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Infinite Random Graphs with a view towards Quantum Gravity

PhD thesis

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Abstract

In this thesis we study random planar graphs and some of the tools and techniques used to address some related combinatorial problems. We give an account of generating function methods, mainly focusing on some analytic aspects of generating functions. Namely, we discuss the so-called *singularity analysis* process, a technique that allows the *transfer* of the singular behaviour of certain functions to the asymptotic behaviour of their Taylor coefficients. Furthermore, we collect a set of theorems for the study of solutions of certain functional equations, which are frequent in combinatorial problems.

As an application of random graph theory, we discuss the dynamical triangulation model and the causal dynamical triangulation model of two-dimensional quantum gravity.

Finally, we study the Ising model on certain infinite random trees, constructed as “thermodynamic” limits of Ising systems on finite random trees. We give a detailed description of the distribution of infinite spin configurations. As an application, we study the magnetization properties of such systems and prove that they exhibit no spontaneous magnetization. The basic reason is that the infinite tree has a certain one dimensional feature despite the fact that we prove its Hausdorff dimension to be 2. Furthermore, we obtain results on the spectral dimension of the trees.

Abstract

I denne afhandling vil vi studere plane grafer samt nogle af de metoder og teknikker, der finder anvendelse i relaterede kombinatoriske problemer. Vi giver en beskrivelse af genererende funktions-metoden, med fokus hovedsageligt på analytiske aspekter af genererende funktioner, i det vi behandler den såkaldte singularites analyse proces, en teknik, der tillader, at vi kan overføre den singulære opførsel af visse funktioner til den asymptotiske opførsel af deres Taylor koefficienter. Yderligere samler vi et antal sætninger til at studere løsninger tile særlige funktionalligninger, som ofte optræder i kombinatoriske problemer.

Som en anvendelse af stokastik graf-teori diskuterer vi den dynamiske trianguleringsmodel og den kausale dynamiske trianguleringsmodel i 2-dimensional kvantegravitation.

Endeligt studerer vi Ising-modellen på visse vendelige stokastiske træer, konstrueret som en "termodynamisk" grænse af Ising-systemer på endelige stokastiske træer. Vi giver en detaljeret beskrivelse af fordelingen af vendelige spin-konfigurationer. Som en anvendelse studerer vi magnetiseringsegenskaber for sådanne systemer og beviser, at de ikke udviser spontane magnetiseringer. Grunden hertil er essentielt at det vendelige træ har ensærlig 1-dimensional egenskab, på trods af, at dets Hausdorff-dimension kan vises at være 2. Yderligere opnår vi resultater om den spektrale dimension for det træ.

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Contents

Notation

In this note we collect some notational conventions which will be used throughout the thesis.

Let $f, g : M \to \mathbb{R}$ (or $f, g : M \to \mathbb{C}$), with $M \subseteq \mathbb{R}$ (or $M \subseteq \mathbb{C}$). Let $a \in M$ a point in the set M.

Then we have the following *asymptotic estimates*. We use the notation

$$f(x) \sim g(x), \qquad (x \to a)$$

to indicate that the function f and g are asymptotic as x approaches a, that is

$$\lim_{x \to a} \frac{f(x)}{g(x)} = 1.$$

The big-O notation $O(\cdot)$,

$$f(x) = O(g(x)), \qquad (x \to a)$$

means that there exists a positive constant C such that

$$|f(x)| \leq C\,|g(x)| \qquad x \in U \cap M,$$

for some neighborhood U of a.

We use the little-o notation $o(\cdot)$

$$f(x) = o(g(x)), \qquad (x \to a)$$

meaning that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = 0.$$

For all tha above cases, we will often omit to specify what the point a is, but it should be clear from the context.

Introduction

The study of planar graphs was initiated in the 1960s by Tutte in a fundamental series of paper [84, 85, 86, 87]. Since then, planar graphs have been extensively studied in different branches of science.

In theoretical physics, they were introduced in a well-known work of 't Hooft [83] and Brézin et al. [22], in connection with the study of Feynman diagrams of certain field theories. In particular, in [83] the author showed that, in the large N limit of an $SU(N)$ gauge field theory, only planar diagrams are relevant.

More recently, planar graphs have been used to model systems possessing random behaviour, such as social networks [1], membranes, see e.g. [32], and fluctuating surfaces in two-dimensional quantum gravity [4].

In this thesis we study random planar graphs and some of the tools and techniques used to address some related combinatorial problems. In particular, we focus our attention on two models of random planar graphs in quantum gravity and on a model of a certain class of infinite random trees coupled with Ising models.

The models of quantum gravity via random graphs discussed here are the dynamical triangulations (DT) model and the causal dynamical triangulations (CDT) model. The basic idea underlying these two model is to represent fluctuating geometries in quantum gravity in terms of random triangulations of the spacetime. The DT model was first introduced in 1985 when it appeared in [2, 31, 60] as a triangulation technique of Euclidean surfaces. In particular, in [2, 60] it was used as a regularization scheme for Polyakov string theory.

The CDT model was first proposed in [6] as a model of triangulated Lorentzian surfaces that includes the notion of causality from the start. This idea was implemented restricting the class of triangulations of spacetime to those that can be sliced perpendicular to the time direction.

As will be seen, both the DT and CDT model for two-dimensional surfaces reduces to the combinatorial problem of counting non-equivalent triangulations of a given surface. In both cases, the problem is analytically tractable and explicit expressions of partition functions can be produced.

From this point of view, statistical mechanical models on random planar graphs can be seen as the discrete realization of the coupling between matter fields and gravity. Probably, one the most well-known of these systems is the Ising model on planar random lattice. This was studied and exactly solved by Kazakov et al. in [59, 18, 21], using matrix model techniques.

In this thesis we study the Ising model on certain infinite random trees, constructed as "thermodynamic" limits of Ising systems on random finite trees. These are subject to a certain genericity condition for which reason we call them *generic Ising trees*. Using tools developed in [39, 42] we prove for such ensembles that spontaneous magnetization is absent. Furthermore, the technique used will allow us to calculate the Hausdorff and spectral dimension of the underlying tree structures.

The thesis is organized as follows. In the first part of Chapter 1 we concentrate on some general aspects of random graphs. In Sec. 1.1 we recall some basic graph theoretic notions and fix the notation that will be used throughout the thesis. Then, after giving a definition of random graph, we will describe the mechanism that will allow us to obtain infinite random graphs by a limiting procedure on graphs of finite size. One can associate to infinite graphs two notions of dimension, the Hausdorff and the spectral dimension, which in some sense give an indication of the geometry of the graph. Namely, the former is a measure of the volume growth rate of a geodesic ball on the graph with respect to its radius. The latter is related to the connectivity of the graph, and will be defined in terms of simple random walk on graphs. These concepts will be given a precise meaning in Sec. 1.1.3 and 1.1.4.

In the second part of the chapter, Sec. 1.2, we will collect some explicit examples of infinite random graphs. A first class of examples consists of random tree models, namely planar trees, generic trees and labeled trees. For all of them the infinite size limit is discussed. As will be seen, the other two examples, the causal triangulation and the planar quadrangulation, are closely related to the tree models, and some of their properties can be transported from the latter to the former.

In Chapter 2 we discuss generating function techniques. In Sec. 2.1 we give the basic notions about generating functions, which we then apply to an explicit enumeration problem, in Sec. 2.1.3. The remainder of the chapter is dedicated to the analytic aspects of generating functions. In Sec. 2.2 we discuss the so-called *singularity analysis*, a technique developed by Flajolet and Odlyzko in [47]. It consists of a set of theorems that allow the *transfer* of the singular behaviour of certain functions to the asymptotic behaviour of their Taylor coefficients. In Sec. 2.2.2 we collect a set of theorems about solutions of a certain class of functional equations, which are frequent in combinatorial problems. In particular, we will see that solutions of this type of equations often present a square root behaviour near their singularity points, therefore, under certain conditions, they can be studied via the singular analysis process.

Chapter 3 is dedicated to the study of two-dimensional quantum gravity models with random graph techniques. The discussion in the first two sections of this chapter are purely formal and they are intended to give the reader an idea of the path-integral formalism and its application to two-dimensional quantum gravity. In Sec. 3.3.1 we outline Regge's construction [78] of the discrete version of General Relativity. This is the starting point for the description of the quantum gravity path-integral in terms of triangulations of spacetime. The dynamical triangulations model, in Sec 3.3.2, and the causal dynamical triangulations model, in Sec. 3.3.3, are discussed.

