Centre for Symmetry and Deformation – Department of Mathematical Sciences
PhD School of Science – Faculty of Science – University of Copenhagen

Dissertation

On dimension and shape theory for C^*-algebras

Hannes Thiel

Submitted: October 2012

Advisor: Mikael Rørdam
University of Copenhagen, Denmark

Assessment committee: Søren Eilers (chair)
University of Copenhagen, Denmark

Joachim Cuntz
University of Münster, Germany

Ilan Hirshberg
Ben Gurion University of the Negev, Israel

Hannes Thiel
Department of Mathematical Sciences
University of Copenhagen
Universitetsparken 5
DK-2100 København Ø
Denmark

thiel@math.ku.dk
http://math.ku.dk/~thiel

PhD thesis submitted to the PhD School of Science, Faculty of Science, University of Copenhagen, Denmark in October 2012.

ISBN 978-87-7078-995-0

Abstract

This thesis deals with the structure theory of C^*-algebras. We first introduce the abstract notion of a noncommutative dimension theory by proposing a natural set of axioms for such theories, see Article A.

We then establish an unexpected connection between the generator problem and dimension theory. The generator problem asks to determine the minimal number of generators for a given C^*-algebra. We define the generator rank by not only asking if a k-tuple exists that generates the C^*-algebra, but also if the set of such tuples is dense. We show that the generator rank behaves like a dimension theory. As an application we obtain that every AF-algebra is singly generated, and that the set of generators is dense, see Article B. We also provide a solution of the generator problem for unital, separable $\mathcal{Z}$-stable C^*-algebras by showing that such algebras are singly generated, see Article C.

In the second part, we study shape theory for C^*-algebras. We first give a characterization of semiprojectivity for commutative C^*-algebras, see Article D. Then, we show that a C^*-algebra is an inductive limits of projective C^*-algebras if and only if it has trivial shape. It follows that a C^*-algebra is projective if and only if it is semiprojective and contractible, see Article E. The equivariant version of semiprojectivity was recently introduced by Phillips. We answer a couple of natural questions related to this notion. In particular, we show that equivariant semiprojectivity is preserved when restricting to a cocompact subgroup, see Article F.

Finally, we study comparison relations for positive elements in a C^*-algebra in connection to comparison relations for the associated open projections in the universal von Neumann algebra. As an application, we give a new picture of the Cuntz semigroup, see Article G.

Resumé

Denne afhandling omhandler strukturteori for C^*-algebraer. Vi starter med at introducere ikkekommutativ dimensionsteori ved at foreslå aksiomer for sådanne teorier, jf. Artikel A.

Dernæst etablerer vi en uventet forbindelse mellem frembringerproblemet og dimensionsteori. Frembringerproblemet omhandler bestemmelsen af det minimale antal frembringere for en given C^*-algebra. Vi definerer frembringerrangen ved at spørge ikke kun om k frembringere findes men også om mængden af sådanne k-tupler er tæt. Vi viser at frembringerrangen opfører sig som en dimensionsteori. Som en anvendelse opnår vi at enhver AF-algebra er frembragt af ét element og at mængden af sådanne frembringere er tæt, jf. Artikel B. Vi giver også en løsning til frembringerproblemet for enhedsbærende, seperable, $\mathcal{Z}$-stabile C^*-algebraer ved at vise at sådanne algebraer er frembragt af ét element, jf. Artikel C.

I den anden del af afhandlingen studerer vi gestaltteori for C^*-algebraer. Vi giver først en karakterisation af semiprojektivitet for kommutative C^*-algebraer, jf. Artikel D. Dernæst viser vi at en C^*-algebra er en direkte limes af projektive C^*-algebraer hvis og

kun hvis den har triviel gestalt. Det følger heraf at en C^*-algebra er projektiv hvis og kun hvis den er semiprojektiv og kontraktibel, jf. Artikel E. Den ækvivariante version af semiprojektivitet blev for nylig introduceret af Phillips. Vi besvarer et par oplagte spørgsmål relateret til dette begreb. Specielt viser vi at ækvivariant semiprojektivitet bevares under restriktion til kokompakte undergrupper, jf. Artikel F.

Endelig studeres sammenligningsrelationer for positive elementer i en C^*-algebra i forbindelse med sammenligningsrelationer for de associerede åbne projektioner i den universelle von Neumann-algebra. Som en anvendelse heraf gives et nyt billede af Cuntz-semigruppen, jf. Artikel G.

Preface

This text constitutes my dissertation for the PhD degree in mathematics. It contains the results of my research carried out as a PhD student at the Department of Mathematics at the University of Copenhagen from November 2009 to October 2012.

The general subject of the thesis is the structure theory of C^*-algebras, and the two main themes are noncommutative dimension and shape theory. The thesis comprises seven articles and a chapter with additional material.

I started my studies by comparing different regularity properties of C^*-algebras connected to the classification problem. During that time, Eduard Ortega was a postdoc in Copenhagen, and we studied different properties of the Cuntz semigroup of a C^*-algebra. Together with Mikael Rørdam we established a connection between comparison relations of positive elements and analogous comparison relations of the associated support projections. The results appeared in the joint article *The Cuntz semigroup and comparison of open projections*, which is included as Appendix G.

In May 2010, I participated in the workshop "Semiprojectivity and Asymptotic Morphisms" in Copenhagen. There was a problem session where the experts presented the open questions of the field. This was very inspiring, and together with Adam Sørensen I started to attack one of the problems, which asked to characterize the spectra of commutative, semiprojective C^*-algebras. We managed to solve the problem, thus confirming a conjecture of Blackadar, and the results appeared in the joint article *A characterization of semiprojectivity for commutative C^*-algebras*, which is included as Appendix D.

Another open problem that was presented at the workshop in May 2010 is whether every C^*-algebra is an inductive limit of semiprojective C^*-algebras. While the general problem remains open, I obtained a natural characterization of the class of algebras that are inductive limits of projective C^*-algebras. It follows from this characterization that a C^*-algebra is projective if and only if it is semiprojective and contractible, which confirms a conjecture of Loring. The results are contained in the article *Inductive limits of projective C^*-algebras*, which is included as Appendix E.

In the spring of 2011, I visited the Centre de Recerca Matemática (CRM) at the Universitat Autónoma de Barcelona for the Research program "The Cuntz semigroup and the classification of C^*-algebras". During that time, I started to work on the generator problem for C^*-algebras, and together with Wilhelm Winter I obtained a solution to the problem for unital, separable $\mathcal{Z}$-stable C^*-algebras by showing that such algebras are singly generated. The results are contained in the joint article *The generator problem for $\mathcal{Z}$-stable C^*-algebras*, which is included as Appendix C.

In November 2011, I organized the master class "The nuclear dimension of C^*-algebras" in Copenhagen. While preparing lecture notes with a survey on noncommutative dimension theories I realized that there exist properties that all these theories share. This encouraged me to define an abstract notion of a noncommutative dimension theory, by proposing a natural set of axioms for such theories. This is contained in the article *The topological dimension of type I C^*-algebras*, which is included as Appendix A.

In the spring of 2012, Christopher Phillips visited Copenhagen. He recently defined the notion of equivariant semiprojectivity which takes group actions into account. Together with him and Adam Sørensen I studied this notion and we answered a couple of natural questions. In particular, we give a characterization when the trivial action of a group is semiprojective. The results are contained in the joint article *Semiprojectivity with and without a group action*, which is included as Appendix F.

In January 2012, I attended the workshop "Set theory and C^*-algebras" at the American Institute of Mathematics (AIM) in Palo Alto. I participated in a research session on the generator problem for non-$\mathcal{Z}$-stable C^*-algebras, which turned into an ongoing joint project with Karen Strung, Aaron Tikuisis, Joav Orovitz and Stuart White. The joint work with them was also the starting point for me to study the denseness of generators in a C^*-algebra. This led to my definition of the generator rank for C^*-algebras. It turns out that the generator rank has many of the permanence properties that noncommutative dimension theories satisfy, which establishes an unexpected connection between the generator problem and dimension theory. The results are contained in the article *The generator rank for C^*-algebras*, which is included as Appendix B.

The mentioned articles are attached in a rough systematic order, starting with articles about dimension theory (which includes the articles about the generator problem), and then articles about shape theory. This is why the articles do not appear in chronological order.

Acknowledgements

I want to thank my friends and colleagues in Copenhagen for many interesting discussions and activities, which created a very inspiring working environment through these years. Thanks are due to Sara Arklint, Rasmus Bentmann, Thomas Danielsen, James Gabe, Søren Knudby, Tim de Laat, Eduard Ortega, Henrik Petersen, Maria Ramirez-Solano, and Adam Sørensen. I want to give a special thanks to my advisor Mikael Rørdam for his support and many stimulating discussions.

I spend five months at the Universitat Autónoma de Barcelona, and I want to thank Francesc Perera, Ramon Antoine and Joan Bosa for their kind hospitality and many fruitful discussions.

I thank my co-authors Eduard Ortega, Christopher Phillips, Mikael Rørdam, Adam Sørensen, and Wilhelm Winter. I benefited greatly from our joint work.

Thanks are due to the Marie Curie Research Training Network EU-NCG, the Danish National Research Foundation through the Centre for Symmetry and Deformation, and the NordForsk Research Network "Operator Algebras and Dynamics" for supporting my work.

Lastly, I want to thank my family. I am indebted to Bernd, my parents, my sister, my grandfather and my beloved late grandmother for their support and love.

Hannes Thiel
Copenhagen, October 2012

Contents

CHAPTER 1

Introduction

1.1. Dimension theory and the generator problem for C^*-algebras

In the first part of the thesis, which consists of the Articles A, B and C, we study the theory of dimension for C^*-algebras and we establish a connection to the generator problem for C^*-algebras.

The covering dimension of a space is an assignment that extends our intuitive understanding that a point is zero-dimensional, a line is one-dimensional, the plane is two-dimensional, etc. There also exist other dimension theories for spaces, e.g., the small and large inductive dimension, but they all agree on the class of separable, metric spaces. In fact, it is the subject of axiomatic dimension theory to show that certain natural axioms for a dimension theory force it to agree with the (covering) dimension, see e.g. [**Nis74**] or [**Cha94**].

When moving to noncommutative spaces, there is no longer only one natural dimension theory. Instead, the theory ramifies into different concepts that suit individual purposes. The first generalization of dimension theory to C^*-algebras was the *stable rank* as introduced in 1983 by Rieffel, [**Rie83**, Definition 1.4]. It generalizes the characterization of dimension in terms of fragility of maps. Herman and Vaserstein, [**HV84**], showed that the stable rank for unital C^*-algebras coincides with the *Bass stable rank*, which is a purely algebraic notion that can be defined for every unital ring, and which was introduced to study non-stable K-theory. This explains why the stable rank for C^*-algebras has many applications in the computation of K-theory.

The *real rank* was introduced in 1991 by L. G. Brown and Pedersen, [**BP91**]. Like the stable rank, it generalizes the concept of fragility of maps. However, while the stable rank considers the analog of maps into $\mathbb{C}^n$, the real rank considers the analog of maps into $\mathbb{R}^n$. Despite this seemingly insignificant difference, the real and stable rank behave much differently.

More recently, the *decomposition rank* was introduced by Kirchberg and Winter, [**KW04**, Definition 3.1], and the *nuclear dimension* was introduced by Winter and Zacharias, [**WZ10**, Definition 2.1]. These notions are mainly used in connection with Elliott's classification program for simple, nuclear C^*-algebras.

It is a recurrent theme that (noncommutative) spaces of low dimension enjoy certain rigidity or regularity properties. For instance, zero-dimensional compact, metrizable spaces are very rigid in the sense that two such spaces are homeomorphic whenever they are shape equivalent. When trying to generalize such a statement to C^*-algebras, one has to find the "right" non-commutative dimension theory. For instance, the analog statement easily holds for the nuclear dimension. Indeed, a separable C^*-algebra has nuclear dimension zero if and only if it is an AF-algebra, and two AF-algebras are isomorphic whenever they are shape equivalent.

On the other side, the analog statement does not hold for the real rank, i.e., there exist separable, real rank zero C^*-algebras that are shape equivalent, yet not isomorphic, as shown by Dadarlat, [**Dad00**]. The (counter)example of Dadarlat consists of a nuclear and a non-nuclear C^*-algebra. It is unknown if a counterexample also exists among nuclear C^*-algebras, and we think that a positive answer to the following question would be an interesting way of unifying many of the known results about the classification of nuclear, real rank zero C^*-algebras:

QUESTION 1.1.1. Are separable, nuclear, real rank C^*-algebras isomorphic whenever they are shape equivalent?

In Appendix A, *The topological dimension of type I C^*-algebras*, we introduce the abstract notion of a noncommutative dimension theory on a class $\mathcal{C}$ of C^*-algebras as an assignment $d\colon \mathcal{C} \to \overline{\mathbb{N}} = \{0, 1, 2, \ldots, \infty\}$ such that six natural axioms are satisfied. These axioms are noncommutative analogs of properties of the covering dimension. Axioms (D1)-(D4) mean that a noncommutative dimension theory behaves well when passing to ideals, quotients, direct sums or minimal unitizations. The less obvious axioms are (D5) and (D6). The former means that a noncommutative dimension theory behaves well with respect to approximation by sub-C^*-algebras. Recall that a family of sub-C^*-algebras $A_i \subset A$ is said to approximate A if for every finite set $F \subset A$ and $\varepsilon > 0$ there exists an index i such that F is contained in A_i up to ε. Then axiom (D5) means that $d(A) \leq k$ whenever A is approximated by $A_i \subset A$ with $d(A_i) \leq k$. It follows that a noncommutative dimension theory behaves well with respect to inductive limits, i.e, $d(A) \leq \liminf_i d(A_i)$ for every inductive limit $A \cong \varinjlim_i A_i$, as shown in Proposition 2 of Appendix A.

Lastly, axiom (D6) is the noncommutative analog of the Mardešić factorization theorem, see Remark 1 in Appendix A and [**Nag70**, Corollary 27.5, p.159] or [**Mar60**, Lemma 4].

We also introduce a notion of Morita-invariance for dimension theories by requiring that $d(A) = d(B)$ for any two Morita-equivalent C^*-algebras A and B, see Definition A.2.

The dimension theories mentioned above are indeed noncommutative dimension theories in the sense of Definition A.1. The decomposition rank and nuclear dimension are Morita-invariant, while the real and stable rank are not. We also show that the *topological dimension* as introduced by L. G. Brown and Pedersen, [**BP09**], is a noncommutative dimension theory for type I C^*-algebras.

In Article B, *The generator rank for C^*-algebras*, we establish an unexpected connection between noncommutative dimension theory and the generator problem.

Let A denote a C^*-algebra. We say that a tuple $\mathbf{a} = (a_1, \ldots, a_k) \in A^k$ generates A if there exists no proper sub-C^*-algebra of A containing all elements $a_1, \ldots, a_k$. For technical reasons, one often restricts to tuples of self-adjoint elements, denoted by A^k_{sa}, and we let $\mathrm{Gen}_k(A)_{\mathrm{sa}}$ denote the set of k-tuples $\mathbf{a} \in A^k_{\mathrm{sa}}$ that generate A, see Notation B.2.1. The generator problem asks to determine the minimal number of self-adjoint generators for A. Let us denote this invariant by $\mathrm{gen}(A)$.

Note that two self-adjoint elements $a, b \in A$ generate the same sub-C^*-algebra as the element $a + ib$. It follows that A is singly generated, i.e., A contains an element whose *-polynomials form a dense subset of A, if and only if $\mathrm{gen}(A) \leq 2$.

The invariant $\mathrm{gen}(\_)$ does not define a noncommutative dimension theory. Indeed, it may increase when passing to inductive limits as is shown in the introduction of Article B. This makes it hard to directly compute the minimal number of self-adjoint generators. For example, we see no obvious way of computing $\mathrm{gen}(\_)$ on the class of AF-algebras.

To obtain a better behaved theory, we consider a "stable" version of the generator problem. More precisely, instead of asking for the minimal k such that there exists a generating k-tuple, i.e., such that $\mathrm{Gen}_k(A)_{\mathrm{sa}} \neq \emptyset$, we want to determine the minimal number k such that $\mathrm{Gen}_k(A)_{\mathrm{sa}}$ is dense in A^k_{sa}. In Definition B.2.2, we use this idea to define the *generator rank* of A, denote by $\mathrm{gr}(A)$, as the minimal k such that $\mathrm{Gen}_{k+1}(A)_{\mathrm{sa}}$ is dense in A^{k+1}_{sa} (notice the index shift). In Proposition B.2.7, we show that $\mathrm{Gen}_k(A)_{\mathrm{sa}} \subset A^k_{\mathrm{sa}}$ is a G_δ-subset for each k (although not necessarily dense). It follows that A has generator rank one if and only if A is singly generated and the set of generators forms a dense G_δ-subset (also called generic set).

We show that the generator rank for the class of separable C^*-algebras satisfies axioms (D1),(D2) and (D4)-(D6) of our definition of a noncommutative dimension theory. The remaining axiom seems to be surprisingly hard to show. We verified it for some particular classes of C^*-algebras, and it would be very interesting to know if it holds in general:

QUESTION 1.1.2. Given two separable C^*-algebras A, B, do we have $\mathrm{gr}(A \oplus B) = \max\{\mathrm{gr}(A), \mathrm{gr}(B)\}$?

It is easy to see that a C^*-algebra has generator rank one if it is finite-dimensional (as a vector space). It follows that every AF-algebra has generator rank one, see Corollary B.3.2. In particular, this solves the generator problem for AF-algebra by showing that they are all singly generated. In Theorem B.4.23, we compute the generator rank of homogeneous C^*-algebras. The result shows that a unital, separable AH-algebra has generator rank one if it has slow dimension growth or if it tensorially absorbs a UHF-algebra, see Corollary B.4.30.

However, all AH-algebras in the mentioned classes are $\mathcal{Z}$-stable, i.e., they tensorially absorb the Jian-Su algebra $\mathcal{Z}$. Therefore, the partial solution to the generator problem in Corollary B.4.30 was already covered by a more general result in Article C, *The generator problem for $\mathcal{Z}$-stable C^*-algebras*, which is co-authored with Wilhelm Winter. The main result in that paper is Theorem C.3.7, which shows that every unital, separable, $\mathcal{Z}$-stable C^*-algebra is singly generated.

Note that an AF-algebra is $\mathcal{Z}$-stable if (and only if) it has no finite-dimensional representation. This is, however, not enough to immediately deduce that all AF-algebra are singly generated.

1.2. Shape theory for C^*-algebras

In the second part of the thesis, which consists of the Articles D, E and F, we study shape theory for C^*-algebras.

Shape theory is a tool to study global properties of a space by disregarding its local behavior. It agrees with homotopy theory on spaces with good local behaviour, i.e., without singularities, and one usually considers homotopy only for such spaces. One can consider shape theory as the natural extension of homotopy theory from spaces without singularities to general spaces.

One approach to shape theory is to approximate a space by nicer spaces, the building blocks. Then, one studies the original space through its approximating sequence. The building blocks for commutative shape theory are absolute neighborhood retracts (ANRs), and to approximate a (compact, metric) space X by building blocks means to write X as an inverse limit of ANRs. It is a classical result that this is always possible, i.e., that every compact, metric space is an inverse limit of ANRs (even polyhedra).

Noncommutative shape theory was introduced by Effros and Kaminker, [**EK86**], and developed to its modern form by Blackadar, [**Bla85**]. The building blocks are the semiprojective C^*-algebras, which are defined by dualizing the concept of an absolute neighborhood extensor, which is equivalent to the concept of an ANR; the precise definitions are recalled in subsections 2.1 and 2.2 of Article D, and section 2 of Article E. To approximate a C^*-algebra by building blocks means to write it as an inductive limit of semiprojective C^*-algebras. This raises the following question:

QUESTION 1.2.1 (Blackadar). Is every C^*-algebra an inductive limit of semiprojective C^*-algebras?

This problem remains open, although we provide a positive answer for a certain class of C^*-algebras in Article E. To develop a shape theory for all C^*-algebras, Blackadar relaxes the requirements of approximating a C^*-algebra using the notion of a semiprojective *-homomorphism, see e.g. E.2.1. He showed that every separable C^*-algebra can be approximated in this way, see [**Bla85**, Theorem 4.3].

In Theorem 1.2 of Article D, *A characterization of semiprojectivity for commutative C^*-algebras*, which is co-authored with Adam Sørensen, we show that a unital, separable, commutative C^*-algebra is semiprojective if and only if its spectrum is a one-dimensional ANR. This verifies a conjecture of Blackadar.

Both implication of Theorem D.1.2 are non-trivial. Assume first that X is a compact, metric space such that $C(X)$ is semiprojective. It follows from [**Bla85**, Proposition 2.11] that X is an ANR, and it remains to show that $\dim(X) \leq 1$. We show that every compact ANR of dimension at least two contains a copy of one of the three spaces Y_1, Y_2, Y_3 of "smaller and smaller circle", as described in Remark D.3.4. If a space X contains a copy of Y_1, Y_2 or Y_3, then $C(X)$ cannot be semiprojective. The argument is analogous to the proof showing that $C(X)$ is not semiprojective if X contains a copy of the disc. We note that our more involved argument with the spaces of "smaller and smaller circles" is necessary, since a compact ANR of dimension at least two need not contain a copy of the disc, as shown by Bing and Borsuk, [**BB64**].

For the converse implication, we have to show that $C(X)$ is semiprojective if X is a one-dimensional ANR. To establish this result, we prove a structure theorem for compact, one-dimensional ANRs, see Theorem D.4.17.

To extend our characterization to non-unital commutative C^*-algebras, we study the structure of non-compact, one-dimensional ANRs and their compactifications. The structure theorem for such spaces is Theorem D.6.1, and we provide a detailed proof in chapter 2.

In Article E, *Inductive limits of projective C^*-algebras*, we show that a C^*-algebra has trivial shape, i.e., is shape equivalent to the zero C^*-algebra, if and only if it is an inductive limit of projective C^*-algebras, see Theorem E.4.4. This is the noncommutative analog of the well-known fact in commutative shape theory that a (compact, metric) space has trivial shape if and only if it is an inverse limit of absolute retracts (ARs). Every contractible C^*-algebra has trivial shape and is therefore an inductive limit of projective C^*-algebras. This also provides a positive answer to Question 1.2.1 for a large class of C^*-algebras.

The main application of these results is to show that a C^*-algebra is (weakly) projective if and only if it is (weakly) semiprojective and has trivial shape, see Theorem E.5.6. It follows that a C^*-algebra is projective if and only if it is semiprojective and contractible, which confirms a conjecture of Loring, see Corollary E.5.7.

In Article F, *Semiprojectivity with and without a group action*, which is co-authored with Christopher Phillips and Adam Sørensen, we study the notion of equivariant semiprojectivity, which was introduced by Phillips in [**Phi12**]. It is defined by applying the usual definition of semiprojectivity to the category of G-algebras with G-equivariant *-homomorphisms.

We show that equivariant semiprojectivity is preserved when restricting the action to a co-compact subgroup, see Theorem F.3.10. It follows that a compact group can only act semiprojectively on a C^*-algebra that is semiprojective in the usual sense. This is no longer true for non-compact groups. Indeed, we construct an example of a semiprojective action of the group of integers on a C^*-algebra that is not semiprojective in the usual sense, see Example F.3.12.

For a semiprojective action of a finite group on a unital C^*-algebra, we show that the crossed product is semiprojective, see Theorem F.5.1. If the action is also saturated, then the fixed point algebra is Morita equivalent to the crossed product, which implies that it also semiprojective, see Proposition F.6.2. For a semiprojective action of a non-compact group, we show in Theorem F.6.4 that the fixed point algebra is trivial. This allows us to characterize the semiprojectivity of trivial group actions: The trivial action of a group G on a C^*-algebra A is equivariantly semiprojective if and only if G is compact and A is (non-equivariantly) semiprojective, see Corollary F.6.5.

Finally, in Article G, *The Cuntz semigroup and comparison of open projections*, which is co-authored with Eduard Ortega and Mikael Rørdam, we study comparison relations of positive elements in a C^*-algebra in connection to comparison relations of the associated open projections in the universal von Neumann algebra. Let a, b be two positive elements in a C^*-algebra A. We show that a and b are Blackadar equivalent, i.e., that there exists $x \in A$ such that $\overline{aAa} = \overline{xAx^*}$ and $\overline{x^*Ax} = \overline{bAb}$, if and only if the associated support projections $p_a, p_b \in A^{**}$ are equivalent in the sense Peligrad and Zsido, [**PZ00**, Definition 1.1], which in turn happens precisely if $\overline{aA}$ and $\overline{bA}$ are isomorphic as (right) Hilbert A-modules, see Proposition G.4.3.

This inspired us to define a Cuntz comparison relation for open projections, see Definition G.3.9, such that two positive elements a, b are Cuntz equivalent in the usual sense if and only if the associated support projections are Cuntz equivalent, which in turn happens precisely if the Hilbert A-modules $\overline{aA}$ and $\overline{bA}$ are equivalent in the sense of Coward, Elliott, Ivanescu, [**CEI08**]. We thus give a new new picture of the Cuntz semigroup of a C^*-algebra in terms of open projections.

CHAPTER 2

Additional material

In this chapter, we provide additional material for the article in Appendix D, *A characterization of semiprojectivity for commutative C^*-algebras*, which is co-authored with Adam P. W. Sørensen.

2.1. Structure of non-compact, one-dimensional ANRs

The goal of this section is to provide a thorough proof of Theorem Appendix D.6.1, which appears in this section as Theorem 2.1.17, and which shows that if the one-point compactification of a one-dimensional[1], locally compact, separable, metric ANR is an ANR, then so is every finite-point compactification. This result was obtained together with Adam Sørensen, and it is certainly known to experts of the field, although we could not locate it in the literature.

To obtain Theorem 2.1.17, we carry out a detailed study of the structure of non-compact, one-dimensional ANRs and their compactifications. The result is applied in Appendix D to study the structure of non-unital, commutative, semiprojective C^*-algebras. It allows one to extend the characterization of semiprojectivity for commutative C^*-algebra from the unital to the non-unital setting.

We first recall some basic notions. For more details on continuum theory, we refer the reader to Nadler's book, [**Nad92**].

2.1.1. A *continuum* is a compact, connected, metric space, and a *generalized continuum* is a locally compact, connected, metric space. A *Peano continuum* is a locally connected continuum, and a *generalized Peano continuum* is a locally connected generalized continuum. By a *finite graph* we mean a graph with finitely many vertices and edges, or equivalently a compact, one-dimensional CW-complex. By a *finite tree* we mean a contractible finite graph.

2.1.2 (Theory of ends). We briefly recall the basics of the theory of ends. The definitions and results with proofs can all be found in the survey of Eilers, [**Eil94**]. To get some intuition, note that the space $[0, 1)$ has one end, while $(0, 1)$ has two ends.

A topological space X is called a *Raum* if it is connected, locally connected, locally compact, σ-compact and Hausdorff. We will mostly consider separable, metric Räume[2], and these are exactly the separable, generalized Peano continua. A decreasing sequence of non-empty, open, connected subsets G_k of a Raum X is said to *determine an end* of X if each $\partial(G_k)$ is compact and $\bigcap_{k\geq 1} \overline{G_k} = \emptyset$. If $(G_k)_k$ and $(H_k)_k$ are two sequences determining ends, then the following conditions are equivalent:

(1) for all k: $G_k \cap H_k \neq \emptyset$,

[1] We say that a space is one-dimensional if $\dim(X) \leq 1$. So, although it sounds weird, a one-dimensional space can also be zero-dimensional. It would probably be more precise to speak of "at most one-dimensional" space, however the usage of the term "one-dimensional space" is well established.

[2] Räume is the plural of Raum.

(2) for all k there exists some l such that $G_l \subset H_k$,
(3) for all k there exists some l such that $H_l \subset G_k$.

If $(G_k)_k$ and $(H_k)_k$ satisfy these conditions, then they are called equivalent, which is denoted by $(G_k)_k \approx (H_k)_k$. This defines an equivalence relation, and each equivalence class is called an *end* of X. We denote the set of ends of X by $E(X)$.

2.1.3 (Compactifications). A compactification of a space X is a pair (Y, ι_Y) where Y is a compact space, $\iota\colon X \to Y$ is an embedding and $\iota(X)$ is dense in Y. Usually the embedding is understood and one denotes a compactification just by the space Y.

A non-compact space can have many different compactifications, and one usually restricts attention to Hausdorff compactifications, which exist precisely if X is completely regular[3]. Given two compactifications (Y, ι_Y) and (Z, ι_Z) of X we write $(Y, \iota_Y) \geq (Z, \iota_Z)$ if there exists a surjective map $\theta : Y \to Z$ such that $\theta \circ \iota_Y = \iota_Z$. This defines a partial order on the Hausdorff compactifications of X. The Stone-Čech compactification βX is maximal with respect to this order. The one-point compactification αX is Hausdorff if and only if X locally compact, and in that case αX is the smallest Hausdorff compactification of X. The Stone-Čech and one-point compactification are functors in the sense that every map $f\colon X \to Y$ has a (unique) extension $\beta f\colon \beta X \to \beta Y$ and $\alpha f\colon \alpha X \to \alpha Y$.

2.1.4 (Freudenthal compactification). Let X be a Raum. There is a topology on $F(X) := X \cup E(X)$ making it into a compact, Hausdorff space such that the natural inclusion $X \subset F(X)$ is an embedding. This is called the *Freudenthal compactification* of X. Its remainder $F(X) \setminus X$ is zero-dimensional, and the Freudenthal compactification is the largest compactification with zero-dimensional remainder.

A proper[4] map $f\colon X \to Y$ extends naturally to a map $F(f)\colon (F(X), E(X)) \to (F(Y), E(Y))$. If two proper maps $f, g\colon X \to Y$ are proper homotopic, then the induced maps $F(f), F(g)\colon (F(X), E(X)) \to (F(Y), E(Y))$ are homotopic. Since $E(X)$ is zero-dimensional, this implies that $F(f)$ and $F(g)$ agree on $E(X)$. It follows that two properly homotopy equivalent spaces X, Y have homeomorphic spaces of ends $E(X)$ and $E(Y)$.

In the next result 2.1.5 we give a concrete realization of the homeomorphism $E(X) \cong E(Y)$ in the case of a proper strong deformation retraction. The result is certainly known, but we could not find it in the literature.

Proposition 2.1.5. *Let X, Y be two Räume, and let $r\colon X \to Y \subset X$ be a proper strong deformation retract. Then, the ends of X and Y are in natural one-one-correspondence via the map $\Phi\colon E(Y) \to E(X)$ that sends $[(G_k)_k]$ to $[(r^{-1}(G_k))_k]$.*

Proof. In this proof, we will use that $\overline{f^{-1}(A)} \subset f^{-1}(\overline{A})$ for a continuous map f.

Let $(G_k)_k$ be a sequence determining an end in Y. Let us check that $(r^{-1}(G_k))_k$ determines an end in Y:

- $r^{-1}(G_k)$ is non-empty, since $r(r^{-1}(G_k)) = G_k$ is non-empty,
- $r^{-1}(G_k)$ is open, since r is continuous,
- $r^{-1}(G_k)$ is connected, since r is a homotopy equivalence,
- To see that $\partial(r^{-1}(G_k))$ is compact, consider the following computation:

$$\partial(r^{-1}(G_k)) = \overline{r^{-1}(G_k)} \cap \overline{r^{-1}(G_k^c)} \subset r^{-1}(\overline{G_k}) \cap r^{-1}(\overline{G_k^c}) = r^{-1}(\partial G_k).$$

[3]A space is completely regular if it is Hausdorff and $T_{3\frac{1}{2}}$, i.e., any disjoint point and closed subset are functionally separated. In the literature, completely regular spaces are sometimes called "Tychonoff" or "completely T_3-space". Some authors also use completely regular and $T_{3\frac{1}{2}}$ in the exact opposite way as we do here.

[4]A map $f : X \to Y$ is called proper if preimages of compact sets are compact.

Since r is proper, the preimage $r^{-1}(\partial G_k)$ of the compact set ∂G_k is compact again. Then, the closed subset $\partial(r^{-1}(G_k)) \subset r^{-1}(\partial G_k)$ is also compact.

- Since G_k is a decreasing sequence, so is $r^{-1}(G_k)$.
- Lastly, let us check that $\bigcap_{k\geq 1}\overline{r^{-1}(G_k)}$ is empty:

$$\bigcap_{k\geq 1}\overline{r^{-1}(G_k)} \subset \bigcap_{k\geq 1} r^{-1}(\overline{G_k}) = r^{-1}(\bigcap_{k\geq 1}\overline{G_k}) = \emptyset.$$

Given sequences $(G_k)_k$ and $(H_k)_k$ that determine ends in Y, we have:

$$\begin{aligned}(G_k)_k \approx (H_k)_k &\Leftrightarrow \forall k\ :\ G_k \cap H_k \neq \emptyset \\ &\Leftrightarrow \forall k\ :\ r^{-1}(G_k)\cap r^{-1}(H_k) = r^{-1}(G_k\cap H_k) \neq \emptyset \\ &\Leftrightarrow (r^{-1}(G_k))_k \approx (r^{-1}(H_k))_k.\end{aligned}$$

This shows that Φ is well-defined and injective.

It remains to show that Φ is surjective. Given a sequence $(G_k)_k$ determining an end in X, it is not necessarily true that $(r(G_k))_k$ is a sequence determining an end in Y (for instance, $r(G_k)$ might not be open). Therefore, we use another approach to the theory of ends, as developed by Ball, [**Bal75**]. One considers sequences of points in X, and calls such a sequence $\mathbf{x} = (x_l)_l$ admissible if:

(i) no subsequence of $\mathbf{x}$ converges in X,
(ii) no compact subset of X separates two infinite subsequences of $\mathbf{x}$.

When considered in the Freudenthal compactification $F(X)$, each admissible sequence converges to a unique point in $E(X) = F(X)\setminus X$. Moreover, each point in $E(X)$ is the limit of an admissible sequence. Consider an end $e = [(G_k)_k] \in E(X)$. Then an admissible sequence $\mathbf{x}$ converges to e if and only if for each k the sequence $\mathbf{x}$ is eventually in G_k. Let us denote this by $\mathbf{x} \approx (G_k)_k$. In this way, the ends of X can naturally be identified with equivalence classes of admissible sequences in X.

A proper map $f : X \to Y$ sends an admissible sequence $\mathbf{x} = (x_l)_l$ in X to an admissible sequence $f(\mathbf{x}) = (f(x_l))_l$ in Y. This induces a continuous map $E(X) \to E(Y)$, and this is the same as the restriction to $E(X)$ of the natural extension $F(f)\colon F(X) \to F(Y)$ that was mentioned in 2.1.4.

In our situation, let $(G_k)_k$ be a sequence determining an end in X. We want to find a sequence $(H_l)_l$ determining an end in Y such that $(G_k)_k \approx (r^{-1}(H_l))_l$. Let $\mathbf{x}$ be an admissible sequence in X with $\mathbf{x} \approx (G_k)_k$. Consider the admissible sequence $r(\mathbf{x})$ in Y. It corresponds to an end of Y, and so there exists a sequence $(H_l)_l$ determining an end of Y with $r(\mathbf{x}) \approx (H_l)_l$. This means that the sequence $r(\mathbf{x})$ eventually lies in each set H_l. It follows that the sequence $\mathbf{x}$ eventually lies in each set $(r^{-1}(H_l))$, or put differently $\mathbf{x} \approx (r^{-1}(H_l))_l$. Then $(G_k)_k \approx \mathbf{x} \approx (r^{-1}(H_l))_l$. This shows that Φ is surjective. □

2.1.6 (Finite-point compactifications). Let X be a space. A compactification $\gamma(X)$ of X is called a **finite-point compactification** if the remainder $\gamma(X)\setminus X$ is finite. Since we only want to work with Hausdorff compactifications, we will restrict our attention to locally compact, Hausdorff spaces. Finite-point compactifications preserve many topological properties. Let us discuss some facts that will be used later:

- **Connectedness**: Every finite-point compactification of a connected space is again connected.

  Indeed, assume $A \subset \gamma(X)$ is clopen. Then $A\cap X \subset X$ is clopen. Since X is connected, either $A \cap X = \emptyset$ or $A\cap X = X$. We may assume $A\cap X = \emptyset$ (otherwise consider $\gamma(X)\setminus A$ instead of A). Then $A \subset (\gamma(X)\setminus X)$. This can only be open in

$\gamma(X)$ if $A = \emptyset$, which shows that $\gamma(X)$ contains no non-trivial clopen sets, hence is connected.

- **Local connectedness:** Every finite-point compactification of a connected, locally connected space is again locally connected (and connected).

  This follows from [**dGM67**, Theorem 4.1], which states the following: Let $\kappa(X)$ be a compactification of a connected, locally connected space X. If $\kappa(X)\setminus X$ contains no continuum consisting of more than one point, then $\kappa(X)$ is locally connected. The condition holds in particular for finite-point compactifications.

- **Dimension**: Every finite-point compactification $\gamma(X)$ of a locally compact, Hausdorff space X satisfies $\dim(X) = \dim(\gamma(X))$.

  This follows from a standard argument in dimension theory. For instance, we may use that $\dim(Y) = \max(\dim(U), \dim(Y \setminus U))$ for a normal space Y with $U \subset Y$ an open subset, see e.g. [**Nag70**, Theorem 9.11, p.54]. For a finite-point compactification $\gamma(X)$, X is an open subset of $\gamma(X)$, and $\gamma(X)$ is compact and Hausdorff, hence normal. Further, $\gamma(X) \setminus X$ is finite, and therefore $\dim(\gamma(X) \setminus X) = 0$. Thus, $\dim(X) = \dim(\gamma(X))$.

- **Metrizability**: Every finite-point compactification of a locally compact, separable, metric space is again metrizable.

  Let $\gamma(X)$ be a finite-point compactification of a locally compact, separable, metric space X. A metric space is separable if and only if is second-countable. Thus, there exists a countable basis $\{U_k\}$ for the topology of X. A second countable, locally compact, Hausdorff space is σ-compact, i.e., there exists an increasing sequence of compact sets $K_1 \subset K_2 \subset \ldots X$ with $\bigcup_i K_i = X$. Then, the following forms a countable basis for the topology of $\gamma(X)$:

  $$\{U_k\} \cup \{(X \setminus K_i) \cup F \mid i \geq 1, F \subset (\gamma(X) \setminus X)\}.$$

  Thus, $\gamma(X)$ is second-countable, and therefore metrizable by Urysohn's metrization theorem.

Putting all these facts together, we obtain the following:

LEMMA 2.1.7. *Let X be a one-dimensional, locally compact, locally connected, connected, separable, metric space. Then every finite-point compactification of X is a one-dimensional Peano continuum.*

2.1.8 (Docility at infinity). It is a natural question, when the Freudenthal compactification of an ANR is again an ANR. This was studied by Sher, [**She76**], who defined a space X to be *contractible at infinity*, if for each compact set $A \subset X$ there exists a compact set $B \subset X$ such that each component of $X \setminus B$ is contractible in $X \setminus A$. For ANRs, this is equivalent with several other natural conditions, and Sher calls an ANR *docile at infinite* if it satisfies these conditions.

The main result of [**She76**] says: An ANR X is docile at infinity if and only if $F(X)$ is an ANR and $E(X)$ is unstable in $F(X)$. We will see below that a one-dimensional ANR X is docile at infinite if and only if it has only "dendritic ends" (see Definition 2.1.12 below).

2.1.9 (Dendrites). A *dendrite* is a Peano continuum that does not contain a simple closed curve (i.e., there is no embedding of the circle S^1 into it). Every dendrite is one-dimensional. Let us recall some of the many equivalent characterizations of a dendrite.

Let X be a Peano continuum. Then X is a dendrite if and only if one (or equivalently all) of the following conditions holds:

(1) X is one-dimensional and contractible.
(2) X is tree-like. (A compact, metric space X is called tree-like if for every $\varepsilon > 0$ there exists a finite tree T and a map $f\colon X \to T$ onto T such that $\operatorname{diam}(f^{-1}(y)) < \varepsilon$ for all $y \in T$.)
(3) X is dendritic, i.e., any two points of X are separated by the omission of a third point, see 2.1.10 below.
(4) X is hereditarily unicoherent. (A continuum X is called unicoherent if for each two subcontinua $Y_1, Y_2 \subset X$ with $X = Y_1 \cup Y_2$, the intersection $Y_1 \cap Y_2$ is a continuum, i.e., connected. A continuum is called hereditarily unicoherent if all its subcontinua are unicoherent.)

For more information about dendrites see [**Nad92**, Chapter 10], [**Lel76**], [**CC60**].

2.1.10 (Dendritic spaces). A connected space is called *dendritic* if each pair of distinct points can be separated by the omission of some third point. A dendritic Peano continuum is a dendrite, see 2.1.9. Similarly to the case of a dendrite, there are several equivalent characterizations when a generalized Peano continuum is dendritic.

Let X be a generalized Peano continuum. Then X is dendritic if and only if one (or equivalently all) of the following conditions holds:

(1) X contains no simple closed curve.
(2) X is one-dimensional and contractible.
(3) X is hereditarily unicoherent.

For the proofs and further results see [**FQ06**].

It is possibly false that connected subsets of dendritic spaces are again dendritic but we are not aware of any published counterexample. However, if X is a connected, locally connected, locally compact, Hausdorff spaces, then Ward, [**War58**, Corollary to Theorem 3], proved that X is dendritic if and only if each subcontinuum of X is a dendrite. We obtain the following lemma.

LEMMA 2.1.11 (see [**War58**]). *Let X be a dendritic Raum. Then every connected, open subset of X is dendritic.*

PROOF. Assume X is a dendritic Raum. A Raum is in particular connected, locally connected, locally compact and Hausdorff, so by the result of Ward each subcontinuum of X is a dendrite. Let $Y \subset X$ be a connected, open subset. Then Y is as also locally connected, locally compact and Hausdorff. Every subcontinuum of Y is also a subcontinuum of X and therefore a dendrite. Using Ward's result in the converse direction we get that Y is dendritic. □

This implies the following: If $(G_k)_k$ is determining an end, and some G_k is dendritic, then all $G_k, G_{k+1}, \dots$ are dendritic, i.e., the open sets determining the end are eventually dendritic. Further, if $(H_k)_k$ is another sequence determining an end, and $(G_k)_k \approx (H_k)_k$, then the sets of H_k are eventually dendritic as well. This justifies the following definition:

DEFINITION 2.1.12. Let X be a Raum, and $(G_k)_k$ a sequence determining an end of X. Then $(G_k)_k$ is called *dendritic* if there exists some k such that G_k is dendritic. An end is called dendritic if one (or equivalently all) of its representatives are dendritic.

PROPOSITION 2.1.13. *Let X be a one-dimensional, connected, locally compact, separable, metric ANR. Then the following are equivalent:*

(1) X is contractible at infinity,
(2) all ends of X are dendritic.

By [**She76**, Theorem 4.2]*, the above conditions are also equivalent to $F(X)$ being an ANR with $E(X) \subset F(X)$ unstable.*

Proof. "(1) $\Rightarrow$ (2)": Assume X is contractible at infinity, and let $(G_k)_k$ be a sequence determining an end in X. Consider the compact set $A = \emptyset$. There exists another compact set $B \subset X$ such that each component of $X \setminus B$ is contractible in $X \setminus A = X$. Since $\bigcap_{k \geq 1} \overline{G_k} = \emptyset$, there exists some k such that the connected set G_k is contained in $X \setminus B$, and therefore contractible in X. As shown by Cannon and Conner, [**CC06**, Corollary 3.3], the inclusion $G_k \subset X$ induces an injective map on fundamental groups. It follows that the fundamental group of G_k is trivial, which implies that no circle embeds into G_k. As mentioned in 2.1.10, this implies that G_k is dendritic.

"(2) $\Rightarrow$ (1)": Let $A \subset X$ be compact. For each end $e \in E(X)$, choose a sequence $(G_k^{(e)})_k$ with $e = [(G_k^{(e)})_k]$. By passing to subsequences we may assume that all $G_k^{(e)}$ are dendritic and disjoint from A. Each set $G_k^{(e)}$ naturally defines an open set $\widetilde{G}_k^{(e)} \subset F(X)$ as follows:

$$\widetilde{G}_k^{(e)} = G_k^{(e)} \cup \{[(H_l)_l] \in E(X) \mid \text{ eventually } H_l \subset G_k^{(e)}\}.$$

The open sets $\widetilde{G}_1^{(e)}$ (for $e \in E(X)$) form an open cover containing $E(X)$. Since $E(X)$ is compact, there exists a finite finite set $I \subset E(X)$ such that the subcover $\{\widetilde{G}_1^{(e)} \mid e \in I\}$ still covers $E(X)$. Set $B := X \setminus \left(\bigcup_{e \in I} \widetilde{G}_1^{(e)}\right)$. This is a compact set, and $X \setminus B$ has dendritic components which are therefore already contractible in themselves. □

We will use the following result several times in proofs below. See [**Bor67**] for related and more general results.

Proposition 2.1.14 (Borsuk, [**Bor32**, Satz 9], see also [**Bor67**, IV.6.1, p.90]). *Let A be a space with closed subsets $A_1, A_2 \subset A$ such that $A = A_1 \cup A_2$. Set $A_0 := A_1 \cap A_2$. If A_1, A_2 and A_0, are ANRs, then so is A.*

Lemma 2.1.15. *Let X be an ANR, and let $F \subset X$ be a finite subset. Then the quotient space X/F is an ANR.*

Proof. Let $F = \{x_1, \ldots, x_k\}$, and let $f : X \to Y$ be the quotient map. Set $y_0 := f(x_i)$, the collapsed point. We will use the following theorems of Hanner, [**Han51**], see also [**Bor67**, IV.10., p.96f]:

- First theorem of Hanner: Every open subset of an ANR is again an ANR.
- Second theorem of Hanner: If $X = \bigcup_{k \in \mathbb{N}} G_k$, for open sets $G_k \subset X$, and each G_k is an ANR, then so is X.

It follows that the open set $X \setminus F$ is an ANR. Note that $X \setminus F \cong Y \setminus \{y_0\}$. Therefore, the open subset $Y \setminus \{y_0\} \subset Y$ is an ANR.

For each i, we may find a neighborhood $U_i \subset X$ of x_i such that the U_i are pairwise disjoint. Set $V_i := f(U_i)$ and note that in fact $V_i \cong U_i$ since U_i contains only one of the collapsed points. Note that $V_i \subset Y$ need not be open. However, the set $V := V_1 \cup \ldots \cup V_k$ will be an open neighborhood of y_0. Moreover, each set V_i is a closed subset of V.

Let us see that V is an ANR. This follows from iterated application of [**Bor32**, Satz 9], see 2.1.14. To start, note that $V_1 \cap V_2 = \{y_0\}$, and so $V_1 \cup V_2$ is an ANR. For $i < k$, assume $V_1 \cup \ldots \cup V_i$ is an ANR. Since $(V_1 \cup \ldots \cup V_i) \cap V_{i+1} = \{y_0\}$, it follows that

$V_1 \cup \ldots \cup V_{i+1}$ is an ANR. By induction, V is an ANR. Then, by the second theorem of Hanner, $Y = (Y \setminus \{y_0\}) \cup V$ is an ANR. □

LEMMA 2.1.16. *Let X be a one-dimensional, connected, locally compact, separable, metric ANR. Then the following are equivalent:*

(1) The one-point compactification αX is an ANR.
(2) X has only finitely many ends, and each end is dendritic.
(3) Every finite-point compactification of X is ANR.

PROOF. The Lemma holds if X is compact, for then it has no compactifications other than X, and it also has no ends. So assume from now on that X is non-compact. By 2.1.7, every finite-point compactification of X is a one-dimensional Peano continuum.

"(1) ⇒ (2)": Assume αX is an ANR, and denote the attached point at infinity by x_∞. By Theorem 4.12 in Appendix D, the core of αX is a finite graph. Let $Y \subset \alpha X$ be a finite graph that contains $\mathrm{core}(\alpha X)$ and x_∞. Such a finite graph always exists: If $x_\infty \in \mathrm{core}(\alpha X)$, then simply use $Y = \mathrm{core}(\alpha X)$. Otherwise there is an arc A connecting x_∞ to the core and one may use $Y = \mathrm{core}(\alpha X) \cup A$.

By Proposition 4.16 in Appendix D, there is a strong deformation retract $r\colon \alpha X \to Y$. Since X is connected, $r^{-1}(x_\infty) = \{x_\infty\}$. Set $Y_0 := Y \setminus \{x_\infty\}$. Then r restricts to a proper strong deformation retract from X onto Y_0. By Proposition 2.1.5, this identifies the ends of Y_0 and X via the map $\Phi\colon [(G_k)_k] \mapsto [(r^{-1}(G_k))_k]$.

Since Y is a finite graph, Y_0 has only finitely many ends. It follows that also X has only finitely many ends. Further, each end of X has a representative $r^{-1}(G_k)$, where $(G_k)_k$ is a sequence determining an end of Y_0. The ends of Y_0 are easily understood and for large enough k, $G_k \simeq (0,1)$, the open interval. Since r is a homotopy equivalence, and G_k is contractible, so is $r^{-1}(G_k)$. It follows that $r^{-1}(G_k)$ is dendritic.

"(2) ⇒ (3)": It follows from [**She76**, Theorem 4.2], see Proposition 2.1.13, that $F(X)$ is an ANR. Let $\gamma(X)$ be a finite-point compactification. Since $F(X)$ is the largest compactification with zero-dimensional remainder, there is a unique surjective map $\varphi\colon F(X) \to \gamma(X)$, which restricts to the identity from $X \subset F(X)$ to $X \subset \gamma(X)$. Since $E(X)$ is finite, the space $\gamma(X)$ can be obtained from $F(X)$ by the successive collapsing of finitely points (namely, for each $y \in \gamma(X) \setminus X$, the points $\varphi^{-1}(y)$ get identified). It follows from successive application of 2.1.15 that $\gamma(X)$ is an ANR.

Finally, the implication "(3) ⇒ (1)" is clear. □

Next, we remove the connectedness assumption in Lemma 2.1.16, and thus obtain the main result of this section.

THEOREM 2.1.17. *Let X be a one-dimensional, locally compact, separable, metric ANR. Then the following are equivalent:*

(1) The one-point compactification αX is an ANR.
(2) X has only finitely many compact components and also only finitely many components $C \subset X$ such that αC is not a dendrite.
(3) Every finite-point compactification of X is an ANR.
(4) Some finite-point compactification of X is an ANR.

PROOF. "(1) ⇒ (2)": Assume αX is an ANR. Theorem 4.2 of [**dGM67**] states the following: A locally connected, rim-compact (e.g. locally compact) Hausdorff space Y has a locally connected compactification if and only if at most finitely many of the components of Y are compact. Consequently, since αX is an ANR and every ANR is locally connected, X has only finitely many compact components.

Let us look at the non-compact components of X. By [**Jac52**, Theorem 2], a separable ANR has at most countably many components, each of which is open in X and an ANR itself. Thus, each component of X is a one-dimensional ANR. It follows from Lemma 2.1.7 that the one-point compactification of any component of X is a one-dimensional Peano continuum.

We consider X embedded into αX. Let x_∞ be the point at infinity. Since αX is ANR, it is contractible at x_∞, and so there is a contractible neighborhood U of x_∞.Then all except finitely many components of X lie in U. Let $C \subset X$ be a component of X that lies in U. We want to show that its one-point compactification αC is a dendrite. Since C is non-compact, $C \cup \{x_\infty\} \subset \alpha X$ is homeomorphic to αC, which is a one-dimensional Peano continuum. It follows that αC is a dendrite, since $\alpha C \subset U$ and U is contractible.

"(2) $\Rightarrow$ (3)": Let $\gamma(X)$ be a finite-point compactification of X. Since X has only finitely many compact components, $\gamma(X)$ has only finitely many components (namely the compact components of X and for each point in $\gamma(X) \setminus X$ at most one other component). By the mentioned result of Jackson, [**Jac52**, Theorem 2], $\gamma(X)$ is an ANR if and only if each of its finitely many components is ANR. Therefore, without loss of generality, we may assume from now one that $\gamma(X)$ is connected. If X was (connected) compact, then $\gamma(X) = X$ and there is nothing to show, so that we may also assume X has no compact components.

Let $Y_1, \ldots, Y_n$ be the finitely many (non-compact) components of X whose one-point compactification is not a dendrite, and let D_j $(j \in J)$ be the other (non-compact) components of X. For each $i = 1, \ldots, n$, consider the closure $\overline{Y_i} \subset \gamma(X)$. This is a finite-point compactification of Y_i, and therefore, by Lemma 2.1.16, each $\overline{Y_i}$ is an ANR.

For each $j \in J$, the one-point compactification of D_j is a dendrite. Therefore, the closure $\overline{D_j} \subset \gamma(X)$ is homeomorphic to αD_j and so $\overline{D_j} \cap (\gamma(X) \setminus X)$ contains exactly one point.

Let $y_1, \ldots, y_m$ be the points in $\gamma(X) \setminus X$. For each $k = 1, \ldots, m$, define:

$$J_k := \{j \in J \mid x_k \in \overline{D_j}\}.$$

The sets $J_1, \ldots, J_m$ are disjoint with $J = J_1 \cup \ldots \cup J_m$. For each $k = 1, \ldots, m$, consider the set

$$E_k := \bigcup_{j \in J_k} D_j.$$

The closure of E_k in $\gamma(X)$ is $\overline{E_k} = E_k \cup \{x_k\}$. Let us check that this is a dendrite. It is enough to show it is dendritic, i.e., that any two different points $x, y \in \overline{E_k}$ can be separated by the omission of a third point. We show this by considering two cases:

- **Case 1:** There exists some j such that both points lie in $\overline{D_j}$ (possibly, one of the points is x_∞). In that case we use that $\overline{D_j}$ is dendritic, so that x and y can be separated by the omission of some point in $\overline{D_j}$.
- **Case 2:** There does not exist some j such that $x, y \in \overline{D_j}$. In that case both points are different from x_∞, and they can be separated by the omission of x_∞.

Finally, we have

$$\gamma(X) = \overline{D_1} \cup \ldots \cup \overline{D_k} \cup \overline{E_1} \cup \ldots \cup \overline{E_l}.$$

Note that these sets only intersection within the finite set $\gamma(X) \setminus X$, so that by iterated application of Proposition 2.1.14 we get that $\gamma(X)$ is an ANR.

The implication "(3) $\Rightarrow$ (4)" is clear.

"(4) $\Rightarrow$ (1)": Let $\gamma(X)$ be a finite-point compactification of X that is an ANR. There is a unique surjective map $\varphi\colon \gamma(X) \to \alpha X$, which restricts to the identity from $X \subset \gamma(X)$ to $X \subset \alpha X$ and otherwise collapses all points in $\gamma(X) \setminus X$ to one point. It follows from Lemma 2.1.15 that αX is an ANR. $\square$

Articles

On the following pages, we have attached the seven articles that the author has written or co-authored during his PhD.

A. The topological dimension of type I C^*-algebras.
The article is included on pages 19 - 38. A preprint of an earlier version from October 2012 is also publicly available at `arxiv.org/abs/1210.4314`.

B. The generator rank for C^*-algebras.
The article is included on pages 39 - 66. A preprint from October 2012 is also publicly available at `arxiv.org/abs/1210.6608`.

C. The generator problem for $\mathcal{Z}$-stable C^*-algebras.
The article is co-authored with Wilhelm Winter, and it will appear in Trans. Am. Math. Soc. It is included on pages 67-81. A preprint from January 2012 is also publicly available at `arxiv.org/abs/1201.3879`.

D. A characterization of semiprojectivity for commutative C^*-algebras.
The article is co-authored with Adam P.W. Sørensen, and it will appear in Proc. Lond. Math. Soc. It is included on pages 83 - 108. A preprint of an earlier version from January 2011 is also publicly available at `arxiv.org/abs/1101.1856`.

E. Inductive limits of projective C^*-algebras.
The article is included on pages 109 - 131. A preprint of an earlier version from May 2011 is also publicly available at `arxiv.org/abs/1105.1979`.

F. Semiprojectivity with and without a group action.
The article is co-authored with N. Christopher Phillips and Adam P.W. Sørensen, and it is included on pages 133 - 166.

G. The Cuntz semigroup and comparison of open projections.
The article is co-authored with Eduard Ortega and Mikael Rørdam, and it appeared in J. Funct. Anal. **260** (2011), 3474-3493. It is included on pages 167 - 186. A preprint of an earlier version from August 2010 is also publicly available at `arxiv.org/abs/1008.3497`.

THE TOPOLOGICAL DIMENSION OF TYPE I C^*-ALGEBRAS

HANNES THIEL

Abstract. While there is only one natural dimension concept for separable, metric spaces, the theory of dimension in noncommutative topology ramifies into different important concepts. To accommodate this, we introduce the abstract notion of a noncommutative dimension theory by proposing a natural set of axioms. These axioms are inspired by properties of commutative dimension theory, and they are for instance satisfied by the real and stable rank, the decomposition rank and the nuclear dimension.

We add another theory to this list by showing that the topological dimension, as introduced by Brown and Pedersen, is a noncommutative dimension theory of type I C^*-algebras. We also give estimates of the real and stable rank of a type I C^*-algebra in terms of its topological dimension.

1. Introduction

The covering dimension of a topological space is a natural concept that extends our intuitive understanding that a point is zero-dimensional, a line is one-dimensional etc. While there also exist other dimension theories for topological spaces (e.g., small and large inductive dimension), they all agree for separable, metric spaces.

This is in contrast to noncommutative topology where the concept of dimension ramifies into different important theories, such as the real and stable rank, the decomposition rank and the nuclear dimension. Each of these concepts has been studied in its own right, and they have applications in many different areas. A low dimension in each of these theories can be considered as a regularity property, and such regularity properties play an important role in the classification program of C^*-algebras, see [Rør06], [ET08], [Win12] and the references therein.

In Section 3 of this paper we introduce the abstract notion of a noncommutative dimension theory as an assignment $d\colon \mathcal{C} \to \overline{\mathbb{N}}$ from a class of C^*-algebras to the extended natural numbers $\overline{\mathbb{N}} = \{0, 1, 2, \ldots, \infty\}$ satisfying a natural set of axioms, see Definition 1. These axioms are inspired by properties of the theory of covering dimension, see Remark 1, and they hold for the theories mentioned above. Thus, the proposed axioms do not define a unique dimension theory of C^*-algebras, but rather they collect the essential properties that such theories (should) satisfy.

Besides the very plausible axioms (D1)-(D4), we also propose (D5) which means that the property of being at most n-dimensional is preserved under approximation by sub-C^*-algebras, see 3. This is the noncommutative analog of the notion of "likeness", see 4 and [Thi11, 3.1 - 3.3]. This axiom implies that dimension does not increase when

Date: 25 October 2012.

2010 *Mathematics Subject Classification.* Primary 46L05, 46L85; Secondary 54F45, 55M10.

Key words and phrases. C^*-algebras, dimension theory, stable rank, real rank, topological dimension, type I C^*-algebras.

This research was supported by the Danish National Research Foundation through the Centre for Symmetry and Deformation.

passing to the limit of an inductive system of C^*-algebras, i.e., $d(\varinjlim A_i) \leq \liminf d(A_i)$, see Proposition 2.

Finally, axiom (D6) says that every *separable* sub-C^*-algebra $C \subset A$ is contained in a *separable* sub-C^*-algebra $D \subset A$ such that $d(D) \leq d(A)$. This is the noncommutative analog of Mardešić's factorization theorem, which says that every map $f\colon X \to Y$ from a compact space X to a compact, *metrizable* space Y can be factorized through a compact, *metrizable* space Z with $\dim(Z) \leq \dim(Y)$, see Remark 1 and [Nag70, Corollary 27.5, p.159] or [Mar60, Lemma 4].

In Section 4 we show that the topological dimension as introduced by Brown and Pedersen, [BP09], is a dimension theory in the sense of Definition 1 for the class of type I C^*-algebras. The idea of the topological dimension is to simply consider the dimension of the primitive ideal space of a C^*-algebra. This will, however, run into problems if the primitive ideal space is not Hausdorff. One therefore has to restrict to (locally closed) Hausdorff subsets, and taking the supremum over the dimension of these Hausdorff subsets defines the topological dimension, see Definition 4.

In Section 5 we show how to estimate the real and stable rank of a type I C^*-algebra in terms of its topological dimension.

Section 5 of this article is based on the diploma thesis of the author, [Thi09], which was written under the supervision of Wilhelm Winter at the University of Münster in 2009. Sections 3 and 4 are based upon unpublished notes by the author for the master-class "The nuclear dimension of C^*-algebras", held at the University of Copenhagen in November 2011.

2. Preliminaries

We denote by $\mathcal{C}^*$ the category of C^*-algebras with *-homomorphism as morphisms. In general, by a morphism between C^*-algebras we mean a *-homomorphism.

We write $J \lhd A$ to indicate that J is an ideal in A, and by an ideal of a C^*-algebra we understand a closed, two-sided ideal. Given a C^*-algebra A, we denote by A_+ the set of positive elements. We denote the minimal unitization of A by $\widetilde{A}$. The primitive ideal space of A will be denoted by $\mathrm{Prim}(A)$, and the spectrum by $\widehat{A}$. We refer the reader to Blackadar's book, [Bla06], for details on the theory of C^*-algebras.

If $F, G \subset A$ are two subsets of a C^*-algebra, and $\varepsilon > 0$, then we write $F \subset_\varepsilon G$ if for every $x \in F$ there exists some $y \in G$ such that $\|x - y\| < \varepsilon$. Given elements a, b in a C^*-algebra, we write $a =_\varepsilon b$ if $\|a - b\| < \varepsilon$. Given $a, b \in A_+$, we write $a \ll b$ if b acts as a unit for a, i.e., $ab = a$, and we write $a \ll_\varepsilon b$ if $ab =_\varepsilon a$.

We denote by M_k the C^*-algebra of k-by-k matrices, and by $\mathbb{K}$ the C^*-algebra of compact operators on an infinite-dimensional, separable Hilbert space. We denote by $\overline{\mathbb{N}} = \{0, 1, 2, \ldots, \infty\}$ the extended natural numbers.

1. As pointed out in [Bla06, II.2.2.7, p.61], the full subcategory of commutative C^*-algebras is dually equivalent to the category $\mathcal{SP}_*$ whose objects are pointed, compact Hausdorff spaces and whose morphisms are pointed, continuous maps.

For a locally compact, Hausdorff space X, let αX be its one-point compactification. Let X^+ be the space with one additional point x_∞ attached, i.e., $X^+ = X \sqcup \{x_\infty\}$ if X is compact, and $X^+ = \alpha X$ if X is not compact. In both cases, the basepoint of X^+ is the attached point x_∞.

2. Let X be a space, and let $\mathcal{U}$ be a cover of X. The *order* of $\mathcal{U}$, denoted by $\operatorname{ord}(\mathcal{U})$, is the largest integer k such that some point $x \in X$ is contained in k different elements of $\mathcal{U}$ (and $\operatorname{ord}(\mathcal{U}) = \infty$ if no such k exists). The *covering dimension* of X, denoted by $\dim(X)$, is the smallest integer $n \geq 0$ such that every finite, open cover of X can be refined by a finite, open cover that has order at most $n+1$ (and $\dim(X) = \infty$ if no such n exists). We refer the reader to chapter 2 of Nagami's book [Nag70] for more details.

It was pointed out by Morita, [Mor75], that in general this definition of covering dimension should be modified to consider only *normal*, finite, open covers. However, for normal spaces (e.g. compact spaces) every finite, open cover is normal, so that we may use the original definition.

The *local covering dimension* of X, denoted by $\operatorname{locdim}(X)$, is the smallest integer $n \geq 0$ such that every point $x \in X$ is contained in a closed neighborhood F such that $\dim(F) \leq n$ (and $\operatorname{locdim}(X) = \infty$ if no such n exists). We refer the reader to [Dow55] and [Pea75, Chapter 5] for more information about the local covering dimension.

It was noted by Brown and Pedersen, [BP09, Section 2.2 (ii)], that $\operatorname{locdim}(X) = \dim(\alpha X)$ for a locally compact, Hausdorff space X. We propose that the natural dimension of a pointed space $(X, x_\infty) \in \mathcal{SP}_*$ is $\dim(X) = \operatorname{locdim}(X \setminus \{x_\infty\})$. Then, for a commutative C^*-algebra A, the natural dimension is $\operatorname{locdim}(\operatorname{Prim}(A))$.

If $G \subset X$ is an open subset of a locally compact space, then $\operatorname{locdim}(G) \leq \operatorname{locdim}(X)$, see [Dow55, 4.1]. It was also shown by Dowker that this does not hold for the usual covering dimension (of non-normal spaces).

3. A family of sub-C^*-algebras $A_i \subset A$ is said to *approximate* a C^*-algebra A (in the literature there also appears the formulation that the A_i "locally approximate" A), if for every finite subset $F \subset A$, and every $\varepsilon > 0$, there exists some i such that $F \subset_\varepsilon A_i$. Let us mention some facts about approximation by subalgebras:

(1) If $A_1 \subset A_2 \subset \ldots \subset A$ is an increasing sequence of sub-C^*-algebras with $A = \overline{\bigcup_k A_k}$, then A is approximated by the family $\{A_k\}$.

(2) If A is approximated by a family $\{A_i\}$, and $J \lhd A$ is an ideal, then J is approximated by the family $\{A_i \cap J\}$. In particular, if $A = \overline{\bigcup_k A_k}$, then $J = \overline{\bigcup_k (A_k \cap J)}$. Similarly, A/J is approximated by the family $\{A_i/(A_i \cap J)\}$.

(3) If A is approximated by a family $\{A_i\}$, and $B \subset A$ is a hereditary sub-C^*-algebra, then B might *not* be approximated by the family $\{A_i \cap B\}$. Nevertheless, B is approximated by algebras that are isomorphic to hereditary sub-C^*-algebras of the algebras A_i, see Proposition 4.

4. Let $\mathcal{P}$ be some property of C^*-algebras. We say that a C^*-algebra A is $\mathcal{P}$*-like* (in the literature there also appears the formulation A is "locally $\mathcal{P}$") if A is approximated by subalgebras with property $\mathcal{P}$, see [Thi11, 3.1 - 3.3]. This is motivated by the concept of $\mathcal{P}$-likeness for commutative spaces, as defined in [MS63, Definition 1] and further developed in [MM92].

We will work in the category $\mathcal{SP}_*$ of pointed, compact spaces, see 1. Let $\mathcal{P}$ be a non-empty class of spaces. Then, a space $X \in \mathcal{SP}_*$ is said to be $\mathcal{P}$-like if for every finite, open cover $\mathcal{U}$ of X there exists a (pointed) map $f\colon X \to Y$ onto some space $Y \in \mathcal{P}$ and a finite, open cover $\mathcal{V}$ of Y such that $\mathcal{U}$ is refined by $f^{-1}(\mathcal{V}) = \{f^{-1}(V) \mid V \in \mathcal{V}\}$.

Note that we have used $\mathcal{P}$ to denote both a class of spaces and a property that spaces might enjoy. These are just different viewpoints, as we can naturally assign to a property the class of spaces with that property, and vice versa to each class of spaces the property of lying in that class.

For commutative C^*-algebras, the notion of $\mathcal{P}$-likeness for C^*-algebras coincides with that for spaces. More precisely, it is shown in [Thi11, Proposition 3.4] that for a space $(X, x_\infty) \in \mathcal{SP}_*$ and a collection $\mathcal{P} \subset \mathcal{SP}_*$, the following are equivalent:

(a) (X, x_∞) is $\mathcal{P}$-like,
(b) $C_0(X \setminus \{x_\infty\})$ is approximated by sub-C^*-algebras $C_0(Y \setminus \{y_\infty\})$ with $(Y, y_\infty) \in \mathcal{P}$.

We note that the definition of covering dimension can be rephrased as follows. Let $\mathcal{P}_k$ be the collection of all k-dimensional polyhedra (polyhedra are defined by combinatoric data, and their dimension is defined by this combinatoric data). Then a compact space X satisfies $\dim(X) \leq k$ if and only if it is $\mathcal{P}_k$-like. This motivates (D5) in Definition 1 below.

5. For the definition of continuous trace C^*-algebras we refer to [Bla06, Definition IV.1.4.12, p.333]. It is known that a C^*-algebra A has continuous trace if and only if its spectrum $\widehat{A}$ is Hausdorff and it satisfies Fell's condition, i.e., for every $\pi \in \widehat{A}$ there exists a neighborhood $U \subset \widehat{A}$ of π and some $a \in A_+$ such that $\rho(a)$ is a rank-one projection for each $\rho \in U$, see [Bla06, Proposition IV.1.4.18, p.335].

6. A C^*-algebra A is called a *CCR algebra* (sometimes called a liminal algebra) if for each of its irreducible representations $\pi \colon A \to B(H)$ we have that π takes values inside the compact operators $K(H)$.

A *composition series* for a C^*-algebra A is a collection of ideals $J_\alpha \lhd A$, indexed over all ordinal numbers $\alpha \leq \mu$ for some μ, such that $A = J_\mu$ and:

(i) if $\alpha \leq \beta$, then $J_\alpha \subset J_\beta$,
(ii) if α is a limit ordinal, then $J_\alpha = \overline{\bigcup_{\gamma<\alpha} J_\gamma}$.

The C^*-algebras $J_{\alpha+1}/J_\alpha$ are called the successive quotients of the composition series.

A C^*-algebra is called a *type I algebra* (sometimes also called postliminal) if it has a composition series with successive quotients that are CCR algebras. As it turns out, this is equivalent to having a composition series whose successive quotients have continuous trace.

For information about type I C^*-algebras and their rich structure we refer the reader to Chapter IV.1 of Blackadar's book, [Bla06], and Chapter 6 of Pedersen's book, [Ped79].

3. Dimension theories for C^*-algebras

In this section, we introduce the notion of a noncommutative dimension theory by proposing a natural set of axioms that such theories should satisfy. These axioms hold for many well-known theories, in particular the real and stable rank, the decomposition rank and the nuclear dimension, see Remark 2, and this will also be discussed more thoroughly in a forthcoming paper. In Section 4 we will show that the topological dimension is a dimension theory for type I C^*-algebras.

Our axioms of a noncommutative dimension theory are inspired by facts that the theory of covering dimension satisfies, see Remark 1.

In Definition 2 we introduce the notion of Morita-invariance for dimension theories. If a dimension theory is only defined on a subclass of C^*-algebras, then there is a natural extension of the theory to all C^*-algebras, see Proposition 5. We will show that this extension preserves Morita-invariance.

We denote by $\mathcal{C}^*$ the category of C^*-algebras, and we will use $\mathcal{C}$ to denote a class of C^*-algebras. We may think of $\mathcal{C}$ as a full subcategory of $\mathcal{C}^*$.

Definition 1. Let $\mathcal{C}$ be a class of C^*-algebras that is closed under *-isomorphisms, and closed under taking ideals, quotients, finite direct sums, and minimal unitizations. A *dimension theory* for $\mathcal{C}$ is an assignment $d\colon \mathcal{C} \to \overline{\mathbb{N}} = \{0, 1, 2, \ldots, \infty\}$ such that $d(A) = d(A')$ whenever A, A' are isomorphic C^*-algebras in $\mathcal{C}$, and moreover the following axioms are satisfied:

(D1) $d(J) \leq d(A)$ whenever $J \lhd A$ is an ideal in $A \in \mathcal{C}$,
(D2) $d(A/J) \leq d(A)$ whenever $J \lhd A \in \mathcal{C}$,
(D3) $d(A \oplus B) = \max\{d(A), d(B)\}$, whenever $A, B \in \mathcal{C}$,
(D4) $d(\widetilde{A}) = d(A)$, whenever $A \in \mathcal{C}$.
(D5) If $A \in \mathcal{C}$ is approximated by subalgebras $A_i \in \mathcal{C}$ with $d(A_i) \leq n$, then $d(A) \leq n$.
(D6) Given $A \in \mathcal{C}$ and a separable sub-C^*-algebra $C \subset A$, there exists a separable C^*-algebra $D \in \mathcal{C}$ such that $C \subset D \subset A$ and $d(D) \leq d(A)$.

Note that we do not assume that $\mathcal{C}$ is closed under approximation by sub-C^*-algebra, so that the assumption $A \in \mathcal{C}$ in (D5) is necessary. Moreover, in axiom (D6), we do not assume that the separable subalgebra C lies in $\mathcal{C}$.

Remark 1. The axioms in Definition 1 are inspired by well-known facts of the local covering dimension of commutative spaces, see 2.

Axiom (D1) and (D2) generalize the fact that the local covering dimension does not increase when passing to an open (resp. closed) subspace, see [Dow55, 4.1, 3.1], and axiom (D3) generalizes the fact that $\operatorname{locdim}(X \sqcup Y) = \max\{\operatorname{locdim}(X), \operatorname{locdim}(Y)\}$. Axiom (D4) generalizes that $\operatorname{locdim}(X) = \operatorname{locdim}(\alpha X)$, where αX is the one-point compactification of X.

Axiom (D5) generalizes the fact that a (compact) space is n-dimensional if it is $\mathcal{P}_n$-like for the class $\mathcal{P}_n$ of n-dimensional spaces, see 4. Note also that Proposition 2 generalizes the fact that $\dim(\varprojlim X_i) \leq \liminf_i \dim(X_i)$ for an inverse system of compact spaces X_i.

Axiom (D6) is a generalization of the following factorization theorem, due to Mardešić, see [Nag70, Corollary 27.5, p.159] or [Mar60, Lemma 4]: Given a compact space X and a map $f\colon X \to Y$ to a compact, metrizable space Y, there exists a compact, metrizable space Z and maps $g\colon X \to Z, h\colon Z \to Y$ such that g is onto, $\dim(Z) \leq \dim(X)$ and $f = h \circ g$. This generalizes (D6), since a unital, commutative C^*-algebra $C(X)$ is separable if and only if X is metrizable.

Axioms (D5) and (D6) are also related to the following concept which is due to Blackadar, [Bla06, Definition II.8.5.1, p.176]: A property $\mathcal{P}$ of C^*-algebras is called *separably inheritable* if:

1. For every C^*-algebra A with property $\mathcal{P}$ and separable sub-C^*-algebra $C \subset A$, there exists a separable sub-C^*-algebra $D \subset A$ that contains C and has property $\mathcal{P}$.
2. Given an inductive system (A_k, φ_k) of separable C^*-algebras with injective connecting morphisms $\varphi_k\colon A_k \to A_{k+1}$, if each A_k has property $\mathcal{P}$, then does the inductive limit $\varinjlim A_k$.

Thus, for a dimension theory d, the property "$d(A) \leq n$" is separably inheritable.

Axioms (D5) and (D6) imply that $d(A) \leq n$ if and only if A can be written as an inductive limit (with injective connecting morphisms) of separable C^*-algebras B with $d(B) \leq n$. This allows us to reduce essentially every question about dimension theories to the case of separable C^*-algebras.

By explaining the analogs of (D1)-(D6) for pointed, compact spaces, we have shown the following:

Proposition 1. *Let $\mathcal{C}^*_{\mathrm{ab}}$ denote the class of commutative C^*-algebras. Then, the assignment $d\colon \mathcal{C}^*_{\mathrm{ab}} \to \overline{\mathbb{N}}$, $d(A) := \mathrm{locdim}(\mathrm{Prim}(A))$, is a dimension theory.*

Remark 2. We do not suggest that the axioms of Definition 1 uniquely define a dimension theory. This is clear since the axioms do not even rule out the assignments that give each C^*-algebra the same value.

More interestingly, the following well-known theories are dimension theories for the class of all C^*-algebras:

(1) The stable rank as defined by Rieffel, [Rie83, Definition 1.4].
(2) The real rank as introduced by Brown and Pedersen, [BP91].
(3) The decomposition rank of Kirchberg and Winter, [KW04, Definition 3.1].
(4) The nuclear dimension of Winter and Zacharias, [WZ10, Definition 2.1].

Indeed, for the real and stable rank, (D1) and (D2) are proven in [EH95, Théorème 1.4] and [Rie83, Theorems 4.3, 4.4]. Axiom (D3) is easily verified, and (D4) holds by definition. It is shown in [Rie83, Theorem 5.1] that (D5) holds in the special case of an approximation by a countable inductive limit, but the same argument works for general approximations and also for the real rank. Finally, it is noted in [Bla06, II.8.5.5, p.178] that (D6) holds.

For the nuclear dimension, axioms (D1), (D2), (D3), (D6) and (D4) follow from Propositions 2.5, 2.3, 2.6 and Remark 2.11 in [WZ10], and (D5) is easily verified. For the decomposition rank, (D5) is also easily verified, and axiom (D6) follows from [WZ10, Proposition 2.6] adapted for c.p.c. approximations instead of c.p. approximations. The other axioms (D1)-(D4) follow from Proposition 3.8, 3.11 and Remark 3.2 of [KW04] for separable C^*-algebras. Using axioms (D5) and (D6) this can be extended to all C^*-algebras.

Thus, the idea of Definition 1 is to collect the essential properties that many different noncommutative dimension theories satisfy. Our way of axiomatizing noncommutative dimension theories should therefore not be confused with the work on axiomatizing the dimension theory of metrizable spaces, see e.g. [Nis74] or [Cha94], since these works pursue the goal of finding axioms that uniquely characterize covering dimension.

Proposition 2. *Let $d\colon \mathcal{C} \to \overline{\mathbb{N}}$ be a dimension theory, and let $(A_i, \varphi_{i,j})$ be an inductive system with $A_i \in \mathcal{C}$ and such that the limit $A := \varinjlim A_i$ also lies in $\mathcal{C}$. Then $d(A) \leq \liminf_i d(A_i)$.*

Proof. See [Bla06, II.8.2.1, p.156] for details about inductive systems and inductive limits. For each i, let $\varphi_{\infty,i}\colon A_i \to A$ denote the natural morphism into the inductive limit. Then the subalgebra $\varphi_{\infty,i}(A_i) \subset A$ is a quotient of A_i, and therefore $d(\varphi_{\infty,i}(A_i)) \leq d(A_i)$ by (D2). If $J \subset I$ is cofinal, then A is approximated by the collection of subalgebras $(\varphi_{\infty,i}(A_i))_{i\in J}$. It follows from (D5) that $d(A)$ is bounded by $\sup_{i\in J} d(A_i)$. Since this holds for each cofinal subset $J \subset I$, we obtain:

$$d(A) \leq \inf\{\sup_{i\in J} d(A_i) \mid J \subset I \text{ cofinal}\} = \liminf_i d(A_i),$$

as desired. □

Lemma 1. *Let A be a C^*-algebra, let $B \subset A$ be a full, hereditary sub-C^*-algebra, and let $C \subset A$ be a separable sub-C^*-algebra. Then there exists a separable sub-C^*-algebra $D \subset A$ containing C such that $D \cap B \subset D$ is full, hereditary.*

Proof. The proof is inspired by the proof of [Bla78, Proposition 2.2], see also [Bla06, Theorem II.8.5.6, p.178]. We inductively define separable sub-C^*-algebras $D_k \subset A$. Set

$D_1 := C$, and assume D_k has been constructed. Let $S_k := \{x_1^k, x_2^k, \ldots\}$ be a countable, dense subset of D_k. Since B is full in A, there exist for each $i \geq 1$ finitely many elements $a_{i,j}^k, c_{i,j}^k \in A$ and $b_{i,j}^k \in B$ such that

$$\|x_i^k - \sum_j a_{i,j}^k b_{i,j}^k c_{i,j}^k\| < 1/k.$$

Set $D_{k+1} := C^*(D_k, a_{i,j}^k, b_{i,j}^k, c_{i,j}^k, i, j \geq 1)$. Then define $D := \overline{\bigcup_k D_k}$, which is a separable sub-C^*-algebra of A containing C.

Note that $D \cap B \subset D$ is a hereditary sub-C^*-algebra, and let us check that it is also full. We need to show that the linear span of $D(D \cap B)D$ is dense in D. Let $d \in D$ and $\varepsilon > 0$ be given. Note that $\bigcup_k S_k$ is dense in D. Thus, we may find k and i such that $\|d - x_i^k\| < \varepsilon/2$. We may assume $k \geq 2/\varepsilon$. By construction, there are elements $a_{i,j}^k, c_{i,j}^k \in D_{k+1}$ and $b_{i,j}^k \in B \cap D_{k+1}$ such that $\|x_i^k - \sum_j a_{i,j}^k b_{i,j}^k c_{i,j}^k\| < 1/k$. It follows that the distance from d to the closed linear span of $D(D \cap B)D$ is at most ε. Since d and ε were chosen arbitrarily, this shows that $D \cap B \subset D$ is full. □

Proposition 3. *Let $d\colon \mathcal{C}^* \to \overline{\mathbb{N}}$ be a dimension theory. Then the following statements are equivalent:*

(1) For all C^*-algebras A, B: If $B \subset A$ is a full, hereditary sub-C^*-algebra, then $d(A) = d(B)$.
(2) For all C^*-algebras A, B: If A and B are Morita equivalent, then $d(A) = d(B)$.
(3) For all C^*-algebras A: $d(A) = d(A \otimes \mathbb{K})$.

Moreover, each of the statements is equivalent to the (a priori weaker) statement where the appearing C^*-algebras are additionally assumed to be separable.

If d satisfies the above conditions, and $B \subset A$ is a (not necessarily full) hereditary sub-C^*-algebra, then $d(B) \leq d(A)$.

Proof. For each of the statements $(1), (2), (3)$, let us denote the statement where the appearing C^*-algebras are assumed to be separable by $(1s), (2s), (3s)$ respectively. For example:

$(3s)$ For all separable C^*-algebras A: $d(A) = d(A \otimes \mathbb{K})$.

The implications "$(1) \Rightarrow (1s)$", "$(2) \Rightarrow (2s)$", and "$(3) \Rightarrow (3s)$" are clear. The implication "$(2s) \Rightarrow (3s)$" follows since A and $A \otimes \mathbb{K}$ are Morita equivalent, and "$(1s) \Rightarrow (3s)$" follows since $A \subset A \otimes \mathbb{K}$ is a full, hereditary sub-C^*-algebra.

It remains to show the implication "$(3s) \Rightarrow (1)$". Let A be a C^*-algebra, and let $B \subset A$ be a full, hereditary sub-C^*-algebra. We need to show $d(A) = d(B)$. To that end, we will construct separable sub-C^*-algebras $A' \subset A$ and $B' \subset B$ that approximate A and B, respectively, and such that $d(A') = d(B') \leq \min\{d(A), d(B)\}$. Together with (D5), this implies $d(A) = d(B)$.

So let $F \subset A$ and $G \subset B$ be finite sets. We may assume $G \subset F$. We want to find A' and B' with the mentioned properties and such that $F \subset A'$ and $G \subset B'$.

We inductively define separable sub-C^*-algebras $C_k, D_k \subset A$ and $E_k \subset B$ such that:

(a) $C_k \subset D_k$ and $D_k \cap B \subset D_k$ is full,
(b) $D_k \cap B \subset E_k$ and $d(E_k) \leq d(B)$,
(c) $E_k, D_k \subset C_{k+1}$ and $d(C_{k+1}) \leq d(A)$.

We start with $C_1 := C^*(F) \subset A$. If C_k has been constructed, we apply Lemma 1 to find D_k satisfying (a). If D_k has been constructed, we apply (D6) to $D_k \cap B \subset B$ to find

E_k satisfying (b). If E_k has been constructed, we apply axiom (D6) to $C^*(D_k, E_k) \subset A$ to find C_{k+1} satisfying (c).

Then let $A' := \overline{\bigcup_k C_k} = \overline{\bigcup_k D_k}$, and $B' := \overline{\bigcup_k (D_k \cap B)} = \overline{\bigcup_k E_k}$. The situation is shown in the following diagram:

$$\begin{array}{ccccccccccc} C_k & \subset & D_k & \subset & C^*(D_k, E_k) & \subset & C_{k+1} & \subset & \dots & \subset & A' \\ & & \cup & & \cup & & & & & & \\ & & D_k \cap B & \subset & E_k & \subset & \dots & & \dots & \subset & B' \end{array}$$

Let us verify that A' and B' have the desired properties. First, since $d(C_k) \leq d(A)$ for all k, we get $d(A') \leq d(A)$ from (D5). Similarly, we get $d(B') \leq d(B)$. For each k we have that $D_k \cap B \subset D_k$ is a full, hereditary sub-C^*-algebra, and therefore the same holds for $B' \subset A'$. Since A' and B' are separable (and hence σ-unital), we may apply Brown's stabilization theorem, [Bro77, Theorem 2.8], and obtain $A' \otimes \mathbb{K} \cong B' \otimes \mathbb{K}$. Together with the assumption (3s), we obtain $d(A') = d(A' \otimes \mathbb{K}) = d(B' \otimes \mathbb{K}) = d(B')$. This finishes the construction of A' and B', and we deduce $d(A) = d(B)$ from (D5).

Lastly, if d satisfies condition (1), and $B \subset A$ is a (not necessarily full) hereditary sub-C^*-algebra, then B is full, hereditary in the ideal $J \lhd A$ generated by B. By (D1) and condition (1) we have $d(B) = d(J) \leq d(A)$. □

Definition 2. A dimension theory $d\colon \mathcal{C}^* \to \overline{\mathbb{N}}$ is called *Morita-invariant* if it satisfies the conditions of Proposition 3.

Given positive elements a, b in a C^*-algebra, recall that we write $a =_\sigma b$ if $\|a - b\| < \sigma$. We write $a \ll_\sigma b$ if $ab =_\sigma a$.

Lemma 2. *For every $\varepsilon > 0$ there exists $\delta > 0$ with the following property: Given a C^*-algebra A, and contractive elements $a, b \in A_+$ with $a =_\delta b$, there exists a partial isometry $v \in A^{**}$ such that:*

(1) $v(a - \delta)_+ v^* \in bAb$.
(2) If $d \in A_+$ is contractive with $d \ll_\sigma a$, then $vdv^* =_{4\sigma+\varepsilon} d$.

Proof. To simplify the proof, we will fix $\delta > 0$ and verify the statement for $\varepsilon = \varepsilon(\delta)$ with the property that $\varepsilon(\delta) \to 0$ when $\delta \to 0$.

Fix $\delta > 0$. Let A be a C^*-algebra, and let $a, b \in A_+$ be contractive elements such that $a =_\delta b$. Without loss of generality we may assume that A is unital. It is well-known that there exists $s \in A$ such that $s(a - \delta)_+ s^* \in bAb$, see [Rør92, Proposition 2.4]. One could follow the proof to obtain an estimate similar to that in statement (2). It is, however, easier to find $v \in A^{**}$ such that (1) and (2) hold, and for our application in Proposition 4 it is sufficient that v lies in A^{**}.

It follows from $a =_\delta b$ that $a - \delta \leq b$, and hence:

$$(a - \delta)_+^2 = (a - \delta)_+^{1/2}(a - \delta)(a - \delta)_+^{1/2} \leq (a - \delta)_+^{1/2} b (a - \delta)_+^{1/2}.$$

Set $z := b^{1/2}(a - \delta)_+^{1/2}$. Then:

$$|z| = ((a - \delta)_+^{1/2} b (a - \delta)_+^{1/2})^{1/2}, \qquad |z^*| = (b^{1/2}(a - \delta)_+ b^{1/2})^{1/2},$$

and we let $z = v|z|$ be the polar decomposition of z, with $v \in A^{**}$. We claim that v has the desired properties. First, note that $v((a-\delta)_+^{1/2} b (a-\delta)_+^{1/2})v^* = b^{1/2}(a-\delta)_+ b^{1/2} \in bAb$, and therefore also $v(a - \delta)_+ v^* \in bAb$, which verifies property (1).

For property (2), let us start by estimating the distance from a to z and $|z|$. It is known that there exists an assignment $\sigma \mapsto \varepsilon_1(\sigma)$ with the following property: Whenever x, y are positive, contractive elements of a C^*-algebra, and $x =_\sigma y$, then $x^{1/2} =_{\varepsilon_1(\sigma)} y^{1/2}$, and moreover $\varepsilon_1(\sigma) \to 0$ as $\sigma \to 0$. We may assume $\sigma \leq \varepsilon_1(\sigma)$, and we will use this to simplify some estimates below.

Then, using $(a-\delta)_+ =_\delta a$ and so $(a-\delta)_+^{1/2} =_{\varepsilon_1(\sigma)} a^{1/2}$ at the second step,

$$z = b^{1/2}(a-\delta)_+^{1/2} =_{\varepsilon_1(\delta)} b^{1/2}a^{1/2} =_{\varepsilon_1(\delta)} a. \tag{3.1}$$

For $|z|$ we compute, using $(a-\delta)_+^{1/2} b (a-\delta)_+^{1/2} =_{3\varepsilon_1(\delta)} a^2$ at the second step,

$$|z| = ((a-\delta)_+^{1/2} b (a-\delta)_+^{1/2})^{1/2} =_{\varepsilon_1(3\varepsilon_1(\delta))} (a^2)^{1/2} = a. \tag{3.2}$$

Let $d \in A_+$ be contractive with $d \ll_\sigma a$. Then $ada =_{2\sigma} d$, and we may estimate the distance from vdv^* to d as follows:

$$vdv^* =_{2\sigma} vadav^* \overset{(3.2)}{=\!=}_{2\varepsilon_1(3\varepsilon_1(\delta))} v|z|d|z|v^* = zdz \overset{(3.1)}{=\!=}_{4\varepsilon_1(\delta)} ada =_{2\sigma} d.$$

Thus, $\|vdv^* - d\| \leq 4\sigma + 2\varepsilon_1(3\varepsilon_1(\delta)) + 4\varepsilon_1(\delta)$, and this distance converges to 4σ when $\delta \to 0$. This completes the proof. □

Proposition 4. *Let A be a C^*-algebra, and let $B \subset A$ be a hereditary sub-C^*-algebra. Assume A is approximated by sub-C^*-algebras $A_i \subset A$. Then B is approximated by subalgebras that are isomorphic to hereditary sub-C^*-algebras of the algebras A_i, i.e., given a finite set $F \subset B$ and $\varepsilon > 0$, there exists a sub-C^*-algebra $B' \subset B$ such that $F \subset_\varepsilon B'$ and B' is isomorphic to a hereditary sub-C^*-algebra of A_i for some i.*

Proof. Let $F \subset B$ and $\varepsilon > 0$ be given. We let $\gamma = \varepsilon/36$, which is justified by the estimates that we obtain through the course of the proof. Without loss of generality, we may assume that F consists of positive, contractive elements.

There exists $b \in B_+$ such that b almost acts as a unit on the elements of F in the sense that $x \ll_\gamma b$ for all $x \in F$. Let $\delta > 0$ be the tolerance we get from Lemma 2 for γ. We may assume $\delta \leq \gamma$, and to simplify the computations below we will often estimate a distance by γ, even if it could be estimated by δ.

By assumption, the algebras A_i approximate A. Thus, there exists i such that there is a positive, contractive element $a \in A_i$ with $a =_\delta b$, and such that for each $x \in F$ there exists a positive, contractive $x' \in A_i$ with $x' =_\delta x$. Then:

$$x'(a-\delta)_+ =_{3\delta} xb =_\gamma x =_\delta x',$$

and so $x' \ll_{5\gamma} (a-\delta)_+$, since $\delta \leq \gamma$. In general, if two positive, contractive elements s, t satisfy $s \ll_\sigma t$, then $s =_{2\sigma} tst \ll_\sigma t$. Thus, if for each $x \in F$ we set $x'' := (a-\delta)_+ x' (a-\delta)_+$, then we obtain:

$$x =_\gamma x' =_{10\gamma} x'' \ll_{5\gamma} (a-\delta)_+. \tag{3.3}$$

Since $a =_\delta b$, we obtain from Lemma 2 a partial isometry $v \in A^{**}$ such that $v(a-\delta)_+v^* \in bAb$. Let $A' := (a-\delta)_+ A_i (a-\delta)_+$, which is a hereditary sub-C^*-algebra of A_i. The map $x \mapsto vxv^*$ defines an isomorphism from A' onto $B' := vA'v^*$. Since B is hereditary, B' is a sub-C^*-algebra of B. Let us estimate the distance from F to B'.

