{D}}_{\mathscr{C}om}([{n_1 \atop m_1}], [{n_2 \atop m_2}])$		$x \in pl\mathcal{D}_{\mathscr{C}om}([{n_1 \atop m_1}], [{n_2 \atop m_2}])$ built via a specific procedure described in Def. 3.5	Def. 3.5
$pl\mathcal{D}_{cact}(n_1, n_2)$		$x \in pl\mathcal{D}_+([{n_1 \atop 0}], [{n_2 \atop 0}])$, all white vertices of Γ connected, Γ is embeddable into the plane, all loops irreducible, every bound. segm. of white vert. is part of exactly one loop, one constant loop per genus	Def. 4.4

Table 1. Definitions of the (sub)complexes of $l\mathcal{D}$

$\mathscr{D}$	Complex?	Name	Definition of $x \in \mathscr{D}$	Place defined in the paper
$\mathscr{C}_d$	Yes		the degree of a looped diagram $(\Gamma, \langle \gamma_1^1, \ldots, \gamma_1^{t_1} \rangle, \ldots, \langle \gamma_n^1, \ldots, \gamma_n^{t_n} \rangle)$ is shifted by $-d \cdot \chi(\Gamma, \partial_{out})$	Def. 1.12
$\mathscr{C}^{t_1, \cdots, t_{n_1}}$	No	*irreducible looped diagrams of type* $(t_1, \cdots, t_{n_1})$	$x = (\Gamma, \langle \gamma_1^1, \ldots, \gamma_1^{t_1} \rangle, \ldots, \langle \gamma_n^1, \ldots, \gamma_n^{t_n} \rangle)$ (for the notation cf. Section 1.4) such that all the loops γ_i^j are irreducible	Def. 1.20
$\mathscr{C}^{cst}$	Yes	*partly constant diagrams*	$x \in \mathscr{C}^{t_1, \cdots, t_{n_1}}$ with at least one $t_i = 0$ (spanned by those $x = (\Gamma, \gamma_1, \cdots, \gamma_{n_1})$ with one of the γ_i constant)	Def. 1.13
$\mathscr{C}^{>0}$	Yes	*non-constant diagrams*	$x \in \mathscr{C}^{t_1, \cdots, t_{n_1}}$ with all $t_i > 0$ (split complement of $\mathscr{C}^{cst}$)	Def. 1.16
$i\mathscr{C}$	Yes	*Products of irreducible looped diagrams*	$\mathscr{D} = \prod_{t_1, \cdots, t_{n_1}} \mathscr{C}^{t_1, \cdots, t_{n_1}}$, i.e. infinite sums of elements, in general composition is not well-defined	Def. 1.21

Table 2. Constructions applied to complexes $\mathscr{C} \subseteq l\mathcal{D}(\left[\begin{smallmatrix} n_1 \\ m_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} n_2 \\ m_2 \end{smallmatrix}\right])$

References

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Paper C

NATURAL OPERATIONS ON THE HIGHER HOCHSCHILD HOMOLOGY OF COMMUTATIVE ALGEBRAS

ANGELA KLAMT

Abstract. We give the definition of higher (co)Hochschild homology of dg-functors in the sense of Pirashvili and define their formal operations in the sense of Wahl, which give a complex of operations on the higher Hochschild homology of commutative algebras. In certain cases we obtain smaller models of the operations and identify them with the dual of the chains on the mapping space of simplicial sets.

Introduction

Given a simplicial set $X_\bullet$ and a commutative algebra A one can associate to this data a chain complex $CH_{X_\bullet}(A)$, the *higher Hochschild complex of A with respect to $X_\bullet$* defined in [Pir00], where the classical Hochschild complex is the one associated to the standard simplicial decomposition of the circle. In this paper, we are interested in the chain complex of natural operations on the higher Hochschild complex of given types of algebras such as commutative algebras, Poisson algebras or commutative Frobenius algebras. Following the approach of [Wah12], we approximate this chain complex by a complex of *formal operations* which we identify in certain cases. Our methods differ from [Wah12] in that we only work with strictly associative algebras. This allows us to use simplicial techniques to give easier proofs of many results in [Wah12] in the case of strictly associative algebras.

Let $\mathcal{E}$ be a *commutative PROP*, i.e. a symmetric monoidal dg-category with objects the natural numbers equipped with a symmetric monoidal dg-functor $i : \mathscr{C}om \to \mathcal{E}$ which is the identity on objects (where $\mathscr{C}om$ is the commutative PROP which is the $\mathbb{Z}$-linearization of the category of finite ordinals). An $\mathcal{E}$-algebra is a strong symmetric monoidal functor $\Phi : \mathcal{E} \to \mathrm{Ch}$. Let $X_\bullet$ be a simplicial finite set. The higher Hochschild complex of Φ with respect to $X_\bullet$ (in the sense of [Pir00]) denoted by $CH_{X_\bullet}(\Phi(1))$ is the total complex of a simplicial chain complex which in simplicial degree k is given by $\Phi(1)^{\otimes |X_k|}$ where $|X_k|$ denotes the cardinality of the set X_k. The boundary maps are induced by the boundary maps of the simplicial set. Similarly, one can define the higher Hochschild homology for any dg-functor $\Phi : \mathcal{E} \to \mathrm{Ch}$ (not necessary strong symmetric monoidal) by taking the total complex of the simplicial chain complex with simplicial degree k equal to $\Phi(|X_k|)$. Again, the boundaries are induced by the boundary maps of $X_\bullet$ which act on Φ via the functor $i : \mathscr{C}om \to \mathcal{E}$ (see Definition 2.3). This defines a functor

$$C_{X_\bullet}(-) : Fun(\mathcal{E}, \mathrm{Ch}) \to \mathrm{Ch}$$

and the construction can be extended to arbitrary simplicial sets using homotopy colimits. When restricted to strong symmetric monoidal functors, $C_{X_\bullet}(\Phi)$ is isomorphic to the higher Hochschild complex $CH_{X_\bullet}(\Phi(1))$. On the other hand the higher Hochschild construction can be defined via an enriched tensor product which then, working in the model category of topological spaces instead of chain complexes looks similar to the definition of topological Chiral homology.

In the first part of this paper we work in the category of chain complexes and are interested in the natural transformations of the (iterated) higher Hochschild homology of $\mathcal{E}$-algebras with respect to simplicial sets $X_\bullet$ and $Y_\bullet$ denoted by $\mathrm{Nat}^{\otimes}_{\mathcal{E}}(X_\bullet, Y_\bullet) = \mathrm{Hom}(CH_{X_\bullet}(-), CH_{Y_\bullet}(-))$. We define the complex of formal operations as the complex $\mathrm{Nat}_{\mathcal{E}}(X_\bullet, Y_\bullet) = \mathrm{Hom}(C_{X_\bullet}(-), C_{Y_\bullet}(-))$, i.e. we test on all functors and not only on the strong symmetric monoidal ones. Hence, there is a restriction from $\mathrm{Nat}_{\mathcal{E}}(X_\bullet, Y_\bullet)$ to $\mathrm{Nat}^{\otimes}_{\mathcal{E}}(X_\bullet, Y_\bullet)$. Analogously to [Wah12, Theorem 2.9] we give conditions on the PROP implying that the restriction is injective/surjective/a quasi-isomorphism (see Theorem 3.4).

The higher Hochschild complex is invariant under quasi-isomorphisms of functors and quasi-isomorphic to its reduced version (see Section 2.2). It actually can be defined as a functor $C_{X_\bullet}(\Phi)(-) : \mathrm{Ch} \to \mathrm{Ch}$ and so we can consider the iterated Hochschild complex. We show that for two simplicial sets $X_\bullet$ and $X'_\bullet$ there is a quasi-isomorphism between $C_{X_\bullet}(C_{X'_\bullet}(\Phi))$ and $C_{X_\bullet \amalg X'_\bullet}(\Phi)$ (see Theorem 2.13) and so the general case is covered (up to quasi-isomorphism) by taking the higher Hochschild complex once.

We similarly define the higher coHochschild complex $D_{X_\bullet}(\Phi)$ of a coalgebra (see Definition 2.4) and in general of dg-functors $\Psi : \mathcal{E}^{op} \to \mathrm{Ch}$. The formal operations between these are defined as $\mathrm{Nat}^{D}_{\mathcal{E}}(X_\bullet, Y_\bullet) := \mathrm{Hom}(D_{X_\bullet}(-), D_{Y_\bullet}(-))$. Our first technical theorem connects the two complexes of formal operations to a third more computable complex:

Theorem A (Theorem 3.2 and Theorem 3.6)**.** For any commutative PROP $\mathcal{E}$ and simplicial sets $X_\bullet$ and $Y_\bullet$ there are isomorphisms of chain complexes

$$\mathrm{Nat}_{\mathcal{E}}(X_\bullet, Y_\bullet) \cong D_{X_\bullet}(C_{Y_\bullet}(\mathcal{E}(-,-))) \cong \mathrm{Nat}^{D}_{\mathcal{E}}(Y_\bullet, X_\bullet).$$

For ordinary Hochschild homology this has been proved by Wahl in [Wah12, Theorem 2.1].

In the second part of the paper we consider the case $\mathcal{E} = \mathscr{C}om$. Under two types of conditions on $X_\bullet$ and $Y_\bullet$ we identify the complex $\mathrm{Nat}_{\mathscr{C}om}(X_\bullet, Y_\bullet)$ with other, better known complexes.

First, working over a field $\mathbb{F}$, for $X_\bullet$ arbitrary and $Y_\bullet$ a simplicial set that is weakly equivalent to a simplicial finite set, a quasi-isomorphism of functors $C^*(Y_\bullet^{\times -}) \simeq A^{\otimes -} : \mathscr{C}om \to \mathrm{Ch}$ induces a quasi-isomorphism

$$\mathrm{Nat}_{\mathscr{C}om}(X_\bullet, Y_\bullet) \simeq CH_{X_\bullet}(A)^*$$

(see Proposition 4.2). In particular if $\mathbb{Q} \subset \mathbb{F}$, the deRham algebra $\Omega^\bullet(Y_\bullet; \mathbb{F})$ fulfills this property (see Appendix A) and therefore

$$\mathrm{Nat}_{\mathscr{C}om}(X_\bullet, Y_\bullet) \simeq CH_{X_\bullet}(\Omega^\bullet(Y_\bullet; \mathbb{F}))^*.$$

Our second computation of $\mathrm{Nat}_{\mathscr{C}om}(X_\bullet, Y_\bullet)$ is when the dimension of the simplicial set $X_\bullet$ is smaller than the connectivity of $Y_\bullet$. Using Bousfield's spectral sequence (see [Bou87]), we get a quasi-isomorphism between $\mathrm{Nat}_{\mathscr{C}om}(X_\bullet, Y_\bullet)$ and the simplicial chains on the topological mapping space $\hom_{Top}(|X_\bullet|, |Y_\bullet|)$. We show moreover, that this quasi-isomorphism preserves some extra structure close to a comultiplication:

Theorem B (See Theorem 4.11)**.** For an arbitrary simplicial set $Y_\bullet$ and a finite simplicial set $X_\bullet$ such that $dim(X_\bullet) \leq Conn(Y_\bullet)$, there is weak equivalence

$$\overline{C}_*(Hom_{Top}(|X_\bullet|, |Y_\bullet|)) \simeq \mathrm{Nat}_{\mathscr{C}om}(X_\bullet, Y_\bullet).$$

If we take homology with coefficients in a field $\mathbb{F}$, the comultiplication on the homology $H_*(\mathrm{Nat}_{\mathscr{C}om}(X_\bullet, Y_\bullet); \mathbb{F})$ induced by the one on $H_*(Hom_{Top}(|X_\bullet|, |Y_\bullet|); \mathbb{F})$ commutes with restriction to the filtration of Nat, i.e.

$$\begin{tikzcd}
H_*(\mathrm{Nat}_{\mathscr{C}om}(X_\bullet, Y_\bullet); \mathbb{F}) \arrow[r] \arrow[d] & H_*(\mathrm{Nat}_{\mathscr{C}om}(X_\bullet, Y_\bullet); \mathbb{F}) \otimes H_*(\mathrm{Nat}_{\mathscr{C}om}(X_\bullet, Y_\bullet); \mathbb{F}) \arrow[d] \\
H_*(\mathrm{Nat}^{2m}(X_\bullet, Y_\bullet); \mathbb{F}) \arrow[r, "H_*(\Delta_{2m})"] & H_*(\mathrm{Nat}^{m}(X_\bullet, Y_\bullet); \mathbb{F}) \otimes H_*(\mathrm{Nat}^{m}(X_\bullet, Y_\bullet); \mathbb{F})
\end{tikzcd}$$

and

$$\begin{tikzcd}
H_*(\mathrm{Nat}_{\mathscr{C}om}(X_\bullet, Y_\bullet); \mathbb{F}) \arrow[r] \arrow[d] & H_*(\mathrm{Nat}_{\mathscr{C}om}(X_\bullet, Y_\bullet); \mathbb{F}) \otimes H_*(\mathrm{Nat}_{\mathscr{C}om}(X_\bullet, Y_\bullet); \mathbb{F}) \arrow[d] \\
H_*(\mathrm{Nat}^{2m+1}(X_\bullet, Y_\bullet); \mathbb{F}) \arrow[r, "H_*(\Delta_{2m+1})"] & H_*(\mathrm{Nat}^{m+1}(X_\bullet, Y_\bullet); \mathbb{F}) \otimes H_*(\mathrm{Nat}^{m}(X_\bullet, Y_\bullet); \mathbb{F})
\end{tikzcd}$$

commute.

Here, $\mathrm{Nat}^m(X_\bullet, Y_\bullet)$ is the filtration of $\mathrm{Nat}_{\mathscr{C}om}(X_\bullet, Y_\bullet)$ by its cosimplicial degree. The families of maps $\Delta_{2m} : \mathrm{Nat}^{2m}(X_\bullet, Y_\bullet) \to \mathrm{Nat}^m(X_\bullet, Y_\bullet) \otimes \mathrm{Nat}^m(X_\bullet, Y_\bullet)$ and $\Delta_{2m+1} : \mathrm{Nat}^{2m+1}(X_\bullet, Y_\bullet) \to \mathrm{Nat}^{m+1}(X_\bullet, Y_\bullet) \otimes \mathrm{Nat}^m(X_\bullet, Y_\bullet)$ come from a comultiplication on the cosimplicial simplicial abelian group underlying $\mathrm{Nat}_{\mathscr{C}om}(X_\bullet, Y_\bullet)$.

The proof of Theorem B is similar to the proof of [PT03, Theorem 2] and [GTZ10a, Proposition 2.4.2] but since we are in a kind of dual situation and we do not know a reference for the theorem in this situation, we need to check the compatibility of the maps again.

The last part of the paper is an attempt to carry over the techniques to a much broader generality. We work with $\mathscr{M}$ the monoidal model category of chain complexes (with the projective model structure) or topological spaces (with the mixed model structure) and $\mathcal{E}$ a small category enriched over $\mathscr{M}$. We define the Hochschild construction of an enriched functor $\Phi : \mathcal{E} \to \mathscr{M}$ with respect to an enriched functor $A : \mathcal{E}^{op} \to \mathrm{Ch}$ as a specific model for the derived tensor product. More explicitly, for chain complexes and a functor $A : \mathcal{E} \to \mathrm{Ch}$ with an h–projective replacement $B_A \to A$ (see Def. 5.2) we define

$$C_A(\Phi) := \Phi \underset{\mathcal{E}}{\otimes} B_A$$

and similarly the coHochschild construction as

$$D_A(\Psi) = \hom_{\mathcal{E}^{op}}(B_A, \Psi).$$

In particular the condition of being an h-projective resolution implies that $C_A(-)$ is a model of the left derived functor of $\Phi \underset{\mathcal{E}}{\otimes} A$ and $D_A(-)$ a model of the right derived functor of $\hom_{\mathcal{E}^{op}}(A, -)$. Hence for any dg-category $\mathcal{E}'$ with a map $\mathcal{E} \to \mathcal{E}'$, the Hochschild construction defines a functor $C_A(-) : Fun(\mathcal{E}', \mathrm{Ch}) \to \mathrm{Ch}$ and given $A, A' : \mathcal{E}' \to \mathrm{Ch}$ we can define the formal transformations $\mathrm{Nat}_{\mathcal{E}'}(A, A')$ as all transformations of these functors. We can prove more general versions of Theorem A and in particular deduce:

Corollary C (see Cor. 5.15)**.** Let $\mathcal{E}$ and $\mathcal{E}'$ be small categories cofibrantly enriched over Ch together with a functor $\mathcal{E} \to \mathcal{E}'$ and let $A, A' : \mathcal{E}^{op} \to \mathrm{Ch}$ be two enriched functors. Then

$$\mathrm{Nat}_{\mathcal{E}'}(A, A') \simeq D_A C_{A'}(\mathcal{E}'(-,-)).$$

The paper is organized as follows: In Section 1 we fix notations and conventions on (double) chain complexes and simplicial sets. More details are given in Appendix B, which we will refer to if needed. In Section 2.1 we give the definitions of the higher Hochschild and coHochschild complexes which are the main subject of the paper. In Section 2.2 we establish basic properties of these using the simplicial structure given in our situation. In Section 2.3 we show that the iterated higher (co)Hochschild complex up to quasi-isomorphism is covered by the single one by applying it with respect to the disjoint union of simplicial sets. In Section 3 we define the formal operations of the (co)Hochschild construction and state Theorem A. We also explain the connection to monoidal functors and define the Δ_{2m} and Δ_{2m+1} maps which are used in Theorem B. In Section 4 we fix the commutative PROP and state examples where the formal operations can be identified with the dual of the higher Hochschild complex of some algebra. The proof of the examples is given in Appendix A. Finally, in the last part of this section we establish the details from [Bou87] to give a proof of Theorem B. Section 5 deals with the more general setup in monoidal model categories. In Section 5.2 we define the (co)Hochschild complex in chain complexes and topological spaces and state the analog of Theorem A for general dg-categories. In Section 5.4.1 we explain how to see higher Hochschild homology in this more general setup and how to deduce Theorem A from the more general theorems given before.

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Contents

1. Homological algebra setup

1.1. **Chain complexes and double complexes.** Throughout this paper we will use chain and double chain complexes as dg-categories. In this section we give the sign conventions and notations used later on.

Notation 1.1 (Sign Conventions)**.** In this paper Ch means the category of $\mathbb{Z}$-graded chain complexes over $\mathbb{Z}$, unless otherwise specified. For two chain complexes A_* and B_*

we fix the differential on $A_k \otimes B_l$ to be $d_A \otimes id + (-1)^k id \otimes d_B$. A dg-category $\mathscr{C}$ is a category enriched over Ch, i.e. it has morphism spaces $\mathscr{C}(a,b)$ which are chain complexes together with chain maps $k \to \mathscr{C}(a,a)$ and $\mathscr{C}(a,b) \otimes \mathscr{C}(b,c) \to \mathscr{C}(a,c)$ which fulfill the unit and associativity conditions. Note that by this convention postcomposition with morphisms acts from the right. For an abelian category $\mathscr{A}$ the dg-category $\mathrm{Ch}(\mathscr{A})$ has chain complexes C_* in $\mathscr{A}$ as objects. A morphism f of degree k in $\mathrm{Ch}(C_*, D_*)$ is a family of maps $(f_p) : C_p \to D_{p+k}$ in $\mathscr{A}$. The differential on $\mathrm{Ch}(C_*, D_*)$ is defined as $d(f)_i = (-1)^i (d_D \circ f_i - f_{i-1} \circ d_C) : C_i \to D_{i+d-1}$ (note that by the convention of functions acting from the right the sign differs from the usual one). For $\mathscr{A}$ the category of abelian groups, we define the dual $(A_*)^* := \mathrm{Ch}(A_*, \mathbb{Z})$ with $\mathbb{Z}$ the trivial complex concentrated in degree 0. For an element $f \in A_k^*$, i.e. $f_i = 0$ for $i \neq k$, one gets $d(f)_i = 0$ for $i \neq k+1$ and $d(f) = (-1)^{|f|} f \circ d_A$.

A map $f : C_* \to D_*$ in $\mathrm{Ch}(\mathscr{A})$ is called a chain map if it is a degree zero cycle in $\mathrm{Ch}(C_*, D_*)$. A chain map is a quasi-isomorphism if it induces an isomorphism on homology. Two chain maps $f, g : C_* \to D_*$ are chain homotopic if there is a degree one map $s \in \mathrm{Ch}(C_*, D_*)_1$ such that $d \circ s + s \circ d = f - g$. A chain map $f : C_* \to D_*$ is a chain homotopy equivalence if there exists a map $g : D_* \to C_*$ such that $f \circ g$ and $g \circ f$ are homotopic to the identity of D_* and C_*, respectively.

For a dg-category $\mathscr{C}$ a dg-functor $\Phi : \mathscr{C} \to \mathrm{Ch}$ is an enriched functor, i.e. the structure maps

$$c_\Phi : \Phi(a) \otimes \mathscr{C}(a,b) \to \Phi(b)$$

are chain maps.

Notation 1.2. A *double chain complex* $C_{*,*}$ is for each p, q an abelian group $C_{p,q}$ with maps $d_h : C_{p,q} \to C_{p-1,q}$ and $d_v : C_{p,q} \to C_{p,q-1}$ such that $d_h \circ d_h = 0$, $d_v \circ d_v = 0$ and $d_h \circ d_v = d_v \circ d_h$. By this, a double chain complex can be viewed as a chain complex of chain complexes in two ways: The first one has in each degree p the chain complex $B_p = C_{p,*}$ (i.e. the differential $d_h : B_p \to B_{p-1}$ is the horizontal one). The second one has in degree q the chain complex $D_q = C_{*,q}$ and the differential is the vertical one.

These two ways of seeing double chain complexes as objects in the abelian category of chain complexes induce two structures of a dg-category on the category of double chain complexes.

More precisely, we define the dg-category $d\,\mathrm{Ch}^h$ to have as objects double chain complexes $C_{*,*}$. A morphism of degree k in $d\,\mathrm{Ch}^h(C_{*,*}, D_{*,*})$ is a map $f : C_{*,*} \to D_{*+k,*}$ which is a chain map with respect to d_v (i.e. $d_v \circ f = f \circ d_v$). The differential of f is given by $d^h(f)_{p,q} = (-1)^p (d_D^h \circ f_{p,q} - f_{p-1,q} \circ d_C^h)$.

Similarly, the category $d\,\mathrm{Ch}^v$ is the category with the same objects but inheriting the enriched structure with respect to the vertical differential. An element $f \in d\,\mathrm{Ch}^v(C_{*,*}, D_{*,*})$ is a map $f : C_{*,*} \to D_{*,*+k}$ which is a chain map with respect to d_h and has differential $d^v(f)_{p,q} = (-1)^q (d_D^v \circ f_{p,q} - f_{p,q-1} \circ d_C^v)$.

Chain maps of double chain complexes, i.e. maps of degree zero commuting with both differentials, are precisely the degree zero cycles of the morphism complexes of either category.

We want to define the total complex of a double complex such that it gives a dg-functor $d\,\mathrm{Ch}^v \to \mathrm{Ch}$. This is done as follows: For a double complex $C_{p,q}$ define $\mathrm{Tot}^{\prod}(C)$ to be the *product double complex* with

$$\mathrm{Tot}^{\prod}(C)_n = \prod_{p+q=n} C_{p,q}$$

and the *direct sum double complex*

$$\mathrm{Tot}^{\oplus}(C)_n = \bigoplus_{p+q=n} C_{p,q}$$

both with differential $d_{p,q} = d_p^h + (-1)^p d_q^v$. Note that for a first or third quadrant double complex both complexes agree.

We define $\mathrm{sTot}^{\prod}(C)$ and $\mathrm{sTot}^{\oplus}(C)$ to be the *switched double complexes* with the role of the horizontal and vertical direction switched. As an abelian group $\mathrm{sTot}^{\prod}(C) = \mathrm{Tot}^{\prod}(C)$ and $\mathrm{sTot}^{\oplus}(C) = \mathrm{Tot}^{\oplus}(C)$ but the differentials are $d_{p,q} = (-1)^q d_p^h + d_q^v$. In both cases, the switched and unswitched complex are isomorphic via the isomorphism

$$x \in C_{p,q},\ x \mapsto (-1)^{pq} x. \tag{1.1}$$

For $f \in d\,\mathrm{Ch}^v(C_{*,*}, D_{*,*})$ of degree $|f|$ (i.e. $f_{p,q} : C_{p,q} \to D_{p,q+|f|}$) and $x \in C_{p,q}$ we define$(\mathrm{Tot}(f)(x))_{p,q} = f_{p,q}(x_{p,q})$ and $\mathrm{sTot}(f)(x) = (-1)^{|f|p} f(x)$ for both the product and the direct sum total complexes. With these definitions both functors are dg-functors $d\,\mathrm{Ch}^v \to \mathrm{Ch}$ (see Proposition B.1).

For a double complex $C_{p,q}$ we define its filtration by columns as

$$F_s = \prod_{p \leq s} C_{p,q}.$$

For a right half plane (first and fourth quadrant) double complex, the product becomes a direct sum, whereas for a left half plane (second and third quadrant) double complex it can be non-finite. This filtration yields the spectral sequence of double complexes (see Appendix B.3). The spectral sequence of a right half plane double complex converges to the direct sum total complex, the one of a left half plane double complex converges conditionally to the product total complex. We show that this implies that for $C_{p,q}$ and $D_{p,q}$ both either right or left half plane double complexes and a chain map $f : C_{p,q} \to D_{p,q}$ which is a quasi-isomorphism in $d\,\mathrm{Ch}^v$ (i.e. a quasi-isomorphism with homology taken in the vertical direction), f induces a quasi-isomorphism of their direct sum or product complexes, respectively (see Corollary B.12). If on the other hand $f : C_{p,q} \to D_{p,q}$ is a quasi-isomorphism in $d\,\mathrm{Ch}^h$ (i.e. in the category with the horizontal differential) the spectral sequence argument used in the previous case does not work. However, if f is a chain homotopy equivalence in $d\,\mathrm{Ch}^h$ it still induces a quasi-isomorphism of the direct sum or product total complexes, respectively (see Corollary B.14).

1.2. **Simplicial sets.** In this section we recall the sign conventions and notation for (co)simplicial sets.

Denote by Δ the simplex category with objects totally ordered finite sets and morphisms order preserving maps. Let $\mathscr{A}$ be an abelian category. A simplicial object $A_\bullet$ in $\mathscr{A}$ is a functor $A_\bullet : \Delta^{op} \to \mathscr{A}$. We denote the boundary maps by d_i and the degeneracy maps s_i.

The chain complex $C_*(A_\bullet) \in \mathrm{Ch}(\mathscr{A})$ is given by A_k in the k-th degree and differential $d = \sum_{i=0}^n (-1)^i d_i$.