Chapter 4 is mainly based on [44], which is a joint work of the author with Bergfinnur Durhuus. Here, we study in detail a class of infinite random trees coupled with Ising models. As will be seen, these trees are closely related to the generic trees discussed in Ch. 1, being subject to an analogous genericity condition. For this reason we call them *generic Ising trees*. In Sec. 4.3 we give a detailed description of the distribution of infinite spin configurations. As an application, we study in 4.5 the magnetization properties of such systems and prove that they exhibit no spontaneous magnetization. Furthermore, in Sec. 4.4 the values of the Hausdorff and spectral dimensions of the underlying trees are calculated.

Finally, some concluding remarks on possible future developments are collected in the conclusion at the end of this thesis.

Chapter 1

Infinite Random Graphs

In this chapter we want to introduce the reader to a general approach to the study of infinite random graphs.

The chapter is divided into two sections. The first will be devoted to the general setting which will be used throughout this thesis. After recalling the basic definitions in graph theory, we explain the concept of random graph and the general mechanism that will allow us to obtain infinite random graphs and information about their geometry (Hausdorff and spectral dimension). In physics terms, this can be seen as a way to rigorously define statistical mechanical ensembles and their thermodynamic limits.

In the second section we collect some explicit examples. In particular, the method described in the first section will be applied to two classes of trees (generic and labeled trees) which, as will be seen, also provide some information on two other types of graphs, the so called uniform infinite causal triangulation and uniform infinite quadrangulation.

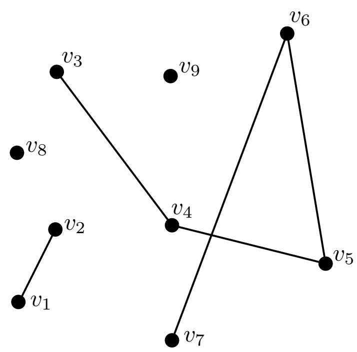

Figure 1.1: Graph $G(V,E)$ with vertex set $V(G) = \{v_1, \ldots, v_9\}$ and size $|G| = 5$.

1.1 The general method

1.1.1 Basic definitions

A *graph* G is defined by its *vertex* set $V(G)$, whose elements will be denoted by v, w, etc., and its *edge* set $E(G)$, formed by unordered pairs $e = (v, w)$ of different vertices[1]. We shall often use the notation $v \in G$ and $e \in G$, instead of $v \in V(G)$ and $e \in E(G)$, respectively.

The number σ_v of edges connecting to a vertex v is called the *degree* of v. The *size* of a graph G is defined as the number of edges in G and is denoted by $|G|$, i.e. $|G| = \sharp E(G)$, where $\sharp M$ is used to denote the number of elements in a set M. We will deal with graphs of both finite and infinite size, but all graphs are assumed to be locally finite, i.e. the degree σ_v of each vertex v in G is assumed to be finite.

A *path* γ in G is a sequence of different edges

$$\gamma = \{(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)\}, \tag{1.1}$$

[1] In the literature, these graphs are often referred to as *undirected graphs*, to be distinguished from *directed graphs*, where edges are given with an orientation. They are often called *simple graphs*, whereas *multigraphs* is used when different edges with same endpoints and loops are allowed.

where v_0 and v_k are called the *end vertices* or *end points* of the path. If the end vertices coincide, i.e. $v_0 = v_k$, the path is called a *circuit* originating at v_0. The *length* $|\gamma|$ of a path is naturally defined as the number of edges in γ.

The notion of path in a graph can be used to define some other basic concepts. A graph G is called *connected* if there exists a path γ between any two vertices $v, v' \in G$, otherwise we call *connected component* each connected subgraph of G. The *graph distance*, or *geodesic distance*, d_g between any two vertices v and $v' \in G$ is then defined as the length of the shortest path between v and v', i.e.

$$d_g(v,v') = \min\left\{|\gamma| \;\middle|\; \gamma \text{ has endpoints } v,\, v' \right\}, \tag{1.2}$$

with the conventions $d_g(v,v) = 0$ and $d_g(v,v') = \infty$ if v and v' belong to different connected components of G. Given a connected graph G, a real number $R \geq 0$ and a vertex $v \in V(G)$, we denote by $B_R(G,v)$ the *closed ball* of radius R centered at v, i.e. $B_R(G,v)$ is the subgraph of G spanned by the vertices at graph distance $\leq R$ from v, hence its vertex and edge set are

$$\begin{aligned} V(B_R(G,v)) &= \left\{w \in V(G) \;\middle|\; d_g(v,w) \leq R\right\}, \\ E(B_R(G,v)) &= \left\{(w,w') \in E(G) \;\middle|\; w,w' \in V(B_R(G,v))\right\}. \end{aligned} \tag{1.3}$$

A *rooted graph* is a graph which contains a distinguished oriented edge $e = \langle r, r' \rangle$, called the *root edge*, whose initial vertex r is called the *root vertex*, or simply *root*. For a rooted graph, the ball $B_R(G,r)$ of radius R centered at the root r will be simply denoted by $B_R(G)$.

In this work we will consider *planar graphs*, i.e. graphs that can be embedded in the plane $\mathbb{R}^2$ (or into the 2-sphere) without crossings. More precisely, a planar graph is a graph G together with an injection $\phi : V(G) \to \mathbb{R}^2$ and an association ψ to each edge $(v,v') \in E(G)$ of an arc $\psi(v,v')$ in $\mathbb{R}^2$ connecting $\phi(v)$ and $\phi(v')$, such that each arc contains at most the

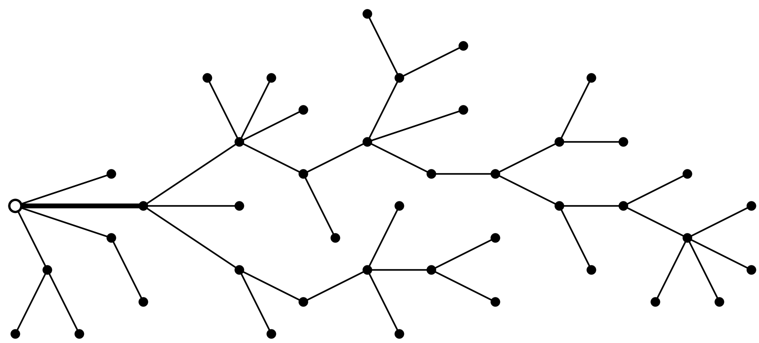

Figure 1.2: Rooted planar tree. The empty disk denotes the root vertex and the bold edges the root edge.

endpoints of another arc.

Two planar graphs are considered identical if one can be continuously deformed into the other in $\mathbb{R}^2$. The *faces* of a planar graph are the connected components of the complement of the edges.

A *tree* is a connected graph without circuits. Note that a tree can always be embedded into the plane (or S^2) without crossings, hence it is a planar graph. On the other hand, a tree admits different embeddings. In the following, we use the terminology *planar tree* to indicate a tree together with an embedding. For instance, two different embeddings of the same tree define two different planar trees.

Throughout this thesis we will deal mainly with *rooted planar trees*, hence we often refer to them simply as trees.

We define the *height* $h(\tau)$ of a finite tree τ as the maximal distance from the root of one of its vertices, that is

$$h(\tau) = \max \left\{ d_g(r, v) \mid v \in V(\tau) \right\}, \tag{1.4}$$

where r denotes the root vertex of τ.

1.1.2 Finite and infinite random graphs

A *random graph* $(\mathcal{G}, \mu)$ is a set of graphs $\mathcal{G}$ equipped with a probability measure μ.

Let $\mathcal{G}$ the set of (finite and infinite) planar rooted graphs and $\mathcal{G}_N$ the subset of finite graphs of size N. Denoting with $\mathcal{G}_\infty$ the set of infinite graphs, the set $\mathcal{G}$ can be decomposed as follows

$$\mathcal{G} = \left(\bigcup_{N=1}^{\infty} \mathcal{G}_N \right) \cup \mathcal{G}_\infty, \tag{1.5}$$

.

The set $\mathcal{G}$ can be equipped with a notion of distance between two graphs as follows. For two planar rooted graphs G and $G' \in \mathcal{G}$, the distance between them is given by

$$d(G, G') = \inf \left\{ \frac{1}{1+R} \mid B_R(G) = B_R(G') \right\}, \tag{1.6}$$

where $B_R(G)$ is defined in (1.3).