For each $x \in F$, we have computed in (3.3) that $x'' \ll_{5\gamma} (a-\delta)_+$, which implies $x'' \ll_{6\gamma} a$. From statement (2) of Lemma 2 we deduce $vx''v^* =_{25\gamma} x''$. Altogether, the distance between x and $vx''v^*$ is at most 36γ. Since $vx''v^* \in B'$, and since we chose $\gamma = \varepsilon/36$, we have $F \subset_\varepsilon B'$, as desired. □

Proposition 5. *Let $d\colon \mathcal{C} \to \overline{\mathbb{N}}$ be a dimension theory. For any C^*-algebra A define:*

$$\widetilde{d}(A) := \inf\{k \in \mathbb{N} \mid A \textit{ is approximated by sub-}C^*\textit{-algebras } B \in \mathcal{C} \textit{ with } d(B) \leq k\}, \tag{3.4}$$

where we define the infimum of the empty set to be $\infty \in \overline{\mathbb{N}}$.

Then $\widetilde{d}\colon \mathcal{C}^* \to \overline{\mathbb{N}}$ is a dimension theory that agrees with d on $\mathcal{C}$.

If, moreover, $\mathcal{C}$ is closed under stable isomorphism, and $d(A) = d(A \otimes \mathbb{K})$ for every (separable) $A \in \mathcal{C}$, then $\widetilde{d}$ is Morita-invariant.

Proof. If $A \in \mathcal{C}$, then clearly $\widetilde{d}(A) \leq d(A)$, and the converse inequality follows from axiom (D5). Axioms (D1)-(D5) for $\widetilde{d}$ are easy to check.

Let us check axiom (D6) for $\widetilde{d}$. Assume A is a C^*-algebra, and assume $C \subset A$ is a separable sub-C^*-algebra. Set $n := \widetilde{d}(A)$, which we may assume is finite. We need to find a separable sub-C^*-algebra $D \subset A$ such that $C \subset D$ and $\widetilde{d}(D) \leq n$.

We first note the following: For a finite set $F \subset A$, and $\varepsilon > 0$ we can find a separable sub-C^*-algebra $A(F,\varepsilon) \subset A$ with $d(A(F,\varepsilon)) \leq n$ and $F \subset_\varepsilon A(F,\varepsilon)$. Indeed, by definition of $\widetilde{d}$ we can first find a sub-C^*-algebra $B \subset A$ with $d(B) \leq n$ and a finite subset $G \subset B$ such that $F \subset_\varepsilon G$. Applying (D6) to $C^*(G) \subset B$, we may find a separable sub-C^*-algebra $A(F,\varepsilon) \subset B$ with $d(A(F,\varepsilon)) \leq n$ and $C^*(G) \subset A(F,\varepsilon)$, which implies $F \subset_\varepsilon A(F,\varepsilon)$.

We will inductively define separable sub-C^*-algebras $D_k \subset A$ and countable dense subsets $S_k = \{x_1^k, x_2^k, \ldots\} \subset D_k$ as follows: We start with $D_1 := C$ and choose any countable dense subset $S_1 \subset D_1$. If D_l and S_l have been constructed for $l \leq k$, then set:

$$D_{k+1} := C^*(D_k, A(\{x_i^j \mid i,j \leq k\}, 1/k)) \subset A,$$

and choose any countable dense subset $S_{k+1} = \{x_1^{k+1}, x_2^{k+1}, \ldots\} \subset D_{k+1}$.

Set $D := \overline{\bigcup_k D_k} \subset A$, which is a separable C^*-algebra containing C. Let us check that $\widetilde{d}(D) \leq n$, which means that we have to show that D is approximated by sub-C^*-algebras $B \in \mathcal{C}$ with $d(B) \leq n$.

Note that $\{x_i^j\}_{i,j\geq 1}$ is dense in D. Thus, if a finite subset $F \subset D$, and $\varepsilon > 0$ is given, we may find k such that $F \subset_{\varepsilon/2} \{x_i^j \mid i,j \leq k\}$, and we may assume $k > 2/\varepsilon$. By construction, D contains the sub-C^*-algebra $B := A(\{x_i^j \mid i,j \leq k\}, 1/k)$, which satisfies $d(B) \leq n$ and $\{x_i^j \mid i,j \leq k\} \subset_{1/k} B$. Then $F \subset_\varepsilon B$, which completes the proof that $\widetilde{d}(D) \leq n$.

Lastly, assume $\mathcal{C}$ is closed under stable isomorphism, and assume $d(A) = d(A \otimes \mathbb{K})$ for every separable $A \in \mathcal{C}$. This implies the following: If A is a separable C^*-algebra in $\mathcal{C}$, and $B \subset A$ is a hereditary sub-C^*-algebra, then B lies in $\mathcal{C}$ and $d(B) \leq d(A)$.

We want to check condition (3) of Proposition 3 for $\widetilde{d}$. Thus, let a separable C^*-algebra A be given. We need to check $\widetilde{d}(A) = \widetilde{d}(A \otimes \mathbb{K})$.

If $\widetilde{d}(A) = \infty$, then clearly $\widetilde{d}(A \otimes \mathbb{K}) \leq \widetilde{d}(A)$. So assume $n := \widetilde{d}(A) < \infty$, which means that A is approximated by algebras $A_i \subset A$ with $d(A_i) \leq n$. Then $A \otimes \mathbb{K}$ is approximated by the subalgebras $A_i \otimes \mathbb{K} \subset A \otimes \mathbb{K}$, and $d(A_i \otimes \mathbb{K}) = d(A_i) \leq n$ by assumption. Then $\widetilde{d}(A \otimes \mathbb{K}) \leq n = \widetilde{d}(A)$.

Conversely, if $\widetilde{d}(A \otimes \mathbb{K}) = \infty$, then $\widetilde{d}(A) \leq \widetilde{d}(A \otimes \mathbb{K})$. So assume $n := \widetilde{d}(A \otimes \mathbb{K}) < \infty$, which means that $A \otimes \mathbb{K}$ is approximated by algebras $A_i \subset A$ with $d(A_i) \leq n$. Consider the hereditary sub-C^*-algebra $A \otimes e_{1,1} \subset A \otimes \mathbb{K}$, which is isomorphic to A. By Proposition 4, $A \otimes e_{1,1}$ is approximated by subalgebras $B_j \subset A \otimes e_{1,1}$ such that each B_j is isomorphic to a hereditary sub-C^*-algebras of A_i, for some $i = i(j)$. It follows $d(B_j) \leq n$, and then $\widetilde{d}(A) = \widetilde{d}(A \otimes e_{1,1}) \leq n = \widetilde{d}(A \otimes \mathbb{K})$. Together we get $\widetilde{d}(A) = \widetilde{d}(A \otimes \mathbb{K})$, as desired. □

4. Topological dimension

One could try to define a dimension theory by simply considering the dimension of the primitive ideal space of a C^*-algebra. This will, however, run into problems if the primitive ideal space is not Hausdorff. Brown and Pedersen, [BP09], suggested a way of dealing with this problem by restricting to (locally closed) Hausdorff subsets of $\mathrm{Prim}(A)$, and taking the supremum over the dimension of these Hausdorff subsets. This defines the topological dimension of a C^*-algebra, see Definition 4.

In this section we will show that the topological dimension is a dimension theory in the sense of Definition 1 for the class of type I C^*-algebras. It follows from the work of Brown and Pedersen that axioms (D1)-(D4) are satisfied, and we verify axiom (D5) in Proposition 8. We use transfinite induction over the length of a composition series of the type I C^*-algebra to verify axiom (D6), see Proposition 9.

See 6 for a short reminder on type I C^*-algebras. For more details, we refer the reader to Chapter IV.1 of Blackadar's book, [Bla06], and Chapter 6 of Pedersen's book, [Ped79].

Definition 3 (Brown, Pedersen, [BP07, 2.2 (iv)])**.** Let X be a topological space. We define:

(1) A subset $C \subset X$ is called *locally closed* if there is a closed set $F \subset X$ and an open set $G \subset X$ such that $C = F \cap G$.
(2) X is called *almost Hausdorff* if every non-empty closed subset F contains a non-empty relatively open subset $F \cap G$ (so $F \cap G$ is locally closed in X) which is Hausdorff.

7. We could consider locally closed subsets as "well-placed" subsets. Then, being almost Hausdorff means having enough "well-placed" Hausdorff subsets.

For a C^*-algebra A, the locally closed subsets of $\mathrm{Prim}(A)$ correspond to ideals of quotients of A (equivalently to quotients of ideals of A) up to canonical isomorphism, see [BP07, 2.2(iii)]. Therefore, the primitive ideal space of every type I C^*-algebra is almost Hausdorff, since every non-zero quotient contains a non-zero ideal that has continuous trace, see [Ped79, Theorem 6.2.11, p. 200], and the primitive ideal space of a continuous trace C^*-algebra is Hausdorff.

Definition 4 (Brown, Pedersen, [BP07, 2.2(v)])**.** Let A be a C^*-algebra. If $\mathrm{Prim}(A)$ is almost Hausdorff, then the *topological dimension* of A, denoted by $\mathrm{topdim}(A)$, is:

$$\mathrm{topdim}(A) := \sup\{\mathrm{locdim}(S) \mid S \subset \mathrm{Prim}(A) \text{ locally closed, Hausdorff}\}. \tag{4.1}$$

We will now show that the topological dimension satisfies the axioms of Definition 1. The following result immediately implies (D1)-(D4).

Proposition 6 (Brown, Pedersen, [BP07, Proposition 2.6])**.** *Let $(J_\alpha)_{\alpha\leq\mu}$ be a composition series for a C^*-algebra A. Then* $\mathrm{Prim}(A)$ *is almost Hausdorff if and only if* $\mathrm{Prim}(J_{\alpha+1}/J_\alpha)$ *is almost Hausdorff for each $\alpha < \mu$, and if this is the case, then:*

$$\mathrm{topdim}(A) = \sup_{\alpha<\mu} \mathrm{topdim}(J_{\alpha+1}/J_\alpha). \tag{4.1}$$

The following result is implicit in the papers of Brown and Pedersen, e.g. [BP09, Theorem 5.6].

Proposition 7. *Let A be a C^*-algebra, and let $B \subset A$ be a hereditary sub-C^*-algebra. If* $\mathrm{Prim}(A)$ *is locally Hausdorff, then so is* $\mathrm{Prim}(B)$*, and then* $\mathrm{topdim}(B) \leq \mathrm{topdim}(A)$*. If B is even full hereditary, then* $\mathrm{topdim}(B) = \mathrm{topdim}(A)$*.*

Proof. In general, if $B \subset A$ is a hereditary sub-C^*-algebra, then $\mathrm{Prim}(B)$ is homeomorphic to an open subset of $\mathrm{Prim}(A)$. In fact, $\mathrm{Prim}(B)$ is canonically homeomorphic to the primitive ideal space of the ideal generated by B, and this corresponds to an open subset of $\mathrm{Prim}(A)$.

Note that being locally Hausdorff is a property that passes to locally closed subsets, and so it passes from $\mathrm{Prim}(A)$ to $\mathrm{Prim}(B)$. Further, every locally closed, Hausdorff subset $S \subset \mathrm{Prim}(B)$ is also locally closed (and Hausdorff) in $\mathrm{Prim}(A)$. It follows $\mathrm{topdim}(B) \leq \mathrm{topdim}(A)$.

If B is full, then $\mathrm{Prim}(B) \cong \mathrm{Prim}(A)$ and therefore $\mathrm{topdim}(B) = \mathrm{topdim}(A)$. □

Lemma 3. *Let A be a continuous trace C^*-algebra, and let $n \in \mathbb{N}$. If A is approximated by sub-C^*-algebras with topological dimension at most n, then* $\mathrm{topdim}(A) \leq n$*.*

Proof. Since $\mathrm{Prim}(A)$ is Hausdorff, we have $\mathrm{topdim}(A) = \mathrm{locdim}(\mathrm{Prim}(A))$. Thus, it is enough to show that every $x \in \mathrm{Prim}(A)$ has a neighborhood U with $\dim(U) \leq n$. This will allow us to reduce the problem to the situation that A has a global rank-one projection, i.e., that there exists a full, abelian projection $p \in A$, see [Bla06, IV.1.4.20, p.335], which we do as follows:

Let $x \in \mathrm{Prim}(A)$ be given. Since A has continuous trace, there exists an open neighborhood $U \subset \mathrm{Prim}(A)$ of x and an element $a \in A_+$ such that $\rho(a)$ is a rank-one projection for every $\rho \in U$, see 5. Then there exists a closed, compact neighborhood $Y \subset \mathrm{Prim}(A)$ of x that is contained in U. Let $J \lhd A$ be the ideal corresponding to $\mathrm{Prim}(A) \setminus Y$. The image of a in the quotient A/J is a full, abelian projection. Since A is approximated by subalgebras $B \subset A$ with $\mathrm{topdim}(B) \leq n$, A/J is approximated by the subalgebras $B/(B \cap J)$ with $\mathrm{topdim}(B/(B \cap J)) \leq \mathrm{topdim}(B) \leq n$. If we can show that this implies $\dim(Y) = \mathrm{topdim}(A/J) \leq n$, then every point of $\mathrm{Prim}(A)$ has a closed neighborhood of dimension $\leq n$, which means $\mathrm{topdim}(A) = \mathrm{locdim}(\mathrm{Prim}(A)) \leq n$.

We assume from now on that A has continuous trace with a full, abelian projection $p \in A$. Thus, $pAp \cong C(X)$ where $X := \mathrm{Prim}(A)$ is a compact, Hausdorff space. Assume A is approximated by subalgebras $A_i \subset A$ with $\mathrm{topdim}(A_i) \leq n$. It follows from Proposition 4 that the hereditary sub-C^*-algebra pAp is approximated by subalgebras B_j such that each B_j is isomorphic to a hereditary sub-C^*-algebra of A_i, for some $i = i(j)$. By Proposition 7, $\mathrm{topdim}(B_j) \leq \mathrm{topdim}(A_{i(j)}) \leq n$ for each j.

Thus, $C(X)$ is approximated by commutative subalgebras $C(X_j)$ with $\dim(X_j) = \mathrm{topdim}(C(X_j)) \leq n$. It follows from Proposition 1 that $\dim(X) \leq n$, as desired. □

Proposition 8. *Let A be a type I C^*-algebra, and let $n \in \mathbb{N}$. If A is approximated by sub-C^*-algebras with topological dimension at most n, then* $\mathrm{topdim}(A) \leq n$*.*

Proof. Let $(J_\alpha)_{\alpha \leq \mu}$ be a composition series for A such that each successive quotient has continuous trace, and assume A is approximated by subalgebras $A_i \subset A$ with $\mathrm{topdim}(A_i) \leq n$.

Then $J_{\alpha+1}/J_\alpha$ is approximated by the subalgebras $(A_i \cap J_{\alpha+1})/(A_i \cap J_\alpha)$, see 3. Since $\mathrm{topdim}((A_i \cap J_{\alpha+1})/(A_i \cap J_\alpha)) \leq \mathrm{topdim}(A_i) \leq n$, we obtain from the above Lemma 3 that $\operatorname{topdim}(J_{\alpha+1}/J_\alpha) \leq n$. By Proposition 6, $\operatorname{topdim}(A) = \sup_{\alpha<\mu} \operatorname{topdim}(J_{\alpha+1}/J_\alpha) \leq n$, as desired. □

Remark 3. It is noted in [BP07, Remark 2.5(v)] that a weaker version of Proposition 8 would follow from [Sud04]. However, the statement is formulated as an axiom there, and it is not clear that the formulated axioms are consistent and give a dimension theory that agrees with the topological dimension.

We will now prove that the topological dimension of type I C^*-algebras satisfies the Mardešić factorization axiom (D6). We start with two lemmas.

Lemma 4. *Let A be a continuous trace C^*-algebra, and let $C \subset A$ be a separable sub-C^*-algebra. Then there exists a separable, continuous trace sub-C^*-algebra $D \subset A$ that contains C, and such that the inclusion $C \subset D$ is proper, and* $\operatorname{topdim}(D) \leq \operatorname{topdim}(A)$.

Proof. Let us first reduce to the case that A is σ-unital, and the inclusion $C \subset A$ is proper. To this end, consider the hereditary sub-C^*-algebra $A' := CAC \subset A$. Since C is separable, it contains a strictly positive element which is then also strictly positive in A'. Moreover, having continuous trace passes to hereditary sub-C^*-algebras, see [Ped79, Proposition 6.2.10, p.199]. Thus, A' is σ-unital and $C \subset A'$ is proper. Moreover, $\operatorname{topdim}(A') \leq \operatorname{topdim}(A)$ by Proposition 7.

Thus, by replacing A with CAC, we may assume from now on that A is σ-unital and that the inclusion $C \subset A$ is proper. Set $X := \operatorname{Prim}(A)$. By Brown's stabilization theorem, [Bro77, Theorem 2.8], there exists an isomorphism $\Phi\colon A \otimes \mathbb{K} \to C_0(X) \otimes \mathbb{K}$. Let $e_{ij} \in \mathbb{K}$ be the canonical matrix units, and consider the following C^*-algebra:

$$E := C^*\Big(\bigcup_{i,j} e_{1i}\Phi(C \otimes \mathbb{K})e_{j1}\Big) \subset C_0(X) \otimes e_{11}.$$

The following diagram shows some of the C^*-algebras and maps that we will construct below:

$$\begin{array}{ccccccc}
A \otimes e_{11} & \subset & A \otimes \mathbb{K} & \xrightarrow[\cong]{\Phi} & C_0(X) \otimes \mathbb{K} & & \\
\cup & & \cup & & \cup & & \\
D & \subset & \Phi^{-1}(D') & \xrightarrow[\cong]{} & C_0(Z_0) \otimes \mathbb{K} & = & D' \\
\cup & & \cup & & \cup & & \\
C \otimes e_{11} & \subset & C \otimes \mathbb{K} & \xrightarrow[\cong]{} & \Phi(C \otimes \mathbb{K}) & &
\end{array}$$

Note that E is separable and commutative. Thus, there exists a separable sub-C^*-algebra $C_0(Y) \subset C_0(X)$ such that $E = C_0(Y) \otimes e_{11}$. We constructed E such that $\Phi(C \otimes \mathbb{K}) \subset C_0(Y) \otimes \mathbb{K}$.

The inclusion $C_0(Y) \subset C_0(X)$ is induced by a pointed, continuous map $f\colon X^+ \to Y^+$, see 1. Recall that a compact, Hausdorff space M is metrizable if and only if $C(M)$ is separable. Thus, Y^+ is compact, metrizable.

By Mardešić's factorization theorem, see [Nag70, Corollary 27.5, p.159] or [Mar60, Lemma 4], there exists a compact, metrizable space Z with $\dim(Z) \leq \dim(X)$ and continuous (surjective) maps $g\colon X \to Z$ and $h\colon Z \to Y$ such that $f = h \circ g$. Set $Z_0 := Z \setminus \{g(\infty)\}$, and note that g^* induces an embedding $C_0(Z_0) \subset C_0(X)$. Moreover, $C_0(Z_0)$ is separable, since Z is compact, metrizable.

Consider $D' := C_0(Z_0) \otimes \mathbb{K}$. We have that D' is a separable, continuous trace C^*-algebra such that $\Phi(C \otimes \mathbb{K}) \subset C_0(Y) \otimes \mathbb{K} \subset D'$, and $\operatorname{topdim}(D') = \dim(Z) \leq \dim(X) = \operatorname{topdim}(A)$. We think of C as included in $C \otimes \mathbb{K}$ via $C \cong C \otimes e_{11}$. Set

$$D := (1_{\widetilde{A}} \otimes e_{11})(\Phi^{-1}(D'))(1_{\widetilde{A}} \otimes e_{11}),$$

which is a hereditary sub-C^*-algebra of $\Phi^{-1}(D') \cong D'$. Hence, D is a separable, continuous trace C^*-algebra with $\operatorname{topdim}(D) \leq \operatorname{topdim}(D') \leq \operatorname{topdim}(A)$. By construction, $C \otimes e_{11} \subset D$, and this inclusion is proper since $D \subset A \otimes e_{11}$ and the inclusion $C \otimes e_{11} \subset A \otimes e_{11}$ is proper. □

Lemma 5. *Let A be a C^*-algebra, let $J \lhd A$ be an ideal, and let $C \subset A$ be a sub-C^*-algebra. Assume $K \subset J$ is a sub-C^*-algebra that contains $C \cap J$ and such that the inclusion $C \cap J \subset K$ is proper. Then K is an ideal in the sub-C^*-algebra $C^*(K, C) \subset A$ generated by K and C. Moreover, there is a natural isomorphism $C^*(K, C)/K \cong C/(C \cap J)$.*

Proof. Set $B := A/J$ and denote the quotient morphism by $\pi \colon A \to B$. Set $D := \pi(C) \subset B$. Clearly, $C^*(K, C)$ contains both K and C, and it is easy to see that the restriction of π to $C^*(K, C)$ maps onto D. The situation is shown in the following commutative diagram, where the top and bottom rows are exact:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J & \longrightarrow & A & \xrightarrow{\pi} & B & \longrightarrow & 0 \\
 & & \cup & & \cup & & \cup & & \\
 & & K & \longrightarrow & C^*(K,C) & \longrightarrow & D & & \\
 & & \cup & & \cup & & \| & & \\
0 & \longrightarrow & C \cap J & \longrightarrow & C & \longrightarrow & D & \longrightarrow & 0
\end{array}$$

Let us show that K is an ideal in $C^*(K, C)$. Since $C^*(K, C)$ is generated by elements of K and C, it is enough to show that xy and yx lie in K whenever $x \in K$ and $y \in K$ or $y \in C$. For $y \in K$ that is clear, so assume $y \in C$.

Since $C \cap J \subset K$ is proper, for any $\varepsilon > 0$ there exists $c \in C \cap J$ such that $\|cxc - x\| < \varepsilon$. Then $\|xy - cxcy\|, \|yx - ycxc\| < \varepsilon\|y\|$. Moreover, $cxcy \in K$ and $ycxc \in K$ since $cy, yc \in C \cap J \subset K$. For $\varepsilon > 0$ was arbitrary, it follows that $xy, yx \in K$. This shows that the middle row in the above diagram is also exact. □

Proposition 9. *Let A be a C^*-algebra, let $J \lhd A$ be an ideal of type I, and let $C \subset A$ be a separable sub-C^*-algebra. Then there exists a separable sub-C^*-algebra $D \subset A$ such that $C \subset D$ and* $\operatorname{topdim}(D \cap J) \leq \operatorname{topdim}(J)$.

Proof. Let $(J_\alpha)_{\alpha \leq \mu}$ be a composition series for J with successive quotients that have continuous trace. To simplify notation, we will write $B[\alpha, \beta)$ for $(B \cap J_\beta)/(B \cap J_\alpha)$ and $B[\alpha, \infty)$ for $B/(B \cap J_\alpha)$ whenever $B \subset A$ is a subalgebra and $\alpha \leq \beta \leq \mu$ are ordinals. In particular, $A[0, \beta) = J_\beta$ and $A[\alpha, \infty) = A/J_\alpha$. We prove the statement of the proposition by transfinite induction over μ, which we carry out in three steps.

Step 1: The statement holds for $\mu = 0$. This follows since J is assumed to have a composition series with length 0 and so $J = \{0\}$ and we can simply set $D := C$.

Step 2: If the statement holds for a finite ordinal n, then it also holds for $n + 1$.

To prove this, assume J has a composition series $(J_\alpha)_{\alpha \leq n+1}$. Let $d := \operatorname{topdim}(J)$. Given $C \subset A$ separable, we want to find a separable subalgebra $D \subset A$ with $C \subset D$ and

$\operatorname{topdim}(D[0, n+1)) \leq d$. The following commutative diagram, whose rows are short exact sequences, contains the algebras and maps that we will construct below:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A[0,1) & \longrightarrow & A & \longrightarrow & A[1,\infty) & \longrightarrow & 0 \\ & & \cup & & \cup & & \cup & & \\ 0 & \longrightarrow & E[0,1) & \longrightarrow & E & \longrightarrow & E' & \longrightarrow & 0 \\ & & \cup & & \cup & & \cup & & \\ 0 & \longrightarrow & C[0,1) & \longrightarrow & C & \longrightarrow & C[1,\infty) & \longrightarrow & 0 \end{array}$$

Consider $A[1,\infty)$ together with the ideal $A[1,n+1) = J[1,n+1)$. Note that $A[1,n+1)$ has the canonical composition series $(A[1,\alpha))_{1\leq\alpha\leq n+1}$ of length n. By assumption of the induction, the statement holds for n, and so there is a separable sub-C^*-algebra $E' \subset A[1,\infty)$ such that $C[1,\infty) \subset E'$ and $\operatorname{topdim}(E' \cap A[1,n+1)) \leq \operatorname{topdim}(A[1,n+1)) \leq d$. Find a separable sub-$C^*$-algebra $E \subset A$ such that $C \subset E$ and $E[1,\infty) = E'$.

We apply Lemma 4 to the inclusion $E[0,1) \subset A[0,1)$ to find a separable sub-C^*-algebra $K \subset A[0,1)$ containing $E[0,1)$ and such that the inclusion $E[0,1) \subset K$ is proper, and $\operatorname{topdim}(K) \leq \operatorname{topdim}(A[0,1)) \leq d$. Set $D := C^*(K,E) \subset A$, which is a separable C^*-algebra with $C \subset D$. By Lemma 5, D is an extension of E by K, and therefore Proposition 6 gives:

$$\begin{aligned} \operatorname{topdim}(D[0,n+1)) &= \max\{\operatorname{topdim}(D[0,1)), \operatorname{topdim}(D[1,n+1))\} \\ &= \max\{\operatorname{topdim}(K), \operatorname{topdim}(E' \cap A[1,n+1))\} \\ &\leq d. \end{aligned}$$

Step 3: Assume λ is a limit ordinal, and n is finite. If the statement holds for all $\alpha < \lambda$, then it holds for $\lambda + n$.

We will prove this by distinguishing the two sub-cases that λ has cofinality at most ω, or cofinality bigger than ω. We start the construction for both cases together. Later we will treat them separately. Let $d := \operatorname{topdim}(J)$.

We will inductively define ordinals $\alpha_k < \mu$ and sub-C^*-algebras $D_k, E_k \subset A$ with the following properties:

(1) $\alpha_1 \leq \alpha_2 \leq \ldots$,
(2) $D_k \subset E_k$ and $\operatorname{topdim}(E_k[\lambda, \lambda+n)) \leq d$,
(3) $E_k \subset D_{k+1}$ and $\operatorname{topdim}(D_{k+1}[0, \alpha_{k+1})) \leq d$.

In both cases 3a and 3b below, we construct E_k from D_k as follows: Given D_k, consider $D_k[\lambda,\infty) \subset A[\lambda,\infty)$ and the ideal $A[\lambda,\lambda+n) \lhd A[\lambda,\infty)$ which has a composition series of length n. Since $n < \lambda$, we get by assumption of the induction that there exists a separable subalgebra $E'_k \subset A[\lambda,\infty)$ such that $D_k[\lambda,\infty) \subset E'_k$ and $\operatorname{topdim}(E'_k \cap A[\lambda,\lambda+n)) \leq d$. Let $E_k \subset A$ be any separable C^*-algebra such that $D_k \subset E_k$ and $E_k[\lambda,\infty) = E'_k$.

Case 3a: Assume λ has cofinality at most ω, i.e., there exist ordinals $0 = \lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda$ such that $\lambda = \sup_k \lambda_k$.

In this case, we let $\alpha_k := \lambda_k$, and we set $D_0 := C$. Given D_k, we construct E_k as described above. Given E_k, we get D_{k+1} satisfying (3) by assumption of the induction.

Case 3b: Assume λ has cofinality larger than ω.

We start by setting $\alpha_0 := 0$ and $D_0 := C$. Given D_k, we construct E_k as described above. Given E_k, we define α_{k+1} as follows:

$$\alpha_{k+1} := \inf\{\alpha \mid \alpha_k \leq \alpha \leq \lambda, \text{ and } E_k[0,\alpha) = E_k[0,\lambda)\}.$$

Since λ has cofinality larger than ω and E_k is separable, we have $\alpha_{k+1} < \lambda$. Hence, we get D_{k+1} satisfying (3) by assumption of the induction.

From now on we treat the cases 3a and 3b together. Set $D := \overline{\bigcup_k D_k} = \overline{\bigcup_k E_k}$. This is a separable sub-C^*-algebra of A with $C \subset D$. Since $D[\lambda, \lambda+n) = \overline{\bigcup_k E_k[\lambda, \lambda + n)}$ and $\mathrm{topdim}(E_k[\lambda, \lambda + n)) \leq d$ for all k, we get $\mathrm{topdim}(D[\lambda, \lambda + n)) \leq d$ from Proposition 8.

One checks that $D[0, \lambda) = \overline{\bigcup_k D_k[0, \alpha_k)}$. Since $\mathrm{topdim}(D_k[0, \alpha_k)) \leq d$ for all k, we get $\mathrm{topdim}(D[0, \lambda)) \leq d$, again by Proposition 8.

Then Proposition 6 gives:

$$\mathrm{topdim}(D[0, \lambda + n)) = \max\{\mathrm{topdim}(D[0, \lambda)), \mathrm{topdim}(D[\lambda, \lambda + n))\} \leq d.$$

This completes the proof. □

Corollary 1. *The topological dimension of type I C^*-algebras satisfies the Mardešić factorization axiom (D6), i.e., given a type I C^*-algebra A and a separable sub-C^*-algebra $C \subset A$, there exists a separable C^*-algebra $D \subset A$ such that $C \subset D \subset A$ and $\mathrm{topdim}(D) \leq \mathrm{topdim}(A)$.*

This following theorem is the main result of this paper. It follows immediately from the above Corollary 1, Proposition 6 and Proposition 8.

Theorem 1. *The topological dimension is a noncommutative dimension theory in the sense of Definition 1 for the class of type I C^*-algebras.*

8. Let us extend the topological dimension from the class of type I C^*-algebras to all C^*-algebras, as defined in Proposition 5. This dimension theory $\widetilde{\mathrm{topdim}} : \mathcal{C}^* \to \overline{\mathbb{N}}$ is Morita-invariant since $\mathrm{topdim}(A) = \mathrm{topdim}(A \otimes \mathbb{K})$ for any type I C^*-algebra A.

If $\widetilde{\mathrm{topdim}}(A) < \infty$, then A is in particular approximated by type I sub-C^*-algebras. This implies that A is nuclear, satisfies the universal coefficient theorem (UCT), see [Dad03, Theorem 1.1], and is not properly infinite. It is possible that this dimension theory is connected to the decomposition rank and nuclear dimension, although the exact relation is not clear.

Let us show that the (extended) topological dimension behaves well with respect to tensor products. First, if A, B are separable, type I C^*-algebras, then $\mathrm{Prim}(A \otimes B) \cong \mathrm{Prim}(A) \times \mathrm{Prim}(B)$, see [Bla06, IV.3.4.25, p.390]. This implies:

$$\mathrm{topdim}(A \otimes B) \leq \mathrm{topdim}(A) + \mathrm{topdim}(B).$$

Next, assume A, B are C^*-algebras with $\widetilde{\mathrm{topdim}}(A) = d_1 < \infty$ and $\widetilde{\mathrm{topdim}}(B) = d_2 < \infty$. This means that A is approximated by separable, type I algebras $A_i \subset A$ with $\mathrm{topdim}(A_i) \leq d_1$, and similarly B is approximated by separable, type I algebras $B_j \subset B$ with $\mathrm{topdim}(B_j) \leq d_2$. Then $A \otimes B$ is approximated by the algebras $A_i \otimes B_j$, and we have seen that $\mathrm{topdim}(A_i \otimes B_j) \leq d_1 + d_2$. Thus:

$$\widetilde{\mathrm{topdim}}(A \otimes B) \leq \widetilde{\mathrm{topdim}}(A) + \widetilde{\mathrm{topdim}}(B).$$

Note that we need not specify the tensor product, since $\widetilde{\mathrm{topdim}}(A) < \infty$ implies that A is nuclear.

5. Dimension theories of type I C^*-algebras

In this section we study the relation of the topological dimension of type I C^*-algebras to other dimension theories. It was shown by Brown, [Bro07, Theorem 3.10], how to compute the real and stable rank of a CCR algebra A in terms of the topological dimension of certain canonical algebras A_k associated to A. We use this to obtain a general estimate of the real and stable rank of a CCR algebra in terms of its topological dimension, see Corollary 2. Using the composition series of a type I C^*-algebra, we will obtain similar (but weaker) estimates for general type I C^*-algebras, see Theorem 3.

Let A be a C^*-algebra. We denote by $\mathrm{rr}(A)$ its real rank, see [BP91], by $\mathrm{sr}(A)$ its stable rank, and by $\mathrm{csr}(A)$ its connected stable rank, see [Rie83, Definition 1.4, 4.7] We denote by A_k the successive quotient of A that corresponds to the irreducible representations of dimension k.

If t is a real number, we denote by $\lfloor t \rfloor$ the largest integer $n \leq t$, and by $\lceil t \rceil$ the smallest integer $n \geq t$.

Theorem 2 (Brown, [Bro07, Theorem 3.10])**.** *Let A be a CCR algebra with* $\mathrm{topdim}(A) < \infty$*. Then:*

(1) If $\mathrm{topdim}(A) \leq 1$*, then* $\mathrm{sr}(A) = 1$.
(2) If $\mathrm{topdim}(A) > 1$*, then* $\mathrm{sr}(A) = \sup_{k\geq 1} \max\{\left\lceil \frac{\mathrm{topdim}(A_k)+2k-1}{2k} \right\rceil, 2\}$.
(3) If $\mathrm{topdim}(A) = 0$*, then* $\mathrm{rr}(A) = 0$.
(4) If $\mathrm{topdim}(A) > 0$*, then* $\mathrm{rr}(A) = \sup_{k\geq 1} \max\{\left\lceil \frac{\mathrm{topdim}(A_k)}{2k-1} \right\rceil, 1\}$.

We may draw the following conclusion:

Corollary 2. *Let A be a CCR algebra. Then:*

$$\mathrm{sr}(A) \leq \left\lfloor \frac{\mathrm{topdim}(A)}{2} \right\rfloor + 1, \tag{5.1}$$

$$\mathrm{csr}(A) \leq \left\lfloor \frac{\mathrm{topdim}(A) + 1}{2} \right\rfloor + 1, \tag{5.2}$$

$$\mathrm{rr}(A) \leq \mathrm{topdim}(A). \tag{5.3}$$

Proof. If $\mathrm{topdim}(A) = \infty$, then the statements hold. So we may assume $\mathrm{topdim}(A) < \infty$, whence we may apply [Bro07, Theorem 3.10], see Theorem 2.

Let us show (5.1). If $\mathrm{topdim}(A) \leq 1$, then $\mathrm{sr}(A) = 1 \leq \lfloor \mathrm{topdim}(A)/2 \rfloor + 1$. If $d := \mathrm{topdim}(A) \geq 2$, then we use $\mathrm{topdim}(A_k) \leq d$ to compute:

$$\mathrm{sr}(A) \leq \sup_k \max\{\left\lceil \frac{d + 2k - 1}{2k} \right\rceil, 2\} \leq \max\{\left\lceil \frac{d+1}{2} \right\rceil, 2\} \leq \left\lfloor \frac{d}{2} \right\rfloor + 1.$$

Now (5.2) follows from (5.1) since $\mathrm{csr}(A) \leq \mathrm{sr}(A \otimes C([0,1]))$ in general, by [Nis86, Lemma 2.4], and $\mathrm{topdim}(A \otimes C([0,1])) \leq \mathrm{topdim}(A) + 1$, see 8.

To show (5.3), we again use [Bro07, Theorem 3.10], see Theorem 2. If $\mathrm{topdim}(A) = 0$, then $\mathrm{rr}(A) = 0 \leq \mathrm{topdim}(A)$. If $d := \mathrm{topdim}(A) \geq 1$, then we use $\mathrm{topdim}(A_k) \leq d$ to compute:

$$\mathrm{rr}(A) \leq \sup_k \max\{\left\lceil \frac{d}{2k-1} \right\rceil, 1\} \leq \max\{\lceil d \rceil, 1\} \leq d,$$

which completes the proof. □

Remark 4. What makes type I C^*-algebras so accessible is the presence of composition series with successive quotients that are easier to handle (i.e., of continuous trace or CCR), see 6. They allow us to prove statements by transfinite induction, for which one has to consider the case of a successor and limit ordinal. Let us see that for statements about dimension theories one only needs to consider successor ordinals.

Let $(J_\alpha)_{\alpha\leq\mu}$ be a composition series, and d a dimension theory. If α is a limit ordinal, then $J_\alpha = \overline{\bigcup_{\gamma<\alpha} J_\gamma}$, and we obtain:

$$d(J_\alpha) \leq_{(D5)} \sup_{\gamma<\alpha} d(J_\gamma) \leq_{(D1)} \sup_{\gamma<\alpha} d(J_\alpha),$$

and thus $d(J_\alpha) = \sup_{\gamma<\alpha} d(J_\gamma)$.

Thus, any reasonable estimate about dimension theories that holds for $\gamma < \alpha$ will also hold for α. It follows that we only need to consider a successor ordinal α, in which case $A = J_\alpha$ is an extension of $B = J_\alpha/J_{\alpha-1}$ by $I = J_{\alpha-1}$. By assumption the result is true for I and has to be proved for A (using that B has continuous trace or is CCR). This idea is used to prove the next theorem.

Theorem 3. *Let A be a type I C^*-algebra. Then:*

$$\mathrm{sr}(A) \leq \left\lfloor \frac{\mathrm{topdim}(A)+1}{2} \right\rfloor + 1, \tag{5.4}$$

$$\mathrm{rr}(A) \leq \mathrm{topdim}(A) + 2. \tag{5.5}$$

Proof. We will prove (5.4) by transfinite induction over the length μ of a composition series $(J_\alpha)_{\alpha\leq\mu}$ for A with successive quotients that are CCR algebras.

Set $d := \mathrm{topdim}(A)$. Assume the statement holds for some ordinal μ, and let us show it also holds for $\mu+1$. Consider the ideal $I := J_\mu$ inside $A = J_{\mu+1}$. We obtain the following, where the first estimate follows from [Rie83, Theorem 4.11], and the second estimate follows by assumption of the induction for I and Corollary 2 for the CCR algebra A/I:

$$\begin{aligned}
\mathrm{sr}(A) &\leq \max\{\mathrm{sr}(I), \mathrm{sr}(A/I), \mathrm{csr}(A/I)\} \\
&\leq \max\{\left\lfloor \frac{d+1}{2} \right\rfloor + 1, \left\lfloor \frac{d}{2} \right\rfloor + 1, \left\lfloor \frac{d+1}{2} \right\rfloor + 1\} \\
&= \left\lfloor \frac{d+1}{2} \right\rfloor + 1.
\end{aligned}$$

Let μ be a limit ordinal, and assume the statement holds for $\alpha < \mu$. This means that $\mathrm{sr}(J_\alpha) \leq \left\lfloor \frac{\mathrm{topdim}(J_\alpha)+1}{2} \right\rfloor + 1$ for all $\alpha < \mu$. As explained in Remark 4, we obtain the desired estimate for μ as follows:

$$\mathrm{sr}(J_\mu) = \sup_{\alpha<\mu} \mathrm{sr}(J_\alpha) \leq \sup_{\alpha<\mu} \left\lfloor \frac{\mathrm{topdim}(J_\alpha)+1}{2} \right\rfloor + 1 = \left\lfloor \frac{\mathrm{topdim}(J_\mu)+1}{2} \right\rfloor + 1.$$

Finally, (5.5) follows from (5.4), using the estimate $\mathrm{rr}(A) \leq 2\,\mathrm{sr}(A) - 1$, which holds for all C^*-algebras, see [BP91, Proposition 1.2]. □

Remark 5. It follows from [Rie83, Proposition 1.7], Corollary 2, and Theorem 3 that we may estimate the stable rank of a C^*-algebra A in terms of its topological dimension as follows:

(1) $\mathrm{sr}(A) = \left\lfloor \frac{\mathrm{topdim}(A)}{2} \right\rfloor + 1$, if A is commutative.

(2) $\mathrm{sr}(A) \leq \left\lfloor \frac{\mathrm{topdim}(A)}{2} \right\rfloor + 1$, if A is CCR.

(3) $\mathrm{sr}(A) \leq \left\lfloor \frac{\mathrm{topdim}(A)+1}{2} \right\rfloor + 1$, if A is type I.

This also shows that the inequality for the stable rank in Corollary 2 cannot be improved (the same is true for the estimates of real rank and connected stable rank).

To see that the estimate of Theorem 3 for the stable rank cannot be improved either, consider the Toeplitz algebra $\mathcal{T}$. We have $\mathrm{sr}(\mathcal{T}) = 2$, while $\mathrm{topdim}(\mathcal{T}) = 1$.

Acknowledgments

I thank Søren Knudby for valuable comments and feedback. I thank Mikael Rørdam for interesting discussions and valuable comments, especially on the results in Section 3. I also thank Wilhelm Winter for inspiring discussions on noncommutative dimension theory.

References

[Bla78] B. Blackadar, *Weak expectations and nuclear C^*-algebras*, Indiana Univ. Math. J. **27** (1978), 1021–1026.

[Bla06] ______, *Operator algebras. Theory of C^*-algebras and von Neumann algebras*, Encyclopaedia of Mathematical Sciences 122. Operator Algebras and Non-Commutative Geometry 3. Berlin: Springer. XX, 2006.

[BP91] L. G. Brown and G. K. Pedersen, *C^*-algebras of real rank zero*, J. Funct. Anal. **99** (1991), no. 1, 131–149.

[BP07] ______, *Ideal structure and C^*-algebras of low rank*, Math. Scand. **100** (2007), no. 1, 5–33.

[BP09] ______, *Limits and C^*-algebras of low rank or dimension*, J. Oper. Theory **61** (2009), no. 2, 381–417.

[Bro77] L. G. Brown, *Stable isomorphism of hereditary subalgebras of C^*-algebras*, Pac. J. Math. **71** (1977), 335–348.

[Bro07] ______, *On higher real and stable ranks for CCR C^*-algebras*, preprint, arXiv:0708.3072, 2007.

[Cha94] M. G. Charalambous, *Axiomatic characterizations of the dimension of metric spaces*, Topology Appl. **60** (1994), no. 2, 117–130.

[Dad03] M. Dadarlat, *Some remarks on the universal coefficient theorem in KK-theory*, Operator algebras and mathematical physics. Proceedings of the conference, Constanţa, Romania, July 2–7, 2001, 65-74, 2003.

[Dow55] C. H. Dowker, *Local dimension of normal spaces*, Q. J. Math., Oxf. II. Ser. **6** (1955), 101–120.

[EH95] N. Elhage Hassan, *Real rank of certain extensions. (Rang réel de certaines extensions.)*, Proc. Am. Math. Soc. **123** (1995), no. 10, 3067–3073.

[ET08] G. A. Elliott and A. S. Toms, *Regularity properties in the classification program for separable amenable C^*-algebras*, Bull. Am. Math. Soc., New Ser. **45** (2008), no. 2, 229–245.

[KW04] E. Kirchberg and W. Winter, *Covering dimension and quasidiagonality*, Int. J. Math. **15** (2004), no. 1, 63–85.

[Mar60] S. Mardešić, *On covering dimension and inverse limits of compact spaces*, Ill. J. Math. **4** (1960), 278–291.

[MM92] S. Mardešić and V. Matijević, *$\mathcal{P}$-like spaces are limits of approximate $\mathcal{P}$-resolutions*, Topology Appl. **45** (1992), no. 3, 189–202.

[Mor75] K. Morita, *Cech cohomology and covering dimension for topological spaces*, Fundam. Math. **87** (1975), 31–52.

[MS63] S. Mardešić and J. Segal, *ϵ-mappings onto polyhedra*, Trans. Am. Math. Soc. **109** (1963), 146–164.

[Nag70] K. Nagami, *Dimension theory. With an appendix by Yukihiro Kodama*, Pure and Applied Mathematics. Vol. 37. New York-London: Akademic Press 1970. XI, 1970.

[Nis74] T. Nishiura, *An axiomatic characterization of covering dimension in metrizable spaces*, TOPO 72 - General Topology Appl., 2nd Pittsburgh internat. Conf. 1972, Lect. Notes Math. 378, 341-353, 1974.

[Nis86] V. Nistor, *Stable range for tensor products of extensions of $\mathbb{K}$ by $C(X)$*, J. Oper. Theory **16** (1986), 387–396.

[Pea75] A. R. Pears, *Dimension theory of general spaces*, Cambridge etc.: Cambridge University Press. XII, 1975.

[Ped79] G. K. Pedersen, *C^*-algebras and their automorphism groups*, London Mathematical Society Monographs. 14. London - New York -San Francisco: Academic Press. X, 1979.

[Rie83] M. A. Rieffel, *Dimension and stable rank in the K-theory of C^*-algebras*, Proc. Lond. Math. Soc., III. Ser. **46** (1983), 301–333.
[Rør92] M. Rørdam, *On the structure of simple C^*-algebras tensored with a UHF-algebra. II*, J. Funct. Anal. **107** (1992), no. 2, 255–269.
[Rør06] ———, *Structure and classification of C^*-algebras*, Zürich: European Mathematical Society (EMS), 2006.
[Sud04] T. Sudo, *A topological rank for C^*-algebras*, Far East J. Math. Sci. (FJMS) **15** (2004), no. 1, 71–86.
[Thi09] H. Thiel, *One-dimensional C^*-algebras and their K-theory*, diploma thesis, University of Münster; available at www.math.ku.dk/ thiel, 2009.
[Thi11] ———, *Inductive limits of projective C^*-algebras*, preprint, arXiv:1105.1979, 2011.
[Win12] W. Winter, *Nuclear dimension and $\mathcal{Z}$-stability of pure C^*-algebras*, Invent. Math. **187** (2012), no. 2, 259–342.
[WZ10] W. Winter and J. Zacharias, *The nuclear dimension of C^*-algebras*, Adv. Math. **224** (2010), no. 2, 461–498.

Department of Mathematical Sciences, University of Copenhagen, Universitetsparken 5, DK-2100, Copenhagen Ø, Denmark
E-mail address: thiel@math.ku.dk

THE GENERATOR RANK FOR C^*-ALGEBRAS

HANNES THIEL

ABSTRACT. The invariant that assigns to a C^*-algebra its minimal number of generators lacks natural permanence properties. In particular, it may increase when passing to ideals or inductive limits. It is therefore hard to compute this invariant directly.

To obtain a better behaved theory, we not only ask if k generators exist, but also if such tuples are dense. This defines the generator rank, which we show has many of the permanence properties that are also satisfied by other noncommutative dimension theories. In particular, it does not increase when passing to ideals, quotients or inductive limits.

The definition of the generator rank is analogous to that of the real rank, and we show that the latter always dominates the generator rank. The most interesting value of the generator rank is one, which means exactly that the generators form a generic set, that is, a dense G_δ-subset. We compute the generator rank of homgeneous C^*-algebras, which allows us to deduce that certain AH-algebras have generator rank one. For example, every AF-algebra has generator rank one and therefore contains a dense set of generators.

1. INTRODUCTION

The generator problem for C^*-algebras is to determine which C^*-algebras are singly generated. More generally, for a given C^*-algebra A one wants to determine the minimal number of generators, i.e., the minimal k such that A contains k elements that are not contained in any proper sub-C^*-algebra. For a more detailed discussion of the generator problem, we refer the reader to the recent paper by Wilhelm Winter and the author, [TW12], where it is also shown that every unital, separable $\mathcal{Z}$-stable C^*-algebra is singly generated, see [TW12, Theorem 3.7].

Given a C^*-algebra A, let us denote by $\mathrm{gen}(A)$ the minimal number of *self-adjoint* generators for A, and set $\mathrm{gen}(A)=\infty$ if A is not finitely generated, see [Nag]. The restriction to self-adjoint elements is mainly for convenience. It only leads to a minor variation of the original generator problem, since two self-adjoint elements a, b generate the same sub-C^*-algebra as the element $a+ib$. In particular, A is singly generated if and only if it is generated by two self-adjoint elements, that is, if and only if $\mathrm{gen}(A)\leq 2$. For a compact, metric space X, it is easy to see that $\mathrm{gen}(C(X))\leq k$ if and only if X can be embedded into $\mathbb{R}^k$.

The problem with computing the minimal number of self-adjoint generators is that it does not behave well with respect to inductive limits, i.e., in general we do not have $\mathrm{gen}(A)\leq\liminf_n \mathrm{gen}(A_n)$ if $A=\varinjlim A_n$ is an inductive limit. This is unfortunate

Date: 24 October 2012.

2010 *Mathematics Subject Classification.* Primary 46L05, 46L85; Secondary 54F45, 55M10.

Key words and phrases. C^*-algebras, dimension theory, real rank, generator rank, generator problem, single generation.

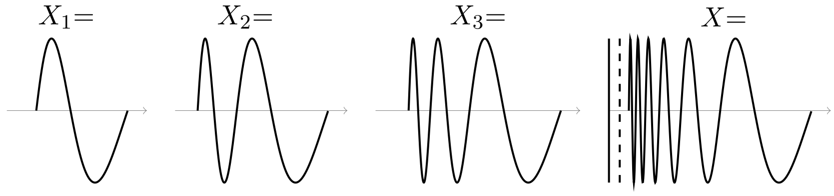
This research was supported by the Danish National Research Foundation through the Centre for Symmetry and Deformation.

since many C^*-algebras are given as inductive limits, e.g., AF-algebras or approximately homogeneous algebras (AH-algebras).

To see an example where the minimal number of generators increasing when passing to an inductive limit, let $X \subset \mathbb{R}^2$ be the topologists sine-curve given by:

$$X = \{0\} \times [-1, 1] \cup \{(t, \sin(\frac{1}{t})) \mid t \in (0, 1/2\pi]\}.$$

Then X can be embedded into $\mathbb{R}^2$ but not into $\mathbb{R}^1$, and therefore $\mathrm{gen}(C(X)) = 2$. However, X is an inverse limit of spaces X_n that are each homeomorphic to the interval, i.e., $X_n \cong [0, 1]$. Therefore $C(X) \cong \varinjlim_n C(X_n)$, with $\mathrm{gen}(C(X)) = 2$, while $\mathrm{gen}(C(X_n)) = 1$ for all n. The spaces X and X_1, X_2, X_3 are shown below.



By considering the spaces $X \times [0, 1]$ and $X_n \times [0, 1]$, one obtains an example of singly generated C^*-algebras A_n such that their inductive limit is not singly generated.

To get a better behaved theory, instead of counting the minimal number of self-adjoint generators, we will count the minimal number of "stable" self-adjoint generators. This is the underlying idea of our definition of the generator rank of a C^*-algebra, see Definition 2.2. More precisely, let A^k_{sa} denote the space of self-adjoint k-tuples in A, and let $\mathrm{Gen}_k(A)_{\mathrm{sa}} \subset A^k_{\mathrm{sa}}$ be the subset of tuples that generate A, see Notation 2.1. We say that A has generator rank at most k, denoted by $\mathrm{gr}(A) \leq k$, if $\mathrm{Gen}_{k+1}(A)_{\mathrm{sa}}$ is dense in A^{k+1}_{sa}. This definition is analogous to that of the real rank, see Remark 2.3, and this also explains the index shift of the definition.

Thus, while "$\mathrm{gen}(A) \leq k$" records that $\mathrm{Gen}_k(A)_{\mathrm{sa}}$ is not empty, "$\mathrm{gr}(A) \leq k - 1$" records that $\mathrm{Gen}_k(A)_{\mathrm{sa}}$ is dense. This indicates why the generator rank is usually much larger than the minimal number of self-adjoint generators. The payoff, however, is that the generator rank is much easier to compute.

The paper is organized as follows: In Section 2, we define the generator rank, see Definition 2.2, and we derive some of its general properties. We show that the set of generating tuples, $\mathrm{Gen}_k(A)_{\mathrm{sa}}$, always forms a G_δ-subset of A^k_{sa}, see Proposition 2.7. It follows, that $\mathrm{gr}(A) \leq 1$ if and only if the set of generators in A forms a generic set, i.e., a dense G_δ-subset.

For an inductive limit $A = \varinjlim A_n$, we obtain

$$\mathrm{gr}(A) \leq \liminf_n \mathrm{gr}(A_n),$$

which shows that the generator rank is indeed better behaved than the theory of counting the minimal number of generators. As an immediate consequence, we get that every AF-algebra has generator rank at most one, see Corollary 3.3. Thus, every AF-algebra contains a generic set of generators.

We show that the generator rank does not increase when passing to ideals or quotients, see Theorem 2.14 and Proposition 2.12. We also provide an estimate of the generator rank of an extension of C^*-algebras in terms of the generator rank of ideal and quotient, see Theorem 2.17.

This shows that the generator rank has many properties which are also satisfied by other "dimension theories" for C^*-algebras, such as the real and stable rank, the nuclear dimension, and the decomposition rank. The concept of a "noncommutative dimension theory" was recently introduced in [Thi11, Definition 2.1] by proposing six axioms that such theories should satisfy. Our results show that the generator rank for separable C^*-algebras satisfies five of these axioms, see Remark 2.16. The remaining axiom would mean that $\mathrm{gr}(A\oplus B) = \max\{\mathrm{gr}(A), \mathrm{gr}(B)\}$, which seems to be surprisingly hard to show.

In Section 3, we study the generator rank on the class of separable C^*-algebras with real rank zero. For such algebras, we show that the remaining axiom holds, i.e., that the generator rank behaves well with respect to direct sums, see Proposition 3.1. We then show that AF-algebras have generator rank at most one, see Corollary 3.3. We proceed by showing the estimate $\mathrm{gr}(A \otimes M_n) \leq \left\lceil \frac{\mathrm{gr}(A)}{n^2} \right\rceil$ for unital C^*-algebras with real rank zero and stable rank one, see Theorem 3.6. Therefore, given such an algebra with finite generator rank, its tensor product with an infinite UHF-algebra has generator rank one. More generally, we show that every separable, real rank zero C^*-algebra that tensorially absorbs a UHF-algebra has generator rank at most one, see Proposition 3.8.

In Section 4, we first compute the generator rank of commutative C^*-algebras as $\mathrm{gr}(C(X)) = \dim(X \times X)$ for a compact, metric space X, see Proposition 4.7. We then compute the codimension of the subspace $\mathrm{Gen}_k(M_n)_{\mathrm{sa}} \subset (M_n)^k_{\mathrm{sa}}$, see Lemma 4.20. This allows us to compute the generator rank of homogeneous C^*-algebras, see Theorem 4.23. In particular, if X is a compact, metric space, and $n \geq 2$, then:

$$\mathrm{gr}(C(X, M_n)) = \left\lceil \frac{\dim(X) + 1}{2n - 2} \right\rceil.$$

This allows us to show that a unital, separable AH-algebra has generator rank one if it is either simple with slow dimension growth, or when it tensorially absorbs a UHF-algebra, see Corollary 4.30.

Throughout, we will use the following notation. For a C^*-algebra A, we let A_{sa} (resp. A_+, A^{-1}) denote the set of self-adjoint (resp. positive, resp. invertible) elements in A. We denote by $\widetilde{A}$ the minimal unitization of A. By a morphism between C^*-algebras we always mean a *-homomorphism. We write $J \lhd A$ to indicate that J is an ideal in A, and by an ideal of a C^*-algebra we understand a closed, two-sided ideal. The primitive ideal space of A will be denoted by $\mathrm{Prim}(A)$. We write M_k for the C^*-algebra of k-by-k matrices $M_k(\mathbb{C})$.

Given $a, b \in A$, and $\varepsilon > 0$, we write $a =_\varepsilon b$ if $\|a - b\| < \varepsilon$. If a, b are positive, then we write $a \ll b$ if $a = ab$, and we write $a \ll_\varepsilon b$ if $a =_\varepsilon ab$. If $F, G \subset A$ are two subsets, and $a \in A$, we write $a \in_\varepsilon G$ if $\mathrm{dist}(a, G) < \varepsilon$, and write $F \subset_\varepsilon G$ if $x \in_\varepsilon G$ for every $x \in F$.

We use bold letters to denote tuples of elements, e.g., $\mathbf{a} = (a_1, \ldots, a_k) \in A^k$.

2. The generator rank

In this section, we define the generator rank of a C^*-algebra, see Definition 2.2, in analogy to the real rank, see Remark 2.4. We then prove general properties of the generator rank, in particular that it behaves well with respect to approximation by subalgebras and inductive limits, see Proposition 2.13. We show that the generator rank does not increase when passing to ideals or quotients, see Theorem 2.14 and Proposition 2.12.

We also provide an estimate of the generator rank of an extension of C^*-algebras in terms of the generator rank of ideal and quotient, see Theorem 2.17.

The concept of a noncommutative dimension theory was introduced in [Thi11, Definition 2.1] by proposing six axioms that such theories should satisfy. Our results show that the generator rank for separable C^*-algebras satisfies five of these axioms, see Remark 2.16, and we conjecture that it also satisfies the missing axiom.

Notation 2.1. Let A be a C^*-algebra. Recall that we use bold letters to denote tuples of elements, e.g., $\mathbf{a} = (a_1, \ldots, a_k) \in A^k$. We denote by $C^*(\mathbf{a})$ the sub-C^*-algebra of A generated by the elements of $\mathbf{a}$.

For $k \geq 1$, we write A^k_{sa} for $(A_{\mathrm{sa}})^k$, the space of self-adjoint k-tuples. We denote the set of generating (self-adjoint) k-tuples by:

$$\begin{aligned} \mathrm{Gen}_k(A) &:= \{\mathbf{a} \in A^k \mid A = C^*(\mathbf{a})\}, \\ \mathrm{Gen}_k(A)_{\mathrm{sa}} &:= \mathrm{Gen}_k(A) \cap A^k_{\mathrm{sa}}. \end{aligned}$$

We equip A^k with the usual norm, i.e., $\|\mathbf{a}\| := \max\{\|a_1\|, \ldots, \|a_k\|\}$ for a tuple $\mathbf{a} \in A^k$.

Definition 2.2. Let A be a unital C^*-algebra. The **generator rank** of A, denoted by $\mathrm{gr}(A)$, is the smallest integer $k \geq 0$ such that $\mathrm{Gen}_{k+1}(A)_{\mathrm{sa}}$ is dense in A^{k+1}_{sa}. If no such n exists, we set $\mathrm{gr}(A) = \infty$.

Given a non-unital C^*-algebra A, set $\mathrm{gr}(A) := \mathrm{gr}(\widetilde{A})$.

Remark 2.3. The definition of the generator rank is analogous to that of the real rank as given by Brown and Pedersen, [BP91]. Let us recall the definition.

Let A be a unital C^*-algebra. One uses the following notation:

$$\begin{aligned} \mathrm{Lg}_k(A) &:= \{\mathbf{a} \in A^k \mid \sum_{i=1}^k a_i^* a_i \in A^{-1}\}, \\ \mathrm{Lg}_k(A)_{\mathrm{sa}} &:= \mathrm{Lg}_k(A) \cap A^k_{\mathrm{sa}}. \end{aligned}$$

The abbreviation "Lg" stands for "left generators", and the reason is that a tuple $\mathbf{a} \in A^k$ lies in $\mathrm{Lg}_k(A)$ if and only if the elements $a_1, \ldots, a_k$ generate A as a (not necessarily closed) left ideal, i.e., $Aa_1 + \ldots + Aa_k = A$.

Rieffel introduced the (topological) stable rank of A, denoted by $\mathrm{sr}(A)$, as the smallest integer $k \geq 1$ such that $\mathrm{Lg}_k(A)$ is dense in A^k, see [Rie83, Definition 1.4]. Considering the analogous question for tuples of self-adjoint elements, Brown and Pedersen defined the real rank of A, denoted by $\mathrm{rr}(A)$, as the smallest integer $k \geq 0$ such that $\mathrm{Lg}_{k+1}(A)_{\mathrm{sa}}$ is dense in A^{k+1}_{sa}, see [BP91]. Note the index shift in the definition of the real rank (as opposed to the definition of stable rank). It leads to nicer formulas, e.g., $\mathrm{rr}(C(X)) = \dim(X)$. We use the same index shift in Definition 2.2 since the generator rank is more closely connected to the real rank than to the stable rank, as we will see now.

Remark 2.4. Let A be a C^*-algebra. We may consider the following variant of the generator rank, defined as the smallest integer $k \geq 1$ such that $\mathrm{Gen}_k(A)$ is dense in A^k. Let us denote this value by $\mathrm{gr}'(A)$. Since the generator rank $\mathrm{gr}(A)$ is defined in analogy to the real rank (using tuples of self-adjoint elements), one might expect that the invariant gr' (using tuples of not necessarily self-adjoint elements) has a closer connection to the stable rank.

This is, however, not the case. For instance, while the estimate $\mathrm{rr}(A) \leq \mathrm{gr}(A)$ always holds, see Proposition 2.5, we will below see an example of a C^*-algebra where $\mathrm{sr}(A) \not\leq$

$\mathrm{gr}'(A)$. Moreover, unlike the real and stable rank, the invariants gr and gr′ are very closely tied together:

$$\mathrm{gr}'(A) = \left\lceil \frac{\mathrm{gr}(A)+1}{2} \right\rceil .$$

To prove this formula, consider the map $\Phi\colon A_{\mathrm{sa}}^{2k} \to A^k$ that sends $(a_1, \ldots, a_{2k}) \in A_{\mathrm{sa}}^{2k}$ to $(a_1 + ia_{k+1}, \ldots, a_k + ia_{2k}) \in A^k$. In general, two self-adjoint elements $c, d \in A_{\mathrm{sa}}$ generate the same sub-C^*-algebra as the element $c+id$. It follows $C^*(\mathbf{a}) = C^*(\Phi(\mathbf{a})) \subset A$ for every $\mathbf{a} \in A_{\mathrm{sa}}^{2k}$, and so Φ maps $\mathrm{Gen}_{2k}(A)_{\mathrm{sa}}$ onto $\mathrm{Gen}_k(A)$. Thus, for every $k \geq 1$, $\mathrm{gr}(A) \leq 2k-1$ if and only if $\mathrm{gr}'(A) \leq k$, from which the formula follows.

Assume now that A is unital. If $\mathbf{a} \in A_{\mathrm{sa}}^k$ generates A as a C^*-algebra, then it also generates A as a left ideal. Indeed, assume p is a polynomial such that $\|1 - p(\mathbf{a})\| < 1$. Then $p(\mathbf{a})$ is invertible, and we denote its inverse by $v \in A$. Write p as a sum of polynomials, $p = \sum_{i=1}^k p_i$, where each p_i is of the form $p_i(\mathbf{x}) = q_i(\mathbf{x}) \cdot x_i$ for some other polynomial q_i. Then:

$$1 = v \cdot p(\mathbf{a}) = \sum_i (v \cdot q_i(\mathbf{a}))\, a_i,$$

which shows $\mathbf{a} \in \mathrm{Lg}_k(A)_{\mathrm{sa}}$.

Thus, for every $k \geq 1$, the following inclusion holds:

$$\mathrm{Gen}_k(A)_{\mathrm{sa}} \subset \mathrm{Lg}_k(A)_{\mathrm{sa}},$$

which immediately implies Proposition 2.5 below.

The analog inclusion $\mathrm{Gen}_k(A) \subset \mathrm{Lg}_k(A)$ does *not* hold. For a counterexample, consider $A = C([0,1]^2, M_3)$. Then $\mathrm{sr}(A) = 2$ and so $\mathrm{Lg}_1(A)$ is not dense in A. On the other hand, $\mathrm{Gen}_1(A)$ is dense in A since $\mathrm{gr}(A) = 1$, see Theorem 4.23. This also shows that $\mathrm{sr}(A) \not\leq \mathrm{gr}'(A)$.

Proposition 2.5. *Let A be a C^*-algebra. Then* $\mathrm{rr}(A) \leq \mathrm{gr}(A)$.

Proof. This follows immediately from the the definition of real and generator rank together with the inclusion $\mathrm{Gen}_k(\widetilde{A})_{\mathrm{sa}} \subset \mathrm{Lg}_k(\widetilde{A})_{\mathrm{sa}}$ for every $k \geq 1$, which is shown in Remark 2.4. □

While many C^*-algebras have real rank zero, the case of generator rank zero is very special:

Lemma 2.6. *Let A be a C^*-algebra. Then* $\mathrm{gr}(A) = 0$ *if and only if A is a separable, commutative C^*-algebra with zero-dimensional spectrum.*

Proof. If $\mathrm{gr}(A) = 0$, then A contains a generating self-adjoint element, and so A is separable and commutative. Thus, it remains to show that $\mathrm{gr}(A) = 0$ if and only if $\dim(\mathrm{Prim}(A)) = 0$ under the assumption that A is separable and commutative. Note that for every locally compact, second countable, Hausdorff space X, $\dim(X) = 0$ if and only if $\dim(X \times X) = 0$. Therefore, the result follows from Proposition 4.7. □

It is easy to see that $\mathrm{Lg}_k(A) \subset A^k$ and $\mathrm{Lg}_k(A)_{\mathrm{sa}} \subset A_{\mathrm{sa}}^k$ are open subsets. For the sets of generating tuples, we have the following result:

Proposition 2.7. *Let A be a C^*-algebra, and let $k \in \mathbb{N}$. Then* $\mathrm{Gen}_k(A) \subset A^k$ *and* $\mathrm{Gen}_k(A)_{sa} \subset A_{sa}^k$ *are G_δ-subsets.*

Proof. We show that $\mathrm{Gen}_k(A) \subset A^k$ is a G_δ-subset. The empty set is clearly G_δ, so we may assume $\mathrm{Gen}_k(A) \neq \emptyset$, which in turn implies that A is separable. Let $a_1, a_2, \ldots$ be a dense sequence in A. Define:

$$U_n := \{\mathbf{x} \in A^k \mid a_1, \ldots, a_n \in_{1/n} C^*(\mathbf{x})\}.$$

Let us check that $U_n \subset A^k$ is open. So let $\mathbf{x} \in U_n$. Then there exist polynomials $p_1, \ldots, p_n$ such that $a_i =_{1/n} p_i(\mathbf{x})$ for $i = 1, \ldots, n$. We may consider each p_i as a function $A^k \to A$, which is clearly continuous. Therefore, for each i, there exists $\delta_i > 0$ such that $a_i =_{1/n} p_i(\mathbf{y})$ for all $\mathbf{y} \in A^k$ with $\mathbf{y} =_{\delta_i} \mathbf{x}$. Then the open ball around $\mathbf{x}$ with radius $\min\{\delta_1, \ldots, \delta_n\}$ is contained in U_n, which is therefore open.

One checks that $\mathrm{Gen}_k(A) = \bigcap_{n\geq 1} U_n$, which completes the proof for $\mathrm{Gen}_k(A)$. The result for $\mathrm{Gen}_k(A)_{\mathrm{sa}}$ is proved analogously. □

Remark 2.8. Let A be a unital C^*-algebra. It is a consequence of Remark 2.4 and Proposition 2.7 that $\mathrm{gr}(A) \leq 1$ if and only if the set of generators $\mathrm{Gen}_1(A)$ forms a generic subset of A, i.e., if and only if $\mathrm{Gen}_1(A) \subset A$ is a dense G_δ-set.

Our main tool to construct generators is the following Lemma 2.9, which was obtained together with Karen Strung, Aaron Tikuisis, Joav Orovitz and Stuart White at the workshop "Set theory and C^*-algebras" at the AIM in Palo Alto, January 2012.

It reduces the problem of showing $\mathrm{gr}_{\mathrm{sa}}(A) \leq k$. Instead of proving that every tuple can be approximated arbitrarily closely by tuples that generate the whole C^*-algebra, it is enough to show that every tuple can be approximated by tuples that approximately generate a single given element of A.

Lemma 2.9. *Let A be a separable C^*-algebra, let $k \in \mathbb{N}$, and let $S \subset A^k$ be a closed subset. Assume that for every $\mathbf{x} \in S$, every $\varepsilon > 0$, and every $z \in A$ there exists $\mathbf{y} \in S$ such that $\mathbf{y} =_\varepsilon \mathbf{x}$ and $z \in_\varepsilon C^*(\mathbf{y})$. Then $\mathrm{Gen}_k(A) \cap S \subset S$ is dense, i.e., for every $\mathbf{x} \in S$ and $\varepsilon > 0$ there exists $\mathbf{y} \in S$ such that $\mathbf{y} =_\varepsilon \mathbf{x}$ and $A = C^*(\mathbf{y})$.*

Proof. Let $a_2, a_3, \ldots \in A$ be a dense sequence of A, where each element is repeated infinitely many times, and set $a_1 = 0$. We inductively find tuples $\mathbf{y}_k \in S$ and numbers $\delta_n > 0$ with the following properties:

(1) $\|\mathbf{y}_n - \mathbf{y}_{n-1}\| < \min\{\delta_1/2^{n-1}, \delta_2/2^{n-2}, \ldots, \delta_{n-1}/2\}$,
(2) $a_n \in_{1/n} C^*(\mathbf{y}')$ whenever $\mathbf{y}' =_{\delta_n} \mathbf{y}_n$.

Set $\mathbf{y}_1 := \mathbf{x}$ and $\delta_1 := \varepsilon$. Then (2) is trivially satisfied.

Assume $\mathbf{y}_i$ and δ_i have been constructed for $i \leq n-1$. By assumption, we can find $\mathbf{y}_n \in S$ satisfying (1) and such that $a_n \in_{1/n} C^*(\mathbf{y}_n)$. Then there exists a polynomial p such that $a_n =_{1/n} p(\mathbf{y}_n)$. We may consider p as a function $A^k \to A$, which is continuous. Therefore, there exists $\delta_n > 0$ satisfying (2).

Condition (1) ensures in particular that $\mathbf{y}_n$ is a Cauchy sequence. Set $\mathbf{y} := \lim_n \mathbf{y}_n \in S$, and let us check that it has the desired properties.

For each n, repeated application of (1) gives:

$$\|\mathbf{y} - \mathbf{y}_n\| < \sum_{i\geq 1} \delta_n/2^i = \delta_n.$$

Thus, $\|\mathbf{y} - \mathbf{x}\| = \|\mathbf{y} - \mathbf{y}_1\| < \delta_1 = \varepsilon$. Moreover, condition (2) ensures that $a_n \in_{1/n} C^*(\mathbf{y})$ for all k. It follows that $a_n \in C^*(\mathbf{y})$, since a_n was assumed to appear infinitely many times in the sequence $a_1, a_2, \ldots$. Since the sequence a_n is dense in A, it follows $A = C^*(\mathbf{y})$, as desired. □

2.10. When we want to estimate the generator rank of an ideal, we have to be careful about adjoining a unit. Mainly for technical reasons, we introduce the following variant of the generator rank:

(2.1) $\quad \mathrm{gr}_{\mathrm{sa}}(A) \leq k :\Leftrightarrow \mathrm{Gen}_{k+1}(A)_{\mathrm{sa}} \subset (A_{\mathrm{sa}})^{k+1}$ is dense .

By definition, $\mathrm{gr}(A) = \mathrm{gr}_{\mathrm{sa}}(\widetilde{A})$. The connection between the generator rank and its variant is summarized in the next result.

Lemma 2.11. *Let A be a C^*-algebra. Then:*

$$\mathrm{gr}(A) = \max\{\mathrm{rr}(A), \mathrm{gr}_{sa}(A)\}.$$

Proof. If A is unital, the statement follows from Proposition 2.5. So assume A is non-unital, and denote by $1 \in \widetilde{A}$ the adjoint unit. Let $\pi\colon \widetilde{A} \to \mathbb{C}$ be the quotient morphism. It induces a natural morphism $\widetilde{A}^k \to \mathbb{C}^k$, which we also denote by π. Let $\sigma\colon \mathbb{C} \to \widetilde{A}$ denote the canonical split of π. We denote the induced morphism $\sigma\colon \mathbb{C}^k \to \widetilde{A}^k$ also by σ.

Let us show $\mathrm{gr}_{\mathrm{sa}}(A) \leq \mathrm{gr}_{\mathrm{sa}}(\widetilde{A}) = \mathrm{gr}(A)$. Let $k := \mathrm{gr}(A) + 1$, and we may assume this is finite. We want to verify the conditions of Lemma 2.9 for $S = A^k_{\mathrm{sa}}$. So let $\mathbf{x} \in A^k_{\mathrm{sa}}$, $\varepsilon > 0$ and $z \in A$ be given. By assumption, there exists $\mathbf{y} \in \widetilde{A}^k_{\mathrm{sa}}$ with $\mathbf{x} =_{\varepsilon/2} \mathbf{y}$ and a polynomial q such that $z =_{\varepsilon/2} q(\mathbf{y})$.

There is a unique decomposition $\mathbf{y} = \mathbf{a} + \mathbf{r}$ for $\mathbf{r} = \sigma \circ \pi(\mathbf{x}) \in (\mathbb{C}1)^k$ and $\mathbf{a} \in A^k_{\mathrm{sa}}$. Then $\|\mathbf{r}\| < \varepsilon/2$, and therefore $\mathbf{x} =_{\varepsilon/2} \mathbf{y} =_{\varepsilon/2} \mathbf{a}$. Note that $p(\mathbf{y}) = p(\mathbf{a} + \mathbf{r})$ has the form $q(\mathbf{a}) + \lambda 1$ for some polynomial q and a constant $\lambda \in \mathbb{C}$. Since $q(\mathbf{a}) \in A^k$ and $p(\mathbf{y}) =_{\varepsilon/2} z \in A^n$, we get $|\lambda| \leq \varepsilon/2$. Then $q(\mathbf{a}) =_{\varepsilon} z$, and so $z \in_{\varepsilon} C^*(\mathbf{a})$, which shows that $\mathbf{a}$ has the desired properties. It follows from Lemma 2.9 that $\mathrm{Gen}_k(A)_{\mathrm{sa}} \subset A^k_{\mathrm{sa}}$ is dense, and so $\mathrm{gr}_{\mathrm{sa}}(A) \leq \mathrm{gr}(A)$, as desired.

It was shown in Proposition 2.5 that $\mathrm{rr}(A) \leq \mathrm{gr}(A)$. Thus, it remains to show $\mathrm{gr}(A) \leq \max\{\mathrm{rr}(A), \mathrm{gr}_{\mathrm{sa}}(A)\}$.

Let $k := \max\{\mathrm{rr}(A), \mathrm{gr}_{\mathrm{sa}}(A)\} + 1$, and we may assume this is finite. Let $\mathbf{x} \in \widetilde{A}^k_{\mathrm{sa}}$ and $\varepsilon > 0$ be given. By assumption, there is $\mathbf{x}' \in \mathrm{Lg}_k(\widetilde{A})_{\mathrm{sa}}$ with $\mathbf{x}' =_{\varepsilon/2} \mathbf{x}$. Since $\mathrm{Lg}_k(\widetilde{A})_{\mathrm{sa}}$ is open, there exists $\delta > 0$ such that $\mathbf{b} \in \mathrm{Lg}_k(\widetilde{A})_{\mathrm{sa}}$ whenever $\mathbf{b} \in \widetilde{A}^k_{\mathrm{sa}}$ satisfies $\mathbf{b} =_{\delta} \mathbf{x}'$. We may assume $\delta < \varepsilon/2$.

There is a unique decomposition $\mathbf{x}' = \mathbf{a} + \mathbf{r}$ for $\mathbf{r} = \sigma \circ \pi(\mathbf{x}) \in (\mathbb{C}1)^k$ and $\mathbf{a} \in A^k_{\mathrm{sa}}$. By assumption, there exists $\mathbf{a}' \in A^k_{\mathrm{sa}}$ with $\mathbf{a}' =_{\delta} \mathbf{a}$ and such that $C^*(\mathbf{a}') = A$. Set $\mathbf{y} := \mathbf{a}' + \mathbf{r}$. Note that $\mathbf{y} =_{\delta} \mathbf{x}'$, and therefore $\mathbf{y} \in \mathrm{Lg}_k(\widetilde{A})_{\mathrm{sa}}$. It follows $1 \in C^*(\mathbf{y})$, and so $a'_i = y_i - r_i 1 \in C^*(\mathbf{y})$ for $i = 1, \ldots, k$. Thus, $\mathbf{y}$ generates $\widetilde{A}$. Moreover, $\mathbf{y} =_{\varepsilon} \mathbf{x}$. We have shown that $\mathrm{Gen}_k(\widetilde{A})_{\mathrm{sa}}$ is dense in $\widetilde{A}^k_{\mathrm{sa}}$, and so $\mathrm{gr}(A) \leq k - 1$, as desired. □

Proposition 2.12. *Let A be a C^*-algebra, and let $J \lhd A$ be an ideal. Then* $\mathrm{gr}(A/J) \leq \mathrm{gr}(A)$.

Proof. Note that J is also an ideal in $\widetilde{A}$, and $\widetilde{A}/J \cong \widetilde{A/J}$. Let $\pi\colon \widetilde{A} \to \widetilde{A/J}$ denote the quotient morphism. It induces a surjective morphism $\widetilde{A}^k_{\mathrm{sa}} \to \widetilde{A/J}^k_{\mathrm{sa}}$, which sends $\mathrm{Gen}_k(\widetilde{A})_{\mathrm{sa}}$ into $\mathrm{Gen}_k(\widetilde{A/J})_{\mathrm{sa}}$. Thus, if $\mathrm{Gen}_k(\widetilde{A})_{\mathrm{sa}} \subset \widetilde{A}^k_{\mathrm{sa}}$ is dense, then so is $\mathrm{Gen}_k(\widetilde{A/J})_{\mathrm{sa}} \subset \widetilde{A/J}^k_{\mathrm{sa}}$. This shows $\mathrm{gr}(A/J) \leq \mathrm{gr}(A)$. □

One immediate consequence of the key lemma Lemma 2.9 is that the generator rank behaves well with respect to approximation by sub-C^*-algebras and inductive limits.

Recall that a collection $A_i \subset A$ of sub-C^*-algebras is said to **approximate** A if for every finite subset $F \subset A$ and for every $\varepsilon > 0$, there exists i such that $F \subset_\varepsilon A_i$.

Proposition 2.13. *Let A be a separable C^*-algebra, and let $k \geq 0$. Assume A is approximated by sub-C^*-algebras $A_i \subset A$ with $\mathrm{gr}(A_i) \leq k$. Then $\mathrm{gr}(A) \leq k$.*

Moreover, if $A = \varinjlim A_n$ is an inductive limit, then $\mathrm{gr}(A) \leq \liminf_n \mathrm{gr}(A_n)$.

Proof. Assume A is approximated by a collection $A_i \subset A$ with $\mathrm{gr}(A_i) \leq k$. For each i, we have $\mathrm{rr}(A_i) \leq k$ by Proposition 2.5. It follows $\mathrm{rr}(A) \leq k$, since the real rank behaves well with respect to approximation by subalgebras, as noted in [Thi12, Remark 2]. Thus, by Lemma 2.11, it is enough to show $\mathrm{gr}_{\mathrm{sa}}(A) \leq k$.

We want to verify the conditions of Lemma 2.9 for $S = A_{\mathrm{sa}}^{k+1}$. So let $\mathbf{x} \in A_{\mathrm{sa}}^{k+1}$, $\varepsilon > 0$ and $z \in A$ be given. Since the A_i approximate A, there exists an index i such that there is $\mathbf{x}' \in (A_i)_{\mathrm{sa}}^{k+1}$ with $\mathbf{x}' =_{\varepsilon/2} \mathbf{x}$, and such that there is $z' \in A_i$ with $z' =_\varepsilon z$. Since $\mathrm{gr}(A_i) \leq k$, there exists $\mathbf{y} \in \mathrm{Gen}_{k+1}(A_i)_{\mathrm{sa}}$ with $\mathbf{y} =_{\varepsilon/2} \mathbf{x}'$. Then $\mathbf{y} =_\varepsilon \mathbf{x}$ and $z \in_\varepsilon C^*(\mathbf{y})$, as desired. It follows from Lemma 2.9 that $\mathrm{gr}_{\mathrm{sa}}(A) \leq k$.

This result, together with Proposition 2.12, implies the estimate for an inductive limit. The argument is standard for dimension theories, see [Thi12, Proposition 2], but for the convenience of the reader we include a short proof.

Assume $A = \varinjlim A_n$. For each n, let B_n be the image of A_n in the inductive limit A. Then B_n is a quotient of A_n, and therefore $\mathrm{gr}(B_n) \leq \mathrm{gr}(A_n)$, by Proposition 2.12. Note that A is approximated by the collection $(B_n)_{n\in J}$ whenever $J \subset \mathbb{N}$ is cofinal. In that case, it follows from the above result that $\mathrm{gr}(A) \leq \sup_{n\in J} \mathrm{gr}(B_n)$. Since this holds for every cofinal subset $J \subset \mathbb{N}$, we obtain:

$$\mathrm{gr}(A) \leq \inf\{\sup_{n\in J} \mathrm{gr}(B_n) \mid J \subset \mathbb{N} \text{ cofinal}\} = \liminf_n \mathrm{gr}(B_n) \leq \liminf_n \mathrm{gr}(A_n),$$

as desired. □

Theorem 2.14. *Let A be a C^*-algebra, and let $J \lhd A$ be an ideal. Then $\mathrm{gr}(J) \leq \mathrm{gr}(A)$.*

Proof. Note that J is also an ideal in $\widetilde{A}$ and $\mathrm{gr}(\widetilde{A}) = \mathrm{gr}(A)$. Thus, we may assume from now on that A is unital,

It is known that the real rank behaves well with respect to ideals, i.e., $\mathrm{rr}(J) \leq \mathrm{rr}(A)$, see [EH95, Théoràme 1.4]. We have $\mathrm{rr}(A) \leq \mathrm{gr}(A)$ by Proposition 2.5. Thus, by Lemma 2.11, it remains to show $\mathrm{gr}_{\mathrm{sa}}(J) \leq \mathrm{gr}(A)$. Let $k := \mathrm{gr}(A) + 1$, and we may assume this is finite. Then A and J are separable, and so there exists a sequential, quasi-central approximate unit $(h_\alpha) \subset J_+$, see [AP77, Corollary 3.3] and [Arv77]. We may assume $\|h_\alpha\| \leq 1$.

For a vector $\mathbf{a} = (a_1, \ldots, a_k) \in A^k$, we will use the following notation:

$$|\mathbf{a}| := \sum_i |a_i| = \sum_i (a_i^* a_i)^{1/2},$$
$$\mathbf{a}^{(\alpha)} := (h_\alpha^{1/2} a_1 h_\alpha^{1/2}, \ldots, h_\alpha^{1/2} a_k h_\alpha^{1/2}).$$

For $a \in A_+$, we denote by $\mathrm{Her}(a) := aAa$ the hereditary sub-C^*-algebra generated by a. We will consider the sequence algebra $Q := \prod_\alpha A / \bigoplus_\alpha A$. For an element $s \in A$, we denote by $\langle s \rangle \in Q$ the image of the constant sequence. We denote by $\langle h_\alpha \rangle \in Q$ the image of the sequence (h_α). Note that $\langle h_\alpha \rangle$ commutes with $\langle s \rangle$ in Q.

To show $\mathrm{gr}_{\mathrm{sa}}(J) \leq k - 1$, we want to verify the conditions of Lemma 2.9. So let $\mathbf{x} \in J_{\mathrm{sa}}^k$, $\varepsilon > 0$ and $z \in J$ be given. In 5 steps, we will construct $\mathbf{y} \in J_{\mathrm{sa}}^k$ such that $\mathbf{y} =_\varepsilon \mathbf{x}$ and $z \in_\varepsilon C^*(\mathbf{y})$.

Step 1: We will find $\mathbf{x}' \in J_{\mathrm{sa}}^k$ and $\delta > 0$ such that $\mathbf{x}' =_{\varepsilon/2} \mathbf{x}$ and $z \in_{\varepsilon/4} \mathrm{Her}((|\mathbf{x}'| - \delta)_+)$.

By assumption, there exists $\mathbf{a} \in \mathrm{Gen}_k(A)_{\mathrm{sa}}$ with $\mathbf{a} =_{\varepsilon/4} \mathbf{x}$. Then $|\mathbf{a}|$ is invertible in A, and so $|\mathbf{a}| \geq 3\delta$ for some $\delta > 0$. Choose an index α_0 large enough such that for all $\alpha \geq \alpha_0$ the following conditions hold:

$$z \ll_{\varepsilon/8} \frac{1}{\delta}(3\delta h_\alpha - 2\delta)_+, \tag{2.2}$$

$$\mathbf{x}^{(\alpha)} =_{\varepsilon/4} \mathbf{x}, \tag{2.3}$$

$$|\mathbf{a}^{(\alpha)}| =_\delta h_\alpha^{1/2} |\mathbf{a}| h_\alpha^{1/2}. \tag{2.4}$$

Then, using (2.4) at the second step, we get:

$$3\delta h_\alpha - 2\delta \leq h_\alpha^{1/2} |\mathbf{a}| h_\alpha^{1/2} - 2\delta \leq |\mathbf{a}^{(\alpha)}| - \delta.$$

In general, if two commuting, self-adjoint elements c, d satisfy $c \leq d$, then $c_+ \leq d_+$, but this does not necessarily hold if c and d do not commute. Thus, we may not deduce $(3\delta h_\alpha - 2\delta)_+ \leq (|\mathbf{a}^{(\alpha)}| - \delta)_+$. However, using the sequence algebra Q, we will show that this holds up to an arbitrarily small tolerance for sufficiently large α. Indeed, the elements $\langle 3\delta h_\alpha - 2\delta \rangle$ and $\langle |\mathbf{a}^{(\alpha)}| - \delta \rangle$ commute, and therefore:

$$\langle (3\delta h_\alpha - 2\delta)_+ \rangle \leq \langle (|\mathbf{a}^{(\alpha)}| - \delta)_+ \rangle.$$

Therefore, for $\alpha \geq \alpha_0$ large enough we have:

$$\frac{1}{\delta}(3\delta h_\alpha - 2\delta)_+ \in_{\varepsilon/8} \mathrm{Her}((|\mathbf{a}^{(\alpha)}| - \delta)_+). \tag{2.5}$$

For such α, we set $\mathbf{x}' := \mathbf{a}^{(\alpha)}$. Let us verify that $\mathbf{x}'$ has the desired properties. From $\mathbf{a} =_{\varepsilon/4} \mathbf{x}$ we get $\mathbf{a}^{(\alpha)} =_{\varepsilon/4} \mathbf{x}^{(\alpha)}$, and we deduce, using (2.3) at the third step:

$$\mathbf{x}' = \mathbf{a}^{(\alpha)} =_{\varepsilon/4} \mathbf{x}^{(\alpha)} =_{\varepsilon/4} \mathbf{x}.$$

Moreover, it follows from (2.2) and (2.5) that $z \in_{\varepsilon/4} \mathrm{Her}((|\mathbf{x}'| - \delta)_+)$, as desired.

Step 2: Since $z \in_{\varepsilon/4} \mathrm{Her}((|\mathbf{x}'| - \delta)_+)$, there exists a polynomial p such that:

$$z \ll_{\varepsilon/4} p(\mathbf{x}') \cdot (|\mathbf{x}'| - \delta)_+.$$

Set $M = 2 \cdot \max\{\|p(\mathbf{x}')\|, \|(|\mathbf{x}'| - \delta)_+\|\}$. Let $\eta > 0$ be such that for all $\mathbf{b} \in A_{\mathrm{sa}}^k$ with $\mathbf{b} =_\eta \mathbf{x}'$ we have:

$$z \ll_{\varepsilon/4} p(\mathbf{b}) \cdot (|\mathbf{b}| - \delta)_+, \tag{2.6}$$

$$|\mathbf{b}| =_\delta |\mathbf{x}'|, \tag{2.7}$$

$$\|p(\mathbf{b})\| \leq M, \tag{2.8}$$

$$\|(|\mathbf{b}| - \delta)_+\| \leq M. \tag{2.9}$$

We may assume $\eta < \varepsilon/2$ and $\eta < \delta$.

Step 3: Since $\mathrm{gr}(A) \leq k - 1$, there exists $\mathbf{c} \in A_{\mathrm{sa}}^k$ with $\mathbf{c} =_\eta \mathbf{x}'$ and a polynomial q such that

$$z =_{\varepsilon/(4M^2)} q(\mathbf{c}). \tag{2.10}$$

Then q can be decomposed as a finite sum of polynomials, $q = \sum_{d=1}^N q_d$, where q_d is homogeneous of degree d, i.e., $q_d(t\mathbf{c}) = t^d q_d(\mathbf{c})$ for every $t \in \mathbb{R}_+$. Set

$$L := \max\{\|q_1(\mathbf{c})\|, \dots, \|q_N(\mathbf{c})\|\}. \tag{2.11}$$

Step 4: We show that for every tolerance $\sigma > 0$ we have $(|\mathbf{c}^{(\alpha)}| - \delta)_+ \ll_\sigma h_\alpha$ for all α large enough.

To that end, let us verify that $\lim_{\alpha\to\infty} \|(|\mathbf{c}^{(\alpha)}| - \delta)_+ \cdot (1 - h_\alpha)\| = 0$, which means:

$$\Big\langle (|\mathbf{c}^{(\alpha)}| - \delta)_+ \Big\rangle \cdot \Big\langle 1 - h_\alpha \Big\rangle = 0,$$

in Q.

We have $\mathbf{c} =_\eta \mathbf{x}'$, and therefore $|\mathbf{y}| =_\delta |\mathbf{x}'|$ by (2.7). Let $\pi\colon A \to B$ denote the quotient morphism. Since $\pi(|\mathbf{x}'|) = 0$, we get $\pi(|\mathbf{c}| - \delta) \leq 0$ and so $\pi((|\mathbf{c}| - \delta)_+) = 0$. Thus, $\langle (|\mathbf{c}| - \delta)_+ \rangle \cdot \langle 1 - h_\alpha \rangle = 0$, in Q.

For $i = 1, \ldots, k$ we have $\langle |h_\alpha^{1/2} c_i h_\alpha^{1/2}| \rangle = \langle |c_i|^{1/2} h_\alpha |c_i|^{1/2} \rangle \leq \langle c_i \rangle$. It follows that $\langle |\mathbf{c}^{(\alpha)}| \rangle \leq \langle |\mathbf{c}| \rangle$, and since these two elements commute, we get get $\langle (|\mathbf{c}^{(\alpha)}| - \delta)_+ \rangle \leq \langle (|\mathbf{c}| - \delta)_+ \rangle$. It follows $\langle (|\mathbf{c}^{(\alpha)}| - \delta)_+ \rangle \langle 1 - h_\alpha \rangle = 0$, as desired.

Step 5: We choose the index α large enough satisfying (2.12), (2.13) and (2.14) below for $\mathbf{y} := \mathbf{c}^{(\alpha)}$. For (2.12), this is possible since $\mathbf{c} =_\eta \mathbf{x}' \in J^k_{\mathrm{sa}}$ and so $\mathbf{c}^{(\alpha)} =_\eta (\mathbf{x}')^{(\alpha)}$ for all α, and $\lim_{\alpha\to\infty} (\mathbf{x}')^{(\alpha)} = \mathbf{x}'$. For (2.13), this is possible by Step 4. Finally, for (2.14) this is possible since (h_α) is quasi-central.

$$\mathbf{y} =_\eta \mathbf{x}' \tag{2.12}$$

$$(|\mathbf{y}| - \delta)_+ \ll_{\varepsilon/(4MNL)} h_\alpha^d, \text{ for } d = 1, \ldots, N. \tag{2.13}$$

$$q(\mathbf{y}) = \sum_{d=1}^N q_d(\mathbf{y}) =_{\varepsilon/(4M^2)} \sum_{d=1}^N h_\alpha^d \cdot q_d(\mathbf{c}). \tag{2.14}$$

Let us check that $\mathbf{y} = \mathbf{c}^{(\alpha)}$ has the desired properties.