Definition 1.3 (cf. [Wei95, Chapter 8.3])**.** The *normalized chain complex* $N_*(A_\bullet) \in \mathrm{Ch}(\mathscr{A})$ is defined to be

$$N_n(A_\bullet) = \bigcap_{i=0}^{n-1} ker(d_i : A_n \to A_{n-1}).$$

The degenerate subcomplex $D_*(A)$ is given by

$$D_n(A_\bullet) = \bigcup im(s_i).$$

A cosimplicial object $B^\bullet$ in an abelian category $\mathscr{A}$ is a functor $B^\bullet : \Delta \to \mathscr{A}$. We denote the boundary maps by d^i and the degeneracy maps s^i. The cochain complex $C^*(B^\bullet) \in co\,\mathrm{Ch}(\mathscr{A})$ is given by B^k in the k-th degree and it has differential $d = \sum_{i=0}^{n+1} (-1)^{i+k} d^i$. By our sign conventions in Notation 1.1, given a simplicial abelian group $A_\bullet$ with dual cosimplicial abelian $(A_\bullet)^*$, we get $C^*((A_\bullet)^*) = (C_{-*}(A_\bullet))^*$.

For a simplicial set $X_\bullet$ we write $C_*(X_\bullet)$ for the chain complex given by the chain complex associated to the linearization of $X_\bullet$, i.e. $C_*(X_\bullet) := C_*(\mathbb{Z}[X_\bullet])$. Since this linearization is never applied to simplicial abelian groups, we use the same notation in both cases.

Definition 1.4. The *normalized cochain complex* $N^*(B^\bullet) \in co\,\mathrm{Ch}(\mathscr{A})$ is given by

$$N^n(B^\bullet) = \bigcap_{i=0}^{n-1} ker(s^i : B^n \to B^{n-1}).$$

The degenerate subcomplex $D^*(B^\bullet)$ is defined to be

$$D^n(B^\bullet) = \bigcup im(d_i).$$

Notation 1.5. We define the reduced Moore complex $\overline{C}_*$ of a simplicial object $A_\bullet$ as

$$\overline{C}_*(A) := C_*(A_\bullet)/D_*(A_\bullet)$$

and the reduced Moore cocomplex $\overline{C}^*$ of a cosimplicial object $B^\bullet$ as

$$\overline{C}^*(B^\bullet) := N^*(B^\bullet).$$

Proposition 1.6 ([Wei95, Lemma 8.3.7 and Theorem 8.3.8],[Fre12, Lemma 4.2.5])**.** *For a simplicial object $A_\bullet$ in an abelian category $\mathscr{A}$ we have*

$$C_*(A_\bullet) \cong N_*(A_\bullet) \oplus D_*(A_\bullet)$$

and there is a natural chain homotopy equivalence

$$N_*(A_\bullet) \simeq^h C_*(A_\bullet).$$

Together, we have

$$\overline{C}_*(A_\bullet) \simeq^h C_*(A_\bullet).$$

Dually, for a cosimplicial object $B^\bullet$ we have

$$C^*(B^\bullet) \cong N^*(B^\bullet) \oplus D^*(B^\bullet)$$

and a natural chain homotopy equivalence

$$\overline{C}^*(B^\bullet) = N^*(B^\bullet) \simeq^h C^*(B^\bullet).$$

For a simplicial abelian group $A_\bullet$ and its dual cosimplicial abelian group $A_\bullet^*$,

$$(C_*(A_\bullet))^* \cong C^*(A_\bullet^*) \text{ and } (\overline{C}_*(A_\bullet))^* \cong \overline{C}^*(A_\bullet^*).$$

2. Higher Hochschild homology

Our definitions of the higher Hochschild complex and coHochschild complex of commutative algebras are analogous to the definition of the Hochschild complex for A_∞-algebras given by Wahl and Westerland in [WW11] and the coHochschild complex defined in [Wah12]. Many of the statements proved in this article have been proven by the aforementioned authors in their case. The proofs generalize but sometimes also simplify by the tools of simplicial sets we have in our setup. Furthermore, the definition of the Hochschild complex for functors in the ungraded setup already occurs in [Pir00].

2.1. **Definition.**

Notation 2.1. Let $FinSet$ be the category of all finite sets with all maps between them and $FinOrd$ the category of sets $n = \{1, \dots, n\}$ with all maps between those. For this paper we fix an equivalence of categories $S : FinSet \to FinOrd$. Given a finite ordered set, from now on we denote the cardinality and the set by the same symbol.

Notation 2.2. Denote by $\mathscr{C}om(-,1)$ the unital commutative operad which has one operation of degree zero in each degree $m \geq 0$. Let $\mathscr{C}om(m,n)$ be the induced linearized Prop. We note that as categories $\mathscr{C}om \cong \mathbb{Z}[FinOrd]$, i.e. the category with the same objects but the linearized homomorphism sets and that we have the embedding functor $L : FinOrd \to \mathscr{C}om$ which is the identity on objects.

Let $\mathcal{E}$ be a symmetric monoidal dg-category equipped with a functor $i : \mathscr{C}om \to \mathcal{E}$. We assume that i is a bijection on objects.

Definition 2.3. Let Φ be a dg-functor from $\mathcal{E}$ to Ch. Let $Y_\bullet$ be a simplicial finite set (i.e. Y_k is finite for each k). We define the *higher Hochschild complex of* Φ with respect to $Y_\bullet$ as $C_{Y_\bullet}(\Phi) : \mathcal{E} \to \mathrm{Ch}$ via

$$C_{Y_\bullet}(\Phi) : \mathcal{E} \xrightarrow{F_{Y_\bullet}(\Phi)} \mathrm{Ch}^{\Delta^{op}} \xrightarrow{C_*} d\,\mathrm{Ch} \xrightarrow{\mathrm{sTot}^{\oplus}} \mathrm{Ch} \tag{2.1}$$

where $F_{Y_\bullet}(\Phi)$ sends a set n to the simplicial chain complex

$$F_{Y_\bullet}(\Phi)(n) : \Delta^{op} \xrightarrow{Y_\bullet} FinSet \xrightarrow{S} FinOrd \xrightarrow{L(-\amalg n)} \mathscr{C}om \xrightarrow{i} \mathcal{E} \xrightarrow{\Phi} \mathrm{Ch}\,.$$

The *reduced Higher Hochschild complex of* Φ is defined via

$$\overline{C}_{Y_\bullet}(\Phi) : \mathcal{E} \xrightarrow{F_{Y_\bullet}(\Phi)} \mathrm{Ch}^{\Delta^{op}} \xrightarrow{\overline{C}_*} d\,\mathrm{Ch} \xrightarrow{\mathrm{sTot}^{\oplus}} \mathrm{Ch} \tag{2.2}$$

where $\overline{C}_*$ is the reduced chain complex functor defined in Definition 1.3. The construction so far is functorial in $Y_\bullet$ so we can generalize to arbitrary simplicial sets as follows:

If $Y_\bullet$ is any simplicial set we define

$$C_{Y_\bullet}(\Phi)(n) := \operatorname*{colim}_{\substack{K_\bullet \to Y_\bullet,\\ K_\bullet \text{ finite}}} C_{K_\bullet}(\Phi)(n)$$

and

$$\overline{C}_{Y_\bullet}(\Phi)(n) := \operatorname*{colim}_{\substack{K_\bullet \to Y_\bullet,\\ K_\bullet \text{ finite}}} \overline{C}_{K_\bullet}(\Phi)(n)$$

as the colimit over all simplicial finite subsets of $Y_\bullet$.

Following [Pir00] there is a different definition using enriched tensor products. Given a simplicial finite set $X_\bullet$ we define

$$\mathscr{L}_{X_\bullet,m}(n) = \bigoplus_k \mathscr{C}om(n, X_k \amalg m)[k]$$

with differential $d : \mathscr{C}om(n, X_k \amalg m) \to \mathscr{C}om(n, X_{k-1} \amalg m)$ given by postcomposition with $d' = \sum_{i=0}^{k} (-1)^i d_i$ where the $d_i \in \mathscr{C}om(X_k \amalg m, X_{k-1} \amalg m)$ are the maps induced by the simplicial boundary maps $d_i : X_k \to X_{k-1}$. $\mathscr{L}_{X_\bullet,m}$ is a covariant functor $\mathscr{C}om \to \mathrm{Ch}$. If $X_\bullet$ is an arbitrary simplicial set, we set

$$\mathscr{L}_{X_\bullet,m} = \operatorname*{colim}_{\substack{K_\bullet \to X_\bullet,\\ K_\bullet \text{ finite}}} \mathscr{L}_{K_\bullet,m}.$$

Then we have an isomorphism

$$C_{X_\bullet}(\Phi)(m) \cong \mathscr{L}_{X_\bullet,m} \underset{\mathscr{C}om}{\otimes} \Phi$$

where the right hand side is the enriched tensor product as defined in Definition 5.1. Throughout the first part of the paper we will not work with this definition, but we return to this description in Section 5.

Definition 2.4. Let Ψ be a functor from $\mathcal{E}^{op}$ to Ch and let $Y_\bullet$ be a simplicial finite set. We define the *higher coHochschild complex of* Ψ with respect to $Y_\bullet$ as $D_{Y_\bullet}(\Psi) : \mathcal{E}^{op} \to \mathrm{Ch}$ via

$$D_{Y_\bullet}(\Psi) : \mathcal{E}^{op} \xrightarrow{G_{Y_\bullet}(\Psi)} \mathrm{Ch}^\Delta \xrightarrow{r \circ C^*} d\,\mathrm{Ch} \xrightarrow{\mathrm{Tot}^\Pi} \mathrm{Ch} \tag{2.3}$$

where $G_{Y_\bullet}(\Psi)$ sends a set n to the cosimplicial chain complex

$$G_{Y_\bullet}(\Psi)(n) : \Delta \xrightarrow{Y_\bullet} FinSet^{op} \xrightarrow{S} FinOrd^{op} \xrightarrow{L(-\amalg n)} \mathscr{C}om^{op} \xrightarrow{i} \mathcal{E}^{op} \xrightarrow{\Psi} \mathrm{Ch}\,.$$

C^* is the Moore functor defined in Definition 1.4 and r turns a cochain object into a chain object with the opposite grading (i.e. sending a cochain complex A^i to a chain complex A_{-i}).

Similar to above we can define the *reduced Higher coHochschild complex of* Ψ to be

$$D_{Y_\bullet}(\Psi) : \mathcal{E}^{op} \xrightarrow{G_{Y_\bullet}(\Psi)} \mathrm{Ch}^\Delta \xrightarrow{r \circ \overline{C}^*} d\,\mathrm{Ch} \xrightarrow{\mathrm{Tot}^\Pi} \mathrm{Ch}\,. \tag{2.4}$$

Again, if $Y_\bullet$ is any simplicial set we define

$$D_{Y_\bullet}(\Psi)(n) := \lim_{\left(\substack{K_\bullet \to Y_\bullet, \\ K_\bullet \text{ finite}}\right)^{op}} D_{K_\bullet}(\Psi)(n)$$

and

$$\overline{D}_{Y_\bullet}(\Psi)(n) := \lim_{\left(\substack{K_\bullet \to Y_\bullet, \\ K_\bullet \text{ finite}}\right)^{op}} \overline{D}_{K_\bullet}(\Psi)(n)$$

as the limit over all finite sets.

Again, we can define $D_{Y_\bullet}(\Psi)$ in terms of $\mathscr{L}_{Y_\bullet}$ equivalently via

$$D_{Y_\bullet}(\Psi)(m) \cong \hom_{\mathscr{C}om^{op}}(\mathscr{L}_{Y_\bullet,m}, \Psi)$$

where $\hom_{\mathscr{C}om^{op}}(-,-)$ denotes the enriched hom as defined in Definition 5.1.

Remark 2.5. For a simplicial finite set the functor $C_{Y_\bullet}(\Phi)$ can be described more explicitly:

$$C_{Y_\bullet}(\Phi)(n)_j = \bigoplus_{k+l=j} \Phi(Y_k \amalg n)_l$$

with the differential of $x \in \Phi(Y_k \amalg n)$ given by

$$d(x) = d_\Phi(x) + (-1)^{|x|} \sum_{i=0}^{k} (-1)^i \Phi(d_i \amalg id_n)(x).$$

Here, $d_i : Y_k \to Y_{k-1}$ is the face map of the simplicial set.

The reduced functor is given as the quotient

$$\overline{C}_{Y_\bullet}(\Phi)(n)_j = \bigoplus_{k+l=j} \Phi(Y_k \amalg n)_l / U_k$$

with

$$U_k = \sum_{i=0}^{k-1} im(s_i \amalg id_n)$$

where the $s_i : Y_{k-1} \to Y_k$ are the degeneracy maps of the simplicial set. Similarly, we have

$$D_{Y_\bullet}(\Psi)(n)_j = \prod_{l-k=j} \Psi(Y_k \amalg n)_l.$$

For $y \in D_{Y_\bullet}(\Psi)$ the differential is

$$d(y)_k = \sum_{i=0}^{k+1} (-1)^{i+k+1} \Psi(d_i \amalg id_n)(y_{k+1}) + (-1)^k d_\Psi(y_k)$$
$$= (-1)^k (d_\Psi(y_k) - \sum_{i=0}^{k+1} (-1)^i \Psi(d_i \amalg id_n)(y_{k-1})).$$

The reduced complex is the subcomplex

$$\overline{D}_{Y_\bullet}(\Psi)(n)_j = \prod_{l-k=j} \bigcap_{i=0}^{k-1} ker(\Psi(s^i \amalg n))_l.$$

Remark 2.6. Taking $\mathcal{E} = \mathscr{C}om$ and $\Phi : \mathscr{C}om \to \mathrm{Ch}$ strong symmetric monoidal (i.e. $\Phi(n) \otimes \Phi(m) \cong \Phi(n+m)$ in a natural and symmetric way), $\Phi(1)$ is a commutative differential graded algebra. $\Phi(n)$ is isomorphic to $A^{\otimes n}$ and in this case our definition of the higher Hochschild complex for a simplicial finite set agrees (up to sign twist) with the higher Hochschild complex $CH_{X_\bullet}(A)$ defined by Pirashvili in [Pir00] (see also [GTZ10b]). For an element $x \in A^{\otimes X_k}$, the isomorphism correcting the sign is given by $x \mapsto (-1)^{|x|k} x$. For $\mathcal{E}$ arbitrary, a strong monoidal functor $\Phi : \mathcal{E} \to \mathrm{Ch}$ induces a strong monoidal functor $\Phi \circ i : \mathscr{C}om \to \mathrm{Ch}$ by precomposing with the inclusion of $\mathscr{C}om$ into $\mathcal{E}$ and the higher Hochschild complex of Φ agrees with the higher Hochschild complex of $\Phi \circ i$.

Remark 2.7. Let $S^1_\bullet$ be the simplicial set with two non-degenerate simplices p and t lying in degree 0 and 1, respectively. We then have $S^1_k = \{y_0^k, \cdots, y_k^k\}$ with $y_0^k = (s_0)^k(p)$ and $y_j^k = s_{k-1}s_{k-2}\cdots \widehat{s_j} \cdots s_0(t)$. This is a simplicial model of the circle. Moreover, it is isomorphic to the simplicial set with $S^1_k = \{0, \cdots, k\}$ and

$$d_i(j) = \begin{cases} j & \text{for } i \leq j \\ j-1 & \text{for } i > j \end{cases} \text{ and } d_n(j) = \begin{cases} j & \text{for } j \neq n \\ 0 & \text{for } j = n. \end{cases}$$

Given a dg-functor $\Phi : \mathcal{E} \to \mathrm{Ch}$ we have $C_{S^1_\bullet}(\Phi)(n) \cong C(\Phi)(n)$, where $C(\Phi)(n)$ is the Hochschild complex of the functor Φ defined in [WW11]. The isomorphism corrects the sign, for $x \in \Phi((k+1)+n)$ it is given by $x \mapsto (-1)^k x$. Similarly, for $\Psi : \mathcal{E}^{op} \to \mathrm{Ch}$, $D_{S^1_\bullet}(\Psi)(n) \cong D(\Psi)(n)$ with $D(\Psi)(n)$ the coHochschild complex defined in [Wah12]. Again, the sign twist between the two definitions for $y \in \Psi((k+1)+n)$ is given by $y \mapsto (-1)^k y$.

2.2. Basic properties of the higher Hochschild and coHochschild functors.

Proposition 2.8. *Let $Y_\bullet$ be a simplicial set. For a dg-functor $\Phi : \mathcal{E} \to \mathrm{Ch}$ the functors $C_{Y_\bullet}(\Phi)$ and $\overline{C}_{Y_\bullet}(\Phi) : \mathcal{E} \to \mathrm{Ch}$ are dg-functors. Similarly, for $\Psi : \mathcal{E}^{op} \to \mathrm{Ch}$ a dg-functor, $D_{Y_\bullet}(\Psi)$ and $\overline{D}_{Y_\bullet}(\Psi) : \mathcal{E}^{op} \to \mathrm{Ch}$ are dg-functors.*

Proof. We only prove the case of simplicial finite sets, the general case follows by similar arguments about colimits and limits.

To break up the proof into steps, we have to equip the categories $\mathrm{Ch}^{\Delta^{op}}$ and Ch^{Δ} with dg-structures: Similar to the definition of $d\,\mathrm{Ch}^v$ we take the levelwise dg-structure on $\mathrm{Ch}^{\Delta^{op}}$. An element in $\mathrm{Ch}^{\Delta^{op}}$ is a simplicial chain complex, i.e. a double graded family of abelian groups $A_{\bullet,*}$ such that the simplicial structure maps and the differentials $d^{\mathrm{Ch}} : A_{\bullet,*} \to A_{\bullet,*+1}$ commute. For two simplicial chain complexes A and B, we define the complex $\mathrm{Ch}^{\Delta^{op}}(A,B)$ in degree k to consist of maps $f : A_{\bullet,*} \to B_{\bullet,*+k}$ such that f commutes with all simplicial structure maps. The differential of f is given as $d(f)_{p,q} = (-1)^q (d_B^{\mathrm{Ch}} \circ f_{p,q} - f_{p,q-1} \circ d_A^{\mathrm{Ch}})$. Similarly, we equip the category of cosimplicial chain complexes Ch^{Δ} with a levelwise dg-structure.

Now we have to show:

(1) The functors $F_{Y_\bullet}(\Phi) : \mathcal{E} \to \mathrm{Ch}^{\Delta^{op}}$ and $G_{Y_\bullet}(\Psi) : \mathcal{E}^{op} \to \mathrm{Ch}^{\Delta}$ are dg-functors.
(2) The functors $C_* : \mathrm{Ch}^{\Delta^{op}} \to d\,\mathrm{Ch}^v$, $\overline{C}_* : \mathrm{Ch}^{\Delta^{op}} \to d\,\mathrm{Ch}^v$, $C^* : \mathrm{Ch}^{\Delta^{op}} \to d\,\mathrm{Ch}^v$ and $\overline{C}^* : \mathrm{Ch}^{\Delta^{op}} \to d\,\mathrm{Ch}^v$ are dg-functors.
(3) $\mathrm{Tot}^{\prod}$ and $\mathrm{sTot}^{\oplus} : d\,\mathrm{Ch}^v \to \mathrm{Ch}$ are dg-functors.

2. follows from the definition of the dg-structure on $\mathrm{Ch}^{\Delta^{op}}$ and $d\,\mathrm{Ch}^v$, since both are levelwise and the (co)simplicial structure maps become the horizontal differential of the double complex. 3. is proved in Proposition B.1.

To show 1. we compute:

$$F_{Y_\bullet}(\Phi)(n) \otimes \mathcal{E}(n,m) \to F_{Y_\bullet}(\Phi)(m)$$

for all simplicial degrees and show that this is a dg-map. The map

$$\Phi(Y_i \amalg n) \otimes \mathcal{E}(n,m) \to \Phi(Y_i \amalg m)$$

is induced by the degree zero embedding $\mathcal{E}(n,m) \hookrightarrow \mathcal{E}(Y_i \amalg n, Y_i \amalg m)$ sending a map f to $id_{Y_i} \amalg f$. This embedding is a chain map since it commutes with the boundary maps. Moreover, the map

$$\Phi(Y_i \amalg n) \otimes \mathcal{E}(Y_i \amalg n, Y_i \amalg m) \to \Phi(Y_i \amalg m)$$

is a chain map since Φ is a dg-functor. Therefore, the composition

$$\Phi(Y_i \amalg n) \otimes \mathcal{E}(n,m) \to \Phi(Y_i \amalg n) \otimes \mathcal{E}(Y_i \amalg n, Y_i \amalg m) \to \Phi(Y_i \amalg m)$$

is a chain map. A similar computation shows that $G_{Y_\bullet}(\Psi)$ is a dg-functor, too.

Combining the three steps, we have shown that all mentioned functors are compositions of dg-functors, i.e. dg-functors themselves. □

The previous proposition allows us to iterate the Hochschild and coHochschild constructions. Properties of this are given in Section 2.3.

Proposition 2.9. *Let $Y_\bullet$ be a simplicial set.*

(1) If $\Phi \simeq \Phi' : \mathcal{E} \to \mathrm{Ch}$ are quasi-isomorphic functors, then $C_{Y_\bullet}(\Phi) \simeq C_{Y_\bullet}(\Phi')$ are also quasi-isomorphic functors. The same holds for the reduced complexes. In particular, the limit in the definition of $C_{Y_\bullet}(\Phi)$ for $Y_\bullet$ a simplicial non-finite set is a homotopy colimit.
(2) If $\Psi \simeq \Psi' : \mathcal{E}^{op} \to \mathrm{Ch}$ are quasi-isomorphic functors, then $D_{Y_\bullet}(\Psi) \simeq D_{Y_\bullet}(\Psi')$ are also quasi-isomorphic functors. The same holds for the reduced complexes. In particular the limit involved in the construction for simplicial non-finite sets is a homotopy limit.

Proof. The proof is analogous to [WW11, Proposition 5.7] and [Wah12, Corollary 1.5]. Natural transformations of functors $\Phi \to \Phi'$ and $\Psi \to \Psi'$ induce natural transformations of functors $C_{Y_\bullet}(\Phi) \to C_{Y_\bullet}(\Phi')$ and $D_{Y_\bullet}(\Psi) \to D_{Y_\bullet}(\Psi')$, respectively, by composing with the natural transformations in the according steps of the construction. We are left to show that these natural transformations are quasi-isomorphisms.

(1) For a simplicial finite set we have an induced map of simplicial chain complexes $F_{Y_\bullet}(\Phi)(n) \to F_{Y_\bullet}(\Phi')(n)$. This map is a quasi-isomorphism in each degree, i.e. a quasi-isomorphism of simplicial chain complexes. Applying C_* (or $\overline{C}_*$) gives us a degreewise quasi-isomorphism $C_*(F_{Y_\bullet}(\Phi))(n) \to C_*(F_{Y_\bullet}(\Phi'))(n)$, i.e. a quasi-isomorphism in $d\,\mathrm{Ch}^v$. By Corollary B.12 this yields a quasi-isomorphism of the direct sum total complex. Similarly, by the proof of [Pir00, Theorem 2.4] the complexes $\mathscr{L}_{Y_\bullet,m}$ are levelwise projective $\mathscr{C}om$-modules and hence the levelwise tensor product

$$\mathscr{L}_{Y_\bullet,m}(n) \underset{\mathscr{C}om}{\otimes} \Phi \cong \operatorname*{colim}_{\substack{K_\bullet \to Y_\bullet, \\ K_\bullet \text{ finite}}} \Phi(Y_n \amalg m)$$

preserves quasi-isomorphisms. Hence a quasi-isomorphims $\Phi \simeq \Phi'$ again implies a quasi-isomorphism in $d\,\mathrm{Ch}^v$ and the result follows.

(2) Again, we get a degreewise quasi-isomorphism $G_{Y_\bullet}(\Psi)(n) \to G'_{Y_\bullet}(\Psi')(n)$ (respectively of the limits) and thus an induced quasi-isomorphism of product double complexes.

□

The next proposition is similar to [Wah12, Section 1.2]. However, we will give a simplified proof making use of the simplicial tools that we have in our situation.

Proposition 2.10. *Let $Y_\bullet$ be a simplicial set.*

(1) For $\Phi : \mathcal{E} \to \mathrm{Ch}$ the split projection $C_{Y_\bullet}(\Phi)(n) \to \overline{C}_{Y_\bullet}(\Phi)(n)$ is a natural quasi-isomorphism for all $n \in \mathcal{E}$.

(2) For $\Psi : \mathcal{E}^{op} \to \mathrm{Ch}$ the split inclusion $\overline{D}_{Y_\bullet}(\Phi)(n) \to D_{Y_\bullet}(\Phi)(n)$ is a natural quasi-isomorphism for all $n \in \mathcal{E}$.

Proof. Since homotopy limits and colimits preserve quasi-isomorphisms, it is enough to prove the theorem for simplicial finite sets.

(1) By Proposition 1.6 the projection

$$C_*(F_{Y_\bullet}(\Phi))(n) \to \overline{C}_*(F_{Y_\bullet}(\Phi))(n)$$

is a natural chain homotopy equivalence in $d\,\mathrm{Ch}^h$.

By Corollary B.14 this induces a filtered quasi-isomorphism between the chain complexes $\mathrm{sTot}^{\oplus}(C_*(F_{Y_\bullet}(\Phi))$ and $\mathrm{sTot}^{\oplus}(\overline{C}_*(F_{Y_\bullet}(\Phi))$ and hence a filtered quasi-isomorphism between $C_{Y_\bullet}(\Phi)(n)$ and $\overline{C}_{Y_\bullet}(\Phi)(n)$.

(2) Again, the map

$$i^* : \overline{C}^*(G_{Y_\bullet}(\Psi))(n) \to C^*(G_{Y_\bullet}(\Psi))(n)$$

is a natural chain homotopy equivalence in $d\,\mathrm{Ch}^h$ and induces by Corollary B.14 the stated quasi-isomorphism.

□

Proposition 2.11 (Duality). *Let $\Phi : \mathcal{E} \to \mathrm{Ch}$ be a dg-functor and $X_\bullet$ a simplicial set. Then*

$$(C_{X_\bullet}(\Phi))^* \cong D_{X_\bullet}(\Phi^*)$$

where $\Phi^* : \mathcal{E}^{op} \to \mathrm{Ch}$ is the dual functor, i.e. $\Phi^*(m) = (\Phi(m))^*$. The same holds in the reduced case.

Proof. Since dualizing takes colimits to limits, it is enough to check this on simplicial finite sets.

Let $X_\bullet$ be a simplicial finite set. As abelian groups we have

$$\begin{aligned}(C_{X_\bullet}(\Phi)(n))^* &= \left(\bigoplus_k C_k(\Phi(X_\bullet \amalg n))\right)^* \\ &= \prod_k (C_k(\Phi(X_\bullet \amalg n)))^* \\ &= \prod_k C^k(\Phi(X_\bullet \amalg n)^*) && \text{by Proposition 1.6} \\ &= D_{X_\bullet}(\Phi^*)(n)\end{aligned}$$

and similar in the reduced case. We check that this fits with the differential. For $y = (y_l) \in \prod_k \Phi(X_k \amalg n)^*$ of total degree r, i.e. $y_l + l = r$ for all l, we get

$$\begin{aligned} d_{C^*}(y)_k &= (-1)^r (y \circ d_C)_k \\ &= (-1)^r (y_k \circ d_\Phi + (-1)^{|y_{k-1}|} (\sum_i (-1)^i y_{k-1} \circ \Phi(d_i)) \\ &= (-1)^{|y_k|+k} y_k \circ d_\Phi + (-1)^{k+1} (\sum_i (-1)^i y_{k-1} \circ \Phi(d_i)) \end{aligned}$$

On the other hand

$$\begin{aligned} d_D(y)_k &= (d_D(y))_k \\ &= (-1)^k (d_{\Phi^*}(y_k) - \sum_i (-1)^i \Phi^*(d_i)(y_{k-1})) \\ &= (-1)^k ((-1)^{|y_k|} y_k \circ d_\Phi - \sum_i (-1)^i y_{k-1} \circ \Phi(d_i)). \end{aligned}$$

So the differentials agree.