Further, we associate to each graph $G \in \mathcal{G}_N$ a weight $w(G) \geq 0$, thus a probability distribution $\mu_N(G)$ on $\mathcal{G}_N$ is defined as

$$\mu_N(G) = \frac{1}{Z_N} w(G), \tag{1.7}$$

where the normalizing factor Z_N, the *partition function*, is given by

$$Z_N = \sum_{G \in \mathcal{G}_N} w(G). \tag{1.8}$$

Note that the probability measures μ_N, $N \in \mathbb{N}$, can be naturally regarded as probability measures on $\mathcal{G}$, since, for each N,

$$\mu_N(\mathcal{G} \setminus \mathcal{G}_N) = 0. \tag{1.9}$$

Infinite random graphs will be defined by a probability measure μ on $\mathcal{G}_\infty$, obtained as a limit of a sequence of measures μ_N, $N \in \mathbb{N}$, viewed as measures on $\mathcal{G}$. More precisely, the measure μ on the set of infinite graphs $\mathcal{G}_\infty$ can be obtained as a weak limit on μ_N for $N \to \infty$, in the sense that

$$\int_{\mathcal{G}} f(G)\, d\mu_N(G) \xrightarrow{N\to\infty} \int_{\mathcal{G}} f(G)\, d\mu(G).$$

for all bounded functions f on $\mathcal{G}$, which are continuous w.r.t. the metric d in eq. (1.6).

In most of the cases presented here we will make use of the following result about weak convergence of probability measures, that can be found in [17]. For our purpose it is most usefully stated as follows. Let ν_N, $N \in \mathbb{N}$, a sequence of probability measures on a metric space M and $\mathcal{U}$ a family of both open and closed subsets of M such that

i) any finite intersection of sets in $\mathcal{U}$ belongs to $\mathcal{U}$,

ii) any open subset of M may be written as a finite or countable union of sets from $\mathcal{U}$

iii) the sequence $(\nu_N(A))$ is convergent for all sets $A \in \mathcal{U}$.

If the sequence (ν_N) is *tight*, that is for each $\varepsilon > 0$ there exists a compact subset C of M such that

$$\nu_N(M \setminus C) < \varepsilon \quad \text{for all } N, \tag{1.10}$$

then the sequence (ν_N) is weakly convergent. We refer the reader to [17] for a proof.

In the following section we define two notions of dimension that can be associated to infinite graphs, and in some sense, provide an indication of their geometric properties.

1.1.3 Hausdorff dimension

Roughly speaking, the Hausdorff dimension of a graph G is a measure of the volume growth rate of a geodesic ball $B_R(G)$ with respect to its radius R.

More precisely, given an infinite connected graph G, if the limit

$$d_H = \lim_{R\to\infty} \frac{\ln |B_R(G,v)|}{\ln R} \tag{1.11}$$

exists, we call d_H the *Hausdorff dimension* of G. It is clear that this definition only makes sense for infinite graphs, d_H being always 0 for finite graphs. It is easily seen that the existence of the limit as well as its value do not depend on the vertex v. We shall always choose $v \equiv r$, the root vertex.

For an ensemble of infinite graphs $(\mathcal{G}_\infty, \mu)$, we define the *annealed* Hausdorff dimension by

$$\bar{d}_H = \lim_{R\to\infty} \frac{\ln \langle\, |B_R(G)|\, \rangle_\mu}{\ln R}\,, \tag{1.12}$$

provided the limit exists, where $\langle \cdot \rangle_\mu$ denotes the expectation value w.r.t. μ.

If there exists a subset $\mathcal{G}_0$ of $\mathcal{G}_\infty$ such that $\mu(\mathcal{G}_0) = 1$ and such that every $G \in \mathcal{G}_0$ has Hausdorff dimension d_H we say that the Hausdorff dimension of $(\mathcal{G}_\infty, \mu)$ is almost surely (a.s.) d_H.

1.1.4 Spectral dimension

There exists another notion of dimension of a graph, which is related to the connectivity of the graph. It is called spectral dimension, and in order to define it we first need to introduce the concept of simple random walk on a graph.

A *walk* on a graph G is a sequence $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$ of (not necessarily distinct) edges in G. Note that it differs from the previous definition of path, eq. (1.1), since a walk can self-intersect. We shall denote such a walk by $\omega : v_0 \to v_k$ and call v_0 the *origin* and v_k the *end* of the walk. Moreover, the number k of edges in ω will be denoted by $|\omega|$. To each such walk ω we associate a weight

$$\pi_G(\omega) = \prod_{i=0}^{|\omega|-1} \sigma_{\omega(i)}^{-1}$$

where $\omega(i)$ is the i'th vertex in ω. Denoting by $\Pi_n(G, v_0)$ the set of walks of length n originating at vertex v_0 we have

$$\sum_{\omega \in \Pi_n(G,v_0)} \pi_G(\omega) = 1\,.$$

i.e. π_G defines a probability distribution on $\Pi_n(G, v_0)$. We call π_G the *simple random walk* on G.

For an infinite connected graph G and $v \in V(G)$ we denote by $\pi_t(G, v)$ the *return probability* of the simple random walk to v at time t, that is

$$\pi_t(G, v) = \sum_{\substack{\omega : v \to v \\ |\omega| = t}} \pi_G(\omega)\,.$$

If the limit

$$d_s = -2 \lim_{t\to\infty} \frac{\ln \pi_t(G, v)}{\ln t} \tag{1.13}$$

exists, we call d_s the *spectral dimension* of G. As before, the definition is valid only for infinite graphs, since for finite graphs the return probability is a positive constant for $t \to \infty$. Again in this case, the existence and value of the limit are independent of v.

The *annealed* spectral dimension of an ensemble $(\mathcal{G}_\infty, \mu)$ of rooted infinite graphs is defined as

$$\bar{d}_s = -2 \lim_{t\to\infty} \frac{\ln \langle\, \pi_t(G, r)\, \rangle_\mu}{\ln t} \tag{1.14}$$

provided the limit exists. As above, we say that the spectral dimension of $(\mathcal{G}_\infty, \mu)$ is almost surely d_s, if the set of graphs with spectral dimension different from d_s has vanishing μ-measure.

The Hausdorff and spectral dimensions do not necessarily agree. In fact we have examples where they do (as the hyper-cubic lattice $\mathbb{Z}^d$ where $d_H = d_s = d$) and other where they do not, as in the case studied in [41].

However d_H and d_s are closely related, and under certain conditions the inequality

$$d_H \geq d_s \geq \frac{2d_H}{1+d_H} \tag{1.15}$$

can be proved [30]. This is for example the case for the random combs [41]. Moreover for some trees, such as the uniform spanning tree on $\mathbb{Z}^2$ [11] and the generic tree [42], one finds that the second inequality is actually an identity, i.e.

$$d_s = \frac{2d_H}{1+d_H}. \tag{1.16}$$

1.2 Examples of infinite random graphs

In this section we collect some explicit examples of infinite random graphs, defined by the limiting procedure described in Sec. 1.1.2. The following examples are also intended to illustrate the close relation between certain types of trees and other planar graphs. In particular, we shall see that rooted planar trees can be bijectively mapped onto the so-called sliced triangulations, whereas infinite labeled trees can be used to construct infinite planar quadrangulations. In both cases, the maps between those objects are obtained as generalizations of the so-called CVS bijection [80].

1.2.1 Uniform infinite planar tree

Let $\mathcal{T}$ be the set of both finite and infinite rooted planar trees, with root of degree 1. We denote by $\mathcal{T}_N$ the subset of $\mathcal{T}$ of trees of size N,

$$\mathcal{T}_N = \left\{ \tau \in \mathcal{T} \;\middle|\; |\tau| = N \right\}, \tag{1.17}$$

and by $\mathcal{T}_l'$ the set of trees with maximal height equal to l.

According to these definitions we have the decomposition

$$\mathcal{T} = \left(\bigcup_{N=1}^{\infty} \mathcal{T}_N \right) \cup \mathcal{T}_\infty, \tag{1.18}$$

where $\mathcal{T}_\infty$ denotes the set of infinite trees.

A probability measure on the set $\mathcal{T}_N$ is defined setting the weight $w(\tau) = 1$ for each tree $\tau \in \mathcal{T}_N$. Hence, we have

$$\mu_N(\tau) = C_N^{-1} \quad \text{for } \tau \in \mathcal{T}_N, \qquad \mu_N(\mathcal{T} \setminus \mathcal{T}_N) = 0, \tag{1.19}$$

where $C_N = \sharp\mathcal{T}_N$ is the number of rooted planar trees with vertex of degree 1 and N edges. It is a well-known fact (see e.g. [4]) that the number $C_N = \sharp\mathcal{T}_N$ of trees of size N is given by

$$C_N = \frac{(2N-2)!}{N!(N-1)!}. \tag{1.20}$$

We will prove this result in Sec. 2.1.3 as an easy application of the generating function method.

The existence of the limiting measure μ has been proved in [39]. This result is summarized as follows.