Since $\eta \leq \varepsilon/2$, we get, using (2.12) for the first estimate, and using Step 1 for the second estimate:

$$\mathbf{y} =_{\varepsilon/2} \mathbf{x}' =_{\varepsilon/2} \mathbf{x}.$$

It remains to check $z \in_\varepsilon C^*(\mathbf{y})$. Since $\mathbf{y} =_\eta \mathbf{x}'$, we get from Step 2 that the estimates (2.6)-(2.9) hold for $\mathbf{y}$. We compute, using (2.14), (2.8) and (2.9) at the first step, using (2.13), (2.8) and (2.11) at the third step, (2.10) at the fifth step, and (2.6) at the last step:

$$\begin{aligned}
p(\mathbf{y}) \cdot (|\mathbf{y}| - \delta)_+ \cdot q(\mathbf{y}) &=_{\varepsilon/4} p(\mathbf{y}) \cdot (|\mathbf{y}| - \delta)_+ \cdot \sum_{d=1}^N h_\alpha^d \cdot q_d(\mathbf{c}) \\
&= p(\mathbf{y}) \cdot \sum_{d=1}^N (|\mathbf{y}| - \delta)_+ \cdot h_\alpha^d \cdot q_d(\mathbf{c}) \\
&=_{\varepsilon/4} p(\mathbf{y}) \cdot \sum_{d=1}^N (|\mathbf{y}| - \delta)_+ \cdot q_d(\mathbf{c}) \\
&= p(\mathbf{y})(|\mathbf{y}| - \delta)_+ \cdot \sum_{d=1}^N q_d(\mathbf{y}) \\
&=_{\varepsilon/4} p(\mathbf{y})(|\mathbf{y}| - \delta)_+ \cdot z \\
&=_{\varepsilon/4} z
\end{aligned}$$

Since $p(\mathbf{y}) \cdot (|\mathbf{y}| - \delta)_+ \cdot q(\mathbf{y}) \in C^*(\mathbf{y})$, we have verified $z \in_\varepsilon C^*(\mathbf{y})$, as desired.

By Lemma 2.9, it follows $\mathrm{gr}_{\mathrm{sa}}(J) \leq k - 1$, as desired. □

In [Thi12, Definition 1], the concept of a non-commutative dimension theory was formalized by proposing a set of axioms. These axioms are generalizations of properties of the dimension of locally compact, Hausdorff spaces, and it was shown that they are satisfied by many theories, in particular the real and stable rank, the topological dimension, the decomposition rank and the nuclear dimension.

Definition 2.15 ([Thi12, Definition 1])**.** Let $\mathcal{C}$ be a class of C^*-algebras that is closed under *-isomorphisms, and closed under taking ideals, quotients, finite direct sums, and minimal unitizations. A **dimension theory** for $\mathcal{C}$ is an assignment $d\colon \mathcal{C} \to \overline{\mathbb{N}} = \{0, 1, 2, \dots, \infty\}$ such that $d(A) = d(A')$ whenever A, A' are isomorphic C^*-algebras in $\mathcal{C}$, and moreover the following axioms are satisfied:

(D1) $d(J) \leq d(A)$ whenever $J \lhd A$ is an ideal in $A \in \mathcal{C}$,
(D2) $d(A/J) \leq d(A)$ whenever $J \lhd A \in \mathcal{C}$,
(D3) $d(A \oplus B) = \max\{d(A), d(B)\}$, whenever $A, B \in \mathcal{C}$,
(D4) $d(\widetilde{A}) = d(A)$, whenever $A \in \mathcal{C}$.
(D5) If $A \in \mathcal{C}$ is approximated by subalgebras $A_i \in \mathcal{C}$ with $d(A_i) \leq n$, then $d(A) \leq n$.
(D6) Given $A \in \mathcal{C}$ and a separable sub-C^*-algebra $C \subset A$, there exists a separable C^*-algebra $D \in \mathcal{C}$ such that $C \subset D \subset A$ and $d(D) \leq d(A)$.

Note that we do not assume that $\mathcal{C}$ is closed under approximation by sub-C^*-algebra, so that the assumption $A \in \mathcal{C}$ in (D5) is necessary. Moreover, in axiom (D6), we do not assume that the separable subalgebra C lies in $\mathcal{C}$.

Remark 2.16. Let us consider the generator rank on the class of separable C^*-algebras.

We have verified axioms (D1) in Theorem 2.14, (D2) in Proposition 2.12 and (D5) in Proposition 2.13. Note that (D4) holds by definition, and (D6) is superfluous if we only consider separable C^*-algebras.

The question remains whether axiom (D3) holds, that is, whether $\mathrm{gr}(A \oplus B) = \max\{\mathrm{gr}(A), \mathrm{gr}(B)\}$, and this turns out to be surprisingly difficult. We can only verify it in specific cases, namely for C^*-algebras of real rank zero, see Proposition 3.1, or for homogeneous C^*-algebras, see Corollary 4.27. We conjecture that (D3) for the generator rank holds in general.

The next result gives an estimate of the generator rank of an extension of C^*-algebras in terms of the generator rank of ideal and quotient. We remark that no such estimate is known for the real rank.

Theorem 2.17. *Let A be a C^*-algebra, and let $J \lhd A$ be an ideal. Then:*

$$\mathrm{gr}(A) \leq \mathrm{gr}(J) + \mathrm{gr}(A/J) + 1.$$

Proof. Since J is also an ideal in $\widetilde{A}$, and $\mathrm{gr}(A) = \mathrm{gr}(\widetilde{A})$, we may assume that A is unital.

Set $B := A/J$. Let $k := \mathrm{gr}(J)+1$ and $l := \mathrm{gr}(B)+1$, which we may assume are finite. Let $\pi\colon A \to B$ denote quotient morphism. It induces a natural morphism $A^l \to B^l$, which we also denote by π. Given $\mathbf{x} \in A^k_{\mathrm{sa}}$, $\mathbf{y} \in A^l_{\mathrm{sa}}$ and $\varepsilon > 0$, we want to find $\mathbf{x}' \in A^k_{\mathrm{sa}}$, $\mathbf{y}' \in A^l_{\mathrm{sa}}$ such that $\mathbf{x}' =_\varepsilon \mathbf{x}$, $\mathbf{y}' =_\varepsilon \mathbf{y}$ and $A = C^*(\mathbf{x}', \mathbf{y}')$.

Let $\mathbf{b} := \pi(\mathbf{y})$. Since $\mathrm{gr}(B) \leq l-1$, we may find $\mathbf{b}' \in \mathrm{Gen}_l(B)_{\mathrm{sa}}$ with $\mathbf{b}' =_\varepsilon \mathbf{b}$. Let $\mathbf{y}' \in A^l_{\mathrm{sa}}$ be a lift of $\mathbf{b}'$ with $\mathbf{y}' =_\varepsilon \mathbf{y}$. For $i = 1, \dots, k$, choose an element $a_i \in C^*(\mathbf{y}')$ such that $\pi(a_i) = \pi(x_i)$. Set $\mathbf{a} = (a_1, \dots, a_k) \in A^k_{\mathrm{sa}}$. Note that $\mathbf{x} - \mathbf{a} \in J^k_{\mathrm{sa}}$. Since $\mathrm{gr}_{\mathrm{sa}}(J) \leq \mathrm{gr}(J) \leq k-1$, we may find $\mathbf{c} \in \mathrm{Gen}_k(J)_{\mathrm{sa}}$ with $\mathbf{c} =_\varepsilon \mathbf{x} - \mathbf{a}$. Set $\mathbf{x}' := \mathbf{a} + \mathbf{c}$. Then $\mathbf{x}'$ and $\mathbf{y}'$ have the desired properties. $\square$

3. The generator rank of real rank zero C^*-algebras

In this section, we restrict our attention to separable C^*-algebras with real rank zero. On this class of C^*-algebras, the generator rank is a dimension theory in the sense of Definition 2.15, see Remark 2.16 and Proposition 3.1.

We then show that AF-algebras have generator rank at most one, see Corollary 3.3.

In Theorem 3.6, we prove the estimate $\mathrm{gr}(A \otimes M_n) \leq \left\lceil \frac{\mathrm{gr}(A)}{n^2} \right\rceil$ under the additional assumption that A is unital and has stable rank. This shows that for such algebras the generator rank decreases when tensoring with matrix algebras of higher and higher dimension. Thus, if A is a separable, unital, real rank zero, stable rank one C^*-algebra with finite generator rank, then $\mathrm{gr}(A \otimes B) = 1$ for any infinite UHF-algebra B. We generalize this by showing that every separable, real rank zero C^*-algebra that tensorially absorbs a UHF-algebra has generator rank at most one, see Proposition 3.8.

Proposition 3.1. *Let A, B be C^*-algebras of real rank zero. Then* $\mathrm{gr}(A \oplus B) = \max\{\mathrm{gr}(A), \mathrm{gr}(B)\}$.

Proof. We have $\mathrm{gr}(A), \mathrm{gr}(B) \leq \mathrm{gr}(A \oplus B)$ by Proposition 2.12 (or Theorem 2.14), and it therefore remains to show $\mathrm{gr}(A \oplus B) \leq \max\{\mathrm{gr}(A), \mathrm{gr}(B)\}$. Since $A \oplus B$ is an ideal in $\widetilde{A} \oplus \widetilde{B}$, we obtain $\mathrm{gr}(A \oplus B) \leq \mathrm{gr}(\widetilde{A} \oplus \widetilde{B})$ from Theorem 2.14. We have $\mathrm{gr}(A) = \mathrm{gr}(\widetilde{A})$ and $\mathrm{gr}(B) = \mathrm{gr}(\widetilde{B})$ by definition, and thus it remains to show $\mathrm{gr}(\widetilde{A} \oplus \widetilde{B}) \leq \max\{\mathrm{gr}(\widetilde{A}), \mathrm{gr}(\widetilde{B})\}$. We may therefore assume that A and B are unital.

So let A, B be unital, real rank zero C^*-algebras. Let $k := \max\{\mathrm{gr}(A), \mathrm{gr}(B)\} + 1$, and we may assume this is finite. We want to verify the conditions of Lemma 2.9. Let $\mathbf{a} \in A^k_{\mathrm{sa}}$, $\mathbf{b} \in B^k_{\mathrm{sa}}$, $\varepsilon > 0$ and $x \in A$, $y \in B$ be given. We need to find $\mathbf{c} \in A^k_{\mathrm{sa}}$, $\mathbf{d} \in B^k_{\mathrm{sa}}$ such that $\mathbf{c} =_\varepsilon \mathbf{a}$, $\mathbf{d} =_\varepsilon \mathbf{b}$ and $(x \oplus y) \in_\varepsilon C^*(\mathbf{c} \oplus \mathbf{d})$, where we use the notation $\mathbf{a}' \oplus \mathbf{b}' = (a'_1 \oplus b'_1, \ldots, a'_k \oplus b'_k) \in (A \oplus B)^k$ for the direct sum of tuples.

Since $\mathrm{rr}(A) = \mathrm{rr}(B) = 0$, we may perturb a_1 and b_1 to be invertible, self-adjoint and have disjoint (finite) spectra. More precisely, there are $a'_1 \in A_{\mathrm{sa}}$ and $b'_1 \in B_{\mathrm{sa}}$ such that $a'_1 =_{\varepsilon/2} a_1$, $b'_1 =_{\varepsilon/2} b_1$ and $\sigma(a'_1) \cap \sigma(b'_1) = \emptyset$ and $0 \notin \sigma(a'_1), 0 \notin \sigma(b'_1)$. Let $\delta_0 > 0$ be smaller than the distance between any two points in $\sigma(a'_1) \cup \sigma(b'_1) \cup \{0\}$. Define continuous functions $f, g \colon \mathbb{R} \to [0,1]$ such that:

(1) f has value 1 on a $\delta_0/4$-neighborhood of $\sigma(a'_1)$, and has value 0 on a $\delta_0/4$-neighborhood of $\sigma(b'_1) \cup \{0\}$.
(2) g has value 1 on a $\delta_0/4$-neighborhood of $\sigma(b'_1)$, and has value 0 on a $\delta_0/4$-neighborhood of $\sigma(a'_1) \cup \{0\}$.

Let $\delta > 0$ be such that:

(1) Whenever $c_1 \in A_{\mathrm{sa}}$ satisfies $c_1 =_\delta a'_1$, then the spectrum $\sigma(c_1)$ is contained in a $\delta_0/4$-neighborhood of $\sigma(a'_1)$.
(2) Whenever $d_1 \in B_{\mathrm{sa}}$ satisfies $d_1 =_\delta b'_1$, then the spectrum $\sigma(d_1)$ is contained in a $\delta_0/4$-neighborhood of $\sigma(b'_1)$.

We may assume $\delta < \varepsilon/2$.

By assumption, there exists $\mathbf{c} \in \mathrm{Gen}_k(A)_{\mathrm{sa}}$ with $\mathbf{c} =_\delta (a'_1, a_2, \ldots, a_k)$, and there exists $\mathbf{d} \in \mathrm{Gen}_k(B)_{\mathrm{sa}}$ with $\mathbf{d} =_\delta (b'_1, b_2, \ldots, b_k)$. Then $\mathbf{c} \oplus \mathbf{d} =_\varepsilon \mathbf{a} \oplus \mathbf{b}$, and we claim that $\mathbf{c} \oplus \mathbf{d}$ generates $A \oplus B$.

So let $x \oplus y \in A \oplus B$, and $\eta > 0$ be given. Since $\mathbf{c}$ generates A, there exists a polynomial p such that $x =_\eta p(\mathbf{c})$. Similarly, there exists a polynomial q such that $y =_\eta q(\mathbf{y})$. By construction, $f(c_1) = 1_A$, $f(d_1) = 0_B$, $g(c_1) = 0_A$ and $g(d_1) = 1_B$. It follows:

$$p(\mathbf{c}) \oplus q(\mathbf{d}) = f(c_1 \oplus d_1)p(\mathbf{c} \oplus \mathbf{d}) + g(c_1 \oplus d_1)q(\mathbf{c} \oplus \mathbf{d}) \in C^*(\mathbf{c} \oplus \mathbf{d}).$$

Since $x \oplus y \in A \oplus B$, and $\eta > 0$ were arbitrary, this shows $\mathbf{c} \oplus \mathbf{d} \in \mathrm{Gen}_k(A \oplus B)_{\mathrm{sa}}$.

Thus, $\mathrm{Gen}_k(A \oplus B)_{\mathrm{sa}}$ is dense in $(A \oplus B)^k_{\mathrm{sa}}$, and so $\mathrm{gr}(A \oplus B) \leq k - 1$, as desired. □

Lemma 3.2. *For $n \geq 2$, we have* $\mathrm{gr}(M_n) = 1$.

Proof. Let $a, b \in (M_n)_{\mathrm{sa}}$ and $\varepsilon > 0$ be given. Then there exists a unitary such that uau^* is diagonal. Let $a' =_\varepsilon uau^*$ be self-adjoint and diagonal such that the spectrum $\sigma(a')$ contains n different non-zero entries. Let $b' =_\varepsilon ubu^*$ be a self-adjoint element such that the off-diagonal entries $b'_{i,i+1}$ are non-zero for $i = 1, \ldots, n-1$. It is easily checked that $C^*(a', b') = M_n$.

Since conjugation by u is isometric, we have $u^*a'u =_\varepsilon a$ and $u^*b'u =_\varepsilon b$. Moreover, $C^*(u^*a'u, u^*b'u) = C^*(a', b') = M_n$. Thus, we have approximated the pair (a, b) by a generating pair, and since $\varepsilon > 0$ was arbitrary, we get that $\mathrm{Gen}_2(M_n)_{\mathrm{sa}} \subset (M_n)^2_{\mathrm{sa}}$ is dense, as desired. □

Corollary 3.3. *Let A be a separable AF-algebra. Then* $\mathrm{gr}(A) \leq 1$.

Proof. It follows from Lemma 3.2 and Proposition 3.1 that $\mathrm{gr}(B) \leq 1$ for every finite-dimensional C^*-algebra B. Then, the result for AF-algebras follows directly from Proposition 2.13. □

We now turn towards the problem of estimating the generator rank of $A \otimes M_n$. An important ingredient is the fact that we can approximately diagonalize matrices, which is conceptualized by the following notion of Xue:

Definition 3.4 (Xue, [Xue10, Definition 3.1])**.** Let A be a C^*-algebra, and $n \geq 2$. An element $a \in A \otimes M_n$ is said to be **approximately diagonalizable** if for any $\varepsilon > 0$ there exist a unitary $u \in A \otimes M_n$ and $d_1, \ldots, d_n \in A$ such that $\|uau^* - \mathrm{diag}(d_1, \ldots, d_n)\| < \varepsilon$.

We call A **approximately diagonal**, if for any $n \geq 2$, every self-adjoint element in $A \otimes M_n$ is approximately diagonalizable.

Proposition 3.5 (Zhang, [Zha90, Corollary 3.6])**.** *Every real rank zero C^*-algebra is approximately diagonal.*

Theorem 3.6. *Let A be a unital C^*-algebra with real rank zero and stable rank one, and $n \in \mathbb{N}$. If $\mathrm{gr}(A) \geq 1$, then:*

$$\mathrm{gr}(A \otimes M_n) \leq \left\lceil \frac{\mathrm{gr}(A)}{n^2} \right\rceil.$$

In particular, if $\mathrm{gr}(A) \leq n^2$, then $\mathrm{gr}(A \otimes M_n) \leq 1$.

Proof. The inequality clearly holds if $n = 1$, so we may assume $n \geq 2$. Then also $\mathrm{gr}(A \otimes M_n) \geq 1$.

Assume $\mathrm{gr}(A) \leq (d-1) \cdot n^2$ for some $d \geq 2$. We need to show $\mathrm{gr}(A \otimes M_n) \leq d - 1$. So let $c^{(k)} = (c^{(k)}_{i,j})$, $k = 1, \ldots, d$ be d self-adjoint matrices in $A \otimes M_n$. In several steps, we will show how to approximate these matrices by matrices $\bar{c}^{(k)}$ that generate $A \otimes M_n$.

We let $e_{ij} \in M_n$ denote the canonical matrix units. To simplify notation, we set $a := c^{(1)}$ and $b := c^{(2)}$. In step 1, we will show that we may assume a is diagonal. In step 2, we will show that we may also assume that the entries $b_{1,n}, \ldots, b_{n-1,n}$ are positive, and in step 3 we show that we may further assume that the diagonal entries of a are invertible and have finite, disjoint spectra.

In step 1 and 2, we will find a unitary $u \in A \otimes M_n$ and consider the conjugated elements $uc^{(k)}u^*$. Note that it is enough to find generators close to these new elements $uc^{(k)}u^*$, since conjugation by u is isometric, and the elements $c^{(1)}, \ldots, c^{(d)}$ generate $A \otimes M_n$ if and only if $uc^{(1)}u^*, \ldots, uc^{(d)}u^*$ do.

Step 1: We show that we may assume $a = c^{(1)}$ is diagonal.

By Proposition 3.5, we may approximately diagonalize a, i.e., for every tolerance $\varepsilon > 0$ there exists a unitary $u \in A \otimes M_n$ and $d_1, \ldots, d_n \in A$ such that $\|uau^* - \operatorname{diag}(d_1, \ldots, d_n)\| < \varepsilon$. As explained above, it is enough to find a generating tuple close to the conjugated matrices $uau^*, uc^{(2)}u^*, \ldots, uc^{(d)}u^*$. Thus, by considering the conjugated elements, we may from now on assume that a is diagonal.

Step 2: We show that we may assume the $n-1$ off-diagonal entries $b_{1,n}, \ldots, b_{n-1,n}$ of $b = c^{(2)}$ are positive and invertible.

Since $\operatorname{sr}(A) = 1$, for each $b_{1,n}, \ldots, b_{n-1,n}$ we may find an invertible element in A that is arbitrarily close. Thus, by perturbing b, we may assume each $b_{i,n}$ is invertible in A.

For $i = 1, \ldots, n-1$, set $u_i := (b_{i,n}^* b_{i,n})^{-1/2} b_{i,n}^*$, which is a unitary in A. Note that $|b_{i,n}| := (b_{i,n}^* b_{i,n})^{1/2}$ is a positive, invertible element, and $|b_{i,n}| = u_i b_{i,n}$.

Set $u_n := 1$ and define a diagonal, unitary matrix as $u := \operatorname{diag}(u_1, \ldots, u_n) \in A \otimes M_n$. As explained above, it is enough to find a generating tuple close to the conjugated matrices $uau^*, ubu^*, uc^{(3)}u^*, \ldots, uc^{(d)}u^*$. Note that uau^* is still diagonal, and that the entries $(ubu^*)_{i,n} = u_i b_{i,n} u_n^* = |b_{i,n}|$ are positive and invertible for $i = 1, \ldots, n-1$. Again, by considering the conjugated elements, we may from now on assume that a is diagonal, and $b_{1,n}, \ldots, b_{n-1,n}$ of $b = c^{(2)}$ are positive, invertible.

Step 3: We show that we may also assume that the diagonal entries of a are invertible and have finite, disjoint spectra.

Let $a = \operatorname{diag}(a_1, \ldots, a_n)$ for elements $a_i \in A$. Since $\operatorname{rr}(A) = 0$, we may perturb each a_i to be an invertible, self-adjoint element with finite spectrum. By perturbing the elements a_i further, we may assume that their spectra are also disjoint.

From now on, we will assume that $a = c^{(1)}$ and $b = c^{(2)}$ have the additional properties as exsplained in step 1-3.

Step 4: Let $\delta_0 > 0$ be smaller than the distance between any two points in $\sigma(a) \cup \{0\}$. For $k = 1, \ldots, n$, define a continuous function $f_k \colon \mathbb{R} \to [0,1]$ such that:

(1) f_k has value 1 on a $\delta_0/4$-neighborhood of $\sigma(a_k)$,
(2) f_k has value 0 on a $\delta_0/4$-neighborhood of $\bigcup_{i \neq k} \sigma(a_i) \cup \{0\}$

Let $\delta > 0$ be such that the spectrum $\sigma(a'_k)$ is contained in a $\delta_0/4$-neighborhood of $\sigma(a_k)$ whenever $a'_k \in A_{\mathrm{sa}}$ satisfies $a'_k =_\delta a_k$. In that case, we have

$$f_k(a_i) = \begin{cases} 1_A & \text{, if } k = i \\ 0 & \text{, if } k \neq i \end{cases},$$

and so we may recover the diagonal matrix units from such $a' = \operatorname{diag}(a'_1, \ldots, a'_n)$ as $1_A \otimes e_{k,k} = f_k(a')$. We may assume $\delta < \varepsilon$.

Step 5: We consider the following elements of A:

(1) The n self-adjoint elements $a_1, \ldots, a_n$,
(2) the $(n-1)^2$ self-adjoint elements corresponding to the entries of the upper-left $(n-1, n-1)$ corner of b,
(3) the $(n-1)$ positive elements $b_{1,n}, \ldots, b_{n-1,n}$,

(4) the self-adjoint element $b_{n,n}$,
(5) and the $(d-2)n^2$ self-adjoint elements corresponding to the entries of the $d-2$ self-adjoint matrices $c^{(3)}, \ldots, c^{(k)} \in A \otimes M_n$.

Together, this gives $n + (n-1)^2 + 1 + (d-2)n^2 = (d-1)n^2 - n + 2$ self-adjoint and $n-1$ positive elements of A.

Since $\mathrm{gr}(A) \leq (d-1)n^2$, we may find $(d-1)n^2 + 1$ self-adjoint elements of A that together generate A, and which we collect as follows:

(1) A diagonal, self-adjoint matrix $\bar{a}$,
(2) A self-adjoint matrix $\bar{b} \in A \otimes M_n$, whose entries $\bar{b}_{1,n}, \ldots, \bar{b}_{n-1,n}$ are self-adjoint,
(3) and $(d-2)$ self-adjoint matrices $\bar{c}^{(3)}, \ldots, \bar{c}^{(k)} \in A \otimes M_n$.

such that $\bar{a} =_\delta a$, $\bar{b} =_\varepsilon b$ and $\bar{c}^{(k)} =_\varepsilon c^{(k)}$ for $k = 3, \ldots, d$. For $i = 1, \ldots, n-1$, we may ensure that $\bar{b}_{i,n}$ is positive and invertible by choosing it close enough to $b_{i,n}$.

Let $D := C^*(\bar{a}, \bar{b}, \bar{c}^{(3)}, \ldots, \bar{c}^{(d)}) \subset A \otimes M_n$. We claim that $D = A \otimes M_n$.

As explained in Step 4, we have chosen δ such that we can recover the diagonal matrix units from $\bar{a}$ as $1_A \otimes e_{k,k} = f_k(\bar{a}) \in D$ for $k = 1, \ldots, n$. Following ideas from Olsen, Zame, [OZ76], we consider the elements $g_i := (1_A \otimes e_{i,i})b'(1_A \otimes e_{n,n})$ for $i = 1, \ldots, n-1$. Note that $g_i = \bar{b}_{i,n} \otimes e_{i,n}$ is an element of D. Then:

$$g_i^* g_i = (\bar{b}_{i,n})^2 \otimes e_{n,n} \in D.$$

Since $(\bar{b}_{i,n})^2$ is positive and invertible, we have:

$$(\bar{b}_{i,n})^{-1} \otimes e_{n,n} \in C^*(g_i^* g_i) \subset D.$$

Then:

$$1_A \otimes e_{i,n} = g_i \cdot \left((\bar{b}_{i,n})^{-1} \otimes e_{n,n}\right) \in D.$$

It follows that D contains all matrix units $1_A \otimes e_{i,j}$. Since the entries of the matrices $\bar{a}, \bar{b}, \bar{c}^{(3)}, \ldots, \bar{c}^{(k)}$ generate A, we get $D = A \otimes M_n$, as desired. □

Remark 3.7. Let us observe that Theorem 3.6 can be complemented by a partial converse inequality: Whenever A is a unital C^*-algebra, and $\mathrm{Gen}_k(M_n(A))_{\mathrm{sa}}$ is dense in $M_n(A)^k_{\mathrm{sa}}$, then necessarily $\mathrm{Gen}_{kn^2}(A)_{\mathrm{sa}}$ is dense in $A^{kn^2}_{\mathrm{sa}}$. It follows:

$$\left\lceil \frac{\mathrm{gr}(A)+1}{n^2} \right\rceil - 1 \leq \mathrm{gr}(A \otimes M_n).$$

Thus, if A is a unital C^*-algebra with real rank zero and stable rank one, $\mathrm{gr}(A) \geq 1$, and $n \in \mathbb{N}$, then:

$$\left\lceil \frac{\mathrm{gr}(A)+1}{n^2} \right\rceil - 1 \leq \mathrm{gr}(A \otimes M_n) \leq \left\lceil \frac{\mathrm{gr}(A)}{n^2} \right\rceil.$$

This can probably be improved.

Proposition 3.8. *Let A be a separable, real rank zero C^*-algebra that tensorially absorbs a UHF-algebra. Then* $\mathrm{gr}(A) \leq 1$.

Proof. We will first reduce to the case that A is unital. Let $p_1, p_2 \ldots \in A$ be an approximate unit of projections. Consider the corners $A_n := p_n A p_n$. Then each A_n is a unital, separable, real rank zero C^*-algebra that tensorially absorbs a UHF-algebra. If we can show $\mathrm{gr}(A_n) \leq 1$, then $\mathrm{gr}(A) \leq 1$ by Proposition 2.13.

So we may assume that A is unital. To simplify the proof, we will assume that A absorbs the 2^∞ UHF-algebra, denoted by M_{2^∞}. For other UHF-algebras, the proof is analogous but notationally more involved. Since M_{2^∞} is strongly self-absorbing, see [TW07], there exists a *-isomorphism $\Phi\colon A \to A \otimes M_{2^\infty}$ that is approximately unitarily equivalent to the inclusion $\iota\colon A \to A \otimes M_{2^\infty}$ given by $\iota(x) = x \otimes 1$. This means there exists a sequence of unitaries $u_n \in A \otimes M_{2^\infty}$ such that $\lim_n u_n \Phi(a) u_n^* = \iota(a)$ for all $a \in A$.

Since $A \cong A \otimes M_{2^\infty}$, it is enough to show $\mathrm{gr}(A \otimes M_{2^\infty}) \leq 1$. We need to approximate any pair $a, b \in A \otimes M_{2^\infty}$ of self-adjoint elements by a pair that generates $A \otimes M_{2^\infty}$, and we first reduce the problem to the case that a, b lie in the image of ι.

As explained in the proof of Theorem 3.6, for any unitary $u \in A \otimes M_{2^\infty}$, it is enough to find generators close to the conjugated elements uau^*, ubu^*. Using the unitaries u_n implementing the approximate unitary equivalence between Φ and ι, we see that $u_n a u_n^* \to \iota(\Phi(a))$ and $u_n b u_n^* \to \iota(\Phi(b))$. If we can find generators a', b' close to $\iota(\Phi(a)), \iota(\Phi(b))$, say $a' =_\nu \iota(\Phi(a))$ and $b' =_\nu \iota(\Phi(b))$ for some $\nu > 0$, then for n large enough we have $u_n^* a' u_n =_\nu a$ and $u_n^* b' u_n =_\nu b$, and moreover the pair $u_n^* a' u_n$ and $u_n^* b' u_n$ generates $A \otimes M_{2^\infty}$.

So, let $a, b \in A$ be self-adjoint elements, and $\varepsilon > 0$ be given. We need to find a generating pair of self-adjoint elements $c, d \in A \otimes M_{2^\infty}$ such that $c =_\varepsilon a \otimes 1$ and $d =_\varepsilon b \otimes 1$. Since $\mathrm{rr}(A) = 0$, we may assume that a is invertible and has finite spectrum, i.e., $a = \sum_i \lambda_i p_i$ for some $\lambda_i \in \mathbb{R} \setminus \{0\}$ and pairwise orthogonal projections $p_i \in A$ that sum to 1_A.

Let $x_1, x_2 \ldots \in A$ be a sequence of positive elements that generates A and such that $\|x_k\| \leq \varepsilon/2^k$. Let μ the smaller than the distance between any two values in $\sigma(a) \cup \{0\}$. We may assume $\mu < \varepsilon$. We picture M_{2^∞} as $M_{2^\infty} = \bigotimes_{k=1}^\infty M_2$, and we let $e_{ij}^{(k)}$, $i, j = 1, 2$, be the matrix units of the k-th copy of M_2. For $p \geq 1$, we let $1^{(\geq p)}$ denote the unit of the factor $\bigotimes_{k=p}^\infty M_2$. Define $c, d \in A \otimes M_{2^\infty}$ as:

$$c := \sum_{k\geq 1} \left[\left(\sum_i (\lambda_i + \frac{\mu}{2^k}) p_i \right) \otimes e_{22}^{(1)} \otimes \cdots \otimes e_{22}^{(k-1)} \otimes e_{11}^{(k)} \otimes 1^{(\geq k+1)} \right],$$

$$d := b \otimes 1 + \sum_{k \geq 1} x_k \otimes e_{22}^{(1)} \otimes \cdots \otimes e_{22}^{(k-1)} \otimes \left(e_{12}^{(k)} + e_{21}^{(k)} \right) \otimes 1^{(\geq k+1)}$$

One checks $c =_\varepsilon a \otimes 1$ and $d =_\varepsilon b \otimes 1$. Set $D := C^*(c, d) \subset A \otimes M_{2^\infty}$, and let us check $D = A \otimes M_{2^\infty}$.

For $k \geq 1$, let $f_k, g_k \colon \mathbb{R} \to [0, 1]$ be continuous functions such that:

(1) f_k takes value 1 on $\{\lambda_i + \frac{\mu}{2^k} \mid i \geq 1\}$, and value 0 on $\{\lambda_i + \frac{\mu}{2^l} \mid i \geq 1, l \neq k\} \cup \{0\}$.
(2) g_k takes value 1 on $\{\lambda_i + \frac{\mu}{2^l} \mid i \geq 1, l \neq k\}$, and value 0 on $\{\lambda_i + \frac{\mu}{2^k} \mid i \geq 1\} \cup \{0\}$.

Then

$$e_{22}^{(1)} \otimes \cdots \otimes e_{22}^{(k-1)} \otimes e_{11}^{(k)} \otimes 1^{(\geq k+1)} = f_k(c) \in D,$$
$$e_{22}^{(1)} \otimes \cdots \otimes e_{22}^{(k-1)} \otimes e_{22}^{(k)} \otimes 1^{(\geq k+1)} = g_k(c) \in D.$$

We follow ideas of Olsen, Zame, [OZ76]. For $k \geq 1$, consider the element $y_k := f_k(c) d g_k(c) \in D$. By construction of d, we have:

$$y_k = x_k \otimes e_{22}^{(1)} \otimes \cdots \otimes e_{22}^{(k-1)} \otimes e_{12}^{(k)} \otimes 1^{(\geq k+1)}.$$

Then

$$y_k^* y_k = x_k^2 \otimes e_{22}^{(1)} \otimes \cdots \otimes e_{22}^{(k-1)} \otimes e_{22}^{(k)} \otimes 1^{(\geq k+1)} \in D.$$

Since x_k^2 is positive and invertible, we have $(x_k)^{-1} \otimes e_{22}^{(1)} \otimes \cdots \otimes e_{22}^{(k)} \otimes 1^{(\geq k+1)} \in D$, and then:

$$1_A \otimes e_{22}^{(1)} \otimes \cdots \otimes e_{22}^{(k-1)} \otimes e_{12}^{(k)} \otimes 1^{(\geq k+1)} \in D.$$

It follows that D contains all matrix units of $1 \otimes M_{2^\infty}$, and therefore $1 \otimes M_{2^\infty} \subset D$. Then $x_k \otimes 1 \in D$ for all k. Since the x_k generate A, we get $A \otimes 1 \subset D$, and then $D = A \otimes M_{2^\infty}$, as desired. □

Remark 3.9. We do not know of any example of a separable, real rank zero C^*-algebra A for which $\mathrm{gr}(A) \geq 2$, and it is possible that no such algebra exists. Let us consider the following weaker question: Do all separable, unital, simple, real rank zero, stable rank one C^*-algebras have generator rank at most one, or are they at least singly generated?

This is certainly a hard problem, since a positive answer to it would give a positive solution to the generator problem for von Neumann algebras, which asks whether every von Neumann algebra acting on a separable Hilbert space is singly generated, see [Kad67, Problem 14] and [Ge03]. The generator problem for von Neumann algebras has been reduced to the case of a type II_1-factor, see [Wil74]. Every II_1-factor M acting on a separable Hilbert space contains a separable unital, simple, real rank zero, stable rank one C^*-algebras $A \subset M$ such that $A'' = M$. If A is singly generated (as a C^*-algebra), then so is M (as a von Neumann algebra).

4. The generator rank of homogeneous C^*-algebras

In this section, we first compute the generator rank of commutative C^*-algebras, see Proposition 4.7. The main result is Theorem 4.23, which shows how to compute the generator rank of homogeneous C^*-algebras. To obtain these results, we have to compute the codimension of $\mathrm{Gen}_k(M_n)_{\mathrm{sa}} \subset (M_n)_{\mathrm{sa}}^k$, see Lemma 4.20.

For spaces X, Y we denote by $E(X, Y)$ the space of continuous embeddings.

4.1. Let us discuss the generator rank of commutative C^*-algebras.

Let X be a compact, metric space, and $k \in \mathbb{N}$. We may identify $C(X)_{\mathrm{sa}}^{k+1} \cong C(X, \mathbb{R}^{k+1})$. By the Stone-Weierstrass theorem, an element $\mathbf{a} \in C(X)_{\mathrm{sa}}^{k+1}$ generates $C(X)$ if and only if $\mathbf{a}(x) = (a_1(x), \ldots, a_{k+1}(x)) \neq 0$ for all $x \in X$, and $\mathbf{a}$ separates the points of X.

It follows that $\mathrm{gr}(C(X)) \leq k$ if and only if $E(X, \mathbb{R}^{k+1} \setminus \{0\})$ is dense in $C(X, \mathbb{R}^{k+1})$.

We will break this problem into two parts:

(1) When is $C(X, \mathbb{R}^{k+1} \setminus \{0\}) \subset C(X, \mathbb{R}^{k+1})$ dense?
(2) When is $E(X, \mathbb{R}^{k+1} \setminus \{0\}) \subset C(X, \mathbb{R}^{k+1} \setminus \{0\})$ dense?

These questions have been studied and answered in a more general setting. We recall the results in a way that will also be used to compute the generator rank of homogeneous C^*-algebras.

4.2. Let X be a compact, metric space, let $q \geq 1$.We give $C(X, \mathbb{R}^q)$ the topology induced by the supremum norm $\|f\| := \sup_{x \in X} |f(x)|$.

If $Y \subset \mathbb{R}^q$ is closed, then it follows from compactness of X that $C(X, \mathbb{R}^q \setminus Y)$ is an open subset of $C(X, \mathbb{R}^q)$. We want to see when it is also dense. The goal is Proposition 4.4 below, which is a classical result that can be obtained in several ways. A particular version appeared in [BE91, Theorem 1.3].

Recall that $Y \subset \mathbb{R}^q$ is said to be "codimension three" if $\dim(Y) \leq q-3$. For the notion of "tameness" of embeddings we refer the reader to the survey by Edwards, [Edw75]. We note that every codimension three submanifold $M \subset \mathbb{R}^q$, $q \geq 5$, is tamely embedded. It follows that $Y \subset \mathbb{R}^q$ is a codimension three, tame embedding, if Y is a countable union of codimension three submanifolds of $\mathbb{R}^q$, $q \geq 5$. To derive Proposition 4.4, we will use the following result:

Proposition 4.3 (Dranishnikov, [Dra91])**.** *Let X be a compact, metric space, and $Y \subset \mathbb{R}^q$ be a codimension three, tame compact subset. Then the following are equivalent:*

(1) $C(X, \mathbb{R}^q \setminus Y) \subset C(X, \mathbb{R}^q)$ *is dense,*
(2) $\dim(X \times Y) < q$.

Proposition 4.4. *Let X be a compact, metric space, let $q \geq 5$, and let $Y \subset \mathbb{R}^q$ be a closed subset that is the countable union of codimension three submanifolds of $\mathbb{R}^q$. Then the following are equivalent:*

(1) $C(X, \mathbb{R}^q \setminus Y) \subset C(X, \mathbb{R}^q)$ *is dense (and open),*
(2) $\dim(X) < q - \dim(Y)$.

Proof. We noted in 4.2 that $C(X, \mathbb{R}^q \setminus Y) \subset C(X, \mathbb{R}^q)$ is always open.

Set $d := \dim(Y)$. Let $Y_1, Y_2, \ldots \mathbb{R}^q$ be codimension three submanifolds such that $Y = \bigcup_k Y_k$. Then $\dim(Y) = \sup_k \dim(Y_k)$, and so there exist an index l with $\dim(Y_l) = d$. Choose a compact subset $Z \subset Y_l$ such that $Z \cong [0,1]^d$. It follows from [Edw75] that Z is tamely embedded in $\mathbb{R}^q$.

Let us show "(1) ⇒ (2)". Since $C(X, \mathbb{R}^q \setminus Y) \subset C(X, \mathbb{R}^q \setminus Z)$, we get $\dim(X \times Z) < q$ from Proposition 4.3. Then (2) follows, since $\dim(X \times [0,1]^d) = \dim(X) + d$.

To show "(2) ⇒ (1)", assume $\dim(X) < q - \dim(Y)$. Let $f \in C(X, \mathbb{R}^q)$, and $\varepsilon > 0$ be given. We let $B \subset \mathbb{R}^q$ be the closed ball of radius $(1+\varepsilon)\|f\|$. Then $Y \cap B$ is codimension three, tame compact subset. Moreover, $\dim(X \times (Y \cap B)) \leq \dim(X) + \dim(Y) < q$. By Proposition 4.3, we may find $g \in C(X, \mathbb{R}^q \setminus (Y \cap B))$ with $g =_\varepsilon f$. By construction, $g \in C(X, \mathbb{R}^q \setminus Y)$. Since f and ε were arbitrary, this shows that $C(X, \mathbb{R}^q \setminus Y) \subset C(X, \mathbb{R}^q)$ is dense, as desired. □

4.5. Let X be a compact, metric space with $\dim(X) = d$. It is a classical result that X can be embedded into $\mathbb{R}^{2d+1}$, and even more, the embeddings $E(X, \mathbb{R}^{2d+1})$ are dense in $C(X, \mathbb{R}^{2d+1})$, see e.g. [Eng95, Theorem 1.11.4, p.95].

The converse is not quite true, and it is connected to the question whether $\dim(X \times X) = 2\dim(X)$. It is known that the dimension of $X \times X$ can only take the values $2\dim(X) - 1$ or $2\dim(X)$. Thus, we may divide the class of all compact, metric, finite-dimensional spaces into two classes, see the Definition before 3.17 in [Dra01]:

(1) If $\dim(X \times X) = 2\dim(X)$, we say X is "of basic type".
(2) If $\dim(X \times X) < 2\dim(X)$, we say X is "of exceptional type".

If $\dim(X) \leq 1$, then X of basic type, and for every $d \geq 2$ there exists a space X of exceptional type with $\dim(X) = d$.

It is shown by Spież, [Spi90, Theorem 2], that X is of exceptional type if and only if $E(X, \mathbb{R}^{2d})$ is dense in $C(X, \mathbb{R}^{2d})$. As shown in [DRS91, Theorem 1.1], one may deduce that the following are equivalent:

(1) $\dim(X \times X) < q$.
(2) $E(X, \mathbb{R}^q) \subset C(X, \mathbb{R}^q)$ is dense.

This result was generalized to topological manifolds by Luukkainen. By a q-manifold we mean a separable, metric space M such that every point $x \in M$ has a neighborhood homeomorphic to $\mathbb{R}^q$. It follows from [Luu81, Theorem 5.1] that $E(C, M) \subset C(X, M)$ is dense if $q \geq 2d+1$. This is complemented by [Luu91, Theorem 2.5] which shows that X is of exceptional type if and only if $E(X,M) \subset C(X,M)$ is dense for some (and hence all) $2d$-manifolds.

This shows the following result:

Proposition 4.6 (Lukkainen, [Luu81], [Luu91])**.** *Let X be a compact, metric space, and M a manifold. Then the following are equivalent:*

(1) $\dim(X \times X) < \dim(M)$*,*
(2) $E(X,M) \subset C(X,M)$ *is dense.*

Proposition 4.7. *Let X be a compact, metric space. Then* $\mathrm{gr}(C(X)) = \dim(X \times X)$*.*

More generally, if A is a separable, commutative C^*-algebra, then

$$\mathrm{gr}(A) = \dim(\mathrm{Prim}(A) \times \mathrm{Prim}(A)).$$

Proof. Let X be a compact, metric space, and let $k \in \mathbb{N}$. As explained in 4.1, we have $\mathrm{gr}(C(X)) \leq k$ if and only if the following conditions hold:

(1) $C(X, \mathbb{R}^{k+1} \setminus \{0\}) \subset C(X, \mathbb{R}^{k+1})$ is dense.
(2) $E(X, \mathbb{R}^{k+1} \setminus \{0\}) \subset C(X, \mathbb{R}^{k+1} \setminus \{0\})$ is dense.

It follows from Proposition 4.4 that (1) is equivalent to $\dim(X) \leq k$, and it follows from Proposition 4.6 that (2) is equivalent to $\dim(X \times X) \leq k$.

Since $\dim(X) \leq \dim(X \times X)$, condition (2) implies condition (1), and we deduce:

$$\mathrm{gr}(C(X)) \leq k \quad \Leftrightarrow \quad \dim(X \times X) \leq k,$$

from which the result follows.

Now let A be a non-unital, separable, commutative C^*-algebra, and set $X := \mathrm{Prim}(A)$. Then $\widetilde{A}$ has primitive ideal space αX, the one-point compactification of X. Then $\mathrm{gr}(A) = \mathrm{gr}(\widetilde{A}) = \dim((\alpha X) \times (\alpha X))$, and the result follows since $\dim((\alpha X) \times (\alpha X)) = \dim(X \times X)$. ☐

Lemma 4.8. *Let X, Y be compact, metric spaces. Set $Z = X \sqcup Y$. Then:*

$$\dim(Z \times Z) \leq \max\{\dim(X \times X), \dim(Y \times Y)\}.$$

Moreover, if both X and Y are of exceptional type (see 4.5), then so is $Z = X \sqcup Y$.

Proof. For any compact, metric space M, we use the notation $M^k := M \times \ldots_k \times M$ for the k-fold Cartesian power. It is shown in [Dra01, Theorem 3.16] that M is of exceptional type if and only if $\dim(M^k) = k\dim(X) - k + 1$ for $k \geq 1$, and that M is of basic type if and only if $\dim(M^k) = k\dim(X)$ for $k \geq 1$. If M, N are two compact spaces, then $\dim(M \times N) \leq \dim(M) + \dim(N)$, by the product theorem of covering dimension.

Now let X, Y be two compact, metric space, and set $Z = X \sqcup Y$. Note that we have $\dim(Z) = \max\{\dim(X), \dim(Y)\}$. We distinguish two cases:

Case 1: Assume $\dim(X) \neq \dim(Y)$. Without loss of generality we may assume $\dim(X) < \dim(Y)$. Then $\dim(Z) = \dim(Y)$. Moreover, $\dim(X \times Z) \leq \dim(X) + \dim(Y) \leq \dim(Y^2)$, and $\dim(X^2) \leq 2\dim(X) \leq \dim(Y^2)$. Thus, we may estimate:

$$\dim(Z^2) = \max\{\dim(X^2), \dim(X \times Y), \dim(Y^2)\} \leq \dim(Y^2),$$

and we also obtain the desired inequality $\dim(Z^2) \leq \max\{\dim(X^2), \dim(Y^2)\}$.

If Y is of exceptional type, then

$$\dim(Z^2) \leq \dim(Y^2) = 2\dim(Y) - 1 = 2\dim(Z) - 1,$$

showing that Z is of exceptional type.

Case 2: Assume $\dim(X) = \dim(Y)$, and set $d := \dim(X)$. We may estimate:

$$\begin{aligned}\dim(Z^2) &= \max\{\dim(X^2), \dim(X \times Y), \dim(Y^2)\} \\ &\leq \max\{2\dim(X), \dim(X) + \dim(Y), 2\dim(Y)\} \\ &\leq 2d.\end{aligned}$$

If at least one of X or Y is of basic type, then $\max\{\dim(X^2), \dim(Y^2)\} = 2d$, showing the desired inequality.

If both X and Y are of expectional type, then $\dim(X^2) = \dim(Y^2) = 2d - 1$, and we have the following estimate:

$$\dim(X^2 \times Y) \leq \dim(X^2) + \dim(Y) = 3d - 1,$$

and similarly $\dim(X \times Y^2) \leq 3d - 1$.

Then:

$$\begin{aligned}\dim(Z^3) &= \max\{\dim(X^3), \dim(X^2 \times Y), \dim(X \times Y^2), \dim(Y^3)\} \\ &\leq \max\{3d - 2, 3d - 1, 3d - 1, 3d - 2\} \\ &\leq 3d - 1.\end{aligned}$$

If Z were of basic type, then $\dim(Z^3) = 3\dim(Z)$. Thus, Z is of exceptional type, and so $\dim(Z^2) = 2\dim(Z) - 1 = 2d - 1 = \max\{\dim(X^2), \dim(Y^2)\}$. □

Proposition 4.9. *Let A and B be separable, commutative C^*-algebras. Then* $\mathrm{gr}(A \oplus B) = \max\{\mathrm{gr}(A), \mathrm{gr}(B)\}$.

Proof. We have $\mathrm{gr}(A), \mathrm{gr}(B) \leq \mathrm{gr}(A \oplus B)$ by Proposition 2.12. For the converse inequality, note that $A \oplus B$ is an ideal in $\widetilde{A} \oplus \widetilde{B}$. Then, using Theorem 2.14 at the first step, using Proposition 4.7 and Lemma 4.8 at the second step:

$$\mathrm{gr}(A \oplus B) \leq \mathrm{gr}(\widetilde{A} \oplus \widetilde{B}) \leq \max\{\mathrm{gr}(\widetilde{A}), \mathrm{gr}(\widetilde{B})\} = \max\{\mathrm{gr}(A), \mathrm{gr}(B)\},$$

as desired. □

We now turn to the computation of the generator rank of homogeneous C^*-algebras. We first recall the well-known structure theory of such algebras.

Definition 4.10 (Fell, [Fel61, 3.2])**.** Let A be a C^*-algebra and $n \geq 1$. Then A is called n**-homogeneous** if all its irreducible representations are n-dimensional. We further say that A is **homogeneous** if it is n-homogeneous for some n.

4.11. Let us recall a general construction: Assume $\mathfrak{B} = (E \xrightarrow{p} X)$ is a locally trivial fibre bundle (over a locally compact, Hausdorff space X) whose fiber has the structure of a C^*-algebra. Let

$$\Gamma_0(\mathfrak{B}) = \{f \colon X \to E \mid p \circ f = id_X, (x \to \|f(x)\|) \in C_0(X)\} \tag{4.1}$$

be the sections of $\mathfrak{B}$ that vanish at infinity. Then $\Gamma_0(\mathfrak{B})$ has a natural structure of a C^*-algebra, with the algebraic operations defined fibrewise, and norm $\|f\| := \sup_{x \in X} \|f(x)\|$.

If the bundle has fibre M_n (a so-called M_n-bundle), then $A := \Gamma_0(\mathfrak{B})$ is n-homogeneous and $\mathrm{Prim}(A) \cong X$. Thus, every M_n-bundle defines an n-homogeneous C^*-algebra. The converse does also hold:

Proposition 4.12 (Fell, [Fel61, Theorem 3.2]). *Let A be a C^*-algebra, $n \in \mathbb{N}$. Then the following are equivalent:*

(1) A is n-homogeneous,

(2) $A \cong \Gamma_0(\mathfrak{B})$ for a locally trivial M_n-bundle $\mathfrak{B}$.

4.13. Let A be a n-homogeneous C^*-algebra, and set $X := \mathrm{Prim}(A)$. Then A is naturally a $C_0(X)$-algebra, with each fiber isomorphic to M_n. For the definition and results of $C_0(X)$-algebras, we refer the reader to § 1 of [Kas88] or § 2 of [Dad09]. We use the same notation as in [TW12, 2.4].

4.14. Let A be a C^*-algebra, and $k \in \mathbb{N}$. Let us denote the automorphism group of A by $\mathrm{Aut}(A)$. We define a natural action Υ of $\mathrm{Aut}(A)$ on A_{sa}^k. Given $\alpha \in \mathrm{Aut}(A)$ and $\mathbf{a} = (a_1, \ldots, a_k) \in A_{\mathrm{sa}}^k$ we set:

$$\Upsilon(\alpha)(\mathbf{a}) = \alpha \cdot \mathbf{a} := (\alpha(a_1), \ldots, \alpha(a_k)).$$

For $\alpha \in \mathrm{Aut}(A)$ and $\mathbf{a} \in A_{\mathrm{sa}}^k$, note that $\alpha \cdot \mathbf{a} = \mathbf{a}$ if and only if $\alpha(x) = x$ for all $x \in C^*(\mathbf{a})$. Thus, the restriction of Υ to $\mathrm{Gen}_k(A)_{\mathrm{sa}}$ is free.

For $A = M_n$, every automorphism is inner. The kernel of the map $U_n \to \mathrm{Aut}(M_n)$ is the group of central unitary matrices $\mathbb{T} \cdot 1 \subset U_n$. Let $PU_n := U_n/(\mathbb{T} \cdot 1)$ be the projective unitary group. It follows that $\mathrm{Aut}(M_n) \cong PU_n$, which is a compact Lie group of dimension $n^2 - 1$.

Lemma 4.15. *Let A be a simple C^*-algebra, let $k \in \mathbb{N}$, and let $\mathbf{a}, \mathbf{b} \in \mathrm{Gen}_k(A)_{sa}$. Then $\mathbf{a} \oplus \mathbf{b} \in \mathrm{Gen}_k(A \oplus A)_{sa}$ if and only if $\mathbf{a}$ and $\mathbf{b}$ lie in different orbits of the action of $\mathrm{Aut}(A)$ on $\mathrm{Gen}_k(A)_{sa}$.*

Proof. If $\mathbf{a} = \alpha \cdot \mathbf{b}$ for some $\alpha \in \mathrm{Aut}(A)$, then $C^*(\mathbf{a} \oplus \mathbf{b}) = \{(x, \alpha(x)) \mid x \in A\} \cong A$, and so $\mathbf{a} \oplus \mathbf{b}$ does not generate $A \oplus A$.

Conversely, let $D := C^*(\mathbf{a} \oplus \mathbf{b}) \subset A \oplus A$, and assume that $D \neq A \oplus A$. Let $\pi_i \colon A \oplus A \to A$ be the surjective morphisms on the two summands, $i = 1, 2$. Then $\pi_1(D) = A$. Note that $\ker(\pi_1) = 0 \oplus A$. If $\ker(\pi_1) \cap D \neq 0$, then $\ker(\pi_1) \cap D = 0 \oplus A$, and so $D = A \oplus A$, a contradiction. Therefore, $\pi_1 \colon D \to A$ is an isomorphism. Similarly $\pi_2 \colon D \to A$ is an isomorphism. Then $\alpha := \pi_2 \circ \pi_1^{-1}$ is the desired automorphism of A that satisfies $\alpha \cdot \mathbf{a} = \mathbf{b}$. □

Proposition 4.16. *Let X be a compact, metric space, let A be a simple, separable C^*-algebra, and let B a continuous $C(X)$-algebra with fibers isomorphic to A. Let $k \in \mathbb{N}$, and $\mathbf{b} \in B_{sa}^k$. Then $\mathbf{b} \in \mathrm{Gen}_k(B)_{sa}$ if and only if the following two conditions are satisfied:*

(1) $\mathbf{b}$ pointwise generates A, i.e., for each $x \in X$ we have $\mathbf{b}(x) = (b_1(x), \ldots, b_k(x)) \in \mathrm{Gen}_k(A)_{sa}$.

(2) $\mathbf{b}$ separates the points of X in the sense that for distinct $x, y \in X$ the tuples $\mathbf{b}(x)$ and $\mathbf{b}(y)$ lie in different orbits of the action of $\mathrm{Aut}(A)$ on $\mathrm{Gen}_k(A)_{sa}$.

Proof. Let us first assume that $\mathbf{b} \in \mathrm{Gen}_k(B)_{\mathrm{sa}}$. For $x \in X$, let $\pi_x \colon B \to A$ be the surjective morphism to the fiber at x. This maps $\mathrm{Gen}_k(B)_{\mathrm{sa}}$ into $\mathrm{Gen}_k(A)_{\mathrm{sa}}$. This shows (1). For distinct points $x, y \in X$, consider the surjective morphism $\varphi := \pi_x \oplus \pi_y \colon B \to A \oplus A$. Since φ maps $\mathrm{Gen}_k(B)_{\mathrm{sa}}$ into $\mathrm{Gen}_k(A \oplus A)_{\mathrm{sa}}$, we get $\mathbf{b}(x) \oplus \mathbf{b}(y) = \varphi(\mathbf{b}) \in \mathrm{Gen}_k(A \oplus A)_{\mathrm{sa}}$, and so (2) follows from Lemma 4.15.

The converse follows from [TW12, Lemma 3.2], which is proved using the factorial Stone-Weierstrass conjecture. □

Notation 4.17. For $n, k \in \mathbb{N}$, set:

$$E_n^k := (M_n)_{\text{sa}}^k, \qquad G_n^k := \text{Gen}_k(M_n)_{\text{sa}} \subset E_n^k.$$

Lemma 4.18. *Let $n, k \in \mathbb{N}$. Then $G_n^k \subset E_n^k$ is open.*

Proof. In general, if A is any C^*-algebra, and $\text{Gen}_l(A)_{\text{sa}}$ contains a non-empty open subset, for some l, then $\text{Gen}_k(A)_{\text{sa}}$ is open (but possibly empty) for all k. To see this, let $U \subset \text{Gen}_l(A)_{\text{sa}}$ be a non-empty, open subset. Let $\mathbf{a}$ be any element of $\text{Gen}_k(A)_{\text{sa}}$. Then there exist polynomials $p_1, \dots, p_l$ such that $(p_1(\mathbf{a}), \dots, p_l(\mathbf{a})) \in U$. For $i = 1, \dots, l$, let $q_i := (p_i + p_i^*)/2$, considered as a polynomial that maps A_{sa}^k to A_{sa}. Consider the continuous map $q\colon A_{\text{sa}}^k \to A_{\text{sa}}^l$ given by $q(\mathbf{x}) := (q_1(\mathbf{x}), \dots, q_l(\mathbf{x}))$. Then $q(\mathbf{a}) \in U$, and so $q^{-1}(U)$ is an open subset of $\text{Gen}_k(A)_{\text{sa}}$ containing $\mathbf{a}$.

Let $e_{i,j} \in M_n$ denote the matrix units. Consider the self-adjoint element $a = \sum_{s=1}^n \frac{s}{n} e_{s,s}$, and $b = \sum_{s=1}^{n-1}(e_{s,s+1} + e_{s+1,s})$. Then every pair $(a', b') \in (M_n)_{\text{sa}}^2$ close enough to (a, b) generates M_n. Indeed, if a' is close enough to a, then it generates a maximal abelian subalgebra of M_n. If b' is close enough to b, then it does not commute with the elements of $C^*(a')$, and it follows that the commutant of $C^*(a', b')$ is $\mathbb{C} \cdot 1$. This implies $C^*(a', b') = M_n$, as desired. □

Lemma 4.19. *Let $n, k \in \mathbb{N}$. Let Υ denote the action of PU_n on E_n^k, as defined in 4.14. Let $\mathbf{a} \in E_n^k$. Then the following are equivalent:*

(1) The stabilizer subgroup of $\mathbf{a}$ is trivial.
(2) $\mathbf{a} \in G_n^k$.

Proof. Let $\alpha \in PU_n$, and let it be represented by a unitary $u \in M_n$. It was noted in 4.14 that $\alpha \cdot \mathbf{a} = \mathbf{a}$ if and only if $\alpha(x) = x$ for all $x \in C^*(\mathbf{a})$. This shows the implication "(2) ⇒ (1)".

For the converse, assume $\mathbf{a} \in E_n^k \setminus G_n^k$. Let $B := C^*(\mathbf{a})$, the sub-C^*-algebras generated by $\mathbf{a}$. Since B is a proper sub-C^*-algebra, the commutant B' is non-trivial. It follows that there exists a unitary $u \in B'$ which induces a non-trivial automorphism on M_n, while it stabilizes $\mathbf{a}$, and so the stabilizer subgroup of $\mathbf{a}$ is non-trivial. □

Lemma 4.20. *Let $n, k \geq 2$. Then $E_n^k \setminus G_n^k$ is a closed subset which is a finite union of codimension three submanifolds and $\dim(E_n^k \setminus G_n^k) = kn^2 - (k-1)(2n-2)$.*

If, moreover, X is a compact, metric space, then the following are equivalent:

(1) $C(X, G_n^k) \subset C(X, E_n^k)$ is dense (and open),
(2) $\dim(X) < (k-1)(2n-2)$.

Proof. Set $Y := E_n^k \setminus G_n^k$, which is closed by Lemma 4.18. The action Υ of PU_n on E_n^k, as defined in 4.14, is a smooth action of a compact Lie group on a manifold, and we consider its orbit type decomposition.

In general, if a compact Lie group G acts smoothly on a manifold M, and $H \leqslant G$ is a closed subgroup, then $M_{(H)}$ denotes the set of points $x \in M$ such that the stabilizer subgroup of x is conjugate to H. Then M decomposes into orbit types $M = \bigcup M_{(H)}$. Let $\pi\colon M \to M/G$ denote the quotient map to the orbit space. Then, for each H, the image of $\pi(M_{(H)}) \subset M/G$ has a unique manifold structure, and the restriction of π to $M_{(H)}$ is a submersion. We refer the reader to [Mei03] for more details.

By Lemma 4.19, G_n^k is the submanifold corresponding to the trivial stabilizer subgroup. Thus, we have to consider $M_{(H)}$ for subgroups $H \leqslant PU_n$ with $H \neq \{1\}$.

Let $V(M_n)$ denote the space of sub-C^*-algebras of M_n. Then PU_n naturally acts on $V(M_n)$. Let $\Psi\colon E_n^k \to V(M_n)$ be the map that sends $\mathbf{a}$ to $\Psi(\mathbf{a}) := C^*(\mathbf{a}) \subset M_n$. It is easily checked that Ψ is PU_n-equivariant.

There are only finitely many orbits of the action of PU_n on $V(M_n)$, which we may label by multi-indices $\omega = (d_1, m_2, \ldots, d_s, m_s)$ for $d_i, m_i \in \mathbb{N}$ and $\sum_i m_i d_i \leq n$. The multi-index $\omega = (d_1, m_2, \ldots, d_s, m_s)$ is assigned to a sub-C^*-algebra $B \subset M_n$ if $B \cong M_{d_1} \oplus \ldots \oplus M_{d_s}$, and the copy $M_{d_i} \subset B$ has multiplicity m_i in the embedding $B \subset M_n$. Two indices can be assigned to the same orbit, and one could put more restrictions on the indices to get a unique assignment. For our considerations, however, this is unimportant, since we are only interested in the dimension of certain submanifolds, and it is no problem if we consider the same submanifold several times.

Let $\omega = (d_1, m_2, \ldots, d_s, m_s)$ be a multi-index, and let $B \subset M_n$ be a sub-C^*-algebra whose orbit in $V(M_n)$ has index ω. For $\mathbf{a} \in E_n^k$, we have $\Psi(\mathbf{a}) = B$ if and only if $\mathbf{a} \in \mathrm{Gen}_k(B)_{\mathrm{sa}} \subset \mathrm{Gen}_k(M_n)_{\mathrm{sa}} = E_n^k$. By Corollary 3.3, we have $\mathrm{gr}(B) \leq 1$, and since $k \geq 2$ we get that $\mathrm{Gen}_k(B)_{\mathrm{sa}} \subset B_{\mathrm{sa}}^k$ is dense. Moreover, it follows from Lemma 4.18 that $\mathrm{Gen}_k(B)_{\mathrm{sa}} \subset B_{\mathrm{sa}}^k$ is open. Thus, $\Psi^{-1}(B) = \mathrm{Gen}_k(B)_{\mathrm{sa}}$ is a dense, open subset of the manifold B_{sa}^k, and therefore:

$$\dim(\Psi^{-1}(B)) = \dim(B_{\mathrm{sa}}^k) = k \sum_i d_i^2.$$

Let $K(B) \subset PU_n$ be the stabilizer subgroup of B. If B is a unital subalgebra of M_n, then every unitary of B stabilizes B, and so we obtain a natural map $U(B) \to K(B)$ with kernel $U(B) \cap \mathbb{T} \cdot 1$. If B is a non-unital subalgebra, let $r \in M_n$ denote the projection such that $1_B + r = 1$. For every unitary $u \in B$, the element $u + r$ is a unitary in M_n that stabilizes B. Thus, in the unital case, $K(B)$ has a subgroup isomorphic to $U(B)/\mathbb{T}$, and in the non-unital case, $K(B)$ has a subgroup isomorphic to $U(B)$. Therefore, we get the rough (but for our purposes sufficient) estimate:

$$\dim(K(B)) \geq \dim U(B) - 1 = \sum_i d_i^2 - 1.$$

Let $V(\omega) \subset V(M_n)$ be the orbit with index ω. Then:

$$\dim(V(\omega)) = \dim(PU_n) - \dim(K(B)) \leq n^2 - 1 - [\sum_i d_i^2 - 1] = n^2 - \sum_i d_i^2.$$

Therefore:

$$\begin{aligned}\dim(\Psi^{-1}(V(\omega))) &= \dim(\Psi^{-1}(B)) + \dim(V(\omega)) \\ &\leq k \sum_i d_i^2 + n^2 - \sum_i d_i^2 \\ &= n^2 + (k-1) \sum_i d_i^2.\end{aligned}$$

Note that $\omega = (n, 1)$ labels the one-point orbit of $M_n \subset M_n$, and so $\Psi^{-1}(V(n,1)) = G_n^k$. Among $\omega \neq (n, 1)$, $n^2 + (k-1)\sum_i d_i^2$ has its maximum value for the partition $d_1 = n-1, d_2 = 1$ (and $m_1 = m_2 = 1$). Thus, for $\omega \neq (n, 1)$:

$$\dim(\Psi^{-1}(V(\omega))) \leq n^2 + (k-1)[(n-1)^2 + 1] = kn^2 - (k-1)(2n-2).$$

Since $E_n^k \setminus G_n^k = \bigcup_{\omega \neq (n,1)} \Psi^{-1}(V(\omega))$, we get:

$$\dim(E_n^k \setminus G_n^k) \leq kn^2 - (k-1)(2n-2).$$

The partition $\omega' := (n-1,1,1,1)$ labels the orbit of $B = M_{n-1} \oplus \mathbb{C} \subset M_n$, and one checks $K(B) \cong U_{n-1}$, so that $\dim(V(\omega')) = n^2-1-(n-1)^2$ and then $\dim(\Psi^{-1}(V(\omega'))) = kn^2-(k-1)(2n-2)$. Since $\Psi^{-1}(V(\omega')) \subset E_n^K \setminus G_n^k$, we get $\dim(E_n^K \setminus G_n^k) \geq kn^2-(k-1)(2n-2)$, and so $\dim(E_n^K \setminus G_n^k) = kn^2-(k-1)(2n-2)$, as desired.

Now the assertion of the equivalent conditions follows from our dimension computations and Proposition 4.4. □

Lemma 4.21. *Let A be a unital, separable n-homogeneous C^*-algebra, $n \geq 2$, and let $k \geq 2$. Let $X := \mathrm{Prim}(A)$, and for $x \in X$ let $\pi_x\colon A \to M_n$ be a morphism onto the fiber at x. Let S be the set of elements $\mathbf{a} \in A^k_{sa}$ such that for all $x \in \mathrm{Prim}(A)$, the tuple $\pi_x(a_1), \dots, \pi_x(a_k)$ generates M_n. Then the following are equivalent:*

(1) $S \subset A^k_{sa}$ *is dense (and open),*
(2) $\dim(X) < (k-1)(2n-2)$.

Proof. There are finitely many closed subsets $X_1, \dots, X_r \subset X$ such that the M_n-bundle associated to A is trivial over each X_i. For each i, let $\Phi_i\colon A \to C(X_i, M_n)$ be a trivialization. This induces a surjective map $A^k_{\mathrm{sa}} \to C(X_i, M_n)^k_{\mathrm{sa}} \cong C(X_i, E_n^k)$, we also denote by Φ_i, and which is open by then open mapping theorem., Note that $\Phi_i(S) = C(X, G_n^k)$.

It follows that $S = \bigcap_i \Phi_i^{-1}(C(X, G_n^k))$ is dense if and only if each $C(X_i, G_n^k) \subset C(X_i, E_n^k)$ is dense. By Lemma 4.20, this is equivalent to $\dim(X_i) < (k-1)(2n-2)$ for each i, and since the X_i form a finite, closed cover of X, this is equivalent to $\dim(X) < (k-1)(2n-2)$. □

4.22. Let A be a unital, separable n-homogeneous C^*-algebra, and let $k \in \mathbb{N}$. Let $X := \mathrm{Prim}(A)$, and for $x \in X$ let $\pi_x\colon A \to M_n$ be a morphism onto the fiber at x. This induces a natural map $A^k_{\mathrm{sa}} \to E_n^k$, which we also denote by π_x.

Let us define a map $\Psi\colon A^k_{\mathrm{sa}} \to C(X, E_n^k/PU_n)$ by $\Psi(\mathbf{a})(x) := PU_n \cdot \pi(\mathbf{a})$. Note that the map from A to its fiber at some point $x \in X$ is not unique. It is, however, unique up to an automorphism of M_n, which shows that Ψ is well-defined.

Restricting Ψ to the subset S as defined in Lemma 4.21, gives a map $\Psi\colon S \to C(X, G_n^k/PU_n)$. Proposition 4.16 shows that $\mathrm{Gen}_k(A)_{\mathrm{sa}} = \Psi^{-1}(E(X, G_n^k/PU_n))$.

Theorem 4.23. *Let A be a unital, separable n-homogeneous C^*-algebra, $n \geq 2$. Set $X := \mathrm{Prim}(A)$, the primitive ideal space of A. Then:*

$$\mathrm{gr}(A) = \left\lceil \frac{\dim(X)+1}{2n-2} \right\rceil.$$

Proof. Since M_n is a quotient of A, we get $\mathrm{gr}(A) \geq \mathrm{gr}(M_n) = 1$ by Proposition 2.12 and Lemma 3.2. We also have $\left\lceil \frac{\dim(X)+1}{2n-2} \right\rceil \geq 1$ for every value of $\dim(X)$. Thus, it is enough to show that for every $k \geq 2$ the following holds:

$$\mathrm{gr}(A) \leq k-1 \quad \Leftrightarrow \quad \dim(X) < (k-1)(2n-2).$$

Let $S \subset A^k_{\mathrm{sa}}$ be defined as in Lemma 4.21.

Assume $\mathrm{gr}(A) \leq k-1$, i.e., $\mathrm{Gen}_k(A)_{\mathrm{sa}} \subset A^k_{\mathrm{sa}}$ is dense. Since $\mathrm{Gen}_k(A)_{\mathrm{sa}} \subset S$, it follows from Lemma 4.21 that $\dim(X) < (k-1)(2n-2)$.

Conversely, assume $\dim(X) < (k-1)(2n-2)$. Again by Lemma 4.21, we have that $S \subset A^k_{\mathrm{sa}}$ is dense. Consider the map $\Psi\colon S \to C(X, G_n^k/PU_n)$, as defined in 4.22. One checks that this map is continuous and open. Note that G_n^k/PU_n is a manifold of dimension $kn^2-(n^2-1) = (k-1)n^2+1$. It follows from Proposition 4.6 that

$E(X, G_n^k/PU_n) \subset C(X, G_n^k/PU_n)$ is dense if and only if $\dim(X \times X) < (k-1)n^2 + 1$. But this follows from the assumption on $\dim(X)$, using $n \geq 2$ at the third step:

$$\dim(X \times X) \leq 2\dim(X) < 2(k-1)(2n-2) \leq (k-1)n^2 + 1.$$

Then $\Psi^{-1}(E(X, G_n^k/PU_n)) \subset S$ is dense. It follows from Proposition 4.16 that $\mathrm{Gen}_k(A)_{\mathrm{sa}} = \Psi^{-1}(E(X, G_n^k/PU_n))$, which shows $\mathrm{gr}(A) \leq k-1$, as desired. □

Lemma 4.24. *Let A be a separable n-homogeneous C^*-algebra. Set $X := \mathrm{Prim}(A)$. If $\dim(X) < \infty$, then A is isomomorphic to an ideal in a unital, separable n-homogeneous C^*-algebra B with $\dim(\mathrm{Prim}(B)) = \dim(X)$.*

Proof. For $n = 1$ this is clear. For $n \geq 2$, this follows from Proposition 2.9 and Lemma 2.10 in [Phi07], since $\dim(X) < \infty$ implies that the M_n-bundle associated to A has finite type. □

Corollary 4.25. *Let A be a separable n-homogeneous C^*-algebra. Set $X := \mathrm{Prim}(A)$. If $n = 1$ (i.e., A is commutative), then*

$$\mathrm{gr}(A) = \dim(X \times X).$$

If $n \geq 2$, then:

$$\mathrm{gr}(A) = \left\lceil \frac{\dim(X) + 1}{2n - 2} \right\rceil.$$

Proof. For $n = 1$, this follows from Proposition 4.7. So assume $n \geq 2$. If A is unital, the formula follows from Theorem 4.23, so we may assume A is non-unital. We first show the inequality $\mathrm{gr}(A) \geq \left\lceil \frac{\dim(X)+1}{2n-2} \right\rceil$. Given a compact subset $Y \subset X$, let $A(Y)$ denote the unital, separable, n-homogeneous quotient of A corresponding to Y. It follows from Theorem 4.23 and Proposition 2.12 that:

$$\left\lceil \frac{\dim(Y) + 1}{2n - 2} \right\rceil = \mathrm{gr}(A(Y)) \leq \mathrm{gr}(A).$$

The desired inequality follows, since $\dim(X)$ is equal to the maximum of $\dim(Y)$ when Y is running over the compact subsets of X.

The converse inequality is clear if $\dim(X) = \infty$, so assume $\dim(X) < \infty$. By Lemma 4.24, A is an ideal in a unital, separable n-homoeneous C^*-algebra B with $\dim(\mathrm{Prim}(B)) = \dim(X)$. It follows from Theorem 2.14 and Theorem 4.23 that:

$$\mathrm{gr}(A) \leq \mathrm{gr}(B) = \left\lceil \frac{\dim(X) + 1}{2n - 2} \right\rceil,$$

as desired. □

Lemma 4.26. *Let A, B be two separable homogeneous C^*-algebras. Then $\mathrm{gr}(A \oplus B) = \max\{\mathrm{gr}(A), \mathrm{gr}(B)\}$.*

Proof. We have $\mathrm{gr}(A), \mathrm{gr}(B) \leq \mathrm{gr}(A \oplus B)$ by Proposition 2.12. We need to show the converse inequality. Assume A is n-homogeneous, and B is m-homogeneous. Set $X := \mathrm{Prim}(A)$, and $Y := \mathrm{Prim}(B)$. If $n = m$, then $A \oplus B$ is n-homogeneous with $\mathrm{Prim}(A \oplus B) = X \sqcup Y$. If $n = 1$, the result follows from Proposition 4.9.

If $n \geq 2$, then Corollary 4.25 shows:

$$\begin{aligned}\mathrm{gr}(A \oplus B) &= \left\lceil \frac{\dim(X \sqcup Y) + 1}{2n-2} \right\rceil \\ &= \max\left\{ \left\lceil \frac{\dim(X) + 1}{2n-2} \right\rceil, \left\lceil \frac{\dim(Y) + 1}{2n-2} \right\rceil \right\} \\ &= \max\{\mathrm{gr}(A), \mathrm{gr}(B)\}.\end{aligned}$$

For general C^*-algebras A, B one has $\mathrm{Gen}_k(A \oplus B)_{\mathrm{sa}} \subset \mathrm{Gen}_k(A)_{\mathrm{sa}} \oplus \mathrm{Gen}_k(B)_{\mathrm{sa}}$, and the inclusion might be strict. However, if A is n-homogeneous, and B is m-homogeneous with $n \neq m$, then no non-zero quotient of A is isomorphic to a quotient of B, and therefore:

$$\mathrm{Gen}_k(A \oplus B)_{\mathrm{sa}} = \mathrm{Gen}_k(A)_{\mathrm{sa}} \oplus \mathrm{Gen}_k(B)_{\mathrm{sa}},$$

from which the desired equality follows. □

In the same way as Lemma 4.26, one proves the following result:

Corollary 4.27. *Let $A_1, \ldots, A_k$ be separable, homogeneous C^*-algebras. Then:*

$$\mathrm{gr}(\bigoplus_i A_i) = \max_i \mathrm{gr}(A_i).$$

Remark 4.28. Let A be a unital, separable, n-homogeneous C^*-algebra, $n \geq 2$, and set $X := \mathrm{Prim}(A)$. It follows from Theorem 4.23 that the generator rank of A only depends on $\dim(X)$ (and n), but not on $\dim(X \times X)$. Thus, whether X is of basic or exceptional type does not matter for the computation of the generator rank of A.

Remark 4.29. Let $d \geq 1$ and $n \geq 2$, and set $A = C([0,1]^d, M_n)$. Recall that we denote by $\mathrm{gen}(A)$ the minimal number of self-adjoint generators for A. It follows from [Nag], [BE91] and Theorem 4.23 that:

$$\mathrm{gen}(A) = \left\lceil \frac{d-1}{n^2} + 1 \right\rceil, \qquad \mathrm{rr}(A) = \left\lceil \frac{d}{2n-1} \right\rceil, \qquad \mathrm{gr}(A) = \left\lceil \frac{d+1}{2n-2} \right\rceil.$$

This shows that the generator rank is more closely connected to the real rank than to the minimal number of generators.

The generator problem for simple C^*-algebras asks whether every unital, separable, simple C^*-algebra A is singly generated, i.e, whether A contains a generating element. We might consider a strengthened version that asks if $\mathrm{gr}(A) \leq 1$, i.e., whether the generating elements in A are dense. It follows from the work of Villadsen that this strengthened generator problem has a negative answer. Indeed, there exist simple AH-algebras of arbitrarily high real rank, see [Vil99]. Let A be such an AH-algebra with $\mathrm{rr}(A) = \infty$. Then $\mathrm{gr}(A) = \infty$, by Proposition 2.5.

Let A be a unital, separable, n-homogeneous C^*-algebra. Then A is an inductive limit of unital, separable, n-homogeneous C^*-algebras with finite-dimensional primitive ideal space. It follows from Theorem 4.23 and Proposition 2.13 that the tensor product of A with an infinite UHF-algebra has generator rank one. Using also Corollary 4.27, we may draw the following conclusion:

Corollary 4.30. *Let A be a unital, separable AH-algebra. Assume either that A is simple with slow dimension growth, or that A tensorially absorbs a UHF-algebra. Then $\mathrm{gr}(A) \leq 1$, and so the generators of A form a generic subset.*

Acknowledgments

The author thanks James Gabe and Mikael Rørdam for valuable comments and feedback.

This paper grew out of joint work with Karen Strung, Aaron Tikuisis, Joav Orovitz and Stuart White that started at the workshop "Set theory and C^*-algebras" at the AIM in Palo Alto, January 2012. In particular, the key Lemma 2.9 was obtained in that joint work. The author benefited from many fruitful discussions with Strung, Tikuisis, Orovitz and White, and he wants to thank them for their support of this paper.

References

[AP77] C. A. Akemann and G. K. Pedersen, *Ideal perturbations of elements in C^*-algebras*, Math. Scand. **41** (1977), 117–139.

[Arv77] W. Arveson, *Notes on extensions of C^*-algebras*, Duke Math. J. **44** (1977), 329–355.

[BE91] E. J. Beggs and D. E. Evans, *The real rank of algebras of matrix valued functions*, Int. J. Math. **2** (1991), no. 2, 131–138.

[BP91] L. G. Brown and G. K. Pedersen, *C^*-algebras of real rank zero*, J. Funct. Anal. **99** (1991), no. 1, 131–149.

[Dad09] M. Dadarlat, *Continuous fields of C^*-algebras over finite dimensional spaces*, Adv. Math. **222** (2009), no. 5, 1850–1881.

[Dra91] A. N. Dranishnikov, *On intersection of compacta in Euclidean space. II*, Proc. Am. Math. Soc. **113** (1991), no. 4, 1149–1154.

[Dra01] ———, *Cohomological dimension theory of compact metric spaces*, Topology Atlas Invited Contributions vol. 6 (2001), issue 1, 7-73. Available at http://at.yorku.ca/t/a/i/c/43.htm, 2001.

[DRS91] A. N. Dranishnikov, D. Repovš, and E. V. Shchepin, *On intersections of compacta of complementary dimensions in Euclidean space*, Topology Appl. **38** (1991), no. 3, 237–253.

[Edw75] R. D. Edwards, *Demension theory, I*, Geom. Topol., Proc. Conf. Park City 1974, Lect. Notes Math. 438, 195-211, 1975.

[EH95] N. Elhage Hassan, *Real rank of certain extensions. (Rang réel de certaines extensions.)*, Proc. Am. Math. Soc. **123** (1995), no. 10, 3067–3073.

[Eng95] R. Engelking, *Theory of dimensions, finite and infinite*, Sigma Series in Pure Mathematics. 10. Lemgo: Heldermann. VIII, 1995.

[Fel61] J. M. G. Fell, *The structure of algebras of operator fields*, Acta Math. **106** (1961), 233–280.

[Ge03] L. M. Ge, *On "Problems on von Neumann algebras by R. Kadison, 1967"*, Acta Math. Sin., Engl. Ser. **19** (2003), no. 3, 619–624.

[Kad67] R. Kadison, *Problems on von Neumann algebras*, unpublished manuscript, presented at Conference on Operator Algebras and Their Applications, Louisiana State Univ., Baton Rouge, La., 1967.

[Kas88] G. G. Kasparov, *Equivariant KK-theory and the Novikov conjecture*, Invent. Math. **91** (1988), no. 1, 147–201.

[Luu81] J. Luukkainen, *Approximating continuous maps of metric spaces into manifolds by embeddings*, Math. Scand. **49** (1981), 61–85.

[Luu91] ———, *Embeddings of n-dimensional locally compact metric spaces to $2n$-manifolds*, Math. Scand. **68** (1991), no. 2, 193–209.

[Mei03] E. Meinreken, *Group actions on manifolds*, Lecture Notes, University of Toronto, Spring 2003, 2003.

[Nag] M. Nagisa, *Single generation and rank of C^*-algebras*, preprint.

[OZ76] C. L. Olsen and W. R. Zame, *Some C^*-algebras with a single generator*, Trans. Am. Math. Soc. **215** (1976), 205–217.

[Phi07] N. C. Phillips, *Recursive subhomogeneous algebras*, Trans. Am. Math. Soc. **359** (2007), no. 10, 4595–4623.

[Rie83] M. A. Rieffel, *Dimension and stable rank in the K-theory of C^*-algebras*, Proc. Lond. Math. Soc., III. Ser. **46** (1983), 301–333.

[Spi90] S. Spież, *The structure of compacta satisfying* $\dim(X \times X) < 2\dim X$, Fundam. Math. **135** (1990), no. 2, 127–145.

[Thi11] H. Thiel, *Inductive limits of projective C^*-algebras*, preprint, arXiv:1105.1979, 2011.

[Thi12] ______, *The topological dimension of type I C^*-algebras*, preprint, arXiv:1210.4314, 2012.
[TW07] A. S. Toms and W. Winter, *Strongly self-absorbing C^*-algebras.*, Trans. Am. Math. Soc. **359** (2007), no. 8, 3999–4029.
[TW12] H. Thiel and W. Winter, *The generator problem for $\mathcal{Z}$-stable C^*-algebras*, preprint, arXiv:1201.3879, 2012.
[Vil99] J. Villadsen, *On the stable rank of simple C^*-algebras*, J. Am. Math. Soc. **12** (1999), no. 4, 1091–1102.
[Wil74] P. Willig, *Generators and direct integral decompositions of W^*-algebras*, Tohoku Math. J., II. Ser. **26** (1974), 35–37.
[Xue10] Y. Xue, *Approximate diagonalization of self-adjoint matrices over $C(M)$*, preprint, arXiv:1002.3962, 2010.
[Zha90] S. Zhang, *Diagonalizing projections in multiplier algebras and in matrices over a C^*-algebra*, Pac. J. Math. **145** (1990), no. 1, 181–200.

Department of Mathematical Sciences, University of Copenhagen, Universitetsparken 5, DK-2100, Copenhagen Ø, Denmark
E-mail address: thiel@math.ku.dk

THE GENERATOR PROBLEM FOR $\mathcal{Z}$-STABLE C^*-ALGEBRAS

HANNES THIEL AND WILHELM WINTER

ABSTRACT. The generator problem was posed by Kadison in 1967, and it remains open until today. We provide a solution for the class of C^*-algebras absorbing the Jiang-Su algebra $\mathcal{Z}$ tensorially. More precisely, we show that every unital, separable, $\mathcal{Z}$-stable C^*-algebra A is singly generated, which means that there exists an element $x \in A$ that is not contained in any proper sub-C^*-algebra of A.

To give applications of our result, we observe that $\mathcal{Z}$ can be embedded into the reduced group C^*-algebra of a discrete group that contains a non-cyclic, free subgroup. It follows that certain tensor products with reduced group C^*-algebras are singly generated. In particular, $C^*_r(F_\infty) \otimes C^*_r(F_\infty)$ is singly generated.

1. INTRODUCTION

By an operator algebra we mean a *-subalgebra of $B(H)$ that is either closed in the norm topology (a concrete C^*-algebra) or the weak operator topology (a von Neumann algebra). One way of realizing an operator algebra is to take a subset of $B(H)$ and consider the smallest operator algebra containing it.

In a trivial way, every operator algebra can be obtained this way. The situation becomes interesting if one imposes restrictions on the generating set, and one natural possibility is to require that it consists of only one element, i.e., to consider operator algebras that are generated by a single operator. It is an old problem to determine which operator algebras arise this way.

More generally, one tries to compute the minimal number of elements that generate a given operator algebra, see 2.1. It is often convenient to consider self-adjoint generators. Note that two self-adjoint elements a, b generate the same operator algebra as the element $a+ib$. Thus, if we ask whether an operator algebra is singly generated, it is equivalent to ask whether it is generated by two self-adjoint elements.

In the case of von Neumann algebras, the generator problem was included in Kadison's famous 'Problems on von Neumann algebras', [Kad67]. This problem list has turned out to be very influential, yet its original form remains unpublished. It is indirectly available in an article by Ge, [Ge03], where a brief summary of the developments around Kadison's famous problems is given.

Date: April 23, 2012.

2000 *Mathematics Subject Classification.* Primary 46L05, 46L85; Secondary 46L35.

Key words and phrases. C^*-algebras, generator problem, single generation, $\mathcal{Z}$-stability.

This research was partially supported by the Centre de Recerca Matemàtica, Barcelona. The first named author was partially supported by the Danish National Research Foundation through the Centre for Symmetry and Deformation, Copenhagen. The second named author was partially supported by EPSRC Grants EP/G014019/1 and EP/I019227/1.

Question 1.1 (Kadison, [Kad67, Problem 14], see also [Ge03])**.** Is every separably-acting[1] von Neumann algebra singly generated?

As noted in [She09], there exist singly generated von Neumann algebras that are not separably-acting. However, the separably-acting von Neumann algebras are the natural class for which one might expect single generation. The answer to Question 1.1 is still open in general, but many authors have contributed to show that large classes of separably-acting von Neumann algebras are singly generated.

We just mention an incomplete list of results. It starts with von Neumann, [vN31], who showed that the abelian operator algebras named after him are generated by a single self-adjoint element, thus implicitly raising the generator problem. Some thirty years later, this was extended by Pearcy, [Pea62], who showed that all von Neumann algebras of type I are singly generated. Then Wogen, [Wog69, Theorem 2], proved that all properly infinite von Neumann algebras are singly generated, thus reducing the generator problem to the type II_1 case.

Later, this was further reduced to the case of a II_1-factor by Willig, [Wil74], and then to the case of a finitely-generated II_1-factor by Sherman, [She09, Theorem 3.8]. This means that Question 1.1 has a positive answer if every separably-acting, finitely generated II_1-factor is singly generated.

There are many properties known to imply that a II_1-factor is singly generated. We just mention that Ge and Popa, [GP98, Theorem 6.2], show that every tensorially non-prime[2] II_1-factor is singly generated. Our main result Theorem 3.5 can be considered as a partial C^*-algebraic analog of this result.

Let us also mention that the free group factors $W^*(F_k)$ are the outstanding examples of separably-acting von Neumann algebra for which it is not known whether they are singly generated.

In the case of C^*-algebras, the generator problem is more subtle. There is already no obvious class of C^*-algebras for which one conjectures that they are singly generated. Every singly generated C^*-algebra is separable[3]. However, the converse is false, and counterexamples can be found among the commutative C^*-algebras.

In fact, the C^*-algebra $C_0(X)$ is generated by n self-adjoint elements if and only if X can be embedded into $\mathbb{R}^n$. Thus, $C_0(X)$ is singly generated if and only if X is planar, i.e., can be embedded into the plane $\mathbb{R}^2$.

It is easy to see that a C^*-algebra A is generated by n self-adjoint elements if and only if its minimal unitization $\widetilde{A}$ is generated by n self-adjoint elements. Therefore, we will mostly consider the generator problem for separable, unital C^*-algebra. In that case, taking the tensor product with a matrix algebra has the effect of reducing the necessary number of generators. If A is generated by $n^2 + 1$ self-adjoint elements, then $A \otimes M_n$ is singly generated, see e.g. [Nag04, Theorem 3].

One derives the principle that a C^*-algebra needs less generators if it is 'more non-commutative'. Consequently, one might expect a (separable) C^*-algebra to be singly generated if it is 'maximally non-commutative'. As a non-unital instance

[1]A von Neumann algebra is called 'separably-acting', or just 'separable', if it is a subalgebra of $B(l^2\mathbb{N})$, or equivalently if it has a separable predual.

[2]A II_1-factor M is called tensorially non-prime if it is isomorphic to a tensor product, $M_1 \bar{\otimes} M_2$, of two II_1-factors M_1, M_2.

[3]A C^*-algebra is called 'separable' if it contains a countable, norm-dense subset

of this principle, we note that the stabilization, $A \otimes \mathbb{K}$, of a separable unital C^*-algebra A is singly generated, [OZ76, Theorem 8]. In the unital case, there are at least three natural cases when one considers a C^*-algebra A to be 'maximally non-commutative', which are the following:

(1) A contains a simple, unital, nonelementary sub-C^*-algebra,
(2) A contains a sequence of pairwise orthogonal, full elements,
(3) A has no finite-dimensional irreducible representations.

In general, the implications $(1) \Rightarrow (2) \Rightarrow (3)$ hold; it is not known if the converses are true.

Conditions (2) and (3) can also be considered for possibly non-unital C^*-algebras, and we let (2^*) be the weaker statement that A contains *two* orthogonal, full elements. The implication '$(3) \Rightarrow (2)$' holds exactly if the implication '$(3) \Rightarrow (2^*)$' holds.

The Global Glimm halving problem asks the following: Given a (possibly non-unital) C^*-algebra A that satisfies condition (3), does there exist a full map from the cone over M_2 to A? It is not known whether the Global Glimm halving problem has a positive answer, but if it does then it shows that implication '$(3) \Rightarrow (2)$' holds, since the cone over M_2 contains two orthogonal, full elements.

Let us remark that the analogs of conditions $(1)-(3)$ for von Neumann algebras are all equivalent. In fact, if a von Neumann algebra M has no finite-dimensional representations, then the hyperfinite II_1-factor $\mathcal{R}$ unitally embeds into M.

Historically, the generator problem for C^*-algebras is mostly asked for C^*-algebras that are simple ore more generally have no finite-dimensional representations:

Question 1.2. Is every simple, separable, unital C^*-algebra singly generated?

Question 1.3. Is a separable, unital C^*-algebra singly generated provided it has no finite-dimensional irreducible representations?

The answers to both questions are open. A positive answer to Question 1.3 implies a positive answer to Question 1.2, of course. The converse is not clear.

Let us mention some results that solve the generator problem for particular classes of separable C^*-algebras. It was shown by Topping, [Top68], that every UHF-algebra is singly generated. This was generalized by Olsen and Zame, [OZ76, Theorem 9], who showed that the tensor product, $A \otimes B$, of any separable, unital C^*-algebra A with a UHF-algebra B is singly generated.

Later, it was shown by Li and Shen, [LS10, Theorem 3.1], that every unital, approximately divisible[4] C^*-algebra is singly generated. This generalizes the result of Olsen and Zame, since the tensor product with a UHF-algebra is always approximately divisible.