□

2.3. **Iterated Hochschild functors.** Since the higher (co)Hochschild construction of a dg-functor is again a dg-functor, we can iterate it. In the first part of this section we show that the iteration of the higher Hochschild construction with respect to two simplicial sets $X_\bullet$ and $Y_\bullet$ is the same as applying the construction once with respect to the disjoint union $X_\bullet \amalg Y_\bullet$. This is the analog of the last part of [GTZ10b, Prop. 2] which there was proved for algebras. In the second part we apply the Hochschild construction to strong symmetric monoidal functors $\Phi : \mathcal{E} \to \mathrm{Ch}$ and show that $C_{X_\bullet}(C_{Y_\bullet}(\Phi)) \cong C_{X_\bullet}(\Phi) \otimes C_{Y_\bullet}(\Phi)$.

2.3.1. *Disjoint Union.*

Lemma 2.12. *For two simplicial finite sets $Y_\bullet$ and $Y'_\bullet$ and a functor $\Phi : \mathcal{E} \to \mathrm{Ch}$ and for each $n \in \mathcal{E}$ there is a chain homotopy equivalence of complexes in $d\,\mathrm{Ch}^h$*

$$C_*(F_{Y_\bullet \amalg Y'_\bullet}(\Phi)(n)) \xrightarrow{\simeq_h} \mathrm{sTot}^{\oplus}_{1,2}\, C_*(F_{Y_\bullet}(C_*(F_{Y'_\bullet}(\Phi)(n)))) \tag{2.5}$$

where $\mathrm{sTot}^{\oplus}_{1,2} : tri\,\mathrm{Ch} \to d\,\mathrm{Ch}$ applies the functor $\mathrm{sTot}^{\oplus}$ in the first two directions of a triple chain complex. The map is natural in n and Φ.

Proof. If we apply the Eilenberg-Zilber Theorem (cf. Theorem B.7) to the bisimplicial chain complex $A_{\bullet,\bullet}$ defined via

$$\Delta^{op} \times \Delta^{op} \xrightarrow{Y_\bullet \times id} FinSet \times \Delta^{op} \xrightarrow{id \times Y'_\bullet} FinSet \times FinSet$$
$$\xrightarrow{S \times S} FinOrd \times FinOrd \xrightarrow{\amalg} FinOrd \xrightarrow{L(-\amalg n)} \mathscr{C}om \xrightarrow{i} \mathcal{E} \xrightarrow{\Phi} \mathrm{Ch},$$

we get a chain homotopy equivalence between $C_*(\mathrm{diag}_\bullet A_{\bullet,\bullet}) = C_*(F_{Y_\bullet \amalg Y'_\bullet}(\Phi)(n))$ and $\mathrm{Tot}^{\oplus} C_* C_*(A_{\bullet,\bullet}) = \mathrm{Tot}^{\oplus} C_*(F_{Y_\bullet}(C_*(F_{Y'_\bullet}(\Phi)(n))$ which is natural in n and Φ. Postcomposing with the isomorphism between $\mathrm{Tot}^{\oplus}$ and $\mathrm{sTot}^{\oplus}$ described in Equation (1.1) is also natural in n, since a map $n \to n'$ induces the identity in the first two directions of the triple chain complex and these are the only directions involved in the sign. Hence, the whole construction is natural in Φ. □

Since the order of taking total complexes does not matter, we obtain

$$\mathrm{sTot}^{\oplus}(\mathrm{sTot}^{\oplus}_{1,2}\, C_*(F_{Y_\bullet}(C_*(F_{Y'_\bullet}(\Phi)(n)))) \cong C_{Y_\bullet}(C_{Y'_\bullet}(\Phi)(n)).$$

Moreover, by definition $\mathrm{sTot}^{\oplus}(C_*(F_{Y_\bullet \amalg Y'_\bullet}(\Phi)(n)))$ is equal to $C_{Y_\bullet \amalg Y'_\bullet}(\Phi)(n)$.

Theorem 2.13. *For any simplicial sets $Y_\bullet$ and $Y'_\bullet$ and a dg-functor $\Phi : \mathcal{E} \to \mathrm{Ch}$ there is a quasi-isomorphism of functors*

$$C_{Y_\bullet}(C_{Y'_\bullet}(\Phi)) \xrightarrow{\simeq} C_{Y_\bullet \amalg Y'_\bullet}(\Phi).$$

Proof. We first show the theorem for simplicial finite sets. By Corollary B.14 and the aforementioned equalities the chain homotopy from Lemma 2.12 induces the requested quasi-isomorphism in this case. Since tensor product commutes with directed colimits, the non-finite case follows. □

Everything we did above dualizes to the cohomological case, hence we deduce:

Lemma 2.14. *For $n \in \mathcal{E}$, two simplicial finite sets $Y_\bullet$ and $Y'_\bullet$ and a functor $\Psi : \mathcal{E}^{op} \to \mathrm{Ch}$ there is a chain homotopy equivalence in $d\,\mathrm{Ch}^h$*

$$C^*(G_{Y_\bullet \amalg Y'_\bullet}(n)) \xrightarrow{\simeq_h} \mathrm{Tot}^{\prod}_{1,2} C^*(G_{Y_\bullet}(C^*(G_{Y'_\bullet}(\Psi)(n)))) \tag{2.6}$$

where $\mathrm{Tot}^{\prod}_{1,2} : tri\,\mathrm{Ch} \to d\,\mathrm{Ch}$ takes the totalization in the first two directions of a triple chain complex.

Since taking the product total space of a trisimplicial space is associative (i.e. it does not matter which two directions one pairs first), we have

$$\mathrm{Tot}^{\prod}(\mathrm{Tot}_{1,2}\, C^*(G_{Y_\bullet}(C^*(G_{Y'_\bullet}(\Psi)(n))) \cong D_{Y_\bullet}(D_{Y'_\bullet}(\Psi)(n)).$$

Moreover, by definition $\mathrm{Tot}^{\prod}(C^*(G_{Y_\bullet \amalg Y'_\bullet}(\Psi)(n)))$ is equal to $D_{Y_\bullet \amalg Y'_\bullet}(\Psi)(n)$.

Theorem 2.15. *For any simplicial sets $Y_\bullet$ and $Y'_\bullet$ and a dg-functor $\Psi : \mathcal{E}^{op} \to \mathrm{Ch}$ there is a quasi-isomorphism of functors $D_{Y_\bullet}(D_{Y'_\bullet}(\Psi)) \xrightarrow{\simeq} D_{Y_\bullet \amalg Y'_\bullet}(\Psi)$.*

Proof. For simplicial finite sets this is again a direct consequence of Corollary B.14.

For arbitrary simplicial sets, it follows from the fact that $D_{Y_\bullet}$ commutes with limits. □

2.3.2. *Symmetric monoidal functors.* For a symmetric monoidal category $\mathcal{E}$ a functor $\Phi : \mathcal{E} \to \mathrm{Ch}$ is called symmetric monoidal if there are maps $\Phi(n) \otimes \Phi(m) \to \Phi(n+m)$ which are natural in n and m and compatible with the symmetries, the associators and unitors in $\mathcal{E}$ and Ch. The functor Φ is called strong if these maps are isomorphisms and h-strong if they are quasi-isomorphisms.

In this case we want to give an easier description of the iterated Hochschild construction. For a functor $\Phi : \mathcal{E} \to \mathrm{Ch}$ or a functor $\Psi : \mathcal{E}^{op} \to \mathrm{Ch}$ and a collection of simplicial sets $\{X^1_\bullet, \dots, X^n_\bullet\}$ we have functors

$$C_{X^n_\bullet}(\cdots C_{X^1_\bullet}(\Phi)\cdots) : \mathcal{E} \to \mathrm{Ch}$$

and

$$D_{X^n_\bullet}(\cdots D_{X^1_\bullet}(\Psi)\cdots) : \mathcal{E}^{op} \to \mathrm{Ch},$$

respectively.

Notation 2.16. To simplify notation, we write

$$C_{X^n_\bullet,\dots,X^1_\bullet}(\Phi)(m) := C_{X^n_\bullet}(\cdots C_{X^1_\bullet}(\Phi)\cdots)(m)$$

and

$$D_{X^n_\bullet,\dots,X^1_\bullet}(\Psi)(m) := D_{X^n_\bullet}(\cdots D_{X^1_\bullet}(\Psi)\cdots)(m)$$

and similarly in the reduced cases.

Lemma 2.17. *Fix a collection $\{X^1_\bullet, \dots, X^{n_1}_\bullet\}$ of simplicial finite sets and $m_1 \in \mathcal{E}$.*

(1) *For* $\Phi : \mathcal{E} \to \mathrm{Ch}$ *we have*

$$C_{X_\bullet^{n_1},\ldots,X_\bullet^1}(\Phi)(m_1) \cong \bigoplus_{k_1,\ldots,k_{n_1}} \Phi(X_{k_1}^1 \amalg \cdots X_{k_n}^n \amalg m_1)$$

and

$$\overline{C}_{X_\bullet^{n_1},\ldots,X_\bullet^1}(\Phi)(m_1) \cong \bigoplus_{k_1,\ldots,k_{n_1}} \Phi(X_{k_1}^1 \amalg \cdots X_{k_n}^n \amalg m_1)/U_{k_1,\ldots,k_{n_1}}$$

with

$$U_{k_1,\ldots,k_{n_1}} = \sum_{j=1}^{n_1} \sum_i im\Phi(id \amalg s_i^{X_{k_j}^j} \amalg id).$$

For $x \in \Phi(X_{k_1}^1 \amalg \cdots X_{k_n}^n \amalg m_1)$ *the differential is given by*

$$d_C(x) = d_\Phi(x) + \sum_{j=1}^{n_1} (-1)^{|x|+\sum_{t=1}^{j-1} k_t} \sum_i (-1)^i \Phi(id \amalg d_i^{X_{k_j}^j} \amalg id)(x).$$

(2) *For* $\Psi : \mathcal{E}^{op} \to \mathrm{Ch}$ *we have*

$$D_{X_\bullet^{n_1},\ldots,X_\bullet^1}(\Psi)(m_1) \cong \prod_{k_1,\ldots,k_{n_1}} \Psi(X_{k_1}^1 \amalg \cdots X_{k_n}^n \amalg m_1)$$

and

$$\overline{D}_{X_\bullet^{n_1},\ldots,X_\bullet^1}(\Psi)(m_1) \cong \prod_{k_1,\ldots,k_{n_1}} \bigcap_{j=1}^{n_1} \bigcap_{i=0}^{k_j-1} ker(id \amalg s_i^{X_{k_j}^j} \amalg id).$$

For $y \in D_{X_\bullet^{n_1},\ldots,X_\bullet^1}(\Psi)(m_1)$ *the differential is computed via*

$$\begin{aligned} &d_D(y)_{k_{n_1},\ldots,k_1} = (-1)^{\sum_{t=1}^{n_1} k_t} d_\Psi(y_{k_{n_1},\ldots,k_1}) \\ &- \sum_{j=1}^{n_1} (-1)^{\sum_{t=j}^{n_1} k_t} \sum_i (-1)^i \Psi(id \amalg d_i^{X_{k_j}^j} \amalg id)(y_{k_{n_1},\ldots,k_j-1,\ldots,k_1}). \end{aligned}$$

The following proposition is the analog of [WW11, Prop. 5.10].

Proposition 2.18. *Let* $\{X_\bullet^1, \ldots, X_\bullet^n\}$ *be a collection of simplicial sets and* $m \in \mathbb{N}$. *If* $\Phi : \mathcal{E} \to \mathrm{Ch}$ *is symmetric monoidal there are natural maps*

$$\lambda : C_{X_\bullet^1}(\Phi)(0) \otimes \cdots \otimes C_{X_\bullet^n}(\Phi)(0) \otimes \Phi(1)^{\otimes m} \to C_{X_\bullet^n,\ldots,X_\bullet^1}(\Phi)(m)$$

and

$$\overline{\lambda} : \overline{C}_{X_\bullet^1}(\Phi)(0) \otimes \cdots \otimes \overline{C}_{X_\bullet^n}(\Phi)(0) \otimes \Phi(1)^{\otimes m} \to \overline{C}_{X_\bullet^n,\ldots,X_\bullet^1}(\Phi)(m).$$

These maps are quasi-isomorphisms if Φ *is h-strong. For simplicial finite sets these are isomorphisms if* Φ *is strong.*

Proof. In the case of simplicial finite sets the proof works completely analogously to the proof of [WW11, Prop. 5.10]. For arbitrary simplicial sets we need to show that the maps are quasi-isomorphisms if Φ is (h-)strong. Commuting colimits and tensor products proves the statement for arbitrary simplicial sets. $\square$

3. Formal and natural operations of the higher Hochschild complex functors

3.1. **Formal operations.** The definitions and results in this section are analog to those in [Wah12, Section 2].

Recall from Notation 2.16 that for a collection of simplicial sets $\{X^1_\bullet, \dots, X^{n_1}_\bullet\}$ and a natural number $m_1 \in \mathbb{N}$ we denote the iterated Hochschild and coHochschild constructions by $C_{X^{n_1}_\bullet,\dots,X^1_\bullet}(\Phi)(m_1)$ and $D_{X^{n_1}_\bullet,\dots,X^1_\bullet}(\Psi)(m_1)$, respectively. Thus, we have functors

$$C_{X^{n_1}_\bullet,\dots,X^1_\bullet}(-)(m_1) : Fun(\mathcal{E}, \mathrm{Ch}) \to \mathrm{Ch}$$

and

$$D_{X^{n_1}_\bullet,\dots,X^1_\bullet}(-)(m_1) : Fun(\mathcal{E}^{op}, \mathrm{Ch}) \to \mathrm{Ch}\,.$$

Definition 3.1. For collections $\{X^1_\bullet, \dots, X^{n_1}_\bullet\}$ and $\{Y^1_\bullet, \dots, Y^{n_2}_\bullet\}$ of simplicial sets and natural numbers m_1 and m_2 the chain complex of *formal operations* $\mathrm{Nat}_{\mathcal{E}}$ between these functors is defined to be

$$\begin{aligned}&\mathrm{Nat}_{\mathcal{E}}(\{X^1_\bullet, \dots, X^{n_1}_\bullet\}, m_1; \{Y^1_\bullet, \dots, Y^{n_2}_\bullet\}, m_2)\\ &\quad := \hom(C_{X^{n_1}_\bullet,\dots,X^1_\bullet}(-)(m_1), C_{Y^{n_2}_\bullet,\dots,Y^1_\bullet}(-)(m_2))\end{aligned}$$

and similarly in the reduced setup

$$\begin{aligned}&\overline{\mathrm{Nat}}_{\mathcal{E}}(\{X^1_\bullet, \dots, X^{n_1}_\bullet\}, m_1; \{Y^1_\bullet, \dots, Y^{n_2}_\bullet\}, m_2)\\ &\quad := \hom(\overline{C}_{X^{n_1}_\bullet,\dots,X^1_\bullet}(-)(m_1), \overline{C}_{Y^{n_2}_\bullet,\dots,Y^1_\bullet}(-)(m_2)).\end{aligned}$$

We now state the main theorem used to compute the formal operations.

Theorem 3.2. *There are isomorphisms of chain complexes*

$$\begin{aligned}&\mathrm{Nat}_{\mathcal{E}}(\{X^1_\bullet, \dots, X^{n_1}_\bullet\}, m_1; \{Y^1_\bullet, \dots, Y^{n_2}_\bullet\}, m_2)\\ &\quad \cong D_{X^{n_1}_\bullet,\dots,X^1_\bullet}(C_{Y^{n_2}_\bullet,\dots,Y^1_\bullet}\mathcal{E}(-,-)(m_2))(m_1)\end{aligned}$$

and

$$\begin{aligned}&\overline{\mathrm{Nat}}_{\mathcal{E}}(\{X^1_\bullet, \dots, X^{n_1}_\bullet\}, m_1; \{Y^1_\bullet, \dots, Y^{n_2}_\bullet\}, m_2)\\ &\quad \cong \overline{D}_{X^{n_1}_\bullet,\dots,X^1_\bullet}(\overline{C}_{Y^{n_2}_\bullet,\dots,Y^1_\bullet}\mathcal{E}(-,-)(m_2))(m_1).\end{aligned}$$

The theorem can be proved completely along the lines of the proof of [Wah12, Theorem 2.1]. However, in Section 5.4.1, this is deduced from the more general setup of Theorem 5.14. Moreover, we can deduce the following corollary:

Corollary 3.3. *We have quasi-isomorphisms of chain complexes*

$$\begin{aligned}&\mathrm{Nat}_{\mathcal{E}}(\{X^1_\bullet, \dots, X^{n_1}_\bullet\}, m_1; \{Y^1_\bullet, \dots, Y^{n_1}_\bullet\}, m_2)\\ &\quad \simeq \overline{\mathrm{Nat}}_{\mathcal{E}}(\{X^1_\bullet, \dots, X^{n_1}_\bullet\}, m_1; \{Y^1_\bullet, \dots, Y^{n_1}_\bullet\}, m_2)\\ &\quad \simeq \mathrm{Nat}_{\mathcal{E}}(X^1_\bullet \amalg \dots X^{n_1}_\bullet \coprod_{m_1} *, Y^1_\bullet \amalg \dots Y^{n_1}_\bullet \coprod_{m_2} *)\\ &\quad \simeq \overline{\mathrm{Nat}}_{\mathcal{E}}(X^1_\bullet \amalg \dots X^{n_1}_\bullet \coprod_{m_1} *, Y^1_\bullet \amalg \dots Y^{n_1}_\bullet \coprod_{m_2} *).\end{aligned}$$

Proof. We write $F_X = \{X^1_\bullet, \dots, X^{n_1}_\bullet\}$, $F_Y = \{Y^1_\bullet, \dots, Y^{n_1}_\bullet\}$, $\amalg X = X^1_\bullet \amalg \dots X^{n_1}_\bullet \coprod_{m_1} *$ and $\amalg Y = Y^1_\bullet \amalg \dots Y^{n_2}_\bullet \coprod_{m_2} *$. Then the previous theorem reduces the corollary to showing that

$$\begin{aligned}D_{F_X}C_{F_Y}(\mathcal{E}'(-,-))(m_2)(m_1) \qquad &\simeq \overline{D}_{F_X}\overline{C}_{F_Y}(\mathcal{E}'(-,-))(m_2)(m_1)\\ \simeq D_{\amalg X}C_{\amalg Y}(\mathcal{E}'(-,-)) \qquad &\simeq \overline{D}_{\amalg X}\overline{C}_{\amalg Y}(\mathcal{E}'(-,-)).\end{aligned}$$

By the first part of Lemma 2.10 and Theorem 2.13 all the involved Hochschild constructions are quasi-isomorphic. Moreover, by Lemma 2.9 the coHochschild construction preserves quasi-isomorphism and again by Lemma 2.10 and Theorem 2.15 all the different coHochschild constructions above are quasi-isomorphic. Hence, the corollary follows. □

3.2. **Restriction to natural transformations of algebras.** This section is similar to [Wah12, Section 2.2]. For simplicity we restrict to simplicial finite sets, but all the results hold up to quasi-isomorphism in the general case, too.

Denote by $C^{\otimes}_{X_\bullet}(-)(m)$ the restriction of the higher Hochschild construction to strong symmetric monoidal functors and similar for the iterated functor. By Proposition 2.18 we have an isomorphism

$$C^{\otimes}_{X^1_\bullet}(-)(0) \otimes \cdots \otimes C^{\otimes}_{X^n_\bullet}(-)(0) \otimes (-)(1)^{\otimes m} \to C^{\otimes}_{X^n_\bullet,\dots,X^1_\bullet}(-)(m).$$

Denote by

$$\begin{aligned}&\mathrm{Nat}^{\otimes}_{\mathcal{E}}(\{X^1_\bullet,\dots,X^{n_1}_\bullet\},m_1;\{Y^1_\bullet,\dots,Y^{n_2}_\bullet\},m_2)\\ &\qquad := \mathrm{Hom}(C^{\otimes}_{X^{n_1}_\bullet,\dots,X^1_\bullet}(-)(m_1), C^{\otimes}_{Y^{n_2}_\bullet,\dots,Y^1_\bullet}(-)(m_2))\end{aligned}$$

the natural transformations between those functors. There is a restriction map

$$\begin{aligned}r : \;&\mathrm{Nat}_{\mathcal{E}}(\{X^1_\bullet,\dots,X^{n_1}_\bullet\},m_1;\{Y^1_\bullet,\dots,Y^{n_2}_\bullet\},m_2)\\ &\to \mathrm{Nat}^{\otimes}_{\mathcal{E}}(\{X^1_\bullet,\dots,X^{n_1}_\bullet\},m_1;\{Y^1_\bullet,\dots,Y^{n_2}_\bullet\},m_2).\end{aligned}$$

For a symmetric monoidal functor $\Phi : \mathcal{E} \to \mathrm{Ch}$ we define $U(\Phi) = \Phi(1)$ the forgetful functor. Then we define ρ to be the map

$$\rho : \mathcal{E}(m_1, m_2) \to \mathrm{Hom}(U^{\otimes m_1}, U^{\otimes m_2})$$

which associates to an element of $\mathcal{E}$ its action on all $\mathcal{E}$-algebras. Completely similarly to [Wah12, Theorem 2.9] one can show

Theorem 3.4. *For a commutative Prop $\mathcal{E}$ The restriction map $r : \mathrm{Nat}_{\mathcal{E}} \to \mathrm{Nat}^{\otimes}_{\mathcal{E}}$ is injective (resp. surjective) if and only if $\rho : \mathcal{E}(m_1, m_2) \to \mathrm{Hom}(U^{\otimes m_1}, U^{\otimes m_2})$ is injective (resp. surjective).*

Remark 3.5. By [Wah12, Example 2.11], ρ (and therefore r) is always injective if $\mathcal{E}$ is the Prop associated to an operad. Moreover, by [Fre09] for $\mathbb{F}$ a field of characteristic 0 and $\mathcal{E}$ a prop coming from an operad over the category of (ungraded) vector spaces over $\mathbb{F}$ one gets an equivalence $\mathcal{E}(s,t) \cong \hom((U_V)^{\otimes r}, (U_V)^{\otimes s})$, with U_V the forgetful functor from $\mathcal{E}$-algebras to vector spaces (for more details cf. [Wah12, Example 2.13]).

Unfortunately, this also implies that in this case ρ cannot be an isomorphism. To see so, note that since $\mathcal{E}(s,t) \cong \hom((U_V)^{\otimes r}, (U_V)^{\otimes s})$ we would want to get an isomorphism $\hom((U_V)^{\otimes r}, (U_V)^{\otimes s}) \cong \hom(U^{\otimes r}, U^{\otimes s})$, i.e. a map $A^{\otimes r} \to A^{\otimes s}$ commuting with algebra morphisms must be determined by what it does in degree 0. This is not true, since a morphism of algebras over $\mathscr{C}om$ needs to respect the twist map. Therefore, there cannot be a morphism f sending an element b of odd degree to an element $f(b)$ of even degree (or the other way round). To see so, note that $(f \otimes f)(b \otimes b)$ lies in $A_{\mathrm{even}} \otimes A_{\mathrm{even}}$, i.e. the twist acts as the identity. Precomposing with the twist, i.e. applying the twist to $b \otimes b$ is the same as multiplication with -1, i.e. $\tau((f \otimes f)(b \otimes b)) = -(f \otimes f)(\tau(b \otimes b))$ and hence f does not commute with the twist map. Thus, we can define a natural map $\nu : A \otimes A \to A$ with

$$\nu(a \otimes b) = \begin{cases} a \cdot b & \text{if } a \text{ or } b \text{ is of even degree} \\ 2a \cdot b & \text{if both } a \text{ and } b \text{ are of odd degree.} \end{cases}$$

Hence, we found a transformation ν which is not determined on degree zero and thus ρ cannot be an isomorphism in this case.

3.3. **Formal operations of the coHochschild construction.** We can define formal operations on the coHochschild construction in a similar way to those on the Hochschild construction.

We define

$$\begin{aligned}&\mathrm{Nat^D}_{\mathcal{E}}(\{X_\bullet^1,\dots,X_\bullet^{n_1}\},m_1;\{Y_\bullet^1,\dots,Y_\bullet^{n_2}\},m_2)\\&\qquad:=\hom(D_{X_\bullet^{n_1},\dots,X_\bullet^1}(-)(m_1),D_{Y_\bullet^{n_2},\dots,Y_\bullet^1}(-)(m_2))\end{aligned}$$

and similarly in the reduced setup

$$\begin{aligned}&\overline{\mathrm{Nat^D}}_{\mathcal{E}}(\{X_\bullet^1,\dots,X_\bullet^{n_1}\},m_1;\{Y_\bullet^1,\dots,Y_\bullet^{m}\},m_2)\\&\qquad:=\hom(\overline{D}_{X_\bullet^{n_1},\dots,X_\bullet^1}(-)(m_1),\overline{D}_{Y_\bullet^{n_2},\dots,Y_\bullet^1}(-)(m_2)).\end{aligned}$$

Theorem 3.6. *There are isomorphisms of chain complexes*

$$\begin{aligned}&\mathrm{Nat^D}_{\mathcal{E}}(\{X_\bullet^1,\dots,X_\bullet^{n_1}\},m_1;\{Y_\bullet^1,\dots,Y_\bullet^{m}\},m_2)\\&\qquad\cong D_{Y_\bullet^{n_2},\dots,Y_\bullet^1}(C_{X_\bullet^{n_1},\dots,X_\bullet^1}\mathcal{E}(-,-)(m_1))(m_2)\end{aligned}$$

and similar in the reduced case. In particular,

$$\begin{aligned}&\mathrm{Nat}_{\mathcal{E}}(\{X_\bullet^1,\dots,X_\bullet^{n_1}\},m_1;\{Y_\bullet^1,\dots,Y_\bullet^{n_2}\},m_2)\\&\qquad\cong \mathrm{Nat^D}_{\mathcal{E}}(\{Y_\bullet^1,\dots,Y_\bullet^{n_2}\},m_2;\{X_\bullet^1,\dots,X_\bullet^{n_1}\},m_1).\end{aligned}$$

Again, in Section 5.4.1, this is deduced from the more general setup of Theorem 5.16.