Theorem 1.2.1. *The sequence of measures (μ_N) on $\mathcal{T}$ converges weakly as $N \to \infty$ to a Borel probability measure μ concentrated on $\mathcal{T}_\infty$.*

The random tree $(\mathcal{T}_\infty, \mu)$, called *uniform infinite tree*, exhibits some peculiar properties that we now illustrate.

First we note that, setting

$$A(\tau_0) = \left\{ \tau \in \mathcal{T} \;\middle|\; B_R(\tau) = \tau_0 \right\}, \tag{1.21}$$

where $\tau_0 \in \mathcal{T}'_R$, the μ-volume of this set turns out to be

$$\mu\left(A(\tau_0)\right) = M\,2^{M+1}\,4^{-|\tau_0|}, \tag{1.22}$$

with M denoting the number of vertices $(v_1, \ldots, v_M)$ in τ_0 at maximal distance R from the root.

More interestingly, choosing a sequence $(\tau_1, \ldots, \tau_M)$ of M trees in $\mathcal{T}$, whose root's neighbor is identified with $(v_1, \ldots, v_M)$, respectively, the conditional probability measure $d\mu\left(\tau_1, \ldots, \tau_M | A(\tau_0)\right)$ is given by

$$d\mu(\tau_1, \ldots, \tau_M | A(\tau_0)) = \mu\left(A(\tau_0)\right)^{-1} \sum_{i=1}^{M} d\mu(\tau_i) \prod_{j \neq i} d\rho(\tau_j), \tag{1.23}$$

where ρ is concentrated on finite trees and defined by

$$\rho(\tau) = 4^{-|\tau|}. \tag{1.24}$$

This expression, together with the fact that μ and ρ are concentrated, respectively, on infinite and finite trees, gives us some information about the shape of the uniform infinite tree. In fact, it turns out that with probability 1 only one tree τ_i is infinite and that the trees $\tau_1, \ldots, \tau_M$ are independently distributed. In other words, the limiting measure μ is concentrated on the set of infinite trees τ containing only one infinite path, called the *spine*, originating at the root. Finally, τ is obtained by attaching finite trees, the *branches*, at the vertices of the spine, see Fig. 1.3.

The explicit expression (1.22) for the infinite measure μ can be used to calculate the average number $\langle d_r \rangle_\mu$ of vertices at distance r from the root, and accordingly the average volume $\langle |B_r| \rangle_\mu$ of the ball of radius r centered at the root (see [39]). We have the following result.

Theorem 1.2.2. *For $r \geq 1$ we have*

$$\langle d_r \rangle_\mu = 2r - 1 \tag{1.25}$$

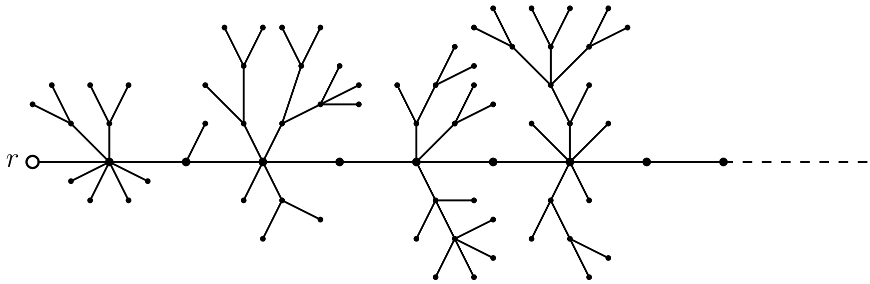

Figure 1.3: The uniform infinite planar tree, with the spine starting at root r and finite branches.

and

$$\langle |B_r| \rangle_\mu = r^2, \tag{1.26}$$

As a consequence, it follows from (1.12) that the annealed Hausdorff dimension of the uniform infinite tree is $\bar{d}_H = 2$.

The description of the infinite tree emerging in the uniformly distributed tree case, although surprising, is far from exceptional. In fact, as we will see in the following examples as well as in the case studied in chapter 4, a large class of random trees shows the same behaviour in the infinite size limit.

1.2.2 Generic trees

The uniform infinite planar tree presented in the previous section is a special case of a larger class of trees, called *generic trees*, which we review in this section.

Let $\mathcal{T}$, $\mathcal{T}_N$ and $\mathcal{T}_\infty$ be the sets of trees as defined in the previous section. We attach to each vertex $v \in V(\tau)$, except the root r, of a tree $\tau \in \mathcal{T}$ a non-negative *branching weight* $p_{\sigma_v - 1}$, depending on the degree σ_v of v.

The weight $w(\tau)$ associated to each tree $\tau \in \mathcal{T}_N$ is defined as the product of the branching weights of τ, i.e. $w(\tau) \equiv \prod_{v_i \in \tau \setminus r} p_{\sigma_{v_i} - 1}$. Hence the probability measure μ_N on $\mathcal{T}_N$ is given by

$$\mu_N(\tau) = \frac{1}{Z_N} \prod_{v \in \tau \setminus r} p_{\sigma_v - 1} \quad \tau \in \mathcal{T}_N, \tag{1.27}$$

and the partition function

$$Z_N = \sum_{\tau \in \mathcal{T}_N} \prod_{v \in \tau \setminus r} p_{\sigma_v - 1}. \tag{1.28}$$

We further assume $p_0 \neq 0$, since Z_N vanishes otherwise, and $p_n > 0$ for some $n \geq 2$, otherwise only linear chains would contribute. It is clear that the uniform planar tree

Next, we define the generating function for the branching weights as

$$P(z) = \sum_{n=0}^{\infty} p_n z^n, \tag{1.29}$$

which we assume to have radius of convergence $\rho > 0$, and the generating function for the finite volume partition functions as

$$Z(x) = \sum_{N=1}^{\infty} Z_N x^N, \tag{1.30}$$

whose radius of convergence is denoted by x_0.

The partition function $Z(x)$ is known (see [42]) to satisfy the functional equation

$$Z(x) = x\, P\left(Z(x)\right), \tag{1.31}$$

which is sufficient to determine the analytic function $Z(x)$. From this equation it follows that

$$Z_0 = \lim_{x \nearrow x_0} Z(x) \tag{1.32}$$

is finite and $\leq \rho$.

The *genericity assumption* that will define the set of generic trees states that

$$Z_0 < \rho. \tag{1.33}$$

In particular, all sets of branching weights such that $\rho = +\infty$ define an ensemble of generic trees.

Under the assumption (1.33), the equation (1.31) is sufficient to determine the singular behaviour of $Z(x)$ in the vicinity of x_0:

$$Z(x) = Z_0 - \sqrt{\frac{2P(Z_0)}{x_0 P''(Z_0)}}\sqrt{x_0 - x} + O(x - x_0). \tag{1.34}$$

The expression (1.34) can be further used to determine the asymptotic behaviour of the partition function Z_N for large N, which turns out to be

$$Z_N \sim N^{-\frac{3}{2}}\, x_0^{-N}. \tag{1.35}$$

This result can be explained with a so-called *transfer theorem* that we shall discuss in Ch. 2. In general, the transfer theorem allows to relate the behaviour of a certain class of functions near their singularity points to the asymptotic behaviour of their coefficients.

The result in eq. (1.35) is the fundamental step to prove the following theorem.

Theorem 1.2.3. *Under the genericity assumption (1.33), the probability measure μ_N converges to a probability measure μ on $\mathcal{T}$ concentrated on the subset $\mathcal{T}_\infty$.*

Details of the proofs of these results can be found in [42]. We omit them here, since a similar but slightly more involved case will be considered in Ch. 4.

The infinite tree defined by the limiting measure μ on $\mathcal{T}$ is called an *infinite generic tree*. As in the case considered in the previous section, an infinite generic tree consists of only one spine and finite branches.

The annealed Hausdorff and spectral dimensions of the ensemble $(\mathcal{T}, \mu)$

can be computed and are found to be, respectively,

$$\bar{d}_H = 2 \qquad \text{and} \qquad \bar{d}_s = \frac{4}{3}. \tag{1.36}$$

Actually this result can be further improved. In fact, using an upper and lower bound argument [43] on $|B_R(\tau)|$, $\tau \in \mathcal{T}$, one finds that for any generic random tree

$$d_H = 2 \qquad \text{a.s..} \tag{1.37}$$

Moreover, it can proved that [10]

$$d_s = \frac{4}{3} \qquad \text{a.s.} \tag{1.38}$$

using a continuous time random walk argument (see also [40]).