In this article we prove that every separable, unital, $\mathcal{Z}$-stable C^*-algebra is singly generated, see Theorem 3.7. This generalizes the result of Li and Shen, since every approximately divisible C^*-algebra is $\mathcal{Z}$-stable, see [TW08, Theorem 2.3]. The notion of $\mathcal{Z}$-stability has proven to be very important in the classification program of nuclear C^*-algebras, see e.g. [Win07] or [ET08], and it is has been

[4]A unital C^*-algebra A is 'approximately divisible' if for every $\varepsilon > 0$ and finite subset $F \subset A$ there exists a finite-dimensional, unital sub-C^*-algebra $B \subset A$ such that B has no characters and $\|xb - bx\| \leq \varepsilon \|b\|$ for all $x \in F, b \in B$.

shown that many nuclear, simple C^*-algebras are $\mathcal{Z}$-stable, see e.g. [Win10]. $\mathcal{Z}$-stability is also relevant in the non-nuclear context; for example, unital $\mathcal{Z}$-stable C^*-algebras satisfy Kadison's similarity property, see [JW11].

This paper proceeds as follows:

In Section 2 we set up our notation and give some basic facts about the generator rank, see 2.1, and $C_0(X)$-algebras, see 2.4.

Section 3 contains the proof of our main result, which states that the tensor product $A \otimes_{\text{max}} B$ of two separable, unital C^*-algebras is singly generated, if A satisfies condition (2) from above (e.g. A is simple and non-elementary) and B admits a unital embedding of the Jiang-Su algebra $\mathcal{Z}$, see Theorem 3.5.

We derive that every separable, unital, $\mathcal{Z}$-stable C^*-algebra is singly generated, see Theorem 3.7. Our main result can be considered as a (partial) C^*-algebraic analog of a theorem of Ge and Popa, [GP98, Theorem 6.2], which shows that a tensor product, $M\bar{\otimes}N$, of two II_1-factors M, N is singly generated. In fact, we can reprove their theorem with our methods, see Corollary 3.11.

In Section 4 we give further applications of our main theorem to tensor products with reduced group C^*-algebras. We first observe that $\mathcal{Z}$ embeds unitally into $C^*_r(F_\infty)$, the reduced group C^*-algebra of the free group on infinitely many generators, see Lemma 4.1. Consequently, if a discrete group Γ contains a non-cyclic free subgroup, then $\mathcal{Z}$ embeds unitally into $C^*_r(\Gamma)$, see Proposition 4.2.

We deduce that tensor products of the form $A \otimes_{\text{max}} C^*_r(\Gamma)$ are singly generated if A is a separable, unital C^*-algebra satisfying condition (2) from above, and Γ is a group containing a non-cyclic free subgroup, see Corollary 4.4. For example, $C^*_r(F_\infty) \otimes C^*_r(F_\infty)$ is singly generated, although this C^*-algebra is not $\mathcal{Z}$-stable, see Example 4.5.

2. Preliminaries

By a morphism between C^*-algebras we mean a *-homomorphism, and by an ideal of a C^*-algebra we understand a closed, two-sided ideal. If A is a C^*-algebra, then we denote by $\widetilde{A}$ its minimal unitization. Often, we write M_k for the C^*-algebra of k-by-k matrices $M_k(\mathbb{C})$.

2.1. Let A be a C^*-algebra, and $A_{\text{sa}} \subset A$ the subset of self-adjoint elements. We say that a set $S \subset A_{\text{sa}}$ generates A, denoted $A = C^*(S)$, if the smallest sub-C^*-algebra of A containing S is A itself. We denote by $\operatorname{gen}(A)$ the smallest number $n \in \{1, 2, 3, \ldots, \infty\}$ such that A contains a generating subset $S \subset A_{\text{sa}}$ of cardinality n, and we call $\operatorname{gen}(A)$ the **generating rank** of A.

We stress that for the definition of $\operatorname{gen}(A)$, the generators are assumed to be self-adjoint. Two self-adjoint elements a, b generate the same C^*-algebra as the (non-self-adjoint) element $a + ib$. Therefore, a C^*-algebra A is said to be singly generated if $\operatorname{gen}(A) \leq 2$.

For more details on the generating rank we refer the reader to Nagisa, [Nag04], where also the following simple facts are noted for C^*-algebras A and B:

(1) $\operatorname{gen}(\widetilde{A}) = \operatorname{gen}(A)$,

(2) $\mathrm{gen}(C^*(A,B)) \leq \mathrm{gen}(A) + \mathrm{gen}(B)$, if A, B are sub-C^*-algebras of a common C^*-algebra, and where $C^*(A,B)$ denotes the sub-C^*-algebra they generate together,
(3) $\mathrm{gen}(A \oplus B) = \max\{\mathrm{gen}(A), \mathrm{gen}(B)\}$ if at least one of the algebras is unital.

Let $I \lhd A$ be an ideal in a C^*-algebra A. It is easy to see that the generating rank of the quotient A/I is not bigger than the generating rank of A, i.e., $\mathrm{gen}(A/I) \leq \mathrm{gen}(A)$, and the generating rank of A can be estimated as $\mathrm{gen}(A) \leq \mathrm{gen}(I) + \mathrm{gen}(A/I)$. The following result gives an estimate for $\mathrm{gen}(I)$, and it is probably well-known to experts; since we could not locate it in the literature, we include a short proof.

Proposition 2.2. *Let A be a C^*-algebra, and $I \lhd A$ an ideal. Then* $\mathrm{gen}(I) \leq \mathrm{gen}(A) + 1$.

Proof. We may assume $\mathrm{gen}(A)$ is finite. So let $a_1, \ldots, a_k$ be a set of self-adjoint generators for A. Then A and I are separable, and so I contains a strictly positive element h. It follows that $C^*(h)$ contains a quasi-central approximate unit, see [AP77, Corollary 3.3] and [Arv77]. It is straightforward to show that I is generated by the $k+1$ elements $h, ha_1h, \ldots, ha_kh$. □

The following result is attributed to Kirchberg in [Nag04].

Theorem 2.3 (Kirchberg). *Every separable, unital, properly infinite C^*-algebra is singly generated.*

Proof. We sketch a proof based on the proof of [OZ76, Theorem 9]. Let A be a separable, unital, properly infinite C^*-algebra. Then there exist isometries $s_1, s_2, \ldots \in A$ with pairwise orthogonal ranges (i.e., A contains a unital copy of the Cuntz algebra $\mathcal{O}_\infty$).

Let $a_1, a_2, \ldots \in A$ be a sequence of (positive) generators for A such that their spectra satisfy $\sigma(a_k) \subset [1/2 \cdot 1/4^k, 1/4^k]$. A generator for A is given by:

$$x := \sum_{k \geq 1} (s_k a_k s_k^* + 1/2^k s_k).$$

As in in the proof of [OZ76, Theorem 9], one can show that $\sigma(x) \subset \{0\} \cup \bigcup_{k\geq 1}[1/2 \cdot 1/4^k, 1/4^k]$. Let $B := C^*(x) \subset A$. Proceeding inductively, one shows that $a_k, s_k \in B$. We only sketch this for $k = 1$. Set $p := s_1 s_1^*$. Let f_n be a sequence of polynomials converging uniformly to 1 on $[1/8, 1/4]$ and to 0 on $[0, 1/16]$. Then $f_n(x)$ converges to an element $y \in B$ of the form $y = p + pb(1-p)$ for some $b \in A$. We compute $yy^* = p(1_A + b(1-p)b^*)p$. Then for a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$ and $f(t) = 1$ for $t \geq 1$, we get $f(yy^*) = p \in B$. Then $s_1 a_1 s_1^* = pxp \in B$ and $s_1 = 2 \cdot px(1-p) \in B$, and then also $a_1 \in B$. □

2.4. Let X be a locally compact σ-compact Hausdorff space. A $C_0(X)$-algebra is a C^*-algebra A together with a morphism $\eta\colon C_0(X) \to Z(M(A))$, from the commutative C^*-algebra $C_0(X)$ to the center of the multiplier algebra of A, such that for any approximate unit $(u_\lambda)_\Lambda$ of $C_0(X)$, $\eta(u_\lambda)a \to a$ for any $a \in A$, or equivalently, the closure of $\eta(C_0(X))A$ is all of A. Thus, if X is compact, then η is necessarily unital. We will usually suppress reference to the structure map, and simply write fa or $f \cdot a$ instead of $\eta(f)a$ for the product of a function $f \in C_0(X)$ and an element $a \in A$.

Let $Y \subset X$ be a closed subset, and $U := X \setminus Y$ its complement (an open subset). Then $C_0(U) \cdot A$ is an ideal of A, denoted by $A(U)$. The quotient $A/A(U)$ is denoted by $A(Y)$.

Given a point $x \in X$, we write $A(x)$ for $A(\{x\})$, and we call this C^*-algebra the fiber of A at x. For an element $a \in A$, we denote by $a(x)$ the image of a in the fiber $A(x)$. For each $a \in A$, we may consider the map $\check{a}\colon x \mapsto \|a(x)\|$. This is a real-valued, upper-semicontinuous function on X, vanishing at infinity. The $C_0(X)$-algebra A is called continuous if $\check{a}$ is a continuous function for each $a \in A$.

For more information on $C_0(X)$-algebras we refer the reader to [Kas88, §1] or the more recent [Dad09, §2].

2.5. The Jiang-Su algebra $\mathcal{Z}$ was constructed in [JS99]; it may be regarded as a C^*-algebraic analog of the hyperfinite II_1-factor. It can be obtained as an inductive limit of prime dimension drop algebras $\mathcal{Z}_{p,q} := \{f\colon [0,1] \to M_p \otimes M_q \mid f(0) \in 1_p \otimes M_q, f(1) \in M_p \otimes 1_q\}$.

For more details, we refer the reader to [Win11], where $\mathcal{Z}$ is characterized in an entirely abstract manner, and to [Rør04] and [RW10], where it is shown that the generalized dimension drop algebra $\mathcal{Z}_{2^\infty,3^\infty} := \{f\colon [0,1] \to M_{2^\infty} \otimes M_{3^\infty} \mid f(0) \in 1 \otimes M_{3^\infty}, f(1) \in M_{2^\infty} \otimes 1\}$ embeds unitally into $\mathcal{Z}$; in fact, $\mathcal{Z}$ can be written as a stationary inductive limit of $\mathcal{Z}_{2^\infty,3^\infty}$.

3. Results

Lemma 3.1. *Let A be a separable, unital C^*-algebra. Then $\mathrm{gen}(A \otimes \mathcal{Z}_{2^\infty,3^\infty}) \leq 5$.*

Proof. Consider the ideal $I := C_0(0,1) \otimes M_{6^\infty}$ in $B := A \otimes \mathcal{Z}_{2^\infty,3^\infty}$. The quotient B/I is isomorphic to $(A \otimes M_{2^\infty}) \oplus (A \otimes M_{3^\infty})$. Thus, we have a short exact sequence:

$$A \otimes C_0(0,1) \otimes M_{6^\infty} \longrightarrow A \otimes \mathcal{Z}_{2^\infty,3^\infty} \longrightarrow (A \otimes M_{2^\infty}) \oplus (A \otimes M_{3^\infty})$$

It follows from [OZ76] that the tensor product of a unital, separable C^*-algebra with a UHF-algebra is singly generated. In particular, $\mathrm{gen}(A \otimes M_{2^\infty}), \mathrm{gen}(A \otimes M_{3^\infty}) \leq 2$. Thus, the quotient satisfies $\mathrm{gen}(B/I) = \max\{\mathrm{gen}(A \otimes M_{2^\infty}), \mathrm{gen}(A \otimes M_{3^\infty})\} \leq 2$, see 2.1.

Note that I is an ideal in the C^*-algebra $C := A \otimes C(S^1) \otimes M_{2^\infty}$. We have $\mathrm{gen}(C) \leq 2$, and then $\mathrm{gen}(I) \leq \mathrm{gen}(C) + 1 \leq 3$, by Proposition 2.2. Then, the extension is generated by at most $2 + 3 = 5$ self-adjoint elements. □

The following is a Stone-Weierstrass type result. We prove it using the factorial Stone-Weierstrass conjecture, which states that a sub-C^*-algebra $B \subset A$ exhausts A if it separates the factorial states of A. The factorial Stone-Weierstrass conjecture was proved for separable C^*-algebras independently by Longo, [Lon84], and Popa, [Pop84].

See 2.4 for a short introduction to $C_0(X)$-algebras.

Lemma 3.2. *Let A be a separable, continuous $C_0(X)$-algebra, and $B \subset A$ a sub-C^*-algebra such that the following two conditions are satisfied:*

(i) For each $x \in X$, B exhausts the fiber $A(x)$,

(ii) B separates the points of X by full elements, i.e., for each distinct pair of points $x_0, x_1 \in X$ there exists some $b \in B$ such that $b(x_1)$ is full in $B(x_1) = A(x_1)$ and $b(x_0) = 0$.

Then $A = B$.

Condition (ii) is for instance satisfied if B contains the image of the structure map $\eta\colon C_0(X) \to Z(M(A))$.

Proof. Set $Y := \mathrm{Prim}(Z(M(A)))$, and identify $Z(M(A))$ with $C(Y)$. Let $\pi\colon A \to B(H)$ be a non-degenerate factor representation. Then π extends to a representation $\tilde{\pi}\colon M(A) \to B(H)$. It is straightforward to show $\pi(A)'' = \tilde{\pi}(M(A))''$, so that $\tilde{\pi}$ is a factor representation of $M(A)$. For any $c \in Z(M(A))$, we have $c \in \pi(A)' \cap \tilde{\pi}(M(A))'' = \mathbb{C} \cdot 1_H$. Thus, there exists a point $y \in Y$ such that $\tilde{\pi}(c) = c(y) \cdot 1_H$ for all $c \in Z(M(A))$. Since $\eta(C_0(X))$ contains an approximate unit for A, we have that $\tilde{\pi} \circ \eta$ is non-zero. Thus, there exists a point $x \in X$ such that $\tilde{\pi} \circ \eta(f) = f(x) \cdot 1_H$ for all $f \in C_0(X)$. This means that $\tilde{\pi} \circ \eta$ vanishes on the ideal $A(X \setminus \{x\})$, so that π factors through the fiber $A(x)$.

Let us show that $B \subset A$ separates the factors states of A. So let φ_1, φ_2 be two different, non-degenerate factors states of A. We have shown above that there are two points $x_1, x_2 \in X$ such that φ_i factors through $A(x_i)$, and we denote by $\bar{\varphi}_i\colon A(x_i) \to \mathbb{C}$ the induced factor state on $A(x_i)$, for $i = 1, 2$. We distinguish two cases:

Case 1: $x_1 = x_2$. In this case, since $\varphi_1 \neq \varphi_2$, there exists an element $a \in A$ such that $\varphi_1(a) \neq \varphi_2(a)$. By condition (i), there exists some element $b \in B$ such that $b(x_1) = a(x_1)$. Note that $\varphi_i(b) = \bar{\varphi}_i(b(x_1)) = \bar{\varphi}_i(a(x_1)) = \varphi_i(a)$, for $i = 1, 2$. Thus, b separates the two states.

Case 2: $x_1 \neq x_2$. In this case, by condition (ii), there exists an element $b \in B$ such that $b(x_2)$ is full in $A(x_2)$ and $b(x_1) = 0$. Since $\varphi_2 \neq 0$, there exists an element $a \in A$ such that $|\varphi_2(a)| = |\bar{\varphi}_2(a(x_2))| \geq 1$.

Since $b(x_2)$ is full, there exist finitely many elements $g_i, h_i \in A(x_2)$ such that $\|a(x_2) - \sum_i c_i b(x_2) d_i\| < 1$. By condition (i), there exist elements $\tilde{g}_i, \tilde{h}_i \in B$ such that $\tilde{g}_i(x_2) = g_i$ and $\tilde{h}_i(x_2) = h_i$. Set $b' := \sum_i \tilde{c}_i b \tilde{d}_i$. Then $|\varphi_2(b')| = |\bar{\varphi}_2(b'(x_2))| > 0$, while $b'(x_1) = 0$. This shows that b' separates the two states.

It follows that B separates the factor states of A, and therefore $B = A$ by the factorial Stone-Weierstrass conjecture, proved independently by Longo, [Lon84], and Popa, [Pop84]. □

Lemma 3.3. *Let A be a unital C^*-algebra with $\mathrm{gen}(A) \leq 3$. Then there exist a positive element $x \in A \otimes \mathcal{Z}_{2,3}$ and two positive, full elements $y', z' \in \mathcal{Z}_{2,3}$ such that $A \otimes \mathcal{Z}_{2,3}$ is generated by x and $1 \otimes y'$, and further y' and z' are orthogonal.*

Proof. We consider $\mathcal{Z}_{2,3}$ as the C^*-algebra of continuous functions from $[0,1]$ to M_6 with the boundary conditions

$$f(0) = \begin{pmatrix} Y & & \\ & Y & \\ & & Y \end{pmatrix} \quad f(1) = \begin{pmatrix} Z & \\ & QZQ^* \end{pmatrix},$$

where $Y \in M_2$ and $Z \in M_3$ are arbitrary matrices, and $Q \in M_3$ is the following fixed permutation matrix:

$$Q = \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}.$$

This means that $f(0), f(1) \in M_6$ have the following form:

$$
f(0) = \begin{pmatrix} \mu_{11} & \mu_{12} & & & & \\ \mu_{21} & \mu_{22} & & & & \\ & & \mu_{11} & \mu_{12} & & \\ & & \mu_{21} & \mu_{22} & & \\ & & & & \mu_{11} & \mu_{12} \\ & & & & \mu_{21} & \mu_{22} \end{pmatrix} \quad f(1) = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} & & & \\ \lambda_{21} & \lambda_{22} & \lambda_{23} & & & \\ \lambda_{31} & \lambda_{22} & \lambda_{33} & & & \\ & & & \lambda_{33} & \lambda_{31} & \lambda_{32} \\ & & & \lambda_{13} & \lambda_{11} & \lambda_{12} \\ & & & \lambda_{23} & \lambda_{21} & \lambda_{22} \end{pmatrix},
$$

for numbers $\mu_{i,j}, \lambda_{i,j} \in \mathbb{C}$.

Note that $\mathcal{Z}_{2,3}$ is naturally a continuous $C([0,1])$-algebra, with fibers $\mathcal{Z}_{2,3}(0) \cong M_2$, $\mathcal{Z}_{2,3}(1) \cong M_3$, and $\mathcal{Z}_{2,3}(t) \cong M_6$ for points $t \in (0,1) \subset [0,1]$.

Let $a, b, c \in A$ be a set of invertible, positive generators for A. Denote by $e_{i,j}$ the matrix units in M_6. To shorten notation, for indices i, j set $f_{i,j} := e_{i,j} + e_{j,i}$. For $t \in [0,1]$ we define the following element of $A \otimes M_6$:

$$
\begin{aligned}
x_t := & a \otimes (e_{1,1} + (1-t) \cdot e_{3,3} + e_{5,5}) \\
& + b \otimes (f_{1,2} + (1-t) \cdot f_{3,4} + f_{5,6}) \\
& + c \otimes (e_{2,2} + (1-t) \cdot e_{4.4} + e_{6,6}) \\
& + 1_A \otimes (t \cdot f_{2,3} + t \cdot f_{4,5} + \delta(t) \cdot f_{1,3})
\end{aligned}
$$

where $\delta\colon [0,1] \to [0,1]$ is a continuous function on $[0,1]$ that takes the value 0 at the endpoints 0 and 1, and is strictly positive at each point $t \in (0,1)$, e.g., δ could be given by $\delta(t) = 1/4 - (t - 1/2)^2$. We also define for $t \in [0,1]$ two elements of M_6:

$$
\begin{aligned}
y'_t := & e_{1,1} + (1-t) \cdot e_{3,3} + e_{5,5} \\
z'_t := & e_{2,2} + (1-t) \cdot e_{4,4} + e_{6,6}
\end{aligned}
$$

It is easy to check that the assignment $x\colon t \mapsto x_t$ defines an element $x \in A \otimes \mathcal{Z}_{2.3}$. Similarly, we get two elements $y', z' \in \mathcal{Z}_{2.3}$ defined via $t \mapsto y'_t$ and $t \mapsto z'_t$. In matrix form, these elements look as follows:

$$
x_t := \left(\begin{array}{cc|cc|cc} a & b & \delta(t) & & & \\ b & c & t & & & \\ \hline \delta(t) & t & (1-t)a & (1-t)b & & \\ & & (1-t)b & (1-t)c & t & \\ \hline & & & t & a & b \\ & & & & b & c \end{array}\right)
$$

$$
y'_t := \left(\begin{array}{cc|cc|cc} 1 & & & & & \\ & & & & & \\ \hline & & (1-t) & & & \\ & & & & & \\ \hline & & & & 1 & \\ & & & & & \end{array}\right) \quad z'_t := \left(\begin{array}{cc|cc|cc} & & & & & \\ & 1 & & & & \\ \hline & & & & & \\ & & & (1-t) & & \\ \hline & & & & & \\ & & & & & 1 \end{array}\right)
$$

Set $y := 1 \otimes y'$, and let $D := C^*(x+1, y)$ be the sub-C^*-algebra of $E := A \otimes \mathcal{Z}_{2,3}$ generated by the two self-adjoint elements $x + 1$ and y. Since $x \geq 0$, we get that both 1 and x lie in $C^*(x+1)$. It follows that $D = C^*(1, x, y)$, and we will show that $D = E$. Note that E has a natural continuous $C([0,1])$-algebra structure (induced by the one of $\mathcal{Z}_{2,3}$), with fibers $E(0) \cong A \otimes M_2$, $E(1) \cong A \otimes M_3$, and $E(t) \cong A \otimes M_6$ for points $t \in (0,1) \subset [0,1]$.

Let $J := E((0,1)) \lhd E$ be the natural ideal corresponding to the open set $(0,1) \subset [0,1]$. Note that $J \cong A \otimes C_0((0,1)) \otimes M_6$, and J is naturally a continuous $C_0((0,1))$-algebra. We will show in two steps that D exhausts the ideal J (i.e., $D \cap J = J$) and the quotient E/J (i.e., $D/(D \cap J) = E/J$).

Step 1: We want to apply Lemma 3.2 to the $C((0,1))$-algebra J with sub-C^*-algebra $D \cap J$. To verify condition (ii), note that the C^*-algebra generated by y' contains $C_0((0,1)) \otimes e_{3,3}$. Therefore, $D \cap J$ contains $1_A \otimes C_0((0,1)) \otimes e_{3,3}$, which separates the points of $(0,1)$. Since $1_A \otimes e_{3,3} \in E(t) \cong A \otimes M_6$ is full, condition (ii) of Lemma 3.2 holds and it remains to verify condition (i).

We need to show that $D \cap J$ exhausts all fibers of J. Fix some $t \in (0,1)$, and set $D_t := C^*(1, x_t, y_t) \subset A \otimes M_6$. To simplify notation, we write $\bar{e}_{i,j}$ for the matrix units $1_A \otimes e_{i,j} \in A \otimes M_6$. We need to show that D_t is all of $A \otimes M_6$. This will follow if D_t contains all $\bar{e}_{i,j}$, and for this it is enough to show that the off-diagonal matrix units $\bar{e}_{i,i+1}$ are in D_t, for $i = 1, \ldots, 5$.

The spectrum of y_t is $\{0, 1-t, 1\}$. Applying functional calculus to y_t we obtain that the following three elements lie in D_t:

$$\begin{aligned} u &:= \bar{e}_{1,1} + \bar{e}_{5,5} \\ v &:= \bar{e}_{3,3} \\ w &:= 1 - v - u = \bar{e}_{2,2} + \bar{e}_{4,4} + \bar{e}_{6,6} \end{aligned}$$

Then, we proceed as follows:

1. $\bar{e}_{1,3} = \delta(t)^{-1} u x_t v \in D_t$ and so $\bar{e}_{1,1}, \bar{e}_{5,5} \in D_t$.
2. $g := b \otimes e_{1,2} = \bar{e}_{1,1} x_t w \in D_t$. It follows $b \otimes e_{1,1} = (gg^*)^{1/2} \in D_t$, cf. [OZ76]. Then $b^{-1} \otimes e_{1,1} \in C^*(b \otimes e_{1,1}) \subset D_t$ and so $\bar{e}_{1,2} = (b^{-1} \otimes e_{1,1}) \cdot g \in D_t$ and $\bar{e}_{2,2} \in D_t$.
3. $b \otimes e_{3,4} = (1-t)^{-1} \bar{e}_{3,3} x_t (w - \bar{e}_{2,2}) \in D_t$. Arguing as above, it follows that $\bar{e}_{3,4} \in D_t$, and then $\bar{e}_{4,4}, \bar{e}_{6,6} \in D_t$.
4. $\bar{e}_{2,3} = t^{-1} \bar{e}_{2,2} x_t \bar{e}_{3,3} \in D_t$.
5. $\bar{e}_{4,5} = t^{-1} \bar{e}_{4,4} x_t \bar{e}_{5,5} \in D_t$.
6. $b \otimes e_{5,6} = \bar{e}_{5,5} x_t \bar{e}_{6,6} \in D_t$ and so $\bar{e}_{5,6} \in D_t$.

This shows that $D \cap J$ exhausts the fibers of J. We may apply Lemma 3.2 and deduce $D \cap J = J$, which finishes step 1.

Step 2: We want to show that D/J exhausts $E/J = E(\{0,1\}) \cong A \otimes (M_2 \oplus M_3)$. Let us denote the matrix units in M_2 by $e^{(0)}_{i,j}$, $i = 1, 2$, and the matrix units in M_3 by $e^{(1)}_{i,j}$, $i = 1, 2, 3$. To simplify notation, we write $\bar{e}^{(k)}_{i,j}$ for the matrix units $1_A \otimes e^{(k)}_{i,j} \in A \otimes (M_2 \oplus M_3)$. Let us denote the image of x and y in D/J by v and w:

$$\begin{aligned} v &= a \otimes (e^{(0)}_{1,1} + e^{(1)}_{1,1}) + b \otimes (e^{(0)}_{1,2} + e^{(0)}_{2,1} + e^{(1)}_{1,2} + e^{(1)}_{2,1}) + c \otimes (e^{(0)}_{2,2} + e^{(1)}_{2,2}) + \bar{e}^{(1)}_{2,3} + \bar{e}^{(1)}_{3,2} \\ &= \begin{pmatrix} a & b \\ b & c \end{pmatrix} \oplus \begin{pmatrix} a & b & \\ b & c & 1 \\ & 1 & \end{pmatrix} \end{aligned}$$

$$w = \bar{e}^{(0)}_{1,1} + \bar{e}^{(1)}_{1,1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}.$$

As in step 1, it is enough to show that D/J contains the off-diagonal matrix units $\bar{e}^{(0)}_{1,2}$, $\bar{e}^{(1)}_{1,2}$ and $\bar{e}^{(1)}_{2,3}$. We argue as follows:

1. $g := wv(1-w) = b \otimes (e^{(0)}_{1,2} + e^{(1)}_{1,2}) \in D/J$. As in step 1, it follows that $b \otimes (e^{(0)}_{1,1} + e^{(1)}_{1,1}) = (gg^*)^{1/2} \in D/J$. Then $b^{-1} \otimes (e^{(0)}_{1,1} + e^{(1)}_{1,1}) \in D/J$, and so $\bar{e}^{(0)}_{1,2} + \bar{e}^{(1)}_{1,2} = (b^{-1} \otimes (e^{(0)}_{1,1} + e^{(1)}_{1,1})) \cdot g \in D/J$. It follows that $\bar{e}^{(0)}_{2,2} + \bar{e}^{(1)}_{2,2} \in D/J$.
2. $\bar{e}^{(1)}_{3,3} = 1 - w - (\bar{e}^{(0)}_{2,2} + \bar{e}^{(1)}_{2,2}) \in D/J$.
3. $\bar{e}^{(1)}_{2,3} = v\bar{e}^{(1)}_{3,3} \in D/J$, and so $\bar{e}^{(1)}_{2,2} \in D/J$.
4. $b \otimes e^{(1)}_{1,2} = wv\bar{e}^{(1)}_{2,2} \in D/J$. Again, this implies $\bar{e}^{(1)}_{1,2} \in D/J$ and so $\bar{e}^{(1)}_{1,1} \in D/J$.
5. $\bar{e}^{(0)}_{1,1} = w - \bar{e}^{(1)}_{1,1} \in D/J$.
6. $\bar{e}^{(0)}_{2,2} = 1 - w - \bar{e}^{(1)}_{2,2} - \bar{e}^{(1)}_{3,3} \in D/J$.
7. $b \otimes e^{(0)}_{1,2} = \bar{e}^{(0)}_{1,1} v \bar{e}^{(0)}_{2,2} \in D/J$. Again, this implies $\bar{e}^{(0)}_{1,2} \in D/J$.

This finishes step 2.

We have seen that $A \otimes \mathcal{Z}_{2,3}$ is generated by $x+1$ and y. Moreover, z' is full, positive and orthogonal to y'. □

Lemma 3.4. *Let A be a separable, unital C^*-algebra. Then there exist a positive element $x \in A \otimes \mathcal{Z}_{2^\infty,3^\infty}$ and two positive, full elements $y', z' \in \mathcal{Z}_{2^\infty,3^\infty}$ such that $A \otimes \mathcal{Z}_{2^\infty,3^\infty}$ is generated by x and $y := 1 \otimes y'$, and further y' and z' are orthogonal.*

Proof. Let $B := A \otimes \mathcal{Z}_{2^\infty,3^\infty}$. Note that $\mathcal{Z}_{2^\infty,3^\infty} \otimes \mathcal{Z}_{2,3}$ is naturally a $C([0,1] \times [0,1])$-algebra. Then, the quotient corresponding to the diagonal $\{(t,t) \mid t \in [0,1]\} \subset [0,1] \times [0,1]$ is isomorphic to $\mathcal{Z}_{2^\infty,3^\infty}$, and we denote the resulting surjective morphism by $\pi\colon \mathcal{Z}_{2^\infty,3^\infty} \otimes \mathcal{Z}_{2,3} \to \mathcal{Z}_{2^\infty,3^\infty}$. We proceed in two steps.

Step 1: We show that $\mathrm{gen}(B) \leq k+1$ implies $\mathrm{gen}(B) \leq k$ for $k \geq 2$. So assume B is generated by the self-adjoint, invertible elements $a_1, \ldots, a_{k+1}$. The sub-C^*-algebra $C := C^*(a_{k-1}, a_k, a_{k+1}) \subset B$ is unital and satisfies $\mathrm{gen}(C) \leq 3$. Consider the C^*-algebra $B \otimes \mathcal{Z}_{2,3}$. By Lemma 3.3, the sub-$C^*$-algebra $C \otimes \mathcal{Z}_{2,3}$ is generated by two self-adjoint elements, say b, c.

One readily checks that $B \otimes \mathcal{Z}_{2,3}$ is generated by the k self-adjoint elements $a_1 \otimes 1, \ldots, a_{k-2} \otimes 1, b, c$. Since $B = A \otimes \mathcal{Z}_{2^\infty,3^\infty}$ is isomorphic to a quotient of $B \otimes \mathcal{Z}_{2,3} = A \otimes \mathcal{Z}_{2^\infty,3^\infty} \otimes \mathcal{Z}_{2,3}$, we obtain $\mathrm{gen}(B) \leq \mathrm{gen}(B \otimes \mathcal{Z}_{2,3}) \leq k$.

Step 2: By Lemma 3.1, we have $\mathrm{gen}(B) \leq 5$. Applying Step 1 several times, we obtain $\mathrm{gen}(B) \leq 3$.

It follows from Lemma 3.3 that there exists a positive element $\tilde{x} \in B \otimes \mathcal{Z}_{2,3}$ and two positive, full elements $\tilde{y}', \tilde{z}' \in \mathcal{Z}_{2,3}$ such that $B \otimes \mathcal{Z}_{2,3}$ is generated by $\tilde{x}$ and $1 \otimes \tilde{y}'$, and further $\tilde{y}'$ and $\tilde{z}'$ are orthogonal.

Consider the surjective morphism $\mathrm{id} \otimes \pi\colon A \otimes \mathcal{Z}_{2^\infty,3^\infty} \otimes \mathcal{Z}_{2,3} \to A \otimes \mathcal{Z}_{2^\infty,3^\infty}$. One checks that the elements $x := (\mathrm{id} \otimes \pi)(\tilde{x}) \in A \otimes \mathcal{Z}_{2^\infty,3^\infty}$, and $y' := \pi(\tilde{y}')$, $z' := \pi(\tilde{z}') \in \mathcal{Z}_{2^\infty,3^\infty}$ have the desired properties. □

Theorem 3.5. *Let A, B be two separable, unital C^*-algebras. Assume the following:*

(1) A contains a sequence $a_1, a_2, \ldots$ of full, positive elements that are pairwise orthogonal,
(2) B admits a unital embedding of the Jiang-Su algebra $\mathcal{Z}$.

Then $A \otimes_{max} B$ is singly generated. Every other tensor product $A \otimes_\lambda B$ is a quotient of $A \otimes_{max} B$, and therefore is also singly generated.

Proof. There exists a unital embedding of $\mathcal{Z}_{2^\infty,3^\infty}$ in $\mathcal{Z}$, so we may assume that there is a unital embedding of $\mathcal{Z}_{2^\infty,3^\infty}$ in B. We may assume that the elements $a_1, a_2, \ldots \in A$ are contractive.

Choose a sequence $b_1, b_2, \ldots \in B$ of contractive, positive elements that is dense in the set of all contractive, positive elements of B.

Consider the sub-C^*-algebra $A \otimes \mathcal{Z}_{2^\infty,3^\infty} \subset A \otimes_{\max} B$. By Lemma 3.4, there exist a positive element $x \in A \otimes \mathcal{Z}_{2^\infty,3^\infty}$ and two full, positive elements $y', z' \in \mathcal{Z}_{2^\infty,3^\infty}$ such that $A \otimes \mathcal{Z}_{2^\infty,3^\infty}$ is generated by x and $y := 1 \otimes y'$, and further y' and z' are orthogonal.

Define the following two elements of $A \otimes_{\max} B$:

$$v := x, \qquad w := 1 \otimes y' - \sum_{k \geq 1} 1/2^k \cdot a_k \otimes (z' b_k z').$$

Let $D := C^*(v, w)$ be the sub-C^*-algebra of $A \otimes_{\max} B$ generated by v and w. We claim that $D = A \otimes B$.

Step 1: We show $A \otimes \mathcal{Z}_{2^\infty,3^\infty} \subset D$. Note that the two elements $1 \otimes y'$ and $\sum_{k\geq 1} 1/2^k \cdot a_k \otimes (z' b_k z')$ are positive and orthogonal. It follows that $1 \otimes y'$ is the positive part of w, and therefore $1 \otimes y' \in D$. Therefore, $C^*(v, 1 \otimes y') = A \otimes \mathcal{Z}_{2^\infty,3^\infty} \subset D$.

Step 2: We show $1 \otimes B \subset D$. We have $g := \sum_{k\geq 1} 1/2^k \cdot a_k \otimes (z' b_k z') \in D$. It follows from Step 1 that $a_k \otimes 1 \in D$, and so $a_k^2 \otimes (z' b_k z') = 2^k \cdot (a_k \otimes 1) g \in D$. Since a_k^2 is full, there exist finitely many elements $c_i, d_i \in A$ such that $1_A = \sum_i c_i a_k^2 d_i$. By Step 1, we have $c_i \otimes 1, d_i \otimes 1 \in D$. Then $1 \otimes (z' b_k z') = \sum_i (c_i \otimes 1)(a_k^2 \otimes (z' b_k z'))(d_i \otimes 1) \in D$, for each k.

Let $b \in B$ be a contractive, positive element. Then $b = \lim_j b_{k(j)}$ for certain indices $k(j)$. Then $1 \otimes (z' b z') = \lim_j 1 \otimes (z' b_{k(j)} z') \in D$. It follows that the hereditary sub-C^*-algebra $1 \otimes \overline{z' B z'}$ is contained in D. Since z' is full in $\mathcal{Z}_{2^\infty,3^\infty}$, there exist finitely many elements $c_i, d_i \in \mathcal{Z}_{2^\infty,3^\infty}$ such that $1_B = \sum_i c_i z' d_i$. We have seen that $1 \otimes z' b z' \in D$ for any $b \in B$. Then $1 \otimes b z' = \sum_i (1 \otimes c_i)(1 \otimes z' d_i b z') \in D$ for any $b \in B$. Similarly $1 \otimes b = \sum_i (1 \otimes b c_i z')(1 \otimes d_i) \in D$ for any $b \in B$, as desired.

It follows from Steps 1 and 2 that for each $a \in A$ and $b \in B$ the simple tensor $a \otimes b$ is contained in D. The conclusion follows since $A \otimes_{\max} B$ is the closure of the linear span of simple tensors. □

Corollary 3.6. *Let A, B be two separable, unital C^*-algebras that both admit a unital embedding of the Jiang-Su algebra $\mathcal{Z}$. Then $A \otimes_{max} B$ is singly generated.*

Proof. It is easy to verify that condition (i) of Theorem 3.5 is fulfilled if A admits a unital embedding of $\mathcal{Z}$. □

Theorem 3.7. *Let A be a unital, separable C^*-algebra. Then $A \otimes \mathcal{Z}$ is singly generated.*

Proof. Note that $A \otimes \mathcal{Z} \cong (A \otimes \mathcal{Z}) \otimes \mathcal{Z}$. It is clear that both $A \otimes \mathcal{Z}$ and $\mathcal{Z}$ admit unital embeddings of $\mathcal{Z}$. Then apply the above Corollary 3.6. □

Corollary 3.8. *Let A be a separable C^*-algebra. Then* $\mathrm{gen}(A \otimes \mathcal{Z}) \leq 3$.

Proof. Let $\widetilde{A}$ be the minimal unitization of A. It follows from Theorem 3.7 that $\mathrm{gen}(\widetilde{A} \otimes \mathcal{Z}) \leq 2$. Since $A \otimes \mathcal{Z}$ is an ideal in $\widetilde{A} \otimes \mathcal{Z}$, we get $\mathrm{gen}(A \otimes \mathcal{Z}) \leq \mathrm{gen}(\widetilde{A} \otimes \mathcal{Z}) + 1 \leq 3$ from Proposition 2.2, as desired. □

Our results allow us to give new proofs for results about single generation of certain von Neumann algebras.

Proposition 3.9. *Assume M, N are separably-acting von Neumann algebras that both admit a unital embedding of the hyperfinite II_1-factor. Then $M \bar{\otimes} N$ is singly generated.*

Proof. Consider the GNS-representation $\pi\colon \mathcal{Z} \to B(H)$ of the Jiang-Su algebra with respect to its tracial state. The weak closure, $\pi(\mathcal{Z})''$, is isomorphic to the hyperfinite II_1-factor $\mathcal{R}$. Thus, there exists a weakly dense, unital copy of $\mathcal{Z}$ inside $\mathcal{R}$.

Choose weakly dense, separable, unital C^*-algebras $A_0 \subset M$, and similarly $B_0 \subset N$. Consider $\mathcal{Z} \subset \mathcal{R} \subset M$ and set $A := C^*(A_0, \mathcal{Z}) \subset M$. Similarly set $B := C^*(B_0, \mathcal{Z}) \subset N$.

Then A and B are separable, unital C^*-algebras that both contain unital copies of the Jiang-Su algebra. By Corollary 3.6, $A \otimes_{\max} B$ is singly generated.

Consider the sub-C^*-algebra $C := C^*(A \bar{\otimes} 1, 1 \bar{\otimes} B) \subset M \bar{\otimes} N$. Then C is a quotient of $A \otimes_{\max} B$, and therefore singly generated. Since C is weakly dense in $M \bar{\otimes} N$, we obtain that $M \bar{\otimes} N$ is singly generated, as desired. □

Remark 3.10. We note that a von Neumann algebra M admits a unital embedding of $\mathcal{R}$ if and only if M has no (non-zero) finite-dimensional representations.

The analogous statement for C^*-algebras would be that a C^*-algebra A admits a unital embedding of $\mathcal{Z}$ if and only if A has no (non-zero) finite-dimensional representations. It was shown by Elliott and Rørdam, [ER06], that this is true for C^*-algebras of real rank zero. However, in [DHTW09] a simple, separable, unital, non-elementary AH-algebra is constructed into which $\mathcal{Z}$ does not embed.

As a particular case of Proposition 3.9 we obtain the following result of Ge and Popa.

Corollary 3.11 (Ge, Popa, [GP98, Theorem 6.2]). *Assume M, N are separably-acting II_1-factors. Then $M \bar{\otimes} N$ is singly generated.*

4. Applications

In this section we show that the Jiang-Su algebra $\mathcal{Z}$ embeds unitally into the reduced group C^*-algebras, $C^*_r(\Gamma)$, of groups Γ that contain a non-cyclic free subgroup, see Proposition 4.2. We only consider discrete groups, and we let F_k denote the free group with k generators ($k \in \{2, 3 \ldots, \infty\}$).

We can apply Theorem 3.5 to show that certain tensor products of the form $A \otimes_{\max} C^*_r(\Gamma)$ are singly generated, see Corollary 4.4. In particular, $C^*_r(F_\infty) \otimes C^*_r(F_\infty)$ is singly generated, although it is not $\mathcal{Z}$-stable, see Example 4.5.

4.1. It was shown by Robert, [Rob10], that the Jiang-Su algebra $\mathcal{Z}$ embeds unitally into $C^*_r(F_\infty)$. A key observation is that $C^*_r(F_\infty)$ has strict comparison of positive elements. This follows from the work of Dykema and Rørdam on reduced free product C^*-algebras, see [DR98] and [DR00].

Dykema and Rørdam study the comparison of projections, but this can be generalized to obtain results about the comparison of positive elements, as noted by Robert, [Rob10]. In particular, [DR98, Lemma 5.3] and [DR00, Theorem 2.1] can be generalized, and it follows that $C^*_r(F_\infty)$ has strict comparison of positive elements.

Proposition 4.2. *If Γ is a discrete group that contains F_∞ as a subgroup, then $\mathcal{Z}$ embeds unitally into $C^*_r(\Gamma)$.*

Proof. In general, for any subgroup Γ_1 of a discrete group Γ, we have a unital embedding $C^*_r(\Gamma_1) \subset C^*_r(\Gamma)$. Hence, if F_∞ is a subgroup of Γ, then $C^*_r(\Gamma)$ contains a unital copy of $C^*_r(F_\infty)$, which in turn contains a unital copy of $\mathcal{Z}$. □

Remark 4.3. Every non-cyclic free group F_k ($k \geq 2$) contains F_∞ as a subgroup. In general, by the Nielsen-Schreier theorem, every subgroup of a free group is again free. Thus, if a, b are free elements, then the the elements $a^k b^k$ generate a subgroup $\Gamma = \langle a^k b^k, k \geq 1\rangle$ that is free, and since none of the elements $a^k b^k$ is contained in the subgroup generated by the other elements, we have $\Gamma \cong F_\infty$.

Thus, when we ask which discrete groups contain F_∞ as a subgroup, we are equivalently asking which groups Γ contain a non-cyclic free subgroup. It is a necessary condition that Γ is non-amenable. The converse implication is known as the von Neumann conjecture, but this was disproved in 1980 by Ol'shanskij.

A counterexample are the so-called Tarski monster groups, in which every non-trivial proper subgroup is cyclic of some fixed prime order. Clearly, such a group cannot contain F_∞ as a subgroup, and it is Ol'shanskij's contribution to show that Tarski monster groups exist and are non-amenable.

On the other hand, every group with the weak Powers property, as defined in [BN88], has a non-cyclic free subgroup. A proof can be found in [dlH07], which also lists classes of groups that have the (weak) Powers property. We just mention that all free products $\Gamma_1 * \Gamma_2$ with $|\Gamma_1| \geq 2, |\Gamma_2| \geq 3$ have the Powers property, and therefore Proposition 4.2 applies.

We may derive the following from Theorem 3.5 and Proposition 4.2:

Corollary 4.4. *Let A be a separable, unital C^*-algebra that contains a countable sequence of pairwise orthogonal, full elements (e.g., A is simple and nonelementary), and let Γ be a group that contains a non-cyclic free subgroup. Then $A \otimes_{max} C^*_r(\Gamma)$ is singly generated.*

Example 4.5. Let Γ_1, Γ_2 be two groups that contain non-cyclic free subgroups. Then $C^*_r(\Gamma_1 \times \Gamma_2) \cong C^*_r(\Gamma_1) \otimes_{\max} C^*_r(\Gamma_2)$ is singly generated. For example, for any $k, l \in \{2, 3, \ldots, \infty\}$, the C^*-algebra $C^*_r(F_k) \otimes_{\max} C^*_r(F_l)$ is singly generated. In particular, $C^*_r(F_\infty) \otimes_{\max} C^*_r(F_\infty)$ is singly generated.

It was pointed out to the authors by S. Wassermann that $C^*_r(F_k) \otimes C^*_r(F_l)$ is not $\mathcal{Z}$-stable, for any $k, l \in \{2, 3, \ldots, \infty\}$. In fact, if $C^*_r(F_k) \otimes C^*_r(F_l) \cong A \otimes B \otimes C$, then one of the three algebras A, B or C is isomorphic to $\mathbb{C}$. This is a generalization of the fact that $C^*_r(F_k)$ is tensorially prime, and it can be proved similarly.

We note that it is a difficult open problem whether $C^*_r(F_k)$ is singly generated itself.

Question 4.6. Given a non-amenable (discrete) group Γ. Does $C^*_r(\Gamma)$ admit a unital embedding of $\mathcal{Z}$?

For each group Γ, the trivial group-morphism $\Gamma \to \{1\}$ induces a surjective morphism $C^*(\Gamma) \to \mathbb{C}$. Thus, the Jiang-Su algebra can never unitally embed into a full group C^*-algebra. If Γ is amenable, then $C^*_r(\Gamma) \cong C^*(\Gamma)$, and consequently there is no unital embedding of $\mathcal{Z}$ into the reduced group C^*-algebra of an amenable group.

On the other hand, if Γ contains a non-cyclic free subgroup, then Proposition 4.2 gives a positive answer to Question 4.6. As noted in Remark 4.3, not every non-amenable group contains a non-cyclic free subgroup. However, it is known that the reduced group C^*-algebra of a non-amenable group has no finite-dimensional representations, which is a necessary condition for the Jiang-Su algebra to embed.

ACKNOWLEDGMENTS

The first named author thanks Mikael Rørdam for valuable comments, especially on the applications in Section 4.

REFERENCES

[AP77] C.A. Akemann and G.K. Pedersen, *Ideal perturbations of elements in C^*-algebras*, Math. Scand. **41** (1977), 117–139.

[Arv77] W. Arveson, *Notes on extensions of C^*-algebras*, Duke Math. J. **44** (1977), 329–355.

[BN88] F. Boca and V. Niţică, *Combinatorial properties of groups and simple C^*-algebras with a unique trace*, J. Oper. Theory **20** (1988), no. 1, 183–196.

[Dad09] M. Dadarlat, *Continuous fields of C^*-algebras over finite dimensional spaces*, Adv. Math. **222** (2009), no. 5, 1850–1881.

[DHTW09] M. Dadarlat, I. Hirshberg, A.S. Toms, and W. Winter, *The Jiang-Su algebra does not always embed*, Math. Res. Lett. **16** (2009), no. 1, 23–26.

[dlH07] P. de la Harpe, *On simplicity of reduced C^*-algebras of groups*, Bull. Lond. Math. Soc. **39** (2007), no. 1, 1–26.

[DR98] K.J. Dykema and M. Rørdam, *Projections in free product C^*-algebras*, Geom. Funct. Anal. **8** (1998), no. 1, 1–16.

[DR00] ———, *Projections in free product C^*-algebras. II*, Math. Z. **234** (2000), no. 1, 103–113.

[ER06] G.A. Elliott and M. Rørdam, *Perturbation of Hausdorff moment sequences, and an application to the theory of C^*-algebras of real rank zero*, Bratteli, Ola (ed.) et al., Operator algebras. The Abel symposium 2004. Proceedings of the first Abel symposium, Oslo, Norway, September 3–5, 2004. Berlin: Springer. Abel Symposia 1, 97-115, 2006.

[ET08] G.A. Elliott and A.S. Toms, *Regularity properties in the classification program for separable amenable C^*-algebras*, Bull. Am. Math. Soc., New Ser. **45** (2008), no. 2, 229–245.

[Ge03] L.M. Ge, *On 'Problems on von Neumann algebras by R. Kadison, 1967'*, Acta Math. Sin., Engl. Ser. **19** (2003), no. 3, 619–624.

[GP98] L. Ge and S. Popa, *On some decomposition properties for factors of type II_1*, Duke Math. J. **94** (1998), no. 1, 79–101.

[JS99] X. Jiang and H. Su, *On a simple unital projectionless C^*-algebra*, Am. J. Math. **121** (1999), no. 2, 359–413.

[JW11] M. Johanesova and W. Winter, *The similarity problem for $\mathcal{Z}$-stable C^*-algebras*, preprint, arXiv:1104.2067, 2011.

[Kad67] R. Kadison, *Problems on von Neumann algebras*, unpublished manuscript, presented at Conference on Operator Algebras and Their Applications, Louisiana State Univ., Baton Rouge, La., 1967.

[Kas88] G.G. Kasparov, *Equivariant KK-theory and the Novikov conjecture*, Invent. Math. **91** (1988), no. 1, 147–201.

[Lon84] R. Longo, *Solution of the factorial Stone-Weierstrass conjecture. An application of the theory of standard split W^*-inclusions*, Invent. Math. **76** (1984), 145–155.

[LS10] W. Li and J. Shen, *A note on approximately divisible C^*-algebras*, preprint, arXiv:0804.0465, 2010.

[Nag04] M. Nagisa, *Single generation and rank of C^*-algebras*, Kosaki, Hideki (ed.), Operator algebras and applications. Proceedings of the US-Japan seminar held at Kyushu University, Fukuoka, Japan, June 7–11, 1999. Tokyo: Mathematical Society of Japan. Advanced Studies in Pure Mathematics 38, 135-143, 2004.

[OZ76] C.L. Olsen and W.R. Zame, *Some C^*-algebras with a single generator*, Trans. Am. Math. Soc. **215** (1976), 205–217.

[Pea62] C. Pearcy, *W^*-algebras with a single generator*, Proc. Am. Math. Soc. **13** (1962), 831–832.

[Pop84] S. Popa, *Semiregular maximal Abelian $*$-subalgebras and the solution to the factor state Stone-Weierstrass problem*, Invent. Math. **76** (1984), 157–161.

[Rob10] L. Robert, *Classification of inductive limits of 1-dimensional NCCW complexes*, preprint, arXiv:1007.1964, 2010.

[Rør04] M. Rørdam, *The stable and the real rank of $\mathcal{Z}$-absorbing C^*-algebras*, Int. J. Math. **15** (2004), no. 10, 1065–1084.

[RW10] M. Rørdam and W. Winter, *The Jiang-Su algebra revisited*, J. Reine Angew. Math. **642** (2010), 129–155.

[She09] D. Sherman, *On cardinal invariants and generators for von Neumann algebras*, preprint, arXiv:0908.4565, 2009.

[Top68] D.M. Topping, *UHF algebras are singly generated*, Math. Scand. **22** (1968), 224–226.

[TW08] A.S. Toms and W. Winter, *$\mathcal{Z}$-stable ASH algebras*, Can. J. Math. **60** (2008), no. 3, 703–720.

[vN31] J. von Neumann, *Über Funktionen von Funktionaloperatoren*, Ann. of Math. (2) **32** (1931), no. 2, 191–226.

[Wil74] P. Willig, *Generators and direct integral decompositions of W^*-algebras*, Tohoku Math. J., II. Ser. **26** (1974), 35–37.

[Win07] W. Winter, *Localizing the Elliott conjecture at strongly self-absorbing C^*-algebras*, preprint, arXiv:0708.0283, 2007.

[Win10] ———, *Nuclear dimension and $\mathcal{Z}$-stability of pure C^*-algebras*, preprint, arXiv:1006.2731, to appear in Inventiones, 2010.

[Win11] ———, *Strongly self-absorbing C^*-algebras are $\mathcal{Z}$-stable*, J. Noncommut. Geom. **5** (2011), no. 2, 253–264.

[Wog69] W.R. Wogen, *On generators for von Neumann algebras*, Bull. Am. Math. Soc. **75** (1969), 95–99.

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF COPENHAGEN, UNIVERSITETSPARKEN 5, DK-2100, COPENHAGEN Ø, DENMARK

E-mail address: thiel@math.ku.dk

MATHEMATISCHES INSTITUT DER UNIVERSITÄT MÜNSTER, EINSTEINSTR. 62, 48149 MÜNSTER, GERMANY

E-mail address: wwinter@uni-muenster.de

Proc. London Math. Soc. Page 1 of 26
doi:10.1112/plms/pdr051

A characterization of semiprojectivity for commutative C^*-algebras

Adam P. W. Sørensen and Hannes Thiel

Abstract

Given a compact metric space X, we show that the commutative C^*-algebra $C(X)$ is semiprojective if and only if X is an absolute neighbourhood retract of dimension at most 1. This confirms a conjecture of Blackadar.

Generalizing to the non-unital setting, we derive a characterization of semiprojectivity for separable, commutative C^*-algebras. As applications of our results, we prove two theorems about the structure of semiprojective commutative C^*-algebras. Letting A be a commutative C^*-algebra, we show firstly: If I is an ideal of A and A/I is finite-dimensional, then A is semiprojective if and only if I is; and secondly: A is semiprojective if and only if $M_2(A)$ is. This answers two questions about semiprojective C^*-algebras in the commutative case.

1. *Introduction*

A semiprojective C^*-algebra is the non-commutative analogue of an absolute neighbourhood retract (ANR). Indeed if X is a compact metrizable space, then $C(X)$ is semiprojective in the category of commutative C^*-algebras if and only if X is an ANR. The concept of semiprojectivity was first introduced by Effros and Kaminker [**13**]. They wanted to study shape theory for C^*-algebras. Soon after both non-commutative shape theory and semiprojectivity were developed to their modern forms by Blackadar [**3**]. When moving from the world of topology to the world of C^*-algebras, the Gelfand transform was used to 'reverse arrows'. Hence, where a topologist might think of an ANR as a space that behaves well with respect to embeddings out of it, a C^*-algebraist will think of a semiprojective C^*-algebra as one that behaves well with respect to surjections onto it.

Shape theory is a machinery that allows to focus on the global properties of a space by abstracting from its local behaviour. This is done by approximating the space by a system of nicer spaces, and then studying this approximating system instead of the original space. In the topological world, this is carried out by writing a space as an inverse limit of ANRs, and it is a classical result that every compact, metric space can be obtained this way. Dually, one would expect to be able to write every unital, separable C*- algebra as an inductive limit of semiprojective C*-algebras. It is, however, not yet known if this is always possible. Recently, some progress was made on this problem by Loring and Shulman [LS10] and the second named author [Thi11]. One of the main problems is that we do not have a very large supply of semiprojective C*-algebras.

Although semiprojectivity was modelled on ANRs, the first large class of C^*-algebras shown to be semiprojective were the highly non-commutative Cuntz–Krieger algebras; see [**3**]. Since then, these results have been extended to cover all Kirchberg algebras satisfying the universal

Received 1 February 2011; revised 5 August 2011.

2010 *Mathematics Subject Classification* 46L05, 54C55, 55M15, 54F50 (primary), 46L80, 46M10, 54F15, 54D35, 54C56, 55P55 (secondary).

This research was supported by the Danish National Research Foundation through the Centre for Symmetry and Deformation. The second author was partially supported by the Marie Curie Research Training Network EU-NCG.

coefficient theorem, with finitely generated (f.g.) K-theory and free K_1-group (see [25, 26]), and it is conjectured that in fact all Kirchberg algebras with f.g. K-theory are semiprojective.

Yet, the following natural question remained unanswered.

QUESTION 1.1. Which commutative C^*-algebras are semiprojective?

It has long been known that for a commutative C^*-algebra to be semiprojective, its spectrum must be an ANR. It has also been known for a long time that this does not suffice. Indeed both date back to Blackadar's original paper. As an example, the continuous functions on the two disc $C(D^2)$ is not a semiprojective C^*-algebra (see Propositions 3.2 and 3.3).

An important partial answer was obtained by Loring [18, Proposition 16.2.1, p. 125] who showed that all one-dimensional CW-complexes give rise to semiprojective C^*-algebras. In [15], this was extended to the class of one-dimensional non-commutative CW-complexes.

In another direction, Chigogidze and Dranishnikov recently gave a characterization of the commutative C^*-algebras that are projective: They showed in [11, Theorem 4.3] that $C(X)$ is projective in the category of unital, separable C^*-algebras with unital $*$-homomorphisms if and only if X is an absolute retract (AR) and $\dim(X) \leqslant 1$. Inspired by their results, we obtain the following answer to Question 1.1.

THEOREM 1.2. *Let X be a compact, metric space. The following are equivalent:*

(I) *$C(X)$ is semiprojective;*
(II) *X is an ANR and $\dim(X) \leqslant 1$.*

This confirms a conjecture of Blackadar [5, II.8.3.8, p. 163]. Along the way, we obtain some results about the structure of ANRs (see Remark 3.4 for the result about ANRs of dimension at least 2 and Theorem 4.17 for a result about ANRs of dimension 1). We proceed as follows.

In Section 2, we recall the basic concepts of commutative and non-commutative shape theory, in particular the notion of an ANR and of semiprojectivity.

In Section 3, we show the implication '(I) $\Rightarrow$ (II)' of our main result Theorem 1.2. The idea is to use the topological properties of higher dimensional spaces, to show that if $C(X)$ was semiprojective and X an ANR of dimension at least 2, then we could solve a lifting problem known to be unsolvable. In particular, we obtain a characterization of ANRs of dimension at least 2.

In Section 4, we study the structure of compact, one-dimensional ANRs. This section is purely topological. We characterize when a one-dimensional Peano continuum X is an ANR; see Proposition 4.12. As it turns out, one criterion is that X contains a finite subgraph that contains all homotopy information, a (homotopy) core; see Proposition 4.10. This is also equivalent to $K^*(X)$ being f.g., which is a recurring property in connection with semiprojectivity.

The main result of this section is Theorem 4.17, which describes the internal structure of a compact, one-dimensional ANR X. Starting with the homotopy core $Y_1 \subset X$, there is an increasing sequence of subgraphs $Y_1 \subset Y_2 \subset \ldots \subset X$ that exhaust X, and such that Y_{k+1} is obtained from Y_k by simply attaching a line segment at one end to a point in Y_k. This generalizes the classical structure theorem for dendrites (which are precisely the *contractible*, compact, one-dimensional ANRs).

In Section 5, we show the implication '(II) $\Rightarrow$ (I)' of Theorem 1.2. Using the structure Theorem 4.17 for a one-dimensional ANR X, we obtain subgraphs $Y_k \subset X$ such that $X \cong \varprojlim Y_k$. The first graph Y_1 contains all K-theory information, and the subsequent graphs are obtained by attaching line segments. Dualizing, we can write $C(X)$ as an inductive limit $C(X) = \varinjlim C(Y_k)$, with bounding maps induced by retractions.

The main result of this section is Proposition 5.3. Using the structure of inductive limits, it shows that if $C(Y_1)$ is semiprojective, then $C(X)$ is semiprojective. The idea of solving lifting problems (that is, proving semiprojectivity) along inductive limits is central in [**11**], but it has also been used before, for instance, by Blackadar in order to prove that the Cuntz algebra $\mathcal{O}_\infty$ is semiprojective. We wish to point out that Chigogidze and Dranishnikov only needed semiprojectivity, and not projectivity, in many steps of their proofs.

The proof '(II) $\Rightarrow$ (I)' follows from Proposition 5.3 if we can find an initial lift from $C(Y_1)$. For this we use Loring's deep result [**18**] which says that $C(Y)$ is semiprojective for every finite graph Y. We also need Loring's result to write the algebras $C(Y_k)$ as universal C^*-algebras.

In Section 6, we give applications of our main result Theorem 1.2. First, we analyse the structure of non-compact, one-dimensional ANRs. We give a characterization of when the one-point compactification of such spaces is again an ANR; see Proposition 6.1. Using the characterization of semiprojectivity for unital, separable, commutative C^*-algebras given in Theorem 1.2, we derive a characterization of semiprojectivity for non-unital, separable, commutative C^*-algebras; see Proposition 6.2.

In Proposition 6.1, we note that the one-point compactification of the considered spaces is an ANR if and only if every finite-point compactification is an ANR. This allows us to study short exact sequences

$$0 \longrightarrow I \longrightarrow A \longrightarrow F \longrightarrow 0$$

with F finite-dimensional. In [**18**], Loring asked whether semiprojectivity of I implies semiprojectivity of A under the assumption that $F = \mathbb{C}$. Later Blackadar upgraded this to a conjecture (under the additional assumption that the extension is split), [**4**, Conjecture 4.5]. Recently, Enders (private communication) showed that semiprojectivity passes to ideals when the quotient is finite-dimensional. Blackadars conjecture remains open. (After submission of this paper, Eilers and Katsura (private communication) have shown by example that Blackadars' conjecture is false in general.)

However, in Proposition 6.3 we answer Loring's questions in the positive under the additional assumption that, A is commutative but with F being any finite-dimensional C^*-algebra.

We also study the semiprojectivity of C^*-algebras of the form $C_0(X, M_k)$. We derive in Corollary 6.9 that for a separable, commutative C^*-algebra A, the algebra $A \otimes M_k$ is semiprojective if and only if A is semiprojective. Whether or not this holds for general C^*-algebras is open. The question is related to a conjecture by Blackadar [**4**, Conjecture 4.4]. (Eilers and Katsura (private communication) also have a counterexample to this conjecture. They do not, at present, have an example of a non-semiprojective C^*-algebra A with $M_2(A)$ semiprojective.) It is known that semiprojectivity of A implies that $A \otimes M_k$ is semiprojective [**3**, Corollary 2.28; **18**, Theorem 14.2.2, p. 110].

As a final application, we consider the following variant of Question 1.1: When is a commutative C^*-algebra weakly (semi-)projective? In order to study this problem, we analyse the structure of one-dimensional approximative absolute (neighbourhood) retracts, abbreviated AA(N)R. In Proposition 6.15, we show that such spaces are approximated from within by finite trees (finite graphs).

Let $\mathcal{S}_1$ denote the category of separable unital C^*-algebras with unital morphisms. We can then summarize our results, Theorems 1.2 and 6.16, and the result of Chigogidze and Dranishnikov [**11**, Theorem 4.3] as the following theorem.

THEOREM 1.3. *Let X be a compact, metric space with* $\dim(X) \leqslant 1$. *Then:*

(1) *$C(X)$ is projective in $\mathcal{S}_1 \Leftrightarrow X$ is an AR;*
(2) *$C(X)$ is weakly projective in $\mathcal{S}_1 \Leftrightarrow X$ is an AAR;*
(3) *$C(X)$ is semiprojective in $\mathcal{S}_1 \Leftrightarrow X$ is an ANR;*

(4) *$C(X)$ is weakly semiprojective in $\mathcal{S}_1 \Leftrightarrow X$ is an AANR.*

Moreover, $C(X)$ projective or semiprojective already implies $\dim(X) \leqslant 1$.

2. *Preliminaries*

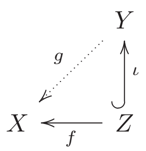
By A, B, C, D we mostly denote C^*-algebras, usually assumed to be separable here, and by a morphism between C^*-algebras we understand a $*$-homomorphism. By an ideal in a C^*-algebra, we mean a closed, two-sided ideal. If A is a C^*-algebra, then we denote by $\tilde{A}$ its minimal unitalization, and by A^+ the forced unitalization. Thus, if A is unital, then $\tilde{A} = A$ and $A^+ \cong A \oplus \mathbb{C}$. We use the symbol $\simeq$ to denote homotopy equivalence.

We write $\mathcal{S}$ for the category of separable C^*-algebras with all morphisms, and $\mathcal{S}_1$ for the category of unital separable C^*-algebras with all unital morphisms. We denote by $\mathcal{SC}$ the full subcategory of $\mathcal{S}$ consisting of (separable) commutative C^*-algebras. Similarly for $\mathcal{SC}_1$.

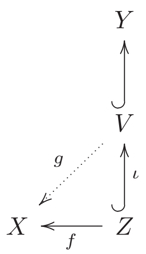
By a map between two topological spaces, we mean a continuous map. Given $\varepsilon > 0$ and subsets $F, G \subset X$ of a metric space, we say F is *ε-contained* in G, denoted by $F \subset_\varepsilon G$, if for every $x \in F$ there exists some $y \in G$ such that $d_X(x, y) < \varepsilon$. Given two maps $\varphi, \psi \colon X \to Y$ between metric spaces and a subset $F \subset X$, we say 'φ and ψ agree on F', denoted $\varphi =^F \psi$, if $\varphi(x) = \psi(x)$ for all $x \in F$. If moreover $\varepsilon > 0$ is given, then we say 'φ and ψ agree up to ε', denoted by $\varphi =_\varepsilon \psi$, if $d_Y(\varphi(x), \psi(x)) < \varepsilon$ for all $x \in X$ (for normed spaces, this is usually denoted by $\|\varphi - \psi\|_\infty < \varepsilon$). We say '$\varphi$ and ψ agree on F up to ε', denoted by $\varphi =^F_\varepsilon \psi$, if $d_Y(\varphi(x), \psi(x)) < \varepsilon$ for all $x \in F$.

2.1. (*Approximative*) *absolute* (*neighbourhood*) *retracts*

Recall that a pair of spaces, (Y, Z), is simply a space Y with a *closed* subset Z. A metric space X is an *absolute retract*, abbreviated by *AR*, if, for all pairs (Y, Z) of metric spaces and maps $f \colon Z \to X$, there exists a map $g \colon Z \to X$ such that $f = g \circ \iota$, where $\iota \colon Z \hookrightarrow Y$ is the inclusion map. This means that the following diagram can be completed to commute:



A metric space X is an approximative absolute retract, abbreviated by AAR, if, for all $\varepsilon > 0$ the diagram can be completed to commute up to ε. A metric space X is an *absolute neighbourhood retract*, abbreviated by *ANR*, if, for all pairs (Y, Z) of metric spaces and maps $f \colon Z \to X$, there exists a closed neighbourhood V of Z and a map $g \colon V \to X$ such that $f = g \circ \iota$ where $\iota \colon Z \hookrightarrow V$ is the inclusion map. This means that the following diagram can be completed to commute:

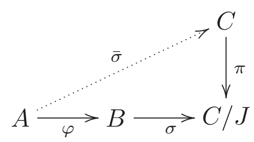


A metric space X is an approximative absolute neighbourhood retract, abbreviated by AANR, if, for all $\varepsilon > 0$ the diagram can be completed to commute up to ε. For details about ARs and ANRs see [**6**]. We only consider compact AARs and AANRs in this paper, and the reader is referred to [**12**] for more details.

We consider shape theory for separable C^*-algebras as developed by Blackadar. See [**3**] for more on projective and semiprojective C^*-algebras. For the notion of weakly semiprojective see [**14**], for weak projectivity see [**19**]. Let us briefly recall the main notions and results.

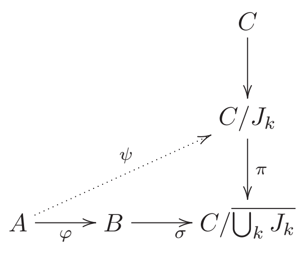
2.2. (*Weakly*) (*semi-*) *projective C^*-algebras*

Let $\mathcal{D}$ be a subcategory of the category of C^*-algebras, closed under quotients. (This means the following: Assume that B is a quotient C^*-algebra of A with quotient morphism $\pi\colon A \to B$. If $A \in \mathcal{D}$, then $B \in \mathcal{D}$ and π is a $\mathcal{D}$-morphism.) A $\mathcal{D}$-morphism $\varphi\colon A \to B$ is called *projective in* $\mathcal{D}$ if, for any C^*-algebra C in $\mathcal{D}$ and $\mathcal{D}$-morphism $\sigma\colon B \to C/J$ to some quotient, there exists a $\mathcal{D}$-morphism $\bar{\sigma}\colon A \to C$ such that $\pi \circ \bar{\sigma} = \sigma \circ \varphi$, where $\pi\colon C \to C/J$ is the quotient morphism. This means that the following diagram can be completed to commute:



A $\mathcal{D}$-morphism is called weakly projective if for all finite subsets F of A and all $\varepsilon > 0$" the diagram can be completed to commute up to ε on F. A C^*-algebra A is called (*weakly*) *projective* in $\mathcal{D}$ if the identity morphism $\mathrm{id}_A\colon A \to A$ is (weakly) projective.

A $\mathcal{D}$-morphism $\varphi\colon A \to B$ is called *semiprojective in* $\mathcal{D}$ if, for any C^*-algebra C in $\mathcal{D}$ and increasing sequence of ideals $J_1 \lhd J_2 \lhd \ldots \lhd C$ and $\mathcal{D}$-morphism $\sigma\colon B \to C/\overline{\bigcup_k J_k}$, there exists an index k and a $\mathcal{D}$-morphism $\bar{\sigma}\colon A \to C/J_k$ such that $\pi_k \circ \bar{\sigma} = \sigma \circ \varphi$, where $\pi_k\colon C/J_k \to C/\overline{\bigcup_k J_k}$ is the quotient morphism. This means that the following diagram can be completed to commute:



A $\mathcal{D}$-morphism is called weakly semiprojective if for all finite subsets F of A and all $\varepsilon > 0$" the diagram can be completed to commute up to ε on F. A C^*-algebra A is called (*weakly*) *semiprojective* in $\mathcal{D}$ if the identity morphism $\mathrm{id}_A\colon A \to A$ is (weakly) semiprojective.

It is well known that if A is separable, then A is semiprojective in the category of all C^*-algebras if and only if it is in $\mathcal{S}$. If $\mathcal{D}$ is the category $\mathcal{S}$, then one drops the reference to $\mathcal{D}$ and simply speaks of (weakly) (semi-)projective C^*-algebras.

A projective C^*-algebra cannot have a unit. For a (separable) C^*-algebra A we get from [**3**, Proposition 2.5] (see also [**18**, Theorem 10.1.9, p. 75]) that the following are equivalent:

(1) A is projective;
(2) $\tilde{A}$ is projective in $\mathcal{S}_1$.

The situation for semiprojectivity is even easier. A unital C^*-algebra is semiprojective if and only if it is semiprojective in $\mathcal{S}_1$. Further, for a separable C^*-algebra A we get from [**3**, Corollary 2.16] (see also [**18**, Theorem 14.1.7, p. 108]) that the following are equivalent:

(1) A is semiprojective;
(2) $\tilde{A}$ is semiprojective;
(3) $\tilde{A}$ is semiprojective in $\mathcal{S}_1$.

2.3. *Connections between (approximative) absolute (neighbourhood) retracts and (weakly) (semi-) projective C^*-algebras*

In general, for a C^*-algebra it is easier to be (weakly) (semi-)projective in a smaller full subcategory, since there are fewer quotients to map into. In particular, if a commutative C^*-algebra is (weakly) (semi-)projective, then it will be (weakly) (semi-)projective with respect to $\mathcal{SC}$. If one compares the definitions carefully, then one gets the following equivalences for a *compact*, metric space X (see [**3**, Proposition 2.11]):

(1) $C(X)$ is projective in $\mathcal{SC}_1 \Leftrightarrow X$ is an AR;
(2) $C(X)$ is weakly projective in $\mathcal{SC}_1 \Leftrightarrow X$ is an AAR;
(3) $C(X)$ is semiprojective in $\mathcal{SC}_1 \Leftrightarrow X$ is an ANR;
(4) $C(X)$ is weakly semiprojective in $\mathcal{SC}_1 \Leftrightarrow X$ is an AANR.

Thus, the notion of (weak) (semi-)projectively is a translation of the concept of an (approximate) absolute (neighbourhood) retract to the world of non-commutative topology. Let us clearly state a point which is used in the proof of the main theorem: If $C(X)$ is (weakly) (semi-)projective in $\mathcal{SC}_1$, then X is an (approximate) absolute (neighbourhood) retract. As we will see, the converse is not true in general. We need an assumption on the dimension of X.

2.4. *Covering dimension*

By $\dim(X)$ we denote the covering dimension of a space X. By definition, $\dim(X) \leqslant n$ if every finite open cover $\mathcal{U}$ of X can be refined by a finite open cover $\mathcal{V}$ of X such that $\operatorname{ord}(\mathcal{V}) \leqslant n+1$. Here $\operatorname{ord}(\mathcal{V})$ is the largest number k such that there exists some point $x \in X$ that is contained in k different elements of $\mathcal{V}$.

To an open cover $\mathcal{V}$ one can naturally assign an abstract simplicial complex $\mathcal{N}(\mathcal{V})$, called the nerve of the covering. It is defined as the family of finite subsets $\mathcal{V}' \subset \mathcal{V}$ with non-empty intersection, in symbols:

$$\mathcal{N}(\mathcal{V}) := \left\{ \mathcal{V}' \subset \mathcal{V} \text{ finite } : \bigcap \mathcal{V}' \neq \emptyset \right\}.$$

Remark. An abstract simplicial complex over a set S is a family C of finite subsets of S such that $X \subset Y \in C$ implies $X \in C$. An element $X \in C$ with $n+1$ elements is called an n-simplex (of the abstract simplicial complex).

A n-simplex of $\mathcal{N}(\mathcal{V})$ corresponds to a choice of n different elements in the cover that have non-empty intersection. Given an abstract simplicial complex C, one can naturally associate to it a space $|C|$, called the geometric realization of C. The space $|C|$ is a polyhedron, in particular it is a CW-complex.

Note that $\operatorname{ord}(\mathcal{V}) \leqslant n+1$ if and only if the nerve $\mathcal{N}(\mathcal{V})$ of the covering $\mathcal{V}$ is an abstract simplicial set of dimension at most n (the dimension of an abstract simplicial set is the largest integer k such that it contains a k-simplex), or equivalently the geometric realization of $|\mathcal{N}(\mathcal{V})|$ is a polyhedron of covering dimension at most n. Note that the covering dimension of a polyhedron, or more generally a CW complex, is the highest dimension of a cell that was attached when building the complex.

Let $\mathcal{U}$ be a finite open covering of a space X, and $\{e_U : U \in \mathcal{U}\}$ be a partition of unity that is subordinate to $\mathcal{U}$. This naturally defines a map $\alpha \colon X \to |\mathcal{N}(\mathcal{U})|$ sending a point $x \in X$ to the (unique) point $\alpha(x) \in |\mathcal{N}(\mathcal{U})|$ that has 'coordinates' $e_U(x)$.

By $\mathrm{locdim}(X)$ we denote the local covering dimension of a space X. By definition $\mathrm{locdim}(X) \leqslant n$ if every point $x \in X$ has a closed neighbourhood D such that $\dim(D) \leqslant n$. If X is paracompact (for example, if it is compact, or locally compact and σ-compact), then $\mathrm{locdim}(X) = \dim(X)$.

See [**24**] for more details on nerves, polyhedra and the (local) covering dimension of a space.

A particularly nice class of one-dimensional spaces are the so-called dendrites. Before we look at them, let us recall some notions from continuum theory. A good reference is Nadler's book [**23**].

REMARK. We say a space is one-dimensional if $\dim(X) \leqslant 1$. So, although it sounds weird, a one-dimensional space can also be zero-dimensional. It would probably be more precise to speak of 'at most one-dimensional' space, however, the usage of the term 'one-dimensional space' is well established.

A *continuum* is a compact, connected, metrizable space, and a *generalized continuum* is a locally compact, connected, metrizable space. A *Peano continuum* is a locally connected continuum, and a *generalized Peano continuum* is a locally connected generalized continuum. By a *finite graph* we mean a graph with finitely many vertices and edges, or equivalently a compact, one-dimensional CW-complex. By a *finite tree* we mean a contractible finite graph.

2.5. *Dendrites*

A *dendrite* is a Peano continuum that does not contain a simple closed curve (that is, there is no embedding of the circle S^1 into it). There are many other characterizations of a dendrite. We collect a few and we shall use them without further mentioning.

Let X be a Peano continuum. Then X is a dendrite if and only if one (or equivalently all) of the following conditions holds:

(1) X is one-dimensional and contractible;
(2) X is tree-like (a (compact, metric) space X is tree-like, if, for every $\varepsilon > 0$, there exists a finite tree T and a map $f \colon X \to T$ onto T such that $\mathrm{diam}(f^{-1}(y)) < \varepsilon$ for all $y \in T$);
(3) X is dendritic (a space X is called dendritic, if any two points of X can be separated by the omission of a third point);
(4) X is hereditarily unicoherent. (A continuum X is called unicoherent if, for each two subcontinua $Y_1, Y_2 \subset X$ with $X = Y_1 \cup Y_2$ the intersection $Y_1 \cap Y_2$ is a continuum (that is, connected). A continuum is called hereditarily unicoherent if all its subcontinua are unicoherent.)

For more information about dendrites , see [**9**, **16**; **23**, Chapter 10].

3. *One implication of the main theorem: necessity*

PROPOSITION 3.1. *Let $C(X)$ be a unital, separable C^*-algebra that is semiprojective. Then X is a compact ANR with* $\dim(X) \leqslant 1$.

Proof. Assume that such a $C(X)$ is given. Then X is a compact, metric space. As noted in Subsection 2.3, semiprojectivity (in $\mathcal{S}_1$) implies semiprojectivity in the full subcategory $\mathcal{SC}_1$ and this means exactly that X is a (compact) ANR. We are left with showing $\dim(X) \leqslant 1$.

Assume otherwise, that is, assume $\dim(X) \geqslant 2$. Since X is paracompact, we have $\operatorname{locdim}(X) = \dim(X) \geqslant 2$. This means that there exists $x_0 \in X$ such that $\dim(D) \geqslant 2$ for each closed neighbourhood D of x_0. For each k consider $D_k := \{y \in X : d(y, x_0) \leqslant 1/k\}$. This defines a decreasing sequence of closed neighbourhoods around x_0 with $\dim(D_k) \geqslant 2$.

If X, Y are spaces, then an injective map $i\colon X \to Y$ is called a topological embedding if the original topology of X is the same as the initial topology induced by the map i. We usually consider a topologically embedded space as a subset with the subset topology.

It was noted in [**11**, Proposition 3.1] that a Peano continuum of dimension at least 2 admits a topological embedding of S^1. Indeed, a Peano continuum that contains no simple arc (that is, in which S^1 cannot be embedded) is a dendrite, and therefore at most one-dimensional. It follows that there are embeddings $\varphi_k\colon S^1 \hookrightarrow D_k \subset X$. Putting these together, we get a map (not necessarily an embedding) $\varphi\colon Y \to X$ where Y is the space of 'smaller and smaller circles':

$$Y = \{(0,0)\} \cup \bigcup_{k \geqslant 1} S((1/2^k, 0), 1/(4 \cdot 2^k)) \subset \mathbb{R}^2,$$

where $S(x, r)$ is the circle of radius r around the point x. We define φ as φ_k on the circle $S((1/k, 0), 1/3k)$. The map $\varphi\colon Y \to X$ induces a morphism $\varphi^*\colon C(X) \to C(Y)$.

Next we construct a C^*-algebra B with a nested sequence of ideals $J_k \lhd B$, such that $C(Y) = B/\overline{\bigcup_k J_k}$ and $\varphi^*\colon C(X) \to C(Y)$ cannot be lifted to some B/J_k. Let $\mathcal{T}$ be the Toeplitz algebra and $\mathcal{T}_1, \mathcal{T}_2, \ldots$ be a sequence of copies of the Toeplitz algebra, and set

$$\begin{aligned} B &:= \left(\bigoplus_{k \in \mathbb{N}} \mathcal{T}_k \right)^+ \\ &= \left\{ (b_1, b_2, \ldots) \in \prod_{k \geqslant 1} \mathcal{T} \text{ such that } (b_k)_k \text{ converges to a scalar multiple of } 1_{\mathcal{T}} \right\}. \end{aligned}$$

The algebras $\mathcal{T}_k$ come with ideals $\mathbb{K}_k \lhd \mathcal{T}_k$ (each $\mathbb{K}_k$ a copy of the algebra of compact operators $\mathbb{K}$). Define ideals $J_k \lhd B$ as follows:

$$\begin{aligned} J_k &:= \mathbb{K}_1 \oplus \ldots \oplus \mathbb{K}_k \oplus 0 \oplus 0 \oplus \ldots \\ &= \{(b_1, \ldots, b_k, 0, 0, \ldots) \in B : b_i \in \mathbb{K}_i \lhd \mathcal{T}_i\}. \end{aligned}$$

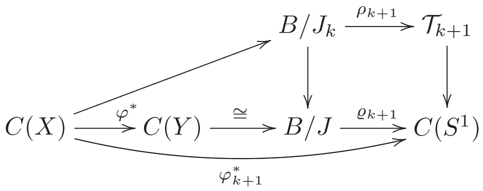
Note

$$B/J_k = C(S^1) \oplus \ldots_{(k)} \oplus C(S^1) \oplus \left(\bigoplus_{l \geqslant k+1} \mathcal{T}_l \right)^+$$

(there are k summands of $C(S^1)$). Also $J_k \subset J_{k+1}$ and $J := \overline{\bigcup_k J_k} = \bigoplus_{k \in \mathbb{N}} \mathbb{K}_k$ and $B/J = (\bigoplus_{l \geqslant 1} C(S^1))^+ \cong C(Y)$.

The semiprojectivity of $C(X)$ gives a lift of $\varphi^*\colon C(X) \to C(Y) = B/J$ to some B/J_k. Consider the projection $\rho_{k+1}\colon B/J_k \to \mathcal{T}_{k+1}$ onto the (k+1)th coordinate, and similarly $\varrho_{k+1}\colon B/J \to C(S^1)$. The composition $C(X) \to C(Y) \cong B/J \to C(S^1)$ is φ_{k+1}^*, the morphism induced by the inclusion $\varphi_{k+1}\colon S^1 \hookrightarrow X$. Note that φ_{k+1}^* is surjective since φ_{k+1} is an inclusion.

The situation is viewed in the following commutative diagram:



The unitary $\mathrm{id}_{S^1} \in C(S^1)$ lifts under φ^*_{k+1} to a normal element in $C(X)$, but it does not lift to a normal element in $\mathcal{T}_{k+1}$. This is a contradiction, and our assumption $\dim(X) \geqslant 2$ must be wrong. □

It is well known that $C(D^2)$, the C^*-algebra of continuous functions on the two-dimensional disc $D^2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leqslant 1\}$, is not weakly semiprojective. For completeness we include the argument that is essentially taken from Loring [**18**, 17.1, p. 131]; see also [**17**].

PROPOSITION 3.2. *The C^*-algebra $C(D^2)$ is not weakly semiprojective.*

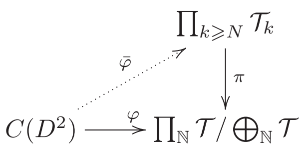
Proof. The $*$-homomorphisms from $C(D^2)$ to a unital C^*-algebra A are in natural one-to-one correspondence with normal contractions in A. Thus, statements about (weak) (semi-)projectivity of $C(D^2)$ correspond to statements about the (approximate) liftability of normal elements. For example, that $C(D^2)$ is projective would correspond to the (wrong) statement that normal elements lift from quotient C^*-algebras. To disprove weak semiprojectivity of $C(D^2)$, one uses a construction of operators that are approximately normal but do not lift in the required way due to an index obstruction.

More precisely, define weighted shift operators t_n on the separable Hilbert space l^2 (with basis $\xi_1, \xi_2, \ldots$) as follows:

$$t_n(\xi_k) = \begin{cases} ((r+1)/2^{n-1})\xi_{k+1} & \text{if } k = r2^{n+1} + s, 0 \leqslant s < 2^{n+1}, \\ \xi_{k+1} & \text{if } k \geqslant 4^n. \end{cases}$$

Each t_n is a finite-rank perturbation of the unilateral shift. Therefore, the t_n lie in the Toeplitz algebra $\mathcal{T}$ and have index -1. The construction of t_n is made so that $\|t_n^* t_n - t_n t_n^*\| = 1/2^{n-1}$.

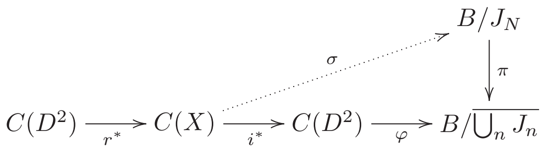
Consider the C^*-algebra $B = \prod_{\mathbb{N}} \mathcal{T} / \bigoplus_{\mathbb{N}} \mathcal{T}$. The sequence $(t_1, t_2, \ldots)$ defines an element in $\prod_{\mathbb{N}} \mathcal{T}$. Let $x = [(t_1, t_2, \ldots)] \in B$ be the equivalence class in B. Then x is a normal element of B, and we let $\varphi\colon C(D^2) \to B$ be the corresponding morphism. We have the following lifting problem:



Assume that $C(D^2)$ is weakly semiprojective. Then the lifting problem can be solved, and $\bar{\varphi}$ defines a normal element $y = (y_N, y_{N+1}, \ldots)$ in $\prod_{k\geqslant N} \mathcal{T}_k$. But the index of each y_l is zero, while the index of each t_l is -1, so that the norm-distance between y_l and t_l is at least 1. Therefore, the distance of $\pi(y)$ and x is at least one, which is a contradiction. Thus, $C(D^2)$ is not weakly semiprojective. □

REMARK 3.3 (Spaces containing a two-dimensional disc). We have seen above that $C(D^2)$ is not weakly semiprojective. Even more is true: Whenever a (compact, metric) space X contains a two-dimensional disc, then $C(X)$ is not weakly semiprojective. This was noted by Loring (private communication). For completeness we include the following argument.

Let $D^2 \subset X$ be a two-dimensional disc with inclusion map $i\colon D^2 \to X$. Since D^2 is an AR, there exists a retraction $r\colon X \to D$, that is, $r \circ i = \mathrm{id}\colon D^2 \to D^2$. Passing to C^*-algebras, we get induced momorphisms $i^*\colon C(X) \to C(D^2), r^*\colon C(D^2) \to C(X)$ such that $i^* \circ r^*$ is the identity on $C(D^2)$. Assume that $C(X)$ is weakly semiprojective. Then any lifting problem for $C(D^2)$ could be solved as follows: Using the weak semiprojectivity of $C(X)$, the morphism $\varphi \circ i^*$ can be lifted. Then $\sigma \circ r^*$ is a lift for $\varphi = \varphi \circ i^* \circ r^*$. The situation is viewed in the following commutative diagram:



This gives a contradiction, as we have shown above that $C(D^2)$ is not weakly semiprojective.

However, that a space does not contain a two-dimensional disc is no guarantee that it has dimension at most 1. These kind of questions are studied in continuum theory, and Bing [**1**] gave examples of spaces of arbitrarily high dimension that are hereditarily indecomposable; in particular they do not contain an arc or a copy of D^2.

Remark. A continuum (that is, compact, connected, metric space) is called decomposable if it can be written as the union of two proper subcontinua. Note that the union is not assumed to be disjoint. For example, the interval $[0, 1]$ is decomposable as it can be written as the union of $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$. A continuum is called hereditarily indecomposable if none of its subcontinua is decomposable. See [**23**] for further information.

These pathologies cannot occur if we restrict to 'nicer' spaces. For example, if a CW-complex does not contain a two-dimensional disc, then it has dimension at most 1. What about ANRs? Bing and Borsuk [**2**] gave an example of a three-dimensional AR that does not contain a copy of D^2. The question for four-dimensional ARs is still open, that is, it is unknown whether there exist high-dimensional ARs (or just ANRs) that do not contain a copy of D^2.

The point we want to make clear is the following: To prove that an ANR is one-dimensional, it is not enough to prove that it does not contain a copy of D^2.

Remark 3.4 (Spaces contained in ANRs of dimension at least 2). Although an ANR with $\dim(X) \geqslant 2$ might not contain a disc, one can show that it must contain (a copy of) one of the following three spaces (see Figure 1):

Space 1 The space Y_1 of distinct 'smaller and smaller circles' as considered in the proof of Proposition 3.1, that is, $Y_1 = \{(0,0)\} \cup \bigcup_{k\geqslant 1} S((1/2^k, 0), 1/(4 \cdot 2^k)) \subset \mathbb{R}^2$.

Space 2 The Hawaiian earrings, that is,
$Y_2 = \bigcup_{k\geqslant 1} S((1/2^k, 0), 1/2^k) \subset \mathbb{R}^2$.

Space 3 A variant of the Hawaiian earrings, where the circles do not just intersect in one point, but have a segment in common. It is homeomorphic to:
$Y_3 = \{(x, x), (x, -x) : x \in [0, 1]\} \cup \bigcup_{k\geqslant 1} \{1/k\} \times [-1/k, 1/k] \subset \mathbb{R}^2$.

To prove this, one uses the same idea as in the proof of Proposition 3.1: If $\dim(X) \geqslant 2$, then there exists a point x_0 where the local dimension is at least 2. Then one can embed into X a sequence of circles that get smaller and smaller and converge to x_0. Note that the circles may

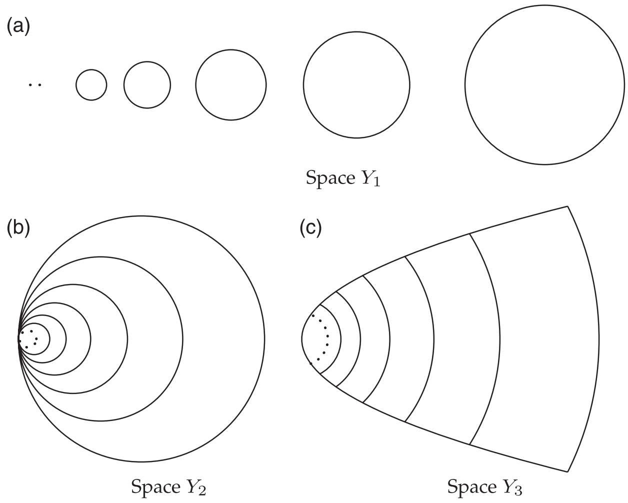


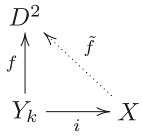
FIGURE 1. *Spaces contained in high-dimensional ANRs.* (a) *Space* Y_1, (b) *Space* Y_2, (c) *Space* Y_3.

intersect or overlap. By passing to subspaces, we can get rid of 'unnecessary' intersections and overlappings, and finally there are only three qualitatively different ways a bunch of 'smaller and smaller' can look like. We skip the details.

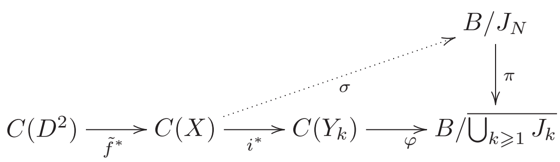
We remark that the converse is also true: If an ANR X contains a copy of one of the spaces Y_1, Y_2 or Y_3, then $dim(X) \geqslant 2$. Assume on the contrary that there is an embedding $\iota : Y_i \to X$ and $dim(X) \leqslant 1$. It follows from [CC06, Corollary 3.3] that ι induces an injective map on fundamental groups, i.e., that $\pi(\iota)\colon \pi(Yi) \to \pi(X)$ is injective. This contradicts the fact that $\pi(Y_i)$ is uncountable, while $\pi(X)$ is countable.

Note that none of the three spaces Y_1, Y_2, Y_3 are semiprojective. Further, no (compact, metric) space X that contains a copy of Y_1, Y_2 or Y_3 can be semiprojective. One uses a similar argument as for an embedded D^2. Assume that for some k, there is an inclusion $i\colon Y_k \hookrightarrow X$. Since Y_k is not an AR, there will in general be no retraction onto it.

Instead, choose an embedding $f\colon Y_k \hookrightarrow D^2$. This map can be extended to a map $\tilde{f}\colon X \to D^2$ on all of X since D^2 is an AR .