3.4. **Coalgebra structures.** From now on we focus on $\mathrm{Nat}_{\mathcal{E}}(X_\bullet,Y_\bullet)$ for two simplicial sets $X_\bullet$ and $Y_\bullet$, which by Corollary 3.3 covers the general case of families of simplicial sets up to quasi-isomorphism.

We are going to describe a coalgebra structure on the quotients of a filtration of the natural transformations of a fixed inner degree. This will give us structure on $\mathrm{Nat}_{\mathcal{E}}(X_\bullet,Y_\bullet)$ if $\mathcal{E}$ is concentrated in degree zero. In Theorem 4.11 in certain cases we describe a weak equivalence between $\mathrm{Nat}_{\mathcal{E}}(X_\bullet,Y_\bullet)$ and the cochains of the topological mapping space between the realizations of $X_\bullet$ and $Y_\bullet$. In homology with field coefficients, we show that the structure on the quotients agrees with the one induced by the coproduct on the chains of the mapping space.

To define the structure on the quotients, for a cosimplicial simplicial set $A_\bullet^\bullet$, we define a coalgebra structure on $C^*C_*(\mathbb{Z}[A_\bullet^\bullet])$ where $\mathbb{Z}[-]:\mathit{FinSet}\to \mathit{Ab}$ is the linearization functor.

Proposition 3.7. *Let $A_\bullet^\bullet$ be a cosimplicial simplicial abelian group. The double complex $C^*C_*(A_\bullet^\bullet)$ is a counital coalgebra. The comultiplication is the degree zero chain map of double complexes given by*

$$\begin{aligned}C^kC_l(A_\bullet^\bullet)&\xrightarrow{\Delta}C^kC_l(diag^\bullet diag_\bullet(A_\bullet^\bullet\otimes A_\bullet^\bullet))\\&\xrightarrow{AW}C^k(diag^\bullet(\bigoplus_{l_1+l_2=l}C_{l_1}A_\bullet^\bullet\otimes C_{l_2}A_\bullet^\bullet))\\&\xrightarrow{EZ^*}\bigoplus_{k_1+k_2=k}\bigoplus_{l_1+l_2=l}C^{k_1}C_{l_1}A_\bullet^\bullet\otimes C^{k_2}C_{l_2}A_\bullet^\bullet\\&=(C^*C_*(A_\bullet^\bullet)\otimes C^*C_*(A_\bullet^\bullet))_l^k\end{aligned}$$

with Δ being the diagonal map. The counit is the constant one map on $C_0C^0(A_\bullet^\bullet)$ and zero elsewhere.

The coalgebra structure commutes with taking the reduced Moore complex $\overline{C}_*$ instead of C_* and the same holds in the cosimplicial directions, i.e. taking $\overline{C}^*$ instead of C^*.

Proof. We need to show the axioms of a coalgebra, i.e. coassociativity and counitality. We note that $(\Delta \otimes \mathrm{id}) \circ EZ^* \circ AW \circ \Delta = EZ^*_{2,3} \circ AW_{2,3} \circ (\Delta \times \mathrm{id}) \circ \Delta$ where we apply the Eilenberg-Zilber and Alexander-Whitney map to the 2nd and 3rd directions of the trisimplicial set $(\Delta \times \mathrm{id}) \circ \Delta(A^\bullet_\bullet)$. It is clear that $(\Delta \times \mathrm{id}) \circ \Delta = (\mathrm{id} \times \Delta) \circ \Delta$. $\square$

We would like to have a comultiplication on the product double complex, but unfortunately since the tensor product does not commute with infinite products, we need to filter the complex.

For a cochain chain complex $D^{*\geq 0}_{*\geq 0}$ (second quadrant double chain complex) the filtration of $\mathrm{Tot}^{\prod} D$ by columns defined in Section 1.1 is given by $(F_m(D))_t = \prod_{\substack{k \geq m+1 \\ l-k=t}} D^k_l$.

For a cosimplicial simplicial group let

$$T^m(A^\bullet_\bullet) = \prod C^* C_*(A^\bullet_\bullet)/F_m(C^*C_*(A^\bullet_\bullet)).$$

This quotient is isomorphic to the chain complex $(\bigoplus_{\substack{k \leq m \\ l-k=t}} C^k C_l(A^\bullet_\bullet), \widetilde{d})$, with the new differential being zero in the cochain direction for $k = m$ and equal to the old differential otherwise.

Since the filtration is decreasing, we get surjective maps

$$T^m(A^\bullet_\bullet) \xrightarrow{p_m} T^{m-1}(A^\bullet_\bullet)$$

which under the isomorphism stated above are just the projections onto the smaller sum. Moreover, $T(A^\bullet_\bullet) = \lim T^m(A^\bullet_\bullet)$.

Proposition 3.8. *Let $A^\bullet_\bullet$ be a cosimplicial simplicial abelian group and Δ the comultiplication defined in Proposition 3.7. Δ induces maps $T^{2m}(A^\bullet_\bullet) \xrightarrow{\Delta_{2m}} T^m(A^\bullet_\bullet) \otimes T^m(A^\bullet_\bullet)$ and $T^{2m+1}(A^\bullet_\bullet) \xrightarrow{\Delta_{2m+1}} T^{m+1}(A^\bullet_\bullet) \otimes T^m(A^\bullet_\bullet)$ such that the diagram*

$$\begin{tikzcd}
T^{2m+1}(A^\bullet_\bullet) \arrow[r, "\Delta_{2m+1}"] \arrow[d, "p_{2m+1}"] & T^{m+1}(A^\bullet_\bullet) \otimes T^m(A^\bullet_\bullet) \arrow[d, "p_{m+1}\otimes id"] \\
T^{2m}(A^\bullet_\bullet) \arrow[r, "\Delta_{2m}"] \arrow[d, "p_{2m}"] & T^m(A^\bullet_\bullet) \otimes T^m(A^\bullet_\bullet) \arrow[d, "id\otimes p_m"] \\
T^{2m-1}(A^\bullet_\bullet) \arrow[r, "\Delta_{2m-1}"] & T^m(A^\bullet_\bullet) \otimes T^{m-1}(A^\bullet_\bullet)
\end{tikzcd} \tag{3.1}$$

commutes.

Proof. We have to check that the map which sends the element $[x] \in T^{2m}(A^\bullet_\bullet)_n$ to $[\Delta(x)] \in (T^m(A^\bullet_\bullet) \otimes T^m(A^\bullet_\bullet))_n$ commutes with the differential. We use the identification under the isomorphism given above and show that there is a map

$$\left(\bigoplus_{\substack{k \leq 2m \\ l-k=t}} C^k C_l(A^\bullet_\bullet), \tilde{d} \right) \to \left(\bigoplus_{\substack{k \leq m \\ l-k=t}} C^k C_l(A^\bullet_\bullet), \tilde{d} \right) \otimes \left(\bigoplus_{\substack{k \leq m \\ l-k=t}} C^k C_l(A^\bullet_\bullet), \tilde{d} \right).$$

For $x \in C^k C_l(A^\bullet_\bullet)$ with $k < 2m$, the differential is just the normal differential on the total complex. The comultiplication maps x to $\Delta(x) \in \bigoplus_{k_1+k_2=k} \bigoplus_{l_1+l_2=l} C^{k_1} C_{l_1}(A^\bullet_\bullet) \otimes$

$C^{k_2}C_{l_2}(A^\bullet_\bullet)$. By the definition of the new differential, the projection

$$\bigoplus_{k_1+k_2=k}\bigoplus_{l_1+l_2=l} C^{k_1}C_{l_1}(A^\bullet_\bullet)\otimes C^{k_2}C_{l_2}(A^\bullet_\bullet)$$

$$\to \bigoplus_{t_1+t_2=l-k}\left(\bigoplus_{\substack{k_1\le m\\ l_1-k_1=t_1}} C^{k_1}C_{l_1}(A^\bullet_\bullet),\tilde{d}\right)\otimes\left(\bigoplus_{\substack{k_2\le m\\ l_2-k_2=t_2}} C^{k_2}C_{l_2}(A^\bullet_\bullet),\tilde{d}\right)$$

is a chain map. Similarly one checks that for $x\in C^{2m}C_l(A^\bullet_\bullet)$, the only non-zero part of the image of $\Delta_m(x)$ lies in $\bigoplus_{l_1+l_2=l} C^mC_{l_1}(A^\bullet_\bullet)\otimes C^mC_{l_2}(A^\bullet_\bullet)$. The cohomological differential vanishes on both sides and thus the map is a chain map.

The computations for Δ_{2m+1} are analogous to the ones above.

By similar considerations one checks that the diagram mentioned in the proposition commutes.

□

To apply the proposition to the complex of formal transformations of fixed inner degree, we first start working with $D_{X_\bullet}C_{Y_\bullet}(\mathcal{E}(-,-))$. This complex is given as a totalization of the triple chain complex $C^*C_*\mathcal{E}(X_\bullet,Y_\bullet)_*$ where $\mathcal{E}(-,-)_r$ are the morphisms of degree r (where we forgot about the differential on $\mathcal{E}$). To avoid confusion, from now on we fix a degree r, i.e. work with the double chain complex $C^*C_*\mathcal{E}(X_\bullet,Y_\bullet)_r$. It is possible to bring the last direction into the picture, but this will not be used in this paper and seems to be rather confusing. So $\mathcal{E}(X_\bullet,Y_\bullet)_r$ is a cosimplicial simplicial abelian group and we can take its total complex. In degree p is given by

$$T(D_{X_\bullet}(C_{Y_\bullet}(\mathcal{E}(-,-)_r))=\prod_{l-k=p}(C^kC_l\mathcal{E}(X_\bullet,Y_\bullet)_r).$$

We filter it as done before to obtain the quotients $T^m(D_{X_\bullet}(C_{Y_\bullet}(\mathcal{E}(-,-)_r))$. Thus, we get maps

$$\Delta_{2m}: T^{2m}(D_{X_\bullet}(C_{Y_\bullet}(\mathcal{E}(-,-)_r)\to T^m(D_{X_\bullet}(C_{Y_\bullet}(\mathcal{E}(-,-)_r)\otimes T^m(D_{X_\bullet}(C_{Y_\bullet}(\mathcal{E}(-,-)_r)$$

and

$$\Delta_{2m+1}: T^{2m+1}(D_{X_\bullet}(C_{Y_\bullet}(\mathcal{E}(-,-)_r)\to T^{m+1}(D_{X_\bullet}(C_{Y_\bullet}(\mathcal{E}(-,-)_r)\otimes T^m(D_{X_\bullet}(C_{Y_\bullet}(\mathcal{E}(-,-)_r)$$

fitting in a diagram of the form (3.1) given in Proposition 3.8. If $\mathcal{E}(-,-)$ is concentrated in degree zero we have $T(D_{X_\bullet}(C_{Y_\bullet}(\mathcal{E}(-,-)_0))=D_{X_\bullet}C_{Y_\bullet}(\mathcal{E}(-,-))$.

On the other hand, for any $\Phi:\mathcal{E}\to\mathrm{Ch}$ and $f_\Phi:\Phi(X_k)_n\to\Phi(X_l)_m$ the map f has degree $(l+m)-(k+n)=(l-k)+(m-n)$. We refer to the first term as the simplicial degree $|f|_{simp}$ of f and to the second as the inner degree $|f|_{inn}$. So the complex $C^*C_*\hom(\Phi(X_\bullet)),\Phi(Y_\bullet))_{inn=r}$ of morphisms of a fixed inner degree r is a double chain complex, too. Taking all homomorphisms which are natural in Φ we get a double chain complex $C^*C_*\hom(-(X_\bullet)),-(Y_\bullet))_{inn=r}$, which by a similar proof to the one of Theorem 3.2 is isomorphic to the double chain complex $C^*C_*\mathcal{E}(X_\bullet,Y_\bullet)_r$. Hence we can define

$$\mathrm{Nat}(X_\bullet,Y_\bullet)_{inn=r}:=\prod_{l-k}C^kC_l\hom(-(X_\bullet)),-(Y_\bullet))_{inn=r}$$

so that the isomorphism above gives us an isomorphism

$$T(D_{X_\bullet}(C_{Y_\bullet}(\mathcal{E}(-,-)_r)\cong\mathrm{Nat}(X_\bullet,Y_\bullet)_{inn=r}.$$

The quotients $\mathrm{Nat}^m(X_\bullet,Y_\bullet)_{inn=r}$ are defined as the images of the respective quotients $T^m(D_{X_\bullet}(C_{Y_\bullet}(\mathcal{E}(-,-)_r)$. By the isomorphisms we get maps

$$\Delta_{2m}:\mathrm{Nat}^{2m}(X_\bullet,Y_\bullet)_{inn=r}\to\mathrm{Nat}^m(X_\bullet,Y_\bullet)_{inn=r}\otimes\mathrm{Nat}^m(X_\bullet,Y_\bullet)_{inn=r}$$

and

$$\Delta_{2m+1} : \mathrm{Nat}^{2m+1}(X_\bullet, Y_\bullet)_{inn=r} \to \mathrm{Nat}^{m+1}(X_\bullet, Y_\bullet)_{inn=r} \otimes \mathrm{Nat}^{m}(X_\bullet, Y_\bullet)_{inn=r}$$

fulfilling the properties proved in Proposition 3.8.

Assembling what we have said, we get:

Proposition 3.9. *For two simplicial finite sets $X_\bullet$ and $Y_\bullet$ the isomorphism in Theorem 3.2 induces isomorphisms*

$$T(D_{X_\bullet}(C_{Y_\bullet}(\mathcal{E}(-,-)_r))) \cong \mathrm{Nat}_{\mathcal{E}}(X_\bullet, Y_\bullet)_{inn=r}.$$

The complex $\mathrm{Nat}_{\mathcal{E}}(X_\bullet, Y_\bullet)_{inn=r}$ admits a filtration with quotients $\mathrm{Nat}^m_{\mathcal{E}}(X_\bullet, Y_\bullet)_{inn=r}$ together with maps Δ_{2m} and Δ_{2m+1} fitting into a commutative diagram of the form (3.1) *given in Proposition 3.8. In particular, in the case when $\mathcal{E}(-,-)$ is concentrated in degree zero,* $\mathrm{Nat}_{\mathcal{E}}(X_\bullet, Y_\bullet)_{inn=0} = \mathrm{Nat}_{\mathcal{E}}(X_\bullet, Y_\bullet)$ *and we obtain a filtration of the space of formal transformations together with maps Δ_{2m} and Δ_{2m+1}.*

4. Formal operations for the commutative PROP

For this section we fix $\mathcal{E} = \mathscr{C}om$.

4.1. **Small models for the formal operations.** In this section we aim for an easier and smaller descriptions of $\mathrm{Nat}_{\mathscr{C}om}(X_\bullet, Y_\bullet)$, which by Theorem 3.2 is isomorphic to $D_{X_\bullet}C_{Y_\bullet}(\mathscr{C}om(-,-))$. We start with rewriting the morphism spaces $\mathscr{C}om(r,s)$. For this, recall the linearization functor $\mathbb{Z}[-] : FinSet \to Ab$ which sends a finite set M to the free abelian group of formal linear combinations of elements of M. This induces a functor from $\mathscr{C}om$ to Ab via the following: An object s is sent to $\mathbb{Z}[s]$ and the map $Com(s,t) = \mathbb{Z}[FinOrd(s,t)] \to Ab(\mathbb{Z}[s], \mathbb{Z}[t])$ is the linear extension of $\mathbb{Z}[-] : FinOrd(s,t) \to Ab(\mathbb{Z}[s], \mathbb{Z}[t])$.

Moreover, taking products inside $\mathbb{Z}[-]$ gives us a functor $\mathbb{Z}[(-)^{\times -}] : \mathscr{C}om^{op} \times \mathscr{C}om \to Ab$ which sends an object (t,s) to $\mathbb{Z}[s^t]$. For an element $g \in FinOrd(t,t')$ we have an induced map $\mathbb{Z}[s^{t'}] \to \mathbb{Z}[s^t]$ sending a tuple $(a_1, \cdots, a_{t'}) \in s^{t'}$ to $(a_{g(1)}, \cdots, a_{g(t)}) \in s^t$. The action of $\mathscr{C}om^{op}$ is again the linear extension of this map $FinOrd(t,t') \to Ab(\mathbb{Z}[s^{t'}], \mathbb{Z}[s^t])$.

Lemma 4.1. *There is an equivalence of functors*

$$\mathscr{C}om(-,-) \cong \mathbb{Z}[(-)^{\times -}] \cong \mathbb{Z}[-]^{\otimes -} : \mathscr{C}om^{op} \times \mathscr{C}om \to Ab.$$

Proof. A morphism in $FinOrd(t,s)$ is given by specifying the image for each point of t independently, i.e. $FinOrd(t,s) \cong FinOrd(1,s)^{\times t} = s^{\times t}$. Since linearization sends products to tensor products we get an isomorphism $\mathscr{C}om(t,s) \cong \mathbb{Z}[s]^{\otimes t}$. By definition, these isomorphisms are natural. □

By the above considerations, for a simplicial finite set $Y_\bullet$ we get a functor

$$C_*(\mathbb{Z}[Y_\bullet^{\times -}]) : \mathscr{C}om^{op} \to \mathrm{Ch}$$

which is the same as the functor $C_*(Y_\bullet^{\times -})$. Its dual is given by $C^*(Y_\bullet^{\times -}) : \mathscr{C}om \to co\,\mathrm{Ch}$.

Proposition 4.2. *Let $X_\bullet$ be an arbitrary simplicial set and $Y_\bullet$ a simplicial finite set. We have an isomorphism*

$$D_{X_\bullet}(C_{Y_\bullet}(\mathscr{C}om(-,-))) \cong (C_{X_\bullet}(D_{Y_\bullet}(\mathbb{Z}[(-)^{\times -}])^*))^* = (C_{X_\bullet}(C^*(Y_\bullet^{\times -})))^*.$$

Moreover, working with coefficients in a field $\mathbb{F}$ and given a commutative cochain algebra A^* in $\mathrm{Ch}(\mathbb{F} - mod)$ such that $C^*(Y_\bullet^{\times -})$ and $(A^*)^{\otimes -}$ are quasi-isomorphic functors from $\mathscr{C}om$ to $\mathrm{Ch}(\mathbb{F} - mod)$ we have

$$D_{X_\bullet}(C_{Y_\bullet}(\mathscr{C}om(-,-)) \simeq C_{X_\bullet}(A^{\otimes -})^* \cong CH_{X_\bullet}(A)^*,$$

where the last term is the dual of the higher Hochschild homology of A.

Proof. By the previous lemma $\mathscr{C}om(-,-)$ is isomorphic to $\mathbb{Z}[(-)^{\times -}]$ as a bifunctor and we therefore obtain $D_{X_\bullet}(C_{Y_\bullet}(\mathscr{C}om(-,-))) \cong D_{X_\bullet}(C_{Y_\bullet}(\mathbb{Z}[(-)^{\times -}]))$. By Proposition 2.11 we have

$$(C_{Y_\bullet}(\mathbb{Z}[(-)^{\times -}])^* \cong D_{Y_\bullet}((\mathbb{Z}[(-)^{\times -}])^*).$$

Moreover, since $(C_{Y_\bullet}(\mathbb{Z}[(-)^{\times k}])) = C_*(Y_\bullet^{\times k})$ is degreewise free and finitely generated for any k (since $Y_\bullet$ is finite in each degree), dualizing twice is isomorphic to the identity, i.e. we get isomorphisms of functors

$$C_{Y_\bullet}(\mathbb{Z}[(-)^{\times -}]) \cong (D_{Y_\bullet}((\mathbb{Z}[(-)^{\times -}])^*)^*.$$

Plugging this into our original expression and using Proposition 2.11, we have

$$D_{X_\bullet}(C_{Y_\bullet}(\mathbb{Z}[(-)^{\times -}]))) = D_{X_\bullet}((D_{Y_\bullet}((\mathbb{Z}[(-)^{\times -}])^*)^*) = (C_{X_\bullet}(D_{Y_\bullet}(((\mathbb{Z}[(-)^{\times -}])^*)^*.$$

Since $D_{Y_\bullet}((\mathbb{Z}[(-)^{\times k}])^*) = C^*(Y_\bullet^{\times k})$, the first chain of isomorphisms follows.

For the second part of the proposition let A^* be a commutative cochain algebra such that $(A^*)^{\otimes -}$ and $C^*(Y_\bullet^{\times -})$ are quasi-isomorphic functors. By Proposition 2.9 we get a quasi-isomorphism $C_{X_\bullet}(C^*(Y_\bullet^{\times -})) \simeq C_{X_\bullet}((A^*)^{\otimes -})$ and likewise for their dual spaces (since we work over a field).

By Remark 2.6, the isomorphism $C_{X_\bullet}(A^{\otimes -}) \cong CH_{X_\bullet}(A)$ holds.

□

In the following remark and proposition, we show that we actually can weaken the conditions on $Y_\bullet$ such that $Y_\bullet$ only needs to be weakly equivalent to a simplicial finite set.

Remark 4.3. For $Y_\bullet$ an arbitrary simplicial set, we still get an isomorphism between the chain complexes $C_{Y_\bullet}(\mathscr{C}om(k,-))$ and $C_*(Y_\bullet^{\times k})$. This is true, since we obtain an isomorphism of chain complexes

$$C_{Y_\bullet}(\mathscr{C}om(k,-)) := \operatorname*{colim}_{\substack{K_\bullet \to Y_\bullet,\\ K_\bullet \text{ finite}}} C_*(\mathbb{Z}[K_\bullet^{\times k}]) \cong C_*(\mathbb{Z}[(\operatorname*{colim}_{\substack{K_\bullet \to Y_\bullet,\\ K_\bullet \text{ finite}}} K_\bullet)^{\times k}]) = C_*(Y_\bullet^{\times k}).$$

Since C_* and $\mathbb{Z}[-]$ are left adjoint functors and hence commute with colimits we have an isomorphism between $\operatorname{colim}_{\substack{K_\bullet \to Y_\bullet,\\ K_\bullet \text{ finite}}} C_*(\mathbb{Z}[K_\bullet^{\times k}])$ and $C_*(\mathbb{Z}[(\operatorname{colim}_{\substack{K_\bullet \to Y_\bullet,\\ K_\bullet \text{ finite}}} K_\bullet)^{\times k}])$. From now on, we do not distinguish between these functors.

Proposition 4.4. *Let $X_\bullet$ and $Y_\bullet$ be arbitrary simplicial sets. For a simplicial set $Y'_\bullet$ being weakly equivalent to $Y_\bullet$ we get a quasi-isomorphism*

$$D_{X_\bullet}C_{Y_\bullet}(\mathscr{C}om(-,-)) \simeq D_{X_\bullet}C_{Y'_\bullet}(\mathscr{C}om(-,-))$$

and the maps

$$T^m(D_{X_\bullet}C_{Y_\bullet}(\mathscr{C}om(-,-))) \to T^m(D_{X_\bullet}C_{Y'_\bullet}(\mathscr{C}om(-,-)))$$

commute with the Δ_k defined in Section 3.4.

The same holds for the reduced functors.

Proof. By Lemma 4.1 and the previous remark, we have $C_{Y_\bullet}(\mathscr{C}om(-,-)) = C_*(\mathbb{Z}[Y_\bullet^{\times -}])$. A weak equivalence $Y_\bullet \simeq Y'_\bullet$ induces a weak equivalence of simplicial abelian groups $\mathbb{Z}[Y_\bullet^{\times r}] \simeq \mathbb{Z}[Y_\bullet'^{\times r}]$ (cf. [GJ09, III 2.14]) commuting with the $\mathscr{C}om^{op}$ action on these functors. By the Dold-Kan correspondence this gives a quasi-isomorphism of functors $C_*(Y_\bullet^{\times -}) \simeq C_*(Y_\bullet'^{\times -}) : \mathscr{C}om^{op} \to \mathrm{Ch}$. By Proposition 2.9, applying $D_{X_\bullet}$ afterward preserves quasi-isomorphism. Moreover, the construction gives us maps of double complexes (before taking $\mathrm{Tot}^{\prod}$). Therefore, we get induced comultiplications on both double complexes as described in Proposition 3.7. By the naturality of the Alexander-Whitney and Eilenberg-Zilber maps, the quasi-isomorphism commutes with the comultiplication on the double complexes and therefore with the induced maps Δ_k on the quotients. □

In the following theorems we give an example where we can apply Proposition 4.2, namely the deRham algebra of a simplicial set which is weakly equivalent to a simplicial finite set. The proof of the theorem is given in Appendix A.

Theorem 4.5. *Let $\mathbb{F}$ be a field such that $\mathbb{Q} \subseteq \mathbb{F}$, $X_\bullet$ an arbitrary simplicial set and $Y_\bullet$ a simplicial set weakly equivalent to a simplicial finite set. Then there is a quasi-isomorphism*

$$\mathrm{Nat}_{\mathscr{C}om}(C_{X_\bullet}, C_{Y_\bullet}) \simeq (CH_{X_\bullet}(\Omega^*(Y_\bullet)))^* \simeq (CH_{X_\bullet}(\Omega^*(|Y_\bullet|)))^*.$$

4.2. **Relationship of the formal operations and the mapping space.** Making use of [Bou87], in this section we prove that under certain conditions there is a weak equivalence between $\mathrm{Nat}_{\mathscr{C}om}(X_\bullet, Y_\bullet)$ and the singular chain complex of the topological mapping space $\hom_{Top}(|X|, |Y|)$. On $C_*(\hom_{Top}(|X|, |Y|))$ we have an actual coalgebra structure with comultiplication being the composition of the induced map of the diagonal and the Alexander-Whitney map (which the cupproduct is dual to). Working with coefficients over a field $\mathbb{F}$, this induces a coalgebra structure on $H_*(\hom_{Top}(|X|, |Y|))$ and thus by the aforementioned isomorphism a coalgebra structure on $H_*(D_{X_\bullet}(C_{Y_\bullet}(\mathscr{C}om(-,-))))$. This restricts to the Δ_k maps on the quotients of the filtration of the space introduced in Section 3.4.

We start with recalling definitions about cosimplicial simplicial sets and introduce the language used in [Bou87].

For a simplicial set $X_\bullet$ its *n-skeleton* $(sk_n X)_\bullet$ is the subsimplicial set of $X_\bullet$ generated by the simplices of degree $\leq n$.

The *standard cosimplicial simplicial set* $\Delta_\bullet^\bullet$ is defined via

$$\Delta_\bullet^\bullet = \hom_\Delta(-,-) : \Delta^{op} \times \Delta \to Set.$$

This means that in cosimplicial degree m it is given by the standard simplicial set $\Delta_\bullet^m = \hom_\Delta(-, m)$. The *$n$-skeleton* $(sk_n\Delta)_\bullet^\bullet$ of $\Delta_\bullet^\bullet$ is in each cosimplicial degree m the n-skeleton of the simplicial set $\Delta_\bullet^m$, i.e. $(sk_n\Delta)_\bullet^m := (sk_n\Delta^m)_\bullet$.