It should be stressed that the property of the infinite generic tree to have only one infinite path is strictly related to the genericity assumption (1.33). Models in which the genericity condition is not assumed to hold have been studied in [58] (see also [29, 23] for further results in this direction). In [58] the authors prove that the so-called *nongeneric trees* exhibit two different phases, called *critical* and *subcritical* phase, the latter being characterized by a limiting measure concentrated on the set of trees with exactly one vertex of infinite degree, provided that the condition of the tree to be locally finite is dropped.

As mentioned above, the uniform infinite tree is a special case of the generic tree, obtained by setting $p_n = 1$, for any $n \in \mathbb{Z}_{\geq 0}$. In fact, from (1.28) one finds

$$Z_N = \sharp \mathcal{T}_N = C_N \tag{1.39}$$

where the Catalan numbers C_N's are given in (1.20). This result, together with eq. (1.30) gives

$$Z(x) = \sum_{N=1}^{\infty} C_N x^N = \frac{1-\sqrt{1-4x}}{2} \quad \text{for } x < x_0 = \frac{1}{4}, \tag{1.40}$$

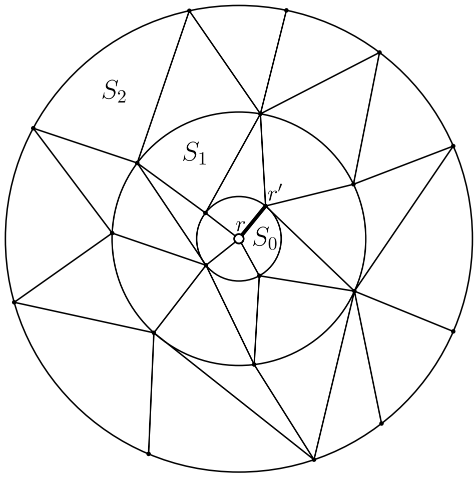

Figure 1.4: A sliced triangulation S of the disc with circles containing vertices at distance 1, 2 and 3 from the root. Here S consists of two annuli S_1, S_2 and the disc $S_0 = B_1(S)$. The bold edge indicates the root edge.

whereas the partition function for the branching weights (1.29) reads

$$P(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \quad \text{for } z < \rho = 1. \tag{1.41}$$

Hence, by (1.32) $Z_0 = \frac{1}{2}$ and the genericity assumption is clearly satisfied.

1.2.3 Uniform infinite causal triangulation (UICT)

In order to define this model we let $\mathcal{G}_N \equiv \mathcal{C}_N$ denote the set of *sliced* triangulations of the disc with N vertices. Here, a triangulation S of the disc is said to be sliced if the subgraph of S spanned by vertices at distance n and $n+1$ from the root r, $n = 1, \ldots, M$, is an annulus S_n such that every triangle in S_n has all vertices in the boundary and not all in the same boundary component of S_n. For $n = 0$ we require that $B_1(S)$ is a disc (see Fig.1.4). Moreover, M denotes the maximal distance of vertices in S from r.

In particular, if the boundary components of S_n contain l_n and l_{n+1} edges, respectively, then the total number of vertices, edges and triangles in S are

$$|V(S)| = 1 + \sum_{n=1}^{M} l_n, \quad |E(S)| = 3\sum_{n=1}^{M} l_n - l_M, \quad |S| = 2\sum_{n=1}^{M} l_n - l_M \tag{1.42}$$

respectively. Here we have assumed $M < \infty$. However, the definition of a sliced surface is also valid for infinite triangulations of the plane, corresponding to $M = \infty$.

We then define ν_N to be the uniform distribution on $\mathcal{C}_N$, i.e. we set $w(S) = 1$ for $S \in \mathcal{C}_N$. Thus, in this case

$$Z_N = \sharp\mathcal{C}_N. \tag{1.43}$$

We claim that

$$\sharp\mathcal{C}_N = \sharp\mathcal{T}_N \tag{1.44}$$

To see this, pick an orientation of the plane and consider $S \in \mathcal{C}_N$. For any vertex v at distance $n \geq 1$ from r, order the edges in $S_n \setminus \partial S_n$ emerging from v from left to right in accordance with the orientation of the plane. Next, delete from S all edges in $\bigcup_{n=1}^{M} \partial S_n$ as well as the rightmost edge emerging from v into S_n for each v as above. Finally, attach a new edge (r_0, r) to the root vertex r. Then the resulting graph is a tree $\beta(S)$ with a unique embedding into the plane such that the root edge (r, r') in S becomes the rightmost edge emerging from r in $\beta(S)$ (see Fig.1.5).

It is a fact, as the reader may easily verify, that $\beta : \mathcal{C}_N \to \mathcal{T}_N$ is a bijection which proves (1.44). In fact, β is a particular case of the so-called Schaeffer (or CVS) bijection applicable for labeled trees [80].

It is easy to check that β extends to the case $M = \infty$ corresponding to infinite sliced triangulations:

$$\beta : \mathcal{C}_\infty \to \mathcal{T}_\infty.$$

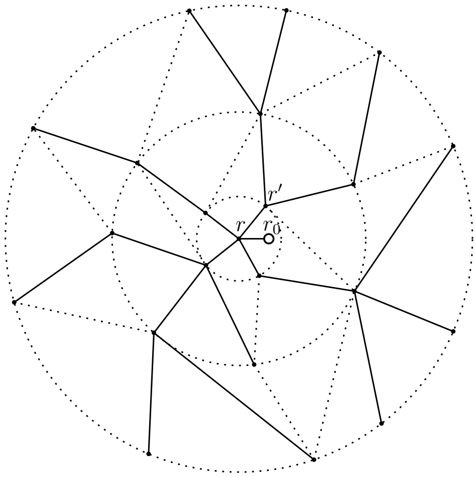

Figure 1.5: The full lines indicate the edges of the tree $\beta(S)$ constructed from the triangulation in Fig.1.4.

Taking this fact into account the following result is an immediate consequence of [39].

Theorem 1.2.4. *The probability distributions* (μ_N) *defined by*

$$\nu_N(S) = (\sharp\mathcal{C}_N)^{-1}, \quad S \in \mathcal{C}_N,$$

converge weakly to a probability distribution ν *on* $\mathcal{C}_\infty$*, which is given by*

$$\nu(A) = \mu(\beta(A))$$

for measurable sets $A \subseteq \mathcal{C}_\infty$*, where* μ *denotes the distribution of the uniform infinite tree.*

We call the ensemble $(\mathcal{C}_\infty, \nu)$ the *uniform infinite causal triangulation* (UICT) [43, 62]

Except for the root $r_0 \in \beta(S)$, the vertices in S and $\beta(S)$ are the same and β preserves the distance from r to $v \in S$. It follows that the Hausdorff dimensions of the two ensemble $(\mathcal{T}_\infty, \mu)$ and $(\mathcal{C}_\infty, \nu)$ are identical.

Theorem 1.2.5. *For the uniform infinite causal triangulation we have*

$$\bar{d}_h = 2$$

and

$$d_H = 2 \quad \text{a.s.}$$

Proof. That $\bar{d}_h = 2$ follows from the remarks above and [39], while the second statement follows from the corresponding result for generic trees established in [43]. □

Clearly, there is no canonical bijective correspondence between walks in S and in $\beta(S)$ and hence results on the spectral dimension for the uniform tree cannot be carried over to the UICT. A result by Benjamini and Schramm [16] states that under rather general circumstances a planar random graph is recurrent, which means that the simple random walk starting at r will return to r with probability 1. It is well known that this is the case if and only if $d_s \leq 2$. Since the result of [16] presupposes a fixed upper bound on vertex degrees for the graphs in question it cannot be applied to the UICT. However, it was shown in [43], by combining the so-called Nash-Williams criterion for recurrency of graphs [74] with the known structure of the distribution ν described above, that the UICT is recurrent with probability 1. Thus we have

Theorem 1.2.6. *For the UICT the spectral dimension fulfills $d_s \leq 2$ almost surely.*

It is generally believed that $d_s = 2$ almost surely. A proof of this is still missing. To our knowledge the best known lower bound is

$$d_s \geq \frac{4}{3} \quad \text{a.s.},$$

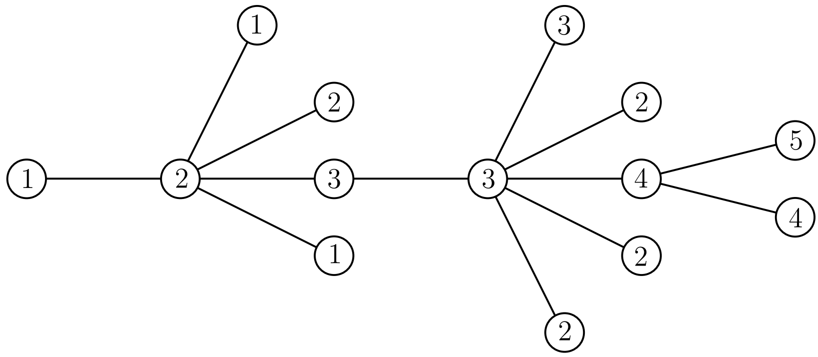

Figure 1.6: A well-labeled tree.

which is obtained by applying the inequality [30]

$$d_s \geq \frac{2d_H}{d_H + 1}$$

to the present situation using Thm.1.2.5.