If $C(X)$ is semiprojective, then any lifting problem as shown in the diagram below can be solved. However, using Toeplitz algebras as in Proposition 3.1, we see that the morphism $f^* = i^* \circ \tilde{f}^*\colon C(D^2) \to C(Y_k)$ is not semiprojective.



Finally, let us note that the C^*-algebras $C(Y_1), C(Y_2)$ and $C(Y_3)$ are weakly semiprojective.

4. *Structure of compact, one-dimensional ANRs*

In this section, we prove structural theorems about compact, one-dimensional ANRs. The results are used in the next section to show that the C^*-algebra of continuous functions on such a space is semiprojective. In Section 6, we study the structure on non-compact, one-dimensional ANRs. We start with some preparatory lemmas. By $\pi(X, x_0)$ we denote the fundamental group of X based at $x_0 \in X$. Statements about the fundamental group often do not depend on the basepoint, and then we simply write $\pi(X)$ to mean that any (fixed) basepoint may be chosen.

Lemma 4.1. *Let X be a Hausdorff space. Assume that X has a simply connected covering space. Then every path in X is homotopic (relative endpoints) to a path that is a piecewise arc.*

Proof. Let $p\colon \widetilde{X} \to X$ be a simply connected, Hausdorff covering space. Let $\alpha\colon [0,1] \to X$ be a path, and $\tilde{\alpha}\colon [0,1] \to \tilde{X}$ be a lift. Then the image of $\tilde{\alpha}$ is a Peano continuum (that is, a compact, connected, locally connected, metric space), and is therefore arcwise connected. Choose any arc $\beta\colon [0,1] \to \tilde{X}$ from $\tilde{\alpha}(0)$ to $\tilde{\alpha}(1)$. The arc may of course be chosen within the image of $\tilde{\alpha}$. Since $\tilde{X}$ is simply connected, the paths $\tilde{\alpha}$ and β are homotopic (relative endpoints). Then $\alpha = p \circ \tilde{\alpha}$ and $p \circ \beta$ are homotopic paths in X.

Since p is locally a homeomorphism, $p \circ \beta$ is a piecewise arc, that is, there exists a finite subdivision $0 = t_0 < t_1 < \ldots < t_N = 1$ such that each restriction $p \circ \beta_{|[t_j, t_{j+1}]}$ is an arc. □

Lemma 4.2. *Let X be a Hausdorff space, and $x_0 \in X$. Assume that X has a simply connected covering space, and $\pi(X, x_0)$ is f.g. Then there exists a finite graph $Y \subset X$ with $x_0 \in Y$ such that $\pi(Y, x_0) \to \pi(X, x_0)$ is surjective.*

Proof. Choose a set of generators $g_1, \ldots, g_k$ for $\pi(X, x_0)$, represented by loops $\alpha_1, \ldots, \alpha_k\colon S^1 \to X$. From the above lemma, we can homotope each α_j to a loop β_j that is a piecewise arc. Then the image of each β_j in X is a finite graph. Consequently, also the union $Y := \bigcup_j \operatorname{im}(\beta_j)$ is a finite graph (containing x_0). By construction each g_j lies in the image of the natural map $\pi(Y, x_0) \to \pi(X, y_0)$. Therefore, this map is surjective. □

Remark 4.3. Let X be a connected, locally pathwise connected space. Then X has a simply connected covering space (also called universal cover) if and only if X is semilocally simply connected (s.l.s.c.) (a space X is called s.l.s.c. (sometimes also called locally relatively simply connected) if, for each $x_0 \in X$, there exists a neighbourhood U of x_0 such that $\pi(U, x_o) \to \pi(X, x_0)$ is zero); see [**7**, Theorem III.8.4, p. 155].

Proposition 4.4. *Let X be an s.l.s.c. Peano continuum and $x_0 \in X$. Then there exists a finite graph $Y \subset X$ with $x_0 \in Y$ such that $\pi(Y, x_0) \to \pi(X, x_0)$ is surjective.*

Proof. Peano continua are connected and locally pathwise connected. Therefore, by Remark 4.3, X has a simply connected covering space. By Cannon and Conner [**8**, Lemma 7.7], $\pi(X, x_0)$ is f.g. (even finitely presented). Now we may apply Lemma 4.2. □

Remark 4.5. The fundamental group of a finite graph is f.g. and free. Thus, the above map $\pi(Y, x_0) \to \pi(X, x_0)$ will in general not be injective.

Even if $\pi(X, x_0)$ is f.g. and free, the constructed map might not be injective. The reason is simply that the constructed graph could contain 'unnecessary' loops (for example, consider a circle embedded into a disc). However, by restricting to a subgraph, one can get $\pi(Y, x_0) \to \pi(X, x_0)$ to be an isomorphism.

Thus, if X is a Hausdorff space that has a simply connected covering space, and $\pi(X, x_0)$ is f.g. and free, then there exists a finite graph $Y \subset X$ such that $\pi(Y, x_0) \to \pi(X, x_0)$ is an isomorphism.

Let us consider a one-dimensional space X. This situation is special, since Cannon and Conner [**8**, Corollary 3.3] have shown that an inclusion $Y \subset X$ of one-dimensional spaces induces an injective map on the fundamental group. Thus, we get the following proposition.

PROPOSITION 4.6. *Let X be a one-dimensional, Hausdorff space, and $x_0 \in X$. Assume that X has a simply connected covering space, and $\pi(X, x_0)$ is f.g. Then there exists a finite graph $Y \subset X$ with $x_0 \in Y$ such that $\pi(Y, x_0) \to \pi(X, x_0)$ is an isomorphism.*

Above we have studied the question: when is there a finite subgraph containing (up to homotopy) all loops of a space. We now turn to the question: when is there canonical such subgraph. It is clear that we can only hope for this to happen if the space is one-dimensional.

We will use results from the master thesis of Meilstrup [**22**] where also the following concept is introduced. A one-dimensional Peano continuum is called a *core continuum* if it contains no proper deformation retracts.

PROPOSITION 4.7 (see [**22**, Corollary 2.6]). *Let X be a one-dimensional Peano continuum. The following are equivalent*:

(1) *X is a core*;
(2) *X has no attached dendrites* (*an attached dendrite is a dendrite $C \subset X$ such that, for some $y \in C$, there is a strong deformation retract $r\colon X \to (X \setminus C) \cup \{y\}$*);
(3) *every point of X is on an essential loop that cannot be homotoped off it*;
(4) *whenever $Y \subset X$ is a subset with $\pi(Y) \to \pi(X)$ surjective* (*hence bijective*), *then $Y = X$.*

Proof. The equivalence of (1)–(3) is proved in [**22**, Corollary 2.6].

'(3) $\Rightarrow$ (4)': Let $Y \subset X$ be a subset with $\pi(Y) \to \pi(X)$ surjective. Let $x \in X$ be any point. Then x is on an essential loop, say α, which cannot be homotoped off it. Since $[\alpha] \in \pi(Y, x)$, there is a loop β with image in Y that is homotopic to α. Therefore, $x \in Y$.

'(4) $\Rightarrow$ (1)': For any subset Y that is a deformation retract of X, the map $\pi(Y) \to \pi(X)$ surjective. □

To proceed further and prove that every one-dimensional Peano continuum contains a core, we need the notion of reduced loop from [**8**, Definition 3.8]. In fact, we shall slighty generalize this to the notion of reduced path. This will help to simplify some proofs below.

DEFINITION 4.8 (see [**8**, Definition 3.8]). A non-constant path $\alpha\colon [0,1] \to X$ is called *reducible*, if there is an open arc $I = (s,t) \subset [0,1]$ such that $f(s) = f(t)$ and the loop $\alpha_{|[s,t]}$ based at $f(s)$ is nullhomotpic. A path is called *reduced* if it is not reducible. A constant path is also called reduced.

By Cannon and Conner [8, Theorem 3.9] every loop is homotopic to a reduced loop, and if the space is one-dimensional, then this reduced loop is even unique (up to re-parametrization of S^1). The analogue for paths is proved in the same way.

Proposition 4.9 (see [8, Theorem 3.9]). *Let X be a space, and $\alpha\colon [0,1] \to X$ be a path. Then α is homotopic (relative endpoints) to a reduced path $\beta\colon [0,1] \to X$ and we may assume that the homotopy takes place inside the image of α, so that also the image of β lies inside the image of α. If X is one-dimensional, then the reduced path is unique up to the re-parametrizing of $[0,1]$.*

Proposition 4.10 (see [22, Theorem 2.4]). *Let X be a non-contractible, one-dimensional Peano continuum. Then there exists a unique strong deformation retract $C \subset X$ that is a core continuum. We call it the core of X and denote it by $\mathrm{core}(X)$. Further:*

(1) $\mathrm{core}(X)$ *is the smallest strong deformation retract of X;*
(2) $\mathrm{core}(X)$ *is the smallest subset $Y \subset X$ such that the map $\pi(Y) \to \pi(X)$ is surjective.*

Proof. Let $\mathrm{core}(X) \subset X$ be the union of all essential, reduced loops in X. In the proof of Meilstrup [22, Theorem 2.4] it is shown that $\mathrm{core}(X)$ is a core continuum and a strong deformation retract of X.

For every strong deformation retract $Y \subset X$ the map $\pi(Y) \to \pi(X)$ is surjective. Thus, to prove the two statements, it is enough to show that $\mathrm{core}(X)$ is contained in every subset $Y \subset X$ such that the map $\pi(Y) \to \pi(X)$ is surjective.

Let $Y \subset X$ be any subset such that the map $\pi(Y) \to \pi(X)$ is surjective, and let α be an essential, reduced loop in X. Then α is homotopic to a loop α' in Y. By the above remark, the image of α' contains the image of α. Thus, Y contains all essential, reduced loops in X, and therefore $\mathrm{core}(X) \subset Y$. □

Remark 4.11. If X is a contractible, one-dimensional Peano continuum (that is, a dendrite), then it can be contracted to any of its points. That is why $\mathrm{core}(X)$ is not defined in this situation. However, to simplify the following statements, we consider the core of a dendrite to be just any fixed point.

If X is a finite graph, then the core is obtained by successively removing all 'loose' edges, that is, vertices that are endpoints and the edge connecting the endpoint to the rest of the graph.

Next, we combine a bunch of known facts with some of our results to obtain a list of equivalent characterizations of when a one-dimensional Peano continuum is an ANR.

Theorem 4.12. *Let X be a one-dimensional Peano continuum. The following are equivalent:*

(1) *X is an ANR;*
(2) *X is locally contractible;*
(3) *X has a simply connected covering space;*
(4) *$\pi(X)$ is f.g.;*
(5) *there exists a finite graph $Y \subset X$ such that $\pi(Y) \to \pi(X)$ is an isomorphism;*
(6) $\mathrm{core}(X)$ *is a finite graph.*

Proof. ‘(1) ⇒ (2)’: Every ANR is locally contractible; see [**6**, V.2.3, p. 101].
‘(2) ⇒ (3)’: By Remark 4.3.
‘(3) ⇒ (4)’: By Cannon and Conner [**8**, Lemma 7.7].
‘(4) ⇒ (1)’: This follows from [**6**, V.13.6, p. 138].
‘(3)+(4) ⇒ (5)’: The proof follows from Proposition 4.6.
‘(5) ⇒ (6)’: By Proposition 4.10(2), $\mathrm{core}(X) \subset Y$. Then $\pi(\mathrm{core}(X)) \to \pi(Y)$ is an isomorphism, and therefore $\mathrm{core}(X) = \mathrm{core}(Y)$. By Remark 4.11 the core of a finite graph is again a finite graph.
‘(6) ⇒ (4)’: The proof follows since $\pi(\mathrm{core}(X)) \to \pi(X)$ is bijective and the fundamental group of a finite graph is f.g. □

Remark 4.13. Let X be a one-dimensional Peano continuum. In the same way as Theorem 4.12, one obtains that the following are equivalent:

(1) X is an AR;
(2) X is contractible;
(3) X is simply connected;
(4) $\pi(X, x_0)$ is zero;
(5) there exists a finite tree $Y \subset X$ such that $\pi(Y, x_0) \to \pi(X, x_0)$ is an isomorphism (for any $x_0 \in Y$);
(6) $\mathrm{core}(X)$ is a point.

Note that X is a dendrite if and only if it is a one-dimensional Peano continuum that satisfies one (or equivalently all) of the above conditions.

Let us proceed with the study of the internal structure of compact, one-dimensional ANRs. We give a structure theorem that says that these spaces can be approximated by finite graphs in a nice way, namely from within. This generalizes a theorem from Nadler's book [**23**] about the structure of dendrites (which are exactly the *contractible* one-dimensional compact ANRs). The point is that compact, one-dimensional ANRs can be approximated from within by finite graphs in exactly the same way as dendrites can be approximated by finite trees (which are exactly the contractible finite graphs).

Lemma 4.14. *Let X be a one-dimensional Peano continuum, and Y be a subcontinuum with $\mathrm{core}(X) \subset Y$. For each $x \in X \setminus Y$ there is a unique point $r(x) \in Y$ such that $r(x)$ is a point of an arc in X from x to any point of Y.*

Proof. This is the analogue of Nadler [**23**, Lemma 10.24, p. 175]. We use ideas from the proof of Meilstrup [**22**, Theorem 2.4]. Let X, Y be given, and $x \in X \setminus Y$.
Pick some point $y \in Y$. Since X is arc-connected, there exists an arc $\alpha\colon [0,1] \to X$ starting at $\alpha(0) = x$ and ending at $\alpha(1) = y$. Let $y_0 = \alpha(\min \alpha^{-1}(Y))$, which is the first point in Y of the arc (starting from x). Note that $y_0 \in Y$ since Y is closed.
Assume that there are two arcs $\alpha_1, \alpha_2\colon [0,1] \to X$ from x to different points $y_1, y_2 \in Y$ such that $\alpha_i([0,1)) \subset X \setminus Y$. We show that this leads to a contradiction. Let β be a reduced path in Y from y_1 to y_2. Define

$$t_1 := \sup\{t \in [0,1] : \alpha_1(t) \in \mathrm{im}(\alpha_2)\},$$
$$t_2 := \sup\{t \in [0,1] : \alpha_2(t) \in \mathrm{im}(\alpha_1)\},$$

so that $x_0 = \alpha_1(t_1) = \alpha_2(t_2)$ is the first point where the arcs α_1, α_2 meet (looking from y_1 and y_2). Connecting $(\alpha_1)_{|[t_1,1]}$ (from x_0 to y_1) with β (from y_1 to y_2) and the inverse of $(\alpha_1)_{|[t_2,1]}$ (from y_2 to x_0), we get a reduced loop containing x_0 which contradicts $x_0 \notin \mathrm{core}(X) \subset Y$. It follows that there exists a unique point $y \in Y$ with the desired properties. □

Definition 4.15 (see [**23**, Definition 10.26, p. 176]). Let X be a one-dimensional Peano continuum, and Y be a subcontinuum with $\mathrm{core}(X) \subset Y$. Define a map $r\colon X \to Y$ by letting $r(x)$ as in the Lemma 4.14 if $x \in X \setminus Y$, and $r(x) = x$ if $x \in Y$. This map is called the *first point map*.

The first point map is continuous, and thus a retraction of X onto Y. This is the analogue of Nadler [**23**, Lemma 10.25, p. 176] and proved the same way.

But more is true: As in the proof of Meilstrup [**22**, Theorem 2.4], one can show that Y is a strong deformation retract of X.

Proposition 4.16. *Let X be a one-dimensional Peano continuum, and Y be a subcontinuum with $\mathrm{core}(X) \subset Y$. Then the first point map is continuous. Further, there is a strong deformation retraction to the first point map.*

Proof. Let X, Y be given. As in the proof of Meilstrup [**22**, Theorem 2.4], the complement $X \setminus Y$ consist of a collection of attached dendrites $\{C_i\}$. That means each $C_i \subset X$ is a dendrite such that $C_i \cap Y$ consists of exactly one point y_i and such that there is a strong deformation retract $r_i\colon X \to (X \setminus C_i) \cup \{y_i\}$. Meilstrup shows that these strong deformation retracts can be assembled to give a strong deformation retract to the first point map r. □

Theorem 4.17. *Let X be a one-dimensional Peano continuum. Then there is a sequence $\{Y_k\}_{k=1}^{\infty}$ such that:*

(1) *each Y_k is a subcontinuum of X;*
(2) $Y_k \subset Y_{k+1}$;
(3) $\lim_k Y_k = X$;
(4) $Y_1 = \mathrm{core}(X)$ *and, for each k, Y_{k+1} is obtained from Y_k by attaching a line segment at a point, that is, $\overline{Y_{k+1} \setminus Y_k}$ is an arc with an end point p_k such that $\overline{Y_{k+1} \setminus Y_k} \cap Y_k = \{p_k\}$;*
(5) *letting $r_k\colon X \to Y_k$ be the first point map for Y_k, we have that $\{r_k\}_{k=1}^{\infty}$ converges uniformly to the identity map on X.*

If X is also ANR, then all Y_k are finite graphs. If X is even contractible (that is, is an AR, or equivalently a dendrite), then $\mathrm{core}(X)$ is just some point, and all Y_k are finite trees.

Proof. This is the analogue of Nadler [**23**, Lemma 10.24, p. 175], and the proof goes through if we use our analogous Lemmas 4.14 and 4.16. □

5. *The other implication of the main theorem: sufficiency*

For this implication we aim to mirror the approach of Chigogidze and Dranishnikov [**11**]. In order to do this, we need universal C^*-algebras. We take all our notation and most of the needed results from [**18**]. So we shall write $C^*(\mathcal{G} \mid \mathcal{R})$ to mean the universal C^*-algebra generated by the elements of $\mathcal{G}$ subject to the relations $\mathcal{R}$. For instance, $C^*(\{p\} \mid p = p^* = p^2) = \mathbb{C}$. An axiomatic approach to C^*-relations and universal C^*-algebras can be found in [**20**].

We first show how to go from $C(X)$ being a universal C^*-algebra to $C(Y)$ being one, where Y is obtained from X by attaching a line segment at one point. This step is not needed in [**11**], since they are able to give a general description of the generators and relations of the relevant spaces. We have not been able to find such generators and relations, and doing so might be of independent interest.

LEMMA 5.1. *Suppose that X is a space, that $C(X) = C^*\langle \mathcal{G} \mid \mathcal{R}\rangle$ and that $\{\hat{g} \mid g \in \mathcal{G}\}$ is a generating set of $C(X)$ that fulfils $\mathcal{R}$. Let Y be the space formed from X by attaching a line segment at a point v, and let $\lambda_g = \hat{g}(v)$. Then $C(Y) = C^*\langle \mathcal{G} \cup \{h\} \mid \mathcal{R}'\rangle$, where*

$$\mathcal{R}' = \mathcal{R} \cup \{gh = \lambda_g h \text{ and } gh = hg \mid g \in \mathcal{G}\} \cup \{0 \leqslant h \leqslant 1\}.$$

Proof. Extending the $\hat{g}$ to Y by letting them be constant on the added line segment and letting $\hat{h}$ be the function that is zero on X and grows linearly to one on the line segment (identifying it with $[0,1]$) shows that that there is a generating family in $C(Y)$ that fulfils $\mathcal{R}'$. It remains to show that whenever we have a family $\{\tilde{g} \mid g \in \mathcal{G} \cup \{h\}\}$ of elements, in some C^*-algebra A, that fulfils $\mathcal{R}$, we get a $*$-homomorphism from $C(Y)$ to A sending $\hat{g}$ to $\tilde{g}$ for all $g \in \mathcal{G} \cup \{h\}$. For this, note that we have the following short exact sequence:

$$0 \longrightarrow C_0((0,1]) \longrightarrow C(Y) \longrightarrow C(X) \longrightarrow 0,$$

which splits. Since $C_0((0,1])$ is the universal C^*-algebra for a positive contraction, we get a morphism $\phi\colon C_0((0,1]) \to A$ taking the identity function to h. By assumption $C(X)$ is universal for $\mathcal{R}$, so we get a morphism $\psi\colon C(X) \to A$ taking $\hat{g}$ to $\tilde{g}$ for all $g \in \mathcal{G}$. If we let $\lambda\colon C(X) \to C(Y)$ denote the splitting, then one easily checks that

$$\phi(t \longmapsto t)\psi(\hat{g}) = \phi((t \longmapsto t)\lambda(\hat{g})),$$

for all $g \in \mathcal{G}$. Standard arguments then show

$$\phi(a)\psi(b) = \phi(a\lambda(b)),$$

for all $a \in C_0((0,1])$ and all $b \in C(X)$. Thus, we get our desired result from [**18**, Theorem 7.2.3, p. 53]. □

We now provide a slightly altered (in both proof and statement) version of Chigogidze and Dranishnikov [**11**, Proposition 4.1].

LEMMA 5.2. *Suppose that X is a one-dimensional finite graph, that $C(X) = C^*\langle \mathcal{G} \mid \mathcal{R}\rangle$, that $\{\hat{g} \mid g \in \mathcal{G}\}$ is a generating set of $C(X)$ that fulfils $\mathcal{R}$ and that $\mathcal{G}$ is finite. Let Y be the space formed from X by attaching a line segment at a point v. Suppose that we have a commutative square*

$$\begin{array}{ccc} C(X) & \xrightarrow{\psi} & C \\ {\scriptstyle\iota}\downarrow & & \downarrow{\scriptstyle\pi} \\ C(Y) & \xrightarrow[\phi]{} & C/J \end{array}$$

where J is an ideal in the unital C^*-algebra C, π is the quotient morphism, ψ and ϕ are unital morphisms, and ι is induced by the retraction from Y onto X, that is, ι takes a function in

$C(X)$ to the function in $C(Y)$ given by

$$\iota(f)(x) = \begin{cases} f(x), & x \in X, \\ f(v), & x \text{ is in the added line segment.} \end{cases}$$

Then, for every $\varepsilon > 0$, we can find a morphism $\chi\colon C(Y) \to C$ such that $\pi \circ \chi = \phi$ and $\|(\chi \circ \iota)(\hat{g}) - \psi(\hat{g})\| \leqslant \varepsilon$ for every $g \in \mathcal{G}$.

Proof. Throughout the proof, we use the notation of Lemma 5.1.

Let $\delta > 0$ be given. We shall construct a δ-representation $\{d_g \mid g \in \mathcal{G} \cup \{h\}\}$ of $\mathcal{R}'$ in C such that $\pi(d_g) = \phi(\iota(\hat{g}))$ for $g \in \mathcal{G}$ and $\pi(d_h) = \phi(\hat{h})$.

Let $q_\kappa\colon X \to X$ be the map that collapses the ball $B_{\kappa/2}(v)$, fixes $X \setminus B_\kappa(v)$ and extends linearly in between. Since there are only finitely many $\hat{g}$, we can find κ_0 such that $\|q_{\kappa_0}^*(\hat{g}) - \hat{g}\| \leqslant \delta/2$, where q_κ^* is the morphism on $C(X)$ induced by q_κ. For simpler notation we let $q = q_{\kappa_0}$, and put $w_g = q^*(\hat{g})$ for all $g \in \mathcal{G}$.

Let f_0 be a positive function in $C(X)$ of norm 1 that is zero on $X \setminus B_{\kappa_0/2}(v)$ and 1 at v. Observe that if $f \in q^*(C_0(X \setminus \{v\}))$, then $ff_0 = 0$. Since $\hat{h} \leqslant \iota(f_0)$ and $\psi(f_0)$ is a lift of $\phi(\iota(f_0))$, we can, by Loring [18, Corollary 8.2.2, p. 63], find a lift $\bar{h}$ of $\phi(\hat{h})$ such that $0 \leqslant \bar{h} \leqslant \psi(f_0)$. We now claim that $\{\psi(\hat{g}) \mid g \in \mathcal{G}\} \cup \{\bar{h}\}$ is a δ-representation of $\mathcal{R}$.

Since the $\bar{g}$ fulfil the relations $\mathcal{R}$ and $\bar{h}$ is a positive contraction, we only need to check that $\psi(\hat{g})$ and $\bar{h}$ almost commute, and that $\psi(\hat{g})\bar{h}$ is almost $\lambda_g\bar{h}$.

First we note that since $0 \leqslant \bar{h} \leqslant \psi(f_0)$ for any $f \in q^*(C_0(X \setminus \{v\}))$, we have

$$\|\psi(f)\bar{h}^{1/2}\|^2 = \|\psi(f)\bar{h}\psi(f)^*\| \leqslant \|\psi(f)\psi(f_0)\psi(f)^*\| = 0.$$

Thus, $\psi(f)\bar{h} = 0$. In particular, we have

$$\psi(w_g - \lambda_g)\bar{h} = 0.$$

Now we have

$$\begin{aligned} \|\psi(\hat{g})\bar{h} - \bar{h}\psi(\hat{g})\| &= \|\psi(\hat{g})\bar{h} - \psi(w_g - \lambda_g)\bar{h} - \bar{h}\psi(\hat{g}) + \bar{h}\psi(w_g - \lambda_g)\| \\ &= \|\psi(\hat{g} - w_g)\bar{h} + \lambda_g\bar{h} - \bar{h}(\psi(\hat{g} - w_g)) - \lambda_g\bar{h}\| \\ &\leqslant \|\bar{h}\|(\|\psi(\hat{g} - w_g)\| + \|\psi(\hat{g} - w_g)\|) \\ &\leqslant 2\|\hat{g} - w_g\| \leqslant 2 \cdot \delta/2 = \delta, \end{aligned}$$

for all $g \in \mathcal{G}$. Likewise we have

$$\begin{aligned} \|\psi(\hat{g})\bar{h} - \lambda_g\bar{h}\| &= \|\psi(\hat{g})\bar{h} - \lambda_g\bar{h} - \psi(w_g - \lambda_g)\bar{h}\| \\ &= \|\psi(\hat{g} - w_g)\bar{h} + \lambda_g\bar{h} - \lambda_g\bar{h}\| \\ &= \|\psi(\hat{g} - w_g)\bar{h}\| \leqslant \|\hat{g} - w_g\| \leqslant \delta/2 \leqslant \delta, \end{aligned}$$

for all $g \in \mathcal{G}$. So $\{\psi(g) \mid g \in \mathcal{G}\} \cup \{\bar{h}\}$ is indeed a δ-representation of $\mathcal{R}'$. Further, we have that $\pi(\psi(\hat{g})) = \phi(\iota(\hat{g}))$ and that $\pi(\bar{h}) = \phi(h)$.

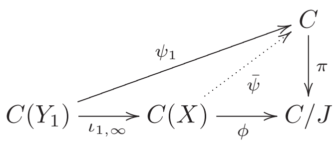
Since X is a one-dimensional finite graph, Y is also a one-dimensional finite graph, so $C(Y)$ is semiprojective by Loring [18, Proposition 16.2.1, p. 125]. By Loring [18, Theorem 14.1.4, p. 106] the relations $\mathcal{R}'$ are then stable. So the fact that we can find a δ-representation for all δ implies that we can find a morphism $\chi\colon C(Y) \to C$ such that $\pi \circ \chi = \phi$ and $\|\chi(\iota(\hat{g})) - \psi(\hat{g})\| \leqslant \varepsilon$ for all $g \in \mathcal{G}$. □

We are now ready to show that some inductive limits have good lifting properties. In particular, if we have an initial lift, then we can lift all that follows.

Proposition 5.3. *Suppose that X is a compact space such that $C(X)$ can be written as an inductive limit $\varinjlim_n C(Y_n) = C(X)$, where each Y_n is a finite graph, Y_{n+1} is just Y_n with a line segment attached at a point (as in Lemma 5.2), and the bonding morphisms $\iota_{n,n+1}\colon C(Y_n) \to C(Y_{n+1})$ are as the morphism in Lemma 5.2, that is, induced by retracting the attached interval to the attaching point.*

If there is a unital morphism $\phi\colon C(X) \to C/J$, where J is an ideal in a unital C^*-algebra C, and a unital morphism $\psi_1\colon C(Y_1) \to C$ such that $\pi \circ \psi_1 = \phi \circ \iota_{1,\infty}$, then there is a unital morphism $\bar{\psi}\colon C(X) \to C$ such that $\pi \circ \bar{\psi} = \phi$.

Proof. We have the following situation:



As Y_1 is a finite graph, $C(Y_1)$ is f.g. Thus, $C(Y_1)$ is a universal C^*-algebra for some finite set of generators and relations, $C(Y_1) = C^*\langle \mathcal{G}_1 \mid \mathcal{R}_1\rangle$, say. In view of Lemma 5.1, we can now assume that $C(Y_n) = C^*\langle \mathcal{G}_n \mid \mathcal{R}_n\rangle$, where $\mathcal{G}_1 \subseteq \mathcal{G}_2 \ldots$, and likewise for the $\mathcal{R}_n$. We also get from Lemma 5.1 that all the $\mathcal{G}_n$ and $\mathcal{R}_n$ are finite.

Since we are given ψ_1, we can, using Lemma 5.2 inductively, for any sequence of positive numbers (ε_n) find morphisms $\psi_n\colon C(Y_n) \to C$ for each $n > 1$ such that $\pi \circ \psi_n = \phi \circ \iota_{n,\infty}$ and such that $\|\psi_n(\hat{g}) - \psi_{n-1}(\hat{g})\| \leqslant \varepsilon_n$ for the generators $\hat{g}$ of $C(Y_n)$.

We now wish to define new morphisms $\chi_n\colon C(Y_n) \to C$ such that $\pi \circ \chi_n = \phi \circ \iota_{n,\infty}$ and χ_{n+1} extends χ_n. To this end, we define, for each $n \in \mathbb{N}$, elements $\{\bar{g}_n \mid g \in \mathcal{G}_n\}$ by

$$\bar{g}_n = \lim_k \psi_{n+k}(\hat{g}).$$

We will assume that $\sum \varepsilon_n < \infty$, so the sequence $(\psi_{n+k}(\hat{g}))$ becomes Cauchy. We claim that, for any $n \in \mathbb{N}$, the elements $\{\bar{g}_n \mid g \in \mathcal{G}_n\}$ in C fulfil $\mathcal{R}_n$. By Loring [**18**, Lemma 13.2.3, p. 103] the set $\{\bar{g}_n \mid g \in \mathcal{G}_n\}$ is an ε-representation of $\mathcal{R}_n$ for all $\varepsilon > 0$ since $\{\psi_{n+k}(\hat{g}) \mid g \in \mathcal{G}_n\}$ is a representation of $\mathcal{R}_n$ for all k. Thus, $\{\bar{g}_n \mid g \in \mathcal{G}_n\}$ is a representation of $\mathcal{R}_n$. Observe that if $m \geqslant n$, then $\bar{g}_m = \bar{g}_n$. Thus, we will drop the subscripts, and simply say that we have elements $\{\bar{g} \mid g \in \bigcup \mathcal{G}_n\}$ such that, for any $n \in \mathbb{N}$, the set $\{\bar{g} \mid g \in \mathcal{G}_n\}$ fulfils $\mathcal{R}_n$. Now we can define the χ_n. We put $\chi_n(\hat{g}) = \bar{g}$ for $g \in \mathcal{G}_n$, and this extends to a morphism since $C(Y_n) \cong C^*\langle \mathcal{G}_n \mid \mathcal{R}_n\rangle$. We get $\chi_{n_1} \circ \iota_{n,n+1} = \chi_n$ and $\pi \circ \chi_n = \phi \circ \iota$ by universality, since it holds on generators.

By the universal property of an inductive limit, we get a morphism $\chi\colon C(X) \to C$ such that $\pi \circ \chi = \phi$. □

Remark 5.4. Using the structure theorem for dendrites [**23**, Theorem 10.27, p. 176] (see Theorem 4.17) and Proposition 5.3, we may deduce that for a dendrite X the C^*-algebra $C(X)$ is projective in $\mathcal{S}_1$ (the category of unital C^*-algebras; see Subsection 2.2). Thus, we recover the implication '(1) ⇒ (2)' of Chigogidze and Dranishnikov [**11**, Theorem 4.3].

To elaborate: Each dendrite X can be approximated from within by finite trees, that is, $C(X) \cong \varinjlim C(Y_k)$ where Y_1 is just a single point and the trees Y_k are obtained by successive attaching of line segments. Since $C(Y_1) = \mathbb{C}$ is projective in $\mathcal{S}_1$, we obtain from Proposition 5.3 that morphisms from $C(X)$ into a quotients can be lifted, that is, $C(X)$ is projective in $\mathcal{S}_1$.

We are now ready to prove our main theorem.

Proof of Theorem 1.2. The implication '(I) $\Rightarrow$ (II) ' is Proposition 3.1.

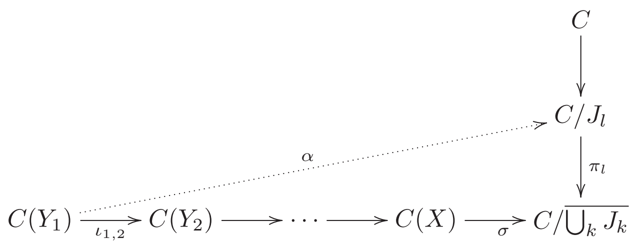
Let us prove '(II) $\Rightarrow$ (I) ': So assume that X is a compact ANR with $\dim(X) \leqslant 1$. Note that X can have at most finitely many components X_i. If we can show that each $C(X_i)$ is semiprojective, then $C(X) = \bigoplus_i C(X_i)$ will be semiprojective (since semiprojectivity is preserved by finite direct sums; see [**18**, Theorem 14.2.1, p. 110]). So we may assume that X is connected.

Then Theorem 4.17 applies, and we may find an increasing sequence $Y_1 \subset Y_2 \subset \ldots \subset X$ of finite subgraphs such that

(1) $\lim_k Y_k = X$, that is, $\overline{\bigcup_k Y_k} = X$ and
(2) Y_{k+1} is obtained from Y_k by attaching a line segment at a point.

Then $C(X) = \varinjlim_k C(Y_k)$ where each bonding morphism $\iota_{k,k+1} \colon C(Y_k) \to C(Y_{k+1})$ is induced by the retraction from Y_{k+1} to Y_k that contracts $Y_{k+1} \setminus Y_k$ to the point $\overline{Y_{k+1} \setminus Y_k} \cap Y_k$. Suppose now that we are given a unital C^*-algebra C, an increasing sequence of ideals $J_1 \lhd J_2 \lhd \ldots \lhd C$ and a unital morphism $\sigma \colon C(X) \to C/\overline{\bigcup_k J_k}$. We need to find a lift $\bar{\sigma} \colon C(X) \to C/J_l$ for some l.

Consider the unital morphism $\sigma \circ \iota_{1,\infty} \colon C(Y_1) \to C/\overline{\bigcup_k J_k}$. By Loring [**18**, Proposition 16.2.1, p. 125], the initial C^*-algebra $C(Y_1)$ is semiprojective. Therefore, we can find an index l and a unital morphism $\alpha \colon C(Y_1) \to C/J_l$ such that $\pi_l \circ \alpha = \sigma \circ \iota_{1,\infty}$. This is viewed in the following commutative diagram:



Now we can apply Proposition 5.3 to find a unital morphism $\bar{\sigma} \colon C(X) \to C/J_l$ such that $\pi_l \circ \bar{\sigma} = \sigma$. This shows that $C(X)$ is semiprojective. $\square$

6. *Applications*

In this section, we give applications of our findings. First, we characterize semiprojectivity of non-unital, separable commutative C^*-algebras. Building on this, we are able to confirm a conjecture of Loring in the particular case of commutative C^*-algebras. Then we will study the semiprojectivity of C^*-algebras of the form $C_0(X, M_k)$. Finally, we will give a partial solution to the problem when a commutative C^*-algebra is weakly (semi-)projective. To keep this article short, we omit most of the proofs in this sections.

To characterize semiprojectivity of non-unital commutative C^*-algebras, we have to study the structure of non-compact, one-dimensional ANRs. We are particularly interested in the one-point compactifications of such spaces. The motivation are the following results: If X is a locally compact, Hausdorff space, then naturally $\widetilde{C_0(X)} \cong C(\alpha X)$, where αX is the one-point compactification of X. Further, a C^*-algebra A is semiprojective if and only if $\tilde{A}$ is semiprojective. Thus, $C_0(X)$ is semiprojective if and only if $C(\alpha X)$ is semiprojective. By our main result Theorem 1.2 this happens precisely if αX is a one-dimensional ANR.

The following result gives a topological characterization of such spaces. We derive a characterization of semiprojectivity for non-unital, separable commutative C^*-algebras; see

Corollary 6.2. We also show that αX is a one-dimensional ANR if and only if every finite-point compactification of X is a one-dimensional ANR. Using this, we can confirm a conjecture about the semiprojective extensions in the commutative case; see Corollaries 6.3 and 6.4.

Remark. A compactification of a space X is a pair (Y, ι_Y) where Y is a compact space, $\iota\colon X \to Y$ is an embedding and $\iota(X)$ is dense in Y. Usually the embedding is understood and one denotes a compactification just by the space Y. A compactification $\gamma(X)$ of X is called a finite-point compactification if the remainder $\gamma(X) \setminus X$ is finite.

Theorem 6.1. *Let X be a one-dimensional, locally compact, separable, metric ANR. The following are equivalent.*

1. *The one-point compactification αX is an ANR.*
2. *The space X has only finitely many compact components and also only finitely many components $C \subset X$ such that αC is not a dendrite .*
3. *Every finite-point compactification of X is an ANR.*
4. *Some finite-point compactification of X is an ANR.*

Corollary 6.2. *Let X be a locally compact, separable , metric space. Then the following are equivalent:*

1. *$C_0(X)$ is semiprojective;*
2. *X is a one-dimensional ANR that has only finitely many compact components, and X has also only finitely many components $C \subset X$ such that αC is not a dendrite.*

Corollary 6.3. *Let A be a separable, commutative C^*-algebra, and $I \lhd A$ be an ideal. Assume that A/I is finite-dimensional, that is, $A/I \cong \mathbb{C}^k$ for some k. Then A is semiprojective if and only if I is semiprojective.*

Proof. Let $A = C_0(X)$ for a locally compact, separable metric space X. Then $I = C_0(Y)$ for an open subset $Y \subset X$. Since A/I is finite-dimensional, $X \setminus Y$ is finite. It follows that also $\alpha X \setminus Y$ is finite, and so the closure $\bar{Y} \subset \alpha X$ is a finite-point compactification of Y. Set $F := \alpha X \setminus \bar{Y}$ (which is also finite). Note that $\bar{Y} \subset \alpha X$ is a component, so that $\alpha X = \bar{Y} \sqcup F$. It follows that αX is an ANR if and only $\bar{Y}$ is. Then we argue as follows:

	$A = C_0(X)$ is semiprojective	
$\Leftrightarrow$	$\tilde{A} = C(\alpha X)$ is semiprojective	
$\Leftrightarrow$	αX is a one-dimensional ANR	[by Theorem 1.2]
$\Leftrightarrow$	$\bar{Y} \subset \alpha X$ is a one-dimensional ANR	[since $\alpha X = \bar{Y} \sqcup F$]
$\Leftrightarrow$	αY is a one-dimensional ANR	[by Theorem 6.1 since $\bar{Y}$ is a finite-point compactification of Y]
$\Leftrightarrow$	$\tilde{I} = C(\alpha Y)$ is semiprojective	[by Theorem 1.2]
$\Leftrightarrow$	$I = C_0(Y)$ is semiprojective	

□

Remark 6.4. Let A be a separable C^*-algebra, and $I \lhd A$ be an ideal so that the quotient is finite-dimensional. We get a short exact sequence:

$$0 \longrightarrow I \longrightarrow A \longrightarrow F \longrightarrow 0.$$

It was conjectured by Blackadar [4, Conjecture 4.5] that, in this situation, A is semiprojective if and only if I is semiprojective. One implication was recently proved by Enders (private communication) who showed that semiprojectivity passes to ideals when the quotient is finite-dimensional.

Our above result, Corollary 6.3, confirms this conjecture in the case where A is commutative.

Let us now study the semiprojectivity of C^*-algebras of the form $C_0(X, M_k)$.

Lemma 6.5. *Let X be a locally compact metric space and let $k \in \mathbb{N}$. If $\phi\colon C_0(X, M_k) \to M_k$ is a morphism, then there is a unitary $u \in M_k$ and a unique point $x \in \alpha X$ such that*

$$\phi = Ad_u \circ \underset{x}{ev} .$$

Proposition 6.6. *Let X be a locally compact, separable, metric space and let $k \in \mathbb{N}$. If $C_0(X, M_k)$ is projective, then αX is an AR.*

Proof. Suppose that we are given a compact metric space Y with an embedding $\iota\colon \alpha X \to Y$. Dualizing and embedding $C_0(X)$ into $C(\alpha X)$, we get the following diagram:

$$\begin{array}{ccc} & & C_0(Y) \\ & & \big\downarrow{\scriptstyle \iota_*} \\ C_0(X) & \longrightarrow & C(\alpha X) \end{array}$$

Tensoring everything by the k by k matrices M_k, we get

$$\begin{array}{ccc} & & C_0(Y, M_k) \\ & & \big\downarrow{\scriptstyle (\iota_*)_k} \\ C_0(X, M_k) & \longrightarrow & C(\alpha X, M_k) \end{array}$$

Since $C_0(X, M_k)$ is projective, there is a morphism $\psi\colon C_0(X, M_k) \to C_0(Y, M_k)$ such that $(\iota_*)_k \circ \psi$ is the inclusion of $C_0(X, M_k)$ into $C(\alpha X, M_k)$.

For each $y \in Y$ Lemma 6.5 tells us that the morphism $ev_y \circ \psi$ has the form $Ad_{u_y} \circ ev_{x_y}$ for some unitary $u_y \in M_k$ and some unique $x_y \in \alpha X$. Hence, we can define a function $\lambda\colon Y \to \alpha X$ such that

$$\underset{y}{ev} \circ \psi = Ad_{u_y} \circ \underset{\lambda(y)}{ev} .$$

This map λ is continuous.

For each $x \in \alpha X$ we have the following commutative diagram:

$$\begin{array}{ccccc} & & C_0(Y, M_k) & \xrightarrow{ev_{\iota(x)}} & M_k \\ & {\scriptstyle \psi} \nearrow & \big\downarrow{\scriptstyle (\iota_*)_k} & & \big\| \\ C_0(X, M_k) & \longrightarrow & C(\alpha X, M_k) & \xrightarrow{ev_x} & M_k \end{array}$$

From this diagram, it follows that if $x \in \alpha X$, then

$$Ad_{u_{\iota(x)}} \circ \underset{\lambda(\iota(x))}{ev} = \underset{\iota(x)}{ev} \circ \psi = \underset{x}{ev} \circ (\iota_*)_k \circ \psi = \underset{x}{ev}.$$

So, for any function $g \in C_0(X)$, we get

$$\underset{\lambda(\iota(x))}{ev} \begin{pmatrix} g & & \\ & \ddots & \\ & & g \end{pmatrix} = (Ad_{u_{\iota(x)}} \circ \underset{\lambda(\iota(x))}{ev}) \begin{pmatrix} g & & \\ & \ddots & \\ & & g \end{pmatrix} = \underset{x}{ev} \begin{pmatrix} g & & \\ & \ddots & \\ & & g \end{pmatrix}.$$

Hence, we must have $\lambda(\iota(x)) = x$.

All in all, we have found a continuous map $\lambda\colon Y \to \alpha X$ such that $\lambda \circ \iota = \mathrm{id}$, that is, the embedded space $\alpha X \subset Y$ is a retract. As the embedding was arbitrary, αX is an AR. □

The proof can be modified to show the following proposition.

PROPOSITION 6.7. *Let X be a locally compact, separable , metric space and let $k \in \mathbb{N}$. If $C_0(X, M_k)$ is semiprojective, then αX is an ANR.*

Using the idea of the proof of Proposition 3.1, one can show the following proposition.

PROPOSITION 6.8. *Let X be a locally compact, separable , metric space, and let $k \in \mathbb{N}$. If $C_0(X, M_k)$ is semiprojective, then* $\dim(X) \leqslant 1$.

COROLLARY 6.9. *Let A be a separable, commutative C^*-algebra, and let $k \in \mathbb{N}$. If $A \otimes M_k$ is projective, then so is A. Analogously, if $A \otimes M_k$ is semiprojective, then so is A.*

Proof. Let $A = C_0(X)$ for a locally compact, separable metric space X.

First, assume that $A \otimes M_k$ is semiprojective. By Proposition 6.8, $\dim(X) \leqslant 1$. This implies that the dimension of αX is at most 1. By Proposition 6.7, αX is an ANR. Then our main Theorem 1.2 shows that $C(\alpha X)$ is semiprojective. Since $C(\alpha X)$ is the unitization of $C_0(X)$, we also have that $C_0(X)$ is semiprojective.

Assume now that $A \otimes M_k$ is projective. It follows that A cannot be unital, for otherwise $A \otimes M_k$ would be unital and that is impossible for projective C^*-algebras. As in the semiprojective case, we deduce $\dim(\alpha X) \leqslant 1$. By Proposition 6.6, αX is an AR. It follows from [**11**, Theorem 4.3] (see also Theorem 1.3) that the $C(\alpha X)$ is projective in $\mathcal{S}_1$. It follows that $C_0(X)$ is projective; see Subsection 2.2. □

We now turn to the question: when is a unital, commutative C^*-algebra weakly (semi-)projective in $\mathcal{S}_1$? The analogue of a weakly (semi-)projective C^*-algebra in the commutative world is an approximative absolute (neighbourhood) retract (abbreviated as AAR and AANR). As mentioned in Subsection 2.3, if $C(X)$ is weakly (semi-)projective, then X is an AA(N)R. We will show below that for one-dimensional spaces the converse is also true.

6.10. *Approximation from within*

Let X be a compact metric space. Consider the following conditions:

(1) for each $\varepsilon > 0$ there exists a map $f\colon X \to Y \subset X$ such that Y is an AR (an ANR), and $d(f) \leqslant \varepsilon$ and
(2) The space X is an AAR (an AANR).

Here, by $d(f) < \varepsilon$ we mean that the distance of x and $f(x)$ is less than ε for all $x \in X$, that is, $d(x, f(x)) < \varepsilon$ for all $x \in X$. The first condition means that X can be approximated from within by ARs (by ANRs). As shown by Clapp [12, Theorem 2.3] (see also [10, Proposition 2.2(a)]) the implication '(1) ⇒ (2)' holds in general.

It was asked by Charatonik and Prajs [10, Question 5.3] whether the converse also holds (at least for continua). They showed that this is indeed the case for hereditarily unicoherent continua [10, Observation 5.4]. In Theorem 6.15, we show that the two conditions are also equivalent for one-dimensional, compact metric spaces.

The following is a standard result from continuum theory.

Proposition 6.11. *Let X be a one-dimensional Peano continuum, and let $\varepsilon > 0$. Then there exists a finite subgraph $Y \subset X$ and a surjective map $f\colon X \to Y \subset X$ such that $d(f) < \varepsilon$.*

Corollary 6.12. *Every one-dimensional Peano continuum is an AANR.*

Proof. Let X be a one-dimensional Peano continuum. By Proposition 6.11, X can be approximated from within by finite subgraphs. A finite graph is an ANR. It follows from [12, Theorem 2.3] (see Subsection 6.10) that X is an AANR. □

The following lemma is a direct translation of Loring [19, Lemma 5.5] to the commutative setting.

Lemma 6.13 (see [19, Lemma 5.5]). *Let X be an compact AAR, and D be any ANR. Then every map $f\colon X \to D$ is inessential, that is, homotopic to a constant map.*

Corollary 6.14. *Every one-dimensional, compact AAR is tree-like.*

Proof. Let X be a one-dimensional, compact AAR. Then X is connected and thus a continuum. In [9, Theorem 1], tree-like continua are characterized as one-dimensional continua such that every map into a finite graph is inessential. Thus, we need to show that every map from X into a finite graph is inessential. This follows from the above lemma since every finite graph is an ANR. □

Theorem 6.15. *Let X be a one-dimensional, compact , metric space. Then the following are equivalent.*

(1) *For each $\varepsilon > 0$ there exists a map $f\colon X \to Y \subset X$ such that Y is a finite tree (a finite graph), and $d(f) \leqslant \varepsilon$.*
(2) *For each $\varepsilon > 0$ there exists a map $f\colon X \to Y \subset X$ such that Y is an AR (an ANR), and $d(f) \leqslant \varepsilon$.*
(3) *The space X is an AAR (an AANR).*

Moreover, in (1) and (2) the map f may be assumed to be surjective.

Proof. '(1) ⇒ (2)' is clear, and '(2) ⇒ (3)' follows from [12, Theorem 2.3]; see Subsection 6.10.

'(3) ⇒ (1)': It was shown by Clapp [12, Theorem 4.5] that, for each embedding of a compact AANR X in the Hilbert cube Q and $\delta > 0$, there exists a compact polyhedron $P \subset Q$ with maps

$f\colon X \to P$ and $g\colon P \to X$ such that $d(f) < \delta$ and $d(g) < \delta$. Note that g maps each component of P onto a Peano subcontinuum of X. Thus, the image $Y := g(P) \subset X$ is a finite union of Peano subcontinua. Moreover, the map $g \circ f : X \to Y \subset X$ satisfies $d(f) < 2\delta$.

Assume that X is a one-dimensional, compact AANR and fix some $\varepsilon > 0$. We apply the result of Clapp for $\delta = \varepsilon/4$ and obtain a compact subspace $Y \subset X$ that is the (disjoint) union of finitely many Peano continua, together with a surjective map $f\colon X \to Y$ such that $d(f) < \varepsilon/2$. Since $Y \subset X$ is closed, $\dim(Y) \leqslant \dim(X) \leqslant 1$. Applying Proposition 6.11 to each component of Y and $\varepsilon/2$, we obtain a finite subgraph $Z \subset Y$ and a surjective map $g\colon Y \to Z$ such that $d(g) < \varepsilon/2$.

We may consider Z as a finite subgraph of X. The map $h := g \circ f\colon X \to Z \subset X$ is surjective and satisfies $d(h) < \varepsilon$. So we have shown the implication for the case where X is an AANR.

Assume additionally that X is an AAR. We have already shown that X can be approximated from within by finite subgraphs. We need to show that the same is true with finite trees.

By Corollary 6.14, X is tree-like. By Lelek [**16**, 2.2 and 2.3], every tree-like continuum is hereditarily unicoherent. A coherent finite graph is a finite tree. It follows that *every* finite subgraph $Z \subset X$ is a finite tree, and so X can be approximated from within by finite subgraphs which automatically are finite trees. □

Corollary 6.16. *Let X be a compact metric space. Then the following implications hold.*

(1) *If X is an AANR and* $\dim(X) \leqslant 1$, *then $C(X)$ is weakly semiprojective $\mathcal{S}_1$.*
(2) *If X is an AAR and* $\dim(X) \leqslant 1$, *then $C(X)$ is weakly projective in $\mathcal{S}_1$.*

Proof. Let X be a one-dimensional, compact AAR (AANR). By Theorem 6.15, X can be approximated from within by finite trees (finite graphs), that is, for each $n \geqslant 1$, there exists a finite tree (graph) $Y_n \subset X$ and a surjective map $f_n\colon X \to Y_n$ with $d(f_n) < 1/n$. We desire to use [**19**, Theorem 4.7] to show $C(X)$ is weakly (semi-)projective in $\mathcal{S}_1$.

The surjective maps f_n induce injective morphisms $f_n^*\colon C(Y_n) \to C(X)$. Consider also the inclusion map $\iota_n\colon Y_n \hookrightarrow X$ and the dual morphism $\iota_n^*\colon C(X) \to C(Y_n)$. Set $\theta_n := f_n^* \circ \iota_n^*\colon C(X) \to C(X)$.

Since $d(f_n)$ tends to zero, the morphisms θ_n converge (pointwise) to the identity morphism. Further, the image of θ_n is equal to the image of f_n^*, and therefore isomorphic to $C(Y_n)$.

As shown by Loring [**18**, Proposition 16.2.1, p. 125], $C(Y)$ is semiprojective (in $\mathcal{S}_1$) if Y is a finite graph. Similarly, $C(Y)$ is projective in $\mathcal{S}_1$ if Y is a finite tree Y (see also [**11**]). Now, it follows from [**19**, Theorem 4.7] (and the analogous result for weakly semiprojective C^*-algebras) that $C(X)$ is weakly (semi-)projective in $\mathcal{S}_1$. □

Remark 6.17. We remark that the converse implications of Corollary 6.16 also hold. As explained in Subsection 2.3, if $C(X)$ is weakly (semi-)projective in $\mathcal{S}_1$, then X is necessarily an approximative absolute (neighbourhood) retract. The dimension condition was recently shown by Enders (private communication).

Thus, $C(X)$ is (weakly) (semi-)projective in $\mathcal{S}_1$ if and only if X is a compact (approximative) absolute (neighbourhood) retract with $\dim(X) \leqslant 1$.

Acknowledgements. We thank Dominic Enders for his comments and inspiring suggestions that helped to improve some of the results in Section 6. We thank Søren Eilers for his valuable comments on the first draft of this paper.

References

1. R. H. Bing, 'Higher-dimensional hereditarily indecomposable continua', *Trans. Amer. Math. Soc.* 71 (1951) 267–273.
2. R. H. Bing and K. Borsuk, 'A 3-dimensional absolute retract which does not contain any disk', *Fund. Math.* 54 (1964) 159–175.
3. B. Blackadar, 'Shape theory for C^*-algebras', *Math. Scand.* 56 (1985) 249–275.
4. B. Blackadar, 'Semiprojectivity in simple C^*-algebras', *Operator algebras and applications*, Proceedings of the US-Japan Seminar Held at Kyushu University, Fukuoka, Japan, 7–11 June, 1999, Advanced Studies in Pure Mathematics 38 (ed. Kosaki, Hideki; Mathematical Society of Japan, Tokyo, 2004) 1–17.
5. B. Blackadar, *Operator algebras. Theory of C^*-algebras and von Neumann algebras*, Encyclopaedia of Mathematical Sciences 122, Operator Algebras and Non-commutative Geometry 3 (Springer, Berlin, 2006) 517 p.
6. K. Borsuk, *Theory of retracts*, Monografie Matematyczne 44 (PWN - Polish Scientific Publishers, Warszawa, 1967) 251 p.
7. G. E. Bredon, *Topology and geometry*, Graduate Texts in Mathematics 139 (Springer, New York, 1993) xiv, 557.
8. J. W. Cannon and G. R. Conner, 'On the fundamental groups of one-dimensional spaces', *Topology Appl.* 153 (2006) 2648–2672.
9. J. H. Case and R. E. Chamberlin, 'Characterizations of tree-like continua', *Pacific J. Math.* 10 (1960) 73–84.
10. J. J. Charatonik and J. R. Prajs, 'AANR spaces and absolute retracts for tree-like continua', *Czechoslovak Math. J.* 55 (2005) 877–891.
11. A. Chigogidze and A. N. Dranishnikov, 'Which compacta are noncommutative ARs?', *Topology Appl.* 157 (2010) 774–778.
12. M. H. Clapp, 'On a generalization of absolute neighborhood retracts', *Fund. Math.* 70 (1971) 117–130.
13. E. G. Effros and J. Kaminker, 'Homotopy continuity and shape theory for C^*-algebras', *Geometric methods in operator algebras*, Proc. US-Jap. Semin., Kyoto/Jap., 1983, Pitman Research Notes in Mathematics Series 123 (Longman Sci. Tech., Harlow, 1986) 152–180.
14. S. Eilers and T. A. Loring, 'Computing contingencies for stable relations', *Internat. J. Math.* 10 (1999) 301–326.
15. S. Eilers, T. A. Loring and G. K. Pedersen, 'Stability of anticommutation relations: an application of noncommutative *CW* complexes', *J. reine angew. Math.* 499 (1998) 101–143.
16. A. Lelek, 'Properties of mappings and continua theory', *Rocky Mountain J. Math.* 6 (1976) 47–59.
17. T. A. Loring, 'Normal elements of C^*-algebras of real rank zero without finite-spectrum approximants', *J. London Math. Soc. (2)* 51 (1995) 353–364.
18. T. A. Loring, *Lifting solutions to perturbing problems in C^*-algebras*, Fields Institute Monographs 8 (American Mathematical Society, Providence, RI, 1997) ix, 165 p.
19. T. Loring, 'Weakly projective C^*-algebras', Preprint, 2009, arXiv:0905.1520.
20. T. A. Loring, 'C^*-algebra relations', *Math. Scand.* 107 (2010) 43–72.
21. T. A. Loring and T. Shulman, 'Noncommutative semialgebraic sets and associated lifting problems', Preprint, 2009, arXiv:0907.2618.
22. M. Meilstrup, 'Classifying homotopy types of one-dimensional Peano continua', Thesis for Master of Science at the Department of Mathematics at the Brigham Young University, 2005.
23. S. B. Nadler Jr., *Continuum theory. An introduction*, Pure and Applied Mathematics 158 (Marcel Dekker, New York, 1992) xii, 328.
24. K. Nagami, *Dimension theory. With an appendix by Yukihiro Kodama*, Pure and Applied Mathematics 37 (Academic Press, New York, 1970) XI, 256.
25. J. Spielberg, 'Semiprojectivity for certain purely infinite C^*-algebras', *Trans. Amer. Math. Soc.* 361 (2009) 2805–2830.
26. W. Szymanski, 'On semiprojectivity of C^*-algebras of directed graphs', *Proc. Amer. Math. Soc.* 130 (2002) 1391–1399.
27. H. Thiel, 'Inductive limits of projective C*-algebras', Preprint, 2011, arXiv:1105.1979.

Adam P. W. Sørensen and Hannes Thiel
Department of Mathematical Sciences
University of Copenhagen
Universitetsparken 5
DK-2100 Copenhagen Ø
Denmark

apws@math.ku.dk

thiel@math.ku.dk

INDUCTIVE LIMITS OF PROJECTIVE C^*-ALGEBRAS

HANNES THIEL

ABSTRACT. We show that a C^*-algebra is an inductive limits of projective C^*-algebras if and only if it has trivial shape, i.e., is shape equivalent to the zero C^*-algebra. In particular, every contractible C^*-algebra is an inductive limit of projectives, and one may assume that the connecting morphisms are surjective. Interestingly, an example of Dadarlat shows that trivial shape does not pass to full hereditary sub-C^*-algebras. It then follows that the same fails for projectivity.

To obtain these results, we develop criteria for inductive limit decompositions, and we discuss the relation with different concepts of approximation.

As a main application of our findings we show that a C^*-algebra is (weakly) projective if and only if it is (weakly) semiprojective and has trivial shape. It follows that a C^*-algebra is projective if and only if it is contractible and semiprojective. This confirms a conjecture of Loring.

1. INTRODUCTION

Shape theory and homotopy theory are tools to study global properties of spaces. However, homotopy theory gives useful results mainly for spaces with good local behavior (spaces without singularities). For such well-behaved spaces both theories agree, and one usually employs homotopy theory which is easier to compute. To study more general spaces with possible singularities, one uses shape theory. The idea is to abstract from the local behavior of a space, and focus on its global behavior, its 'shape'.

One way of doing this, is to approximate a space by nicer spaces, the building blocks. In the commutative world the building blocks are the so-called absolute neighborhood retracts (ANRs). The approximation is organized in an inverse limit structure, and instead of looking at the original space one studies an associated inverse system of ANRs.

After shape theory was successfully used to study spaces, it was introduced to the study of noncommutative spaces (i.e., C^*-algebras) by Effros and Kaminker, [EK86], and shortly after developed to its modern form by Blackadar, [Bla85]. Shape theory works best when restricted to metrizable spaces, and similarly for noncommutative shape theory one restricts attention to separable C^*-algebras.

The building blocks of noncommutative shape theory are the semiprojective C^*-algebras, which are defined in analogy to ANRs. Since the category of commutative C^*-algebras is dual to the category of spaces, the approximation by an inverse system for spaces is turned

Date: January 17, 2012.

2010 *Mathematics Subject Classification.* Primary 46L05, 46L85, 46M10 ; Secondary 46M20, 46M40, 54C56, 55P55 .

Key words and phrases. C^*-algebras, non-commutative shape theory, projectivity, contractible C^*-algebras.

This research was supported by the Marie Curie Research Training Network EU-NCG and by the Danish National Research Foundation through the Centre for Symmetry and Deformation.

into an approximation by an inductive system for C^*-algebras. Then, approximating a C^*-algebra by 'nice' C^*-algebras means to write it as an inductive limit of semiprojective C^*-algebras.

This raises the natural question of whether there are enough building blocks to approximate every space. This is true in the commutative world, as every metric space is an inverse limit of ANRs. The analog for C^*-algebras is still an open problem, first asked by Blackadar:

Question 1.1 (Blackadar, [Bla85, 4.4])**.** Are all separable C^*-algebras inductive limits of semiprojective C^*-algebras?

In this paper we study the related question of which C^*-algebras are inductive limits of projective C^*-algebras. A necessary condition is that such a C^*-algebra has trivial shape, i.e., is shape equivalent to the zero C^*-algebra, since this holds for projective C^*-algebras and is preserved by inductive limits. We will show that the converse is also true, i.e., that a separable C^*-algebra is an inductive limit of projective C^*-algebras if and only if it has trivial shape, see Theorem 4.4. This also gives a positive answer to Question 1.1 for C^*-algebras with trivial shape, a class class which is quite large since it contains for instance all contractible C^*-algebras.

This paper proceeds as follows:

In Section 2 we remind the reader of the basic notions of noncommutative shape theory, in particular the notion of (weak) semiprojectivity and (weak) projectivity.

In Section 3 we discuss different concepts of how a C^*-algebra can be 'approximated' by other C^*-algebras, for instance as an inductive limit. If $\mathcal{C}$ is a class of C^*-algebras, then an inductive limit of algebras in $\mathcal{C}$ is called an $A\mathcal{C}$-algebra. We suggest to use the formulation that A is '$\mathcal{C}$-like' if it can be approximated by sub-C^*-algebras from $\mathcal{C}$, see Definition 3.2 and Proposition 3.4.

Building on a one-sided approximate intertwining argument, due to Elliott in [Ell93, 2.1, 2.3], see Proposition 3.5, we give two criteria to show that a given C^*-algebra is an $A\mathcal{C}$-algebra. We assume that the class $\mathcal{C}$ of building blocks consists of weakly semiprojective C^*-algebras. Then every separable $A\mathcal{C}$-like C^*-algebra is already an $A\mathcal{C}$-algebra, see Theorem 3.9, and every $AA\mathcal{C}$-algebra is already an $A\mathcal{C}$-algebra, see Theorem 3.12.

In Section 4 we study the class of C^*-algebras with trivial shape. We show that these are exactly the C^*-algebras that are inductive limits of projective C^*-algebras, see Theorem 4.4. Moreover, one may assume that the connecting morphisms are surjective, since we show in Proposition 4.9 that every inductive system can be changed so that the connecting morphisms become surjective while the limit is unchanged.

As a corollary, we obtain that every separable, contractible C^*-algebra is an inductive limit of projective C^*-algebras, see Corollary 4.5. We discuss permanence properties of trivial shape, see Theorem 4.6. It follows from an example of Dadarlat that trivial shape does not pass to full hereditary sub-C^*-algebras, see Remark 4.11. We deduce that also projectivity does not pass to full hereditary sub-C^*-algebras, see Proposition 4.12.

In Section 5 we show some non-commutative analogs of results in commutative shape theory. We prove that a C^*-algebra is (weakly) projective if and only if is is (weakly) semiprojective and has trivial shape, see Theorem 5.6. It follows that a C^*-algebra is projective if and only if it is is semiprojective and contractible, see Corollary 5.7. This confirms a conjecture of Loring.

2. Preliminaries

By a morphism between C^*-algebras we mean a *-homomorphism. All considered C^*-algebras are assumed to be separable. By ideals we mean closed, two-sided ideals. We use the symbol '$\simeq$' to denote homotopy equivalence, both for objects and morphisms.

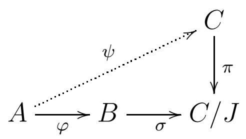
We use the following notations. For $\varepsilon > 0$, a subset F of a C^*-algebra A is said to be **ε-contained** in another subset G, denoted by $F \subset_\varepsilon G$, if for every $x \in F$ there exists some $y \in G$ such that $\|x - y\| < \varepsilon$.

Given two morphisms $\varphi, \psi \colon A \to B$ between C^*-algebras and a subset $F \subset A$ we say φ **and** ψ **agree on** F, denoted $\varphi =^F \psi$, if $\varphi(x) = \psi(x)$ for all $x \in F$. If, moreover, $\varepsilon > 0$ is given, then we say φ **and** ψ **agree on** F **up to** ε, denoted $\varphi =^F_\varepsilon \psi$, if $\|\varphi(x) - \psi(x)\| < \varepsilon$ for all $x \in F$.

We warn the reader that one sometimes defines the above notions of 'ε-containment' and 'agreement up to ε' for the condition that the norm is at most ε (instead of strictly less than ε), e.g. writing $F \subset_e G$ if for every $x \in F$ there exists some $y \in G$ such that only $\|x - y\| \leq \varepsilon$. The difference of notions could be healed by a simple reparametrization, since we always assume $\varepsilon > 0$.

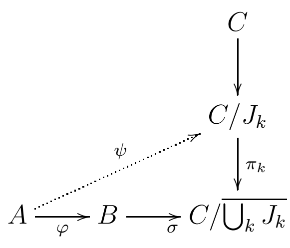
2.1. We consider shape theory for separable C^*-algebra in the sense of Blackadar, see [Bla85]. In this paragraph, which is a shortened version of [ST11, 2.2], we recall the main notions:

A morphism $\varphi \colon A \to B$ is called **(weakly) projective** if for any C^*-algebra C and any morphism $\sigma \colon B \to C/J$ to some quotient (and $\varepsilon > 0$, and finite subset $F \subset A$), there exists a morphism $\psi \colon A \to C$ such that $\pi \circ \psi = \sigma \circ \varphi$ (resp. $\pi \circ \psi =^F_\varepsilon \sigma \circ \varphi$), where $\pi \colon C \to C/J$ is the quotient morphism. This means that the diagram on the right can be completed to commute (up to ε on F).



A C^*-algebra A is called **(weakly) projective** if the identity morphism $\mathrm{id}_A \colon A \to A$ is (weakly) projective.

A morphism $\varphi \colon A \to B$ is called **(weakly) semiprojective** if for any C^*-algebra C, any increasing sequence of ideals $J_1 \lhd J_2 \lhd \cdots \lhd C$ and any morphism $\sigma \colon B \to C/\overline{\bigcup_k J_k}$ (and $\varepsilon > 0$, and finite subset $F \subset A$), there exist an index k and a morphism $\psi \colon A \to C/J_k$ such that $\pi_k \circ \psi = \sigma \circ \varphi$ (resp. $\pi_k \circ \psi =^F_\varepsilon \sigma \circ \varphi$), where $\pi_k \colon C/J_k \to C/\overline{\bigcup_k J_k}$ is the quotient morphism. This means that the diagram on the right can be completed to commute (up to ε on F).

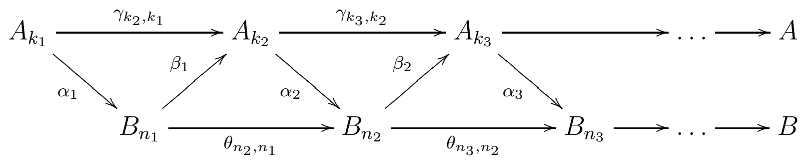


A C^*-algebra A is called **(weakly) semiprojective** if the identity morphism $\mathrm{id}_A \colon A \to A$ is (weakly) semiprojective.

2.2. By an **inductive system** we mean a sequence $A_1, A_2, \dots$ of C^*-algebras together with morphisms $\gamma_k\colon A_k \to A_{k+1}$ for each k. We will denote such a system by $\mathcal{A} = (A_k, \gamma_k)$. If $k < l$, then we let $\gamma_{l,k} := \gamma_{l-1} \circ \dots \circ \gamma_{k+1} \circ \gamma_k\colon A_k \to A_l$ denote the composition of connecting morphisms. By $\varinjlim \mathcal{A}$ or $\varinjlim A_k$ we denote the inductive limit of an inductive system, and by $\gamma_{\infty,k}\colon A_k \to \varinjlim A_k$ we denote the canonical morphism into the inductive limit.

2.3. A **shape system** for A is an inductive system (A_k, γ_k) such that $A \cong \varinjlim A_k$ and such that the connecting morphisms $\gamma_k\colon A_k \to A_{k+1}$ are semiprojective. Blackadar, [Bla85, Theorem 4.3], shows that every separable C^*-algebra has a shape system consisting of finitely generated C^*-algebras.

Two inductive systems $\mathcal{A} = (A_k, \gamma_k)$ and $\mathcal{B} = (B_n, \theta_n)$ are called **(shape) equivalent**, denoted $\mathcal{A} \sim \mathcal{B}$, if there exist an increasing sequences of indices $k_1 < n_1 < k_2 < n_2 < \dots$ and morphisms $\alpha_i\colon A_{k_i} \to B_{n_i}$ and $\beta_i\colon B_{n_i} \to A_{k_{i+1}}$ such that $\beta_i \circ \alpha_i \simeq \gamma_{k_{i+1},k_i}$ and $\alpha_{i+1} \circ \beta_i \simeq \theta_{n_{i+1},n_i}$ for all i. The situation is shown in the following diagram which commutes up to homotopy.



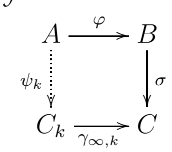
If we have α_i, β_i as above with only $\beta_i \circ \alpha_i \simeq \gamma_{k_{i+1},k_i}$ for all i, then we say $\mathcal{A}$ is **(shape) dominated** by $\mathcal{B}$, denoted $\mathcal{A} \precsim \mathcal{B}$. Of course $\mathcal{A} \sim \mathcal{B}$ implies $\mathcal{A} \precsim \mathcal{B}$ and $\mathcal{B} \precsim \mathcal{A}$, but the converse is false. Nevertheless $\sim$ is an equivalence relation, and $\precsim$ is transitive.

Any two shape systems of a C^*-algebra are equivalent. Given two C^*-algebras A and B we say A is **shape equivalent** to B, denoted $A \sim_{Sh} B$, if they have some shape systems that are equivalent. We say A is **shape dominated** by B, denoted $A \precsim_{Sh} B$, if some shape system of A is dominated by some shape system of B.

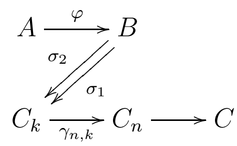
Shape is coarser than homotopy in the following sense: If A and B are homotopy equivalent (denoted $A \simeq B$), then $A \sim_{Sh} B$. Moreover, if A is homotopy dominated by B, then $A \precsim_{Sh} B$.

Theorem 2.4 (Effros, Kaminker, [EK86, 3.2], also Blackadar, [Bla85, Theorem 3.1, 3.3])**.** *Let $\varphi\colon A \to B$ be a semiprojective morphism, and (C_k, γ_k) an inductive system with limit C. Then:*

(1) Let $\sigma\colon B \to C$ be a morphism. Then for k large enough there exist morphisms $\psi_k\colon A \to C_k$ such that $\gamma_{\infty,k} \circ \psi_k \simeq \sigma \circ \varphi$ and such that $\gamma_{\infty,k} \circ \psi_k$ converges pointwise to $\sigma \circ \varphi$. This means that the diagram on the right can be completed to commute up to homotopy.



(2) Let $\sigma_1, \sigma_2\colon B \to C_k$ be two morphisms with $\gamma_{\infty,k} \circ \sigma_1 \simeq \gamma_{\infty,k} \circ \sigma_2$. Then for $n \geq k$ large enough, already the morphisms $\gamma_{n,k} \circ \sigma_1 \circ \varphi$ and $\gamma_{n,k} \circ \sigma_2 \circ \varphi$ are homotopic. The situation is shown in the the diagram on the right.



Remark 2.5. Let us see what the above Theorem 2.4 means for a semiprojective C^*-algebra A. Let (C_k, γ_k) be an inductive system with limit C. Consider the homotopy classes of

morphisms from A to C_k, denoted by $[A, C_k]$. The connecting morphism $\gamma_k \colon C_k \to C_{k+1}$ induces a map $(\gamma_k)_* \colon [A, C_k] \to [A, C_{k+1}]$, and the morphism $\gamma_{\infty,k} \colon C_k \to C$ induces a map $(\gamma_{\infty,k})_* : [A, C_k] \to [A, C]$.

Note that $(\gamma_{\infty,k})_* = (\gamma_{\infty,k+1})_* \circ (\gamma_k)_*$, so that we get a natural map

$$\Phi : \varinjlim [A, C_k] \to [A, \varinjlim C_k] = [A, C].$$

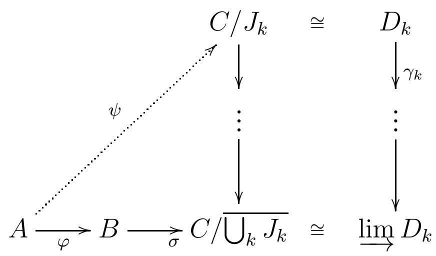
Statement (1) of the above Theorem 2.4 means that Φ is surjective, while statement (2) means exactly that Φ is injective.

Theorem 2.4 is proved using a mapping telescope construction, due to L.G. Brown. The same proof gives the following partial analog of the above result for weakly semiprojective morphisms:

Proposition 2.6. *Let $\varphi \colon A \to B$ be a weakly semiprojective morphism, and (C_k, γ_k) an inductive system with limit C. Let further be given a morphism $\sigma \colon B \to C$, $\varepsilon > 0$ and a finite set $F \subset A$. Then there exist an index k and a morphism $\psi \colon A \to C_k$ such that $\gamma_{\infty,k} \circ \psi =^F_\varepsilon \sigma \circ \varphi$.*

Remark 2.7. Recall from 2.1 that a morphism $\varphi \colon A \to B$ is called weakly semiprojective if the following holds:

(1) Let C be C^*-algebra, and $J_1 \lhd J_2 \lhd \cdots \lhd C$ an increasing sequence of ideals. Let further be given a morphism $\sigma \colon B \to C/\overline{\bigcup_k J_k}$, $\varepsilon > 0$, and a finite subset $F \subset A$. Then there exist an index k and a morphism $\psi \colon A \to C/J_k$ that approximately lifts $\sigma \circ \varphi$, i.e., such that $\pi_k \circ \psi =^F_\varepsilon \sigma \circ \varphi$, where $\pi_k \colon C/J_k \to C/\overline{\bigcup_k J_k}$ is the quotient morphism. This is shown in the left part of the diagram below.



Note that the C^*-algebras C/J_k form an inductive system with inductive limit $C/\overline{\bigcup_k J_k}$. The connecting morphisms $C/J_k \to C/J_{k+1}$ are quotient morphisms and therefore surjective.

Conversely, every inductive system (D_k, γ_k) with surjective connecting morphisms $\gamma_k \colon D_k \to D_{k+1}$ is of the above form. Just set $C := D_1$ and $J_k := \ker(\gamma_{k,1})$. Then $C/J_k \cong D_k$, and $C/\overline{\bigcup_k J_k} \cong \varinjlim_k D_k$. This is shown in the right part of the diagram.

Thus, the definition of weak semiprojectivity of φ can be reformulated as follows:

(2) Let (D_k, γ_k) an inductive system *with surjective connecting morphisms*. Let further be given a morphism $\sigma \colon B \to \varinjlim D_k$, $\varepsilon > 0$ and a finite set $F \subset A$. Then there exist an index k and a morphism $\psi \colon A \to D_k$ that approximately lifts $\sigma \circ \varphi$, i.e., such that $\gamma_{\infty,k} \circ \psi =^F_\varepsilon \sigma \circ \varphi$.

Thus, for the definition of weak semiprojectivity, we consider morphisms into the limit of an inductive system with surjective connecting morphisms, and we ask for approximate lifts. Proposition 2.6 says that one may drop the condition that the connecting morphisms of the inductive system are surjective.

2.8 (Generators for C^*-algebras). Let A be a C^*-algebra. A subset $S \subset A_{\text{sa}}$ of self-adjoint elements is said to generate A, denoted $A = C^*(S)$, if A is the smallest sub-C^*-algebra of

A containing S. The **generating rank** for A, denoted by $\mathrm{gen}(A)$, is the smallest number $n \in \{1, 2, 3, \ldots, \infty\}$ such that A contains a generating set S of n self-adjoint elements.

Note that the generators are assumed to be self-adjoint. If g, h are two self-adjoint elements, then $\{g, h\}$ generates the same sub-C^*-algebra as the element $g + ih$. That is why a C^*-algebra is said to be singly generated if $\mathrm{gen}(A) \leq 2$.

For more details on the generator rank and its behaviour with respect to operations, we refer the reader to Nagisa, [Nag04].

Remark 2.9 (Finitely generated = finitely presented)**.** While it is rather clear what it means that a C^*-algebra is finitely generated, it is not so obvious what it should mean that it is finitely presented. To speak of finite presentation, one needs a theory of universal C^*-algebras defined by generators and relations.

Depending on which relations one considers, one gets different notions of finite presentability. In [Bla85], for instance, only polynomial relations are considered. With this definition, not every finitely generated C^*-algebra is finitely presented.

More generally, one can define a relation to be an element of the universal C^*-algebra generated by a countable number of contractions

$$\mathcal{F}_\infty := C^*(x_1, x_2, \ldots \mid \|x_i\| \leq 1).$$

This definition is used in [Lor97], and it is flexible enough to show that every finitely generated C^*-algebra is already finitely presented, see [ELP98, Lemma 2.2.5].

Thus, in the results of [Lor97] we may replace the assumption of finite presentation by finite generation, e.g. in [Lor97, Lemma 15.2.1, 15.2.2, p.118f]. This can be improved even further, as was shown to the author by Chigogidze and Loring, [CL11]: One may give a version of [Lor97, Lemma 15.2.1, p.118] which does not require the C^*-algebra to be finitely generated, see Lemma 3.8. It follows that [Lor97, Lemma 15.2.2, p.119] remains true if one drops the assumption of finite generation (or presentation) completely, see Corollary 3.10.

3. Approximation and criteria for inductive limits

In this section we will give criteria that allow one to write a C^*-algebra A as an inductive limit of other C^*-algebras that approximate A in a nice way. We start by reviewing the various ways a C^*-algebra can be 'approximated' by other C^*-algebras, see 3.1. If $\mathcal{C}$ is a class of C^*-algebras, then an inductive limit of algebras in $\mathcal{C}$ is called an $A\mathcal{C}$-algebra. We suggest to use the formulation that A is '$\mathcal{C}$-like' if it can be approximated by sub-C^*-algebras from the class $\mathcal{C}$, see Definition 3.2 and Proposition 3.4.

As a basic tool to construct an inductive limit decomposition we use one-sided approximate intertwinings, see Proposition 3.5. These were introduced by Elliott in [Ell93, 2.1, 2.3] and they turned out to be very important in the classification of C^*-algebras, see also chapter 2.3 of Rørdam's book, [Rør02].

Assuming that the class $\mathcal{C}$ consists of weakly semiprojective C^*-algebras, we deduce other criteria to write a C^*-algebra as an inductive limit of building blocks in $\mathcal{C}$. In particular, every $A\mathcal{C}$-like C^*-algebra is an $A\mathcal{C}$-algebras, see Theorem 3.9, and every $AA\mathcal{C}$-algebra is already an $A\mathcal{C}$-algebra, see Theorem 3.12. The latter statement gives a criterion when an 'inductive limit of inductive limits is an inductive limit'.

For example, let $\mathcal{C}$ be the class of finite direct sums of matrices over the circle algebra $C(\mathbb{T})$. Then the mentioned result means that an inductive limit of $A\mathbb{T}$-algebras is itself an

$A\mathbb{T}$-algebra. This is a well-known result, see e.g. [LR95, Proposition 2] which is based on [Ell93, Theorem 4.3].

3.1 (Approximation)**.** The term 'approximation' is used in various contexts. For instance, if $\mathcal{P}$ is some property that C^*-algebras might enjoy, then a C^*-algebra is usually called **approximately** $\mathcal{P}$, or an $A\mathcal{P}$-algebra, if it can be written as an inductive limit of C^*-algebras with property $\mathcal{P}$. In this sense one speaks of 'approximately homogeneous' and 'approximately subhomogeneous' C^*-algebras.

Another concept is approximation by subalgebras. Given a C^*-algebra A, a family $\mathcal{B}$ of sub-C^*-algebras is said to **approximate** A if for every $\varepsilon > 0$ and finite subset $F \subset A$ there exists some algebra $B \in \mathcal{B}$ such that $F \subset_\varepsilon B$. In the literature there appears also the terminology '$\mathcal{B}$ locally approximates A'. Similarly, if $\mathcal{P}$ is some property of C^*-algebras, then a C^*-algebra A that can be approximated by sub-C^*-algebras with property $\mathcal{P}$ is sometimes called 'locally $\mathcal{P}$'. In this sense one speaks of 'locally (sub)homogeneous' C^*-algebras.

However, sometimes the word 'local' might lead to confusion: Consider for instance the property of being contractible. We will show below, see Corollary 4.7, that a C^*-algebra has trivial shape if it is approximated by contractible sub-C^*-algebras. One could phrase this as 'locally contractible C^*-algebras have trivial shape', but this would be in contradiction with the terminology used for spaces. Many locally contractible[1] spaces have non-trivial shape.

The confusion is due to the contravariant duality between spaces and C^*-algebras. If we consider for instance a commutative C^*-algebra $C(X)$, then the elements $f \in C(X)$ are almost constant around each point $x \in X$. Therefore, an approximation of $C(X)$ by sub-C^*-algebras does not captures the local structure of X, it rather captures the global structure of X, its shape. To prevent confusion, we suggest the following definition:

Definition 3.2. If $\mathcal{P}$ is some property that C^*-algebras might enjoy, then a C^*-algebra is called **$\mathcal{P}$-like** if it can be approximated by sub-C^*-algebras with property $\mathcal{P}$.

Remark 3.3 ($\mathcal{P}$-likeness)**.** Using the above definition, Corollary 4.7 would read as: 'A contractible-like C^*-algebra has trivial shape'. This might sound cumbersome, but it is motivated by the concept of $\mathcal{P}$-likeness for spaces, as defined by Mardesic and J. Segal, [MS63, Definition 1], and further developed by Mardešić and Matijević, [MM92]. In Proposition 3.4 we will show that for commutative C^*-algebras both concepts agree.

For a space X, we let $\mathrm{Cov}(X)$ denote the collection of finite, open covers of X. Given $\mathcal{U}_1, \mathcal{U}_1 \in \mathrm{Cov}(X)$, we write $\mathcal{U}_1 \leq \mathcal{U}_2$ if the cover $\mathcal{U}_1$ refines the cover $\mathcal{U}_2$, i.e., for every $U \in \mathcal{U}_1$ there exists some $U' \in \mathcal{U}_2$ such that $U \subset U'$. We refer the reader to chapter 2 of Nagami's book [Nag70] for details.

We are working in the category of pointed spaces and pointed maps since it is the natural setting to study non-unital commutative C^*-algebras, as pointed out in [Bla06, II.2.2.7, p.61]. If we include basepoints and restrict to compact spaces, then the definition of $\mathcal{P}$-likeness from [MM92, Definition 1.2] becomes: Let $\mathcal{P}$ be a non-empty class of pointed, compact, Hausdorff spaces. A pointed, compact, Hausdorff space X is said to be $\mathcal{P}$-like

[1]A space X is called locally contractible if for each point $x \in X$ and every neighborhood U of x there exists a neighborhood V of x such that $V \subset U$ and V is contractible (in itself).

if for every $\mathcal{U} \in \mathrm{Cov}(X)$ there exists a pointed map $f\colon X \to Y$ onto some $Y \in \mathcal{P}$ and $\mathcal{V} \in \mathrm{Cov}(Y)$ such that $f^{-1}(\mathcal{V}) \leq \mathcal{U}$, where $f^{-1}(\mathcal{V}) = \{f^{-1}(V) \mid V \in \mathcal{V}\}$.

If X and all space in $\mathcal{P}$ are pointed, compact, metric spaces, then one can show that X is $\mathcal{P}$-like if and only if for every $\varepsilon > 0$ there exists a pointed map $f\colon X \to Y$ onto some $Y \in \mathcal{P}$ such that the sets $f^{-1}(y)$ have diameter $< \varepsilon$ (for all $y \in Y$). This equivalent formulation is the original definition of $\mathcal{P}$-likeness for compact, metric spaces, [MS63, Definition 1].

Note that we have used $\mathcal{P}$ to denote both a class of spaces and a property that spaces might enjoy. These are just different viewpoints, as we can naturally assign to a property the class of spaces with that property, and vice versa to each class of spaces the property of lying in that class.

Let us use the following notation for the next result: If (X, x_∞) is a pointed space, then $C_0(X, x_\infty) = \{a\colon X \to \mathbb{C} \mid a(x_\infty) = 0\}$ denotes the C^*-algebra of continuous functions on X vanishing at the basepoint.

Proposition 3.4. *Let X be a pointed, compact, Hausdorff space, and let $\mathcal{P}$ be a class of pointed, compact, Hausdorff spaces. Then the following are equivalent:*

(a) X is $\mathcal{P}$-like.
(b) $C_0(X, x_\infty)$ can be approximated by sub-C^*-algebras $C_0(Y, y_\infty)$ with $(Y, y_\infty) \in \mathcal{P}$.

Proof. '(a) ⇒ (b)': Assume we are given $\varepsilon > 0$ and a finite subset $F \subset C_0(X, x_\infty)$. Since X is compact, there exists a finite, open cover $\mathcal{U} \in \mathrm{Cov}(X)$ such that $\|a(x) - a(x')\| < \varepsilon$ whenever $a \in F$ and x, x' lie in some set $U \in \mathcal{U}$. By assumption, there is a pointed map $f\colon X \to Y$ onto some space $Y \in \mathcal{P}$ and $\mathcal{V} \in \mathrm{Cov}(Y)$ such that $f^{-1}(\mathcal{V}) \leq \mathcal{U}$. Note that f induces an inclusion $f^*\colon C_0(Y, y_\infty) \to C_0(X, x_\infty)$.

Choose a partition of unity $\{e_V\}_{V \in \mathcal{V}}$ in Y that is subordinate to $\mathcal{V}$. For each $V \in \mathcal{V}$, choose a point $x_V \in f^{-1}(V)$ such that $x_V = x_\infty$ if $y_\infty \in V$. Given $a \in F$, let us show that $a \in_\varepsilon f^*(C_0(Y, y_\infty)) \subset C_0(X, x_\infty)$. Set $b := \sum_V a(x_V)e_V$, and note that $b(y_\infty) = 0$ since $a(x_V) = 0$ whenever $e_V(y_\infty) \neq 0$. For $x \in X$ we compute:

$$\begin{aligned}
\|a(x) - f^*(b)(x)\| &= \|a(x) - \sum_V a(x_V)e_V(f(x))\| \\
&= \|\sum_V (a(x) - a(x_V))e_V(f(x))\| \\
&< \varepsilon \cdot \|\sum_V e_V(f(x))\| \\
&= \varepsilon.
\end{aligned}$$

To see that the inequality in the computation above holds, note that $e_V(f(x)) \neq 0$ only if $f(x) \in V$, but then $x, x_V \in f^{-1}(V)$ which is contained in some set $U \in \mathcal{U}$, and so $\|a(x) - a(x_V)\| < \varepsilon$.

We get $F \subset_\varepsilon f^*(C_0(Y, y_\infty))$. Since ε and F were arbitrary, this shows that $C_0(X, x_\infty)$ is approximated by sub-C^*-algebras from $\mathcal{P}$, as desired.

'(b) ⇒ (a)': Let $\mathcal{U} = \{U_\alpha\} \in \mathrm{Cov}(X)$ be a finite, open cover of X. We need to find a space $Y \in \mathcal{P}$ together with a pointed, surjective map $f\colon X \to Y$ and $\mathcal{V} \in \mathrm{Cov}(V)$ such that $f^{-1}(\mathcal{V}) \leq \mathcal{U}$.

By passing to a refinement, we may assume that x_∞ is contained in just one U_α, call it U_∞. Since X is a normal space, we may find open sets $V_\alpha \subset X$ such that $V_\alpha \subset \overline{V_\alpha} \subset U_\alpha$ and such that $\{V_\alpha\}$ is a cover of X. By Urysohn's lemma, there are continuous functions $a_\alpha \colon X \to \mathbb{C}$ that are 1 on $\overline{V_\alpha}$ and zero on $X \setminus U_\alpha$. Note that a_α vanishes on x_∞ for $\alpha \neq \infty$, so that $a_\alpha \in C_0(X, x_\infty)$ for $\alpha \neq \infty$.

From (b) we get a sub-C^*-algebra $C_0(Y, y_\infty)$ of $C_0(X, x_\infty)$ that contains the a_α ($\alpha \neq \infty$) up to $1/2$ and such that $(Y, y_\infty) \in \mathcal{P}$. The embedding corresponds to a pointed, surjective map $f \colon (X, x_\infty) \to (Y, y_\infty)$. For $\alpha \neq \infty$, let $b_\alpha \in C_0(Y, y_\infty)$ be elements such that $\|a_\alpha - f^*(b_\alpha)\| < 1/2$.

Define sets $W_\alpha \subset Y$ via:

$$W_\alpha := \{y \in Y \mid \|b_\alpha(y)\| > 1/2\} \quad (\text{ for } \alpha \neq \infty\ ), \qquad W_\infty := Y \setminus f(\bigcup_{\alpha \neq \infty} \overline{V_\alpha}).$$

We compute:

$$\begin{aligned} f^{-1}(W_\alpha) &= \{x \in X \mid \|b_\alpha(f(x))\| > 1/2\} && (\text{ for } \alpha \neq \infty\ ) \\ &\subset \{x \in X \mid \|a_\alpha(x)\| > 0\} \subset U_\alpha \\ f^{-1}(W_\infty) &\subset X \setminus \bigcup_{\alpha \neq \infty} \overline{V_\alpha} \subset U_\infty \\ f^{-1}(W_\alpha) &\supset \{x \in X \mid \|a_\alpha(x)\| \geq 1\} \supset \overline{V_\alpha} && (\text{ for } \alpha \neq \infty\ ) \end{aligned}$$

It follows $f^{-1}(\bigcup_{\alpha \neq \infty} W_\alpha) \supset \bigcup_{\alpha \neq \infty} \overline{V_\alpha}$, and so $\bigcup_{\alpha \neq \infty} W_\alpha \supset f(\bigcup_{\alpha \neq \infty} \overline{V_\alpha}) = Y \setminus W_\infty$. This shows that $\mathcal{W} := \{W_\alpha\}$ is a cover of Y and that $f^{-1}(\mathcal{W}) \leq \mathcal{U}$, as desired. □

The following result formalizes the construction of a (special) one-sided approximate intertwining. The idea goes back to Elliott, [Ell93, 2.3,2.4], see also chapter 2.3 of Rørdam's book, [Rør02]. Note that the version given here does not appear in the literature so far. In particular, we do not require any ordering on the index set of approximating algebras.

Proposition 3.5 (One-sided approximate intertwining). *Let A be a separable C^*-algebra, and A_i ($i \in I$) a collection of separable C^*-algebras together with morphisms $\varphi_i \colon A_i \to A$.*

Assume that the following holds: For every index $i \in I$, and $\varepsilon > 0$, and for every finite subsets $F \subset A_i$, $E \subset \ker(\varphi_i)$ and $H \subset A$, there exists some index $j \in I$ and a morphism $\psi \colon A_i \to A_j$ such that:

- *($\mathfrak{A}1$)* $\varphi_j \circ \psi =^F_\varepsilon \varphi_i$,
- *($\mathfrak{A}2$)* $\psi =^E_\varepsilon 0$,
- *($\mathfrak{A}3$)* $H \subset_\varepsilon \mathrm{im}(\varphi_j) = \varphi_j(A_j) \subset A$.