For two simplicial sets $X_\bullet$ and $Y_\bullet$, *the simplicial mapping space* is the simplicial set whose n-th level is given by

$$Map(X_\bullet, Y_\bullet)_n := \hom_{Set^{\Delta^{op}}}(X_\bullet \times \Delta_\bullet^n, Y_\bullet).$$

For a cosimplicial simplicial set $Z_\bullet^\bullet$ the *cosimplicial realization* $\mathrm{Tot}\, Z_\bullet$ is the simplicial set whose n-th level is given by

$$(\mathrm{Tot}\, Z)_n := \hom_{Set^{\Delta \times \Delta^{op}}}(\Delta_\bullet^\bullet \times \Delta_\bullet^n, Z_\bullet^\bullet) \subset \prod_{k \geq 0} Map(\Delta_\bullet^k, Z_\bullet^k)_n,$$

i.e. cosimplicial maps in the simplicial mapping space. We define a filtration $(\mathrm{Tot}_m\, Z)$ by $(\mathrm{Tot}_m\, Z)_n = \hom_{Set^{\Delta\times\Delta^{op}}}((sk_m\Delta)_\bullet^\bullet \times \Delta_\bullet^n, Z_\bullet^\bullet)$. Hence, we obtain maps $(\mathrm{Tot}_{m+1}\, Z)_\bullet \to (\mathrm{Tot}_m\, Z)_\bullet$ and the cosimplicial realization can be rewritten as the limit $(\mathrm{Tot}\, Z)_\bullet = \lim_m (\mathrm{Tot}_m\, Z)_\bullet$.

The total complex T_*Z of the double complex of a cosimplicial simplicial set $T_*Z = \mathrm{Tot}^{\prod} \overline{C}^*(\overline{C}_*(Z_\bullet^\bullet))$ and its filtrations $T^m = \prod_{k \leq m} \overline{C}^k \overline{C}_l(Z_\bullet^\bullet)$ were already discussed in Section 3.4. Recall that T_*Z is the limit of its filtration, i.e. $T_*Z = \lim_m T_*^m Z$.

Denote by $ev^{[m]}$ the evaluation map $T^m((sk_m\Delta)_\bullet^\bullet \times \mathrm{Tot}_m\, Z)) \to T^m(Z_\bullet^\bullet)$. We define the element c_p as the identity map in $\hom_\Delta(\{p\}, \{p\})$, i.e. $c_p \in \overline{C}_p \overline{C}^p(\Delta_\bullet^\bullet)$. Let $c^m \in T^m(\Delta_\bullet^\bullet)$ be the image of $\sum_{i=0}^m c_i$ under the projection on $T^m(\Delta_\bullet^\bullet)$. The element c^m has trivial differential in $T^m(\Delta_\bullet^\bullet)$. Moreover, note that $\overline{C}_k(\Delta^n) = 0$ if $k > n$.

Following [Bou87] we define the maps λ_m via

$$\begin{aligned} \lambda_m : \overline{C}_*(\mathrm{Tot}_m\, Z) &\xrightarrow{c^m \otimes id} (T^m((sk_m\Delta)_\bullet^\bullet))_0 \otimes \overline{C}_*(\mathrm{Tot}_m\, Z) \\ &\xrightarrow{EZ} (T^m((sk_m\Delta)_\bullet^\bullet \times \mathrm{Tot}_m\, Z))_* \xrightarrow{ev^{[m]}} (T^m(Z_\bullet^\bullet))_*. \end{aligned} \tag{4.1}$$

We know that the reduced cochains have the structure of a coalgebra, i.e. we have maps $\overline{C}_*(\mathrm{Tot}_k\, Z) \xrightarrow{\Delta} \overline{C}_*(\mathrm{Tot}_k\, Z) \otimes \overline{C}_*(\mathrm{Tot}_k\, Z)$. Using the projection maps $p_{k,j} : \overline{C}_*(\mathrm{Tot}_k\, Z) \to \overline{C}_*(\mathrm{Tot}_j\, Z)$ for $k \geq j$, we define $\Delta^{\mathrm{Tot}}_{2m}$ as the composite

$$\overline{C}_*(\mathrm{Tot}_{2m}\, Z) \xrightarrow{\Delta} \overline{C}_*(\mathrm{Tot}_{2m}\, Z) \otimes \overline{C}_*(\mathrm{Tot}_{2m}\, Z) \to \overline{C}_*(\mathrm{Tot}_m\, Z) \otimes \overline{C}_*(\mathrm{Tot}_m\, Z) \tag{4.2}$$

with the last map being $p_{2m,m} \otimes p_{2m,m}$. Similarly, $\Delta^{\mathrm{Tot}}_{2m+1}$ is the composite

$$\overline{C}_*(\mathrm{Tot}_{2m+1}\, Z) \xrightarrow{\Delta} \overline{C}_*(\mathrm{Tot}_{2m+1}\, Z) \otimes \overline{C}_*(\mathrm{Tot}_{2m+1}\, Z) \to \overline{C}_*(\mathrm{Tot}_{m+1}\, Z) \otimes \overline{C}_*(\mathrm{Tot}_m\, Z), \tag{4.3}$$

with the last map being $p_{2m+1,m+1} \otimes p_{2m+1,m}$.

These induce commutative diagrams

$$\begin{tikzcd} \overline{C}_*(\mathrm{Tot}\, Z) \arrow[r, "\Delta"] \arrow[d, "p_{2m}"] & \overline{C}_*(\mathrm{Tot}\, Z) \otimes \overline{C}_*(\mathrm{Tot}\, Z) \arrow[d, "p_m \otimes p_m"] \\ \overline{C}_*(\mathrm{Tot}_{2m}\, Z) \arrow[r, "\Delta^{\mathrm{Tot}}_{2m}"] & \overline{C}_*(\mathrm{Tot}_m\, Z) \otimes \overline{C}_*(\mathrm{Tot}_m\, Z) \end{tikzcd} \tag{4.4}$$

and

$$\begin{tikzcd} \overline{C}_*(\mathrm{Tot}\, Z) \arrow[r, "\Delta"] \arrow[d, "p_{2m+1}"] & \overline{C}_*(\mathrm{Tot}\, Z) \otimes \overline{C}_*(\mathrm{Tot}\, Z) \arrow[d, "p_{m+1} \otimes p_m"] \\ \overline{C}_*(\mathrm{Tot}_{2m+1}\, Z) \arrow[r, "\Delta^{\mathrm{Tot}}_{2m+1}"] & \overline{C}_*(\mathrm{Tot}_{m+1}\, Z) \otimes \overline{C}_*(\mathrm{Tot}_m\, Z). \end{tikzcd} \tag{4.5}$$

We now show that the λ_m commute with the Δ_m.

Lemma 4.6. *The diagrams*

$$\begin{tikzcd} \overline{C}_*(\mathrm{Tot}_{2m}\, Z) \arrow[r, "\Delta^{\mathrm{Tot}}_{2m}"] \arrow[d, "\lambda_{2m}"] & \overline{C}_*(\mathrm{Tot}_m\, Z) \otimes \overline{C}_*(\mathrm{Tot}_m\, Z) \arrow[d, "\lambda_m \otimes \lambda_m"] \\ T^{2m} Z \arrow[r, "\Delta_{2m}"] & T^m Z \otimes T^m Z \end{tikzcd}$$

and

$$\begin{tikzcd} \overline{C}_*(\mathrm{Tot}_{2m+1}\, Z) \arrow[r, "\Delta^{\mathrm{Tot}}_{2m+1}"] \arrow[d, "\lambda_{2m+1}"] & \overline{C}_*(\mathrm{Tot}_{m+1}\, Z) \otimes \overline{C}_*(\mathrm{Tot}_m\, Z) \arrow[d, "\lambda_{m+1} \otimes \lambda_m"] \\ T^{2m} Z \arrow[r, "\Delta_{2m+1}"] & T^{m+1} Z \otimes T^m Z \end{tikzcd}$$

commute.

Proof. We define the maps Δ_{2m} and Δ_{2m+1} on the intermediate spaces used in the definition of λ_{2m} and λ_{2m+1}, respectively (cf. Equation (4.1)).

1. On $\overline{C}_*(\mathrm{Tot}_k\, Z)$ for $k = 2m$ or $k = 2m+1$, the maps $\Delta^{\mathrm{Tot}}_{2m}$ and $\Delta^{\mathrm{Tot}}_{2m+1}$ have been defined in Equation (4.2) and Equation (4.3), respectively.
2. On $(T^k((sk_m\Delta)^{\bullet}_{\bullet}))_0 \otimes \overline{C}_*(\mathrm{Tot}\, Z)$ it is given by $\tau_{2,3} \circ (\Delta_k \otimes \Delta^{Tot}_k)$ where $\tau_{2,3}$ permutes the second and third term.
3. On $T^k(((sk_m\Delta)^{\bullet}_{\bullet} \times \mathrm{Tot}\, Z))_*$ it is given by Δ_k.
4. On $(T^k Z^{\bullet}_{\bullet})_*$ it equals Δ_k, too.

(1) The map $\overline{C}_*(\mathrm{Tot}_m\, Z) \xrightarrow{c \otimes id} T^m(\Delta^{\bullet}_{\bullet}) \otimes \overline{C}_*(\mathrm{Tot}_m\, Z)$ sends $x \mapsto c^m \otimes x$. We show that $\Delta_{2m}(c^{2m}) = c^m \otimes c^m$ and $\Delta_{2m+1}(c^{2m+1}) = c^{m+1} \otimes c^m$. Since the c^m are of total degree zero, permuting them with other elements does not create a sign.

By Proposition 3.7, this reduces to showing that the comultiplication Δ on the double complex $C^*C_*\Delta^\bullet_\bullet$ sends the element $c_p \in C^pC_p\Delta^\bullet_\bullet$ to $\sum_{p_1+p_2=p} c_{p_1} \otimes c_{p_2}$. Recall that Δ_{Tot} is given as

$$\Delta_{Tot} : \overline{C}_p(\Delta^p) \xrightarrow{\Delta_*} \overline{C}_p(\Delta^p \times \Delta^p) \xrightarrow{AW} \bigoplus_{p_1+p_2=p} \overline{C}_{p_1}(\Delta^p) \otimes \overline{C}_{p_2}(\Delta^p)$$
$$\xrightarrow{EZ^*} \bigoplus_{p_1+p_2=p} \bigoplus_{q_1+q_2=p} \overline{C}_{p_1}(\Delta^{q_1}) \otimes \overline{C}_{p_2}(\Delta^{q_2}).$$

Since $\overline{C}_a(\Delta^b) = 0$ for $a > b$, the image lies in $\bigoplus_{p_1+p_2=p} \overline{C}_{p_1}(\Delta^{p_1}) \otimes \overline{C}_{p_2}(\Delta^{p_2})$. In each summand of the reduced complex only one pair of non-degenerate simplices survives, since all summands not belonging to the identity permutation in EZ^* are degenerate.

(2) The map $T^m((sk_m\Delta)^\bullet_\bullet)_0 \otimes \overline{C}_*(\mathrm{Tot}_m Z) \xrightarrow{EZ} T^m((sk_m\Delta)^\bullet_\bullet \times \mathrm{Tot}_m Z)_*$: On both sides we first apply Δ_* and then AW in the simplicial direction. For any two simplicial sets $A_\bullet$ and $B_\bullet$, the diagram

$$\begin{tikzcd}
\overline{C}_k(A_\bullet) \otimes \overline{C}_l(B_\bullet) \arrow[rr, "EZ"] \arrow[d, "AW\circ\Delta_*\otimes AW\circ\Delta_*"] & & \overline{C}_{k+l}(\mathrm{diag}(A_\bullet \times B_\bullet)) \arrow[dd, "AW\circ\Delta_*"] \\
\bigoplus \overline{C}_{k_1}(A_\bullet) \otimes \overline{C}_{k_2}(A_\bullet) \otimes \overline{C}_{l_1}(B_\bullet) \otimes \overline{C}_{l_2}(B_\bullet) \arrow[d, "\tau_{2,3}"] & & \\
\bigoplus \overline{C}_{k_1}(A_\bullet) \otimes \overline{C}_{l_1}(B_\bullet) \otimes \overline{C}_{k_2}(A_\bullet) \otimes \overline{C}_{l_2}(B_\bullet) \arrow[r, "EZ"] & \bigoplus \overline{C}_{r_1}(\mathrm{diag}(A_\bullet \times B_\bullet)) \otimes \overline{C}_{r_2}(\mathrm{diag}(A_\bullet \times B_\bullet)) &
\end{tikzcd}$$

commutes ([FHT01, I.4 b)]). Hence, in our case applying $AW \circ \Delta_*$ commutes with the asked morphism. Since we apply EZ^* to the cosimplicial direction on both sides, which is not affected by the morphism, it preserves the comultiplication.

(3) The map $ev^{[m]}$ commutes with the Δ_{2m} and Δ_{2m+1} by the naturality of EZ^*, AW and Δ_*.

□

We will show that under some conditions the map $\lambda : \overline{C}_*(\mathrm{Tot}\, Z) \to (T(Z^\bullet_\bullet))_*$ induced by the maps $\lambda_m : \overline{C}_*(\mathrm{Tot}_m Z) \to (T^m(Z^\bullet_\bullet))_*$ is a quasi-isomorphism. If this is a case the above commutativity result implies that the comultiplication on the homology with field coefficients induces the Δ_k maps on the quotients:

Lemma 4.7. *Let $\mathbb{F}$ be a field and assume that the map $\lambda : \overline{C}_*(\mathrm{Tot}\, Z) \to (T(Z^\bullet_\bullet))_*$ induced by the maps $\lambda_m : \overline{C}_*(\mathrm{Tot}_m Z) \to (T^m(Z^\bullet_\bullet))_*$ is a quasi-isomorphism. Denote by $\Delta : H_*(T(Z^\bullet_\bullet);\mathbb{F}) \to H_*(T(Z^\bullet_\bullet);\mathbb{F}) \otimes H_*(T(Z^\bullet_\bullet);\mathbb{F})$ the image of the comultiplication on $H_*(\mathrm{Tot}\, Z;\mathbb{F})$ under $H_*(\lambda)$. Then the following diagrams commute:*

$$\begin{tikzcd}
H_*(T(Z^\bullet_\bullet);\mathbb{F}) \arrow[r, "\Delta"] \arrow[d, "p_{2m}"] & H_*(T(Z^\bullet_\bullet);\mathbb{F}) \otimes H_*(T(Z^\bullet_\bullet);\mathbb{F}) \arrow[d, "p_m\otimes p_m"] \\
H_*(T^{2m}(Z^\bullet_\bullet);\mathbb{F}) \arrow[r, "H_*(\Delta_{2m})"] & H_*(T^m(Z^\bullet_\bullet);\mathbb{F}) \otimes H_*(T^m(Z^\bullet_\bullet);\mathbb{F})
\end{tikzcd}$$

and

$$\begin{tikzcd}
H_*(T(Z^\bullet_\bullet);\mathbb{F}) \arrow[r, "\Delta"] \arrow[d, "p_{2m+1}"] & H_*(T(Z^\bullet_\bullet);\mathbb{F}) \otimes H_*(T(Z^\bullet_\bullet);\mathbb{F}) \arrow[d, "p_{m+1}\otimes p_m"] \\
H_*(T^{2m+1}(Z^\bullet_\bullet)) \arrow[r, "H_*(\Delta_{2m+1})"] & H_*(T^{m+1}(Z^\bullet_\bullet);\mathbb{F}) \otimes H_*(T^m(Z^\bullet_\bullet);\mathbb{F})
\end{tikzcd}$$

Proof. During the proof we use the notation $C_* = \overline{C}_*(\operatorname{Tot} Z)$, $C^m_* = \overline{C}_*(\operatorname{Tot}_m Z)$, $B_* = (T(Z^\bullet_\bullet))_*$ and $B^m_* = (T^m(Z^\bullet_\bullet))_*$. To check the commutativity of the diagrams (we only do it for the first one, the second one works the same), we notice that in the diagram

$$\begin{tikzcd}
C_* \arrow[rrr, "\Delta"] \arrow[dr, "p_{2m}"] \arrow[ddd, "\lambda"'] & \arrow[r, phantom, "(1)"] & & C_* \otimes C_* \arrow[dl, "p_m \otimes p_m"'] \arrow[ddd, "\lambda\otimes\lambda"] \\
\arrow[d, phantom, "(2)"] & C^{2m}_* \arrow[r, "\Delta_{2m}"] \arrow[d, "\lambda_{2m}"] \arrow[dr, phantom, "(4)"] & C^m_* \otimes C^m_* \arrow[d, "\lambda_m\otimes\lambda_m"] & \arrow[d, phantom, "(3)"] \\
& B^{2m}_* \arrow[r, "\Delta_{2m}"] & B^m_* \otimes B^m_* & \\
B_* \arrow[ur, "p_{2m}"] & & & B_* \otimes B_* \arrow[ul, "p_m\otimes p_m"']
\end{tikzcd}$$

all the squares commutes. This holds, since (1) commutes by diagram (4.4), (2) and (3) by the definition of λ as the induced map on the limits and (4) by Lemma 4.6.

Since we work over a field (i.e. the Kuenneth Theorem holds) the above diagram induces a diagram on homology

$$\begin{tikzcd}
H_*(C;\mathbb{F}) \arrow[rrr, "\Delta"] \arrow[dr, "H_*(p_{2m})"] \arrow[ddd, "H_*(\lambda)"'] & & & H_*(C;\mathbb{F}) \otimes H_*(C;\mathbb{F}) \arrow[dl, "H_*(p_m)\otimes H_*(p_m)"'] \arrow[ddd, "H_*(\lambda)\otimes H_*(\lambda)"] \\
& H_*(C^{2m};\mathbb{F}) \arrow[r, "H_*(\Delta_{2m})"] \arrow[d, "H_*(\lambda_{2m})"] & H_*(C^m;\mathbb{F}) \otimes H_*(C^m;\mathbb{F}) \arrow[d, "H_*(\lambda_m)\otimes H_*(\lambda_m)"] & \\
& H_*(B^{2m};\mathbb{F}) \arrow[r, "H_*(\Delta_{2m})"] \arrow[d, phantom, "(5)"] & H_*(B^m;\mathbb{F}) \otimes H_*(B^m;\mathbb{F}) & \\
H_*(B;\mathbb{F}) \arrow[ur, "H_*(p_{2m})"] \arrow[rrr, "\Delta"] & & & H_*(B;\mathbb{F}) \otimes H_*(B;\mathbb{F}) \arrow[ul, "H_*(p_m)\otimes H_*(p_m)"']
\end{tikzcd}$$

where the comultiplication $\Delta : H_*(T(Z^\bullet_\bullet);\mathbb{F}) \to H_*(T(Z^\bullet_\bullet);\mathbb{F}) \otimes H_*(T(Z^\bullet_\bullet);\mathbb{F})$ is the map that makes the outer square commute. Since all subsquares beside (5) commute by the above argumentation, the square (5) also commutes and we have proved the lemma. □

Before actually applying Bousfield's spectral sequence theorem and achieving a quasi-isomorphism, we need one more technical lemma:

Lemma 4.8. *For $Y_\bullet$ a Kan complex and $X_\bullet$ a simplicial set, the cosimplicial simplicial set $Y_\bullet^{X_\bullet} = \hom_{Set}(X_\bullet, Y_\bullet)$ is fibrant (in the Reedy model structure on cosimplicial simplicial sets, see* [GJ09, Chapter VII.4]*), i.e. the maps*

$$\hom_{Set}(X_n, Y_\bullet) \to \lim_{[n]\twoheadrightarrow[k]} \hom_{Set}(X_k, Y_\bullet)$$

are Kan fibrations for all n.

Proof. Since $\lim_{[n]\twoheadrightarrow[k]} \hom_{Set}(X_k, Y_\bullet)$ is isomorphic to $\hom_{Set}(\operatorname{colim}_{\substack{[n]\twoheadrightarrow[k]\\ n>k}} X_k, Y_\bullet)$, we have to check that the maps

$$\hom_{Set}(X_n, Y_\bullet) \to \hom_{Set}(\operatorname*{colim}_{\substack{[n]\twoheadrightarrow[k]\\ n>k}} X_k, Y_\bullet),$$

given by precomposition with the map $\operatorname{colim}_{\substack{[n]\twoheadrightarrow[k]\\ n>k}} X_k \to X_n$, are Kan fibrations. Denote $L_nX = \operatorname{colim}_{\substack{[n]\twoheadrightarrow[k]\\ n>k}} X_k$. The map $L_nX \to X_n$ is the embedding of the degenerate n-simplices and thus it is injective.

Rewriting $\hom_{Set}(X_n, Y_\bullet) = \prod_{X_n} Y_\bullet = \prod_{L_nX} Y_\bullet \times \prod_{X_n\setminus L_nX} Y_\bullet$ and analogously $\hom_{Set}(L_nX, Y_\bullet) = \prod_{L_nX} Y_\bullet = \prod_{L_nX} Y_\bullet \times \prod_{X_n\setminus L_nX} *$ the asked map is the product of the identity on the first factor and the map $Y_\bullet \to *$ on the second factor.

The latter maps are fibrations, since $Y_\bullet$ is a Kan-complex. The identity maps are fibrations, too. Since the product of Kan fibrations is a Kan fibration, the lemma is proven. □

For a finite simplicial set $X_\bullet$ denote by $dim(X_\bullet)$ the maximal degree of a non-degenerate simplex in $X_\bullet$.

Let $Y_\bullet$ be a simplicial set. The homotopy groups are defined as those of the topological realization of $Y_\bullet$. We work with unpointed simplicial sets, i.e. unpointed spaces. For $Y_\bullet$ path-connected the homotopy groups are well-defined defined by the choice of an arbitrary base point (up to non-canonical isomorphism). If the fundamental group is abelian and acts trivial on all higher homotopy groups (i.e. $Y_\bullet$ is simple), the isomorphism is canonical.

For a simplicial set $Y_\bullet$ we write $Conn(Y_\bullet)$ for its connectivity, i.e. the smallest $k \geq 0$ such that $\pi_{k+1}(Y) \neq 0$. The homotopy groups of a cosimplicial simplicial set are the cosimplicial set of homotopy groups in each degree, i.e. $(\pi_k(Z^\bullet_\bullet))^n = \pi_k(Z^n_\bullet)$.

For a topological Hausdorff space Y we have an equivalence $S_\bullet(\hom_{Top}(|X|, Y)) \cong Map(X_\bullet, S_\bullet(Y))$.

For the cosimplicial simplicial set $Y_\bullet^{X_\bullet}$, by arguments similar to [PT03, 1.5], the following holds

$$\operatorname{Tot}(Y_\bullet^{X_\bullet}) = Map(X_\bullet, Y_\bullet).$$

Moreover, for a space Y, we conclude

$$\operatorname{Tot}(S_\bullet(Y)^{X_\bullet}) = Map(X_\bullet, S_\bullet(Y)) \cong S_\bullet(\hom_{Top}(|X|, Y)).$$

Theorem 4.9. *Let $X_\bullet$ be a finite simplicial set and $Y_\bullet$ be a simple Kan complex such that $dim(X_\bullet) \leq Conn(Y_\bullet)$. Then the maps λ_m induce a quasi-isomorphism*

$$\lambda : \overline{C}_*(Map(X_\bullet, Y_\bullet)) \to T(Y_\bullet^{X_\bullet}).$$

Proof. The proof goes similar to the argumentation for the pointed case in [Bou87, Example 4.3]. By [Bou87, Lemma 2.3] the map λ is an isomorphism on homology, i.e. a quasi-isomorphism if the spectral sequence of a fibrant cosimplicial simplicial set $Z^\bullet_\bullet$ converges. By [Bou87, Theorem 3.2] for a fibrant cosimplicial simplicial set $Z^\bullet_\bullet$ with each $Z^m_\bullet$ simple, the spectral sequence converges, if the following two conditions hold:

(1) $H^m\pi_{m+n}(Z^\bullet_\bullet) = 0$ if $n \leq 0$.
(2) For all n there are only finitely many m such that $H^m\pi_{m+n}(Z^\bullet_\bullet) \neq 0$.

If we plug in $Z^\bullet_\bullet = Y_\bullet^{X_\bullet}$, with $Y_\bullet$ simple, we get

$$\begin{aligned} H^m\pi_{m+n}(Y_\bullet^{X_\bullet}) &= \pi^m\pi_{m+n}(Hom_{Set}(X_\bullet, Y_\bullet)) \\ =& H^m(Hom_{Set}(X_\bullet, \pi_{m+n}(Y_\bullet))) \cong H^m(C_*(X_\bullet); \pi_{m+n}(Y_\bullet)). \end{aligned}$$

The second last equality holds since homotopy groups commute with products. In the case of $Y_\bullet$ a Kan complex, $Z^\bullet_\bullet$ is fibrant by Lemma 4.8.

Rewriting the conditions for the convergence of the spectral sequence, we obtain

(1) $H^m(C_*(X_\bullet); \pi_{m+n}(Y_\bullet)) = 0$ if $n \leq 0$.
(2) For all n there are only finitely many m, such that $H^m(C_*(X_\bullet); \pi_{m+n}(Y_\bullet)) \neq 0$.

By the universal coefficient Theorem (cf. [Hat02, Theorem 3.2]) we have a (split) short exact sequence

$$\begin{aligned}0 \to & Ext(H_{m-1}(C_*(X_\bullet)), \pi_{m+n}(Y_\bullet)) \to H^m(C_*(X_\bullet); \pi_{m+n}(Y_\bullet)) \\ & \to \hom(H_m(C_*(X_\bullet)), \pi_{m+n}(Y_\bullet)) \to 0\end{aligned}$$

Since $\overline{C}_m(X_\bullet)$ is nonzero only for $m \leq dim(X_\bullet)$, $Ext(H_{m-1}(C_*(X_\bullet)), \pi_{m+n}(Y_\bullet))$ and $\hom(H_m(C_*(X_\bullet)), \pi_{m+n}(Y_\bullet))$ are both zero for $m-1 > dim(X)$. Thus, Condition 2. holds.

To check $H^m(C_*(X_\bullet); \pi_{m+n}(Y_\bullet)) = 0$ if $n \leq 0$, we are left to show that the left term of the short exact sequence vanishes for $m-1 \leq dim(X)$ and the right term vanishes for $m \leq dim(X)$.

By the connectivity of $Y_\bullet$, $\pi_{m+n}(Y_\bullet) = 0$ for $m+n \leq Conn(Y)$. Hence, if we assume $m \leq dim(X)$ and $n \leq 0$, we have $m+n \leq m \leq dim(X) \leq Conn(Y)$, i.e. $\pi_{m+n}(Y_\bullet)$ vanishes for $m \leq dim(X)$. Then the first and third term in the short exact sequence also vanish and $H^m(C_*(X_\bullet); \pi_{m+n}(Y_\bullet)) = 0$ for $m \leq dim(X)$. So far we have shown that the last term of the long exact sequence always vanishes and the first term vanishes as long as $m-1 \neq dim(X)$. Assume $m-1 = dim(X)$. The group $H_{dim(X)}(\overline{C}_*(X_\bullet))$ is free since $\overline{C}_{dim(X)+1}(X_\bullet) = 0$ and thus there are no boundaries divided out when taking homology. So $Ext(H_{dim(X)}(\overline{C}_*(X_\bullet)), \pi_{dim(X)+1+n}(Y_\bullet)) = Ext(H_{dim(X)}(C_*(X_\bullet)), \pi_{dim(X)+1+n}(Y_\bullet))$ vanishes. Taking all this together, we have shown that Condition 1 holds and proved the theorem.