This finishes our discussion of the UICT. For more details the reader should consult [43].

1.2.4 Labeled trees

Let $\mathcal{T}$ be the set of planar rooted tree with root of order 1. A *labeled tree* is a pair (τ, ℓ), with $\tau \in \mathcal{T}$ and $\ell : V(\tau) \to \mathbb{Z}$ is a mapping from the vertex set $V(\tau)$ to the integers $\mathbb{Z}$, i.e. to each vertex v_i of the tree τ it associates an integer ℓ_i, such that

$$|\ell(v) - \ell(w)| \leq 1 \qquad \text{if } (v, w) \in E(\tau). \tag{1.45}$$

If we further assume the label of the root r to be $\ell(r) = k$ and $\ell(v) \geq 1$ for any vertex $v \in V(\tau)$, we call (τ, ℓ) a *k-labeled tree*. 1-labeled trees are often called *well-labeled trees* [26, 28], see Fig. 1.6.

We denote by $W^{(k)}$ the set of k-labeled trees. It can be decomposed, according to the decomposition (1.18), as

$$W^{(k)} = \left(\bigcup_{N=1}^{\infty} W_N^{(k)} \right) \cup W_\infty^{(k)}, \tag{1.46}$$

with a similar notation as above.

As in the previous example, the weight $w(\tau)$ associated to each k-labeled tree is set equal to 1. Thus, the uniform probability measure on the set $W_N^{(k)}$ of k-labeled trees of size N is given by

$$\mu_N^{(k)} = \frac{1}{D_N^{(k)}}, \quad \text{for } \tau \in W_N^{(k)}, \qquad \mu_N^{(k)} \left(W^{(k)} \setminus W_N^{(k)} \right) = 0, \tag{1.47}$$

where $D_N^{(k)} = \sharp W_N^{(k)}$.

A closed expression for the number $D_N^{(k)}$ of k-labeled trees with N links is not known for $k \geq 2$, whereas the corresponding generating function $R^{(k)}(x)$ has been proved to be a solution of the recursion relation

$$R^{(k)} = 1 + x\, R^{(k)} \left(R^{(k-1)} + R^{(k)} + R^{(k+1)} \right), \quad k \geq 1, \tag{1.48}$$

with the convention $R^{(0)} = 0$. This relation can be obtained by decomposing a tree according to the degree and the label of the root's neighbor (see [25] for details). The solution of eq. (1.48) has been given in a closed form in [20]. However the general solution contains much more information than is needed to evaluate the limit $\mu_N^{(k)} \xrightarrow{N\to\infty} \mu^{(k)}$. In fact, the asymptotic behaviour of $D_N^{(k)}$ for large N turns out to be sufficient to achieve this result.

The number of well-labeled trees with N edges can be exactly computed and it is found to be

$$D_N^{(1)} = 2 \cdot 3^N \frac{(2N)!}{N!(N+2)!}. \tag{1.49}$$

This result can be obtained exploiting a bijection between well-labeled planar rooted trees and planar quadrangulations of the 2-sphere with a marked edge [26, 28]. Indeed, it is well-known (see e.g. [69]) that there exists a natural bijection between planar rooted quadrangulations with N faces and rooted planar maps with N edges. The enumeration of the latter was discussed by Tutte in his seminal paper [86], who provided the result mentioned above.

Using (1.49) and the relation (1.48) together with some generating function methods (which we will discuss in Ch. 2), it can be proved that

$$D_N^{(k)} \simeq \frac{2d_k}{\sqrt{\pi}} N^{-\frac{5}{2}} 12^N \quad \text{as } N \to \infty, \tag{1.50}$$

where, for each k, d_k is a positive constant, whose expression we omit (see [25] for details).

As in the previous section, the existence of the limiting measure $\mu^{(k)}$ can be proved [25].

Theorem 1.2.7. *The sequence $\mu_N^{(k)}$, $N \in \mathbb{N}$, converges to a Borel probability measure $\mu^{(k)}$, concentrated on $W_\infty^{(k)}$.*

The uniform probability measure $\mu^{(k)}$ on $W_\infty^{(k)}$ defines the so called *uniform infinite k-labeled tree*.

Also in this case, $\mu^{(k)}$ has a special factorized form, analogous (1.23), which leads to similar conclusions as before. Indeed, one finds that the measure is supported on infinite trees with only one infinite path originating at the root, and the finite branches attached to the spine vertices are independently distributed.

1.2.5 Uniform Infinite Planar Quadrangulation

A *planar quadrangulation* is a connected planar graph whose faces are quadrangles, that is bounded by polygons with four edges. We denote by $\mathcal{Q}$ the set of all quadrangulations.

The main result that we want to record in this section is the existence of a probability measure on the set of infinite quadrangulations of the plane. This is done via a map Q, constructed in [25], between the set of infinite well-labeled trees and the set of planar quadrangulations, which allows to transport the measure obtained in the previous section from the former set to the latter.

The story behind the construction of the map Q goes back to Tutte who showed in [86], as mentioned above, that the number a_N of rooted planar graphs with N edges is equal to

$$a_N = 2 \cdot 3^N \frac{(2N)!}{N!(N+2)!} \tag{1.51}$$

(cf. eq. (1.49)). As already mentioned this is also the number of rooted planar quadrangulation with N faces.

The method developed by Tutte, called *quadratic method*, provided a powerful tool to solve the equations for the generating functions of planar graphs.

Later, Cori and Vauquelin, gave in [28] an explanation of the formula (1.51) exploiting a bijection between rooted planar graphs and well-labeled rooted trees. Their approach was enhanced by Schaeffer in his thesis [80] (see also [26]), who constructed a bijection, called *CVS bijection*, between finite well-labeled rooted trees and quadrangulations, with the following properties:

i) The vertices in a finite well labeled tree and the vertices in the corresponding quadrangulation, except one marked vertex, can be canonically identified.

ii) The label of a vertex in a well labeled tree equals the distance in the corresponding quadrangulation from the vertex to the marked vertex.

Finally, this result has been extended in [25] to a map Q from infinite well-labeled trees to infinite quadrangulations, sharing with the CVS bijec-

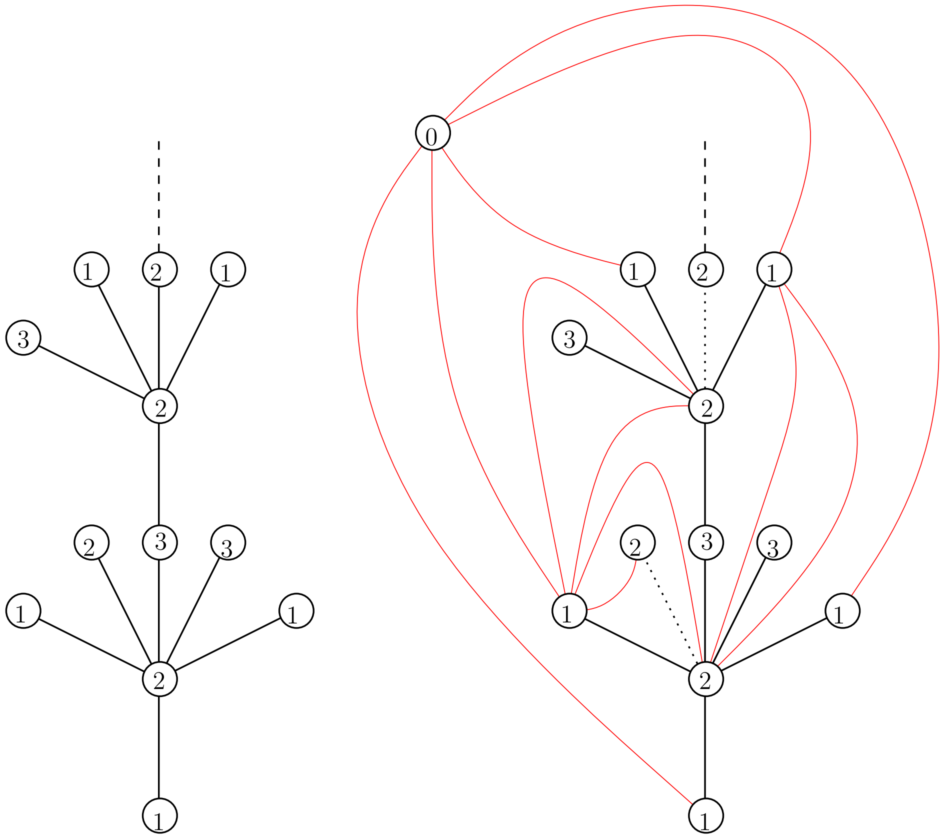

Figure 1.7: An infinite well-labeled tree τ (on the left) and the corresponding quadrangulation $Q(\tau)$ (on the right). The marked vertex has label 0.

tion the properties listed above, see Fig. 1.7. However, the extended map has not been proved to be a bijection, namely it is not known whether it is possible to reconstruct a tree from any infinite quadrangulation.