Then A is isomorphic to an inductive limit of some of the algebras A_i. More precisely, there exist indices $i(1), i(2), \ldots \in I$ and morphisms $\psi_k \colon A_{i(k)} \to A_{i(k+1)}$ such that $A \cong \varinjlim_k (A_{i(k)}, \psi_k)$.

Proof. By induction, we will construct a one-sided approximate intertwining as shown in the following diagram. This diagram does not commute, but it 'approximately commutes'.

$$\begin{array}{ccccccccc} A_{i(1)} & \xrightarrow{\psi_1} & A_{i(2)} & \xrightarrow{\psi_2} & A_{i(2)} & \longrightarrow & \dots & \longrightarrow & B \\ \downarrow{\scriptstyle \varphi_{i(1)}} & & \downarrow{\scriptstyle \varphi_{i(2)}} & & \downarrow{\scriptstyle \varphi_{i(3)}} & & & & \downarrow{\scriptstyle \omega} \\ A & \longrightarrow & A & \longrightarrow & A & \longrightarrow & \dots & \longrightarrow & A \end{array}$$

Property ($\mathfrak{A}$1) is the essential requirement for constructing the one-sided approximate intertwining, i.e., to align some of the algebras A_i into an inductive system with limit B together with a canonical morphism $\omega\colon \varinjlim \mathcal{B} \to A$. Property ($\mathfrak{A}$2) is used to get ω injective, and ($\mathfrak{A}$3) is used to ensure ω is surjective.

More precisely, we proceed as follows: Let $\{x_1, x_2 \dots\} \subset A$ be a dense sequence in A with $x_1 = 0$. We will construct the following:

- indices $i(k) \in I$, for $k \in \mathbb{N}$,
- morphisms $\psi_k\colon A_{i(k)} \to A_{i(k+1)}$, for $k \in \mathbb{N}$,
- finite subsets $F_k^1 \subset F_k^2 \subset \dots \subset A_{i(k)}$, for $k \in \mathbb{N}$,
- finite sets $E_k' \subset \ker(\varphi_{i(k)})$, for $k \in \mathbb{N}$,

such that:

(a) $\psi_k(F_k^l) \subset F_{k+1}^l$, for all $k, l \geq 1$,
(b) $\bigcup_l F_k^l$ is dense in $A_{i(k)}$, for each k,
(c) E_k' contains $E_k := \{x \in F_k^k \ : \ \|\varphi_{i(k)}(x)\| < 1/2^{k-1}\}$ up to $1/2^{k-1}$, for each k,
(d) $\varphi_{i(k+1)} \circ \psi_k =_{1/2^k}^{F_k^k} \varphi_{i(k)}$, for each k,
(e) $\psi_k =_{1/2^k}^{E_k'} 0$, for each k,
(f) $\{x_1, \dots, x_k\} \subset_{1/2^k} \varphi_{i(k)}(F_k^k)$, for each k.

Let us start with any $i(1)$, e.g. $i(1) = 1$. Since $x_1 = 0$, (f) is satisfied. We may find sets F_1^i and E_1' to fulfill properties (a), (b) and (c).

Let us manufacture the induction step from k to $k+1$. We consider the index $i(k)$, the tolerance $1/2^{k+1}$, and the finite sets $F_k^k \subset A_{i(k)}$, $E_k' \subset \ker(\varphi_{i(k)})$, and $\{x_1, \dots, x_{k+1}\} \subset A$. By assumption, there is an index $i(k+1)$, and a morphism $\psi_k\colon A_{i(k)} \to A_{i(k+1)}$ satisfying conditions (d), (e) and (f). Then construct sets F_{k+1}^l and E_{k+1}' to fulfill properties (a), (b) and (c).

Set $B := \varinjlim_k (A_{i(k)}, \psi_k)$. We want to define morphisms $\omega_k\colon A_{i(k)} \to A$ as

$$\omega_k(a) := \lim_s \varphi_{i(s)} \circ \psi_{s,k}(a).$$

This makes sense since $\varphi_{i(s)} \circ \psi_{s,k}(a)$ is a Cauchy sequence (when running over s), which may be checked using properties (b) and (d).

Note that $\omega_l \circ \psi_{l,k} = \omega_k$ for any $l \leq k$. Thus, the morphisms ω_k fit together to define a morphism $\omega\colon B \to A$.

Injectivity of ω: Given any k, and an element $a \in F_k^k$, one computes $\|\omega_k(a) - \varphi_{i(k)}(a)\| \leq 1/2^{k-1}$, using property (d). The construction was made in such a way that we can distinguish two different cases:

- Case 1: $\|\varphi_{i(k)}(a)\| \geq 1/2^{k-1}$. In that case $\omega_k(a) \neq 0$, since above we computed $\|\omega_k(a) - \varphi_{i(k)}(a)\| < 1/2^{k-1}$.

- Case 2: $\|\varphi_{i(k)}(a)\| < 1/2^{k-1}$. In that case, by (c), there exists some $e \in E'_k$ with $\|a - e\| < 1/2^{k-1}$. From (e) we get $\|\psi_k(e)\| < 1/2^k$. We compute: $\|\psi_k(a)\| = \|\psi_k(a - e + e)\| \leq 1/2^{i-k} + 1/2^k \leq 1/2^{k-2}$.

This means: either the given $a \in F_k^k$ has non-zero image in A under the morphism ω_k, or otherwise it has a small image in B under the morphism $\psi_{\infty,k} = \psi_{\infty,k+1} \circ \psi_k$. By considering $\psi_{l,k}(a) \in F_l^l$ for all $l \geq k$ we derive that either $\omega_k(a) = \omega_l(\psi_{l,k}(a)) \neq 0$ or $\|\psi_{\infty,k}(a)\| = \|\psi_{\infty,l} \circ \psi_{l,k}(a)\| \leq 1/2^{k-2}$ for all $k \geq l$. We get that for any $a \in F_k^k$ we have $a \in \ker(\omega_k)$ if and only if $a \in \ker(\psi_{\infty,k})$.

Next we consider $a \in F_k^l$ for $l \geq k$. Then with $b := \psi_{l,k}(a) \in F_l^l$ we deduce:

$$a \in \ker(\omega_k) \quad \Leftrightarrow \quad b \in \ker(\omega_l) \quad \Leftrightarrow \quad b \in \ker(\psi_{\infty,l}) \quad \Leftrightarrow \quad a \in \ker(\psi_{\infty,k})$$

Since $\bigcup_l F_k^l$ is dense in $A_{i(k)}$, we get $\ker(\omega_k) = \ker(\psi_{\infty,k})$. Then $\ker(\omega) = \overline{\bigcup_k \psi_{\infty,k}(\omega_k)} = \overline{\bigcup_k 0} = 0$, and so ω is injective.

Surjectivity of ω: Let $a \in A$ and $\varepsilon > 0$. We want to check that $a \in_\varepsilon \operatorname{im}(\omega)$. Since the sequence $x_1, x_2, \dots$ is dense in A, there exists some l with $\|a - x_l\| < \varepsilon/4$. Let $k \geq l$ be a number with $1/2^{k-1} < \varepsilon/4$. We have seen above that $\omega_k =^{F_k^k}_{1/2^{k-1}} \varphi_{i(k)}$. Then:

$$a \quad \in_{\varepsilon/4} \quad \{x_1, \dots, x_k\} \quad \subset_{1/2^k} \quad \varphi_{i(k)}(F_k^k) \quad \subset_{1/2^{k-1}} \quad \omega_k(F_k^k) \quad \subset \quad \operatorname{im}(\omega).$$

Together, a lies in $\operatorname{im}(\omega)$ up to $\varepsilon/4 + 1/2^i + 1/2^{k-1} < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we deduce $a \in \operatorname{im}(\omega)$, and so ω is surjective. □

3.6. Let us consider a weaker approximation than in Proposition 3.5, where we relax condition ($\mathfrak{A}$3). Let us assume the following situation is given:

Let A be a separable C^*-algebra, and $\{A_i\}_{i \in I}$ a collection of separable C^*-algebras together with morphisms $\varphi_i \colon A_i \to A$, such that the following holds: For every index $i \in I$, and $\varepsilon > 0$, and for every finite subsets $F \subset A_i$ and $E \subset \ker(\varphi_i)$, there exists some index j and a morphism $\psi \colon A_i \to A_j$ such that:

($\mathfrak{A}$1) $\varphi_j \circ \psi =^F_\varepsilon \varphi_i$,
($\mathfrak{A}$2) $\psi =^E_\varepsilon 0$,

and moreover, the following condition holds:

($\mathfrak{A}$3') the collection of sub-C^*-algebras $\operatorname{im}(\varphi_k) = \varphi_k(A_k) \subset A$ approximates A

Condition ($\mathfrak{A}$3) of 3.5 is a statement about the morphism ψ. It roughly says that ψ has 'large' image. The above condition ($\mathfrak{A}$3') is independent of the morphism ψ. It just requires that the collection of all sub-C^*-algebras $\operatorname{im}(\varphi_k)$ is 'large'.

Adopting the proof of Proposition 3.5, we may construct one-sided approximate intertwinings to get the following result: For every $\gamma > 0$ and every finite $H \subset A$, there exists a sub-C^*-algebra $B \subset A$ such that $H \subset_\gamma B$ and B is an inductive limit of some of the algebras A_i.

If we denote by $\mathcal{C} = \{A_i \mid i \in I\}$ the class of approximating algebras, then this means precisely that A is $A\mathcal{C}$-like, i.e., A is approximated by sub-C^*-algebras that are inductive limits of algebras in $\mathcal{C}$.

In general, this does not imply that A is an $A\mathcal{C}$-algebra, i.e., an inductive limit of algebras in $\mathcal{C}$. In fact, not even a $\mathcal{C}$-like C^*-algebra need to be an $A\mathcal{C}$-algebra, as can be seen by the following example.

Example 3.7 (Dadarlat, Eilers, [Dad99])**.** Let us denote by H the class of (direct sums of) homogeneous C^*-algebras. An inductive limit of C^*-algebras in H is called an AH-algebra. In [Dad99], Dadarlat and Eilers construct a C^*-algebra $A = \varinjlim_k A_k$ that is an inductive limit of AH-algebras A_k (so A is an AAH-algebra) but such that A is not an AH-algebra itself. Thus, an $AA\mathcal{C}$-algebra in general need not be an $A\mathcal{C}$-algebra.

Since quotients of homogeneous algebras are again homogeneous, the C^*-algebra A is also H-like. Thus, the example also shows that a $\mathcal{C}$-like algebra in general need not be an $A\mathcal{C}$-algebra.

In the example of Dadarlat and Eilers, each A_k is an inductive limit, $\varinjlim_n A_k^n$, of C^*-algebras A_k^n that have the form $\bigoplus_{i=1}^d M_{d_i}(C(X_i))$ with each X_i a three-dimensional CW-complex. It is well-known that $C(X_i)$ is not weakly semiprojective if X_i contains a copy of the two-dimensional disc, see, e.g., [ST11, Remark 3.3]. It follows that the algebras A_k^n are not weakly semiprojective. This is the crucial point, as will be shown in Theorem 3.9 and Theorem 3.12.

The following Lemma 3.8 is a variant of [Lor97, Lemma 15.2.1, p.118] that avoids the assumption of finite generation, see Remark 2.9. It was shown to the author by Chigogidze and Loring, [CL11]. The result is used in the proof of Theorem 3.9 to 'twist' morphisms from weakly semiprojective C^*-algebras.

We note that Theorem 3.12 and all results in Section 4 and Section 5 can be proved using the original [Lor97, Lemma 15.2.1, p.118] instead of Lemma 3.8.

Lemma 3.8 (Chigogidze and Loring, [CL11], see also Loring, [Lor97, Lemma 15.2.1, p.118])**.** *Suppose A is a weakly semiprojective C^*-algebra. Then for every $\varepsilon > 0$, and every finite subset $F \subset A$, there exists $\delta > 0$ and a finite subset $G \subset A$ such that the following holds: Whenever $\varphi : A \to B$ is a morphism, and $C \subset B$ is a sub-C^*-algebra that contains $\varphi(G)$ up to δ, then there exists a morphism $\psi : A \to C$ such that $\psi =_\varepsilon^F \varphi$.*

Theorem 3.9. *Let $\mathcal{C}$ be a class of weakly semiprojective C^*-algebras. Then every separable $A\mathcal{C}$-like C^*-algebra is already an $A\mathcal{C}$-algebra.*

Proof. Assume A is an $A\mathcal{C}$-like C^*-algebra. We want to apply the one-sided approximate intertwining, Proposition 3.5, to show that A is an $A\mathcal{C}$-algebra. For this we consider the collection of all morphisms $\varphi\colon C \to A$ where C is a C^*-algebra from $\mathcal{C}$ (we may think of this collection as being indexed over $\coprod_{C\in\mathcal{C}} \mathrm{Hom}(C, A)$).

We need to check the requirements for Proposition 3.5. So assume the following data is given: A morphism $\varphi\colon C \to A$ with $C \in \mathcal{C}$, a tolerance $\varepsilon > 0$, and finite subsets $F \subset C$, $E \subset \ker(\varphi)$ and $H \subset A$. We may assume that F contains E. We need to find a C^*-algebra $C' \in \mathcal{C}$ together with a morphism $\varphi'\colon C' \to A$, and a morphism $\psi\colon C \to C'$ such that ($\mathfrak{A}$1), ($\mathfrak{A}$2), and ($\mathfrak{A}$3) are satisfied.

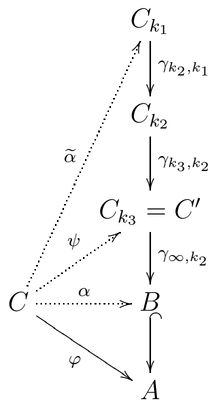
Applying the above variant of [Lor97, Lemma 15.2.1, p.118], see Lemma 3.8, to the weakly semiprojective C^*-algebra C for $\varepsilon/3$ and $F \subset C$, we obtain a $\delta > 0$ and a finite subset $G \subset C$ such that any morphism out of C that maps G up to δ into a given sub-C^*-algebra can be twisted to map exactly into that sub-C^*-algebra while moving F at most by ε. We may assume that $\delta \leq \varepsilon/3$.

Set $H' := H \cup \varphi(G)$, which is a finite subset of A. By assumption, there exists a sub-C^*-algebra $B \subset A$ that contains H' up to δ and which is an $A\mathcal{C}$-algebra, say $B = \varinjlim_k C_k$

with connecting morphisms $\gamma_k\colon C_k \to C_{k+1}$. Since $\varphi(G) \subset_\delta B$, there exists a morphism $\alpha\colon C \to B$ such that $\varphi =^F_{\varepsilon/3} \alpha$.

By Proposition 2.6, the morphism $\alpha\colon C \to B = \varinjlim_k C_k$ has an approximate lift, i.e., there exists an index k_1 and a morphism $\widetilde{\alpha}\colon C \to C_{k_1}$ such that $\alpha =^F_{\varepsilon/3} \gamma_{\infty,k_1} \circ \widetilde{\alpha}$. Then $\varphi =^F_{2\varepsilon/3} \gamma_{\infty,k_1} \circ \widetilde{\alpha}$.

Upon going further down in the inductive limit, we can guarantee properties we need to check. This is shown in the diagram on the right.



Step 1 (in order to guarantee ($\mathfrak{A}2$)): We consider E. Since $\varphi =^F_{2\varepsilon/3} \gamma_{\infty,k_1} \circ \widetilde{\alpha}$ and $E \subset F$, we have $\gamma_{\infty,k_1} \circ \widetilde{\alpha} =^E_{2\varepsilon/3} \varphi =^E 0$. Thus, we may find $k_2 \geq k_1$ such that $\gamma_{k_2,k_1} \circ \widetilde{\alpha} =^E_\varepsilon 0$.

Step 2 (in order to guarantee ($\mathfrak{A}3$)): Since $H \subset_\delta B = \varinjlim_k C_k$, we may find $k_3 \geq k_2$ such that $H \subset_{2\delta} \mathrm{im}(\gamma_{\infty,k_3})$.

Setting $C' := C_{k_2}$, $\varphi' := \gamma_{\infty,k_3}$ and $\psi := \gamma_{k_3,k_1} \circ \widetilde{\alpha}\colon C \to C' = C_{k_3}$, it is easy to check that ($\mathfrak{A}1$), ($\mathfrak{A}2$), and ($\mathfrak{A}3$) are satisfied. □

Corollary 3.10 (Loring, [Lor97, Lemma 15.2.2, p.119])**.** *Let $\mathcal{C}$ be a class of weakly semiprojective C^*-algebras. Then every $\mathcal{C}$-like C^*-algebra is an $A\mathcal{C}$-algebra.*

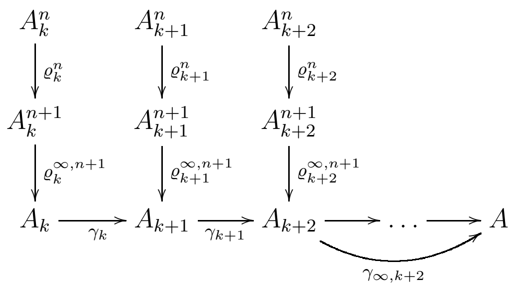
Remark 3.11. Let $\mathcal{C}$ be a class of weakly semiprojective C^*-algebras. If $\mathcal{C}$ is closed under quotients, then every $A\mathcal{C}$-like C^*-algebra is also $\mathcal{C}$-like, and similarly every $AA\mathcal{C}$-algebra is $\mathcal{C}$-like. Then Theorem 3.9 and Theorem 3.12 follow from Loring's local test for inductive limits, [Lor97, Lemma 15.2.2, p.119], see Corollary 3.10.

However, in Section 4 we will consider the class $\mathcal{P}$ of projective C^*-algebras, and this class is not closed under quotients. There even exist $A\mathcal{P}$-like C^*-algebras that are not $\mathcal{P}$-like: Consider for example the commutative C^*-algebra $A = C_0([0,1]^2 \setminus \{(0,0)\})$, which is contractible and hence $A\mathcal{P}$-like (even an $A\mathcal{P}$-algebra) by Corollary 4.5. Every sub-C^*-algebra of A is commutative, and it was shown by Chigogidze and Dranishnikov, [CD10], that every commutative projective C^*-algebra has one-dimensional spectrum. In particular, every commutative projective C^*-algebra has stable rank one, and if A was approximated by such sub-C^*-algebras, then A would have stable rank one as well, which contradicts the fact that the stable rank of A is two.

Therefore, in order to obtain Theorem 4.6 (2) and (3), it is crucial that Theorem 3.12 and Theorem 3.9 also hold for classes $\mathcal{C}$ that are not necessarily closed under quotients.

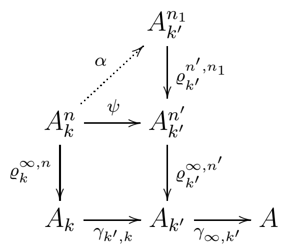
Theorem 3.12. *Let $\mathcal{C}$ be a class of weakly semiprojective C^*-algebras. Then every $AA\mathcal{C}$-algebra is already an $A\mathcal{C}$-algebra.*

Proof. Assume $A \cong \varinjlim_k A_k$ and $\varrho^n_k\colon A^n_k \to A^{n+1}_k$ for algebras $A^n_k \in \mathcal{C}$. Let us denote the connecting morphisms by $\gamma_k\colon A_k \to A_{k+1}$ and $\varrho^n_k\colon A^n_k \to A^{n+1}_k$. We are given the following situation:



We want to use the one-sided approximate intertwining, Proposition 3.5, and we consider the collection of C^*-algebras A_k^n together with morphisms $\varphi_{k,n} := \gamma_{\infty,k} \circ \varrho_k^{\infty,n} \colon A_k^n \to A$ (we may think of this collection as being indexed over $\mathbb{N} \times \mathbb{N}$).

Assume some indices k, n are given together with $\varepsilon > 0$, and with finite sets $F \subset A_k^n$, $E \subset \ker(\varphi_{k,n})$ and $H \subset A$. We may assume $E \subset F$. We need to find k', n' and a morphism $\psi \colon A_k^n \to A_{k'}^{n'}$ that satisfy ($\mathfrak{A}$1) and ($\mathfrak{A}$2) and ($\mathfrak{A}$3).

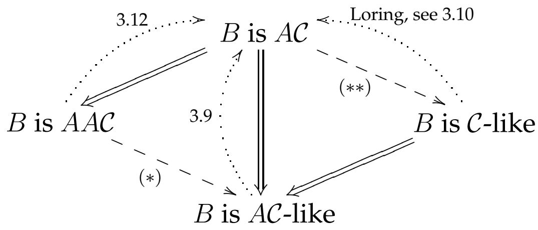


Since $A = \varinjlim_k A_k$ and $\varphi_{k,n} = \gamma_{\infty,k} \circ \varrho_k^{\infty,n} =^E 0$, there exists some $k' \geq k$ such that $\gamma_{k',k} \circ \varrho_k^{\infty,n} =^E_{\varepsilon/3} 0$. We can also ensure that $H \subset_{\varepsilon/2} \operatorname{im}(\gamma_{\infty,k'})$, by further increasing k', if necessary.

Since A_k^n is weakly semiprojective, we may lift the morphism $\gamma_{k',k} \circ \varrho_k^{\infty,n} \colon A_k^n \to A_{k'} = \varinjlim_n A_{k'}^n$ to some $\alpha \colon A_k^n \to A_{k'}^{n_1}$ (for some n_1) such that $\varrho_{k'}^{\infty,n_1} \circ \alpha =^F_{\varepsilon/3} \gamma_{k',k} \circ \varrho_k^{\infty,n}$. This is shown in the diagram on the right.

We have $\varrho_{k'}^{\infty,n_1} \circ \alpha =^E_{\varepsilon/3} \gamma_{k',k} \circ \varrho_k^{\infty,n} =^E_{\varepsilon/3} 0$. As in the proof of Theorem 3.9, by going further down in the inductive limit we may find $n' \geq n_1$ such that $\varrho_{k'}^{n',n_1} \circ \alpha =^E_{\varepsilon} 0$ and $H \subset_\varepsilon \operatorname{im}(\gamma_{\infty,k'} \circ \varrho_{k'}^{\infty,n'})$. Set $\psi := \varrho_{k'}^{n',n_1} \circ \alpha \colon A_k^n \to A_{k'}^{n'}$. It is easy to check that ($\mathfrak{A}$1), ($\mathfrak{A}$2), and ($\mathfrak{A}$3) are satisfied. □

3.13. Let B be a separable C^*-algebra, and $\mathcal{C}$ a class of separable C^*-algebras. The above results give us connections between the four conditions that B is $\mathcal{C}$-like, or $A\mathcal{C}$-like, or an $A\mathcal{C}$-algebra, or an $AA\mathcal{C}$-algebra. This is shown in the diagram below. A dotted arrow indicates that the implication holds under the additional assumption that the algebras in $\mathcal{C}$ are weakly semiprojective. The dashed arrow with $(*)$ indicates that the implication holds if each quotient of an algebra in $\mathcal{C}$ is an $A\mathcal{C}$-algebra, while the dashed arrow with $(**)$ indicates that the implication holds if $\mathcal{C}$ is closed under quotients, see also 3.11.



4. Trivial shape

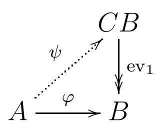
In this section we study C^*-algebras that are shape equivalent to the zero C^*-algebra. Such algebras are said to have **trivial shape**. We will show in Theorem 4.4 that having trivial shape is equivalent to several other natural conditions, most importantly to being an inductive limit of projective C^*-algebras. One may further obtain that the connecting morphisms in such an inductive limit are surjective, see Proposition 4.9.

We prove some natural permanence properties of trivial shape, see Theorem 4.6. However, building on an example of Dadarlat, [Dad11], see 4.11, we show that trivial shape does not necessarily pass to full hereditary sub-C^*-algebras. It follows that also projectivity does not pass to full hereditary sub-C^*-algebras, see Proposition 4.12.
Note that $A \precsim_{Sh} 0$ implies $A \sim_{Sh} 0$, i.e., A is shape dominated by 0 if and only if it is shape equivalent to 0. The following recent result of Loring and Shulman was the inspiration for the main result Theorem 4.4 below. For the definition of the generator rank $\mathrm{gen}(A)$, see 2.8.

Theorem 4.1 (Loring, Shulman, [LS10, Theorem 7.4])**.** *Let A be a C^*-algebra. Then the cone $CA = C_0((0,1]) \otimes A$ is an inductive limit, $\varinjlim_k P_k$, of projective C^*-algebras, P_k, with surjective connecting morphisms $P_k \to P_{k+1}$ and $\mathrm{gen}(P_k) \leq \mathrm{gen}(A) + 1$.*

Lemma 4.2. *Let $\varphi \colon A \to B$ be a projective morphism. Then $\varphi \simeq 0$.*

Proof.
This is a variant of the standard argument for showing that a projective C^*-algebra is contractible. We include it for completeness. Let $\mathrm{ev}_1 \colon CB \to B$ be the evaluation morphism at 1. The projectivity of φ gives us a lift $\psi\colon A \to CB$ such that $\mathrm{ev}_1 \circ \psi = \varphi$. This is indicated in the commutative diagram on the right.

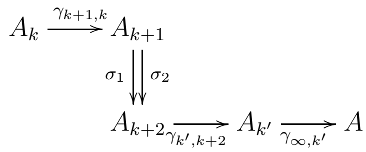


We have $\mathrm{id}_{CB} \simeq 0$ since CB is contractible. Then $\varphi = \mathrm{ev}_1 \circ \mathrm{id}_{CB} \circ \psi \simeq 0$, as desired. □

Lemma 4.3. *Let (A_k, γ_k) a shape system with inductive limit $A := \varinjlim A_k$. Assume that every semiprojective morphism $D \to A$ (from any C^*-algebra D) is null-homotopic. Then for each k there exists $k' \geq k$ such that $\gamma_{k',k} \simeq 0$.*

Proof.

We are given some index k. Note that $\gamma_{k+1,k}$ is semiprojective. Define two morphisms $\sigma_1, \sigma_2 \colon A_{k+1} \to A_{k+2}$ as $\sigma_1 = \gamma_{k+2,k+1}$ and $\sigma_2 = 0$. The morphism $\gamma_{\infty,k+2} \circ \sigma_1 = \gamma_{\infty,k+1}$ is semiprojective, and therefore null-homotopic by assumption. Thus $\gamma_{\infty,k+2} \circ \sigma_1 \simeq 0 = \gamma_{\infty,k+2} \circ \sigma_2$.



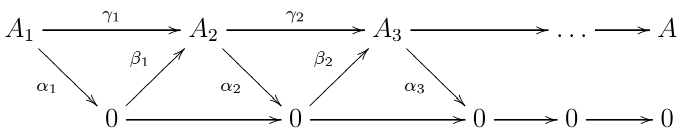
Using the semiprojectivity of $\gamma_{k+1,k}$ it follows from [EK86, 3.2], see Theorem 2.4, that there exists $k' \geq k+2$ such that $\gamma_{k',k} = \gamma_{k',k+2} \circ \sigma_1 \circ \gamma_{k+1,k} \simeq \gamma_{k',k+2} \circ \sigma_2 \circ \gamma_{k+1,k} = 0$. The situation is shown in the diagram on the right. ☐

Theorem 4.4. *Let A be a separable C^*-algebra. Then the following are equivalent:*

(a) $A \sim_{Sh} 0$,
(b) every semiprojective morphism $D \to A$ (from any C^*-algebra D) is null-homotopic,
(c) A is an inductive limit, $\varinjlim A_k$, with projective connecting morphisms $A_k \to A_{k+1}$,
(d) A is an inductive limit, $\varinjlim A_k$, with null-homotopic connecting morphisms $A_k \to A_{k+1}$,
(e) A is an inductive limit of finitely generated, projective C^*-algebras,
(f) A is an inductive limit of finitely generated cones,
(g) A is an inductive limit of contractible C^*-algebras.

Moreover, in conditions (c)-(g), if A is an inductive limit, $\varinjlim A_k$, then we may further assume $\mathrm{gen}(A_k) \leq \mathrm{gen}(A) + 1$.

Proof. Note that 0 has a natural shape system consisting of the zero C^*-algebra at each step. Therefore, $A \sim_{Sh} 0$ means that there exists a shape system (A_k, γ_k) for A and morphisms $\alpha_k \colon A_k \to 0$ and $\beta_k \colon 0 \to A_{k+1}$ such that $\beta_{k+1} \circ \alpha_k \simeq \gamma_k$. This is shown in the following diagram, which homotopy commutes:



'(a) ⇒ (d)': Assume $A \sim_{Sh} 0$. We have just noted that this implies that A has a shape system (A_k) with connecting morphisms $\gamma_k \colon A_k \to A_{k+1}$ that are null-homotopic since they factor through 0 up to homotopy.

'(d) ⇒ (a)': Assume there is an inductive system $\mathcal{A} = (A_k, \gamma_k)$ with $A \cong \varinjlim \mathcal{A}$ and null-homotopic connecting morphisms γ_k. Let $\alpha_k \colon A_k \to 0$ and $\beta_k \colon 0 \to A_{k+1}$ be the zero morphisms. Then $\beta_{k+1} \circ \alpha_k = 0 \simeq \gamma_k$. Conversely also $\beta_k \circ \alpha_k = 0$, so that the inductive systems $\mathcal{A}$ and $(0 \to 0 \to \dots)$ are shape equivalent. This does not show $A \sim_{Sh} 0$ right away since the inductive system $\mathcal{A}$ need not be a shape system. However, thanks to [Bla85, Theorem 4.8], whenever two inductive systems are shape equivalent, then their inductive limit C^*-algebras are shape equivalent.

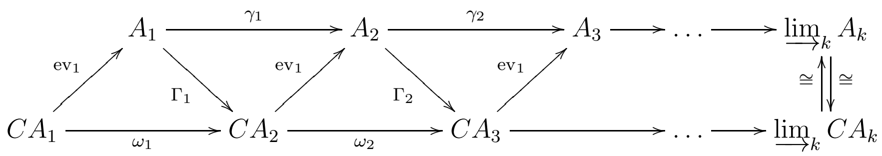
'(d) ⇒ (b)': Assume there is an inductive system $\mathcal{A} = (A_k, \gamma_k)$ with $A \cong \varinjlim \mathcal{A}$ and null-homotopic connecting morphisms γ_k. Let $\varphi \colon D \to A$ be any semiprojective morphism. By [Bla85, Theorem 3.1], see Theorem 2.4, there exists k and a morphism $\psi \colon D \to A_k$ such that

$\varphi \simeq \gamma_{\infty,k} \circ \psi$. However, $\gamma_{\infty,k}$ has a factorization as $\gamma_{\infty,k} = \gamma_{\infty,k+1} \circ \gamma_{k+1,k}$ and is therefore null-homotopic. Then φ is null-homotopic as well.

'(b) ⇒ (f)': By Blackadar, [Bla85, Theorem 4.3], see Lemma 2.3, A has a shape system (A_k, γ_k) with finitely generated algebras A_k and such that $\mathrm{gen}(A_k) \leq \mathrm{gen}(A)$. We may apply Lemma 4.3 inductively to this shape system, and after passing to a suitable subsystem we see that there exists a shape system (A_k, γ_k) of finitely generated C^*-algebras A_k with $\mathrm{gen}(A_k) \leq \mathrm{gen}(A)$ and null-homotopic connecting morphisms γ_k such that $A \cong \varinjlim \mathcal{A}$.

A homotopy $\gamma_k \simeq 0$ induces a natural morphism $\Gamma_k \colon A_k \to CA_{k+1}$ such that γ_k has a factorization $\gamma_k = \mathrm{ev}_1 \circ \Gamma_k$, where CA_{k+1} is the cone over A_{k+1} and ev_1 is evaluation at 1.

Set $\omega_k := \Gamma_k \circ \mathrm{ev}_1 \colon CA_k \to CA_{k+1}$. Consider the inductive system $\mathcal{B} = (CA_k, \omega_k)$. It follows from [LS10, Lemma 7.1] that $\mathrm{gen}(CA_k) \leq \mathrm{gen}(A_k) + 1$, so that CA_k is finitely generated and $\mathrm{gen}(CA_k) \leq \mathrm{gen}(A) + 1$. The systems $\mathcal{A}$ and $\mathcal{B}$ are intertwined, which implies that their inductive limits are isomorphic, so that A is isomorphic to an inductive limit of the finitely generated cones CA_k. The intertwining is shown in the following commutative diagram.



'(f) ⇒ (e)': Assume $A \cong \varinjlim CA_k$ with each A_k finitely generated and $\mathrm{gen}(A_k) \leq \mathrm{gen}(A)$. By the result of Loring and Shulmann, [LS10, Theorem 7.4], see Theorem 4.1, for each k, the cone CA_k can be written as an inductive limit of finitely generated projective C^*-algebras with generator rank at most $\mathrm{gen}(A_k) + 1$. Note that $\mathrm{gen}(A_k) + 1 \leq \mathrm{gen}(A) + 1$ for all k. It follows from Theorem 3.12 that A is isomorphic to an inductive limit of finitely generated, projective C^*-algebras with generator rank at most $\mathrm{gen}(A) + 1$.

'(e) ⇒ (c)', '(e) ⇒ (g)' and '(g) ⇒ (d)' are clear. '(c) ⇒ (d)' follows from Lemma 4.2. □

Corollary 4.5. *Every separable, contractible C^*-algebra is an inductive limit of projective C^*-algebras.*

Theorem 4.6. *The class of separable C^*-algebras with trivial shape is closed under:*

(1) countable direct sums
(2) inductive limits
(3) approximation by sub-C^*-algebras (i.e., likeness, see Definition 3.2)
(4) taking maximal tensor products with any *other (separable) C^*-algebra, i.e., $A \otimes_{max} B$ has trivial shape when A has trivial shape*

Proof. (1): Assume $A_1, A_2, \ldots$ have trivial shape. By condition (g) of the above Theorem 4.4, each A_k can be written as inductive limit of contractible C^*-algebras. Note that countable direct sums of contractible C^*-algebras are again contractible. Hence, $\bigoplus_k A_k$ is an inductive limit of contractible C^*-algebras and thus has trivial shape by (g) of Theorem 4.4.

(2): Assume $A \cong \varinjlim A_k$ with each A_k having trivial shape. By Theorem 4.4, each A_k is an inductive limit of projective C^*-algebras. It follows from Theorem 3.12 that A is an inductive limit of projective C^*-algebras, and so it has trivial shape using Theorem 4.4 again.

(3): Assume a C^*-algebra A is approximated by sub-C^*-algebras $A_i \subset A$. By Theorem 4.4, each A_i is an inductive limit of projective C^*-algebras. This means that A is $A\mathcal{P}$-like for the class $\mathcal{P}$ of projective C^*-algebras. It follows from Theorem 3.9 that A is an $A\mathcal{P}$-algebra, i.e., and inductive limit of projective C^*-algebras, and so A has trivial shape by Theorem 4.4.

(4): Let A be a C^*-algebra with trivial shape, and B any other (separable) C^*-algebra. By condition (f) of Theorem 4.4, we can write A as an inductive limit of cones $CA_k = C_0((0,1]) \otimes A_k$. As noted by Blackadar, [Bla06, II.9.6.5, p.188], maximal tensor products commute with arbitrary inductive limits (while minimal tensor products only commute with inductive limits with injective connecting morphisms). Thus, $A \otimes_{\max} B$ is the inductive limit of $CA_k \otimes_{\max} B = C_0((0,1]) \otimes A_k \otimes_{\max} B = C(A_k \otimes B)$. Using condition (f) of Theorem 4.4 again, we deduce that $A \otimes_{\max} B$ has trivial shape. □

We derive two corollaries. Using the notation from 3.1 and Definition 3.2, they state that a contractible-like C^*-algebra has trivial shape and is approximately contractible, see Corollary 4.7 resp. Corollary 4.8.

Corollary 4.7. *Let A be a separable C^*-algebra that is approximated by contractible sub-C^*-algebras. Then A has trivial shape.*

Corollary 4.8. *Let A be a separable C^*-algebra that is approximated by contractible sub-C^*-algebras. Then A is an inductive limit of contractible C^*-algebras.*

Proposition 4.9. *Let (A_k, γ_k) be an inductive system of separable C^*-algebras. Then there exists an inductive system (B_k, δ_k) with surjective connecting morphisms and such that $\varinjlim A_k \cong \varinjlim B_k$. Moreover, we may assume $B_k = A_k * \mathcal{F}_\infty$ (the free product), where*

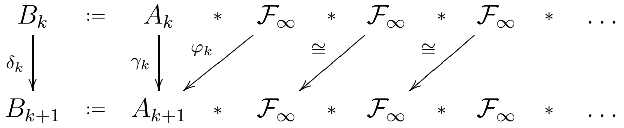
$$\mathcal{F}_\infty := C^*(x_1, x_2, \dots \mid \|x_i\| \leq 1)$$

is the universal C^*-algebra generated by a countable number of contractive generators. If A_k is (semi-)projective, then so is $A_k * \mathcal{F}_\infty$.

Proof. The algebras A_k are separable. Thus, for each k there exists a surjective morphism $\varphi_k \colon \mathcal{F}_\infty \to A_k$. Consider the universal C^*-algebra $\mathcal{G} := C^*(x_{i,j} \mid i,j \in \mathbb{N}, \|x_{i,j}\| \leq 1)$. The only difference between $\mathcal{G}$ and $\mathcal{F}_\infty$ is in the enumeration of generators, and therefore $\mathcal{G} \cong \mathcal{F}_\infty$.

Set $B_k := A_k * \mathcal{G}$ and define a morphism $\psi_k \colon \mathcal{G} \to B_{k+1}$ via $\psi_k(x_{1,j}) := \varphi_{k+1}(x_j)$, and $\psi_k(x_{i,j}) := x_{i-1,j}$ if $i \geq 2$. Define a morphism $\delta_k \colon B_k \to B_{k+1}$ as $\delta_k := \gamma_k * \psi_k$. It is easy to check that δ_k is surjective.

For each i, the elements $x_{i,1}, x_{i,2}, \dots \in \mathcal{G}$ generate a copy of $\mathcal{F}_\infty$. In this way, we may think of $\mathcal{G}$ as a countable free product of copies of $\mathcal{F}_\infty$. Then, the map δ_k looks as follows:



The natural inclusions $\iota_k \colon A_k \to B_k$ intertwine the connecting morphisms γ_k and δ_k, i.e., $\delta_k \circ \iota_k = \iota_{k+1} \circ \gamma_k$. Thus, the morphisms ι_k define a natural morphism $\iota \colon A = \varinjlim A_k \to B := \varinjlim B_k$. Since each ι_k is injective, so is ι.

Let us check that ι is also surjective. Let $b \in B$ and $\varepsilon > 0$ be given. We need to find some $a \in A$ with $b =_\varepsilon \iota(a)$, i.e., $\|b - \iota(a)\| < \varepsilon$. First, we may find an index k and $b' \in B_k$ such that $\delta_{\infty,k}(b') =_{\varepsilon/2} b$. By definition, $B_k = A_k * \mathcal{G}$. This implies that every element of B_k can be approximated by finite polynomials involving the elements of A and the generators $x_{i,j}$. Actually, we only need that b' is approximated up to $\varepsilon/2$ by an element b'' in the sub-C^*-algebra $A_k * C^*(x_{i,j} \mid i \in \{1, 2, \dots, l\}, j \in \mathbb{N}, \|x_{i,j}\| \leq 1)$. Note that $\delta_{k+l,k}(b'')$ lies in the image of ι_{k+l}, say $\delta_{k+l,k}(b'') = \iota_{k+l}(x)$ for $x \in A_{k+l}$. Then $a = \gamma_{\infty,k+l}(x) \in A$ satisfies $b =_\varepsilon \iota(a)$, which completes the proof of surjectivity.

Note that $\mathcal{F}_\infty$ is projective. It follows from [Bla85, Proposition 2.6, 2.31] that $A_k * \mathcal{F}_\infty$ is (semi-)projective, if A_k is so. □

Corollary 4.10. *If a separable C^*-algebra has trivial shape, then it is an inductive limit of projective C^*-algebra with surjective connecting morphisms.*

Remark 4.11 (Dadarlat, [Dad11])**.** Dadarlat gives an example of a commutative C^*-algebra $A = C_0(X, x_0)$ such that $A \otimes \mathbb{K}$ is contractible (in particular has trivial shape), while A is not contractible. In fact, X is a two-dimensional CW-complex with non-trivial fundamental group, so that (X, x_0) does not have trivial shape (in the pointed, commutative category). It follows from [Bla85, Proposition 2.9] that $C_0(X, x_0)$ also does not have trivial shape (as a C^*-algebra).

Thus, while $A \otimes \mathbb{K}$ has trivial shape, the full hereditary sub-C^*-algebra $A \subset A \otimes \mathbb{K}$ does not. This shows that trivial shape does not pass to full hereditary sub-C^*-algebras. From this we may deduce the following result.

Proposition 4.12. *Projectivity does not pass to full hereditary sub-C^*-algebras.*

Proof. Let A be Dadarlat's example of a C^*-algebra with $A \otimes \mathbb{K} \simeq 0$ while $A \not\sim_{Sh} 0$, see [Dad11] and 4.11. By Corollary 4.10, $A \otimes \mathbb{K}$ is an inductive limit of projective C^*-algebra P_k with surjective connecting morphisms $\gamma_k \colon P_k \to P_{k+1}$. Consider the pre-images $Q_k := \gamma_{\infty,k}^{-1}(A) \subset P_k$. Since $A \subset A \otimes \mathbb{K}$ is a full hereditary sub-C^*-algebra, so is $Q_k \subset P_k$.

Note that $A \cong \varinjlim Q_k$. If all algebras Q_k were projective, then A would have trivial shape by Theorem 4.4. Since this is not the case, some algebras Q_k are not projective. □

Remark 4.13. It was recently shown by Eilers and Katsura, [ET11], that also semiprojectivity does not pass to full hereditary sub-C^*-algebras.

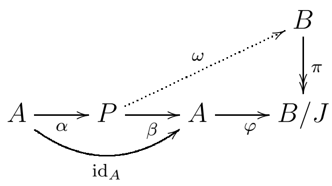
5. Relations among the classes of (weakly) (semi-)projective C^*-algebras

In this section we will study the relation among the four classes of (weakly) semiprojective C^*-algebras and (weakly) projective C^*-algebras. As it turns out, the situation is completely analogous to the commutative setting.

Lemma 5.1. *Let A be a C^*-algebra, P a projective C^*-algebra and $\alpha \colon A \to P$, $\beta \colon P \to A$ two morphisms with $\beta \circ \alpha = \mathrm{id}_A$. Then A is projective.*

Proof. Let B be any C^*-algebra, $J \lhd B$ an ideal, and $\varphi \colon A \to B/J$ a morphism. We need to find a lift $\psi \colon A \to B$.

Since P is projective, there exists a morphism $\omega\colon P \to B$ that lifts $\varphi \circ \beta\colon P \to B/J$, i.e., $\pi \circ \omega = \varphi \circ \beta$. Set $\psi := \omega \circ \alpha\colon A \to B$. Then $\pi \circ \psi = \pi \circ \omega \circ \alpha = \varphi \circ \beta \circ \alpha = \varphi \circ \mathrm{id}_A$. The situation is shown in the diagram on the right.

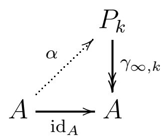


□

Theorem 5.2. *Let A be a semiprojective C^*-algebra of trivial shape. Then A is projective.*

Proof. By Corollary 4.10, A is an inductive limit of projective C^*-algebra P_k with surjective connecting morphisms $\gamma_k\colon P_k \to P_{k+1}$.

The semiprojectivity of A gives an index k and a lift $\alpha\colon A \to P_k$ such that $\gamma_{\infty,k} \circ \alpha = \mathrm{id}_A$. It follows from Lemma 5.1 that A is projective. The situation is shown in the diagram on the right.



□

Since every projective C^*-algebra is contractible, we get the following corollary:

Corollary 5.3. *Let A be a semiprojective C^*-algebra of trivial shape. Then A is contractible.*

Loring, [Lor09, Lemma 5.5], shows that for a weakly projective C^*-algebra A and a semiprojective C^*-algebra D the set $[D, A]$ of homotopy classes of morphisms from D to A is trivial. A variant of this proof shows condition (b) in Theorem 4.4, so that we get the following:

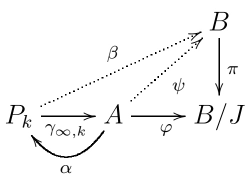
Proposition 5.4 (Loring, [Lor09, Lemma 5.5])**.** *Every weakly projective C^*-algebra has trivial shape.*

This result of Loring shows that a weakly projective C^*-algebra is weakly semiprojective and has trivial shape. We will now show that the converse is also true.

Theorem 5.5. *Let A be a weakly semiprojective C^*-algebra of trivial shape. Then A is weakly projective.*

Proof. Let B be a C^*-algebra, let $J \lhd B$ be an ideal, and $\pi\colon B \to B/J$ the quotient morphism. Let $\varphi\colon A \to B/J$ be a morphism. Let $F \subset A$ be a finite set, and $\varepsilon > 0$. We need to find a lift $\psi\colon A \to B$ such that $\pi \circ \psi =_\varepsilon^F \varphi$.

From Theorem 4.4 we get an inductive system (P_k, γ_k) of projective C^*-algebras P_k with inductive limit A. Considering the identity morphism $\mathrm{id}_A\colon A \to A \cong \varinjlim P_k$ we get from Proposition 2.6 an index k and a morphism $\alpha\colon A \to P_k$ such that $\gamma_{\infty,k} \circ \alpha =_\varepsilon^F \mathrm{id}_A$. Consider the morphism $\varphi \circ \gamma_{\infty,k}\colon P_k \to B/J$. The projectivity of P_k gives us a lift $\beta\colon P_k \to B$ such that $\pi \circ \beta = \varphi \circ \gamma_{\infty,k}$. The situation is shown in the diagram on the right.



Set $\psi := \beta \circ \alpha$. Then $\pi \circ \psi = \pi \circ \beta \circ \alpha = \varphi \circ \gamma_{\infty,k} \circ \alpha =_\varepsilon^F \varphi$ as desired. □

We summarize the results as follows:

Theorem 5.6. *Let A be a C^*-algebra. Then the following are equivalent:*

(a) A is (weakly) projective
(b) A is (weakly) semiprojective and has trivial shape

Corollary 5.7. *Let A be a C^*-algebra. Then the following are equivalent:*

(a) A is projective
(b) A is semiprojective and weakly projective
(c) A is semiprojective and contractible
(d) A is semiprojective and has trivial shape

5.8. Corollary 5.7 confirms a conjecture of Loring. We note that the results in Theorem 5.2, Corollary 5.3, Proposition 5.4 (see [Lor09, Lemma 5.5]), Theorem 5.5, Theorem 5.6 and Corollary 5.7 are in exact analogy with results in commutative shape theory, as shown in the table below.

A (weakly) projective C^*-algebra is the non-commutative analog of an (approximate) absolute retract, and a (weakly) semiprojective C^*-algebra is the non-commutative analog of an (approximate) absolute neighborhood retract. The analogies are shown in the table below. We refer the reader to [ST11, 2.1, 2.2, 2.3] and the references therein for definitions and further discussion.

commutative world (for a compact, metric space X):	noncommutative world (for a separable C^*-algebra A):
analogy of notions	
• X is an absolute retract (AR)	• A is projective (P)
• X is an approximate absolute retract (AAR)	• A is weakly projective (WP)
• X is an absolute neighborhood retract (ANR)	• A is semiprojective (SP)
• X is an approximative absolute neighborhood retract (AANR)	• A is weakly semiprojective (WSP)
analogy of results	
• X is AR $\Leftrightarrow$ X is ANR and $X \simeq$ pt (see [Bor67, IV.9.1])	• A is P $\Leftrightarrow$ A is SP and $A \simeq 0$ (see Theorem 5.2)
• X is AAR $\Leftrightarrow$ X is AANR and $X \sim_{Sh}$ pt (see [Gmu71] and [Bog75])	• A is WP $\Leftrightarrow$ A is WSP and $A \sim_{Sh} 0$ (see [Lor09] and Theorem 5.5)
• if X is ANR, then: $X \sim_{Sh}$ pt $\Leftrightarrow X \simeq$ pt (see [Bor67])	• if A is SP, then: $A \sim_{Sh} 0 \Leftrightarrow A \simeq 0$ (see Corollary 5.3)

6. Questions

Question 6.1. Assume A has trivial shape. Is A an inductive limit, $\varinjlim A_k$, with surjective connecting morphisms of projective C^*-algebras A_k with $\mathrm{gen}(A_k) \leq \mathrm{gen}(A) + 1$?

The result of Loring and Shulmann, [LS10, Theorem 7.4], see Theorem 4.1, shows that Question 6.1 has a positive answer for cones. Furthermore, it follows from Theorem 4.4 that A is an inductive limit, $\varinjlim A_k$, of projective C^*-algebras A_k with $\mathrm{gen}(A_k) \leq \mathrm{gen}(A)+1$,

but the connecting morphisms may not be surjective. Using Proposition 4.9, we can always arrange that the connecting morphisms are surjective, but the approximating algebras are replaced by $A_k * \mathcal{F}_\infty$, which have $\mathrm{gen}(A_k * \mathcal{F}_\infty) = \infty$.
Say A has property $(*)$ if $[D, A] = \mathrm{pt}$ for every semiprojective C^*-algebra D. This means that for each (fixed) semiprojective D, all morphisms from D to A are homotopic. Every C^*-algebra of trivial shape has property $(*)$. We ask if the converse is true:

Question 6.2. Assume A has property $(*)$. Does A have trivial shape?

If A is an inductive limit of semiprojective C^*-algebras, then property $(*)$ for A implies that A has trivial shape. As mentioned in Question 1.1, see [Bla85, 4.4], it is however an open question whether every C^*-algebra is an inductive limit of semiprojective C^*-algebras.

Acknowledgments

I thank Eduard Ortega and Mikael Rørdam for their valuable comments, and especially for their careful reading of all the technical details. I thank Tatiana Shulman and Leonel Robert for discussions and feedback on this paper. I thank George Elliott for interesting discussions on approximate intertwinings.

References

[Bla85] B. Blackadar, *Shape theory for C^*-algebras*, Math. Scand. **56** (1985), 249–275.
[Bla06] ———, *Operator algebras. Theory of C^*-algebras and von Neumann algebras*, Encyclopaedia of Mathematical Sciences 122. Operator Algebras and Non-Commutative Geometry 3. Berlin: Springer. xx, 517 p. , 2006.
[Bog75] S.A. Bogatyi, *Approximative and fundamental retracts*, Math. USSR, Sb. **22** (1975), 91–103.
[Bor67] K. Borsuk, *Theory of retracts*, Monografie Matematyczne. 44. Warszawa: PWN - Polish Scientific Publishers. 251 p. , 1967.
[CD10] A. Chigogidze and A.N. Dranishnikov, *Which compacta are noncommutative ARs?*, Topology Appl. **157** (2010), no. 4, 774–778.
[CL11] A. Chigogidze and T.A. Loring, Private communication, 2011.
[Dad99] Dadarlat, M. and Eilers, S., *Approximate homogeneity is not a local property*, J. Reine Angew. Math. **507** (1999), 1–13.
[Dad11] Dadarlat, M., *A stably contractible C^*-algebra which is not contractible*, preprint, 2011.
[EK86] E.G. Effros and J. Kaminker, *Homotopy continuity and shape theory for C^*-algebras*, Geometric methods in operator algebras, Proc. US-Jap. Semin., Kyoto/Jap. 1983, Pitman Res. Notes Math. Ser. 123, 152-180, 1986.
[Ell93] G.A. Elliott, *On the classification of C^*-algebras of real rank zero*, J. Reine Angew. Math. **443** (1993), 179–219.
[ELP98] S. Eilers, T.A. Loring, and G.K. Pedersen, *Stability of anticommutation relations: An application of noncommutative CW complexes*, J. Reine Angew. Math. **499** (1998), 101–143.
[ET11] S. Eilers and Katsura T., Private communication, 2011.
[Gmu71] A. Gmurczyk, *Approximative retracts and fundamental retracts*, Colloq. Math. **23** (1971), 61–63.
[Lor97] T.A. Loring, *Lifting solutions to perturbing problems in C^*-algebras*, Fields Institute Monographs 8. Providence, RI: American Mathematical Society. ix, 165 p. , 1997.
[Lor09] ———, *Weakly Projective C^*-algebras*, preprint, arXiv:0905.1520, 2009.
[LR95] H. Lin and M. Rørdam, *Extensions of inductive limits of circle algebras*, J. Lond. Math. Soc., II. Ser. **51** (1995), no. 3, 603–613.
[LS10] T.A. Loring and T. Shulman, *Noncommutative Semialgebraic sets and Associated Lifting Problems*, preprint, arXiv:0907.2618, 2010.

[MM92] S. Mardešić and V. Matijević, *$\mathcal{P}$-like spaces are limits of approximate $\mathcal{P}$-resolutions*, Topology Appl. **45** (1992), no. 3, 189–202.

[MS63] S. Mardešić and J. Segal, *ϵ-mappings onto polyhedra*, Trans. Am. Math. Soc. **109** (1963), 146–164.

[Nag70] K. Nagami, *Dimension theory. With an appendix by Yukihiro Kodama*, Pure and Applied Mathematics. Vol. 37. New York-London: Akademic Press 1970. XI,256 p. , 1970.

[Nag04] M. Nagisa, *Single generation and rank of C^*-algebras*, Kosaki, Hideki (ed.), Operator algebras and applications. Proceedings of the US-Japan seminar held at Kyushu University, Fukuoka, Japan, June 7–11, 1999. Tokyo: Mathematical Society of Japan. Advanced Studies in Pure Mathematics 38, 135-143, 2004.

[Rør02] M. Rørdam, *Classification of nuclear, simple C^*-algebras*, Entropy in operator algebras. Berlin: Springer. Encycl. Math. Sci. 126(VII), pp 1–145. , 2002.

[ST11] A.P.W. Sørensen and H. Thiel, *A characterization of semiprojectivity for commutative C^*-algebras*, preprint, arXiv:1101.1856, 2011.

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF COPENHAGEN, UNIVERSITETSPARKEN 5, DK-2100, COPENHAGEN Ø, DENMARK

E-mail address: thiel@math.ku.dk

SEMIPROJECTIVITY WITH AND WITHOUT A GROUP ACTION

N. CHRISTOPHER PHILLIPS, ADAM P. W. SØRENSEN, AND HANNES THIEL

Abstract. The equivariant version of semiprojectivity was recently introduced by the first author. We study properties of this notion, in particular its relation to ordinary semiprojectivity of the crossed product and of the algebra itself.

We show that equivariant semiprojectivity is preserved when the action is restricted to a cocompact subgroup. Thus, if a second countable compact group acts semiprojectively on a C^*-algebra A, then A must be semiprojective. This fails for noncompact groups: we construct a semiprojective action of $\mathbb{Z}$ on a nonsemiprojective C^*-algebra.

We also study equivariant projectivity and obtain analogous results, however with fewer restrictions on the subgroup. For example, if a discrete group acts projectively on a C^*-algebra A, then A must be projective. This is in contrast to the semiprojective case.

We show that the crossed product by a semiprojective action of a finite group on a unital C^*-algebra is a semiprojective C^*-algebra. We give examples to show that this does not generalize to all compact groups.

Equivariant semiprojectivity was introduced in [Phi12], by applying the usual definition of semiprojectivity to the category of unital G-algebras (C^*-algebras with actions of the group G) with unital G-equivariant *-homomorphisms. See Definition 1.1 below. The purpose of [Phi12] was to show that certain actions of compact groups on various specific C^*-algebras are semiprojective. In particular, it is shown that any action of a second countable compact group on a finite dimensional C^*-algebra is semiprojective, and that for $n < \infty$, quasifree actions of second countable compact groups on the Cuntz algebras $\mathcal{O}_n$ are semiprojective.

In this paper we study equivariant semiprojectivity more abstractly. We also introduce equivariant projectivity and carry out a parallel study of it. We extend the definition to allow actions by general locally compact groups, and we consider the nonunital version of equivariant semiprojectivity.

From the work in [Phi12], it is not even clear whether a semiprojective action of a noncompact group could exist. One reason for being skeptical was that the trivial action of $\mathbb{Z}$ on $\mathbb{C}$ is not semiprojective, as was shown by Blackadar ([Bla12]). We give a wide reaching generalization of this result in Corollary 6.5, by showing that if the trivial action of a group on a (nonzero) C^*-algebra is semiprojective then the group must be compact.

Date: 15 October 2012.

1991 *Mathematics Subject Classification.* Primary 46L55. Secondary 22D05, 22D15, 22D30, 46M10, 55P91, 54C55, 55M15.

Key words and phrases. C^*-algebra, equivariant semiprojectivity, projectivity, induced algebra, induction functor, uniformly finitistic, crossed product, fixed point algebra.

This material is based upon work of the first author supported by the US National Science Foundation under Grants DMS-0701076 and DMS-1101742, and on work of all three authors supported by the Danish National Research Foundation through the Centre for Symmetry and Deformation.

There are, however, many nontrivial semiprojective (and even projective) actions of noncompact groups. Indeed, given a countable discrete group G and a semiprojective C^*-algebra A, we show in Proposition 2.4 that the free Bernoulli shift action of G on the free product $\ast_{g\in G} A$ is equivariantly semiprojective.

Our main motivation was to understand how equivariant semiprojectivity (with group action) is related to semiprojectivity (without group action). The following question naturally occurs:

Question 0.1. Assume that (G, A, α) is an equivariantly semiprojective G-algebra (Definition 1.1 below). Is A semiprojective in the usual sense?

We give a positive answer in Corollary 3.11 under the assumption that the group G is compact. If we drop this assumption, then the answer to the question may be negative. Indeed, in Example 3.12 we construct a semiprojective action of $\mathbb{Z}$ on a nonsemiprojective C^*-algebra.

Question 0.1 is a special case of a more natural question:

Question 0.2. Assume that (G, A, α) is a G-algebra that is equivariantly semiprojective (equivariantly projective), and let $H \leqslant G$ be a closed subgroup. Is the restricted H-algebra $(H, A, \alpha|_H)$ equivariantly semiprojective (equivariantly projective)?

The two main results of this paper answer this question positively under certain natural assumptions on the factor space G/H. In the semiprojective case, we get a positive answer (Theorem 3.10) if H is cocompact, that is, if G/H is compact. In the projective case, we get a positive answer if G/H is uniformly finitistic. See Theorem 4.19. This condition on G/H is much less restrictive than compactness.

This paper is organized as follows. In Section 1 we give the definition of equivariant semiprojectivity, Definition 1.1. We also introduce equivariant projectivity (Definition 1.2), and we investigate the relation between the unital and nonunital versions of these definitions (Lemma 1.5, Lemma 1.6, and Proposition 1.7).

In Section 2 we introduce free Bernoulli shifts, and we show in Proposition 2.4 that these provide examples of semiprojective actions of countable discrete groups. We also study the orthogonal Bernoulli shift of G on $\bigoplus_G A = C_0(G, A)$ for (semi)projective C^*-algebras A. It turns out to be a much harder problem to determine when this action is semiprojective, and we give a positive answer only for finite cyclic groups of order 2^n. See Proposition 2.10.

In Section 3 we study Question 0.2 in the semiprojective case. We give a positive answer (Theorem 3.10) when G/H is compact. It follows (Corollary 3.11) that a second countable compact group can only act semiprojectively on a C^*-algebra that is semiprojective in the usual sense. We show that this is not true in general, by constructing in Example 3.12 a semiprojective action of $\mathbb{Z}$ on a nonsemiprojective C^*-algebra. The main ingredient to obtain the results of this section is the induction functor (Definition 3.1), which assigns to each H-algebra an induced G-algebra. We show that this functor is exact (Proposition 3.6) and continuous (Proposition 3.7).

In Section 4 we study Question 0.2 in the projective case. The main result is Theorem 4.19, which gives a positive answer to the question when G/H is right uniformly finitistic in the sense of Definition 4.5 and the left and right uniformities on G agree. The class of uniformly finitistic spaces includes both compact and discrete spaces. We know of no example of a locally compact group that is not uniformly finitistic.

The main technique to obtain the results of this section is an induction functor which uses uniformly continuous functions; see Definition 4.14. In Theorem 4.16, we show that this functor is exact when G/H is right uniformly finitistic and the left and right uniformities on G agree. To prove this, we need conditions under which uniformly continuous functions into a quotient C^*-algebra can be lifted to uniformly continuous functions, and in Theorem 4.8 we provide a satisfying answer that might also be of independent interest.

In Section 5, we study semiprojectivity of crossed products. In Theorem 5.1, we show that for a discrete group G whose group C^*-algebra is semiprojective, semiprojectivity of an action $\alpha\colon G \to \mathrm{Aut}(A)$ on a unital C^*-algebra implies semiprojectivity of the crossed product $A \rtimes_\alpha G$. Example 5.2 shows that this can fail when the group is compact but not finite. At the end of Section 5, we give counterexamples to several other plausible relations between equivariant semiprojectivity for finite groups and semiprojectivity, and state further open problems.

In Section 6, we study semiprojectivity of fixed point algebras. We show that for a saturated, semiprojective action of a finite group G on a unital C^*-algebra A, the fixed point algebra A^G is semiprojective (Proposition 6.2). We show in Example 6.1 that this does not generalize to compact groups. In the case that a noncompact group G acts semiprojectively, we show in Theorem 6.4 that the fixed point algebra is trivial. Thus, the trivial action of a noncompact group on a nonzero C^*-algebra is never semiprojective. We therefore obtain a precise characterization when the trivial action of a group is (semi)projective (Corollary 6.5).

We use the following terminology and notation in this paper. By a topological group we understand a group G together with a Hausdorff topology such that the map $(s,t) \mapsto s \cdot t^{-1}$ is jointly continuous. We mainly consider locally compact topological groups. For such a group, we denote its Haar measure by μ. By the Birkhoff-Kakutani theorem (see for instance Theorem 1.22 of [MZ55]), G is metrizable if and only if it is first countable. Moreover, in that case, the metric d may be chosen to be left invariant, that is, $d(rs, rt) = d(s,t)$ for all $r, s, t \in G$. We will always take our metrics to be left invariant. We usually require G to be second countable.

For a topological group G, by a *G-algebra* we understand a triple (G, A, α) in which A is a C^*-algebra and $\alpha\colon G \to \mathrm{Aut}(A)$ is a continuous action of G on A. Continuity means that for each $a \in A$ the map $s \mapsto \alpha_s(a)$ is continuous. (Such an action is also called strongly continuous.)

By a *G-morphism* between two G-algebras (G, A, α) and (G, B, β) we mean a G-equivariant *-homomorphism, that is, a *-homomorphism $\varphi\colon A \to B$ such that $\beta_s \circ \varphi = \varphi \circ \alpha_s$ for each $s \in G$. We say that a G-algebra (G, A, α) is *separable* if A is a separable C^*-algebra and G is second countable (hence also metrizable).

Given a G-algebra (G, A, α), we denote by A^G its fixed point algebra

$$A^G = \big\{a \in A\colon \alpha_s(a) = a \text{ for all } s \in G\big\}$$

(even when G is not compact), and by $A \rtimes_\alpha G$ the (maximal) crossed product of (G, A, α).

If A is a C^*-algebra, we denote by A^+ its unitization (adding a new identity even if A already has an identity). We let $\widetilde{A}$ be A^+ when A is not unital and be A when A is unital. If A is a G-algebra, then A^+ and $\widetilde{A}$ are both G-algebras in an obvious way.

Subalgebras of C^*-algebras are always assumed to be C^*-subalgebras, and ideals are always closed and two sided.

We use the convention $\mathbb{N} = \{1, 2, \ldots\}$.

1. Equivariant semiprojectivity and equivariant projectivity

In this section we recall the definition of equivariant semiprojectivity. We also give a nonunital version, and we will see in Lemma 1.5, in Lemma 1.6, and in

Proposition 1.7 how the two variants are related. We also introduce equivariant projectivity.

The unital case of the following definition is [Phi12, Definition 1.1].

Definition 1.1. A separable G-algebra (G, A, α) is called *equivariantly semiprojective* if whenever (G, C, γ) is a G-algebra, $J_1 \subset J_2 \subset \cdots$ is an increasing sequence of G-invariant ideals in C, $J = \overline{\bigcup_{n=1}^{\infty} J_n}$, $\pi_n \colon C/J_n \to C/J$ is the quotient *-homomorphism for $n \in \mathbb{N}$, and $\varphi \colon A \to C/J$ is a G-morphism, then there exist $n \in \mathbb{N}$ and a G-morphism $\psi \colon A \to C/J_n$ such that $\pi_n \circ \psi = \varphi$.

When no confusion can arise, we say that A is equivariantly semiprojective, or that α is semiprojective.

We say that a unital G-algebra (G, A, α) is *equivariantly semiprojective in the unital category* if the same condition holds, but under the additional assumption that C and φ are unital, and the additional requirement that one can choose ψ to be unital.

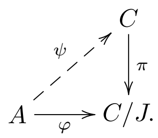
The lifting problem of the definition means that in the right diagram that appears below Definition 1.2, the solid arrows are given, and n and ψ are supposed to exist which make the diagram commute.

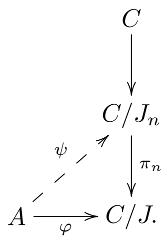
Definition 1.2. A separable (G, A, α) is called *equivariantly projective* if whenever (G, C, γ) is a G-algebra, whenever J is a G-invariant ideal in C with quotient *-homomorphism $\pi \colon C \to C/J$, and whenever $\varphi \colon A \to C/J$ is a G-morphism, then there exists a G-morphism $\psi \colon A \to C$ such that $\pi \circ \psi = \varphi$.

When no confusion can arise, we say that A is equivariantly projective, or that α is projective.

We say that a unital G-algebra is *equivariantly projective in the unital category* if the same condition holds, but under the additional assumption that C and φ are unital, and the additional requirement that one can choose ψ to be unital.

The lifting problem of the definition means that the left diagram on the right can be completed. Again, the solid arrows are given, and ψ is supposed to exist which makes the diagram commute.





When working with semiprojectivity and projectivity, it is often convenient, in the notation of Definition 1.1 and Definition 1.2, to require that the map φ be an isomorphism.

This can also be done in the equivariant case. The proof follows that of [Bla04, Proposition 2.2]. We give the proof since [Bla04] is a survey article and its proof skips some details.

Lemma 1.3. *Let A be a C^*-algebra, let $B \subset A$ be a C^*-subalgebra, and let $I_1 \subset I_2 \subset \cdots \subset A$ be ideals. Then $B \cap \overline{\bigcup_{n=1}^{\infty} I_n} = \overline{\bigcup_{n=1}^{\infty} (B \cap I_n)}$.*

Proof. We have $\bigcup_{n=1}^{\infty} (B \cap I_n) \subset B \cap \overline{\bigcup_{n=1}^{\infty} I_n}$, so $\overline{\bigcup_{n=1}^{\infty} (B \cap I_n)} \subset B \cap \overline{\bigcup_{n=1}^{\infty} I_n}$.

For the reverse, let $b \in B$ and suppose that $b \notin \overline{\bigcup_{n=1}^{\infty} (B \cap I_n)}$. Let ρ be the norm of the image of b in $B / \overline{\bigcup_{n=1}^{\infty} (B \cap I_n)}$. For $n \in \mathbb{N}$, let $\kappa_n \colon B \to B/(B \cap I_n)$ and $\pi_n \colon A \to A/I_n$ be the quotient maps. Then $\|\kappa_n(b)\| \geq \rho$ for all $n \in \mathbb{N}$. The inclusion $\iota \colon B \to A$ induces injective *-homomorphisms $\iota_n \colon B/(B \cap I_n) \to A/I_n$ such that $\pi_n \circ \iota = \iota_n \circ \kappa_n$. Since ι_n is isometric, we have

$$\|(\pi_n \circ \iota)(b)\| = \|\iota_n(\kappa_n(b))\| = \|\kappa_n(b)\| \geq \rho,$$

whence $\operatorname{dist}(b, I_n) \geq \rho$. This is true for all $n \in \mathbb{N}$, so $b \notin \overline{\bigcup_{n=1}^{\infty} I_n}$. $\square$

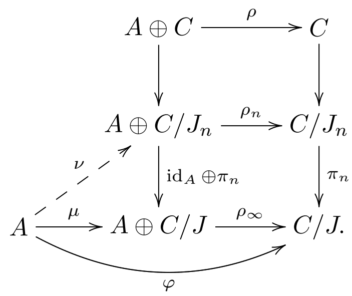
Proposition 1.4. *Let (G, A, α) be a separable G-algebra. Like for usual (semi)projectivity, the definitions of equivariant semiprojectivity (Definition 1.1) and of equivariant projectivity (Definition 1.2) for (G, A, α), in both the unital and nonunital categories, are unchanged if, in the notation of these definitions, we require one or both of the following:*

(1) φ is injective.
(2) φ is surjective.

Proof. We give the proof for equivariant semiprojectivity in the unital category. The other cases are similar but slightly simpler.

Throughout, let the notation be as in Definition 1.1.

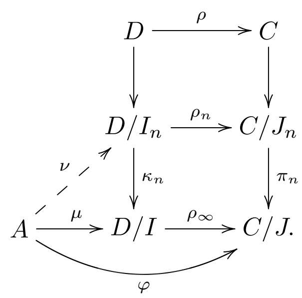
We first prove the result for the restriction (1). So assume that C, $J_1 \subset J_2 \subset \cdots \subset C$, J, quotient maps $\pi_n \colon C/J_n \to C/J$, and $\varphi \colon A \to C/J$, all as in Definition 1.1, are given. The following diagram shows the algebras and maps to be constructed:



Equip $A \oplus C$, $A \oplus C/J_n$ for $n \in \mathbb{N}$, and $A \oplus C/J$ with the direct sum actions of G. Let $\rho \colon A \oplus C \to C$, $\rho_n \colon A \oplus C/J_n \to C/J_n$ for $n \in \mathbb{N}$, and $\rho_\infty \colon A \oplus C/J \to C/J$ be the projections on the second summand. Define $\mu \colon A \to A \oplus C/J$ by $\mu(a) = (a, \varphi(a))$ for $a \in A$. Then μ is a unital injective G-morphism such that $\rho_\infty \circ \mu = \varphi$. By hypothesis, there are $n \in \mathbb{N}$ and a unital G-morphism $\nu \colon A \to A \oplus C/J_n$ such that $\kappa_n \circ \nu = \mu$. Then the map $\psi = \rho_n \circ \nu$ is a unital G-morphism such that $\pi_n \circ \nu = \varphi$. This completes the proof of (1).

We now prove that the condition is equivalent when both restrictions (2) and (1) are applied. It follows that the condition is also equivalent when only (2) is applied.

So let the notation be as before, and assume in addition that φ is injective. The following diagram shows the algebras and maps to be constructed:



Set $D = \pi^{-1}(\varphi(A))$ and $I = D \cap J$. For $n \in \mathbb{N}$ set $I_n = D \cap J_n$. Then $\overline{\bigcup_{n=1}^{\infty} I_n} = I$ by Lemma 1.3.

Let $\rho \colon D \to C$ be the inclusion. Then ρ drops to a *-homomorphism $\rho_n \colon D/I_n \to C/J_n$ for every $n \in \mathbb{N}$, and to a *-homomorphism $\rho_\infty \colon D/I \to C/J$. All these maps are injective. Clearly the range of φ is contained in $\rho_\infty(D/I)$, so there is a *-homomorphism $\mu \colon A \to D/I$ such that $\rho_\infty \circ \mu = \varphi$. This *-homomorphism is injective because φ is and surjective by the definition of D. The hypothesis implies that there are $n \in \mathbb{N}$ and $\nu \colon A \to D/I_n$ such that $\kappa_n \circ \nu = \mu$. Then the map $\psi = \rho_n \circ \nu$ satisfies $\pi_n \circ \nu = \varphi$. □

It is a standard result in the theory of semiprojectivity (contained in Lemma 14.1.6 and Theorem 14.1.7 of [Lor97b]) that for a nonunital C^*-algebra A the following are equivalent:

(1) A is semiprojective.
(2) $\widetilde{A}$ is semiprojective in the unital category.
(3) $\widetilde{A}$ is semiprojective.

In the equivariant case, the equivalence of all three conditions holds when the group G is compact. The proof of the analog of the implications from (1) to (3) and from (2) to (3) breaks down when the trivial action of G on $\mathbb{C}$ is not semiprojective in the nonunital category, but the remaining implications hold in general. The trivial action on $\mathbb{C}$ is always semiprojective in the *unital* category, but we will show in Corollary 6.5 that it is semiprojective in the nonunital category only if G is compact.

Lemma 1.5. *Let (G, A, α) be a separable G-algebra, with A nonunital. Then A is equivariantly (semi)projective if and only if $\widetilde{A}$ is equivariantly (semi)projective in the unital category.*

Proof. We give the proof for equivariant semiprojectivity. The proof for equivariant semiprojectivity is similar but easier. We use the notation of Definition 1.1.

Since A is nonunital, we have $\widetilde{A} = A^+$.

First assume that A is equivariantly semiprojective, and that C and $\varphi \colon A^+ \to C/J$ are unital. By equivariant semiprojectivity of A, there are $n \in \mathbb{N}$ and $\psi_0 \colon A \to C/J_n$ such that $\pi_n \circ \psi_0 = \varphi|_A$. Then the formula $\psi(a + \lambda \cdot 1_{A^+}) = \psi_0(a) + \lambda \cdot 1_{C/J_n}$, for $a \in A$ and $\lambda \in \mathbb{C}$, defines a G-morphism $\psi \colon A^+ \to C/J$ such that $\pi_n \circ \psi = \varphi$. We have shown that $\widetilde{A}$ is equivariantly semiprojective in the unital category.

Now assume that A^+ is equivariantly semiprojective in the unital category, and in the notation of Definition 1.1 take C and $\varphi \colon A \to C/J$ to be not necessarily unital. We have obvious isomorphisms $C^+/J_n \cong (C/J_n)^+$ for $n \in \mathbb{N}$ and $C^+/J \cong (C/J)^+$. (We add a new unit even if C is already unital.) Let $\nu_n \colon C^+/J_n \to \mathbb{C}$ for $n \in \mathbb{N}$, and $\nu_\infty \colon C^+/J \to \mathbb{C}$, be the maps associated with the unitizations. Define a unital G-morphism $\varphi^+ \colon A^+ \to C^+/J$ by $\varphi^+((a + \lambda \cdot 1_{A^+}) = \varphi(a) + \lambda \cdot 1_{C^+/J}$ for $a \in A$ and $\lambda \in \mathbb{C}$. For $n \in \mathbb{N}$, similarly define $\pi_n^+ \colon C^+/J_n \to C^+/J$, giving $\nu_\infty \circ \pi_n^+ = \nu_n$. By hypothesis, there are $n \in \mathbb{N}$ and $\psi_0 \colon A^+ \to C^+/J_n$ such that $\pi_n^+ \circ \psi_0 = \varphi^+$.

We claim that $\psi_0(A) \subset C/J_n$. We have

$$\nu_n \circ \psi_0 = \nu_\infty \circ \pi_n^+ \circ \psi_0 = \nu_\infty \circ \varphi^+,$$

which vanishes on A. The claim follows. So $\psi = \psi_0|_A \colon A \to C/J_n$ is a G-morphism such that $\pi_n \circ \psi = \varphi$. □

Lemma 1.6. *Let (G, A, α) be a separable G-algebra, with A unital. If A is equivariantly semiprojective, then A is equivariantly semiprojective in the unital category. If G is compact, then the converse also holds.*

Proof. The proof is essentially the same as that of Lemma 14.1.6 of [Lor97b]. In the first paragraph of the proof there, B_l should be C_l and it is $1 - \varphi_l(1)$, not $\varphi_l(1) - 1$, that is a projection. In the second paragraph of the proof there, we need the equivariant version of Lemma 14.1.5 of [Lor97b]; it follows from Corollary 1.9 of [Phi12]. □

For compact groups, we now obtain the analog of the equivalence of the first two parts in [Lor97b, Theorem 14.1.7].

Proposition 1.7. *Let G be a second countable compact group, and let A be a separable G-algebra. Then A is equivariantly semiprojective if and only if $\widetilde{A}$ is.*

Proof. Combine Lemma 1.5 and Lemma 1.6. □

The paper [Phi12] contains many examples of equivariantly semiprojective C^*-algebras. In particular, it is shown that for a semiprojective C^*-algebra A and a second countable compact group G, the trivial action of G on A is semiprojective [Phi12, Corollary 1.9]. In the same way one may prove the analog for the projective case, and we include the short argument for completeness. The following lemma is an immediate consequence of [Phi12, Lemma 1.6].

Lemma 1.8. *Let G be a second countable compact group and let $\pi\colon A \to B$ be a surjective G-morphism of G-algebras. Then the restriction of π to the fixed point algebras is surjective, that is, $\pi(A^G) = B^G$.*

Lemma 1.9. *Let G be a second countable compact group, let A be a projective C^*-algebra, and let $\iota\colon G \to \mathrm{Aut}(A)$ be the trivial action. Then (G, A, ι) is equivariantly projective.*

Proof. By Proposition 1.4, it is enough to show that any surjective G-morphism $\pi\colon C \to A$ has an equivariant right inverse. Since G acts trivially on A, we have $A^G = A$, and then $\pi(C^G) = A$ by Lemma 1.8. We can now use the projectivity of A to get a *-homomorphism $\gamma\colon A \to C^G$ such that $\pi \circ \gamma = \mathrm{id}$. Let ψ be the composition of γ with the inclusion of C^G in C. Then ψ is the desired equivariant right inverse of π. □

Remark 1.10. The statement of Lemma 1.8 does not necessarily hold if G is not compact. In fact, for every (second countable) noncompact group G, we construct in the proof of Theorem 6.4 a surjective G-morphism $\pi\colon A \to B$ such that $\pi(A^G) \neq B^G$.

So far, we have only seen semiprojective actions of compact groups. In the next section we will show that every countable discrete group even admits projective actions.

2. Free and orthogonal Bernoulli shifts

In this section, we introduce for every countable discrete group G and C^*-algebra A a natural action, called the free Bernoulli shift, of G on the full free product $\ast_{g\in G} A$. We show (Proposition 2.4) that this action is (semi)projective if A is (nonequivariantly) semiprojective.

We also investigate the orthogonal Bernoulli shift of G on $\bigoplus_{g\in G} A$, that is, the translation action of G on $C_0(G, A)$. It seems to be much more difficult to determine when this action is (semi)projective. In Proposition 2.10 we give a positive answer for the special case that G is finite cyclic of order 2^n.

We use the following notation, roughly as before Remark 3.1.2 of [Lor97b], for the universal C^*-algebra on countably many contractions.

Notation 2.1. Set

$$\mathcal{F}_\infty = C^*\big\langle z_1, z_2, \ldots \mid \|z_j\| \le 1 \text{ for } j \in \mathbb{N}\big\rangle,$$

the universal C^*-algebra on generators $z_1, z_2, \ldots$ with relations $\|z_j\| \le 1$ for $j \in \mathbb{N}$.

Let G be a countable discrete group. Set $P_G = \ast_{g \in G} \mathcal{F}_\infty$. For $s \in G$ let $\iota_s \colon \mathcal{F}_\infty \to \ast_{g\in G}\mathcal{F}_\infty$ be the map which sends $\mathcal{F}_\infty$ to the copy of $\mathcal{F}_\infty$ in P_G indexed by s. We identify P_G with

$$C^*\big\langle \{z_{s,k} \colon s \in G \text{ and } k \in \mathbb{N}\} \mid \|z_{s,k}\| \leq 1 \text{ for } s \in G \text{ and } k \in \mathbb{N}\big\rangle,$$

in such a way that $\iota_s(z_k) = z_{s,k}$ for $s \in G$ and $k \in \mathbb{N}$.

Any separable C^*-algebra A is a quotient of $\mathcal{F}_\infty$. This is just the fact that A contains a countable set of contractive generators. Similarly, for any countable discrete group G, every separable G-algebra A is an equivariant quotient of P_G.

Lemma 2.2. *Let A be a C^*-algebra and let G be a discrete group. For $s \in G$ let $\iota_{A,s} \colon A \to \ast_{g\in G} A$ be the map which sends A to the copy of A in $\ast_{g\in G}A$ indexed by s. Then there exists a unique action $\tau^A \colon G \to \operatorname{Aut}\big(\ast_{g\in G}A\big)$ such that $\tau^A_g(\iota_{A,s}(a)) = \iota_{A,gs}(a)$ for all $g, s \in G$ and $a \in A$. Moreover, for every *-homomorphism $\varphi \colon A \to B$ between C^*-algebras A and B, the corresponding *-homomorphism $\ast_{g\in G}\, \varphi \colon \ast_{g\in G} A \to \ast_{g \in G} B$ is equivariant.*

Proof. This is immediate. □

Definition 2.3. Let A be a separable C^*-algebra and let G be a countable discrete group. The action τ^A of Lemma 2.2 is called the *free Bernoulli shift based on A*. If $A = \mathcal{F}_\infty$, so that $\ast_{g\in G}A = P_G$, we call it the *universal free Bernoulli shift*, and denote it by τ.

Following Notation 2.1, we have $\tau_s(z_{t,k}) = z_{st,k}$ for $s, t \in G$ and $k \in \mathbb{N}$.

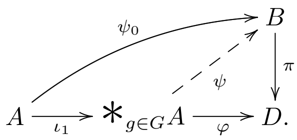
The action $\tau \colon G \to \operatorname{Aut}(P_G)$ is universal for all G-actions, that is, for every separable G-algebra A there exists a surjective G-morphism $P_G \to A$.

Proposition 2.4. *Let A be a separable C^*-algebra and let G be a countable discrete group. If A is (semi)projective, then the free Bernoulli shift of G based on A is (semi)projective.*

Proof. We give the proof when A is projective. The semiprojective case is very similar, but has bigger diagrams.

Let (G, B, β) and (G, D, δ) be G-algebras, let $\pi \colon B \to D$ be a surjective G-morphism, and let $\varphi \colon \ast_{g\in G}A \to D$ be a G-morphism. We prove that there is a G-morphism $\psi \colon \ast_{g \in G} A \to B$ such that $\pi \circ \psi = \varphi$.

Since A is projective, we can find a *-homomorphism $\psi_0 \colon A \to B$ such that $\pi \circ \psi_0 = \varphi \circ \iota_1$. By universality of $\ast_{g\in G}A$, there is a *-homomorphism $\psi \colon \ast_{g\in G}A \to B$ such that $\psi \circ \iota_{A,s} = \beta_s \circ \psi_0$ for all $s \in G$. The following diagram shows some of the maps:



It remains to show that $\pi \circ \psi = \varphi$ and that ψ is G-equivariant.

Let $t \in G$. Using equivariance of π at the second step and equivariance of φ at the fourth step, we get

$$\pi \circ \psi \circ \iota_{A,t} = \pi \circ \beta_t \circ \psi_0 = \delta_t \circ \pi \circ \psi_0 = \delta_t \circ \varphi \circ \iota_1 = \varphi \circ \tau_t \circ \iota_1 = \varphi \circ \iota_{A,t}.$$

Since this is true for all $t \in G$, and since $\bigcup_{t\in G} \iota_{A,t}(A)$ generates $\ast_{g\in G}A$, it follows that $\pi \circ \psi = \varphi$.

To see that ψ is equivariant, let $s, t \in G$. We compute:

$$\beta_s \circ \psi \circ \iota_{A,t} = \beta_s \circ \beta_t \circ \psi_0 = \beta_{st} \circ \psi_0 = \psi \circ \iota_{A,st} = \psi \circ \tau_s \circ \iota_{A,t}.$$

For the same reason as in the previous paragraph, it follows that $\beta_s \circ \psi = \psi \circ \tau_s$, and so ψ is G-equivariant. □

Remark 2.5. Let G be countable discrete group. The universal G-algebra P_G is (nonequivariantly) projective, since it is isomorphic to $\mathcal{F}_\infty$. We can use this to show that if $\alpha\colon G \to \mathrm{Aut}(A)$ is a projective action, then A must be projective. This is a special case of Corollary 4.21.

Using the universal property of P_G and separability of A, we can find a surjective G-morphism $\rho\colon P_G \to A$. Since α is equivariantly projective, we can find a G-morphism $\lambda\colon A \to P_G$ such that $\rho \circ \lambda = \mathrm{id}_A$. If now $\varphi\colon A \to C/J$ is a *-homomorphism, there is a *-homomorphism $\psi\colon P_G \to C$ which lifts $\varphi \circ \rho$. Then $\psi \circ \lambda$ lifts φ.

A more involved argument, which we do not give here, gives a similar result for finite groups and semiprojectivity.

We now turn to what we call the orthogonal Bernoulli shift.

Definition 2.6. Let A be a separable C^*-algebra and let G be a countable discrete group. The *orthogonal Bernoulli shift based on A* is the action $\sigma^A\colon G \to \mathrm{Aut}(C_0(G, A))$ given by $\sigma_s^A(a)(t) = a(s^{-1}t)$ for $a \in C_0(G, A)$ and $s, t \in G$.

We think of $C_0(G, A)$ as $\bigoplus_{g \in G} A$. Then the automorphism σ_s^A sends the summand indexed by $t \in G$ to the summand indexed by st.

By analogy with equivariant semiprojectivity of actions of compact groups on finite dimensional C^*-algebras (Theorem 2.6 of [Phi12]) and projectivity of $C_0((0, 1])$, it seems reasonable to hope that the orthogonal Bernoulli shift based on $C_0((0, 1])$ is projective whenever G is finite. This seems difficult to prove; we have been able to do so only for $G = \mathbb{Z}_{2^n}$, the finite cyclic group of order 2^n. (See Proposition 2.10.) We start with some lemmas.

Lemma 2.7. *Let $n \in \mathbb{N}$. Let B be a C^*-algebra, let $\beta \in \mathrm{Aut}(B)$ satisfy $\beta^{2^n} = \mathrm{id}_B$, let I be a β-invariant ideal in B, let $\pi\colon B \to B/I$ be the quotient map, and let $\alpha \in \mathrm{Aut}(B/I)$ be the induced automorphism. If $x \in B/I$ is selfadjoint and satisfies $\alpha(x) = -x$, then there is a selfadjoint element $y \in B$ such that $\beta(y) = -y$ and $\pi(y) = x$.*

Proof. First lift x to a selfadjoint element $b \in B$. Put

$$y = \frac{1}{2^n}\left[b - \beta(b) + \beta^2(b) - \beta^3(b) + \cdots - \beta^{2^n-1}(b)\right].$$

Then y is selfadjoint, $\beta(y) = -y$, and y is a lift of x. □

Lemma 2.8. *Let $n \in \mathbb{N}$. Let B be a C^*-algebra, let $\beta \in \mathrm{Aut}(B)$ satisfy $\beta^{2^n} = \mathrm{id}_B$, let I be a β-invariant ideal in B, let $\pi\colon B \to B/I$ be the quotient map, and let $\alpha \in \mathrm{Aut}(B/I)$ be the induced automorphism. Let $h_1, h_2 \in B/I$ be positive orthogonal elements such that $\alpha(h_1) = h_2$ and $\alpha(h_2) = h_1$. Then there exist positive orthogonal elements $k_1, k_2 \in B$ such that*

$$\pi(k_1) = h_1, \quad \pi(k_2) = h_2, \quad \beta(k_1) = k_2, \quad \textit{and} \quad \beta(k_2) = k_1.$$

Proof. Put $x = h_1 - h_2$. Since x is selfadjoint and $\alpha(x) = -x$, we can, by Lemma 2.7, lift it to a selfadjoint element $y \in B$ with $\beta(y) = -y$. Let k_1 be the positive part of y, that is, $k_1 = \frac{1}{2}(y + |y|)$. Then $\pi(k_1) = \frac{1}{2}(x + |x|) = h_1$. Put $k_2 = \beta(k_1)$. Routine calculations show that k_2 is the negative part of y, and thus orthogonal to k_1. Essentially the same calculations show that $\beta^2(k_1) = k_1$. It is clear that $\pi(k_2) = h_2$. □

Proposition 2.9. *Let $n \in \mathbb{N}$. Let B be a C^*-algebra, let $\beta \in \mathrm{Aut}(B)$ satisfy $\beta^{2^n} = \mathrm{id}_B$, let I be a β-invariant ideal in B, let $\pi \colon B \to B/I$ be the quotient map, and let $\alpha \in \mathrm{Aut}(B/I)$ be the induced automorphism. Let $h_1, h_2, \ldots, h_{2^n} \in B/I$ be orthogonal positive elements such that $\alpha(h_m) = h_{m+1}$ for $m = 1, 2, \ldots, 2^n - 1$, and such that $\alpha(h_{2^n}) = h_1$. Then they can be lifted to orthogonal positive elements $k_m \in B$ for $m = 1, 2, \ldots, 2^n$ such that $\beta(k_m) = k_{m+1}$ for $m = 1, 2, \ldots, 2^n - 1$ and such that $\beta(k_{2^n}) = k_1$. Moreover, if $\|h_m\| \leq 1$ for $m = 1, 2, \ldots, 2^n$, then we can require that $\|k_m\| \leq 1$ for $m = 1, 2, \ldots, 2^n$.*

Proof. The proof (except for the last statement) is by induction on n. The case $n = 1$ is Lemma 2.8. Let $n > 1$, suppose that we have shown the statement to hold for all natural numbers $l < n$ and all choices of B, β, I, and $h_1, h_2, \ldots, h_{2^l}$, and let B, β, I, and $h_1, h_2, \ldots, h_{2^n}$ be as in the statement.

Set

$$a_1 = h_1 + h_3 + \cdots + h_{2^n - 1} \quad \text{and} \quad a_2 = h_2 + h_4 + \cdots + h_{2^n}.$$

Then $\alpha(a_1) = a_2$, $\alpha(a_2) = a_1$, and $a_1 a_2 = 0$. So, by Lemma 2.8, we can lift a_1 and a_2 to orthogonal positive elements b_1, b_2 such that $\beta(b_1) = b_2$ and $\beta(b_2) = b_1$.

The hereditary subalgebra $\overline{b_1 B b_1} \subset B$ is β^2-invariant and it is easy to check that $\pi\big(\overline{b_1 B b_1}\big) = \overline{a_1 (B/J) a_1}$. Apply the induction hypothesis with $\overline{b_1 B b_1}$ in place of B, with β^2 in place of β, with $\overline{b_1 B b_1} \cap I$ in place of I, and with $h_1, h_3, \ldots, h_{2^n - 1}$ in place of $h_1, h_2, \ldots, h_{2^n}$. We obtain orthogonal positive elements $k_1, k_3, \ldots, k_{2^n - 1} \in \overline{b_1 B b_1}$ such that $\pi(k_m) = h_m$ and $\beta^2(k_m) = k_{m+2}$ for $m = 1, 3, 5, \ldots, 2^n - 1$, and such that $\beta^2(k_{2^n - 1} = k_1$. Set $k_m = \beta(k_{m-1}) \in \overline{b_2 B b_2}$ for $m = 2, 4, 6, \ldots, 2^n$. Then $\pi(k_m) = h_m$ also for $m = 2, 4, 6, \ldots, 2^n$. It is clear that $\beta(k_m) = k_{m+1}$ for $m = 1, 2, \ldots, 2^n - 1$ and that $\beta(k_{2^n}) = k_1$. It only remains to check that the elements k_m are orthogonal for $m = 1, 2, \ldots, 2^n$. The only case needing work is k_l and k_m when one of l and m is even and the other is odd. But then one of k_l and k_m is in $\overline{b_1 B b_1}$ and the other is in $\overline{b_2 B b_2}$, so the desired conclusion follows from $b_1 b_2 = 0$.