□

Plugging in the Kan complex $S_\bullet(Y)$ for a space Y, we get

Corollary 4.10. *For a finite simplicial set $X_\bullet$ and a topological space Y with $dim(X) \leq Conn(Y)$, there is a quasi-isomorphism*

$$\lambda : \overline{C}_*(Hom_{Top}(|X_\bullet|, Y)) \to T(S_\bullet(Y)^{X_\bullet}).$$

Taking coefficients in a field $\mathbb{F}$, the homology of $H_*(Hom_{Top}(|X_\bullet|, Y); \mathbb{F})$ is a coalgebra and thus the homology $H_*(T(S_\bullet(Y)^{X_\bullet}); \mathbb{F})$ is a coalgebra, too. Using this coalgebra structure over a field, we get commuting diagrams

$$\begin{tikzcd} H_*(T(S_\bullet(Y)^{X_\bullet}); \mathbb{F}) \arrow[r] \arrow[d] & H_*(T(S_\bullet(Y)^{X_\bullet}); \mathbb{F}) \otimes H_*(T(S_\bullet(Y)^{X_\bullet}); \mathbb{F}) \arrow[d] \\ H_*(T^{2m}(S_\bullet(Y)^{X_\bullet}); \mathbb{F}) \arrow[r, "H_*(\Delta_{2m})"] & H_*(T^m(S_\bullet(Y)^{X_\bullet}); \mathbb{F}) \otimes H_*(T^m(S_\bullet(Y)^{X_\bullet}); \mathbb{F}) \end{tikzcd}$$

and

$$\begin{tikzcd} H_*(T(S_\bullet(Y)^{X_\bullet}); \mathbb{F}) \arrow[r] \arrow[d] & H_*(T(S_\bullet(Y)^{X_\bullet}); \mathbb{F}) \otimes H_*(T(S_\bullet(Y)^{X_\bullet}); \mathbb{F}) \arrow[d] \\ H_*(T^{2m+1}(S_\bullet(Y)^{X_\bullet}); \mathbb{F}) \arrow[r, "H_*(\Delta_{2m+1})"] & H_*(T^{m+1}(S_\bullet(Y)^{X_\bullet}); \mathbb{F}) \otimes H_*(T^m(S_\bullet(Y)^{X_\bullet}); \mathbb{F}) \end{tikzcd} .$$

Proof. By Theorem 4.9 the map λ is a quasi-isomorphism. Lemma 4.7 gives us the commutativity of the diagrams.

□

Taking everything together, we have:

Theorem 4.11. *For an arbitrary simplicial set $Y_\bullet$ and a finite simplicial set $X_\bullet$ such that $dim(X_\bullet) \leq Conn(Y_\bullet)$, there is weak equivalence*

$$\overline{C}_*(Hom_{Top}(|X_\bullet|, |Y_\bullet|)) \simeq \overline{\mathrm{Nat}}_{\mathscr{C}om}(X_\bullet, Y_\bullet) \simeq \mathrm{Nat}_{\mathscr{C}om}(X_\bullet, Y_\bullet).$$

Taking coefficients in a field $\mathbb{F}$ *the comultiplication on* $H_*(\mathrm{Nat}_{\mathscr{C}om}(X_\bullet, Y_\bullet);\mathbb{F})$ *induced by the one on* $H_*(Hom_{Top}(|X_\bullet|, |Y_\bullet|);\mathbb{F})$ *commutes with restriction to the filtration of* Nat, *i.e.*

$$\begin{tikzcd}
H_*(\mathrm{Nat}_{\mathscr{C}om}(X_\bullet, Y_\bullet);\mathbb{F}) \arrow[r] \arrow[d] & H_*(\mathrm{Nat}_{\mathscr{C}om}(X_\bullet, Y_\bullet);\mathbb{F}) \otimes H_*(\mathrm{Nat}_{\mathscr{C}om}(X_\bullet, Y_\bullet);\mathbb{F}) \arrow[d] \\
H_*(\mathrm{Nat}^{2m}(X_\bullet, Y_\bullet);\mathbb{F}) \arrow[r, "H_*(\Delta_{2m})"] & H_*(\mathrm{Nat}^{m}(X_\bullet, Y_\bullet);\mathbb{F}) \otimes H_*(\mathrm{Nat}^{m}(X_\bullet, Y_\bullet);\mathbb{F})
\end{tikzcd}$$

and

$$\begin{tikzcd}
H_*(\mathrm{Nat}_{\mathscr{C}om}(X_\bullet, Y_\bullet);\mathbb{F}) \arrow[r] \arrow[d] & H_*(\mathrm{Nat}_{\mathscr{C}om}(X_\bullet, Y_\bullet);\mathbb{F}) \otimes H_*(\mathrm{Nat}_{\mathscr{C}om}(X_\bullet, Y_\bullet);\mathbb{F}) \arrow[d] \\
H_*(\mathrm{Nat}^{2m+1}(X_\bullet, Y_\bullet);\mathbb{F}) \arrow[r, "H_*(\Delta_{2m+1})"] & H_*(\mathrm{Nat}^{m+1}(X_\bullet, Y_\bullet);\mathbb{F}) \otimes H_*(\mathrm{Nat}^{m}(X_\bullet, Y_\bullet);\mathbb{F})
\end{tikzcd}$$

commute.

Proof. We only need to check that all quasi-isomorphisms involved respect the Δ_k-maps.

- From Corollary 4.10 we have a quasi-isomorphism of coalgebras
$$\overline{C}_*(Hom_{Top}(|X_\bullet|, |Y_\bullet|)) \to T(S_\bullet(|Y_\bullet|)^{X_\bullet}).$$
- By definition
$$T(S_\bullet(Y)^{X_\bullet}) = T(Hom_{Set}(X_\bullet, S_\bullet(Y)) = \overline{D}_{X_\bullet}(\overline{C}_{S_\bullet(|Y_\bullet|)}(\mathbb{F}[-]^{\otimes -}))).$$
- For any simplicial set $Y_\bullet$, one has $Y_\bullet \simeq S_\bullet(|Y_\bullet|)$, so by Proposition 4.4 we have a quasi-isomorphism
$$\overline{D}_{X_\bullet}(\overline{C}_{S_\bullet(|Y_\bullet|)}(\mathbb{F}[-]^{\otimes -}))) = \overline{D}_{X_\bullet}\overline{C}_{S_\bullet(|Y_\bullet|)}(\mathscr{C}om(-,-)) \simeq \overline{D}_{X_\bullet}\overline{C}_{Y_\bullet}(\mathscr{C}om(-,-)),$$
which commutes with the Δ_k maps.
- By the construction explained before Proposition 3.9 the (quasi-) isomorphism $\overline{D}_{X_\bullet}\overline{C}_{Y_\bullet}(\mathscr{C}om(-,-)) \cong \overline{\mathrm{Nat}}_{\mathscr{C}om}(X_\bullet, Y_\bullet) \simeq \mathrm{Nat}_{\mathscr{C}om}(X_\bullet, Y_\bullet)$ commute with the Δ_k-maps.

Taking these steps together, the theorem is proven. □

5. The general setup in closed monoidal model categories

Let $\mathscr{M}$ be a closed monoidal model category and $\mathcal{E}$ a small category enriched in $\mathscr{M}$. In this section we define the Hochschild and coHochschild construction for $\mathscr{M} = Top$ or Ch and modules over $\mathcal{E}$, i.e. for enriched functors $\Phi : \mathcal{E} \to \mathscr{M}$ and $\Psi : \mathcal{E}^{op} \to \mathscr{M}$, respectively. Then we can also define the formal operations and prove generalizations of Theorem 3.2 and Theorem 3.6 for chain complexes. We hope to achieve results for topological spaces too and state what properties should suffice. This is work in progress. All the functors considered are enriched over $\mathscr{M}$ and so are the categories $\mathcal{E}$ and $\mathcal{E}'$, if not stated otherwise.

5.1. **Derived tensor products and mapping spaces.** We recall definitions and constructions on the derived tensor product and the derived end. We mostly follow Emily Riehl's notes [Rie12].

Definition 5.1. Let $\mathcal{E}$ be a small $\mathscr{M}$-enriched category, $G : \mathcal{E}^{op} \to \mathscr{M}$ and $F : \mathcal{E} \to \mathscr{M}$ enriched functors. The *enriched coend* or *enriched tensor product* of F and G is defined as

$$F \underset{\mathcal{E}}{\otimes} G = \mathrm{coeq}\left(\coprod_{d,d'}(F(d) \otimes G(d')) \otimes \mathcal{E}(d,d') \rightrightarrows \coprod_{d} F(d) \otimes G(d)\right).$$

Similarly, for $F, G : \mathcal{E} \to \mathscr{M}$ the *enriched end* is defined as

$$\hom_{\mathcal{E}}(F,G) = \mathrm{eq}\left(\prod_{d} \hom(F(d), G(d)) \rightrightarrows \prod_{d,d'} \hom(\mathcal{E}(d,d'), \hom(F(d), G(d')))\right).$$

So far, the constructions do not respect weak-equivalences in the functors. This is repaired by taking the homotopy coend or homotopy end (by taking the homotopy coequalizer and equalizer) which are denoted by $- \overset{\mathbb{L}}{\underset{\mathcal{E}}{\otimes}} -$ and $\mathbf{R}\hom_{\mathcal{E}}(-,-)$, respectively. In the unenriched setup, one would use the bar and cobar construction to give an explicit construction (and thus a proof of the existence) of the derived functors. This is not always possible in the enriched setup. For the rest of this section we will work with chain complexes and compactly generated topological spaces equipped with the projective and mixed model structure, respectively (for more details on the model structures cf. Appendix B.4).

Definition 5.2. Let $\mathcal{E}$ be a small category enriched in Ch. We call a functor $B_A : \mathcal{E}^{op} \to$ Ch an *h-projective replacement* of a functor $A : \mathcal{E}^{op} \to$ Ch if there is a levelwise quasi-isomorphism $B_A \xrightarrow{\simeq} A$ such that $\hom_{\mathcal{E}^{op}}(B_A, -)$ is a right derived functor of $\hom_{\mathcal{E}^{op}}(A, -)$ and $- \underset{\mathcal{E}}{\otimes} B_A$ is a left derived functor of $- \overset{\mathbb{L}}{\underset{\mathcal{E}}{\otimes}} A$.

Similarly, for topological spaces we define:

Definition 5.3. Let $\mathcal{E}$ be a small category enriched in Top and $Q(-)$ an enriched levelwise cofibrant replacement functor, i.e. it induces a continuous map between the hom-spaces $\hom(X,Y)$ and $\hom(Q(X), Q(Y))$ for two topological spaces X and Y. We call a functor $B_A : \mathcal{E}^{op} \to Top$ a *topological h-projective replacement* of a functor $A : \mathcal{E}^{op} \to Top$ if there is a levelwise weak equivalence $B_A \xrightarrow{\simeq} A$ and if there are natural weak equivalences such that $\hom_{\mathcal{E}^{op}}(B_A, -)$ is a right derived functor of $\hom_{\mathcal{E}^{op}}(A, -)$ and $Q(-) \underset{\mathcal{E}}{\otimes} B_A$ is a left derived functor of $- \overset{\mathbb{L}}{\underset{\mathcal{E}}{\otimes}} A$ where $Q(-)$ is an enriched levelwise cofibrant replacement functor , i.e. it induces a continuous map between the hom-spaces $\hom(X,Y)$ and $\hom(Q(X), Q(Y))$ for two topological spaces X and Y.

Note that on topological spaces we replace the other tensor functor levelwise cofibrantly. This is a weaker condition than the one we ask in chain complexes. On the other hand, in chain complexes we often can find replacements B_A satisfying the stronger condition.

A functor $B : \mathcal{E} \to \mathscr{M}$ is called *(topological) h-projective* if it is a (topological) h-projective resolution of itself, i.e. if the enriched tensor product realizes the derived tensor product.

In the remainder of this section we want to give constructions of h-projective replacements in spaces. However, these only work under certain conditions.

We start with the definition of the bar construction:

Definition 5.4. For a small $\mathscr{M}$–enriched category $\mathcal{E}$, two functors $G : \mathcal{E}^{op} \to \mathscr{M}$ and $F : \mathcal{E} \to \mathscr{M}$, the *enriched simplicial bar construction* is the simplicial object in $\mathscr{M}$ whose n–th level is defined as

$$B_n(F, \mathcal{E}, G) = \coprod_{d_0, \cdots, d_n \in \mathcal{E}} F(d_0) \otimes \mathcal{E}(d_0, d_1) \otimes \cdots \otimes \mathcal{E}(d_{n-1}, d_n) \otimes G(d_n)$$

with the i–th face map being induced by the composition $\mathcal{E}(d_{i-1}, d_i) \otimes \mathcal{E}(d_i, d_{i+1}) \to \mathcal{E}(d_{i-1}, d_{i+1})$ for $0 < i < n$ and the induced maps on $F(d_0)$ and $G(d_n)$ for $i = 0$ or $i = n$, respectively. The degeneracies are the maps induced by the map $* \to \mathcal{E}(d_i, d_i)$ plugged in at the $i+1$–st position.

For $\mathscr{M} = Top$ the *enriched bar construction* is then the geometric realization, i.e.

$$B(F, \mathcal{E}, G) = |B_\bullet(F, \mathcal{E}, G)|$$

and for $\mathscr{M} = \mathrm{Ch}$ it is the totalization of the double chain complex $C_*(B_\bullet(G, \mathcal{E}, F))$, i.e.

$$B(F, \mathcal{E}, G) = \mathrm{Tot}^{\oplus} C_*(B_\bullet(F, \mathcal{E}, G)).$$

Similarly, for two functors $F, G : \mathcal{E} \to \mathscr{M}$ the *enriched cosimplicial cobar construction* is the cosimplicial object in $\mathscr{M}$ with n–th degree given by

$$C^n(G, \mathcal{E}, F) = \prod_{d_0, \cdots, d_n} \hom(G(d_0) \otimes \mathcal{E}(d_0, d_1) \otimes \cdots \otimes \mathcal{E}(d_{n-1}, d_n), F(d_n))$$

with the face map for $0 \le i < n$ coming from the composition and the n–th face map being induced by applying the extra map to $F(d_{n-1})$.

For $\mathscr{M} = Top$ the *enriched cobar construction* is the totalization, i.e.

$$C(G, \mathcal{E}, F) = \mathrm{Tot}(C^\bullet(G, \mathcal{E}, F))$$

and similarly for $\mathscr{M} = \mathrm{Ch}$ it is the total product complex of the double complex, i.e.

$$C(G, \mathcal{E}, F) = \mathrm{Tot}^{\prod}(C^*(G, \mathcal{E}, F)).$$

Note that for both bar constructions defined above, we can apply the enriched Yoneda Lemma levelwise and get an isomorphism/homeomorphism

$$B(F, \mathcal{E}, G) \cong F \underset{\mathcal{E}}{\otimes} B(\mathcal{E}, \mathcal{E}, G)$$

where $B(\mathcal{E}, \mathcal{E}, G) : \mathcal{E}^{op} \to \mathscr{M}$ takes $x \in \mathcal{E}$ to $B(\mathcal{E}(x, -), \mathcal{E}, G)$. Moreover, taking the category $\mathcal{E}^{op}$ instead of $\mathcal{E}$ and thus $F : (\mathcal{E}^{op})^{op} \to \mathscr{M}$, we get

$$B(F, \mathcal{E}, G) = B(G, \mathcal{E}^{op}, F).$$

Lemma 5.5. *Let $\mathcal{E}$ be an $\mathscr{M}$–enriched category where $\mathscr{M} = \mathrm{Ch}$ or Top. Then we can express the cobar construction via the bar construction as follows:*

$$C(G, \mathcal{E}, F) \cong \hom_{\mathcal{E}}(B(G, \mathcal{E}, \mathcal{E}), F).$$

Proof. We first show that $C^\bullet(G, \mathcal{E}, F) \cong \hom_{\mathcal{E}}(B_\bullet(G, \mathcal{E}, \mathcal{E}), F)$, so in each level we first need to give an isomorphism $C^n(G, \mathcal{E}, F) \cong \hom_{\mathcal{E}}(B_n(G, \mathcal{E}, \mathcal{E}), F)$. We see

$$\begin{aligned} \hom_{\mathcal{E}}(B_n(G, \mathcal{E}, \mathcal{E}), F) &= \prod_{d_0, \cdots, d_n} \hom_{\mathcal{E}}(G(d_0) \otimes \mathcal{E}(d_0, d_1) \otimes \cdots \otimes \mathcal{E}(d_{n-1}, d_n) \otimes \mathcal{E}(d_n, -), F) \\ &\cong \prod_{d_0, \cdots, d_n} \hom(G(d_0) \otimes \mathcal{E}(d_0, d_1) \otimes \cdots \otimes \mathcal{E}(d_{n-1}, d_n), F(d_n)) \end{aligned}$$

where the last step is the enriched Yoneda Lemma. One easily checks that the simplicial structure maps of $B_\bullet$ get mapped to those of $C^\bullet$. Therefore we obtain $C^\bullet(G, \mathcal{E}, F) \cong \hom_{\mathcal{E}}(B_\bullet(G, \mathcal{E}, \mathcal{E}), F)$ and hence, since Tot is the dual of geometric realization (it is the end corresponding to the coend of geometric realization), we have

$$C(G, \mathcal{E}, F) = \mathrm{Tot}\, C^\bullet(G, \mathcal{E}, F) \cong \mathrm{Tot} \hom_{\mathcal{E}}(B_\bullet(G, \mathcal{E}, \mathcal{E}), F) \cong \hom_{\mathcal{E}}(|B_\bullet(G, \mathcal{E}, \mathcal{E})|, F).$$

Thus the lemma is proven for $\mathscr{M} = Top$. Since taking $\mathrm{Tot}^{\oplus}$ is dual to taking $\mathrm{Tot}^{\prod}$, the same holds for $\mathscr{M} = \mathrm{Ch}$. □

We need to give the right setup for the bar construction to be an h-projective replacement:

Definition 5.6. Let $\mathcal{E}$ be a small category enriched over Top or Ch (equipped with the mixed and projective model structure, respectively). The category $\mathcal{E}$ is called *cofibrantly enriched* if

- all the morphism spaces are cofibrant and
- the maps $\mathrm{id} \to \mathcal{E}(a, a)$ are cofibrations for all $a \in \mathcal{E}$.

In the mixed model structure for spaces (cf. Appendix B.4.1) this requires all morphism spaces to be strong homotopy equivalent to CW-complexes and the embeddings of the identity to be Hurewicz cofibrations that are composition of a relative homotopy equivalence and a relative CW-complex (cf. Appendix B.4.2). On chain complexes the conditions are always fulfilled if all the $\mathcal{E}(a, a)$ are bounded below, levelwise projective and the inclusion of the identity is a cofibration.

For $\mathscr{M} = Top$, the following theorem allows us to use the bar construction:

Theorem 5.7 ([Shu06, Theorem 23.12], for more details see [Rie12, Sec. 9.2])**.** *Let $\mathcal{E}$ be a cofibrantly enriched topological category and let Q be an enriched cofibrant replacement. Then an explicit model for the derived functor of the enriched tensor product $- \underset{\mathcal{E}}{\otimes} -$ is given by $- \overset{\mathbb{L}}{\underset{\mathcal{E}}{\otimes}} - = B(Q(-), \mathcal{E}, Q(-))$. An explicit model for the derived enriched end is given by $\mathbf{R}\operatorname{hom}_{\mathcal{E}^{op}}(-, -) = C(Q(-), \mathcal{E}, -)$.*

In particular, for every $A : \mathcal{E}^{op} \to \mathrm{Ch}$ the functor $B(\mathcal{E}, \mathcal{E}, Q(A))$ is a topological h-projective replacement.

Note that we did not need to use a fibrant replacement for the cobar construction, since all spaces are fibrant.

For chain complexes we hope that the enriched bar construction is h-cofibrant, too. However, for the moment we can only show that it preserves weak-equivalences.

Proposition 5.8. *Let $\mathcal{E}$ be cofibrantly enriched over chain complexes and $A : \mathcal{E}^{op} \to \mathrm{Ch}$. Then $B(-, \mathcal{E}, Q(A))$ and $C(A, \mathcal{E}, -)$ preserve weak equivalences. Again, $Q(-)$ is a functorial cofibrant replacement.*

Proof. For the proof assume that A is levelwise cofibrant. We need to check that both $C(A, \mathcal{E}, -)$ and $B(-, \mathcal{E}, A)$ preserve quasi-isomorphisms. A quasi-isomorphism $\Phi \simeq \Phi'$ induces quasi-isomorphisms

$$\begin{aligned}&\operatorname{hom}(A(d_0) \otimes \mathcal{E}(d_0, d_1) \otimes \cdots \otimes \mathcal{E}(d_{n-1}, d_n), \Phi(d_n)) \\ &\qquad\qquad \simeq \operatorname{hom}(A(d_0) \otimes \mathcal{E}(d_0, d_1) \otimes \cdots \otimes \mathcal{E}(d_{n-1}, d_n), \Phi'(d_n))\end{aligned}$$

and

$$A(d_0) \otimes \mathcal{E}(d_0, d_1) \otimes \cdots \otimes \mathcal{E}(d_{n-1}, d_n) \otimes \Phi(d_n) \simeq A(d_0) \otimes \mathcal{E}(d_0, d_1) \otimes \cdots \otimes \mathcal{E}(d_{n-1}, d_n) \otimes \Phi'(d_n)$$

for all $d_i \in \mathcal{E}$ since both A and $\mathcal{E}(-, -)$ were levelwise cofibrant. Thus we can apply Corollary B.12 to compare the total complex of the double complex $C^*(A, \mathcal{E}, \Phi)$ with $C^*(A, \mathcal{E}, \Phi')$ and $B_*(A, \mathcal{E}, \Phi)$ with $B_*(A, \mathcal{E}, \Phi')$ which proves the claim. $\square$

5.2. **Hochschild and coHochschild construction and formal operations.** In this section we fix two small, $\mathscr{M}$-enriched categories $\mathcal{E}$ and $\mathcal{E}'$ for $\mathscr{M} = Top$ or Ch together with an enriched functor $\mathcal{E} \to \mathcal{E}'$. To define the Hochschild and coHochschild construction, we have to distinguish the two cases. First assume that $\mathcal{E}$ is enriched over Ch.

Definition 5.9. For an h-projective functor $B : \mathcal{E}^{op} \to \mathrm{Ch}$ the Hochschild functor $C_B(-) : Fun(\mathcal{E}, \mathrm{Ch}) \to \mathrm{Ch}$ sends a functor $\Phi : \mathcal{E} \to \mathrm{Ch}$ to

$$C_B(\Phi) = \Phi \underset{\mathcal{E}}{\otimes} B.$$

Similarly, the coHochschild functor $D_A(-) : Fun(\mathcal{E}^{op}, \mathrm{Ch}) \to \mathrm{Ch}$ sends a functor $\Psi : \mathcal{E}^{op} \to \mathscr{M}$ to

$$D_B(\Psi) = \operatorname{hom}_{\mathcal{E}^{op}}(B, \Psi).$$

If $\mathcal{E}$ is enriched over Top, we define:

Definition 5.10. For a topological h-projective functor $B : \mathcal{E}^{op} \to Top$ and Q an enriched cofibrant replacement functor the Hochschild functor $C_B(-) : Fun(\mathcal{E}, Top) \to Top$ sends a functor $\Phi : \mathcal{E} \to Top$ to

$$C_B(\Phi) = Q(\Phi) \underset{\mathcal{E}}{\otimes} B.$$

Similarly, the coHochschild functor $D_A(-) : Fun(\mathcal{E}^{op}, Top) \to Top$ sends a functor $\Psi : \mathcal{E}^{op} \to \mathscr{M}$ to

$$D_B(\Psi) = \hom_{\mathcal{E}^{op}}(B, \Psi).$$

In both cases (i.e. if $\mathcal{E}$ is enriched over $\mathscr{M}$ for $\mathscr{M} = \mathrm{Ch}$ or Top), we can define:

Definition 5.11. If A has a (topological) h-projective resolution B_A then we define

$$C_A(-) = C_{B_A}(-)$$

and

$$D_A(-) = D_{B_A}(-).$$

The above give specific models for the derived enriched tensor product $- \overset{\mathbb{L}}{\underset{\mathcal{E}}{\otimes}} A$ and the derived enriched hom functor $\mathbf{R}\hom_{\mathcal{E}^{op}}(A, -)$. Moreover, since a choice of h-projective resolution is involved, the Hochschild and coHochschild constructions are only well-defined up to weak equivalence. In topological spaces if $\mathcal{E}$ is cofibrantly enriched, we can fix a standard model by choosing B_A to be the bar construction. However, for chain complexes we will see that all further definitions also only depend on the choice of h-projective resolution up to weak equivalence.

For another enriched category $\mathcal{E}'$ with a functor $i : \mathcal{E} \to \mathcal{E}'$, a functor $\Phi : \mathcal{E}' \to \mathscr{M}$ the composition $\Phi \circ i$ defines a functor $\mathcal{E} \to \mathrm{Ch}$. For $A : \mathcal{E}^{op} \to \mathscr{M}$ and $\Phi : \mathcal{E}' \to \mathscr{M}$ we write $C_A(\Phi) := C_A(\Phi \circ i)$, i.e. we see $C_A(-) : Fun(\mathcal{E}', \mathscr{M}) \to \mathscr{M}$. We use this to define the formal operations:

Definition 5.12. Let A and A' be two contravariant functors $A, A' : \mathcal{E}^{op} \to \mathscr{M}$ which have h-projective resolutions B_A and $B_{A'}$. Then the formal operations are defined as the mapping space

$$\mathrm{Nat}_{\mathcal{E}'}(A, A') = \hom_{Fun(\mathcal{E}', \mathscr{M})}(C_A(-), C_{A'}(-)).$$

Similarly, the coformal operations are defined as the mapping space

$$\mathrm{Nat}^D_{\mathcal{E}'}(A, A') = \hom_{Fun(\mathcal{E}'^{op}, \mathscr{M})}(D_A(-), D_{A'}(-)).$$

Note that the above definition depends on the choice of h-projective resolution we have chosen for A and A'. However, for chain complexes we will see later that up to weak equivalence the spaces $\mathrm{Nat}_{\mathcal{E}'}(A, A')$ and $\mathrm{Nat}^D_{\mathcal{E}'}(A, A')$ are independent of this choice. To avoid confusion, from now on we assume that we have fixed h-projective resolutions B_A and $B_{A'}$ for A and A', respectively.