Let us define the set of infinite well labeled trees with exactly one spine and with each label occurring finitely many times,

$$\mathcal{S} = \left\{ \tau \in W_\infty^{(1)} \mid \forall k \geq 1,\ N_k(\tau) < +\infty \right\}, \tag{1.52}$$

where N_k is the number of occurrences of the label k. This set has been proved to have $\mu^{(1)}$-measure 1 (cf. [25]). The map Q allows to transport the limiting measure $\mu^{(1)}$ from $\mathcal{S}$ to the set of infinite quadrangulations. Thus we define the probability distribution $\bar{\mu}$ on this set as

$$\bar{\mu}(A) = \mu^{(1)}\left(Q^{-1}(A)\right), \tag{1.53}$$

for any subset $A \subseteq Q(\mathcal{S})$, such that $Q^{-1}(A)$ is measurable in $\mathcal{S}$.

The random object defined by the probability measure $\bar{\mu}$ is called the *uniform infinite planar quadrangulation* (UIPQ).

We record a further result about the Hausdorff dimension of the UIPQ. It can be shown that the number N_k of occurrence of the label k fulfills

$$\langle N_k \rangle_{\mu^{(1)}} \sim k^3, \tag{1.54}$$

provided it is assumed to be finite.

As a consequence of the property *ii)* of the map Q, the average value $\langle N_k \rangle_{\mu^{(1)}}$ represents the average value of the number of vertices at distance k from the marked vertex. It follows that the volume of the ball of radius k grows as k^4. Hence, according to (1.12), the annealed Hausdorff dimension of the UIPQ is $\bar{d}_H = 4$.

Remark 1.2.8. *At the time of the appearance of this result, Krikun [61] proved the following theorem:*

Theorem 1.2.9. *The sequence (ν_N) of uniform probability measures on quadrangulations with N faces converges weakly to a probability measure ν with support on infinite quadrangulations.*

Although the random object defined by the probability distribution ν is a priori *different from the infinite quadrangulation obtained by Chassaing and Durhuus in [25], they have later been proved to be equivalent by Ménard in [68].*

Chapter 2

Generating functions methods

In this chapter we collect a few results about generating functions which will be used throughout the thesis. In particular, in Sec. 2.1 we give the basic notions about generating functions and recall the Lagrange inversion formula. As an application, in Sec. 2.1.3 we give an explicit example, namely the enumeration of rooted planar trees.

In the second part of the chapter, Sec. 2.2, we examine the case when generating functions are also analytic functions in some complex domain. In this situation, many information can be obtained by studying the function near its singularity points, via a so-called singularity analysis process, Sec. 2.2.1.

Finally, in Sec. 2.2.2 this analysis scheme is applied to the solution of a certain type of functional equations, which is frequent in combinatorial problems.

For further details about the topics presented in this chapter we refer the reader to [38, 48, 88].

2.1 An introduction to generating functions

2.1.1 Basic notions

Given a sequence a_n, $n \in \mathbb{N}_0$, of elements in a commutative ring R (in this thesis usually $R = \mathbb{Q}$ or $\mathbb{R}$), the *generating function* $A(x)$ of a_n is the formal power series[1]

$$A(x) = \sum_{n=0}^{\infty} a_n x^n. \tag{2.1}$$

By now, x is just a formal indeterminate, that is we look at power series as purely algebraic objects, which we will manipulate according to some set of rules. In particular, in this section we will not be interested in the analytic properties of the power series, for instance we will not require the series to be convergent. For example we can define the generating function for the number of permutations

$$P(x) = \sum_{n=0}^{\infty} n! x^n = 1 + x + 2x^2 + 6x^3 + \cdots \tag{2.2}$$

even though the series' radius of convergence is 0.

However, later in this chapter, we will look at a generating function as representing an analytic function for $|x| < \rho$, with

$$\rho = \left(\limsup_{n \to \infty} |a_n|^{\frac{1}{n}} \right)^{-1} \tag{2.3}$$

denoting its radius of convergence.

We use the notation

$$[x^n] \, A(x) = a_n \tag{2.4}$$

to extract the coefficient of x^n from the generating function $A(x)$.

[1] In literature generating functions defined in this way are often called *ordinary generating functions* to distinguish them from *exponential generating functions*. We will not need this distinction in the following.

We recall that, as formal power series, generating functions can be manipulated algebraically. Let (a_n), (b_n) be two sequences and $A(x)$, $B(x)$ the corresponding generating functions. We have the following operations.

$$\text{Addition} \qquad A(x) \pm B(x) = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n, \tag{2.5}$$

$$\text{Multiplication} \qquad A(x)B(x) = \sum_{n=0}^{\infty} c_n x^n, \quad \text{with } c_n = \sum_{k=0}^{n} a_k b_{n-k} \tag{2.6}$$

With these two operations, the set of all formal power series becomes a commutative ring, usually denoted by $R[[x]]$. Actually a notion of distance between formal series can also be defined and $R[[x]]$ can be proved to be a complete metric space (see e.g. [48] for further details).

Using the multiplication rule we can define the *reciprocal* of a formal power series $A(x)$ as the formal series $B(x)$, such that $A(x)B(x) = 1$. We have the following result.

Proposition 2.1.1. *A formal power series $A(x) = \sum_{n=0}^{\infty} a_n x^n$ has a reciprocal if and only if $a_0 \neq 0$. In that case it is unique.*

Proof. If the reciprocal $B(x) = \sum_{n=0}^{\infty} b_n x^n$ exists, we have that

$$A(x)B(x) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k b_{n-k} x^n = 1, \tag{2.7}$$

which gives

$$a_0 b_0 = 1 \tag{2.8}$$

$$\sum_{k=0}^{n} a_k b_{n-k} = 0, \qquad n \geq 1. \tag{2.9}$$

The first equation tells us that $a_0 \neq 0$, the second one that

$$b_n = -\frac{1}{a_0} \sum_{k=1}^{n} a_k b_{n-k} \qquad n \geq 1 \tag{2.10}$$

are uniquely determined by induction.

Assuming $a_0 \neq 0$ the equations (2.8), (2.9) determine the b_n's. Hence the reciprocal is uniquely determined by $B(x) = \sum_{n=0}^{\infty} b_n x^n$. □

For a formal power series $A(x)$, we denote its reciprocal by $1/A(x)$, provided it exists.

The *formal derivative* $\mathcal{D}$ of a formal power series $A(x) = \sum_{n=0}^{\infty} a_n x^n$ is defined as follows

$$\mathcal{D}A(x) = A'(x) = \sum_{n=1}^{\infty} n\, a_n x^{n-1} \tag{2.11}$$

We denote by $\mathcal{D}^k$ the kth formal derivative.

Performing operations on the coefficients of a generating function defines operations on the generating function itself. Here we list the basic relations between coefficients and corresponding generating functions. For $A(x) = \sum_{n=0}^{\infty} a_n x^n$ we have

	Coefficients	Generating function
Partial sum	$\sum_{k=0}^{n} a_k$	$\dfrac{A(x)}{1-x}$
Multiplication by $P(n)$	$P(n)a_n$	$P(x\mathcal{D})A(x)$
Scaling	$l^n a_n$	$A(lx)$
Shift by $h \in \mathbb{N}$	a_{n+h}	$\dfrac{A(x) - \sum_{k=0}^{h-1} a_k x^k}{x^h}$

Here by $P(n)$ we mean a polynomial in n and $P(x\mathcal{D})$ denotes the corresponding operator, obtained by substituting n with $x\mathcal{D}$; for instance, for a generic polynomial $P_l(n)$ of degree l we have

$$P_l(n) = \sum_{k=0}^{l} p_k n^k \qquad \longleftrightarrow \qquad P_l(x\mathcal{D}) = \sum_{k=0}^{l} p_k (x\mathcal{D})^k. \tag{2.12}$$

For our purposes, it is worth noting that the multiplication rule for two formal power series can be extended to an arbitrary number of them, as follows. Let $(a_n^{(1)}), \ldots, (a_n^{(k)})$, be k sequences and $A^{(1)}, \ldots, A^{(k)}$ the corresponding generating functions. We then have

$$A^{(1)}(x) \ldots A^{(k)}(x) = \sum_{n=0}^{\infty} \sum_{|j|=n} a_{j_1}^{(1)} \ldots a_{j_k}^{(k)} x^n, \tag{2.13}$$

where the summation is over $j = (j_1, \ldots, j_k) \in \mathbb{N}_0^k$ and $|j| = j_1 + \ldots + j_k = n$.