It remains to prove the last statement. Let $x_1, x_2, \ldots, x_{2^n} \in B$ be the elements produced in the first part. Let $f \colon [0, \infty) \to [0, 1]$ be the function $f(t) = \min(t, 1)$ for $t \geq 0$. Then set $k_m = f(x_m)$ for $m = 1, 2, \ldots, 2^n$. ☐

Proposition 2.10. *Let A be a separable (semi)projective C^*-algebra, let $n \in \mathbb{N}$, and let $\sigma \colon \mathbb{Z}_n \to \mathrm{Aut}\left(\bigoplus_{m=1}^n A\right)$ be the orthogonal Bernoulli shift of Definition 2.6. If n is a power of 2, then σ is (semi)projective.*

Proof. We give the proof when A is projective. The semiprojective case is analogous, but requires Lemma 1.3.

By Proposition 1.4, it is enough to show that for every G-algebra (G, B, β) and every surjective G-morphism $\pi \colon B \to \bigoplus_{m=1}^n A$, there exists a G-morphism $\psi \colon \bigoplus_{m=1}^n A \to B$ such that $\pi \circ \psi = \varphi$.

Let $\alpha = \sigma_1$, the automorphism corresponding to the generator $1 \in \mathbb{Z}_n$, and similarly let $\gamma = \beta_1 \in \mathrm{Aut}(B)$. For $m = 1, 2, \ldots, n$, let $\iota_m \colon A \to \bigoplus_{m=1}^n A$ be the map that sends A to the summand A in $\bigoplus_{m=1}^n A$ indexed by m, and let $\rho_m \colon \bigoplus_{m=1}^n A \to A$ be the surjection onto the m-th summand. Then $a = \sum_{m=1}^n (\iota_m \circ \rho_m)(a)$ for every $a \in \bigoplus_{m=1}^n A$.

Let h be a strictly positive element of A. For $m = 1, 2, \ldots, n$, set $h_m = \iota_m(h)$. Then $h_1, h_2, \ldots, h_n$ are orthogonal positive elements such that $\alpha(h_m) = h_{m+1}$ for $m = 1, 2, \ldots, n - 1$, and such that $\alpha(h_n) = h_1$. By Proposition 2.9, they can be lifted to orthogonal positive elements $k_m \in B$ for $m = 1, 2, \ldots, 2^n$ such that $\gamma(k_m) = k_{m+1}$ for $m = 1, 2, \ldots, n - 1$ and such that $\gamma(k_n) = k_1$.

Let D be the hereditary subalgebra $D = \overline{k_1 B k_1}$. Then $\pi(D) = \iota_1(A)$. Since A is projective, there exists a *-homomorphism $\psi_1 \colon A \to D$ such that $\pi \circ \psi_1 = \iota_1$.

Define $\psi\colon \bigoplus_{m=1}^{n} A \to B$ by

$$\psi(a) = \sum_{m=1}^{n} \big(\gamma^{m-1} \circ \psi_1 \circ \rho_m\big)(a).$$

It is easily checked that ψ has the desired properties. ☐

Question 2.11. Consider the orthogonal Bernoulli shift $G \to \mathrm{Aut}\left(\bigoplus_G C_0((0,1])\right)$ of Definition 2.6. For which groups is this action projective?

In particular, is it projective for $G = \mathbb{Z}_n$ for all $n \in \mathbb{N}$? Is it projective for $G = \mathbb{Z}$?

3. Equivariant semiprojectivity of restrictions to subgroups

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be a semiprojective action. In this section we investigate semiprojectivity of the restriction of α to a subgroup $H \leqslant G$. In Theorem 3.10, we obtain a positive result when H is cocompact. It follows (Corollary 3.11) that a second countable compact group can only act semiprojectively on a C^*-algebra that is semiprojective in the usual sense. Some condition on H is necessary. For instance, in Example 3.12 we construct a semiprojective action of $\mathbb{Z}$ on a nonsemiprojective C^*-algebra.

To obtain these results, we use the induction functor, which assigns in a natural way to each H-algebra an induced G-algebra. We will show that this functor preserves exact sequences (Proposition 3.6) and behaves well with respect to direct limits (Proposition 3.7).

We begin by recalling the definition of the induction functor, from the beginning of Section 2 of [KW99] or the beginning of Section 6 of [Ech10]. In the following definition, one easily checks that the action defined on the algebra $\mathrm{Ind}_H^G(A)$ is continuous, so that $\big(G, \mathrm{Ind}_H^G(A), \mathrm{Ind}_H^G(\alpha)\big)$ is in fact a G-algebra, and that Ind_H^G really is a functor.

Definition 3.1. For a locally compact group G, we let $\mathcal{C}_G$ denote the category whose objects are G-algebras and whose morphisms are G-equivariant *-homomorphisms (also called G-morphisms).

Now let $H \leqslant G$ be a closed subgroup, and let (H, A, α) be an object in $\mathcal{C}_H$. We define an object $\big(G, \mathrm{Ind}_H^G(A), \mathrm{Ind}_H^G(\alpha)\big)$ in $\mathcal{C}_G$ as follows. We take

$$\mathrm{Ind}_H^G(A) = \left\{ f \in C_{\mathrm{b}}(G, A)\colon \begin{array}{c} \alpha_h(f(sh)) = f(s) \text{ for all } s \in G \text{ and } h \in H \\ \text{and } sH \mapsto \|f(s)\| \text{ is in } C_0(G/H) \end{array} \right\}.$$

The induced action $\mathrm{Ind}_H^G(\alpha)\colon G \to \mathrm{Aut}\left(\mathrm{Ind}_H^G(A)\right)$ is given by

$$\big(\mathrm{Ind}_H^G(\alpha)\big)_s(f)(t) = f(s^{-1}t)$$

for $f \in \mathrm{Ind}_H^G(A)$ and $s, t \in G$. If A and B are H-algebras and $\varphi\colon A \to B$ is an H-morphism, then the induced G-morphism $\mathrm{Ind}_H^G(\varphi)\colon \mathrm{Ind}_H^G(A) \to \mathrm{Ind}_H^G(B)$ is given by

$$\mathrm{Ind}_H^G(\varphi)(f)(s) = \varphi(f(s))$$

for $f \in \mathrm{Ind}_H^G(A)$ and $s \in G$.

The induction functor is often defined on a different category than that considered here. The objects are still G-algebras, but the morphisms are equivariant Hilbert bimodules. We refer to Section 6 of [Ech10] and to [EKQR00] for more details.

We next recall the definition of a $C_0(X)$-algebra. See Section 4.5 of [Phi87], Definition 1.5 of [Kas88], or Definition 2.6 of [Bln96]. We recall that if A is a C^*-algebra, then $M(A)$ is its multiplier algebra and $Z(A)$ is its center.

Definition 3.2. Let X be a locally compact Hausdorff space. A *$C_0(X)$-algebra* is a C^*-algebra A together with a *-homomorphism $\eta\colon C_0(X) \to Z(M(A))$, called the structure map, such that

$$\{\eta(f)a\colon f \in C_0(X) \text{ and } a \in A\}$$

is dense in A.

We will usually write fa or $f \cdot a$ instead of $\eta(f)a$ for the product of a function $f \in C_0(X)$ and an element $a \in A$. For an open set $U \subset X$, we set

$$A(U) = \{fa\colon f \in C_0(U) \text{ and } a \in A\},$$

which is an ideal of A. (See Proposition 3.3(2).) For a closed subset $Y \subset X$, we denote by $A(Y)$ the quotient $A/A(X \setminus Y)$.

For $x \in X$ we write $A(x)$ for $A(\{x\})$, and this C^*-algebra is called the *fiber of A at x*. Given $a \in A$, we denote its image in the fiber $A(x)$ by $a(x)$, and we define $\check{a}\colon X \to [0, \infty)$ by $\check{a}(x) = \|a(x)\|$ for $x \in X$. We call A a *continuous* $C_0(X)$-algebra if $\check{a}$ is continuous for each $a \in A$.

If A and B are $C_0(X)$-algebras and $\varphi\colon A \to B$ is a *-homomorphism, then φ is said to be a *$C_0(X)$-morphism* if $\varphi(f \cdot a) = f \cdot \varphi(a)$ for all $f \in C_0(X)$ and $a \in A$.

We recall the following facts about $C_0(X)$-algebras.

Proposition 3.3. *Let X be a locally compact Hausdorff space and let A be a $C_0(X)$-algebra with structure map $\eta\colon C_0(X) \to Z(M(A))$. Then:*

(1) *$A = \{\eta(f)a\colon f \in C_0(X)$ and $a \in A\}$.*
(2) *If $U \subset A$ is open then $A(U)$ is an ideal in A.*
(3) *For $a \in A$, the function $\check{a}$ is an upper semicontinuous function on X which vanishes at infinity.*
(4) *For $a \in A$, we have $\|a\| = \sup_{x \in X} \check{a}(x)$.*

Proof. Part (1) is Proposition 1.8 of [Bln96]. (This is essentially the Cohen Factorization Theorem.)

For (2), it follows from Corollary 1.9 of [Bln96] that $A(U)$ is a closed $C_0(X)$-submodule of A. It now easily follows that $A(U)$ is an ideal.

Part (3) is [Rie89, Proposition 1.2].

Part (4) is Proposition 2.8 of [Bln96]. □

We refer to Section 2 of [Bln96] for more details on $C_0(X)$-algebras.

Proposition 3.4. *Let G be a locally compact group, let $H \leqslant G$ be a closed subgroup, and let (H, A, α) be an H-algebra. Define $\eta\colon C_0(G/H) \to Z(M(\mathrm{Ind}_H^G(A)))$ by*

$$(\eta(g)f)(s) = g(sH) \cdot f(s)$$

for $g \in C_0(G/H)$, $f \in \mathrm{Ind}_H^G(A)$, and $s \in G$. This map makes $\mathrm{Ind}_H^G(A)$ a continuous $C_0(G/H)$-algebra. Moreover:

(1) *If (H, B, β) is a second H-algebra, and $\varphi\colon A \to B$ is an H-morphism, then $\mathrm{Ind}_H^G(\varphi)$ is a morphism of $C_0(G/H)$-algebras.*
(2) *For every $x \in G$, the map $\mathrm{ev}_x\colon \mathrm{Ind}_H^G(A) \to A$, which evaluates a function in $\mathrm{Ind}_H^G(A)$ at x, defines an isomorphism from $\mathrm{Ind}_H^G(A)(xH)$ to A.*

In particular, the fibers of $\mathrm{Ind}_H^G(A)$ as a $C_0(G/H)$-algebra are all isomorphic to A. However, the isomorphism is not canonical. In the proof below, the isomorphism for the fiber at $xH \in G/H$ depends on the choice of the coset representative x.

Proof of Proposition 3.4. It is easy to check that η makes $\mathrm{Ind}_H^G(A)$ a continuous $C_0(G/H)$-algebra, and we omit the details. The proof of (1) is immediate. It remains to prove (2). We abbreviate $\mathrm{Ind}_H^G(A)$ to $\mathrm{Ind}(A)$.

Let $x \in G$. We show that ev_x is surjective. It is immediate that

$$\ker(\mathrm{ev}_x) = \mathrm{Ind}(A)(G/H \setminus \{xH\}),$$

so this will complete the proof.

Since ev_x is a *-homomorphism, it is enough to show that it has dense image in A. So let $a \in A$ and let $\varepsilon > 0$. We want to find $f \in \mathrm{Ind}_H^G(A)$ such that $\|f(x) - a\| < \varepsilon$. Let μ denote the Haar measure of H. Since the action is continuous, there exists an open neighborhood $U \subset H$ of the identity element $1 \in H$, with compact closure, such that $\|\alpha_s(a) - a\| \leq \frac{\varepsilon}{2}$ for all $s \in U$. Let $\chi\colon G \to [0,1]$ be a nonzero continuous function with $\mathrm{supp}(\chi) \subset U$. By scaling, we may assume $\int_H \chi \, d\mu = 1$. We define a function $f\colon G \to A$ by

$$f(s) = \int_H \chi(x^{-1}st) \cdot \alpha_t(a) \, d\mu(t)$$

for $s \in G$. The integral exists for all s, since the integrand is continuous and has compact support. We will now check that f has the desired properties.

For $s \in G$ and $h \in H$ we have, using left invariance of μ at the last step,

$$\alpha_h(f(sh)) = \alpha_h\left(\int_H \chi(x^{-1}sht) \cdot \alpha_t(a) \, d\mu(t)\right) = \int_H \chi(x^{-1}sht) \cdot \alpha_{ht}(a) \, d\mu(t) = f(s).$$

The function $sH \mapsto \|f(s)\|$ has compact support, so $f \in \mathrm{Ind}(A)$. Moreover,

$$\begin{aligned} \|f(x) - a\| &= \left\| \int_H \chi(t) \cdot \alpha_t(a) \, d\mu(t) - \int_H \chi(t) \cdot a \, d\mu(t) \right\| \\ &\leq \int_H \chi(t) \|\alpha_t(a) - a\| \, d\mu(t) \leq \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

This completes the proof that ev_x is surjective. □

The following result is similar to Lemma 3.2 of [TW12]. It is Lemma 2.1(iii) of [Dad09], but the proof given there assumes that X is compact.

Lemma 3.5. *Let A be a $C_0(X)$-algebra with structure map $\eta\colon C_0(X) \to Z(M(A))$. Assume $B \subset A$ is a C^*-subalgebra satisfying the following two conditions:*

(1) For each $x \in X$, the set $\{b(x)\colon b \in B\}$ exhausts the fiber $A(x)$.
(2) $\eta(C_0(X))B \subset B$, that is, B is invariant under multiplication by functions in $C_0(X)$.

Then $A = B$.

Proof. It suffices to show that B is dense in A. Let $a \in A$ and let $\varepsilon > 0$. Using Proposition 3.3(1), choose $f \in C_0(X)$ and $a_0 \in A$ such that $fa_0 = a$. Choose $g \in C_{\mathrm{c}}(X)$ such that $\|f - g\| < \varepsilon/(2\|a_0\|)$. Then $\|ga_0 - a\| < \frac{\varepsilon}{2}$ and $(ga_0)(x) = 0$ for $x \in X \setminus \mathrm{supp}(g)$.

For each point $x \in \mathrm{supp}(g)$, choose $b_x \in B$ such that $b_x(x) = (ga_0)(x)$. By Proposition 3.3(3), there is an open set $U_x \subset X$ with $x \in U_x$ such that for all $y \in U_x$ we have $\|b_x(y) - (ga_0)(y)\| < \frac{\varepsilon}{2}$. Choose $x_1, x_2, \ldots, x_n \in \mathrm{supp}(g)$ such that the sets $U_{x_1}, U_{x_2}, \ldots, U_{x_n}$ cover $\mathrm{supp}(g)$. Choose $h_1, h_2, \ldots, h_n \in C_{\mathrm{c}}(X)$ such that for $k = 1, 2, \ldots, n$ we have $\mathrm{supp}(h_k) \subset U_{x_k}$ and $0 \leq h_k \leq 1$, and such that $\sum_{k=1}^n h_k \leq 1$ and is equal to 1 on $\mathrm{supp}(g)$. Set $b = \sum_{k=1}^n h_k b_{x_k}$. Then $b \in B$. We claim that $\|b - ga_0\| \leq \frac{\varepsilon}{2}$. This will imply that $\|b - a\| < \varepsilon$, and complete the proof.

It suffices to show that $\|b(y) - (ga_0)(y)\| \leq \frac{\varepsilon}{2}$ for $y \in X$. Set $h_0 = 1 - \sum_{k=1}^n h_k$. Then $ga_0 = \sum_{k=0}^n h_k g a_0$. Set $b_k = b_{x_k}$ and $U_k = U_{x_k}$ for $k = 1, 2, \ldots, n$, and set $b_0 = 0$ and $U_0 = X \setminus \mathrm{supp}(g)$. Then for $k = 0, 1, \ldots, n$, we have $\|b_k(y) - (ga_0)(y)\| <$

$\frac{\varepsilon}{2}$ whenever $h_k(y) \neq 0$. Using this fact at the second step, we have

$$\|b(y) - (ga_0)(y)\| \leq \sum_{k=0}^{n} h_k(y)\|b_k(y) - (ga_0)(y)\| \leq \sum_{k=0}^{n} h_k(y) \cdot \frac{\varepsilon}{2} \leq \frac{\varepsilon}{2}.$$

This proves the claim, and completes the proof. □

The following result is Lemma 3.8 of [KW99], but the proof given in [KW99] does not address surjectivity of $\mathrm{Ind}_H^G(\pi)$.

Proposition 3.6. *Let G be a locally compact group, and let $H \leqslant G$ be a closed subgroup. Then the induction functor $\mathrm{Ind}_H^G \colon \mathcal{C}_H \to \mathcal{C}_G$ is exact, that is, given an H-equivariant short exact sequence of H-algebras*

$$0 \longrightarrow I \xrightarrow{\iota} A \xrightarrow{\pi} B \longrightarrow 0,$$

the induced G-equivariant sequence of G-algebras

$$0 \longrightarrow \mathrm{Ind}_H^G(I) \xrightarrow{\mathrm{Ind}_H^G(\iota)} \mathrm{Ind}_H^G(A) \xrightarrow{\mathrm{Ind}_H^G(\pi)} \mathrm{Ind}_H^G(B) \longrightarrow 0$$

is also exact.

Proof. To simplify notation, we abbreviate Ind_H^G to Ind.

We may think of I as an H-invariant ideal in A, so that ι is just the inclusion. It follows that $\mathrm{Ind}(I)$ may be considered as an ideal in $\mathrm{Ind}(A)$, and then $\mathrm{Ind}(\iota)$ is also just the inclusion morphism.

It is straightforward to check that the sequence is exact in the middle, that is, $\ker(\mathrm{Ind}(\pi)) = \mathrm{Ind}(I) \subset \mathrm{Ind}(A)$. Thus, it remains to check that $\mathrm{Ind}(\pi)$ is surjective. Following Proposition 3.4, we consider $\mathrm{Ind}(A)$ and $\mathrm{Ind}(B)$ as $C_0(G/H)$-algebras. We want to apply Lemma 3.5.

Condition (2) of Lemma 3.5 follows immediately from Proposition 3.4(1).

Let us verify condition (1). For $x \in G$, let $\mathrm{ev}_x^A \colon \mathrm{Ind}(A) \to A$ and $\mathrm{ev}_x^B \colon \mathrm{Ind}(B) \to B$ be the evaluation maps at x. By Proposition 3.4(2), these maps are surjective and implement the isomorphisms $\mathrm{Ind}(A)(xH) \cong A$ and $\mathrm{Ind}(B)(xH) \cong B$. We have $\mathrm{ev}_x^B \circ \mathrm{Ind}(\pi) = \pi \circ \mathrm{ev}_x^A$, that is, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Ind}(A) & \xrightarrow{\mathrm{ev}_x^A} & A \\ {\scriptstyle \mathrm{Ind}(\pi)}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathrm{Ind}(B) & \xrightarrow{\mathrm{ev}_x^B} & B. \end{array}$$

Since ev_x^A and π are surjective, it follows that the image of $\mathrm{Ind}(\pi)$ exhausts each fiber of $\mathrm{Ind}(B)$. This verifies condition (1) of Lemma 3.5. So $\mathrm{Ind}(\pi)$ is surjective. □

Proposition 3.7. *Let G be a locally compact group, and let $H \leqslant G$ be a closed subgroup. Then the induction functor $\mathrm{Ind}_H^G \colon \mathcal{C}_H \to \mathcal{C}_G$ is continuous, that is, given an H-equivariant direct system*

$$A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow \cdots,$$

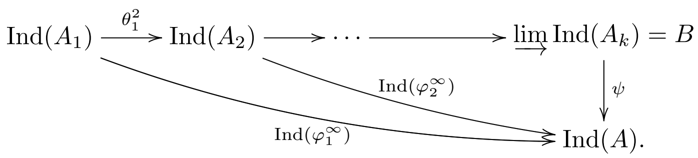
there is a natural isomorphism

$$\mathrm{Ind}_H^G\big(\varinjlim A_k\big) \cong \varinjlim \mathrm{Ind}_H^G(A_k).$$

Proof. To simplify notation, we abbreviate Ind_H^G to Ind. Following Proposition 3.4, we consider the induced algebras as $C_0(G/H)$-algebras.

Denote the connecting H-morphisms by $\varphi_m^n \colon A_m \to A_n$ for $m \leq n$. Let $A = \varinjlim A_k$, and denote the H-morphisms into the direct limit by $\varphi_m^\infty \colon A_m \to A$. Denote the induced G-morphisms by $\theta_m^n \colon \operatorname{Ind}(A_m) \to \operatorname{Ind}(A_n)$, and let $B = \varinjlim \operatorname{Ind}(A_k)$, together with G-morphisms $\theta_m^\infty \colon \operatorname{Ind}(A_m) \to B$.

The maps φ_k^∞ induce G-morphisms $\operatorname{Ind}(\varphi_k^\infty) \colon \operatorname{Ind}(A_k) \to \operatorname{Ind}(A)$, and these induce a G-morphism from the direct limit $\psi \colon B \to \operatorname{Ind}(A)$. The situation is shown in the following commutative diagram:



To show that ψ is surjective, we apply Lemma 3.5.

To verify condition (2) of Lemma 3.5, let $b \in B$ and $f \in C_0(G/H)$ be given. We will show that for every $\varepsilon > 0$ there exists $c \in B$ such that $\|f \cdot \psi(b) - \psi(c)\| < \varepsilon$. Fix $\varepsilon > 0$. By properties of the direct limit, there exist an index k and $a \in \operatorname{Ind}(A_k)$ such that $\|b - \theta_k^\infty(a)\| < \varepsilon / \|f\|$. One checks that $c = \theta_k^\infty(f \cdot a)$ has the desired properties.

To verify condition (1) of Lemma 3.5, we need to show that every fiber of $\operatorname{Ind}(A)$ is exhausted by the image of ψ. We denote by $\operatorname{ev}_x^k \colon \operatorname{Ind}(A_k) \to A_k$ and $\operatorname{ev}_x^\infty \colon \operatorname{Ind}(A) \to A$ the evaluation maps at $x \in G$. Then it is enough to show that $\operatorname{ev}_x^\infty \circ \psi$ is surjective for every $x \in G$.

For each $k \in \mathbb{N}$, we have

$$\operatorname{ev}_x^\infty \circ \psi \circ \theta_k^\infty = \operatorname{ev}_x^\infty \circ \operatorname{Ind}(\varphi_k^\infty) = \varphi_k^\infty \circ \operatorname{ev}_x^k .$$

Since $\operatorname{ev}_x^k \colon \operatorname{Ind}(A_k) \to A_k$ is surjective (by Proposition 3.4(2)), the image of $\operatorname{ev}_x^\infty \circ \psi$ contains the image of φ_k^∞. Thus, the image of $\operatorname{ev}_x^\infty \circ \psi$ contains $\bigcup_{k=1}^\infty \operatorname{ran}(\varphi_k^\infty)$, which is dense in A by properties of the direct limit. It follows that the image of ψ exhausts each fiber of $\operatorname{Ind}(A)$. This verifies Lemma 3.5(1), so ψ is surjective.

To show that ψ is injective, let $b \in B$, and suppose that $\psi(b) = 0$. Let $\varepsilon > 0$; we show that $\|b\| < \varepsilon$. By properties of B as a direct limit, there exist an index $k \in \mathbb{N}$ and $a \in \operatorname{Ind}(A_k)$ such that $\|b - \theta_k^\infty(a)\| < \frac{\varepsilon}{3}$. For $n \geq k$, let $f_n \in C_0(G/H)$ be defined by $f_n(sH) = \|\theta_k^n(a)(sH)\|$. One checks that $(f_n)_{n \in \mathbb{N}}$ is a nonincreasing sequence of functions such that $\lim_{n \to \infty} f_n(sH) < \frac{\varepsilon}{3}$ for each $s \in G$. For $n \in \mathbb{N}$, define a continuous function g_n on the one point compactification $(G/H)^+$ by $g_n(sH) = \max\big(f_n(sH), \frac{\varepsilon}{3}\big)$ for $s \in G$ and $g_n(\infty) = \frac{\varepsilon}{3}$. The functions g_n decrease pointwise to the constant function with value $\frac{\varepsilon}{3}$. Since $(G/H)^+$ is compact, Dini's Theorem (Proposition 11 in Chapter 9 of [Roy88]) implies that the convergence is uniform. So there exists $n \geq k$ such that $\|f_n\| < \frac{2\varepsilon}{3}$. Then $\|\theta_k^n(a)\| = \|f_n\| < \frac{2\varepsilon}{3}$ by Proposition 3.3(4), and thus also $\|\theta_k^\infty(a)\| < \frac{2\varepsilon}{3}$. It follows that $\|b\| < \varepsilon$, as desired.

This completes the proof that ψ is an isomorphism. □

Lemma 3.8. *Let G be a locally compact group, and let $H \leqslant G$ be a closed subgroup. For any H-algebra A, let $\operatorname{ev}_1^A \colon \operatorname{Ind}_H^G(A) \to A$ be the map $\operatorname{ev}_1^A(f) = f(1)$ that evaluates a function at the identity element $1 \in G$. Then ev_1^A is an H-morphism, and is natural in A.*

Proof. We need only check equivariance. Let $\alpha \colon H \to \operatorname{Aut}(A)$ denote the action on A. Let $\gamma = \operatorname{Ind}_H^G(\alpha)$ be the induced action of G on $\operatorname{Ind}_H^G(A)$. For $f \in \operatorname{Ind}_H^G(A)$ and $h \in H$, we have, using the definition of $\operatorname{Ind}_H^G(A)$ at the third step,

$$\operatorname{ev}_1^A(\gamma_h(f)) = (\gamma_h(f))(1) = f(h^{-1}) = \alpha_h(f(1)) = \alpha_h(\operatorname{ev}_1^A(f)),$$

as desired. □

Lemma 3.9. *Let G be a locally compact group, and let $H \leqslant G$ be a closed subgroup such that G/H is compact. Let $\alpha\colon G \to \operatorname{Aut}(A)$ be an action of G on a C^*-algebra A, and let $\beta\colon H \to \operatorname{Aut}(B)$ be an action of H on a C^*-algebra B. Let $\varphi\colon A \to B$ be an H-morphism. Then there is a G-morphism $\eta\colon A \to \operatorname{Ind}_H^G(B)$ such that $\eta(a)(s) = \varphi\big(\alpha_s^{-1}(a)\big)$ for all $a \in A$ and $s \in G$.*

Proof. We only have to prove that the formula for $\eta(a)$ defines an element of $\operatorname{Ind}_H^G(B)$ and that the resulting map from A to $\operatorname{Ind}_H^G(B)$ is G-equivariant. Let $a \in A$.

For the first, since G/H is compact, the function $sH \mapsto \|\eta(a)(s)\|$ is obviously in $C_0(G/H)$. Let $s \in G$ and $h \in H$. Then

$$\beta_h\big(\eta(a)(sh)\big) = \beta_h\big(\varphi\big(\alpha_{sh}^{-1}(a)\big)\big) = \varphi\big(\alpha_h \circ \alpha_{h^{-1}s^{-1}}(a)\big) = \eta(a)(s),$$

as desired.

For the second, let $\gamma = \operatorname{Ind}_H^G(\alpha)$ be the action of G on $\operatorname{Ind}_H^G(B)$. Let $s, t \in G$. Then

$$\gamma_s(\eta(a))(t) = \eta(a)(s^{-1}t) = \varphi\big(\alpha_{t^{-1}}(\alpha_s(a))\big) = \eta(\alpha_s(a))(t),$$

as desired. □

Theorem 3.10. *Let G be a locally compact group, and let $H \leqslant G$ be a closed subgroup such that G/H is compact. Let $\alpha\colon G \to \operatorname{Aut}(A)$ be an action of G on a C^*-algebra A. If α is equivariantly semiprojective, then $\alpha|_H$ is equivariantly semiprojective.*

Proof. Let $\beta\colon H \to \operatorname{Aut}(C)$ be an action of H on a C^*-algebra C. To simplify notation, we abbreviate Ind_H^G to Ind. The maps to be introduced are shown in the diagram below. Let $J_0 \subset J_1 \subset \cdots$ be H-invariant ideals in C, let $J = \overline{\bigcup_{n=0}^\infty J_n}$, let

$$\kappa\colon C \to C/J, \quad \kappa_n\colon C \to C/J_n, \quad \text{and} \quad \pi_n\colon C/J_n \to C/J$$

be the quotient maps, and let $\varphi\colon A \to C/J$ be an H-morphism. Then

$$\operatorname{Ind}(J) = \overline{\bigcup_{n=0}^{\infty} \operatorname{Ind}(J_n)}$$

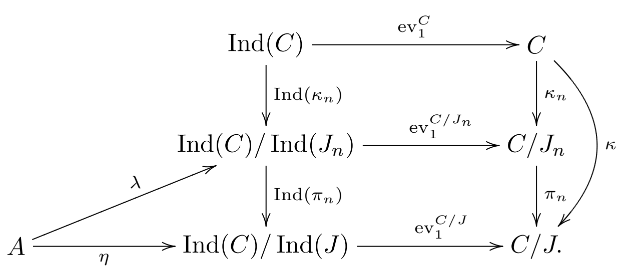
by Proposition 3.7. Moreover, Proposition 3.6 allows us to identify the quotients $\operatorname{Ind}(C)/\operatorname{Ind}(J_n)$ with $\operatorname{Ind}(C/J_n)$ and $\operatorname{Ind}(C)/\operatorname{Ind}(J)$ with $\operatorname{Ind}(C/J)$, with quotient maps

$$\operatorname{Ind}(\kappa)\colon \operatorname{Ind}(C) \to \operatorname{Ind}(C)/\operatorname{Ind}(J),$$
$$\operatorname{Ind}(\kappa_n)\colon \operatorname{Ind}(C) \to \operatorname{Ind}(C)/\operatorname{Ind}(J_n),$$

and

$$\operatorname{Ind}(\pi_n)\colon \operatorname{Ind}(C)/\operatorname{Ind}(J_n) \to \operatorname{Ind}(C)/\operatorname{Ind}(J).$$

Let $\eta\colon A \to \operatorname{Ind}(C)/\operatorname{Ind}(J)$ be as in Lemma 3.9. Since α is equivariantly semiprojective, there exist $n \in \mathbb{N}$ and a G-morphism $\lambda\colon A \to \operatorname{Ind}(C)/\operatorname{Ind}(J_n)$ such that $\operatorname{Ind}(\pi_n) \circ \lambda = \eta$. We now have the following commutative diagram, with the horizontal maps on the right being as in Lemma 3.8:



It is easy to check that $\mathrm{ev}_1^{C/J} \circ \eta = \varphi$. Therefore the map $\psi = \mathrm{ev}_1^{C/J_n} \circ \lambda$ is an H-morphism from A to C/J_n such that $\pi_n \circ \psi = \varphi$. □

Corollary 3.11. *Let G be a second countable compact group, and let A be a G-algebra that is equivariantly semiprojective. Then A is (nonequivariantly) semiprojective.*

In Theorem 3.10, some condition on G/H is necessary, as the following example shows.

Example 3.12. Let $A = C(S^1)$ be the universal C^*-algebra generated by a unitary, and consider the free Bernoulli shift $\tau\colon \mathbb{Z} \to \mathrm{Aut}\left(\ast_{\mathbb{Z}}\, C(S^1)\right)$ of Definition 2.3. This action is semiprojective by Proposition 2.4, but its restriction to the trivial subgroup is not.

Thus, $\mathbb{Z}$ can act semiprojectively on nonsemiprojective C^*-algebras. This is in contrast to the projective case, discussed in Remark 4.23. An analogous example can be constructed for any infinite countable discrete group in place of $\mathbb{Z}$.

4. Equivariant projectivity of restrictions to subgroups

In this section we study the projective analog of the question of Section 3. Given a projective action $\alpha\colon G \to \mathrm{Aut}(A)$, we show in Theorem 4.19 that the restriction of α to a subgroup $H \leqslant G$ is also projective in considerable generality. The condition we have to put on H is that the factor space G/H is uniformly finitistic; see Definition 4.5. The class of uniformly finitistic spaces includes both compact and discrete spaces.

To obtain the results in this section, we use a different induction functor that considers uniformly continuous functions; see Definition 4.14. To show that this functor is exact, we need a criterion for when uniformly continuous functions into quotient C^*-algebras can be lifted to uniformly continuous functions. In Theorem 4.8, we solve this problem in some generality, and we think that this might also be of independent interest.

There are several equivalent ways to define a uniform space. We will mostly use the concept of a uniform cover to define a uniformity on a set. We refer to Isbell's book [Isb64] for the theory of uniform spaces. The basic definitions are in Chapter I. The definition of a uniformity is before item 6 in Chapter I of [Isb64].

If $\mathcal{U}$ and $\mathcal{V}$ are covers of a space X, we write $\mathcal{V} \leq \mathcal{U}$ to mean that $\mathcal{V}$ refines $\mathcal{U}$.

Definition 4.1. Let (X, d) be a metric space. For $\varepsilon > 0$ and $x \in X$, define $U_\varepsilon(x) = \{y \in X : d(x, y) < \varepsilon\}$. The *basic uniform covers* of X are the collections

$$\mathcal{B}(\varepsilon) = \{U_\varepsilon(x) : x \in X\}$$

for $\varepsilon > 0$. A cover $\mathcal{U}$ of X is called *uniform* if there exists $\varepsilon > 0$ such that $\mathcal{B}(\varepsilon) \leq \mathcal{U}$.

The proof of the following result is essentially contained in items 1–3 in Chapter I of [Isb64]. One should note that if (X, d) is a metric space, $\varepsilon_1, \varepsilon_2 > 0$, and $\mathcal{U}_1$ and $\mathcal{U}_2$ are covers of X such that $\mathcal{B}(\varepsilon_1) \leq \mathcal{U}_1$ and $\mathcal{B}(\varepsilon_2) \leq \mathcal{U}_2$, then $\mathcal{B}\big(\min(\varepsilon_1, \varepsilon_2)\big)$ refines both $\mathcal{U}_1$ and $\mathcal{U}_2$, so that the collection of uniform covers in Definition 4.1 is downwards directed. Uniformly continuous functions are defined after Theorem 11 in Chapter I of [Isb64], and equiuniformly continuous families of functions are defined before item 19 in Chapter III of [Isb64]. The usual notion for functions on metric spaces is just that a family F of functions from (X_1, d_1) to (X_2, d_2) is equiuniformly continuous if for every $\varepsilon > 0$ there is $\delta > 0$ such that whenever $x, y \in X_1$ satisfy $d_1(x, y) < \delta$, then for all $f \in F$ we have $d_2(f(x), f(y)) < \varepsilon$.

Proposition 4.2. *Let (X, d) be a metric space. Then the collection of uniform covers in Definition 4.1 is a uniform structure on X. Moreover, for any two metric spaces (X_1, d_1) and (X_2, d_2), the uniformly continuous functions and the equiuniformly continuous families of functions from X_1 to X_2 are the uniformly continuous functions and the equiuniformly continuous families as traditionally defined in terms of the metrics.*

The following theorem is the key result. We warn that the term "subordinate" is used in [Isb64] with a meaning inconsistent with its standard meaning in the context of ordinary partitions of unity.

Theorem 4.3 (Theorem 11 in Chapter IV of [Isb64])**.** *Let X be a uniform space and let $\mathcal{U}$ be a uniform cover of X. Then there is an equiuniformly continuous (but not necessarily locally finite) partition of unity $(h_U)_{U \in \mathcal{U}}$ such that $h_U(x) = 0$ for all $U \in \mathcal{U}$ and $x \in X \setminus U$.*

We recall the following standard definition.

Definition 4.4. Let X be a set, and let $\mathcal{U}$ be a cover of X. The *order* of $\mathcal{U}$, denoted $\operatorname{ord}(\mathcal{U})$, is the least number $n \in \mathbb{N} \cup \{0\}$ such that whenever $U_0, U_1, \ldots, U_n \in \mathcal{U}$ are distinct, then $U_0 \cap U_1 \cap \ldots \cap U_n = \varnothing$. We take $\operatorname{ord}(\mathcal{U}) = \infty$ if no such n exists.

Equivalently, $\operatorname{ord}(\mathcal{U})$ is the largest number n such that n distinct elements of $\mathcal{U}$ have nonempty intersection.

The first part of following definition is found at the very beginning of Chapter V of [Isb64], where the term "large dimension" is used. The second part is Definition 1.7 of [SSG93].

Definition 4.5. Let X be a uniform space. Then the *large uniform dimension of* X, denoted $\Delta\mathrm{d}(X)$, is the least $n \in \{-1, 0, 1, 2, \ldots, \infty\}$ such that every uniform open cover of X can be refined by a uniform open cover of order at most $n + 1$. (We take $\Delta\mathrm{d}(\varnothing) = -1$.)

We say that X is *uniformly finitistic* if every uniform open cover of X can be refined by a uniform open cover of finite order.

An equivalent condition for being uniformly finitistic is that there exists a base for the uniformity consisting of uniform covers of finite order.

If a uniform space X is locally compact and paracompact (in the induced topology), then its covering dimension is bounded by its large uniform dimension, that is, $\dim(X) \leq \Delta\mathrm{d}(X)$. To see this, first note that, with $\operatorname{locdim}(X)$ being the local covering dimension of X, Proposition 5.3.4 in [Pea75] gives $\dim(X) = \operatorname{locdim}(X)$. For a locally compact Hausdorff space X, with X^+ denoting the one point compactification of X, it is a standard result that $\operatorname{locdim}(X) = \dim(X^+)$; for instance, this is easily deduced from Proposition 3.5.6 in [Pea75]. It follows from Theorem V.5 and VI.2 in [Isb64] that $\dim(\gamma X) \leq \Delta\mathrm{d}(X)$ for every compactification γX of X. Thus, if X is locally compact and paracompact, we may combine these results to obtain

$$\dim(X) = \operatorname{locdim}(X) = \dim(X^+) \leq \Delta\mathrm{d}(X),$$

as desired.

The concept of being finitistic was first defined for topological spaces, where it means that every open cover can be refined by an open cover of finite order. This definition is implicit in [Swa59], although the term "finitistic" was only later introduced by Bredon on page 133 of his book [Bre72].

In general, for a uniform space there is no connection between being finitistic and uniformly finitistic. Example (d) after Definition 1.7 of [SSG93] gives a uniformly finitistic space which is not finitistic. Example 2.4 of [Isb59] gives a discrete uniform space, hence obviously finitistic, with a uniform open cover having no uniform open refinement of finite order, thus not uniformly finitistic.

Notation 4.6. Let X be a topological space and let A be a C^*-algebra. We denote by $C_{\mathrm{b}}(X, A)$ the C^*-algebra of all bounded continuous functions from X to A, with the supremum norm. If X is a uniform space, we let $C_{\mathrm{u}}(X, A) \subset C_{\mathrm{b}}(X, A)$ denote the C^*-subalgebra consisting of all bounded uniformly continuous functions from X to A.

Proposition 4.7. *Let X be a uniform space and let A be a C^*-algebra. Then $C_{\mathrm{u}}(X, A)$ is a C^*-algebra.*

Proof. It is easy to check that $C_{\mathrm{u}}(X, A)$ is closed under the algebraic operations. That it is norm closed in $C_{\mathrm{b}}(X, A)$ follows from Corollary 32 in Chapter III of [Isb64]. □

The following theorem is in some sense a dual version of Theorem 1 of [Vid69], on the problem of extending uniformly continuous maps from subspaces. We do not know whether it is necessary that X be uniformly finitistic.

Theorem 4.8. *Let $\pi \colon A \to B$ be a surjective *-homomorphism between two C^*-algebras, and let X be a uniformly finitistic space. Then the induced *-homomorphism $\kappa \colon C_{\mathrm{u}}(X, A) \to C_{\mathrm{u}}(X, B)$ is surjective.*

Proof. It is enough to show that κ has dense range.

Given $b \in C_{\mathrm{u}}(X, B)$ and $\varepsilon > 0$, we will construct $a \in C_{\mathrm{u}}(X, A)$ such that $\|\pi \circ a - b\| < \varepsilon$. Let $\mathcal{U}$ be a uniform cover of X such that whenever $U \in \mathcal{U}$ and $x, y \in U$, then $\|b(x) - b(y)\| < \frac{\varepsilon}{2}$. Since X is uniformly finitistic, we may assume $\mathcal{U}$ has finite order. Set $n = \mathrm{ord}(\mathcal{U})$.

Let $(h_U)_{U \in \mathcal{U}}$ be an equiuniformly continuous partition of unity for $\mathcal{U}$ as in Theorem 4.3. Equiuniform continuity in our situation means that for every $\rho > 0$ there exists a uniform open cover $\mathcal{V}$ of X such that whenever $V \in \mathcal{V}$ and $x, y \in V$, then for all $U \in \mathcal{U}$ we have $|h_U(x) - h_U(y)| < \rho$.

For each $U \in \mathcal{U}$ choose a point $x_U \in U$, and let $a_U \in A$ be a lift of $b(x_U)$ with $\|a_U\| = \|b(x_U)\|$. For $x \in X$, there are at most n sets $U \in \mathcal{U}$ such that $x \in U$, and $h_U(x)$ can be nonzero only for these sets. Therefore the sum in the following definition of a function $a \colon X \to A$ is finite at each point:

$$a(x) = \sum_{U \in \mathcal{U}} h_U(x) \cdot a_U$$

for $x \in X$. Since $\sum_{U \in \mathcal{U}} h_U(x) = 1$, it further follows that $\|a\| \leq \|b\|$, so that a is bounded.

We claim that a is uniformly continuous. We follow an argument in the proof of Theorem 1 of [Vid69]. Let $\rho > 0$. We must find a uniform open cover $\mathcal{V}$ of X such that whenever $V \in \mathcal{V}$ and $x, y \in V$, we have $\|a(x) - a(y)\| < \rho$. We may assume $b \neq 0$. (Otherwise, just take $a = 0$.) Set $\rho_0 = \rho / (2n\|b\|)$. Let $\mathcal{V}$ be a uniform open cover which witnesses equiuniform continuity of $(h_U)_{U \in \mathcal{U}}$ as above, but with ρ_0 in place of ρ. Let $V \in \mathcal{V}$ and let $x, y \in V$. Set

$$\mathcal{U}_0 = \big\{ U \in \mathcal{U} \colon x \in U \text{ or } y \in U \big\}.$$

Then $\mathrm{card}(\mathcal{U}_0) \leq 2n$. Therefore

$$\begin{aligned} \|a(x) - a(y)\| &= \left\| \sum_{U \in \mathcal{U}_0} \big(h_U(x) - h_U(y) \big) \cdot a_U \right\| \\ &\leq 2n \cdot \|b\| \cdot \max_{U \in \mathcal{U}_0} |h_U(x) - h_U(y)| < 2n\|b\|\rho_0 = \rho. \end{aligned}$$

The claim is proved.

It remains to prove that $\|\pi \circ a - b\| < \varepsilon$. Let $x \in X$. Then $\|\pi(a_U) - b(x)\| < \frac{\varepsilon}{2}$ whenever $h_U(x) \neq 0$. Therefore

$$\|(\pi\circ a)(x)-b(x)\| = \left\| \sum_{U\in\mathcal{U}} h_U(x)\big(\pi(a_U) - b(x)\big) \right\| \leq \sum_{U\in\mathcal{U}} h_U(x)\, \|\pi(a_U) - b(x)\| < \frac{\varepsilon}{2}.$$

So $\|\pi \circ a - b\| \leq \frac{\varepsilon}{2} < \varepsilon$, as desired. □

Remark 4.9. The proof of Theorem 4.8 can easily be adopted to the case of bounded continuous maps. More precisely, if $\pi\colon A \to B$ is a surjective *-homomorphism of C^*-algebras, and X is a paracompact space, then the method of proof shows that the induced *-homomorphism $C_{\mathrm{b}}(X, A) \to C_{\mathrm{b}}(X, B)$ is surjective. This is a C^*-algebraic version of the Bartle-Graves Selection Theorem, Theorem 4 of [BG52], which treats the general case in which A and B are Banach spaces. The C^*-algebraic version is much easier to prove since the image of a *-homomorphism is always closed.

Since a C^*-algebra is paracompact, one may also formulate the theorem as follows. Let $\pi\colon A \to B$ be a surjective *-homomorphism between C^*-algebras. Then there exists a continuous function $\sigma\colon B \to A$ (not necessarily linear) such that $\pi \circ \sigma = \mathrm{id}_B$ (that is, σ is a section), and such that there is a constant M such that $\|\sigma(a)\| \leq M \cdot \|a\|$ for all $a \in A$. This also appears in [Lor97a, Theorem 2].

Definition 4.10. Let G be a locally compact group, and let $H \leqslant G$ be a closed subgroup. Let $q\colon G \to G/H$ be the quotient map. For a nonempty open subset $U \subset G$ with $1 \in U$, define $\mathcal{B}_{G,H}(U) = \{q(Us)\colon s \in G\}$, the open cover of G/H by the images in G/H of the right translates of U. Define the *right uniformity* on G/H to consist of all open covers $\mathcal{U}$ of G/H such that there is a nonempty open subset $U \subset G$ with $1 \in U$ for which $\mathcal{B}_{G,H}(U) \leq \mathcal{U}$, and call such covers the *right uniform covers*.

We define the *left uniformity* on G/H and *left uniform covers* of G/H analogously, using the covers by the images in G/H of the left translates $\{sU\colon s \in G\}$ for nonempty open subsets $U \subset G$ with $1 \in U$.

Taking $H = \{1\}$, we see that the inversion map $s \mapsto s^{-1}$ is uniformly continuous if and only if the right and left uniformities on G agree. However, for fixed $t \in G$, both the left translation map $s \mapsto ts$ and the right translation map $s \mapsto st$ are uniformly continuous in the right uniformity (and also in the left uniformity).

Uniform structures on topological groups are discussed on pages 20–22 of [HR79], but from the point of view of neighborhoods of the diagonal rather than uniform open covers.

Clearly the map $q\colon G \to G/H$ is uniformly continuous when both sets are given the right uniformity. In fact, the right uniformity on G/H is the quotient uniformity, as defined before item 5 in Chapter II of [Isb64], of the right uniformity on G. We do not need this fact, so we omit the proof.

Let G be a metrizable topological group. Then G has a left invariant metric determining its topology, by Theorem 1.22 of [MZ55], and analogously it also has right invariant metric. It is easy to check that the uniformity induced by any such metric (as in Proposition 4.2) is equal to the right uniformity of Definition 4.10.

Notation 4.11. Let G be a topological group and let A be a C^*-algebra. We denote by $C_{\mathrm{ru}}(G, A)$ the C^*-algebra of bounded functions $f\colon G \to A$ which are right uniformly continuous. This is just $C_{\mathrm{u}}(G, A)$ as in Notation 4.6 when G is equipped with the right uniformity. We further let $\lambda\colon G \to \mathrm{Aut}(C_{\mathrm{b}}(G, A))$ be the (not necessarily continuous) action given by $\lambda_s(f)(t) = f(s^{-1}t)$ for $f \in C_{\mathrm{b}}(G, A)$ and $s, t \in G$.

Left translation is continuous on the right uniformly continuous functions, not the left uniformly continuous functions. The proof is known and not difficult; we give it here primarily to convince the reader that the statement is correct. We start with a preparatory lemma, which we also need for the left uniformity.

Lemma 4.12. *Let the notation be as in Notation 4.11. Let $f \in C_{\mathrm{b}}(G, A)$. Then $f \in C_{\mathrm{ru}}(G, A)$ if and only if for every $\varepsilon > 0$ there is an open set $V \subset G$ with $1 \in V$ such that whenever $s, t \in G$ satisfy $st^{-1} \in V$, then $\|f(s) - f(t)\| < \varepsilon$. Also, f is left uniformly continuous if and only if for every $\varepsilon > 0$ there is an open set $V \subset G$ with $1 \in V$ such that whenever $s, t \in G$ satisfy $t^{-1}s \in V$, then $\|f(s) - f(t)\| < \varepsilon$.*

Proof. The proofs of the two statements are the same, and we do only the first.

First assume f is right uniformly continuous. Then there is a nonempty open set $V \subset G$ with $1 \in V$ such that whenever $s, t, g \in G$ satisfy $s, t \in Vg$, then $\|f(s) - f(t)\| < \varepsilon$. If now $s, t \in G$ satisfy $st^{-1} \in V$, then $s \in Vt$ and, since $1 \in V$, also $t \in Vt$. Taking $g = t$ above, we get $\|f(s) - f(t)\| < \varepsilon$.

Now assume that f satisfies the condition of the lemma. Let $\varepsilon > 0$, and choose $V \subset G$ as in this condition. Choose an open subset $U \subset G$ such that $1 \in U$ and $s, t \in U$ implies $st^{-1} \in V$. Let $s, t, g \in G$ satisfy $s, t \in Ug$. Then sg^{-1}, $tg^{-1} \in U$, so $st^{-1} = (sg^{-1})(tg^{-1})^{-1} \in V$. Therefore $\|f(s) - f(t)\| < \varepsilon$. □

Lemma 4.13. *Let the notation be as in Notation 4.11. Let $f \in C_{\mathrm{b}}(G, A)$. Then $s \mapsto \lambda_s(f)$ is continuous if and only if $f \in C_{\mathrm{ru}}(G, A)$.*

Proof. First assume f is right uniformly continuous. Let $\varepsilon > 0$. It suffices to find an open subset $V \subset G$ such that $1 \in V$ and whenever $s \in V$ and $t \in G$, then $\|\lambda_{ts}(f) - \lambda_t(f)\| < \varepsilon$. Choose an open subset $V \subset G$ as in Lemma 4.12 with $\frac{\varepsilon}{2}$ in place of ε. Let $s \in V$ and $t \in G$. Then for $g \in G$ we have $(t^{-1}g)(s^{-1}t^{-1}g)^{-1} = s \in V$, so

$$\|\lambda_{ts}(f)(g) - \lambda_t(f)(g)\| = \|f(s^{-1}t^{-1}g) - f(t^{-1}g)\| < \frac{\varepsilon}{2}.$$

Taking the supremum over $g \in G$, we get $\|\lambda_{ts}(f) - \lambda_t(f)\| \leq \frac{\varepsilon}{2} < \varepsilon$.

For the converse, assume that $s \mapsto \lambda_s(f)$ is continuous. We verify the criterion of Lemma 4.12. Let $\varepsilon > 0$. Choose an open subset $V \subset G$ such that $1 \in V$ and whenever $s \in V$ then $\|\lambda_s(f) - f\| < \varepsilon$. Let $s, t \in G$ satisfy $st^{-1} \in V$. Then

$$\|f(s) - f(t)\| = \|f(s) - \lambda_{st^{-1}}(f)(s)\| \leq \|f - \lambda_{st^{-1}}(f)\| < \varepsilon.$$

This completes the proof. □

We now give a definition which is very similar to Definition 3.1, but which uses bounded uniformly continuous functions instead of functions vanishing at infinity.

Definition 4.14. Let G be a locally compact group, and let $H \leqslant G$ be a closed subgroup. Let $\alpha\colon H \to \mathrm{Aut}(A)$ be an action of H on a C^*-algebra A. We define a C^*-algebra $F_H^G(A)$, with not necessarily continuous action $F_H^G(\alpha)\colon G \to \mathrm{Aut}\left(F_H^G(A)\right)$, by

$$F_H^G(A) = \left\{ f \in C_{\mathrm{b}}(G, A) \colon \alpha_h(f(sh)) = f(s) \text{ for all } s \in G \text{ and } h \in H \right\}$$

and

$$\left(F_H^G(\alpha)\right)_s(f)(t) = f(s^{-1}t)$$

for $f \in F_H^G(A)$ and $s, t \in G$. We further define a subalgebra $\mathrm{UInd}_H^G(A) \subset F_H^G(A)$ by

$$\mathrm{UInd}_H^G(A) = \left\{ f \in F_H^G(A) \colon s \mapsto \left(F_H^G(\alpha)\right)_s(f) \text{ is continuous} \right\},$$

and we take $\mathrm{UInd}_H^G(\alpha)$ to be the restriction of $F_H^G(\alpha)$ to this subalgebra.

If A and B are H-algebras and $\varphi\colon A \to B$ is an H-morphism, then the induced G-morphisms

$$F_H^G(\varphi)\colon F_H^G(A) \to F_H^G(B) \quad \text{and} \quad \mathrm{UInd}_H^G(\varphi)\colon \mathrm{UInd}_H^G(A) \to \mathrm{UInd}_H^G(B)$$

are defined by sending f in $F_H^G(A)$ or $\mathrm{UInd}_H^G(A)$ as appropriate to the function $s \mapsto \varphi(f(s))$ for $s \in G$.

We call UInd_H^G the *right uniform induction functor*.

Lemma 4.15. *Let G be a locally compact group, and let $H \leqslant G$ be a closed subgroup. Let the notation be as in Definition 4.14. Then:*

(1) $\mathrm{UInd}_H^G(A) = F_H^G(A) \cap C_{\mathrm{ru}}(G, A)$.
(2) F_H^G is a functor from the category $\mathcal{C}_H$ of H-algebras to the category of C^*-algebras with not necessarily continuous actions of G.
(3) UInd_H^G is a functor from $\mathcal{C}_H$ to $\mathcal{C}_G$.
(4) If G/H is compact, then $\mathrm{UInd}_H^G = F_H^G = \mathrm{Ind}_H^G$.

Proof. Part (1) follows from Lemma 4.13. Part (2) is an algebraic calculation. Part (3) follows from part (1), part (2), and the fact that the formula for $F_H^G(\varphi)$ preserves uniform continuity. Part (4) follows from the observation that the condition in Definition 3.1, that $sH \mapsto \|f(s)\|$ be in $C_0(G/H)$, is automatic when G/H is compact, and the fact that left translation is continuous on $\mathrm{Ind}_H^G(A)$. □

Theorem 4.16. *Let G be a locally compact group and let $H \leqslant G$ be a closed subgroup. Assume that the left and right uniformities on G agree, and that G/H is right uniformly finitistic (using the right uniformity of Definition 4.10). Then the right uniform induction functor $\mathrm{UInd}_H^G\colon \mathcal{C}_H \to \mathcal{C}_G$ is exact, that is, given an H-equivariant short exact sequence of H-algebras*

$$0 \longrightarrow I \xrightarrow{\ \iota\ } A \xrightarrow{\ \pi\ } B \longrightarrow 0,$$

the induced G-equivariant sequence of G-algebras

$$0 \longrightarrow \mathrm{UInd}_H^G(I) \xrightarrow{\mathrm{UInd}_H^G(\iota)} \mathrm{UInd}_H^G(A) \xrightarrow{\mathrm{UInd}_H^G(\pi)} \mathrm{UInd}_H^G(B) \longrightarrow 0$$

is also exact.

We need two further lemmas for the proof.

Lemma 4.17. *Let G be a locally compact group such that the left and right uniformities on G agree. Let $f \in C_{\mathrm{c}}(G)$. Then for every $\varepsilon > 0$ there is an open set $U \subset G$ such that $1 \in U$ and such that whenever $g, h, s, t \in G$ satisfy $s^{-1}t \in U$, then $|f(gsh) - f(gth)| < \varepsilon$.*

Proof. We first claim that there is an open set $V \subset G$ such that $1 \in V$ and such that whenever $s, t \in G$ satisfy $s^{-1}t \in U$, then $|f(s) - f(t)| < \varepsilon$. (By Lemma 4.12, this is just left uniform continuity of f.)

Set $K = \mathrm{supp}(f)$. Let $\varepsilon > 0$. For every $t \in K$, use continuity of f to choose an open neighborhood $Z(t)$ of 1 such that $s \in tZ(t)$ implies $|f(s) - f(t)| < \frac{\varepsilon}{2}$. Further choose an open neighborhood $V(t)$ of 1 such that $x, y \in V(t)$ implies $xy \in Z(t)$ and also such that $x \in V(t)$ implies $x^{-1} \in V(t)$. Choose $t_1, t_2, \ldots, t_n \in K$ such that the sets $t_1 V(t_1)$, $t_2 V(t_2)$, …, $t_n V(t_n)$ cover K. Set $V = \bigcap_{j=1}^n V(t_j)$.

Now suppose $s, t \in G$ satisfy $t^{-1}s \in V$. We show that $|f(s) - f(t)| < \varepsilon$. If $s, t \notin K$, then $f(s) = f(t) = 0$, so this is immediate. If $t \in K$, then there is $j \in \{1, 2, \ldots, n\}$ such that $t \in t_j V(t_j)$. Then $t^{-1}s$, $t_j^{-1}t \in V(t_j)$, so $s = t_j(t_j^{-1}t)(t^{-1}s) \in t_j Z(t_j)$. Therefore

$$|f(s) - f(t)| \leq |f(s) - f(t_j)| + |f(t_j) - f(t)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

as desired. Finally, if $s \in K$, use the definition of V to get $s^{-1}t \in V$, choose j such that $s \in t_j V(t_j)$, and proceed as before. The completes the proof of the claim.

Now we prove the statement of the lemma. By 4.14(g) in Chapter II of [HR79], equality of the left and right uniformities on G implies that there is an open set $U \subset G$ such that $1 \in U$ and such that whenever $g \in G$, then $gUg^{-1} \subset V$. Now let $g, h, s, t \in G$ satisfy $s^{-1}t \in U$. Then $(gsh)^{-1}(gth) = h^{-1}s^{-1}th \in h^{-1}Uh \subset V$, so that $|f(gsh) - f(gth)| < \varepsilon$. □

Lemma 4.18. *Let G be a locally compact group, let $H \leqslant G$ be a closed subgroup, let μ be a left Haar measure on H, and let $L \subset G$ be compact. Then $\sup_{s \in G} \mu(sL \cap H)$ is finite.*

Proof. Let $q\colon G \to G/H$ be the quotient map. Choose a continuous function $f\colon G \to [0,1]$ with compact support and such that $f = 1$ on L. For $s \in G$ define

$$g(s) = \int_H f(sh)\, d\mu(h).$$

Then g is continuous and satisfies $g(sk) = g(s)$ for all $s \in G$ and $k \in H$. Therefore g drops to a continuous function $\overline{g}$ on G/H. If $s \notin \operatorname{supp}(f)H$, then $g(s) = 0$. Therefore $\operatorname{supp}(\overline{g}) \subset q(\operatorname{supp}(f))$, and so is compact. Now

$$\sup_{s \in G} \mu(sL \cap H) \leq \sup_{s \in G} \int_H f(s^{-1}h)\, d\mu(h) = \sup_{x \in G/H} \overline{g}(x) < \infty.$$

This completes the proof. □

Proof of Theorem 4.16. To simplify the notation, we abbreviate the functor UInd_H^G to UInd. As in the proof of Proposition 3.6, it is easy to check that the induced sequence is exact at the right and in the middle. Thus, it remains to check that $\mathrm{UInd}(\pi)\colon \mathrm{UInd}(A) \to \mathrm{UInd}(B)$ is surjective. Let $\alpha\colon H \to \mathrm{Aut}(A)$ and $\beta\colon H \to \mathrm{Aut}(B)$ denote the actions of G.

Let $q\colon G \to G/H$ denote the quotient map.

Let $b \in \mathrm{UInd}(B)$ and let $\varepsilon > 0$. We construct $a \in \mathrm{UInd}(A)$ such that $\|\pi \circ a - b\| < \varepsilon$. The function b is right uniformly continuous by Lemma 4.15(1). The hypothesis on G implies that b is left uniformly continuous. So Lemma 4.12 provides an open neighborhood U of $1 \in G$ such that $t^{-1}s \in G$ implies $\|b(s) - b(t)\| < \frac{\varepsilon}{2}$. Since G is locally compact, we may assume that $\overline{U}$ is compact.

Let V_0 be an open neighborhood of 1 such that $\overline{V_0} \subset U$.

We claim that there is a continuous function $f\colon G \to [0, \infty)$ such that $\operatorname{supp}(f) \subset U$ and such that for every $s \in V_0 H$ we have

$$\int_H f(sh)\, d\mu(h) = 1. \tag{4.1}$$

By left invariance of μ, it suffices to find f such that (4.1) holds for every $s \in V_0$.

Choose an open set $Z \subset G$ with $\overline{V_0} \subset Z \subset \overline{Z} \subset U$, and choose $f_0 \in C_{\mathrm{c}}(G)$ such that

$$0 \leq f_0 \leq 1, \quad \operatorname{supp}(f_0) \subset U, \quad \text{and} \quad f_0|_{\overline{Z}} = 1.$$

Since $q(\overline{V_0})$ is compact, $q(Z)$ is open, and $q(\overline{V_0}) \subset q(Z)$, there exists $f_1 \in C_{\mathrm{c}}(G/H)$ such that

$$0 \leq f_1 \leq 1, \quad \operatorname{supp}(f_1) \subset q(Z), \quad \text{and} \quad f_1|_{q(\overline{V_0})} = 1.$$

Define a continuous function $k\colon G \to [0, \infty)$ by

$$k(s) = \int_H f_0(sh)\, d\mu(h)$$

for $s \in G$. For $s \in Z$, the integrand is equal to 1 on the open set $H \cap s^{-1}Z \subset H$. This set contains 1, so is nonempty, whence $k(s) \neq 0$. Since also $k(sh) = k(s)$ for all $s \in G$ and $h \in H$, we see that $k(s) \neq 0$ for all $s \in ZH$. Therefore the definition

$$f(s) = \begin{cases} f_1(sH)f_0(s)k(s)^{-1} & s \in ZH \\ 0 & s \in G \setminus q^{-1}(\mathrm{supp}(f_1)) \end{cases}$$

is consistent and gives a continuous function $f\colon G \to [0, \infty)$ as required in the claim.

Use the hypothesis on G and 4.14(g) in Chapter II of [HR79] to find an open neighborhood V_1 of 1 such that $sV_1s^{-1} \subset V_0$ for all $s \in G$. This implies, in particular, that

$$HV_1H \subset V_0H. \tag{4.2}$$

Now choose an open neighborhood V of 1 such that $s, t \in V$ implies $s^{-1}t \in V_1$.

Consider the left uniform cover $\mathcal{V} = \{sV : s \in G\}$ of G, and its image $q(\mathcal{V}) = \{(sVH)/H : s \in G\}$ in G/H. Since the left and right uniformities on G agree, $\mathcal{V}$ is a right uniform cover of G, so that $q(\mathcal{V})$ is a right uniform cover of G/H. Since G/H is assumed to be right uniformly finitistic, there exists a uniform cover $\mathcal{W}$ of G/H which refines $q(\mathcal{V})$ and has finite order n. Let $(l_W)_{W \in \mathcal{W}}$ be an equiuniformly continuous partition of unity on G/H for $\mathcal{W}$ as in Theorem 4.3. Then the functions $l_W \circ q$ define an equiuniformly continuous partition of unity on G such that $l_W(x) = 0$ whenever $W \in \mathcal{W}$ and $x \in G \setminus q^{-1}(W)$.

For each $W \in \mathcal{W}$, choose a point $x_W \in q^{-1}(W)$, and let $a_W \in A$ be a lift of $b(x_W)$ with $\|a_W\| = \|b(x_W)\|$. Define a continuous function $g_W : G \to [0, \infty)$ by

$$g_W(s) = l_W(sH) \cdot f(x_W^{-1}s).$$

This function vanishes outside the set $x_W U \cap q^{-1}(W)$. In particular, $\mathrm{supp}(g_W)$ is contained in the compact set $x_W\overline{U}$.

We claim that for every $s \in G$ and $W \in \mathcal{W}$, we have

$$\int_H g_W(sh)\, d\mu(h) = l_W(sH). \tag{4.3}$$

If $s \notin q^{-1}(W)$, then both sides of (4.3) are zero. To prove the claim, we therefore assume $s \in q^{-1}(W)$. Choose $t \in G$ such that $q^{-1}(W) \subset tVH$. Then $s, x_W \in tVH$, so there exist $h, k \in H$ such that $t^{-1}sh$, $t^{-1}x_W k \in V$. So $k^{-1}x_W^{-1}sh \in V_1$. It follows from (4.2) that $x_W^{-1}s \in VH$, and from the choice of f that

$$\int_H f(x_W^{-1}sh)\, d\mu(h) = 1.$$

The claim is proved.

We next claim that there is a function $a\colon G \to A$ defined by

$$a(s) = \sum_{W \in \mathcal{W}} \int_H g_W(sh) \cdot \alpha_h(a_W)\, d\mu(h) \tag{4.4}$$

for $s \in G$. For each $W \in \mathcal{W}$, the integral exists because the integrand is continuous and has compact support. Moreover, for every $s \in G$, from (4.3) we get

$$\left\| \int_H g_W(sh) \cdot \alpha_h(a_W)\, d\mu(h) \right\| \leq l_W(sH) \cdot \|a_W\| \leq l_W(sH) \cdot \|b\|. \tag{4.5}$$

It follows that for each $s \in G$ at most n summands in (4.4) are nonzero. The claim follows. Moreover, $\|a(s)\| \leq \|b\|$ for all $s \in G$.

We claim that a is right uniformly continuous. Since the left and right uniformities agree, it suffices to prove that a is left uniformly continuous. Let $\rho > 0$. By

Lemma 4.18, there is $M > 0$ such that $\mu\big(t\overline{U} \cap H\big) \leq M$ for all $t \in G$. Using equiuniform continuity of $(l_W \circ q)_{W \in \mathcal{W}}$ and Lemma 4.17, choose an open neighborhood Z of 1 which is so small that for every $W \in \mathcal{W}$ and $s, t \in G$ with $t^{-1}s \in Z$, we have

$$|l_W(sH) - l_W(tH)| < \frac{\rho}{4n\|b\| + 1},$$

and also whenever $g, h, s, t \in G$ satisfy $s^{-1}t \in Z$, then

$$|f(gsh) - f(gth)| < \frac{\rho}{4M\|b\| + 1}. \tag{4.6}$$

Now let $s, t \in G$ satisfy $t^{-1}s \in Z$. Then, using $\|a_W\| \leq \|b\|$ for all $W \in \mathcal{W}$,

$$\begin{aligned} \|a(s) - a(t)\| = \Bigg\| \sum_{W \in \mathcal{W}} \bigg(\int_H l_W(sH) f(x_W^{-1}sh) \alpha_h(a_W)\, d\mu(h) \\ - \int_H l_W(tH) f(x_W^{-1}th) \alpha_h(a_W)\, d\mu(h) \bigg) \Bigg\| \\ \leq \|b\| \sum_{W \in \mathcal{W}} |l_W(sH) - l_W(tH)| \int_H f(x_W^{-1}sh)\, d\mu(h) \\ + \|b\| \sum_{W \in \mathcal{W}} l_W(tH) \int_H |f(x_W^{-1}sh) - f(x_W^{-1}th)|\, d\mu(h). \end{aligned}$$

In the first term of the last expression, as in the proof of Theorem 4.8, for any fixed $s, t \in G$, at most $2n$ of the terms are nonzero. Therefore this term is dominated by

$$\|b\| \cdot 2n \left(\frac{\rho}{4n\|b\| + 1} \right) \left(\sup_{W \in \mathcal{W}} \int_H f(x_W^{-1}sh)\, d\mu(h) \right) \leq \left(\frac{2n\|b\|\rho}{4n\|b\| + 1} \right) \cdot 1 < \frac{\rho}{2}.$$

Using $\sum_{W \in \mathcal{W}} l_W(tH) = 1$, $\|a_W\| \leq \|b\|$, the choice of M, and (4.6), we see that the second term is dominated by

$$\left(\frac{\rho}{4M\|b\| + 1} \right) \left[\mu\big(s^{-1}x_W\overline{U} \cap H\big) + \mu\big(t^{-1}x_W\overline{U} \cap H\big) \right] \|b\| \leq \frac{2M\|b\|\rho}{4M\|b\| + 1} < \frac{\rho}{2}.$$

So $\|a(s) - a(t)\| < \rho$. This completes the proof of the claim.

We now claim that $a \in \mathrm{UInd}(A)$. Let $s \in G$ and let $h \in H$. Using left invariance of μ at the last step, we get

$$\begin{aligned} \alpha_h(a(sh)) &= \alpha_s \left(\sum_{W \in \mathcal{W}} \int_H g_W(shk) \cdot \alpha_k(a_W)\, d\mu(k) \right) \\ &= \sum_{W \in \mathcal{W}} \int_H g_W(shk) \cdot \alpha_{hk}(a_W)\, d\mu(k) = a(s). \end{aligned}$$

The claim is proved.

It remains to show that $\|\pi \circ a - b\| < \varepsilon$. Let $s \in G$ and let $h \in H$. For $W \in \mathcal{W}$, we have constructed g_W such that if $s \in G$, $h \in H$, and $g_W(sh) \neq 0$, then $sh \in x_W U$. For such s and h we have $\|b(x_W) - b(sh)\| < \frac{\varepsilon}{2}$ by the choice of U. Using H-equivariance of π for the first equality and $b \in \mathrm{UInd}(B)$ for the third equality, we then get

$$\begin{aligned} \|\pi(\alpha_h(a_W)) - b(s)\| &= \|\beta_h(b(x_W)) - b(s)\| = \|b(x_W) - \beta_{h^{-1}}(b(s))\| \\ &= \|b(x_W) - b(sh)\| < \frac{\varepsilon}{2}. \end{aligned}$$

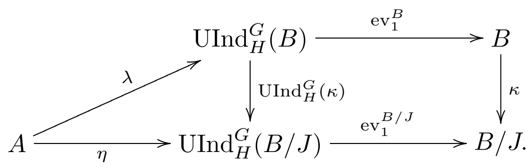
Therefore, using (4.3) and $\sum_{W \in \mathcal{W}} l_W(sH) = 1$ at the first and last steps,

$$\begin{aligned} \|\pi(a(s)) - b(s)\| &= \left\| \sum_{W \in \mathcal{W}} \int_H g_W(sh) \big(\pi(\alpha_h(a_W)) - b(s) \big) \, d\mu(h) \right\| \\ &\leq \frac{\varepsilon}{2} \sum_{W \in \mathcal{W}} \int_H g_W(sh) \, d\mu(h) < \varepsilon, \end{aligned}$$

as desired. □

Theorem 4.19. *Let G be a locally compact group, and let $H \leqslant G$ be a closed subgroup. Suppose that whenever $\varphi\colon A \to B$ is a surjective H-morphism of C^*-algebras, then $\mathrm{UInd}_H^G(\varphi)$ is also surjective. Let α be a projective action of G. Then $\alpha|_H$ is also projective.*

Proof. Let $\beta\colon H \to \mathrm{Aut}(B)$ be an action of H on a C^*-algebra B. The maps to be introduced are shown in the diagram below. Let J be an H-invariant ideal in B, and let $\kappa\colon B \to B/J$ be the quotient map. Let $\varphi\colon A \to B/J$ be an H-morphism. Then $\mathrm{UInd}_H^G(\kappa)\colon \mathrm{UInd}_H^G(B) \to \mathrm{UInd}_H^G(B/J)$ is surjective by hypothesis. We can still define $\eta\colon A \to \mathrm{UInd}_H^G(B)$ by the same formula as in Lemma 3.9, and it is still a G-morphism. It is easy to check that its range, which a priori is in $F_H^G(B)$, is actually in $\mathrm{UInd}_H^G(B)$. Since α is projective, there is a G-morphism $\lambda\colon A \to \mathrm{UInd}_H^G(B)$ such that $I_H^G(\kappa) \circ \lambda = \eta$. We still have H-equivariant maps $\mathrm{ev}_1^B\colon \mathrm{UInd}_H^G(B) \to B$ and $\mathrm{ev}_1^{B/J}\colon \mathrm{UInd}_H^G(B/J) \to B/J$, given by the same formulas as in Lemma 3.8, which give the following commutative diagram:



It is easy to check that $\mathrm{ev}_1^{B/J} \circ \eta = \varphi$. Therefore the map $\psi = \mathrm{ev}_1^B \circ \lambda$ is a H-morphism from A to B such that $\kappa \circ \psi = \varphi$. This completes the proof that $\alpha|_H$ is projective. □

Theorem 4.20. *Let G be a locally compact group, and let $H \leqslant G$ be a closed subgroup. Assume that the left and right uniformities on G agree, and that G/H is right uniformly finitistic. Let α be a projective action of G. Then $\alpha|_H$ is also projective.*

Proof. Combine Theorem 4.16 and Theorem 4.19. □

Corollary 4.21. *Let G be a right uniformly finitistic locally compact group. Assume that the left and right uniformities on G agree. Let A be a G-algebra which is equivariantly projective. Then A is (nonequivariantly) projective.*

Corollary 4.22. *Let G be a locally compact group, and let $H \leqslant G$ be a closed subgroup. Under any of the following conditions, if (G, A, α) is an equivariantly projective G-algebra, then $\alpha|_H$ is also projective:*

(1) G/H is compact.
(2) G is discrete.
(3) G is abelian and G/H is uniformly finitistic.
(4) $G = \mathbb{R}^n$.

Proof. Part (1) follows from Theorem 4.19, Lemma 4.15(4), and Proposition 3.6.

For part (2), it is clear that the cover of G by its one element subsets is both right and left uniform. Therefore the left and right uniformities on G agree. Furthermore, the cover of G/H by its one element subsets is right uniform. So G/H is obviously right uniformly finitistic.

Part (3) follows from equality of the left and right uniformities on an abelian group.

For part (4), it is now only necessary to show that G/H is uniformly finitistic. We know that there exist $k, l \in \mathbb{N} \cup \{0\}$ such that $G/H \cong \mathbb{R}^k \times (S^1)^l$, and it is easy to see that the uniformity on G/H comes from any of the standard product metrics on $\mathbb{R}^k \times (S^1)^l$. It is very easy, using covers by open sets obtained as products of open intervals and arcs of a fixed sufficiently small length, to see that every uniform open cover of $\mathbb{R}^k \times (S^1)^l$ has a uniform refinement of order at most 2^{k+l}. □

In part (4), the true uniform dimension is, of course, $k + l$, but we don't need this.

Remark 4.23. Corollary 4.22(2) implies that there is no projective action of a countable discrete group on a nonprojective C^*-algebra, in contrast to Example 3.12, where it is shown that the discrete group $\mathbb{Z}$ can act semiprojectively on a C^*-algebra which is not semiprojective in the usual sense.

Remark 4.24. The proof of Theorem 4.19 cannot be generalized to cover semiprojectivity. This is clear from Example 3.12. The problem is that there is no analog of Proposition 3.7 for the left uniform induction functor.

Let $\mathbb{N}^+ = \{1, 2, \ldots, \infty\}$ be the one point compactification of $\mathbb{N}$. Set $B = C(\mathbb{N}^+)$, and for $n \in \mathbb{N}$ set

$$J_n = \big\{ b \in B \colon b(k) = 0 \text{ for } k \in \{n+1,\, n+2, \ldots, \infty\} \big\}.$$

Then $\overline{\bigcup_{n=1}^{\infty} J_n} = C_0(\mathbb{N}) \subset B$. Call this ideal J. For $l \in \mathbb{N}$, define $b_l \in B$ by

$$b_l(j) = \begin{cases} 1 & j = l \\ 0 & j \neq l, \end{cases}$$

and define $a \in C_{\mathrm{b}}(\mathbb{Z}, B)$ by $a(n) = b_n$ for $n \in \mathbb{N}$ and $a(n) = 0$ for $n \in \mathbb{Z} \setminus \mathbb{N}$. Then $a \in C_{\mathrm{b}}(\mathbb{Z}, J)$, but the distance from a to any element of $\bigcup_{n=1}^{\infty} C_{\mathrm{b}}(\mathbb{Z}, J_n)$ is at least 1, so $a \notin \overline{\bigcup_{n=1}^{\infty} C_{\mathrm{b}}(\mathbb{Z}, J_n)}$.

We have written everything in terms of bounded continuous functions, but on $\mathbb{Z}$ all continuous functions are uniformly continuous.

5. Semiprojectivity of the crossed product algebra

If (G, A, α) is an equivariantly semiprojective C^*-algebra, can we deduce that the crossed product algebra $A \rtimes_\alpha G$ is semiprojective? We show in Theorem 5.1 that the answer is positive when G is finite and A is unital, and in Example 5.2 that the answer can be negative when G is compact. We then provide examples to show that the converses of both Theorem 5.1 and Corollary 3.11 are false. We end the section with further open problems.

Theorem 5.1. *Let G be a discrete group such that $C^*(G)$ is semiprojective, and let (G, A, α) be an equivariantly semiprojective unital G-algebra. Then $A \rtimes_\alpha G$ is semiprojective (in the usual sense).*

Proof. Lemma 1.6 implies that (G, A, α) is equivariantly semiprojective in the unital category. We will show that $A \rtimes_\alpha G$ is semiprojective in the unital category. Applying Lemma 1.6 again, this time with the group being trivial, we will conclude that $A \rtimes_\alpha G$ is semiprojective.

We regard A as a subalgebra of $A \rtimes_\alpha G$. Also, for $s \in G$ let $u_s \in A \rtimes_\alpha G$ be the standard implementing unitary, so that $u_s a u_s^* = \alpha_s(a)$ for all $a \in A$. The unitaries u_s induce a *-homomorphism $\omega\colon C^*(G) \to A \rtimes_\alpha G$.

By assumption, $C^*(G)$ is semiprojective. Thus, Lemma 1.4 of [Phi12] shows that it suffices to prove that ω is relatively semiprojective in the sense of Definition 1.2 of [Phi12] (but with the group being trivial). Accordingly, let C be a unital C^*-algebra, let $J_1 \subset J_2 \subset \cdots$ be ideals in C, let $J = \overline{\bigcup_{n=1}^\infty J_n}$, let

$$\kappa\colon C \to C/J, \quad \kappa_n\colon C \to C/J_n, \quad \text{and} \quad \pi_n\colon C/J_n \to C/J$$

be the quotient maps, and let $\lambda\colon C^*(G) \to C$ and $\varphi\colon A \rtimes_\alpha G \to C/J$ be unital *-homomorphisms such that $\kappa \circ \lambda = \varphi \circ \omega$.

Define an action $\gamma\colon G \to \mathrm{Aut}(C)$ by $\gamma_s(c) = \lambda(u_s)c\lambda(u_s)^*$ for $c \in C$ and $s \in G$. Then (G, C, γ) is a unital G-algebra, and the ideals J_n are G-invariant.

One checks that $\varphi|_A\colon A \to C/J$ is G-equivariant. Since (G, A, α) is equivariantly semiprojective (in the unital category), there exists $n \in \mathbb{N}$ and a unital G-morphism $\psi_0\colon A \to C/J_n$ such that $\pi_n \circ \psi_0 = \varphi|_A$. Define $v_s = (\kappa_n \circ \lambda)(u_s)$ for $s \in G$. Then (v, ψ_0) is a covariant representation of (G, A, α) in C/J_n, so there exists a unique *-homomorphism $\psi\colon A \rtimes_\alpha G \to C/J_n$ such that $\psi(u_s) = v_s$ and $\psi|_A = \psi_0$. This *-homomorphism is the one required by the definition of relative semiprojectivity. □

The basic examples of countable discrete groups G that satisfy the hypothesis of Theorem 5.1, that is, such that $C^*(G)$ is semiprojective, are finite groups, $\mathbb{Z}$, and the finitely generated free groups. There is no known characterization of those groups G for which $C^*(G)$ is semiprojective.

In Theorem 5.1, some restriction on G is necessary. Even compactness is not enough.

Example 5.2. Let G be an infinite compact group. It follows from Corollary 1.9 of [Phi12] that the trivial action of G on $\mathbb{C}$ is semiprojective. However, the crossed product is $\mathbb{C} \rtimes G = C^*(G)$, which is an infinite direct sum of matrix algebras, so not semiprojective by [Bla04, Corollary 2.10].

Theorem 5.1 gives us an easy way of proving that many actions by $\mathbb{Z}$ are not equivariantly semiprojective.

Example 5.3. Let $\theta \in \mathbb{R}$. Let $\alpha\colon \mathbb{Z} \to \mathrm{Aut}(C(S^1))$ be the action generated by rotation by $\exp(2\pi i\theta)$. Then α is never semiprojective, for any value of θ.

If $\theta \notin \mathbb{Q}$, then the crossed product is a simple $A\mathbb{T}$-algebra, and therefore not semiprojective, for example by [Bla04, Corollary 2.14].

If $\theta \in \mathbb{Q}$, then $A = C(S^1) \rtimes_\alpha \mathbb{Z}$ is Morita equivalent to $C\big((S^1)^2\big)$. Since both A and $C\big((S^1)^2\big)$ are unital and $C\big((S^1)^2\big)$ is not semiprojective, it follows from [Bla85, Corollary 2.29] that A is not semiprojective.

In both cases, it follows from Theorem 5.1 that α is not equivariantly semiprojective.

There are versions of Theorem 5.1 in which one takes the crossed product by only part of the action. As an easy example, consider an action of a product of two groups, and take the crossed product by one of them. We will not explore the possibilities further here.

We end this section with two examples that show that the converses of both Theorem 5.1 and Corollary 3.11 are false, and we give more open problems.

Example 5.4. There is an action α of $\mathbb{Z}_2$ on $\mathcal{O}_2$ such that the crossed product $B = \mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_2$ is not semiprojective. It follows from Theorem 5.1 that this action is not equivariantly semiprojective. Thus, the converse of Corollary 3.11 fails.

We follow [Izu04]; also see Section 6 of [Bla04]. Take α to be as in Lemma 4.7 of [Izu04] or, more generally, as in Theorem 4.8(3) of [Izu04] with the groups Γ_0 and Γ_1 chosen so that at least one of them is not finitely generated, and also such that $\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_2$ satisfies the Universal Coefficient Theorem. The action α is outer, so B is simple by [Kis81, Theorem 3.1] and purely infinite by [JO98, Corollary 4.6]. Therefore it is a Kirchberg algebra (a separable purely infinite simple nuclear C^*-algebra). It does not have finitely generated K-theory, so B is not semiprojective by [Bla04, Corollary 2.11].

Example 5.5. Let $\widehat{\alpha}\colon \mathbb{Z}_2 \to \mathrm{Aut}(B)$ be the dual of the action α of Example 5.4. Then

$$B \rtimes_{\widehat{\alpha}} \mathbb{Z}_2 \cong M_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2,$$

which is semiprojective. However, B was shown in Example 5.4 not to be semiprojective. So Corollary 3.11 implies that $\widehat{\alpha}$ is not equivariantly semiprojective. This shows that the converse of Theorem 5.1 fails.

Example 5.4 also shows if A is semiprojective and $\alpha\colon G \to \mathrm{Aut}(A)$ is an action of a finite group on A, then (G, A, α) need not be equivariantly semiprojective. However, we have neither a proof nor a counterexample for the following question.

Question 5.6. Let G be a finite group, let A be a unital C^*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of G on A. Suppose that A and $A \rtimes_\alpha G$ are both semiprojective. Does it follow that (G, A, α) is equivariantly semiprojective?

If $\alpha\colon G \to \mathrm{Aut}(A)$ is semiprojective, then Theorem 3.10 implies that for any subgroup $H \leqslant G$, the action $\alpha|_H$ is also semiprojective. Thus, by Theorem 5.1, the crossed product $A \rtimes_{\alpha|_H} H$ is also semiprojective. Therefore, if G is not simple, one must probably also consider these intermediate crossed product algebras.

At a conference in August 2010, George Elliott asked if there is a relation between equivariant semiprojectivity and the Rokhlin property. The following question addresses what seems to be a plausible connection.

Question 5.7. Let G be a finite group, and let (G, A, α) be a unital G-algebra. Suppose that A is (nonequivariantly) semiprojective and α has the Rokhlin property. Does it follow that (G, A, α) is equivariantly semiprojective?

Even if this is false in general, it might be true if A is simple, or using an equivariant version of a weak form of semiprojectivity.

6. Semiprojectivity of the fixed point algebra

In this section we study the analog of the question of Section 5 for the fixed point algebra. That is, given an equivariantly semiprojective C^*-algebra (G, A, α), can we deduce that the fixed point algebra A^G is semiprojective?

In Proposition 6.2, we give a positive answer when G is finite, A is unital, and the action is saturated. It is unknown whether one can drop the conditions that A be unital or that the action be saturated.

Some conditions are necessary. In Example 6.1 we give a semiprojective action of a compact (but not finite) group on a unital C^*-algebra such that the fixed point algebra is not semiprojective.

In Theorem 6.4, we show that the fixed point algebra is trivial if a noncompact group acts semiprojectively. This gives a positive answer to the question, but more interestingly it shows that the trivial action of a noncompact group is never semiprojective. We can therefore give a precise characterization when the trivial action of a group is (semi)projective (Corollary 6.5).

Let G be a second countable compact group and let $\alpha\colon G \to \mathrm{Aut}(A)$ be a semiprojective action. Example 5.2 shows that the crossed product $A \rtimes_\alpha G$ need not be semiprojective, but in that example the fixed point algebra is semiprojective. In general, though, the fixed point algebra also need not be semiprojective.

Example 6.1. Let $\alpha\colon S^1 \to \mathrm{Aut}(\mathcal{O}_2)$ be the gauge action on the Cuntz algebra $\mathcal{O}_2$, defined on the standard generators s_1 and s_2 by $\alpha_\zeta(s_j) = \zeta s_j$ for $\zeta \in S^1$ and $j = 1, 2$. This action is equivariantly semiprojective by [Phi12, Corollary 3.12]. However, the fixed point algebra is the 2^∞ UHF algebra, which is not semiprojective, for example by [Bla04, Corollary 2.14].

We obtain a positive result when the group is finite and the action is saturated in the sense of Definition 7.1.4 of [Phi87]. Saturation is a quite weak noncommutative analog of freeness; see the discussion at the beginning of Section 5.2 of [Phi09].

Proposition 6.2. *Let G be a finite group, let A be a unital, separable C^*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be a saturated action of G on A. If α is semiprojective, then A^G is semiprojective.*

Proof. By definition, saturation implies that A^G is strongly Morita equivalent to $A \rtimes_\alpha G$. Theorem 5.1 tells us that $A \rtimes_\alpha G$ is semiprojective, so A^G is semiprojective by [Bla85, Corollary 2.29]. □

Finiteness is needed, since the gauge action in Example 6.1 is saturated. (In fact, it follows from Theorem 5.11 of [Phi09] that this action is hereditarily saturated.) However, we don't know whether saturation is needed.

Question 6.3. Let G be a finite group, let A be a unital C^*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an arbitrary semiprojective action of G on A. Does it follow that A^G is semiprojective?

If G is compact and A unital, then A^G is isomorphic to a unital corner in $A \rtimes_\alpha G$, for example by [Bla06, Theorem II.10.4.18]. If we knew that semiprojectivity passes to arbitrary unital corners (an open problem), we would get a positive answer to Question 6.3.

Theorem 6.4. *Let (G, A, α) be a separable, equivariantly semiprojective G-algebra, and assume G is noncompact. Then the fixed point algebra is trivial, that is, $A^G = \{0\}$.*

Proof. Assume G is a noncompact second countable locally compact group. The action $\alpha\colon G \to \mathrm{Aut}(A)$ induces an action $\overline{\alpha}\colon G \to \mathrm{Aut}(M_2 \otimes A)$ by acting trivially on M_2, that is, $\overline{\alpha}_s(x \otimes a) = x \otimes \alpha_s(a)$ for $x \in M_2$, $a \in A$, and $s \in G$.

Recall that a metric is called *proper* if every closed bounded set is compact. By the main theorem of [Str74], there is a proper left invariant metric d which generates the topology of G. We manufacture an equivariant lifting problem in several steps.

Step 1: Let $(e_{j,k})_{j,k=1,2}$ be the standard system of matrix units for M_2. Let $\lambda \mapsto u_\lambda \in M_2$, for $\lambda \in [0, 1]$, be a continuously differentiable path of unitaries from the identity $u_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ to $u_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Continuous differentiability is required for convenience; it gives us $M \in [0, \infty)$ such that for all $\lambda_1, \lambda_2 \in [0, 1]$ we have $\|u_{\lambda_1} - u_{\lambda_2}\| \leq M|\lambda_1 - \lambda_2|$. For $\lambda \in [0, 1]$ define $\varphi_\lambda\colon A \to M_2 \otimes A$ by $\varphi_\lambda(a) = u_\lambda e_{1,1} u_\lambda^* \otimes a$ for $a \in A$. Thus $\varphi_0(a) = e_{1,1} \otimes a$ and $\varphi_1(a) = e_{2,2} \otimes a$ for $a \in A$. Also, for $\lambda_1, \lambda_2 \in [0, 1]$ and $a \in A$, we have

$$\|\varphi_{\lambda_1}(a) - \varphi_{\lambda_2}(a)\| \leq 2\|u_{\lambda_1} - u_{\lambda_2}\| \cdot \|a\| \leq 2M|\lambda_1 - \lambda_2| \cdot \|a\|. \tag{6.1}$$

It is immediate that

$$\overline{\alpha}_s \circ \varphi_\lambda = \varphi_\lambda \circ \alpha_s. \tag{6.2}$$

for $s \in G$ and $\lambda \in [0, 1]$.

Step 2: Let $G^+ = G \cup \{\infty\}$ denote the one point compactification of G. Let D be the C^*-algebra

$$D = \big\{ f \in C(G^+,\, M_2 \otimes A) \colon f(\infty) \in \mathbb{C}e_{2,2} \otimes A \big\}.$$

For $s \in G$, we take $s \cdot \infty = \infty$. This gives a extension of the action of G on itself by translation to a continuous action of G on G^+. We define an action β of G on D by $\beta_s(f)(t) = \overline{\alpha}_s(f(s^{-1}t))$ for $f \in D$, $s \in G$, and $t \in G^+$. Since G is not compact, the fixed point algebra of this action consists of the constant functions taking values in $\mathbb{C}e_{2,2} \otimes A^G$.

Step 3: For $k = 1, 2, \ldots$, define "stretching" maps $\sigma_k \colon [0, \infty) \to [0, 1]$ by

$$\sigma_k(\lambda) = \min(\lambda/k,\, 1)$$

for $\lambda \in [0, \infty)$. We may extend these to maps from $[0, \infty]$ by setting $\sigma_k(\infty) = 1$ for $k \in \mathbb{N}$. For $\lambda_1, \lambda_2 \in [0, \infty)$, we have

$$|\sigma_k(\lambda_1) - \sigma_k(\lambda_2)| \leq \frac{|\lambda_1 - \lambda_2|}{k}. \tag{6.3}$$

Step 4: For $t \in G$ let $d_0(t) = d(t, 1)$ denote the distance from t to the unit element $1 \in G$, and extend this function to G^+ by setting $d_0(\infty) = \infty$. Since d is proper, the map $d_0 \colon G^+ \to [0, \infty]$ is continuous.