Remark 5.13. If we can choose B_A and $B_{A'}$ such that $C_A(\Phi)$ and $C_{A'}(\Phi)$ are cofibrant, i.e. $C_A(\Phi)$ and $C_{A'}(\Phi)$ are in particular cofibrant replacements of any model of the enriched derived tensor product, then $\mathrm{Nat}_{\mathcal{E}'}(A, A') = \hom(C_A(-), C_{A'}(-)) \cong \mathbf{R}\hom(C_A(-), C_{A'}(-))$ and since $\mathbf{R}\hom(C_A(-), C_{A'}(-)) \simeq \mathbf{R}\hom(- \overset{\mathbb{L}}{\underset{\mathcal{E}}{\otimes}} A, - \overset{\mathbb{L}}{\underset{\mathcal{E}}{\otimes}} A')$, we have

$$\mathrm{Nat}_{\mathcal{E}'}(A, A') \simeq \mathbf{R}\hom(- \overset{\mathbb{L}}{\underset{\mathcal{E}}{\otimes}} A, - \overset{\mathbb{L}}{\underset{\mathcal{E}}{\otimes}} A').$$

Unfortunately, we do not see how to achieve this generalization in the case of chain complexes. However, working in topological spaces with $\mathcal{E}$ cofibrantly enriched, the enriched bar construction $B(Q(\Phi), \mathcal{E}, Q(A))$ is cofibrant, since it is the geometric realization (i.e. a left Quillen functor) of the Reedy cofibrant simplicial space $B_\bullet(Q(\Phi), \mathcal{E}, Q(A))$.

Theorem 5.14. *Let $\mathcal{E}$ and $\mathcal{E}'$ be small categories enriched over* Ch *together with a functor $\mathcal{E} \to \mathcal{E}'$ and let $B, B' : \mathcal{E}^{op} \to \mathrm{Ch}$ be two h-projective functors. Then we have an isomorphism*

$$\mathrm{Nat}_{\mathcal{E}'}(B, B') \cong D_B C_{B'}(\mathcal{E}'(-,-)).$$

For the general case, we can deduce:

Corollary 5.15. *Let $\mathcal{E}$ and $\mathcal{E}'$ be small categories enriched over* Ch *together with a functor $\mathcal{E} \to \mathcal{E}'$ and let $A, A' : \mathcal{E}^{op} \to \mathrm{Ch}$ be two functors which have h-projective resolutions. Then*

$$\mathrm{Nat}_{\mathcal{E}'}(A, A') \simeq D_A C_{A'}(\mathcal{E}'(-,-))$$

and in particular $\mathrm{Nat}_{\mathcal{E}'}(A, A')$ is independent of the choice of h-projective resolutions.

For the coformal transformations we can only prove the theorem in the case of $\mathscr{M} = \mathrm{Ch}$.

Theorem 5.16. *Let $\mathcal{E}$ and $\mathcal{E}'$ be small categories enriched over* Ch *together with a functor $\mathcal{E} \to \mathcal{E}'$ and let B and $B' : \mathcal{E}^{op} \to \mathrm{Ch}$ be two h-projective functors. Then we have an isomorphism*

$$\mathrm{Nat}^D_{\mathcal{E}'}(B, B') \cong D_{B'} C_B(\mathcal{E}'(-,-)).$$

Corollary 5.17. *Let $\mathcal{E}$ and $\mathcal{E}'$ be as above and $A, A' : \mathcal{E}^{op} \to \mathscr{M}$ be two functors which have h-projective resolutions. Then we have a weak equivalence*

$$\mathrm{Nat}^D_{\mathcal{E}'}(A, A') \simeq D_{A'} C_A(\mathcal{E}'(-,-))$$

and again $\mathrm{Nat}^D_{\mathcal{E}'}(A, A')$ is independent of the choice of h-projective resolution.

The corollaries all follow from the fact, that $D_A C_{A'}(\mathcal{E}'(-,-))$ (up to weak equivalence) is independent of the choice of h-projective resolution, since both D_A and $C_{A'}$ are so and D_A preserves weak equivalences (since it is a model of a derived functor).

It is work in progress to achieve similar theorems for topological spaces. The main ingredient is a cofibrant replacement functor Q which is continuous (as explained earlier), strong monoidal and comonadic. Under this conditions one can prove the analog for topological spaces. A nice application would be to compute the formal operations in the case of Chiral homology, which for $Disk_m^{fr}$ the framed m–disk operad and M a framed m–manifold is computed as

$$\int_M \Phi = \Phi \overset{\mathbb{L}}{\underset{Disk_m^{fr}}{\otimes}} \mathbb{E}_M.$$

Since we are not aware whether a cofibrant replacement functor with the above properties exists, we cannot say anything more about that case here.

5.3. **The proofs for chain complexes.**

In this section we give a proof of Theorem 5.14. We first check:

Lemma 5.18. *Let $\mathscr{M}$ be a closed monoidal model category and $\mathcal{E}$ and $\mathcal{E}'$ be small categories enriched over $\mathscr{M}$ together with a functor $\mathcal{E} \to \mathcal{E}'$. Moreover, take $B : \mathcal{E}^{op} \to \mathscr{M}$ and functors $H_1 : Fun(\mathcal{E}, \mathscr{M}) \to Fun(\mathcal{E}, \mathscr{M})$ and $H_2 : Fun(\mathcal{E}, \mathscr{M}) \to \mathscr{M}$. Then we have an isomorphism*

$$\hom(H_1(-) \underset{\mathcal{E}}{\otimes} B, H_2(-)) \cong \hom_{\mathcal{E}^{op}}(B, \hom(H_1(-), H_2(-)))$$

where in both cases the hom*-sets are natural in all $\Phi : \mathcal{E}' \to \mathscr{M}$.*

Proof. Under the tensor hom adjunction, all morphisms in $\hom(H_1(\Phi) \underset{\mathcal{E}}{\otimes} B, H_2(\Phi))$ natural in Φ correspond to $\hom_{\mathcal{E}^{op}}(B, \hom(H_1(\Phi), H_2(\Phi))$ natural in Φ. Since the naturality is only used in the second hom, this is the same as (compatible) maps from B into the set of all elements of $\hom(H_1(\Phi), H_2(\Phi))$ that are natural in Φ. $\square$

Next we check:

Lemma 5.19. *For a functor $H : Fun(\mathcal{E}', \mathrm{Ch}) \to \mathrm{Ch}$ we have an isomorphism*

$$\hom_{Fun(\mathcal{E}',\mathrm{Ch})}(-(e), H) \cong H \circ \mathcal{E}'(e, -).$$

Proof. For each $\Phi : \mathcal{E}' \to \mathrm{Ch}$ we define a map

$$F_\Phi : \mathcal{E}'(e, -) \underset{\mathcal{E}}{\otimes} B \to \hom(\Phi(e), \Phi \underset{\mathcal{E}}{\otimes} B)$$

as the adjoint of the map

$$\widetilde{F}_\Phi : \mathcal{E}'(e, -) \underset{\mathcal{E}}{\otimes} B \otimes \Phi(e) \to \Phi \underset{\mathcal{E}}{\otimes} B$$

given by sending an element $f \otimes b \otimes x$ to $f(x) \otimes b$. One checks that this is defined on equivalence classes of the coend and is natural in Φ.

Given a natural transformation $\nu \in \hom(-(e), (-) \underset{\mathcal{E}}{\otimes} B)$ we can plug in the functor $\mathcal{E}'(e, -)$ and thus get $\nu_{\mathcal{E}'(e,-)} \in \hom(\mathcal{E}'(e, e), \mathcal{E}'(e, -) \underset{\mathcal{E}}{\otimes} B)$. Evaluating at the identity $\mathrm{id} \in \mathcal{E}'(e, e)$, we define

$$G : \hom(-(e), (-) \underset{\mathcal{E}}{\otimes} B) \to \mathcal{E}'(e, -) \underset{\mathcal{E}}{\otimes} B$$

as $G(\nu) = \nu_{\mathcal{E}'(e,-)}(\mathrm{id})$.

We need to show that the two maps are inverse to each other: The composition $G \circ F$ sends an element $f \otimes b \in \mathcal{E}'(e, -) \underset{\mathcal{E}}{\otimes} B$ to $(f \circ \mathrm{id}) \otimes b = f \otimes b$ and thus equals the identity.

To see that $F \circ G$ is the identity we fix a natural transformation $\nu \in \hom(-(e), (-) \underset{\mathcal{E}}{\otimes} B)$ and an element $x \in \Phi(e)$. Define a natural transformation $\mu_x \in \mathcal{E}'(e, -) \to \Phi(-)$ by sending an element $f \in \mathcal{E}'(e, k)$ to $f(x) \in \Phi(k)$. The naturality of ν induces a commutative diagram

$$\begin{tikzcd}
\mathcal{E}'(e,e) \arrow[r, "\nu"] \arrow[d, "\mu_{x,e}"] & \mathcal{E}'(e,-) \underset{\mathcal{E}}{\otimes} B \arrow[d, "\mu_x"] \\
\Phi(e) \arrow[r, "\nu"] & \Phi \underset{\mathcal{E}}{\otimes} B
\end{tikzcd}$$

for every x.

Checking the diagram on $\mathrm{id} \in \mathcal{E}'(e, e)$, the composition via the lower left corner gives $\nu(x)$ whereas the composition via the upper right corner is $G \circ F(\nu)(x)$. Thus $\nu(x) = G \circ F(\nu)(x)$ for all natural transformations ν and for all x, hence $F \circ G = \mathrm{id}$. □

Proof of Theorem 5.14. By the definition, we have that $\mathrm{Nat}_{e'}(B, B')$ is given by the collection of compatible morphisms in $\hom(\Phi \underset{\mathcal{E}}{\otimes} B, \Phi \underset{\mathcal{E}}{\otimes} B')$ which are natural in $\Phi : \mathcal{E}' \to \mathrm{Ch}$. Applying Lemma 5.18 to $H_1 = \mathrm{id}$ and $H_2 = - \underset{\mathcal{E}}{\otimes} B'$ we get

$$\hom_{Fun(\mathcal{E}',\mathrm{Ch})}(C_B(-), C_{B'}(-)) \cong \hom_{\mathcal{E}^{op}}(B, \hom_{Fun(\mathcal{E}',\mathrm{Ch})}(-, - \underset{\mathcal{E}}{\otimes} B')).$$

By the previous lemma we obtain $\hom_{Fun(\mathcal{E}',\mathrm{Ch})}(-, - \underset{\mathcal{E}}{\otimes} B')) \cong \mathcal{E}'(e, -) \underset{\mathcal{E}}{\otimes} B'$ and thus

$$\hom_{\mathcal{E}^{op}}(B, \hom_{Fun(\mathcal{E}',\mathrm{Ch})}(-, - \underset{\mathcal{E}}{\otimes} B')) \cong \hom_{\mathcal{E}^{op}}(B, \mathcal{E}'(-, -) \underset{\mathcal{E}}{\otimes} B') \cong D_B C_{B'}(\mathcal{E}'(-, -))$$

where the last isomorphism is given by the definition of the coHochschild and Hochschild complex. Thus the theorem is proven. □

Next we give a proof of the theorem for the coformal operations:

Proof of Theorem 5.16. We start with describing a map $F_\Psi : \hom_{\mathcal{E}^{op}}(B', \mathcal{E}'(-,-) \underset{\mathcal{E}}{\otimes} B) \to \hom(\hom_{\mathcal{E}^{op}}(B,\Psi), \hom_{\mathcal{E}^{op}}(B',\Psi))$ for all $\Psi : \mathcal{E}^{op} \to \mathrm{Ch}$ that is natural in Ψ. Using the tensor-hom adjunction, we need to construct a map

$$\widetilde{F}_\Psi : \hom_{\mathcal{E}^{op}}(B', \mathcal{E}'(-,-) \underset{\mathcal{E}}{\otimes} B) \otimes \hom_{\mathcal{E}^{op}}(B,\Psi) \to \hom_{\mathcal{E}^{op}}(B',\Psi).$$

For $f \in \hom_{\mathcal{E}^{op}}(B,\Psi)$ we have a map

$$R_f \in \hom_{\mathcal{E}^{op}}(\mathcal{E}'(-,-) \underset{\mathcal{E}}{\otimes} B, \mathcal{E}'(-,-) \underset{\mathcal{E}}{\otimes} \Psi)$$

defined as $R_f = \mathrm{id} \otimes f$. One checks that this is well-defined with the end and coend constructions involved. Moreover, the evaluation ev is an element in $\hom_{\mathcal{E}^{op}}(\mathcal{E}'(-,-) \underset{\mathcal{E}}{\otimes} \Psi, \Psi)$. For $h \in \hom_{\mathcal{E}^{op}}(B', \mathcal{E}'(-,-) \underset{\mathcal{E}}{\otimes} B)$ and $f \in \hom_{\mathcal{E}^{op}}(B,\Psi)$ we define $\widetilde{F}_\Psi(h \otimes f)$ as the enriched composition

$$B' \xrightarrow{h} \mathcal{E}'(-,-) \underset{\mathcal{E}}{\otimes} B \xrightarrow{R_f} \mathcal{E}'(-,-) \underset{\mathcal{E}}{\otimes} \Psi \xrightarrow{ev} \Psi,$$

i.e. $\widetilde{F}_\Psi(h \otimes f) = ev \circ R_f \circ h \in \hom_{\mathcal{E}^{op}}(B',\Psi)$.

The inverse map G is easier to construct. Given $\nu \in \hom(\hom_{\mathcal{E}^{op}}(B,-), \hom_{\mathcal{E}^{op}}(B',-))$ we want to apply ν to the functor $\mathcal{E}'(-,-) \underset{\mathcal{E}}{\otimes} B$. This gives us a map

$$\nu_{(\mathcal{E}'(-,-) \underset{\mathcal{E}}{\otimes} B)} : \hom_{\mathcal{E}^{op}}(B, \mathcal{E}'(-,-) \underset{\mathcal{E}}{\otimes} B) \to \hom_{\mathcal{E}^{op}}(B', \mathcal{E}'(-,-) \underset{\mathcal{E}}{\otimes} B)$$

which we evaluate on the element $w \in \hom_{\mathcal{E}^{op}}(B, \mathcal{E}'(-,-) \underset{\mathcal{E}}{\otimes} B)$ defined by $w(b) = \mathrm{id} \otimes b$. Hence $G = \nu_{(\mathcal{E}'(-,-) \underset{\mathcal{E}}{\otimes} B)}(w) \in \hom_{\mathcal{E}^{op}}(B', \mathcal{E}'(-,-) \underset{\mathcal{E}}{\otimes} B)$.

We need to show that the two maps are inverse: To see that $G \circ F$ is the identity, for $f = \mathrm{id} \in \hom_{\mathcal{E}^{op}}(B,B)$ we need to show that $\widetilde{F}_{(\mathcal{E}'(-,-) \underset{\mathcal{E}}{\otimes} B)}(\mathrm{id} \otimes f) = \mathrm{id}$. Plugging in the definition we see that $ev \circ R_f = \mathrm{id}$ and thus the claim follows.

To show $F \circ G = \mathrm{id}$, we fix a $\Psi : \mathcal{E}^{op} \to \mathrm{Ch}$ and $f \in \hom_{\mathcal{E}^{op}}(B,\Psi)$. We define a map $H_f \in \hom_{\mathcal{E}^{op}}(\mathcal{E}'(-,-) \underset{\mathcal{E}}{\otimes} B, \Psi)$ by $H_f = ev \circ R_f$. Given a natural transformation $\nu \in \hom(\hom_{\mathcal{E}^{op}}(B,-), \hom_{\mathcal{E}^{op}}(B',-))$ we thus get a commutative diagram

$$\begin{tikzcd}
\hom_{\mathcal{E}^{op}}(B, \mathcal{E}'(-,-) \underset{\mathcal{E}}{\otimes} B) \arrow[r, "\nu_{(\mathcal{E}'(-,-)\otimes_{\mathcal{E}} B)}"] \arrow[d, "H_{f_*}"] & \hom_{\mathcal{E}^{op}}(B', \mathcal{E}'(-,-) \underset{\mathcal{E}}{\otimes} B) \arrow[d, "H_{f^*}"] \\
\hom_{\mathcal{E}^{op}}(B,\Psi) \arrow[r, "\nu_\Psi"] & \hom_{\mathcal{E}^{op}}(B,\Psi)
\end{tikzcd}$$

where H_{f_*} is the induced map by postcomposition.

Evaluating the diagram on the element $w = \mathrm{id} \otimes - \in \hom_{\mathcal{E}^{op}}(B, \mathcal{E}'(-,-) \underset{\mathcal{E}}{\otimes} B)$ via the upper horizontal map it is sent to $\nu_{(\mathcal{E}'(-,-)\otimes_{\mathcal{E}} B)}(w) = G(\nu)$ and postcomposition with H_f gives $F(G(\nu))(f)$. On the other hand, $H_f \circ w = f$ and thus composition via the lower left corner is $\nu_\Psi(f)$. Hence $\nu_\Psi(f) = F(G(\nu))(f)$ for all f and ν and thus $F \circ G = \mathrm{id}$. $\square$

5.4. **Applications to higher Hochschild homology.** In this section we want to apply the results proved so far to higher Hochschild and Chiral homology. In both cases we want the categories $\mathcal{E}$ and $\mathcal{E}'$ to be PROPs enriched over Ch or *Top*, i.e. they are enriched symmetric monoidal categories with objects the natural numbers including zero (so we also assume that the structure maps are enriched monoidal functors). Moreover, for simplicity, we always assume that the functor $\mathcal{E} \to \mathcal{E}'$ is the identity on objects.

5.4.1. *Higher Hochschild homology as a derived tensor product.* In this section we focus on the setup to deduce Theorem 3.2 and Theorem 3.6 from Theorem 5.14 and 5.16. From now on we choose $\mathcal{E} = \mathscr{C}om$. Given a simplicial set $X_\bullet$ we want to use the chain complex $\mathscr{L}_{X_\bullet}$ mentioned in Section 2.1 to rewrite the higher Hochschild complex $C_{X_\bullet}(\Phi)$ defined in Definition 2.3 as the complex $C_{\mathscr{L}_{X_\bullet}}(\Phi)$ defined in Definition 5.9 and similarly for $D_{X_\bullet}(\Psi)$ and $D_{\mathscr{L}_{X_\bullet}}(\Psi)$. So we recall and define:

Given a simplicial finite set $X_\bullet$ we define

$$\mathscr{L}_{X_\bullet}(e) = \bigoplus_k \mathscr{C}om(e, X_k)[k]$$

with differential $d : \mathscr{C}om(e, X_k) \to \mathscr{C}om(e, X_{k-1})$ given by postcomposition with $d' = \sum_{i=0}^k (-1)^i d_i$ where the $d_i \in \mathscr{C}om(X_k, X_{k-1})$ are the maps induced by the simplicial boundary maps $d_i : X_k \to X_{k-1}$.

Similarly, for a family of simplicial finite sets $\{X^1_\bullet, \cdots, X^n_\bullet\}$ and a natural number m we define

$$\mathscr{L}_{X^1_\bullet,\ldots,X^n_\bullet,m}(e) = \bigoplus_{k_1,\cdots,k_n} \mathscr{C}om(e, X^1_{k_1} \amalg \cdots \amalg X^n_{k_n} \amalg m)[k_1 + \cdots + k_n]$$

where the differential comes from the boundary maps of the multisimplicial abelian group structure on $\mathscr{C}om(e, X^1_\bullet \amalg \cdots \amalg X^n_\bullet \amalg m)$.

For $X_\bullet$ an arbitrary simplicial set we define

$$\mathscr{L}_{X_\bullet}(e) = \operatorname*{colim}_{\substack{X_\bullet \hookleftarrow K^0_\bullet \hookleftarrow K^1_\bullet \hookleftarrow \cdots \hookleftarrow K^n_\bullet \\ K^i_\bullet \text{ finite}}} \mathscr{L}_{K^n_\bullet}(e) \tag{5.1}$$

and similarly for the iterated construction.

Analogously, we define the complex

$$\overline{\mathscr{L}}_{X^1_\bullet,\ldots,X^n_\bullet,m}(e) = \bigoplus_{k_1,\cdots,k_n} \mathscr{C}om(e, X^1_{k_1} \amalg \cdots \amalg X^n_{k_n} \amalg m)[k_1 + \cdots + k_n]/U_{k_1,\cdots,k_n}$$

with $U_{k_1,\cdots,k_n}$ the image of the simplicial degeneracy maps, i.e. $\overline{\mathscr{L}}_{X^1_\bullet,\ldots,X^n_\bullet,m}(e)$ is the reduced chain complex of the multisimplicial abelian group $\mathscr{C}om(e, X^1_\bullet \amalg \cdots \amalg X^n_\bullet \amalg m)$. Again, for simplicial non-finite sets, we take the similar construction of equation (5.1).

Lemma 5.20. *We have isomorphisms*

$$\begin{aligned}
\Phi \underset{\mathscr{C}om}{\otimes} \mathscr{L}_{X^1_\bullet,\ldots,X^n_\bullet,m} &\cong C_{X^1_\bullet,\ldots,X^n_\bullet}(\Phi)(m), \\
\Phi \underset{\mathscr{C}om}{\otimes} \overline{\mathscr{L}}_{X^1_\bullet,\ldots,X^n_\bullet,m} &\cong \overline{C}_{X^1_\bullet,\ldots,X^n_\bullet}(\Phi)(m), \\
\hom_{\mathscr{C}om^{op}}(\mathscr{L}_{X^1_\bullet,\ldots,X^n_\bullet,m}, \Psi(-)) &\cong D_{X^1_\bullet,\ldots,X^n_\bullet}(\Psi)(m) \qquad \textit{and} \\
\hom_{\mathscr{C}om^{op}}(\overline{\mathscr{L}}_{X^1_\bullet,\ldots,X^n_\bullet,m}, \Psi(-)) &\cong \overline{D}_{X^1_\bullet,\ldots,X^n_\bullet}(\Psi)(m)
\end{aligned}$$

natural in Φ and Ψ, respectively. In particular given two families of simplicial sets $\{X^1_\bullet, \ldots, X^{n_1}_\bullet\}$ and $\{Y^1_\bullet, \ldots, Y^{n_2}_\bullet\}$ together with natural numbers m_1 and m_2 there are isomorphisms

$$\mathrm{Nat}_{\mathcal{E}}(\{X^1_\bullet, \ldots, X^{n_1}_\bullet\}, m_1; \{Y^1_\bullet, \ldots, Y^{n_2}_\bullet\}, m_2) \cong \mathrm{Nat}_{\mathcal{E}}(\mathscr{L}_{X^1_\bullet,\ldots,X^{n_1}_\bullet,m_1}, \mathscr{L}_{Y^1_\bullet,\ldots,Y^{n_2}_\bullet,m_2}) \quad \textit{and}$$
$$\overline{\mathrm{Nat}}_{\mathcal{E}}(\{X^1_\bullet, \ldots, X^{n_1}_\bullet\}, m_1; \{Y^1_\bullet, \ldots, Y^{n_2}_\bullet\}, m_2) \cong \mathrm{Nat}_{\mathcal{E}}(\overline{\mathscr{L}}_{X^1_\bullet,\ldots,X^{n_1}_\bullet,m_1}, \overline{\mathscr{L}}_{Y^1_\bullet,\ldots,Y^{n_2}_\bullet,m_2})$$

where the left hand sides where defined in Definition 3.1 and the right hand sides of the equality is the complex of formal operations defined earlier in this section.

Proof. Plugging in the definition for a simplicial finite set, we get

$$\begin{aligned}\Phi \underset{\mathscr{C}om}{\otimes} \mathscr{L}_{X^1_\bullet,\dots,X^n_\bullet,m} &= \Phi \underset{\mathscr{C}om}{\otimes} \bigoplus_{k_1,\cdots,k_n} \mathscr{C}om(e, X^1_{k_1} \amalg \cdots \amalg X^n_{k_n} \amalg m)[k_1+\cdots+k_n] \\ &\cong \bigoplus_{k_1,\cdots,k_n} \Phi(X^1_{k_1} \amalg \cdots \amalg X^n_{k_n} \amalg m)[k_1+\cdots+k_n] \\ &= C_{X^1_\bullet,\dots,X^n_\bullet}(\Phi)(m)\end{aligned}$$

which follows from the Yoneda lemma. One easily checks that the differentials agree and that the isomorphism factors through the reduced versions. Since the tensor product commutes with colimits, the result for simplicial non-finite sets follows. The proof for the coHochschild construction works similarly. □

Lemma 5.21. *For a family of simplicial sets $\{X^1_\bullet,\dots,X^n_\bullet\}$ and a natural number m the functors $\mathscr{L}_{X^1_\bullet,\dots,X^n_\bullet,m}$ and $\overline{\mathscr{L}}_{X^1_\bullet,\dots,X^n_\bullet,m}$ are h-projective.*

Proof. In Proposition 2.9 we have shown (both in the finite and non-finite case) that the functors $C_{X^1_\bullet,\dots,X^n_\bullet}(-)(m)$, $\overline{C}_{X^1_\bullet,\dots,X^n_\bullet}(-)(m)$, $D_{X^1_\bullet,\dots,X^n_\bullet}(-)(m)$ and $\overline{D}_{X^1_\bullet,\dots,X^n_\bullet}(-)(m)$ preserve quasi-isomorphism and thus so do the functors $-\otimes_{\mathscr{C}om} \mathscr{L}_{X^1_\bullet,\dots,X^n_\bullet,m}$, $-\otimes_{\mathscr{C}om} \overline{\mathscr{L}}_{X^1_\bullet,\dots,X^n_\bullet,m}$, $\hom_{\mathscr{C}om^{op}}(\mathscr{L}_{X^1_\bullet,\dots,X^n_\bullet,m},-)$ and $\hom_{\mathscr{C}om^{op}}(\overline{\mathscr{L}}_{X^1_\bullet,\dots,X^n_\bullet,m},-)$ by the previous lemma. Hence they give a model for the derived tensor product and hom, respectively. □

Now we are finally able to deduce Theorem 3.2 and Theorem 3.6:

Proof of Theorem 3.2 and Theorem 3.6. Since both $\mathscr{L}_{X^1_\bullet,\dots,X^n_\bullet,m}$ and $\overline{\mathscr{L}}_{X^1_\bullet,\dots,X^n_\bullet,m}$ are h-projective, the assumptions of Theorems 5.14 and 5.16 are fulfilled. By the isomorphism in Lemma 5.20, this implies the reduced and non-reduced versions of Theorem 3.2 and Theorem 3.6. □

Remark 5.22. Let A_∞ be the dg-PROP encoding A_∞–algebras, i.e. algebras only associative up to homotopy (see for example [WW11, Sec. 3.1]). We say that $\mathcal{E}'$ is a PROP with A_∞-multiplication, if there is a functor $\mathscr{A}_\infty \to \mathcal{E}'$ which is an isomorphism on objects. For such a PROP $\mathcal{E}'$ with A_∞-multiplication, two enriched functors $\Phi : \mathcal{E}' \to \mathrm{Ch}$ and $\Psi : \mathcal{E}'^{op} \to \mathrm{Ch}$, in [WW11, Def. 5.1] and [Wah12, Def 1.1] the Hochschild and coHochschild complex were defined as

$$C(\Phi)(m) = \Phi(-+m) \underset{A_\infty}{\otimes} \mathscr{L}$$

and

$$D(\Psi)(m) = \hom_{A_\infty^{op}}(\mathscr{L}, \Psi(-+m))$$

for $\mathscr{L}$ a complex of graphs (defined more generally later). For a fixed m we can again change $\mathscr{L}$ slightly to some $\mathscr{L}^m$ and can rewrite

$$C(\Phi)(m) = \Phi(-) \underset{A_\infty}{\otimes} \mathscr{L}^m$$

and

$$D(\Psi)(m) = \hom_{A_\infty^{op}}(\mathscr{L}^m, \Psi(-))$$

and thus can use Theorem 5.14 to reprove [Wah12, Theorem 2.1].