Using left invariance of d, for $s, t \in G$ we get $|d_0(s^{-1}t) - d_0(t)| \leq d_0(s)$. Therefore, for $k \in \mathbb{N}$,

$$\big|\sigma_k(d_0(s^{-1}t)) - \sigma_k(d_0(t))\big| \leq \frac{d_0(s)}{k}. \tag{6.4}$$

For $k \in \mathbb{N}$, we define a *-homomorphism $\omega_k \colon A \to D$ by

$$\omega_k(a)(t) = \varphi_{\sigma_k(d_0(t))}(a)$$

for $a \in A$ and $t \in G^+$. Then for $s \in G$ we have, using density of G in G^+ and (6.2) at the second step, (6.1) at the third step, and (6.4) at the fourth step,

$$\begin{aligned} \|\beta_s(\omega_k(a)) - \omega_k(\alpha_s(a))\| &= \sup_{t \in G^+} \|\overline{\alpha}_s(\varphi_{\sigma_k(d_0(s^{-1}t))}(a)) - \varphi_{\sigma_k(d_0(t))}(\alpha_s(a))\| \\ &= \sup_{t \in G} \|\varphi_{\sigma_k(d_0(s^{-1}t))}(\alpha_s(a))) - \varphi_{\sigma_k(d_0(t))}(\alpha_s(a))\| \\ &\leq \sup_{t \in G} 2M \big|\sigma_k(d_0(s^{-1}t)) - \sigma_k(d_0(t))\big| \cdot \|\alpha_s(a)| \\ &\leq \frac{2M\|a\| d_0(s)}{k}, \end{aligned}$$

that is,

$$\|\beta_s(\omega_k(a)) - \omega_k(\alpha_s(a))\| \leq \frac{2M\|a\| d_0(s)}{k}. \tag{6.5}$$

In particular, we have

$$\lim_{k \to \infty} \|\beta_s(\omega_k(a)) - \omega_k(\alpha_s(a))\| = 0. \tag{6.6}$$

Moreover, for $k \in \mathbb{N}$ and $a \in A$, we have $\omega_k(a)(1) = e_{1,1} \otimes a$, so, using Step 2,

$$\begin{aligned} \operatorname{dist}(\omega_k(a),\, D^G) &\geq \inf_{b \in A} \|e_{1,1} \otimes a - e_{2,2} \otimes b\| \\ &\geq \inf_{b \in A} \|(e_{1,1} \otimes 1)(e_{1,1} \otimes a - e_{2,2} \otimes b)\| = \|a\|. \end{aligned} \tag{6.7}$$

Step 5: Consider the sequence algebra $E = l^\infty(\mathbb{N}, D)$ and for $n \in \mathbb{N}$ the ideals $J_n \lhd E$ defined by

$$J_n = \big\{ (x_k)_{k \in \mathbb{N}} \in E \colon x_k = 0 \text{ for } k \geq n \big\}.$$

Then $J_1 \subset J_2 \subset \cdots$ is an increasing sequence of invariant ideals, and the ideal $J = \overline{\bigcup_{n=1}^{\infty} J_n}$ is equal to $C_0(\mathbb{N}, D) \subset l^\infty(\mathbb{N}, D)$.

Let $\gamma\colon G \to \mathrm{Aut}(E)$ denote the (not necessarily continuous) coordinatewise action of G on E, that is, for $s \in G$ and $(x_k)_{k\in\mathbb{N}} \in E$ we set $\gamma_s((x_k)_{k\in\mathbb{N}}) = (\beta_s(x_k))_{k\in\mathbb{N}}$. We let $F \subset E$ be the C^*-subalgebra on which γ is continuous, that is,

$$F = \big\{x \in E \colon s \mapsto \gamma_s(x) \text{ is continuous}\big\}.$$

Then F is γ-invariant, and we also use γ to denote the restricted action $\gamma\colon G \to \mathrm{Aut}(F)$. By construction, this action is continuous.

Clearly $J \subset F$. Moreover, J_n is G-invariant for all $n \in \mathbb{N}$, so the action of G on F drops to F/J_n. Similarly J is G-invariant and the action drops to F/J. For $n \in \mathbb{N}$, let $\pi_n\colon F/J_n \to F/J$ be the natural quotient G-morphism. We have $F^G = l^\infty(\mathbb{N}, D^G)$, and one checks by direct computation that the fixed point algebra of F/J_n is $(F/J_n)^G = F^G/J_n^G$, which we identify with $l^\infty\big(\{n+1, n+2, \ldots\}, D^G\big)$.

Step 6: For each $a \in A$, consider the sequence $\omega(a) = (\omega_1(a), \omega_2(a), \ldots) \in E$ constructed in Step 4. We claim that $\omega(a) \in F$. To see this, let $a \in A$ and let $s, t \in G$. Then, using (6.5) and $\|\omega_k\| = 1$ at the third step,

$$\begin{aligned}
\|\gamma_s(\omega(a)) - \gamma_t(\omega(a))\| &= \sup_{k\in\mathbb{N}} \|\beta_s(\omega_k(a)) - \beta_t(\omega_k(a))\| \\
&= \sup_{k\in\mathbb{N}} \|\beta_{t^{-1}s}(\omega_k(a)) - \omega_k(\alpha_{t^{-1}s}(a)) + \omega_k(\alpha_{t^{-1}s}(a) - a)\| \\
&\leq \sup_{k\in\mathbb{N}} \frac{2M\|a\|d_0(t^{-1}s)}{k} + \|\alpha_{t^{-1}s}(a) - a\| \\
&= 2M\|a\|d(s,t) + \|\alpha_s(a) - \alpha_t(a)\|.
\end{aligned}$$

Since α is a continuous action, this proves the claim.

Step 7: Define a *-homomorphism $\overline{\omega}\colon A \to F/J$ by sending $a \in A$ to the image of $\omega(a)$ in the quotient F/J. It follows from (6.6) that $\overline{\omega}$ is a G-morphism.

Suppose now that A is equivariantly semiprojective. Then there are $n \in \mathbb{N}$ and a G-morphism $\psi\colon A \to F/J_n$ such that $\pi_n \circ \psi = \overline{\omega}$.

Fix an element $a \in A^G$. We want to show $a = 0$. Since ψ is G-equivariant, $\psi(a) \in (F/J_n)^G$.

Identify $(F/J_n)^G$ with $l^\infty\big(\{n+1, n+2, \ldots\}, D^G\big)$ as at the end of Step 5, and write $\psi(a) = (\psi_{n+1}(a), \psi_{n+2}(a), \ldots)$. Then, using (6.7) at the last step,

$$\begin{aligned}
\|\pi_n(\psi(a)) - \overline{\omega}(a)\| &= \big\|\pi_n\big((\psi_{n+1}(a), \psi_{n+2}(a), \ldots) - (\omega_{n+1}(a), \omega_{n+2}(a), \ldots)\big)\big\| \\
&= \liminf_{k\to\infty} \|\psi_k(a) - \omega_k(a)\| \\
&\geq \inf_{k\in\{n+1, n+2, \ldots\}} \mathrm{dist}(\omega_k(a), D^G) \geq \|a\|.
\end{aligned}$$

For $a \neq 0$ this contradicts $\pi_n(\psi(a)) = \overline{\omega}(a)$. Thus $A^G = \{0\}$. □

We now address the question of when the trivial action of a group G on a C^*-algebra A is (semi)projective.

If G is compact, then it follows from Corollary 4.21 and Corollary 3.11 that A is (semi)projective in the usual sense. Conversely, if A is (semi)projective in the usual sense, then it follows from [Phi12, Corollary 1.9] and Lemma 1.9 that the trivial action of G on A is (semi)projective.

If G is noncompact, then the trivial action on a (non-zero) C^*-algebra is never semiprojective. Indeed, if (G, A, α) is equivariantly semiprojective, and α is trivial, then Theorem 6.4 above shows that $A = A^G = \{0\}$.

We thus obtain the following precise characterization when the trivial action of a group is (semi)projective.

Corollary 6.5. *Let A be separable C^*-algebra, and let G be a second countable locally compact group. Then the trivial action of G on A is (semi)projective if and only if A is (semi)projective and G is compact.*

Acknowledgments

The authors thank Siegfried Echterhoff and Stefan Wagner for valuable discussions on induction of group actions.

References

[BG52] R. G. Bartle and L. M. Graves, *Mappings between function spaces*, Trans. Am. Math. Soc. **72** (1952), 400–413.

[Bla85] B. Blackadar, *Shape theory for C^*-algebras*, Math. Scand. **56** (1985), 249–275.

[Bla04] ———, *Semiprojectivity in simple C^*-algebras*, Kosaki, Hideki (ed.), Operator algebras and applications. Proceedings of the US-Japan seminar held at Kyushu University, Fukuoka, Japan, June 7–11, 1999. Tokyo: Mathematical Society of Japan. Advanced Studies in Pure Mathematics 38, 1-17, 2004.

[Bla06] ———, *Operator algebras. Theory of C^*-algebras and von Neumann algebras*, Encyclopaedia of Mathematical Sciences 122. Operator Algebras and Non-Commutative Geometry 3. Berlin: Springer. XX, 2006.

[Bla12] ———, Private communication, 2012.

[Bln96] E. Blanchard, *Déformations de C^*-algèbres de Hopf*, Bull. Soc. Math. Fr. **124** (1996), no. 1, 141–215 (French).

[Bre72] G. E. Bredon, *Introduction to compact transformation groups.*, Pure and Applied Mathematics, 46. New York-London: Academic Press. XIII, 1972.

[Dad09] M. Dadarlat, *Continuous fields of C^*-algebras over finite dimensional spaces*, Adv. Math. **222** (2009), no. 5, 1850–1881.

[Ech10] S. Echterhoff, *Crossed products, the Mackey-Rieffel-Green machine and applications*, preprint, arXiv:1006.4975, 2010.

[EKQR00] S. Echterhoff, S. Kaliszewski, J. Quigg, and I. Raeburn, *Naturality and induced representations*, Bull. Aust. Math. Soc. **61** (2000), no. 3, 415–438.

[HR79] E. Hewitt and K. A. Ross, *Abstract harmonic analysis. Vol. 1: Structure of topological groups; integration theory; group representations. 2nd ed.*, Grundlehren der mathematischen Wissenschaften. 115. A Series of Comprehensive Studies in Mathematics. Berlin-Heidelberg-New York: Springer-Verlag. IX, 1979.

[Isb59] J. R. Isbell, *On finite-dimensional uniform spaces*, Pac. J. Math. **9** (1959), 107–121.

[Isb64] ———, *Uniform spaces*, Mathematical Surveys. 12. Providence, R.I.: American Mathematical Society (AMS). XI, 1964.

[Izu04] M. Izumi, *Finite group actions on C^*-algebras with the Rohlin property. I*, Duke Math. J. **122** (2004), no. 2, 233–280.

[JO98] J. A Jeong and H. Osaka, *Extremally rich C^*-crossed products and the cancellation property*, J. Aust. Math. Soc., Ser. A **64** (1998), no. 3, 285–301.

[Kas88] G. G. Kasparov, *Equivariant KK-theory and the Novikov conjecture*, Invent. Math. **91** (1988), no. 1, 147–201.

[Kis81] A. Kishimoto, *Outer automorphisms and reduced crossed products of simple C^*-algebras*, Commun. Math. Phys. **81** (1981), 429–435.

[KW99] E. Kirchberg and S. Wassermann, *Permanence properties of C^*-exact groups*, Doc. Math., J. DMV **4** (1999), 513–558.

[Lor97a] T. A. Loring, *Almost multiplicative maps between C^*-algebras*, Doplicher, S. (ed.) et al., Operator algebras and quantum field theory. Proceedings of the conference dedicated to Daniel Kastler in celebration of his 70th birthday, Accademia Nazionale dei Lincei, Roma, Italy, July 1–6, 1996. Cambridge, MA: International Press. 111-122, 1997.

[Lor97b] ———, *Lifting solutions to perturbing problems in C^*-algebras*, Fields Institute Monographs 8. Providence, R.I.: American Mathematical Society. IX, 1997.

[MZ55] D. Montgomery and L. Zippin, *Topological transformation groups*, (Interscience Tracts in Pure and Applied Mathematics). New York: Interscience Publishers, Inc. XI, 1955.

[Pea75] A. R. Pears, *Dimension theory of general spaces*, Cambridge etc.: Cambridge University Press. XII, 1975.

[Phi87] N. C. Phillips, *Equivariant K-theory and freeness of group actions on C^*-algebras*, Lecture Notes in Mathematics, 1274. Berlin etc.: Springer-Verlag. VIII, 1987.

[Phi09] ———, *Freeness of actions of finite groups on C^*-algebras*, de Jeu, Marcel (ed.) et al., Operator structures and dynamical systems. Satellite conference of the 5th European congress of mathematics, Leiden, Netherlands, July 21–25, 2008. Providence, RI: American Mathematical Society (AMS). Contemporary Mathematics 503, 217-257, 2009.

[Phi12] ———, *Equivariant semiprojectivity*, preprint, arXiv:1112.4584, 2012.

[Rie89] M. A. Rieffel, *Continuous fields of C^*-algebras coming from group cocycles and actions.*, Math. Ann. **283** (1989), no. 4, 631–643.

[Roy88] H. L. Royden, *Real analysis. 3rd ed.*, New York: Macmillan Publishing Company; London: Collier Macmillan Publishing. XX, 1988.

[SSG93] J. Segal, S. Spież, and B. Günther, *Strong shape of uniform spaces*, Topology Appl. **49** (1993), no. 3, 237–249.

[Str74] R. A. Struble, *Metrics in locally compact groups*, Compos. Math. **28** (1974), 217–222.

[Swa59] R. G. Swan, *A new method in fixed point theory*, Bull. Am. Math. Soc. **65** (1959), 128–130.

[TW12] H. Thiel and W. Winter, *The generator problem for $\mathcal{Z}$-stable C^*-algebras*, preprint, arXiv:1201.3879, 2012.

[Vid69] G. Vidossich, *A theorem on uniformly continuous extension of mappings defined in finite-dimensional spaces*, Isr. J. Math. **7** (1969), 207–210.

Department of Mathematics, University of Oregon, Eugene OR 97403-1222, USA, and Research Institute for Mathematical Sciences, Kyoto University, Kitashirakawa-Oiwakecho, Sakyo-ku, Kyoto 606-8502, Japan.

E-mail address: ncp@darkwing.uoregon.edu

Department of Mathematical Sciences, Københavns Universitet, Universitetsparken 5, DK-2100 København Ø, Denmark

E-mail address: apws@math.ku.dk

Department of Mathematical Sciences, Københavns Universitet, Universitetsparken 5, DK-2100 København Ø, Denmark

E-mail address: thiel@math.ku.dk

Available online at www.sciencedirect.com

ScienceDirect

Journal of Functional Analysis 260 (2011) 3474–3493

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

The Cuntz semigroup and comparison of open projections ☆

Eduard Ortega [a], Mikael Rørdam [b,*], Hannes Thiel [b]

[a] *Department of Mathematical Sciences, NTNU, NO-7491 Trondheim, Norway*
[b] *Department of Mathematical Sciences, University of Copenhagen, Universitetsparken 5, DK-2100, Copenhagen Ø, Denmark*

Received 24 August 2010; accepted 15 February 2011

Available online 3 March 2011

Communicated by S. Vaes

Abstract

We show that a number of naturally occurring comparison relations on positive elements in a C^*-algebra are equivalent to natural comparison properties of their corresponding open projections in the bidual of the C^*-algebra. In particular we show that Cuntz comparison of positive elements corresponds to a comparison relation on open projections, that we call Cuntz comparison, and which is defined in terms of—and is weaker than—a comparison notion defined by Peligrad and Zsidó. The latter corresponds to a well-known comparison relation on positive elements defined by Blackadar. We show that Murray–von Neumann comparison of open projections corresponds to tracial comparison of the corresponding positive elements of the C^*-algebra. We use these findings to give a new picture of the Cuntz semigroup.

Keywords: C^*-algebras; Cuntz semigroup; von Neumann algebras; Open projections

☆ This research was supported by the NordForsk Research Network "Operator Algebras and Dynamics" (grant #11580). The first named author was partially supported by the Research Council of Norway (project 191195/V30), by MEC-DGESIC (Spain) through Project MTM2008-06201-C02-01/MTM, by the Consolider Ingenio "Mathematica" project CSD2006-32 by the MEC, and by 2009 SGR 1389 grant of the Comissionat per Universitats i Recerca de la Generalitat de Catalunya. The second named author was supported by grants from the Danish National Research Foundation and the Danish Natural Science Research Council (FNU). The third named author was partially supported by the Marie Curie Research Training Network EU-NCG and by the Danish National Research Foundation.

* Corresponding author.
E-mail addresses: eduardo.ortega@math.ntnu.no (E. Ortega), rordam@math.ku.dk (M. Rørdam), thiel@math.ku.dk (H. Thiel).

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2011.02.017

1. Introduction

There is a well-known bijective correspondence between hereditary sub-C^*-algebras of a C^*-algebra and open projections in its bidual. Thus to every positive element a in a C^*-algebra A one can associate the open projection p_a in A^{**} corresponding to the hereditary sub-C^*-algebra $A_a = \overline{aAa}$. Any comparison relation between positive elements in a C^*-algebra that is invariant under the relation $a \cong b$, defined by $a \cong b \Leftrightarrow A_a = A_b$, can in this way be translated into a comparison relation between open projections in the bidual. Vice versa, any comparison relation between open projections corresponds to a comparison relation (which respects $\cong$) on positive elements of the underlying C^*-algebra.

Peligrad and Zsidó defined in [19] an equivalence relation (and also a sub-equivalence relation) on open projections in the bidual of a C^*-algebra as Murray–von Neumann equivalence with the extra assumption that the partial isometry that implements the equivalence gives an isomorphism between the corresponding hereditary sub-C^*-algebras of the given C^*-algebra. Very recently, Lin [17], noted that the Peligrad–Zsidó (sub-)equivalence of open projections corresponds to a comparison relation of positive elements considered by Blackadar in [6].

The Blackadar comparison relation of positive elements is stronger than the Cuntz comparison relation of positive elements that is used to define the Cuntz semigroup of a C^*-algebra. The Cuntz semigroup has recently come to play an influential role in the classification of C^*-algebras. We show that Cuntz comparison of positive elements corresponds to a natural relation on open projections, that we also call Cuntz comparison. It is defined in terms of—and is weaker than—the Peligrad–Zsidó comparison. It follows from results of Coward, Elliott, and Ivanescu [10], and from our results, that the Blackadar comparison relation is equivalent to Cuntz comparison of positive elements when the C^*-algebra is separable and has stable rank one, and consequently that Peligrad–Zsidó comparison is equivalent to our notion of Cuntz comparison of open projections in this case.

The best known and most natural comparison relation for projections in a von Neumann algebra is the one introduced by Murray and von Neumann. It is weaker than the Cuntz and the Peligrad–Zsidó comparison relations. We show that Murray–von Neumann (sub-)equivalence of open projections in the bidual in the separable case is equivalent to tracial comparison of the corresponding positive elements of the C^*-algebra. Tracial comparison is defined in terms of dimension functions arising from lower semicontinuous tracial weights on the C^*-algebra. The proof of this equivalence builds on two results on von Neumann algebras that may have independent interest, and which probably are known to experts: One says that Murray–von Neumann comparison of projections in any von Neumann algebra which is not too big (in the sense of Tomiyama—see Section 5 for details) is completely determined by normal tracial weights on the von Neumann algebra. The other result states that every lower semicontinuous tracial weight on a C^*-algebra extends (not necessarily uniquely) to a normal tracial weight on the bidual of the C^*-algebra.

We use results of Elliott, Robert, and Santiago [11], to show that tracial comparison of positive elements in a C^*-algebra is equivalent to Cuntz comparison if the C^*-algebra is separable and exact, its Cuntz semigroup is weakly unperforated, and the involved positive elements are purely non-compact.

We also relate comparison of positive elements and of open projections to comparison of the associated right Hilbert A-modules. The Hilbert A-module corresponding to a positive element a in A is the right ideal $\overline{aA}$. We show that Blackadar equivalence of positive elements is equivalent to isomorphism of the corresponding Hilbert A-modules, and we recall that Cuntz comparison of positive elements is equivalent to the notion of Cuntz comparison of the corresponding Hilbert A-modules introduced in [10].

2. Comparison of positive elements in a C^*-algebra

We remind the reader about some, mostly well-known, notions of comparison of positive elements in a C^*-algebra. If a is a positive element in a C^*-algebra A, then let A_a denote the hereditary sub-C^*-algebra generated by a, i.e., $A_a = \overline{aAa}$. The *Pedersen equivalence relation* on positive elements in a C^*-algebra A is defined by $a \sim b$ if $a = x^*x$ and $b = xx^*$ for some $x \in A$, where $a, b \in A^+$, and it was shown by Pedersen, that this indeed defines an equivalence relation. Write $a \cong b$ if $A_a = A_b$. The equivalence relation generated by these two relations was considered by Blackadar in [5, Definition 6.1.2]:

Definition 2.1 *(Blackadar comparison).* Let a and b be positive elements in a C^*-algebra A. Write $a \sim_s b$ if there exists $x \in A$ such that $a \cong x^*x$ and $b \cong xx^*$, and write $a \precsim_s b$ if there exists $a' \in A_b^+$ with $a \sim_s a'$.

(It follows from Lemma 4.2 below that $\sim_s$ is an equivalence relation.) Note that $\precsim_s$ is not an order relation on $A^+/{\sim_s}$ since in general $a \precsim_s b \precsim_s a$ does not imply $a \sim_s b$ (see [16, Theorem 9]). If p and q are projections, then $p \sim_s q$ agrees with the usual notion of equivalence of projections defined by Murray and von Neumann, denoted by $p \sim q$.

The relation defining the Cuntz semigroup that currently is of importance in the classification program for C^*-algebras is defined as follows:

Definition 2.2 *(Cuntz comparison of positive elements).* Let a and b be positive elements in a C^*-algebra A. Write $a \precsim b$ if there exists a sequence $\{x_n\}$ in A such that $x_n^* b x_n \to a$. Write $a \approx b$ if $a \precsim b$ and $b \precsim a$.

2.3 *(The Cuntz semigroup).* Let us briefly remind the reader about the ordered Cuntz semigroup $W(A)$ associated to a C^*-algebra A. Let $M_\infty(A)^+$ denote the disjoint union $\bigcup_{n=1}^\infty M_n(A)^+$. For $a \in M_n(A)^+$ and $b \in M_m(A)^+$ set $a \oplus b = \mathrm{diag}(a, b) \in M_{n+m}(A)^+$, and write $a \precsim b$ if there exists a sequence $\{x_k\}$ in $M_{m,n}(A)$ such that $x_k^* b x_k \to a$. Write $a \approx b$ if $a \precsim b$ and $b \precsim a$. Put $W(A) = M_\infty(A)^+/{\approx}$, and let $\langle a \rangle \in W(A)$ be the equivalence class containing a. Let us denote by $\mathrm{Cu}(A)$ the completion of $W(A)$ with respect to countable suprema, i.e., $\mathrm{Cu}(A) := W(A \otimes \mathcal{K})$.

Lastly we define comparison by traces. We shall here denote by $T(A)$ the set of (norm) lower semicontinuous tracial weights on a C^*-algebra A. We remind the reader that a tracial weight on A is an additive function $\tau : A^+ \to [0, \infty]$ satisfying $\tau(\lambda a) = \lambda \tau(a)$ and $\tau(x^*x) = \tau(xx^*)$ for all $a \in A^+$, $x \in A$, and $\lambda \in \mathbb{R}^+$. That τ is lower semicontinuous means that $\tau(a) = \lim \tau(a_i)$ whenever $\{a_i\}$ is a norm-convergent increasing sequence (or net) with limit a. Each $\tau \in T(A)$ gives rise to a lower semicontinuous dimension function $d_\tau : A^+ \to [0, \infty]$ given by $d_\tau(a) = \sup_{\varepsilon>0} \tau(f_\varepsilon(a))$, where $f_\varepsilon : \mathbb{R}^+ \to \mathbb{R}^+$ is the continuous function that is 0 on 0, 1 on $[\varepsilon, \infty)$, and linear on $[0, \varepsilon]$. Any dimension function gives rise to an additive order preserving state on the Cuntz semigroup, and in particular it preserves the Cuntz relation $\precsim$.

Definition 2.4 *(Comparison by traces).* Let a and b be positive elements in a C^*-algebra A. Write $a \sim_{\mathrm{tr}} b$ and $a \precsim_{\mathrm{tr}} b$ if $d_\tau(a) = d_\tau(b)$, respectively, $d_\tau(a) \leqslant d_\tau(b)$, for all $\tau \in T(A)$.

Remark 2.5. Observe that

$$a \precsim_s b \quad\Rightarrow\quad a \precsim b \quad\Rightarrow\quad a \precsim_{\mathrm{tr}} b, \qquad a \sim_s b \quad\Rightarrow\quad a \approx b \quad\Rightarrow\quad a \sim_{\mathrm{tr}} b$$

for all positive elements a and b in any C^*-algebra A. In Section 6 we discuss under which conditions these implications can be reversed.

3. Open projections

The bidual A^{**} of a C^*-algebra A can be identified with the von Neumann algebra arising as the weak closure of the image of A under the universal representation $\pi_u : A \to \mathbb{B}(H_u)$ of A. Following Akemann [1, Definition II.1], and Pedersen [18, Proposition 3.11.9, p. 77], a projection p in A^{**} is said to be *open* if it is the strong limit of an increasing sequence of positive elements from A, or, equivalently, if it belongs to the strong closure of the hereditary sub-C^*-algebra $pA^{**}p \cap A$ of A. We shall denote this hereditary sub-C^*-algebra of A by A_p. (This agrees with the previous definition of A_p if p is a projection in A.) The map $p \mapsto A_p$ furnishes a bijective correspondence between open projections in A^{**} and hereditary sub-C^*-algebras of A. The open projection corresponding to a hereditary sub-C^*-algebra B of A is the projection onto the closure of the subspace $\pi_u(B)H_u$ of H_u. Let $P_o(A^{**})$ denote the set of open projections in A^{**}.

A projection in A^{**} is *closed* if its complement is open.

For each positive element a in A we let p_a denote the open projection in A^{**} corresponding to the hereditary sub-C^*-algebra A_a of A. Equivalently, p_a is equal to the range projection of $\pi_u(a)$, and if a is a contraction, then p_a is equal to the strong limit of the increasing sequence $\{a^{1/n}\}$. Notice that $p_a = p_b$ if and only if $A_a = A_b$ if and only if $a \cong b$. If A is separable, then each hereditary sub-C^*-algebra of A contains a strictly positive element and hence is of the form A_a for some a. It follows that every open projection in A^{**} is of the form p_a for some positive element a in A, whence there is a bijective correspondence between open projections in A^{**} and positive elements in A modulo the equivalence relation $\cong$.

3.1 *(Closure of a projection).* If $K \subseteq P_o(A^{**})$ is a family of open projections, then their supremum $\bigvee K$ is again open. Dually, the infimum of a family of closed projections is again closed. Therefore, if we are given any projection p, then we can define its *closure* $\overline{p}$ as

$$\overline{p} := \bigwedge \bigl\{ q \in P\bigl(A^{**}\bigr) \colon\ q \text{ is closed},\ p \leqslant q \bigr\}.$$

We shall consider various notions of comparisons and equivalences of open projections in A^{**} that, via the correspondence $a \mapsto p_a$, match the notions of comparison and equivalences of positive elements in a C^*-algebra considered in the previous section. First of all we have Murray–von Neumann equivalence $\sim$ and subequivalence $\precsim$ of projections in any von Neumann algebra. We shall show in Section 5 that they correspond to tracial comparison. Peligrad and Zsidó made the following definition:

Definition 3.2 *(PZ-equivalence).* (See [19, Definition 1.1].) Let A be a C^*-algebra, and let p and q be open projections in A^{**}. Then p, q are equivalent in the sense of Peligrad and Zsidó (PZ-equivalent, for short), denoted by $p \sim_{\mathrm{PZ}} q$, if there exists a partial isometry $v \in A^{**}$ such that

$$p = v^*v, \qquad q = vv^*, \qquad vA_p \subseteq A, \qquad v^*A_q \subseteq A.$$

Say that $p \precsim_{\mathrm{PZ}} q$ if there exists $p' \in P_o(A^{**})$ such that $p \sim_{\mathrm{PZ}} p' \leqslant q$.

PZ-equivalence is stronger than Murray–von Neumann equivalence. We will see in Section 6 that it is in general strictly stronger, but the two equivalences do agree for some C^*-algebras and for some classes of projections.

We will now turn to the question of PZ-equivalence of left and right support projections. Peligrad and Zsidó proved in [19, Theorem 1.4] that $p_{xx^*} \sim_{\mathrm{PZ}} p_{x^*x}$ for every $x \in A$ (and even for every x in the multiplier algebra of A). One can ask whether the converse is true. The following result gives a satisfactory answer.

Proposition 3.3. *Let $p, q \in P_o(A^{**})$ be two open projections with $p \sim_{\mathrm{PZ}} q$. If p is the support projection of some element in A, then so is q, and in this case $p = p_{xx^*}$ and $q = p_{x^*x}$ for some $x \in A$.*

Proof. There is a partial isometry v in A^{**} with $p = v^*v$, $vv^* = q$, and $vA_p \subseteq A$. This implies that $vA_pv^* \subseteq A$, so the map $x \mapsto vxv^*$ defines a *-isomorphism from A_p onto A_q. By assumption, $p = p_a$ for some positive element a in A. Upon replacing a by $\|a\|^{-1}a$ we can assume that a is a contraction. Put $b := vav^* \in A^+$. Then

$$p_b = \sup_n \bigl(vav^*\bigr)^{1/n} = \sup_n va^{1/n}v^* = vp_av^* = q.$$

Hence q is a support projection, and moreover for $x := va^{1/2} \in A$ we have $a = x^*x$ and $xx^* = b$. □

Remark 3.4. As noted above, every open projection in the bidual of a *separable* C^*-algebra is realized as a support projection, so that PZ-equivalence of two open projections means precisely that they are the left and right support projections of some element in A.

3.5 *(Compact and closed projections).* We define below an equivalence relation and an order relation on open projections that we shall show to match Cuntz comparison of positive elements (under the correspondence $a \mapsto p_a$). To this end we need to define the concept of compact containment, which is inspired by the notion of a compact (and closed) projection developed by Akemann.

The idea first appeared in [1], although it was not given a name there, and it was later termed in the slightly different context of the atomic enveloping von Neumann algebra in [2, Definition II.1]. Later again, it was studied by Akemann, Anderson, and Pedersen in the context of the universal enveloping von Neumann algebra (see [3, after Lemma 2.4]).

A closed projection $p \in A^{**}$ is called *compact* if there exists $a \in A^+$ of norm one such that $pa = p$. See [3, Lemma 2.4] for equivalent conditions. Note that a compact, closed projection $p \in A^{**}$ must be dominated by some positive element of A (since $pa = p$ implies $p = apa \leqslant a^2 \in A$). The converse also holds (this follows from the result [2, Theorem II.5] transferred to the context of the universal enveloping von Neumann algebra).

Definition 3.6 *(Compact containment).* Let A be a C^*-algebra, and let $p, q \in P_o(A^{**})$ be open projections. We say that p is *compactly contained* in q (denoted $p \Subset q$) if $\overline{p}$ is a compact projection in A_q, i.e., if there exists a positive element a in A_q with $\|a\| = 1$ and $\overline{p}a = \overline{p}$.

Further, let us say that an open projection p is *compact* if it is compactly contained in itself, i.e., if $p \Subset p$.

Proposition 3.7. *An open projection in A^{**} is compact if and only if it belongs to A.*

Proof. Every projection in A is clearly compact.

If p is open and compact, then by definition there exists $a \in (A_p)^+$ such that $\overline{p}a = \overline{p}$. This implies that $p \leqslant \overline{p} \leqslant a \leqslant p$, whence $p = a \in A$. □

Remark 3.8. Note that compactness was originally defined only for *closed* projections in A^{**} (see 3.5). In Definition 3.6 above we also defined a notion of compactness for *open* projections in A^{**} by assuming it to be compactly contained in itself. This should cause no confusion since, by Proposition 3.7, a *compact, open* projection is automatically closed as well as compact in the sense defined for closed projections in 3.5.

Now we can give a definition of (sub-)equivalence for open projections that we term Cuntz (sub-)equivalence, and which in the next section will be shown to agree with Cuntz (sub-)equivalence for positive elements and Hilbert modules in a C^*-algebra. We warn the reader that our definition of Cuntz equivalence (below) does not agree with the notion carrying the same name defined by Lin in [17]. The latter was the one already studied by Peligrad and Zsidó that we (in Definition 3.2) have chosen to call Peligrad–Zsidó equivalence (or PZ-equivalence). Our definition below of Cuntz equivalence for open projections turns out to match the notion of Cuntz equivalence for positive elements, also when the C^*-algebra does not have stable rank one.

Definition 3.9 *(Cuntz comparison of open projections).* Let A be a C^*-algebra, and let p and q be open projections in A^{**}. We say that p is *Cuntz subequivalent* to q, written $p \precsim_{\mathrm{Cu}} q$, if for every open projection $p' \Subset p$ there exists an open projection q' with $p' \sim_{\mathrm{PZ}} q' \Subset q$. If $p \precsim_{\mathrm{Cu}} q$ and $q \precsim_{\mathrm{Cu}} p$ hold, then we say that p and q are *Cuntz equivalent*, which we write as $p \sim_{\mathrm{Cu}} q$.

4. Comparison of positive elements and the corresponding relation on open projections

We show in this section that the Cuntz comparison relation on positive elements corresponds to the Cuntz relation on the corresponding open projections. We also show that the Blackadar relation on positive elements, the Peligrad–Zsidó relation on their corresponding open projections, and isometric isomorphism of the corresponding Hilbert modules are equivalent.

4.1 *(Hilbert modules).* See [4] for a good introduction to Hilbert A-modules. Throughout this note all Hilbert modules are assumed to be right modules and countably generated. Let A be a general C^*-algebra. We will denote by $\mathcal{H}(A)$ the set of isomorphism classes of Hilbert A-modules. Every closed, right ideal in A is in a natural way a Hilbert A-module. In particular, $E_a := \overline{aA}$ is a Hilbert A-module for every element a in A. The assignment $a \mapsto E_a$ defines a natural map from the set of positive elements of A to $\mathcal{H}(A)$.

If E and F are Hilbert A-modules, then E is said to be *compactly contained* in F, written $E \Subset F$, if there exists a positive element x in $\mathcal{K}(F)$, the compact operators of $\mathcal{L}(F)$, such that $xe = e$ for all $e \in E$.

For two Hilbert A-modules E, F we say that $E \precsim_{\mathrm{Cu}} F$ (E is *Cuntz subequivalent* to F) if for every Hilbert A-submodule $E' \Subset E$ there exists $F' \Subset F$ with $E' \cong F'$ (isometric isomorphism). Further declare $E \approx F$ (*Cuntz equivalence*) if $E \precsim_{\mathrm{Cu}} F$ and $F \precsim_{\mathrm{Cu}} E$.

Before relating the Blackadar relation with the Peligrad–Zsidó relation we prove the following lemma restating the Blackadar relation:

Lemma 4.2. *Let A be a C^*-algebra, and let a and b be positive elements in A. The following conditions are equivalent*:

(i) $a \sim_s b$,
(ii) *there exist* $a', b' \in A^+$ *with* $a \cong a' \sim b' \cong b$,
(iii) *there exists* $x \in A$ *such that* $A_a = A_{x^*x}$ *and* $A_b = A_{xx^*}$,
(iv) *there exists* $b' \in A^+$ *with* $a \sim b' \cong b$,
(v) *there exists* $a' \in A^+$ *with* $a \cong a' \sim b$.

Proof. (ii) is just a reformulation of (i), and (iii) is a reformulation of (ii) keeping in mind that $A_c = A_d$ if and only if $c \cong d$.

(iv) $\Rightarrow$ (ii) and (v) $\Rightarrow$ (ii) are trivial.

(iii) $\Rightarrow$ (v): Take $x \in A$ such that $A_a = A_{x^*x}$ and $A_b = A_{xx^*}$. Let $x = v|x|$ be the polar decomposition for x (with v a partial isometry in A^{**}). Then $c \mapsto v^*cv$ defines an isomorphism from $A_{xx^*} = A_b$ onto $A_{x^*x} = A_a$. This isomorphism maps the strictly positive element b of A_b onto a strictly positive element $a' = v^*bv$ of A_a. Hence $b \sim a' \cong a$ as desired.

The proof of (iii) $\Rightarrow$ (iv) is similar. □

The equivalence of (i) and (iv) in the proposition below was noted to hold in Lin's recent paper [17]. We include a short proof of this equivalence for completeness.

Proposition 4.3. *Let A be a C^*-algebra, and let a and b be positive elements in A. The following conditions are equivalent*:

(i) $a \sim_s b$,
(ii) E_a *and* E_b *are isomorphic as Hilbert* A*-modules*,
(iii) *there exists* $x \in A$ *such that* $E_a = E_{x^*x}$ *and* $E_b = E_{xx^*}$,
(iv) $p_a \sim_{\mathrm{PZ}} p_b$.

Proof. (i) $\Rightarrow$ (iv): As remarked earlier, it was shown in [19, Theorem 1.4] that $p_{x^*x} \sim_{\mathrm{PZ}} p_{xx^*}$ for all $x \in A$. In other words, $a \sim b$ implies $p_a \sim_{\mathrm{PZ}} p_b$. Recall also that $p_a = p_b$ when $a \cong b$. These facts prove the implication.

(iv) $\Rightarrow$ (i): If $p_a \sim_{\mathrm{PZ}} p_b$, then by Proposition 3.3, there exist positive elements a' and b' in A such that $p_a = p_{a'}$, $p_b = p_{b'}$, and $a' \sim b'$. Now, $p_a = p_{a'}$ and $p_b = p_{b'}$ imply that $a \cong a'$ and $b \cong b'$, whence (i) follows (see also Lemma 4.2).

(ii) $\Rightarrow$ (iii): Let $\Phi : E_a \to E_b$ be an isomorphism of Hilbert A-modules, i.e., a bijective A-linear map preserving the inner product. Set $x := \Phi(a) \in E_b$. Then

$$\overline{xA} = \overline{\Phi(a)A} = \overline{\Phi(aA)} = E_b,$$

whence $E_b = E_x = E_{xx^*}$. Since Φ preserves the inner product,

$$a^2 = \langle a, a\rangle_{E_a} = \langle \Phi(a), \Phi(a)\rangle_{E_b} = x^*x.$$

Hence $E_a = E_{a^2} = E_{x^*x}$ and $E_b = E_{xx^*}$.

(iii) $\Rightarrow$ (ii): Let $x = v|x|$ be the polar decomposition of x in A^{**}. Note that $E_{|x|} = E_{x^*x}$ and $E_{xx^*} = E_{|x^*|}$. Define an isomorphism $E_{|x|} \to E_{|x^*|}$ by $z \mapsto vz$.

(i) $\Leftrightarrow$ (iii): This follows from the one-to-one correspondence between hereditary sub-C^*-algebras and right ideals: A hereditary sub-C^*-algebra B corresponds to the right ideal $\overline{BA}$, and, conversely, a right ideal R corresponds the hereditary algebra R^*R. In particular, $E_a = \overline{A_aA}$ and $A_a = E_a^*E_a$.

If (i) holds, then, by Lemma 4.2, $A_a = A_{x^*x}$ and $A_{xx^*} = A_b$ for some $x \in A$. This shows that $E_a = \overline{A_aA} = \overline{A_{x^*x}A} = E_{x^*x}$ and, similarly, $E_b = E_{xx^*}$.

In the other direction, if $E_a = E_{x^*x}$ and $E_{xx^*} = E_b$ for some $x \in A$, then $A_a = E_a^*E_a = E_{x^*x}^*E_{x^*x} = A_{x^*x}$ and, similarly, $A_b = A_{xx^*}$, whence $a \sim_s b$. □

4.4. It follows from the proof of (ii) $\Rightarrow$ (iii) of the proposition above that if a is a positive element in a C^*-algebra A and if F is a Hilbert A-module such that $E_a \cong F$, then $F = E_b$ for some positive element b in A. In fact, if $\Phi : E_a \to F$ is an isometric isomorphism, then we can take b to be $\Phi(a)$ as in the before mentioned proof.

4.5. For any pair of positive elements a and b in a C^*-algebra A we have the following equivalences:

$$a \in A_b \quad\Leftrightarrow\quad A_a \subseteq A_b \quad\Leftrightarrow\quad E_a \subseteq E_b \quad\Leftrightarrow\quad p_a \leqslant p_b,$$

as well as the following equivalences:

$$a \in A_b \quad \text{and} \quad b \in A_a \quad\Leftrightarrow\quad a \cong b \quad\Leftrightarrow\quad A_a = A_b \quad\Leftrightarrow\quad E_a = E_b \quad\Leftrightarrow\quad p_a = p_b.$$

As a consequence of Proposition 4.3, Lemma 4.2, and the remark above we obtain the following proposition:

Proposition 4.6. *Let A be a C^*-algebra, and let a and b be positive elements in A. The following conditions are equivalent*:

(i) $a \precsim_s b$,
(ii) *there exists a Hilbert A-module E' such that $E_a \cong E' \subseteq E_b$,*
(iii) *there exists $x \in A$ with $E_a = E_{x^*x}$ and $E_{xx^*} \subseteq E_b$,*
(iv) $p_a \precsim_{\mathrm{PZ}} p_b$.

Lemma 4.7. *Let a and e be positive elements in a C^*-algebra A and assume that e is a contraction. Then the following equivalences hold*

$$ae = a \quad\Leftrightarrow\quad p_ae = p_a \quad\Leftrightarrow\quad \overline{p}_ae = \overline{p}_a.$$

Proof. The two "⇐"-implications are trivial. Suppose that $ae = a$. Let χ be indicator function for the singleton $\{1\}$, and put $q = \chi(e) \in A^{**}$. Then $qe = q$ and q is the largest projection in A^{**} with this property. Moreover, q is the projection onto the kernel of $1 - e$, hence $1 - q$ is the projection onto the range of $1 - e$, i.e., $1 - q = p_{1-e}$. This shows that q is a closed projection. As a and $1 - e$ are orthogonal so are their range projections p_a and p_{1-e}, whence $p_a \leqslant 1 - p_{1-e} = q$. Thus $\overline{p}_a \leqslant q$. This shows that $\overline{p}_a e = \overline{p}_a$. □

Lemma 4.8. *Let A be a C^*-algebra, and let e and a be positive elements in A. If $ae = a$, then $\overline{p}_a \leqslant p_e$.*

Proof. Upon replacing e with $f(e)$, where $f : \mathbb{R}^+ \to \mathbb{R}^+$ is given by $f(t) = \max\{t, 1\}$, we may assume that e is a contraction. If $ae = a$, then $\overline{p}_a e = \overline{p}_a$ by Lemma 4.7, and this implies that $\overline{p}_a \leqslant p_e$. □

We show below that the two previously defined notions of compact containment agree. To do so we introduce a third notion of compact containment:

Definition 4.9. Let a and b be positive elements in a C^*-algebra. Then a is said to be compactly contained in b, written $a \Subset b$, if and only if there exists a positive element e in A_b such that $ea = a$.

Following the proof of Lemma 4.8, the element e above can be assumed to be a contraction.

Proposition 4.10. *Let A be a C^*-algebra, let b be a positive element in A, and let a be a positive element in A_b. Then the following statements are equivalent*:

(i) $E_a \Subset E_b$,
(ii) $a \Subset b$,
(iii) $p_a \Subset p_b$,
(iv) $\overline{p}_a \leqslant p_b$ *and* $\overline{p}_a$ *is compact in* A.

Proof. (i) ⇔ (ii): By definition, (i) holds if and only if there exists a positive element e in $\mathcal{K}(E_b)$, such that e acts as the identity on E_a. We can identify $\mathcal{K}(E_b)$ with A_b, as elements of the latter act on E_b by left-multiplication. Thus (i) is equivalent to the existence of a positive element e in A_b such that $ex = x$ for all $x \in E_a = \overline{aA}$. The latter condition is fulfilled precisely if $ea = a$.

(ii) ⇔ (iii): (iii) holds if and only if there exists a positive element e in A_b such that $\overline{p}_a e = \overline{p}_a$; and (ii) holds if and only if there exists a positive element e in A_b such that $ae = a$. In both cases e can be taken to be a contraction, cf. the proof of Lemma 4.8. The bi-implication now follows from Lemma 4.7.

(ii) and (iii) ⇒ (iv): If $a \Subset b$, then there is a positive contraction e in A_b such that $ae = a$. By Lemma 4.8 this implies that $\overline{p}_a \leqslant p_e \leqslant p_b$. From (iii) we have that $\overline{p}_a$ is compact in A_b which entails that $\overline{p}_a$ also is compact in A.

(iv) ⇒ (iii): This is [3, Lemma 2.5]. □

Remark 4.11. In many cases it is automatic that $\overline{p}$ is compact, and then $p \Subset q$ is equivalent to the condition $\overline{p} \leqslant q$. For example, if A is unital, then all closed projections in A^{**} are compact. More generally, if $a \in A^+$ sits in some corner qAq for a projection $q \in A$, then $\overline{p}_a$ is compact.

Lemma 4.12. *Let a be a positive element in a C^*-algebra A.*

(i) *If E' is a Hilbert A-module that is compactly contained in E_a, then $E' \subseteq E_{(e-\varepsilon)_+}$ for some positive element $e \in A_a$ and some $\varepsilon > 0$.*
(ii) *If q, q' are open projections in A^{**} such that q' is compactly contained in q, then $q' \leqslant p_{(e-\varepsilon)_+}$ for some positive element $e \in A_q$ and some $\varepsilon > 0$.*

Proof. (i): By definition there is a positive element e in $\mathcal{K}(E_a) = A_a$ such that $ex = x$ for all $x \in E'$. This implies that $(e - 1/2)_+ x = \frac{1}{2}x$ for all $x \in E'$, whence $E' \subseteq E_{(e-1/2)_+}$.

(ii): If q' is compactly contained in q, then there is a positive element e in A_q such that $q'e = q'$ (in fact such that $\overline{q'}e = \overline{q'}$). It follows that $q'(e - 1/2)_+ = \frac{1}{2}q'$, and hence that $q' \leqslant p_{(e-1/2)_+}$. □

Proposition 4.13. *Let a and b be positive elements in a C^*-algebra A. Then the following statements are equivalent*:

(i) $a \precsim b$.
(ii) $E_a \precsim_{\mathrm{Cu}} E_b$.
(iii) $p_a \precsim_{\mathrm{Cu}} p_b$.

Proof. The equivalence of (i) and (ii) was first shown in [10, Appendix], see also [4, Theorem 4.33].

(ii) ⇒ (iii): Suppose that $E_a \precsim_{\mathrm{Cu}} E_b$, and let p' be an arbitrary open projection in A^{**} which is compactly contained in p_a. Then, by Lemma 4.12, $p' \leqslant p_{(e-\varepsilon)_+}$ for some positive element e in A_a and some $\varepsilon > 0$. Notice that $(e - \varepsilon)_+ \Subset a$. It follows from Proposition 4.10 that $E_{(e-\varepsilon)_+}$ is compactly contained in E_a. Accordingly, $E_{(e-\varepsilon)_+} \cong F'$ for some Hilbert A-module F' that is compactly contained in E_b. By 4.4, $F' = E_c$ for some positive element c in A. It now follows from Proposition 4.10 and from Proposition 4.3 that

$$p' \leqslant p_{(e-\varepsilon)_+} \sim_{\mathrm{PZ}} p_c \Subset p_b.$$

This shows that $p_a \precsim_{\mathrm{Cu}} p_b$.

(iii) ⇒ (ii): Suppose that $p_a \precsim_{\mathrm{Cu}} p_b$, and let E' be an arbitrary Hilbert A-module which is compactly contained in E_a. Then, by Lemma 4.12, $E' \subseteq E_{(e-\varepsilon)_+}$ for some positive element e in A_a and some $\varepsilon > 0$. It follows from Proposition 4.10 that $p_{(e-\varepsilon)_+}$ is compactly contained in p_a. Accordingly, $p_{(e-\varepsilon)_+} \sim_{\mathrm{PZ}} q'$ for some open projection q' in A^{**} that is compactly contained in p_b. By Proposition 3.3, $q' = p_c$ for some positive element c in A. It now follows from Proposition 4.10 and from Proposition 4.3 that

$$E' \subseteq E_{(e-\varepsilon)_+} \cong E_c \Subset E_b.$$

This shows that $E_a \precsim_{\mathrm{Cu}} E_b$. □

By the definition of Cuntz equivalence of positive elements, Hilbert A-modules, and of open projections, the proposition above immediately implies the following:

3484 E. Ortega et al. / Journal of Functional Analysis 260 (2011) 3474–3493

Corollary 4.14. *For every pair of positive elements a and b in a C^*-algebra A we have the following equivalences*:

$$a \approx b \quad \Leftrightarrow \quad E_a \approx E_b \quad \Leftrightarrow \quad p_a \sim_{\mathrm{Cu}} p_b.$$

We conclude this section by remarking that the pre-order $\precsim_{\mathrm{PZ}}$ on the open projections is not algebraic (unlike the situation for Murray–von Neumann subequivalence). Indeed, if p and q are open projections A^{**} with $p \leqslant q$, then $q - p$ need not be an open projection. For the same reason, $\precsim_{\mathrm{Cu}}$ is not an algebraic order. However, Cuntz comparison is approximately algebraic in the following sense.

Proposition 4.15. *Let A be a C^*-algebra, and let $p, p', q \in A^{**}$ be open projections with $p' \Subset p \precsim_{\mathrm{Cu}} q$. Then there exists an open projection $r \in A^{**}$ such that $p' \oplus r \precsim_{\mathrm{Cu}} q \precsim_{\mathrm{Cu}} p \oplus r$.*

Proof. By Lemma 4.12 (ii) there exists an open projection p'' with $p' \Subset p'' \Subset p$ (take p'' to be $p_{(a-\varepsilon/2)_+}$ in that lemma). By the definition of Cuntz sub-equivalence there exists an open projection q'' such that $p'' \sim_{\mathrm{PZ}} q'' \Subset q$. Since $p'' \sim_{\mathrm{PZ}} q''$ implies $p'' \sim_{\mathrm{Cu}} q''$, there exists an open projection q' with $p' \sim_{\mathrm{PZ}} q' \Subset q''$. Then $r := q - \overline{q'}$ is an open projection.

Since $q' \Subset q''$ implies $\overline{q'} \leqslant q''$, and $q' \leqslant \overline{q'}$, we get

$$p' \oplus r \sim_{\mathrm{PZ}} q' \oplus r \precsim_{\mathrm{PZ}} q = \overline{q'} + r \precsim_{\mathrm{Cu}} q'' \oplus r \sim_{\mathrm{PZ}} p'' \oplus r \precsim p \oplus r$$

as desired. □

Translated, this result says that for positive elements a', a, b in A with $a' \Subset a \precsim b$ there exists a positive element c such that $a' \oplus c \precsim b \precsim a \oplus c$.

To formulate the result in the ordered Cuntz semigroup, we recall that an element $\alpha \in \mathrm{Cu}(A)$ is called *way-below* $\beta \in \mathrm{Cu}(A)$, denoted $\alpha \ll \beta$, if for every increasing sequence $\{\beta_k\}$ in $\mathrm{Cu}(A)$ with $\beta \leqslant \sup_k \beta_k$ there exists $l \in \mathbb{N}$ such that already $\alpha \leqslant \beta_l$. Consequently, in the Cuntz semigroup we get the following almost algebraic order:

Corollary 4.16 *(Almost algebraic order in the Cuntz semigroup). Let A be a C^*-algebra, and let α', α, β in $\mathrm{Cu}(A)$ be such that $\alpha' \ll \alpha \leqslant \beta$. Then there exists $\gamma \in \mathrm{Cu}(A)$ such that $\alpha' + \gamma \leqslant \beta \leqslant \alpha + \gamma$.*

5. Comparison of projections by traces

In this section we show that Murray–von Neumann (sub-)equivalence of open projections in the bidual of a separable C^*-algebra is equivalent to tracial comparison of the corresponding positive elements of the C^*-algebra. For the proof we need to show that every lower semicontinuous tracial weight on a C^*-algebra extends (not necessarily uniquely) to a normal tracial weight on its bidual and that Murray–von Neumann comparison of projections in any von Neumann algebra "that is not too big" is determined by tracial weights. We expect those two results to be known to experts, but in lack of a reference and for completeness we have included their proofs.

Recall that a weight φ on a C^*-algebra A is an additive map $\varphi : A^+ \to [0, \infty]$ satisfying $\varphi(\lambda a) = \lambda\varphi(a)$ for all $a \in A^+$ and all $\lambda \in \mathbb{R}^+$. We say that φ is *densely defined* if the set

$\{a \in A^+\colon\ \varphi(a) < \infty\}$ is dense in A^+. Recall from Section 2 that the set of (norm) lower semicontinuous tracial weights on A in this paper is denoted by $T(A)$.

If M is a von Neumann algebra, then let $W(M)$ denote the set of *normal* weights on M, and let $W_{\mathrm{tr}}(M)$ denote the set of normal tracial weights on M, i.e., weights φ for which $\varphi(x^*x) = \varphi(xx^*)$ for all $x \in M$. The standard trace on $\mathbb{B}(H)$ is an example of a normal tracial weight.

For the extension of weights on a C^*-algebra to its universal enveloping von Neumann algebra, we use the result below from [9, Proposition 4.1 and Proposition 4.4]. For every f in the dual A^* of a C^*-algebra A, let $\widetilde{f}$ denote the unique normal extension of f to A^{**}. (One can equivalently obtain $\widetilde{f}$ via the natural pairing: $\widetilde{f}(z) = \langle f, z\rangle$ for $z \in A^{**}$.)

Proposition 5.1. *(See Combes [9].) Let A be a C^*-algebra, let $\varphi : A^+ \to [0,\infty]$ be a densely defined lower semicontinuous weight. Define a map $\widetilde{\varphi} : (A^{**})^+ \to [0,\infty]$ by*

$$\widetilde{\varphi}(z) := \sup\bigl\{\widetilde{f}(z)\colon\ f \in A^*,\ 0 \leqslant f \leqslant \varphi\bigr\}, \quad z \in \bigl(A^{**}\bigr)^+.$$

Then $\widetilde{\varphi}$ is a normal weight on A^{**} extending φ. Moreover, if φ is tracial, then $\widetilde{\varphi}$ is the unique extension of φ to a normal weight on A^{**}.

Combes did not address the question whether the (unique) normal weight on A^{**} that extends a densely defined lower semicontinuous tracial weight on A is itself a trace. An affirmative answer to this question is included in the proposition below.

Proposition 5.2. *Let A be a C^*-algebra, and let φ be a lower semicontinuous tracial weight on A. Then there exists a normal, tracial weight on A^{**} that extends φ.*

Proof. The closure of the linear span of the set $\{a \in A^+\colon\ \varphi(a) < \infty\}$ is a closed two-sided ideal in A. Denote it by I_φ. The restriction of φ to I_φ is a densely defined tracial weight, which therefore, by Combes' extension result (Proposition 5.1), extends (uniquely) to a normal weight $\widehat{\varphi}$ on I_φ^{**}. The ideal I_φ corresponds to an open central projection p in A^{**} via the identification $I_\varphi = A^{**}p \cap A$, and $I_\varphi^{**} = A^{**}p$. In other words, I_φ^{**} is a central summand in A^{**}. Extend φ to a normal weight $\widetilde{\varphi}$ on the positive elements in A^{**} by the formula

$$\widetilde{\varphi}(z) = \begin{cases} \widehat{\varphi}(z), & \text{if } z \in I_\varphi^{**}, \\ \infty, & \text{otherwise.} \end{cases}$$

It is easily checked that $\widetilde{\varphi}$ is a normal weight that extends φ, and that $\widetilde{\varphi}$ is tracial if we knew that $\widehat{\varphi}$ is tracial. To show the latter, upon replacing A with I_φ, we can assume that φ is densely defined.

We proceed to show that $\widetilde{\varphi}$ is tracial under the assumption that φ is densely defined. To this end it suffices to show that $\widetilde{\varphi}$ is unitarily invariant, i.e., that $\widetilde{\varphi}(uzu^*) = \widetilde{\varphi}(z)$ for all unitaries u in A^{**} and all positive elements z in A^{**}. We first check this when the unitary u lies in $\widetilde{A}$, the unitization of A, which we view as a unital sub-C^*-algebra of A^{**}, and for an arbitrary positive element z in A^{**}. For each f in A^* let $u.f$ denote the functional in A^* given by $(u.f)(a) = f(uau^*)$ for $a \in A$. By the trace property of φ we see that if $f \in A^*$ is such that $0 \leqslant f \leqslant \varphi$, then also $0 \leqslant u.f \leqslant \varphi$, and vice versa since $f = u^*.(u.f)$. It follows that

$$\begin{aligned}\widetilde{\varphi}(uzu^*) &= \sup\{\widetilde{f}(uzu^*)\colon\ f \in A^*,\ 0 \leqslant f \leqslant \varphi\} = \sup\{\widetilde{u.f}(z)\colon\ f \in A^*,\ 0 \leqslant f \leqslant \varphi\} \\ &= \sup\{\widetilde{f}(z)\colon\ f \in A^*,\ 0 \leqslant f \leqslant \varphi\} = \widetilde{\varphi}(z).\end{aligned}$$

For the general case we use Kaplansky's density theorem (see [18, Theorem 2.3.3, p. 25]), which says that the unitary group $U(\widetilde{A})$ is σ-strongly dense in $U(A^{**})$. Thus, given u in $U(A^{**})$ we can find a net (u_λ) in $U(\widetilde{A})$ converging σ-strongly to u. It follows that $(u_\lambda z u_\lambda^*)$ converges σ-strongly (and hence σ-weakly) to uzu^*. As $\widetilde{\varphi}$ is σ-weakly lower semicontinuous (see [6, III.2.2.18, p. 253]), we get

$$\widetilde{\varphi}(uzu^*) = \widetilde{\varphi}\Big(\lim_\lambda u_\lambda z u_\lambda^*\Big) \leqslant \lim_\lambda \widetilde{\varphi}(u_\lambda z u_\lambda^*) = \widetilde{\varphi}(z).$$

The same argument shows that $\widetilde{\varphi}(z) = \widetilde{\varphi}(u^*(uzu^*)u) \leqslant \widetilde{\varphi}(uzu^*)$. This proves that $\widetilde{\varphi}(uzu^*) = \widetilde{\varphi}(z)$ as desired. □

The extension $\widetilde{\varphi}$ in Proposition 5.2 need not be unique if φ is not densely defined. Take for example the trivial trace φ on the Cuntz algebra $\mathcal{O}_2$ (that is zero on zero and infinite elsewhere). Then every normal tracial weight on $\mathcal{O}_2^{**}$ that is infinite on every (non-zero) properly infinite element is an extension of φ, and there are many such normal tracial weights arising from the type I_∞ and type II_∞ representations of $\mathcal{O}_2$. On the other hand, every densely defined lower semicontinuous tracial weight on a C^*-algebra extends uniquely to a normal tracial weight on its bidual by Combes' result (Proposition 5.1) and by Proposition 5.2.

Remark 5.3. Given a C^*-algebra A equipped with a lower semicontinuous tracial weight τ and a positive element a in A. Then we can associate to τ a dimension function d_τ on A (as above Definition 2.4). Let $\widetilde{\tau}$ be (any) extension of τ to a normal tracial weight on A^{**} (cf. Proposition 5.2). Then $d_\tau(a) = \widetilde{\tau}(p_a)$. To see this, assume without loss of generality that a is a contraction. Then p_a is the strong operator limit of the increasing sequence $\{a^{1/n}\}$, whence

$$d_\tau(a) = \lim_{n\to\infty} \tau(a^{1/n}) = \lim_{n\to\infty} \widetilde{\tau}(a^{1/n}) = \widetilde{\tau}(p_a)$$

by normality of $\widetilde{\tau}$.

Corollary 5.4. *Let a and b be positive elements in a C^*-algebra A. If $p_a \precsim p_b$ in A^{**}, then $a \precsim_{\mathrm{tr}} b$ in A; and if $p_a \sim p_b$ in A^{**}, then $a \sim_{\mathrm{tr}} b$ in A.*

Proof. Suppose that $p_a \precsim p_b$ in A^{**}. Then $\omega(p_a) \leqslant \omega(p_b)$ for every tracial weight ω on A^{**}.

Now let $\tau \in T(A)$ be any lower semicontinuous tracial weight, and let d_τ be the corresponding dimension function. By Proposition 5.2, τ extends to a tracial, normal weight $\widetilde{\tau}$ on A^{**}. Using the remark above, it follows that $d_\tau(a) = \widetilde{\tau}(p_a) \leqslant \widetilde{\tau}(p_b) = d_\tau(b)$. This proves that $a \precsim_{\mathrm{tr}} b$. The second statement in the corollary follows from the first statement. □

We will now show that the converse of Corollary 5.4 is true for separable C^*-algebras. First we need to recall some facts about the dimension theory of (projections in) von Neumann algebras. A good reference is the recent paper [23] of David Sherman.

Definition 5.5. (Tomiyama [24, Definition 1], see also [23, Definition 2.3].) Let M be a von Neumann algebra, $p \in P(M)$ a non-zero projection, and κ a cardinal. Say that p is *κ-homogeneous* if p is the sum of κ mutually equivalent projections, each of which is the sum of centrally orthogonal σ-finite projections. Set

$$\kappa_M := \sup\{\kappa\colon\ M \text{ contains a } \kappa\text{-homogeneous element}\}.$$

A projection can be κ-homogeneous for at most one $\kappa \geqslant \aleph_0$; and if $\kappa \geqslant \aleph_0$, then two κ-homogeneous projections are equivalent if they have identical central support (see [24,23]). We shall use these facts in the proof of Proposition 5.7.

But first we show that the enveloping von Neumann algebra A^{**} of a separable C^*-algebra A has $\kappa_{A^{**}} \leqslant \aleph_0$, a property that has various equivalent formulations and consequences (see [23, Propositions 3.8 and 5.1]). This property is useful, since it means that there are no issues about different "infinities". For instance, the set of projections up to Murray–von Neumann equivalence in an arbitrary II_∞ factor M (not necessarily with separable predual) can be identified with $[0,\infty) \cup \{\kappa\colon\ \aleph_0 \leqslant \kappa \leqslant \kappa_M\}$, see [23, Corollary 2.8]. Thus, tracial weights on M need not separate projections up to equivalence. However, if $\kappa_M \leqslant \aleph_0$, then normal, tracial weights on M do in fact separate projections up to Murray–von Neumann equivalence.

Lemma 5.6. *Let A be a separable C^*-algebra. Then $\kappa_{A^{**}} \leqslant \aleph_0$.*

Proof. We show the stronger statement that whenever $\{p_i\}_{i\in I}$ is a family of non-zero pairwise equivalent and orthogonal projections in A^{**}, then $\mathrm{card}(I) \leqslant \aleph_0$. The universal representation π_u of A is given as $\pi_u = \bigoplus_{\varphi\in S(A)} \pi_\varphi$, where $S(A)$ denotes the set of states on A, and where $\pi_\varphi : A \to \mathbb{B}(H_\varphi)$ denotes the GNS-representation corresponding to the state φ. It follows that

$$A^{**} = \pi_u(A)'' \subseteq \bigoplus_{\varphi\in S(A)} \mathbb{B}(H_\varphi).$$

The projections $\{p_i\}_{i\in I}$ are non-zero in at least one summand $\mathbb{B}(H_\varphi)$; but then I must be countable because each H_φ is separable. □

Proposition 5.7. *Let M be a von Neumann algebra with $\kappa_M \leqslant \aleph_0$, and let $p, q \in P(M)$ be two projections. Then $p \precsim q$ if and only if $\omega(p) \leqslant \omega(q)$ for all normal tracial weights ω on M.*

Proof. The "only if" part is obvious. We prove the "if" part and assume accordingly that $\omega(p) \leqslant \omega(q)$ for all normal tracial weights ω on M, and we must show that $p \precsim q$. We show first that it suffices to consider the case where $q \leqslant p$.

There is a central projection z in M such that $zp \precsim zq$ and $(1-z)p \succsim (1-z)q$. We are done if we can show that $(1-z)p \precsim (1-z)q$. Every normal tracial weight on $(1-z)M$ extends to a normal tracial weight on M (for example by setting it equal to zero on zM), whence our assumptions imply that $\omega((1-z)p) \leqslant \omega((1-z)q)$ for all tracial weights ω on $(1-z)M$. Upon replacing M by $(1-z)M$, and p and q by $(1-z)p$ and $(1-z)q$, respectively, we can assume that $p \succsim q$, i.e., that $q \sim q' \leqslant p$ for some projection q' in M. Upon replacing q by q' we can further assume that $q \leqslant p$ as desired.

3488 E. Ortega et al. / Journal of Functional Analysis 260 (2011) 3474–3493

There is a central projection z in M such that zq is finite and $(1-z)q$ is properly infinite (see [14, 6.3.7, p. 414]). Arguing as above it therefore suffices to consider the two cases where q is finite and where q is properly infinite.

Assume first that q is finite. We show that $p = q$. Suppose, to reach a contradiction, that $p - q \neq 0$. Then there would be a normal tracial weight ω on M such that $\omega(q) = 1$ and $\omega(p-q) > 0$. But that would entail that $\omega(p) > \omega(q)$ in contradiction with our assumptions. To see that ω exists, consider first the case where q and $p-q$ are not centrally orthogonal, i.e., that $c_q c_{p-q} \neq 0$. Then there are non-zero projections $e \leqslant q$ and $f \leqslant p-q$ such that $e \sim f$. Choose a normal tracial state τ on the finite von Neumann algebra qMq such that $\tau(e) > 0$. Then τ extends uniquely to a normal tracial weight ω_0 on Mc_q and further to a normal tracial weight ω on M by the recipe $\omega(x) = \omega_0(xc_q)$. Then $\omega(q) = \tau(q) = 1$ and $\omega(p-q) \geqslant \omega_0(f) = \omega_0(e) = \tau(e) > 0$. In the case where q and $p-q$ are centrally orthogonal, take a normal tracial weight ω_0 (for example as above) such that $\omega_0(q) = 1$ and extend ω_0 to a normal tracial weight ω on M by the recipe $\omega(x) = \omega_0(x)$ for all positive elements $x \in Mc_q$ and $\omega(x) = \infty$ whenever x is a positive element in M that does not belong to Mc_q. Then $\omega(q) = 1$ and $\omega(p-q) = \infty$.

Assume next that q is properly infinite. Every properly infinite projection can uniquely be written as a central sum of homogeneous projections (see [24, Theorem 1], see also [23, Theorem 2.5] and the references cited there). By the assumption that $\kappa_M \leqslant \aleph_0$ we get that every properly infinite projection is $\aleph_0$-homogeneous. Therefore q is $\aleph_0$-homogeneous and hence equivalent to its central support projection c_q. Let ω be the normal tracial weight on M which is zero on Mc_q and equal to ∞ on every positive element that does not lie in Mc_q. Then $\omega(p) \leqslant \omega(q) = 0$, which shows that $p \in Mc_q$, and hence $c_p \leqslant c_q$. It now follows that $p \leqslant c_p \leqslant c_q \sim q$, and so $p \precsim q$ as desired. □

We can now show that Murray–von Neumann (sub-)equivalence of open projections in the bidual of a C^*-algebra is equivalent to tracial (sub-)equivalence of the corresponding positive elements in the C^*-algebra.

Theorem 5.8. *Let a and b be positive elements in a separable C^*-algebra A. Then $p_a \precsim p_b$ in A^{**} if and only if $a \precsim_{\mathrm{tr}} b$ in A; and $p_a \sim p_b$ in A^{**} if and only if $a \sim_{\mathrm{tr}} b$ in A.*

Proof. The "only if parts" have already been proved in Corollary 5.4. Suppose that $a \precsim_{\mathrm{tr}} b$. Let ω be a normal tracial weight on A^{**}, and denote by ω_0 its restriction to A. Then ω_0 is a norm lower semicontinuous tracial weight on A, whence

$$\omega(p_a) = d_{\omega_0}(a) \leqslant d_{\omega_0}(b) = \omega(p_b),$$

cf. Remark 5.3. As ω was arbitrary we can now conclude from Lemma 5.6 and Proposition 5.7 that $p_a \precsim p_b$.

The second part of the theorem follows easily from the first part. □

Corollary 5.9. *Let A be a separable C^*-algebra, and p and q be two open projections in A^{**}. Then*

$$p \precsim_{\mathrm{PZ}} q \quad \Rightarrow \quad p \precsim_{\mathrm{Cu}} q \quad \Rightarrow \quad p \precsim q, \qquad p \sim_{\mathrm{PZ}} q \quad \Rightarrow \quad p \sim_{\mathrm{Cu}} q \quad \Rightarrow \quad p \sim q.$$

The first implication in each of the two strings holds without assuming A to be separable.

Proof. Since A is separable there are positive elements a and b such that $p = p_a$ and $q = p_b$. The corollary now follows from Remark 2.5, Proposition 4.3, Proposition 4.13, and Theorem 5.8. □

It should be remarked, that one can prove the corollary above more directly without invoking Remark 2.5.

Remark 5.10. There is a certain similarity of our main results with the following result recently obtained by Robert in [21, Theorem 1]: If a, b are positive elements of a C^*-algebra A, then the following are equivalent:

(i) $\tau(a) = \tau(b)$ for all norm lower semicontinuous tracial weights on A,
(ii) a and b are Cuntz–Pedersen equivalent, i.e., there exists a sequence $\{x_k\}$ in A such that $a = \sum_{k=1}^{\infty} x_k x_k^*$ and $b = \sum_{k=1}^{\infty} x_k^* x_k$ (the sums are norm-convergent).

It is known that Cuntz–Pedersen equivalence and Murray–von Neumann equivalence agree for projections in a von Neumann algebra (see [13, Theorem 4.1]), but they are different for projections in a C^*-algebra.

6. Summary and applications

In the previous sections we have established equivalences and implications between different types of comparison of positive elements and their corresponding open projections and Hilbert modules. The results we have obtained can be summarized as follows. Given two positive elements a and b in a (separable) C^*-algebra A with corresponding open projections p_a and p_b in A^{**} and Hilbert A-modules E_a and E_b, then:

$$(*)\qquad \begin{array}{ccc} a \precsim_s b & \Longleftrightarrow & p_a \precsim_{\mathrm{PZ}} p_b \\ \Downarrow & & \Downarrow \\ a \precsim b & \Longleftrightarrow & p_a \precsim_{\mathrm{Cu}} p_b \\ \Downarrow & & \Downarrow \\ a \precsim_{\mathrm{tr}} b & \Longleftrightarrow & p_a \precsim p_b \end{array} \qquad \begin{array}{ccccc} a \sim_s b & \Longleftrightarrow & p_a \sim_{\mathrm{PZ}} p_b & \Longleftrightarrow & E_a \cong E_b \\ \Downarrow & & \Downarrow & & \Downarrow \\ a \approx b & \Longleftrightarrow & p_a \sim_{\mathrm{Cu}} p_b & \Longleftrightarrow & E_a \sim_{\mathrm{Cu}} E_b \\ \Downarrow & & \Downarrow & & \\ a \sim_{\mathrm{tr}} b & \Longleftrightarrow & p_a \sim p_b & & \end{array}$$

We shall discuss below to what extend the reverse (upwards) implications hold. First we remark how the middle bi-implications yield an isomorphism between the Cuntz semigroup and a semigroup of open projections modulo Cuntz equivalence.

6.1 *(The semigroup of open projections).* Given a C^*-algebra A. We wish to show that its Cuntz semigroup $\mathrm{Cu}(A)$ can be identified with an ordered semigroup of open projections in $(A \otimes \mathcal{K})^{**}$. More specifically, we show $P_{\mathrm{o}}((A \otimes \mathcal{K})^{**})/{\sim_{\mathrm{Cu}}}$ is an ordered abelian semigroup which is isomorphic to $\mathrm{Cu}(A)$.

First we note how addition is defined on the set $P_o((A \otimes \mathcal{K})^{**})/\sim_{\mathrm{Cu}}$. Note that

$$A \otimes \mathbb{B}(\ell^2) \subseteq \mathcal{M}(A \otimes \mathcal{K}) \subseteq (A \otimes \mathcal{K})^{**}.$$

Choose two isometries s_1 and s_2 in $\mathbb{B}(\ell^2)$ satisfying the Cuntz relation $1 = s_1 s_1^* + s_2 s_2^*$, and consider the isometries $t_1 = 1 \otimes s_1$ and $t_2 = 1 \otimes s_2$ in $\mathcal{M}(A \otimes \mathcal{K}) \subseteq (A \otimes \mathcal{K})^{**}$. For every positive element a in $A \otimes \mathcal{K}$ and for every isometry t in $\mathcal{M}(A \otimes \mathcal{K})$ we have $a \sim_s tat^*$ in $A \otimes \mathcal{K}$ and $p_a \sim_{\mathrm{PZ}} tp_a t^* = p_{tat^*}$ in $(A \otimes \mathcal{K})^{**}$. We can therefore define addition in $P_o((A \otimes \mathcal{K})^{**})/\sim_{\mathrm{Cu}}$ by

$$(**) \quad [p]_{\mathrm{Cu}} + [q]_{\mathrm{Cu}} := \left[t_1 p t_1^* + t_2 q t_2^*\right]_{\mathrm{Cu}}, \quad p, q \in P_o\big((A \otimes \mathcal{K})^{**}\big).$$

The relation $\precsim_{\mathrm{Cu}}$ yields an order relation on $P_o((A \otimes \mathcal{K})^{**})/\sim_{\mathrm{Cu}}$, which thus becomes an ordered abelian semigroup.

Proposition 4.13 and Corollary 4.14 applied to the C^*-algebra $A \otimes \mathcal{K}$ yield that the mapping $\langle a \rangle \mapsto [p_a]_{\mathrm{Cu}}$, for $a \in (A \otimes \mathcal{K})^+$, defines an isomorphism

$$\mathrm{Cu}(A) \cong P_o\big((A \otimes \mathcal{K})^{**}\big)/\sim_{\mathrm{Cu}}$$

of ordered abelian semigroups whenever A is a separable C^*-algebra. In more detail, Proposition 4.13 and Corollary 4.14 imply that the map $\langle a \rangle \mapsto [p_a]_{\mathrm{Cu}}$ is well defined, injective, and order preserving. Surjectivity follows from the assumption that A (and hence $A \otimes \mathcal{K}$) are separable, whence all open projections in $(A \otimes \mathcal{K})^{**}$ are of the form p_a for some positive element $a \in A \otimes \mathcal{K}$. Additivity of the map follows from the definition of addition defined in $(**)$ above and the fact that $\langle a \rangle + \langle b \rangle = \langle t_1 a t_1^* + t_2 b t_2^* \rangle$ in $\mathrm{Cu}(A)$.

6.2 *(The stable rank one case).* It was shown by Coward, Elliott, and Ivanescu in [10, Theorem 3] that in the case when A is a separable C^*-algebra with stable rank one, then two Hilbert A-modules are isometrically isomorphic if and only if they are Cuntz equivalent, and that the order structure given by Cuntz subequivalence is equivalent to the one generated by inclusion of Hilbert modules together with isometric isomorphism (see also [4, Theorem 4.29]). Combining those results with Proposition 4.3, Proposition 4.6, Proposition 4.13 and Corollary 4.14 shows that the following holds for all $a, b \in A^+$ and for all $p, q \in P_o(A^{**})$:

(1) $a \precsim b \Leftrightarrow a \precsim_s b$, and $a \approx b \Leftrightarrow a \sim_s b$.
(1)′ $p \precsim_{\mathrm{Cu}} q \Leftrightarrow p \precsim_{\mathrm{PZ}} q$, and $p \sim_{\mathrm{Cu}} q \Leftrightarrow p \sim_{\mathrm{PZ}} q$.
(2) If $a \precsim_s b$ and $b \precsim_s a$, then $a \sim_s b$.
(2)′ If $p \precsim_{\mathrm{PZ}} q$ and $q \precsim_{\mathrm{PZ}} p$, then $p \sim_{\mathrm{PZ}} q$.

Hence the vertical implications between the first and the second row of $(*)$ can be reversed when A is separable and of stable rank one.

The right-implications in (1) and (2) (and hence in (1)′ and (2)′) above do not hold in general. Counterexamples were given by Lin in [16, Theorem 9], by Perera in [20, before Corollary 2.4], and by Brown and Ciuperca in [8, Section 4]. For one such example take non-zero projections p and q in a simple, purely infinite C^*-algebra. Then, automatically, $p \precsim q$, $p \precsim_s q$, $q \precsim_s p$, and $p \approx q$; but $p \sim q$ and $p \sim_s q$ hold (if and) only if p and q define the same K_0-class (which they do not always do).

It is unknown whether (1)–(2)′ hold for residually stably finite C^*-algebras, and in particular whether they hold for stably finite simple C^*-algebras.

6.3 *(Almost unperforated Cuntz semigroup).* We discuss here when the vertical implications between the second and the third row of (∗) can be reversed. This requires both a rather restrictive assumption on the C^*-algebra A, and also an assumption on the positive elements a and b. To define the latter, we remind the reader of the notion of purely non-compact elements from [11, before Proposition 6.4]: The quotient map $\pi_I : A \to A/I$ induces a morphism $\mathrm{Cu}(A) \to \mathrm{Cu}(A/I)$ whenever I is an ideal in A. An element $\langle a\rangle$ in $\mathrm{Cu}(A)$ is *purely non-compact* if whenever $\langle \pi_I(a)\rangle$ is compact for some ideal I, it is properly infinite, i.e., $2\langle \pi_I(a)\rangle = \langle \pi_I(a)\rangle$ in $\mathrm{Cu}(A/I)$. Recall that an element α in the Cuntz semigroup $\mathrm{Cu}(B)$ of a C^*-algebra B is called *compact* if it is *way-below* itself, i.e., $\alpha \ll \alpha$ (see the end of Section 4 for the definition).

It is shown in [11, Theorem 6.6] that if $\mathrm{Cu}(A)$ is almost unperforated and if a and b are positive elements in $A \otimes \mathcal{K}$ such that $\langle a\rangle$ is purely non-compact in $\mathrm{Cu}(A)$, then $\widehat{\langle a\rangle} \leqslant \widehat{\langle b\rangle}$ implies that $\langle a\rangle \leqslant \langle b\rangle$ in $\mathrm{Cu}(A)$. In the notation of [11], and using [11, Proposition 4.2], $\widehat{\langle a\rangle} \leqslant \widehat{\langle b\rangle}$ means that $d_\tau(a) \leqslant d_\tau(b)$ for every (lower semicontinuous, possibly unbounded) 2-quasitrace on A. In the case where A is exact it is known that all such 2-quasitraces are traces by Haagerup's theorem [12] (extended to the non-unital case by Kirchberg [15], and Blanchard and Kirchberg [7, Remark 2.29(i)]) so it follows that $\widehat{\langle a\rangle} \leqslant \widehat{\langle b\rangle}$ if and only if $a \precsim_{\mathrm{tr}} b$. We can thus rephrase [11, Theorem 6.6] (see also [22, Corollary 4.6 and Corollary 4.7]) as follows: Suppose that A is an exact, separable C^*-algebra with $\mathrm{Cu}(A)$ almost unperforated. Then the following holds for all positive elements a, b in $A \otimes \mathcal{K}$:

(3) If $\langle a\rangle \in \mathrm{Cu}(A)$ is purely non-compact, then $a \precsim_{\mathrm{tr}} b \Leftrightarrow a \precsim b$.
(4) If $\langle a\rangle, \langle b\rangle \in \mathrm{Cu}(A)$ are purely non-compact, then $a \sim_{\mathrm{tr}} b \Leftrightarrow a \approx b$.

We wish to rephrase (3) and (4) above for open projections. We must first deal with the problem of choosing which kind of compactness of open projection to be invoked. Compactness of an open projection $p \in A^{**}$ as in Definition 3.6 means that $p \in A$ (see Proposition 3.7). On the other hand, compactness for an element of the Cuntz semigroup $\mathrm{Cu}(A)$ is defined in terms of its ordering. Compactness of p_a implies compactness of $\langle a\rangle \in \mathrm{Cu}(A)$ for every positive element a in $A \otimes \mathcal{K}$. Brown and Ciuperca have shown that the converse holds in stably finite C^*-algebras [8, Corollary 3.3]. Recall that a C^*-algebra is called stably finite if its stabilization contains no infinite projections.

From now on, we restrict our attention to the residually stably finite case, which means that all quotients of the C^*-algebra are stably finite. We define an open projection p in A^{**} to be *residually non-compact* if there is no closed, central projection $z \in A^{**}$ such that pz is a non-zero, compact (open) projection in $A^{**}z$. Here, we identify $A^{**}z$ with the bidual of the quotient A/I, where I is the ideal corresponding to the open, central projection $1 - z$, i.e., $I = A_{1-z} = (1 - z)A^{**}(1 - z) \cap A$.

It follows from Proposition 3.7 that an open projection $p \in A^{**}$ is residually non-compact if and only if there is no closed, central projection $z \in A^{**}$ such that pz is non-zero and belongs to Az. Applying [8, Corollary 3.3] to each quotient of A, we get that $\langle a\rangle \in \mathrm{Cu}(A)$ is purely non-compact if and only if p_a is residually non-compact whenever a is a positive element in $A \otimes \mathcal{K}$.

Thus, for open projections p, q in the bidual of a separable, exact, residually stably finite C^*-algebra A with $\mathrm{Cu}(A)$ almost unperforated, the following hold:

(3)′ If p is residually non-compact, then $p \precsim q \Leftrightarrow p \precsim_{\mathrm{Cu}} q$.
(4)′ If p and q are residually non-compact, then $p \sim q \Leftrightarrow p \sim_{\mathrm{Cu}} q$.

If, in addition, A is assumed to be simple, then an open projection p in A^{**} is residually non-compact if and only if it is not compact, i.e., if and only if $p \notin A$, thus:

(3)″ If $p \notin A$, then $p \precsim q \Leftrightarrow p \precsim_{\mathrm{Cu}} q$.
(4)″ If $p, q \notin A$, then $p \sim q \Leftrightarrow p \sim_{\mathrm{Cu}} q$.

If A is stably finite, and p, q are two Cuntz equivalent open projections in A^{**}, then p is compact if and only if q is compact (see [8, Corollary 3.4]). Together with (3)″ and (4)″ this gives the following new picture of the Cuntz semigroup: Let A be a separable, simple, exact, stably finite C^*-algebra with $\mathrm{Cu}(A)$ almost unperforated. Then

$$\mathrm{Cu}(A) = V(A) \sqcup \big(P_o\big((A \otimes \mathcal{K})^{**}\big) \setminus P(A \otimes \mathcal{K})\big)/\!\sim.$$

In other words, the Cuntz semigroup can be decomposed into the monoid $V(A)$ (of Murray–von Neumann equivalence classes of projections in $A \otimes \mathcal{K}$) and the non-compact open projections modulo Murray–von Neumann equivalence in $(A \otimes \mathcal{K})^{**}$.

In conclusion, let us note that the vertical implications between the second and the third row of $(*)$ cannot be reversed in general. Actually, these implications will fail whenever $\mathrm{Cu}(A)$ is not almost unperforated, which tends to happen when A has "high dimension". These implications can also fail for projections in very nice C^*-algebras. Indeed, if p and q are projections, then $p \sim_{\mathrm{tr}} q$ simply means that $\tau(p) = \tau(q)$ for all traces τ. It is well known that the latter does not imply Murray–von Neumann or Cuntz equivalence even for simple AF-algebras, if their K_0 groups have non-zero infinitesimal elements.

Acknowledgment

We thank Uffe Haagerup for his valuable comments on von Neumann algebras that helped us to shorten and improve some of the proofs in Section 5.

References

[1] C.A. Akemann, The general Stone–Weierstrass problem, J. Funct. Anal. 4 (1969) 277–294.
[2] C.A. Akemann, A Gelfand representation theory for C^*-algebras, Pacific J. Math. 39 (1971) 1–11.
[3] C.A. Akemann, J. Anderson, G.K. Pedersen, Approaching infinity in C^*-algebras, J. Operator Theory 21 (2) (1989) 255–271.
[4] P. Ara, F. Perera, A.S. Toms, K-theory for operator algebras. Classification of C^*-algebras, arXiv:0902.3381, 2009.
[5] B. Blackadar, Comparison theory for simple C^*-algebras, in: Operator Algebras and Applications, vol. 1, in: London Math. Soc. Lecture Note Ser., vol. 135, Cambridge Univ. Press, Cambridge, 1988, pp. 21–54.
[6] B. Blackadar, Operator algebras, in: Theory of C^*-Algebras and von Neumann Algebras, Operator Algebras and Non-Commutative Geometry, III, in: Encyclopaedia Math. Sci., vol. 122, Springer-Verlag, Berlin, 2006.
[7] E. Blanchard, E. Kirchberg, Non-simple purely infinite C^*-algebras: The Hausdorff case, J. Funct. Anal. 207 (2004) 461–513.
[8] N.P. Brown, A. Ciuperca, Isomorphism of Hilbert modules over stably finite C^*-algebras, J. Funct. Anal. 257 (1) (2009) 332–339.
[9] F. Combes, Poids sur une C^*-algèbre, J. Math. Pures Appl. (9) 47 (1968) 57–100.
[10] K.T. Coward, G.A. Elliott, C. Ivanescu, The Cuntz semigroup as an invariant for C^*-algebras, J. Reine Angew. Math. 623 (2008) 161–193.

[11] G.A. Elliott, L. Robert, L. Santiago, The cone of lower semicontinuous traces on a C^*-algebras, Amer. J. Math., in press.
[12] U. Haagerup, Quasitraces on exact C^*-algebras are traces, preprint, 1992.
[13] R.V. Kadison, G.K. Pedersen, Equivalence in operator algebras, Math. Scand. 27 (1970) 205–222.
[14] R.V. Kadison, J.R. Ringrose, Fundamentals of the theory of operator algebras. Volume II: Advanced theory, in: Pure Appl. Math., vol. 100-2, Academic Press/Harcourt Brace Jovanovich, Orlando, 1986, pp. 399–1074, XIV.
[15] E. Kirchberg, On the existence of traces on exact stably projectionless simple C^*-algebras, in: Operator Algebras and Their Applications, Waterloo, ON, 1994/1995, in: Fields Inst. Commun., vol. 13, Amer. Math. Soc., Providence, RI, 1997, pp. 171–172.
[16] H. Lin, Equivalent open projections and corresponding hereditary C^*-subalgebras, J. Lond. Math. Soc. (2) 41 (2) (1990) 295–301.
[17] H. Lin, Cuntz semigroups of C^*-algebras of stable rank one and projective Hilbert modules, arXiv:1001.4558, 2010.
[18] G.K. Pedersen, C^*-Algebras and Their Automorphism Groups, London Math. Soc. Monogr. Ser., vol. 14, Academic Press, London/New York/San Francisco, 1979, X+416 pp.
[19] C. Peligrad, L. Zsidó, Open projections of C^*-algebras: comparison and regularity, in: Operator Theoretical Methods, Timişoara, 1998, Theta Found., Bucharest, 2000, pp. 285–300.
[20] F. Perera, The structure of positive elements for C^*-algebras with real rank zero, Int. J. Math. 8 (3) (1997) 383–405.
[21] L. Robert, On the comparison of positive elements of a C^*-algebra by lower semicontinuous traces, Indiana Univ. Math. J. 58 (6) (2009) 2509–2515.
[22] M. Rørdam, The stable and the real rank of $\mathcal{Z}$-absorbing C^*-algebras, Int. J. Math. 15 (10) (2004) 1065–1084.
[23] D. Sherman, On the dimension theory of von Neumann algebras, Math. Scand. 101 (1) (2007) 123–147.
[24] J. Tomiyama, Generalized dimension function for W^*-algebras of infinite type, Tohoku Math. J. (2) 10 (1958) 121–129.

Bibliography

[Bal75] B. J. Ball, *Proper shape retracts*, Fundam. Math. **89** (1975), 177–189 (English).

[BB64] R. H. Bing and K. Borsuk, *A 3-dimensional absolute retract which does not contain any disk*, Fundam. Math. **54** (1964), 159–175.

[Bla85] B. Blackadar, *Shape theory for C^*-algebras*, Math. Scand. **56** (1985), 249–275.

[Bor32] K. Borsuk, *Über eine Klasse von lokal zusammenhängenden Räumen*, Fundam. Math. **19** (1932), 220–242 (German).

[Bor67] ______, *Theory of retracts*, Monografie Matematyczne. 44. Warszawa: PWN - Polish Scientific Publishers, 1967.

[BP91] L. G. Brown and G. K. Pedersen, *C^*-algebras of real rank zero*, J. Funct. Anal. **99** (1991), no. 1, 131–149.

[BP09] ______, *Limits and C^*-algebras of low rank or dimension*, J. Oper. Theory **61** (2009), no. 2, 381–417.

[CC60] J. H. Case and R. E. Chamberlin, *Characterizations of tree-like continua*, Pac. J. Math. **10** (1960), 73–84.

[CC06] J. W. Cannon and G. R. Conner, *On the fundamental groups of one-dimensional spaces*, Topology Appl. **153** (2006), no. 14, 2648–2672.

[CEI08] K. T. Coward, G. A. Elliott, and C. Ivanescu, *The Cuntz semigroup as an invariant for C^*-algebras*, J. Reine Angew. Math. **623** (2008), 161–193.

[Cha94] M. G. Charalambous, *Axiomatic characterizations of the dimension of metric spaces*, Topology Appl. **60** (1994), no. 2, 117–130.

[Dad00] M. Dadarlat, *Nonnuclear subalgebras of AF algebras*, Am. J. Math. **122** (2000), no. 3, 581–597.

[dGM67] J. de Groot and R. H. McDowell, *Locally connected spaces and their compactifications*, Ill. J. Math. **11** (1967), 353–364 (English).

[Eil94] S. Eilers, *Notes on End Theory*, Unpublished preprint. Available under www.math.ku.dk/ eilers, 1994.

[EK86] E. G. Effros and J. Kaminker, *Homotopy continuity and shape theory for C^*-algebras*, Geometric methods in operator algebras, Proc. US-Jap. Semin., Kyoto/Jap. 1983, Pitman Res. Notes Math. Ser. 123, 152-180, 1986.

[FQ06] T. Fernández and A. Quintero, *Dendritic generalized Peano continua*, Topology Appl. **153** (2006), no. 14, 2551–2559.

[Han51] O. Hanner, *Some theorems on absolute neighborhood retracts*, Ark. Mat. **1** (1951), 389–408.

[HV84] R. H. Herman and L. N. Vaserstein, *The stable range of C^*-algebras*, Invent. Math. **77** (1984), 553–556.

[Jac52] J. R. Jackson, *Some theorems concerning absolute neighbourhood retracts*, Pac. J. Math. **2** (1952), 185–189.

[KW04] E. Kirchberg and W. Winter, *Covering dimension and quasidiagonality*, Int. J. Math. **15** (2004), no. 1, 63–85.

[Lel76] A. Lelek, *Properties of mappings and continua theory*, Rocky Mt. J. Math. **6** (1976), 47–59 (English).

[Mar60] S. Mardešić, *On covering dimension and inverse limits of compact spaces*, Ill. J. Math. **4** (1960), 278–291.

[Nad92] S. B. jun. Nadler, *Continuum theory. An introduction*, Pure and Applied Mathematics (New York, Marcel Dekker). 158. New York: Marcel. XII, 1992.

[Nag70] K. Nagami, *Dimension theory. With an appendix by Yukihiro Kodama*, Pure and Applied Mathematics. Vol. 37. New York-London: Akademic Press 1970. XI, 1970.

[Nis74] T. Nishiura, *An axiomatic characterization of covering dimension in metrizable spaces*, TOPO 72 - General Topology Appl., 2nd Pittsburgh internat. Conf. 1972, Lect. Notes Math. 378, 341-353, 1974.

[Phi12] N. C. Phillips, *Equivariant semiprojectivity*, preprint, arXiv:1112.4584, 2012.

[PZ00] C. Peligrad and L. Zsidó, *Open projection of C^*-algebras: comparison and regularity.*, Gheondea, A. (ed.) et al., Operator theoretical methods. Proceedings of the 17th international conference on operator theory, Timişoara, Romania, June 23-26, 1998. Bucharest: The Theta Foundation. 285-300., 2000.

[Rie83] M. A. Rieffel, *Dimension and stable rank in the K-theory of C^*-algebras*, Proc. Lond. Math. Soc., III. Ser. **46** (1983), 301–333.

[She76] R. B. Sher, *Docility at infinity and compactifications of ANR's*, Trans. Am. Math. Soc. **221** (1976), 213–224 (English).

[War58] L. E. jun. Ward, *On dendritic sets*, Duke Math. J. **25** (1958), 505–513.

[WZ10] W. Winter and J. Zacharias, *The nuclear dimension of C^*-algebras*, Adv. Math. **224** (2010), no. 2, 461–498.