Appendix A. More details for Theorem 4.5

Let $\mathbb{F}$ be a field with $\mathbb{Q} \subseteq \mathbb{F}$. We briefly recall details on the de Rham algebra and give a proof of Theorem 4.5. This is mainly based on material from [FHT01, Chapter 10].

Definition A.1. The *simplicial deRham algebra* is the graded commutative simplicial cochain algebra $\Omega^*_\bullet$ with n-simplices

$$\Omega^*_n = \frac{\mathbb{F}[t_0,\dots,t_n] \otimes \Lambda(dt_0,\dots,dt_n)}{(t_0+\cdots+t_n-1, dt_0+\cdots+dt_n)}, \ |t_i| = 0, \ |dt_i| = 1$$

with differential

$$d(f) = \sum \frac{\partial f}{\partial t_i} dt_i,$$

for $f \in \mathbb{F}[t_0, \dots, t_n]/(\sum t_i - 1)$. The *simplicial de Rham algebra of a simplicial set $X_\bullet$* $\Omega^*(X_\bullet)$ is the graded commutative simplicial cochain algebra

$$\Omega^*(X_\bullet) = \mathrm{Hom}_{\Delta^{op}}(X_\bullet, \Omega^*_\bullet)$$

inheriting the algebra structure from $\Omega^*_\bullet$.

For a topological space this gives the usual de Rham algebra $\Omega^*(X)$ if we take the simplicial set $S_\bullet(X)$, the singular chains of X. Moreover, there is a natural quasi-isomorphism $\Omega^*(X_\bullet) \simeq \Omega^*(|X_\bullet|)$.

Theorem A.2 ([FHT01, Theorem 10.9]). *For each simplicial set there is a zig zag of natural quasi-isomorphisms*

$$\Omega^*(X_\bullet) \to \bullet \leftarrow C^*(X_\bullet).$$

Note that the naturality ensures that we have a zig zag of quasi-isomorphisms of functors between $C^*(Y_\bullet^{\times -})$ and $\Omega^*(Y_\bullet^{\times -}) : \mathscr{C}om^{op} \to \mathrm{Ch}$.

Since $-\otimes-$ is the coproduct in cdga, for two simplicial sets $X_\bullet$ and $Y_\bullet$ the maps induced by the projections $X \times Y \to X$ and $X \times Y \to Y$ induce a map $\Omega^*(X_\bullet) \otimes \Omega^*(Y_\bullet) \to \Omega^*(X_\bullet \times Y_\bullet)$, which is known to be a quasi-isomorphism. By definition, this map is associative, i.e. we get a quasi-isomorphism $\Omega^*(X_\bullet) \otimes \Omega^*(Y_\bullet) \otimes \Omega^*(Z_\bullet) \to \Omega^*(X_\bullet \times Y_\bullet \times Z_\bullet)$. Denote by τ the algebraic twist on $\Omega^*(X_\bullet) \otimes \Omega^*(Y_\bullet)$ and $\Omega^*(\tau)$ the twist induced by the twist on the level of simplicial sets on $\Omega^*(X_\bullet \times Y_\bullet)$. We want to see that the following diagram commutes:

$$\begin{tikzcd}
\Omega^*(X_\bullet) \otimes \Omega^*(Y_\bullet) \arrow[r] \arrow[d, "\tau"] & \Omega^*(X_\bullet \times Y_\bullet) \arrow[d, "\Omega^*(\tau)"] \\
\Omega^*(Y_\bullet) \otimes \Omega^*(X_\bullet) \arrow[r] & \Omega^*(Y_\bullet \times X_\bullet).
\end{tikzcd}$$

Precomposing with the inclusions of $\Omega^*(X_\bullet)$ and $\Omega^*(Y_\bullet)$, respectively, going via the upper right corner and going via the lower left corner commute. Therefore, by the universal property of the coproduct, the diagram commutes. Similarly, taking $\mathrm{id} \otimes \Omega^*(f)$ on $\Omega^*(X_\bullet) \otimes \Omega^*(Y_\bullet)$ and then mapping to $\Omega^*(Y_\bullet \times X_\bullet)$ commutes with first mapping there and then taking $\Omega^*(\mathrm{id} \times f)$. Furthermore, the diagram

$$\begin{tikzcd}
\Omega^*(X_\bullet) \otimes \Omega^*(X_\bullet) \arrow[r] \arrow[d, "\mu"] & \Omega^*(X_\bullet \times X_\bullet) \arrow[d, "\Omega^*(\Delta)"] \\
\Omega^*(X_\bullet) \arrow[r, "="] & \Omega^*(X_\bullet)
\end{tikzcd}$$

commutes and tensoring with the unit agrees with the map induced by forgetting the corresponding copy of $X_\bullet$. Thus we have shown that permuting any two factors or multiplying any two factors is preserved under the quasi-isomorphism $\Omega^*(Y_\bullet^{\times k}) \simeq \Omega^*(Y_\bullet)^{\otimes k}$ and hence we have a quasi-isomorphism of functors

$$C^*(Y_\bullet^{\times -}) \simeq \Omega^*(Y_\bullet^{\times -}) \simeq \Omega^*(Y_\bullet)^{\otimes -} : \mathscr{C}om^{op} \to \mathrm{Ch}\,.$$

Applying Proposition 4.2, we have proved Theorem 4.5.

Appendix B. Algebraic tools

In this appendix we give some additional proofs needed in Section 1 and some background on spectral sequences of double complexes and homotopy limits.

B.1. **Functorality of** Tot **and** sTot**.** Let Ch and $d\,\mathrm{Ch}^v$ be the dg-categories of chain complexes and double chain complexes with respect to the vertical differential, respectively, which were defined in Section 1. There we also defined the (switched) total complex functors $(s)\,\mathrm{Tot}^{\prod} : d\,\mathrm{Ch}^v \to \mathrm{Ch}$ and $(s)\,\mathrm{Tot}^{\oplus} : d\,\mathrm{Ch}^v \to \mathrm{Ch}$.

Proposition B.1. *The functors* $\mathrm{Tot}^{\prod}$, $\mathrm{Tot}^{\oplus}$, $\mathrm{sTot}^{\prod}$, $\mathrm{sTot}^{\oplus}$ $: d\,\mathrm{Ch}^v \to \mathrm{Ch}$ *are dg-functors.*

Proof. The calculations are the same for the direct sum and product complexes. Therefore, we omit the label and handle both cases at the same time. We start with the unswitched double complex: Let f be a map $f : C_{\bullet,\bullet} \to D_{\bullet,\bullet+|f|}$. We compute

$$\begin{aligned} d\,\mathrm{Tot}(f)_{p,q} &= (-1)^{p+q}(d^D \circ f - f \circ d^C) \\ &= (-1)^{p+q}(d_h^D \circ f + (-1)^p d_v^D \circ f - f \circ d_h^C - (-1)^p f \circ d_v^C) \\ &= (-1)^q(d_v^D \circ f - f \circ d_v^C) \\ &= Tot(d_v(f)). \end{aligned}$$

To avoid confusion, we do the second computation on elements, i.e. we look at

$$d(x \otimes \mathrm{sTot}(f)) \mapsto d(\mathrm{sTot}(f)(x)).$$

We get

$$\begin{aligned} & d(x \otimes \mathrm{sTot}(f)) \\ &= dx \otimes \mathrm{sTot}(f) + (-1)^{p+q} x \otimes d(s\,\mathrm{Tot}(f)) \\ &= d_v(x) \otimes \mathrm{sTot}(f) + (-1)^q d_h(x) \otimes \mathrm{sTot}(f) + (-1)^{p+q} x \otimes d(s\,\mathrm{Tot}(f)) \\ &\mapsto \mathrm{sTot}(f)(d_v(x)) + (-1)^q\,\mathrm{sTot}(f)(d_h(x)) + (-1)^{p+q} d(\mathrm{sTot}(f))(x) \\ &= (-1)^{|f|p} f(d_v(x)) + (-1)^{|f|(p+1)+q} f(d_h(x)) + (-1)^{p+q} d(\mathrm{sTot}(f))(x). \end{aligned}$$

On the other hand

$$d(\mathrm{sTot}(f)(x)) = (-1)^{p|f|} d_v(f(x)) + (-1)^{p|f|+(q+|f|)} d_h(f(x)).$$

Comparing the two results, we obtain

$$\begin{aligned} & (-1)^{|f|p} f(d_v(x)) + (-1)^{|f|(p+1)+q} f(d_h(x)) + (-1)^{p+q} d(\mathrm{sTot}(f))(x) \\ & \quad = (-1)^{p|f|} d_v(f(x)) + (-1)^{p|f|+q+|f|} d_h(f(x)) \\ \Leftrightarrow & (-1)^{p+q} d(\mathrm{sTot}(f))(x) \\ & \quad = (-1)^{|f|p}(d_v(f(x)) - f(d_v(x))) + (-1)^{|f|(p+1)+q}(d_h(f(x)) - f(d_h(x)) \end{aligned}$$

and plugging in the assumptions on the right side we get

$$\begin{aligned} \Leftrightarrow & (-1)^{p+q} d(\mathrm{sTot}(f))(x) = (-1)^{|f|p+q}(d_v(f)(x)) + 0 \\ \Leftrightarrow & d(\mathrm{sTot}(f))(x) = (-1)^{(|f|+1)p} d_v(f)(x). \end{aligned}$$

However, computing $\mathrm{sTot}(d_v(f))(x)$ we have

$$\mathrm{sTot}(d_v(f))(x) = (-1)^{|d_v(f)|p} d_v(f)(x) = (-1)^{(|f|+1)p} d_v(f)(x)$$

i.e. for any x we have that $d(\mathrm{sTot}(f))(x) = \mathrm{sTot}(d_v(f))(x)$ hence the claim follows. □

B.2. **The Eilenberg-Zilber Theorem.** In this section we recall the simplicial and cosimplicial Eilenberg-Zilber and Alexander-Whitney maps and the Eilenberg-Zilber Theorem. This is based on [Wei95, Chapter 8.5] and [GM04, Appendix 3].

Definition B.2. For a bisimplicial object $A_{\bullet,\bullet}$ in an abelian category $\mathscr{A}$ the simplicial object $\mathrm{diag}_\bullet(A_{\bullet,\bullet})$ is given by composing the diagonal map $\Delta \to \Delta \times \Delta$ with $A_{\bullet,\bullet}$. Thus $\mathrm{diag}_n(A_{\bullet,\bullet}) = A_{n,n}$.

Similarly, for a bicosimplicial object $B^{\bullet,\bullet}$ in an abelian category $\mathscr{A}$ the cosimplicial object $\mathrm{diag}^\bullet(B^{\bullet,\bullet})$ is the composition of the diagonal map $\Delta \to \Delta \times \Delta$ with $B^{\bullet,\bullet}$. Thus $\mathrm{diag}^n(B^{\bullet,\bullet}) = B^{n,n}$.

We first work in the simplicial setup (cf. [Wei95, Chapter 8.5.]).

Definition B.3 (Alexander-Whitney map)**.** Let $A_{\bullet,\bullet}$ be a bisimplicial object with horizontal boundary maps d_i^h and vertical face maps d_i^v. For $p+q=n$ the map $AW_{p,q} : A_{n,n} \to A_{p,q}$ is defined as

$$d^h_{p+1}\cdots d^h_n \underbrace{d^v_0\cdots d^v_0}_{p}.$$

The sum over all p and q yields a map $AW_n : A_{n,n} \to \mathrm{Tot}^{\oplus}_n C_*C_*(A_{\bullet,\bullet})$ and assembling all these maps gives a chain map

$$AW : C_*(\mathrm{diag}_\bullet A_{\bullet,\bullet}) \to \mathrm{Tot}^{\oplus} C_*C_*(A_{\bullet,\bullet})$$

called the Alexander-Whitney map. The map is natural with respect to morphisms of bisimplicial objects.

Furthermore, the map is well-defined on reduced complexes and we have a commutative square

$$\begin{tikzcd}
C_*(\mathrm{diag}_\bullet A_{\bullet,\bullet}) \arrow[r, "AW"] \arrow[d, "\simeq"] & \mathrm{Tot}^{\oplus} C_*C_*(A_{\bullet,\bullet}) \arrow[d, "\simeq"] \\
\overline{C}_*(\mathrm{diag}_\bullet A_{\bullet,\bullet}) \arrow[r, "AW"] & \mathrm{Tot}^{\oplus} \overline{C}_*\overline{C}_*(A_{\bullet,\bullet}).
\end{tikzcd}$$

Remark B.4. For a trisimplicial object $A_{\bullet,\bullet,\bullet}$ with boundary maps d_i^1, d_i^2 and d_i^3, we can iterate the Alexander-Whitney map. The map $(AW \otimes \mathrm{id}) \circ AW$

$$C_*(\mathrm{diag}_\bullet A_{\bullet,\bullet,\bullet}) \to \mathrm{Tot}^{\oplus} \mathrm{Tot}^{\oplus}_{1,2} C_*C_*(A_{\bullet,\bullet,\bullet})$$

is induced by maps $AW_{p_1,p_2,p_3} : A_{n,n,n} \to A_{p_1,p_2,p_3}$ with $p_1+p_2+p_3 = n$ defined as

$$d^1_{p_1+1}\cdots d^1_n \circ \underbrace{d^2_0\cdots d^2_0}_{p_1} d^2_{p_1+p_2+1}\cdots d^2_n \circ \underbrace{d^3_0\cdots d^3_0}_{p_1+p_2}.$$

This explicit definition of the maps implies the coassociativity of the Alexander-Whitney map, i.e. $(AW \otimes \mathrm{id}) \circ AW = (\mathrm{id} \otimes AW) \circ AW$.

Definition B.5 (Eilenberg-Zilber map)**.** Let $A_{\bullet,\bullet}$ be a bisimplicial object with codegeneracy maps s_i^h and s_i^v. For $p+q=n$ the map $EZ_{p,q} : A_{p,q} \to A_{n,n}$ is defined to be

$$\sum_{(p,q)\text{ shuffles }\mu} (-1)^{sgn(\mu)} s^h_{\mu(n)}\cdots s^h_{\mu(p+1)} s^v_{\mu(p)}\cdots s^v_{\mu(1)}.$$

The sum of these maps gives a chain map

$$EZ : \mathrm{Tot}^{\oplus} C_*C_*(A_{\bullet,\bullet}) \to C_*(\mathrm{diag}_\bullet A_{\bullet,\bullet}),$$

called the Eilenberg-Zilber map. It is natural with respect to morphisms of cosimplicial objects.

The Eilenberg-Zilber map commutes with restriction to the reduced complex, too.

Remark B.6. Via the explicit formulas of the Eilenberg-Zilber map given above, it is easy to check that it is associative.

Theorem B.7 (simplicial Eilenberg-Zilber)**.** *For a bisimplicial object $A_{\bullet,\bullet}$ in an abelian category $\mathscr{A}$ the maps EZ and AW yield chain homotopy equivalences*

$$AW : C_*(\operatorname{diag}_\bullet A_{\bullet,\bullet}) \rightleftarrows \operatorname{Tot}^\oplus C_*C_*(A_{\bullet,\bullet}) : EZ$$

restricting to the reduced complexes.

Dually, for a cosimplicial object $B^{\bullet,\bullet}$ we define

$$AW^* : \operatorname{Tot}^\oplus C^*C^*(B^{\bullet,\bullet}) \to C^*(\operatorname{diag} B^{\bullet,\bullet})$$

$$EZ^* : C^*(\operatorname{diag}^\bullet B^{\bullet,\bullet}) \to \operatorname{Tot}^\oplus C^*C^*(B^{\bullet,\bullet})$$

using the coface maps and codegeneracies, respectively. Again, the maps commute with restriction to the reduced complexes.

Theorem B.8 (cosimplicial Eilenberg-Zilber, [GM04, A.3])**.** *For a bicosimplicial object $B^{\bullet,\bullet}$ in an abelian category $\mathscr{A}$ the maps EZ^* and AW^* yield chain homotopy equivalences*

$$AW^* : \operatorname{Tot}^\oplus C^*C^*(B^{\bullet,\bullet}) \rightleftarrows C^*(\operatorname{diag}^\bullet B^{\bullet,\bullet}) : EZ^*$$

restricting to the reduced complexes.

B.3. **Spectral sequences for double complexes.** We recall the results for spectral sequences of right and left half plane double complexes, i.e. complexes being zero in the left and right halfplane, respectively.

Theorem B.9 ([Wei95, Section 5.6.])**.** *Let $C_{p,q}$ be a right half plane double complex. Then filtration by columns gives a spectral sequence with $E^0_{p,q} = C_{p,q}$, $E^1_{p,q} = H^v_q(C_{p,*}, d_v)$ and $E^2_{p,q} = H^h_p(H^v_q(C_{*,*}), (-1)^q d_h)$ strongly converging to $H_{p+q}(\operatorname{sTot}^\oplus(C_{*,*}))$.*

Now let $C_{p,q}$ be a left half plane double complex. The filtration by columns was defined as $F_s = \operatorname{Tot}^\Pi(C_{*,*})$ for $s > 0$ and

$$F_s = \prod_{p\le s} C_{p,q}.$$

This gives us a descending filtration $\cdots \subseteq F_{s-1} \subseteq F_s \subseteq \cdots$ starting at $F_0 = \operatorname{Tot}^\Pi C_{p,q}$ with $(F_s)_{s+t}/(F_{s-1})_{s+t} = C_{s,t}$. Therefore, by [Wei95, Theorem 5.4.1] the associated spectral sequence has the form

$$E_1^{p,q} = H_q(C_{p,*}, d_2) \Rightarrow H_{p+q}(\operatorname{Tot}^\Pi C_{*,*}).$$

This spectral sequence does not need to converge, but it always converges conditionally:

Theorem B.10. *For any left half plane double complex the spectral sequence associated to the product filtration by columns defined above is conditionally convergent.*

Proof. Applying [Boa99, Theorem 9.2] we have to show that the filtration is exhaustive, complete and Hausdorff. The fact that $\bigcup_s F_s = F_0 = \operatorname{Tot}^\Pi(C_{*,*})$ shows that the filtration is exhaustive. By [Wei95, Section 5.6], the spectral sequence is complete Hausdorff. □

Theorem B.11. *In both situations (i.e. either right or left half plane double complex) a morphism of double complexes inducing an isomorphism on E_r-pages for some $r > 0$ induces an isomorphism of filtered groups on the target.*

Proof. In the case of strong convergence (i.e. right halfplane double complex) this follows by the classical Comparison Theorem ([Wei95, Theorem 5.2.12]).

In the second case, by [Boa99, Proposition 2.4] we have an isomorphism between the E_∞ and RE_∞ pages. Moreover, both spectral sequences converge to the limit by the previous theorem. Applying [Boa99, Theorem 7.2], we get an isomorphism of filtered groups. □

Corollary B.12. *Let $f : C_{p,q} \to D_{p,q}$ be a map of left (respectively right) half plane double complexes. If f is a quasi-isomorphism with respect to the vertical differential (i.e. an isomorphism after taking homology in the vertical direction), f induces a quasi-isomorphism $f : \prod_{p,q} C_{p,q} \to \prod_{p,q} D_{p,q}$ respectively $f : \bigoplus_{p,q} C_{p,q} \to \bigoplus_{p,q} D_{p,q}$.*

Proof. A map f as given in the corollary induces an isomorphism of the E_1-pages of the respective spectral sequences. Thus, by the previous theorem, the claim follows. □

If we have a map which is a quasi-isomorphism with respect to the horizontal differential, we have to ask for stronger properties for it to induce a quasi-isomorphism of total complexes, namely it has to be a chain homotopy equivalence. To deduce this result, we need a small technical lemma from homological algebra:

Lemma B.13. *Let $\mathscr{A}$ and $\mathscr{B}$ be two abelian categories and $F : \mathscr{A} \to \mathscr{B}$ an additive functor. Then the induced functor $F : \mathrm{Ch}(\mathscr{A}) \to \mathrm{Ch}(\mathscr{B})$ preserves chain homotopy equivalences.*

Proof. For f and g inverse chain homotopy equivalences, we have maps K, L such that $d_{n+1} \circ K_n + K_{n+1} \circ d_n = f_n \circ g_n - id$ and $d_{n+1} \circ L_n + L_{n+1} \circ d_n = g_n \circ f_n - id$. Since F is additive, $F(d_{n+1}) \circ F(K)_n + F(K)_{n+1} \circ F(d_n) = F(f)_n \circ F(g)_n - id$ and $F(d_{n+1}) \circ F(L)_n + F(L)_{n+1} \circ F(d_n) = F(g)_n \circ F(f)_n - id$, i.e. $F(f)$ and $F(g)$ are inverse chain homotopy equivalences. □

On the other hand, for a chain map $f : C_{p,q} \to D_{p,q}$ being a chain homotopy equivalence in $d\,\mathrm{Ch}^h$ (as defined above) it also induces a quasi-isomorphism of the respective total complexes (see Corollary B.14).

Corollary B.14. *A chain-homotopy equivalence of (double) complexes $C_{*,*}$ and $\widetilde{C}_{*,*}$ in $d\,\mathrm{Ch}^h$ or $D_{*,*}$ and $\widetilde{D}_{*,*}$ in $d\,\mathrm{Ch}^h$ (i.e. with respect to the horizontal differential) induces quasi-isomorphisms of total complexes $\mathrm{sTot}^{\oplus} C_{*,*} \simeq \mathrm{sTot}^{\oplus} \widetilde{C}_{*,*}$ and $\mathrm{Tot}^{\prod} D_{*,*} \simeq \mathrm{Tot}^{\prod} \widetilde{D}_{*,*}$, respectively.*

Proof. Given chain homotopy equivalences

$$i^C : C_{*,*} \rightleftarrows \widetilde{C}_{*,*} : p^C \text{ and } i^D : D_{*,*} \rightleftarrows \widetilde{D}_{*,*} : p^D,$$

respectively, by the additivity of the homology functor $H_* : \mathrm{Ch} \to Ab$ and Lemma B.13 we get a chain homotopy equivalence after taking homology in the vertical direction, i.e. maps such that both compositions are chain homotopic to the identity:

$$i^C_* : H_*(C_{*,*}, d_2) \rightleftarrows H_*(\widetilde{C}_{*,*}, d_2) : p^C_* \text{ and } i^D_* : H_*(D_{*,*}, d_2) \rightleftarrows H_*(\widetilde{D}_{*,*}, d_2) : p^D_*.$$

This implies that the maps induce isomorphisms after taking homology in the horizontal direction, i.e.

$$i^C_* : H_*(H_*(C_{*,*}, d_2), d_1) \xrightarrow{\cong} H_*(H_*(\widetilde{C}_{*,*}, d_2), d_1)$$

and

$$i^D_* : H_*(H_*(D_{*,*}, d_2), d_1) \xrightarrow{\cong} H_*(H_*(\widetilde{D}_{*,*}, d_2), d_1).$$

So looking at the associated right and left half-plane spectral sequences of the double complexes (cf. Theorem B.9 and B.10, respectively), i^C and i^D are maps of spectral sequences inducing isomorphisms on the E_2-pages. By Theorem B.11, this yields filtered isomorphisms of $\mathrm{sTot}^{\oplus} C_{*,*} \simeq \mathrm{sTot}^{\oplus} \widetilde{C}_{*,*}$ and $\mathrm{Tot}^{\prod} D_{*,*} \simeq \mathrm{Tot}^{\prod} \widetilde{D}_{*,*}$, respectively. □

B.4. **The model structures on topological spaces and chain complexes used in Section 5.** Here we recall the mixed model structure on topological spaces and the projective model structure on unbounded chain complexes over a ring which are used in Section 5. Given a model category $\mathscr{M}$, we denote by (C, W, F) the tuple of the sets of cofibrations, weak equivalences and fibrations.

B.4.1. *The mixed model structure on topological spaces.* In this section we follow the introduction to the mixed model structure on compactly generated topological spaces provided in [MP11, 17.3 and 17.4].

The mixed model structure, which mixes the Quillen and the Hurewicz model structure on topological spaces, was first introduced by Cole in [Col06]. Cole proves more generally that given a category $\mathscr{M}$ with two model structures (C_q, W_q, F_q) and (C_h, W_h, F_h) such that $F_h \subset F_q$ and $W_h \subset W_q$ there is a model structure (C_m, W_m, F_m) with $W_m = W_q$ and $F_m = F_h$ and the cofibrations are those maps $j \in C_h$, which factor as $j = f \circ i$ with $f \in W_h$ and $i \in C_q$.

In topological spaces (C_q, W_q, F_q) is chosen to be the Quillen model structure, so the weak equivalences W_q are the weak homotopy equivalences, i.e. the maps that induce bijections on path components and an isomorphism on homotopy groups for all choices of basepoints. The q–fibrations F_q are the Serre fibrations. Then the q–cofibrations are defined to be those maps, which have the left lifting property with respect to the acyclic q–fibrations.

The Hurewicz (or Strøm) model structure (C_h, W_h, F_h) has as weak equivalences the homotopy equivalences and F_h and C_h the Hurewicz fibrations and cofibrations, respectively.

Then the mixed model structure has as weak equivalences the weak homotopy equivalences and the fibrations the Hurewicz fibrations. Hence the cofibrations are those Hurewicz cofibrations j, that factor as $j = f \circ i$ with f a (strong) homotopy equivalence and i a Quillen cofibration. Since one composes with a homotopy equivalence afterwards, one can assume that $i : A \to X$ is a relative CW-complex.

The following facts where shown in [Col06]:

- Every space is fibrant.
- The cofibrant spaces are exactly those which are strongly homotopy equivalent to CW-complexes.
- The model structure turns Top into a closed monoidal model category.
- There are Quillen adjunctions to both Top with the Quillen and Top with the Hurewicz model structure.

The last two properties imply that we have a strong monoidal Quillen adjunction to simplicial sets.

B.4.2. *The projective model structure on unbounded chain complexes.* On unbounded chain complexes $Ch(R)$ for R a commutative ring, we use the projective model structure. More details can be found in [Hov99, 2.3] and [MP11, 17.3 and 17.4].

The weak equivalences are the quasi-isomorphisms. The fibrations are the levelwise surjective maps, i.e. $p : X_* \to Y_*$ is a fibration if $p_k : X_k \to Y_k$ is surjective for all k. The cofibrations are a little more difficult to describe explicitly and therefore are omitted here. Cofibrant objects are levelwise projective, but the converse is only true for bounded below chain complexes. Every chain complex is fibrant in this model structure.

The model category $Ch(R)$ is a strong monoidal model category. It is cofibrantly generated and thus there is a functorial cofibrant replacement functor.

By [Hov01, Cor. 3.7] the derived tensor product can be computed by only replacing one of the factors cofibrantly.

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