The rules listed above can be obtained by simple application of the addition and multiplication rules mentioned before.

The *composition* $A(B(x))$ of the formal power series $A(x)$ and $B(x)$, with $b_0 = 0$, is defined as

$$A(B(x)) = \sum_{n=0}^{\infty} c_n x^n, \tag{2.14}$$

where the coefficients c_n's are given by

$$c_0 = a_0 \tag{2.15}$$

$$c_n = \sum_{\substack{1 \le k \le n \\ |j|=n}} a_k \prod_{i=1}^{k} b_{j_i}, \qquad n \ge 1 \tag{2.16}$$

with $j = (j_1, \ldots, j_k) \in \mathbb{N}^k$ and $|j| = j_1 + \ldots + j_k$. Note that the coefficients c_n's are given by formally substituting $B(x)$ for x in $A(x) = \sum_{n=0}^{\infty} a_n x^n$. It needs to be stressed that, as formal power series, the composition is valid only if $b_0 = 0$ or if $A(x)$ is a polynomial. Suppose that we want to calculate the m-th coefficient of the composition. We note that if $b_0 = 0$, for any $k > m$, the term $a_k B(x)^k$ contains only powers of x higher than m, therefore only the terms $\sum_{n=0}^{m} a_n B(x)^n$ will contribute to the coefficient c_m. If $b_0 \neq 0$, this would not be true and an infinite number of terms would contribute to c_m, making the composition rule ill-defined.

For instance, consider the formal power series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \tag{2.17}$$

According to the above definition, the composition e^{e^x-1} is a well-defined formal series, whereas e^{e^x} is not.

Given the formal power series $A(x)$ associated to the sequence (a_n), with $a_0 = 0$ and $a_1 \neq 0$, we define the *inverse* power series $A^{-1}(x) = \sum_{n=1}^{\infty} \tilde{a}_n x^n$ by

$$A(A^{-1}(x)) = A^{-1}(A(x)) = x. \tag{2.18}$$

Indeed, using the composition rule given above, it follows from (2.18) that the coefficients $\tilde{a}_n$'s are recursively determined by

$$a_1\tilde{a}_1 = 1, \tag{2.19}$$

$$a_1\tilde{a}_n + \sum_{\substack{2\le k\le n \\ |j|=n}} a_k \prod_{i=1}^{k} \tilde{a}_{j_i} = 0, \qquad n \geq 2. \tag{2.20}$$

The Lagrange inversion formula provides a useful tool to find the explicit representation of coefficients of inverse power series.

2.1.2 Lagrange Inversion Formula

Theorem 2.1.2. *Let a_n, $n \in \mathbb{N}_0$ be a sequence such that $a_0 = 0$ and $a_1 \neq 0$. Let $A(x)$ be the corresponding formal power series, $A^{[-1]}(x)$ its inverse and $H(x)$ an arbitrary formal series. Then the coefficients of $H(A^{-1}(x))$ are given by*

$$[x^n]\, H(A^{[-1]}(x)) = \frac{1}{n}\left[y^{n-1}\right] H'(y)\left(\frac{y}{A(y)}\right)^n \qquad n \geq 1 \tag{2.21}$$

Note that, choosing a power series $\varphi(x) = \sum_{n\geq 0} \varphi_n x^n$, with $\varphi_0 \neq 0$, and setting $A(x) = x/\varphi(x)$, then the inverse series $A^{-1}(x)$ satisfies the equation

$$A^{-]}(x) = x\varphi(A^{-1}(x)). \tag{2.22}$$

We can use this fact to reformulate the theorem in the following way.

Theorem 2.1.3. *Let $\varphi(x) = \sum_{n\geq 0} \varphi_n x^n$ be a formal power series with $\varphi_0 \neq 0$. Then the equation*

$$L(x) = x\varphi(L(x)), \tag{2.23}$$

admits a unique solution $L(x) = \sum_{n\geq 1} l_n x^n$ and for an arbitrary formal series $H(x) = \sum_{n\geq 0} h_n x^n$ the coefficients of $H(L(x))$ are given by

$$[x^n]\, H(L(x)) = \frac{1}{n}\left[y^{n-1}\right] H'(y)\varphi(y)^n. \tag{2.24}$$

Proof. First we note that eq. (2.23) provides a system of polynomial equations for $\{l_n\}$ which determines the unique solution $L(x)$, that is

$$l_1 = \varphi_0, \qquad l_2 = \varphi_0\varphi_1, \qquad l_3 = \varphi_0\varphi_1^2 + \varphi_0^2\varphi_2, \ldots \tag{2.25}$$

Hence, since for fixed n, l_n depends only on the coefficients of φ up to order n, we may assume that φ is a polynomial, without loss of generality.

Further we assume that, for fixed n, $H(x)$ is a polynomial of degree n. If the result is true for this polynomial it will remain true for any formal power series, since the coefficients h_k, for $k > n$, do not contribute to the formula (2.24).

Since $\varphi(0) = \varphi_0 \neq 0$, $\varphi(y)$ stays non-zero in some neighborhood of 0 and function $y/\varphi(y)$ is analytic there. Hence, it follows from eq. (2.23) that $L(x)$ is an analytic function in a neighborhood of $x = 0$. Let C denote a small circle centered at 0 and contained in this neighborhood. Therefore, setting $y = L(x)$ we have, from the analyticity of L, that $C' = L(C)$ is still a small circle around 0. Hence, we have

$$\begin{aligned}\left[y^{n-1}\right] H'(y)\varphi(y)^n &= \frac{1}{2\pi i}\int_C \frac{H'(y)\,\varphi(y)^n}{y^n}\,\mathrm{d}y \\ &= \frac{1}{2\pi i}\int_{C'} \frac{H'(L(x))\varphi(L(x))^n L'(x)}{L(x)^n}\,dx \\ &= \frac{1}{2\pi i}\int_{C'} \frac{H'(L(x))L'(x)}{x^n}\,dx \\ &= [x^n]\, x\frac{\mathrm{d}}{\mathrm{d}x}H(L(x)) \\ &= n\,[x^n]\,H(L(x)).\end{aligned} \tag{2.26}$$

The first equality is the Cauchy formula, the second is the change of variable $y = L(x)$, the third comes from (2.23). The rest follows easily. □

In the next section, we will see how to apply the formula to an enumeration problem.

2.1.3 Enumeration of Rooted Planar Trees

In section 1.2.1 we claimed that the number C_N of rooted planar trees of size N with root of degree 1 equals the $(N-1)$-th Catalan number. We prove it using a standard generating function argument.

Theorem 2.1.4. *The number C_N of rooted planar trees of size N with root of degree 1 is given by*

$$C_N = \frac{(2N-2)!}{N!(N-1)!}. \tag{2.27}$$

Proof. Let $C(x)$ be the generating function for the C_N's, that is

$$C(x) = \sum_{N=1}^{\infty} C_N x^{N-1}. \tag{2.28}$$

Decomposing the tree according to the degree of the root's neighbor

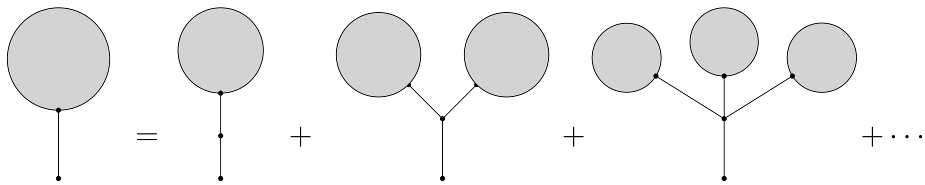

Figure 2.1: Tree decomposition

(see Fig. 2.1), we have the recurrence relation:

$$C_{N+1} = C_N + \sum_{N_1+N_2=N} C_{N_1} C_{N_2} + \sum_{N_1+N_2+N_3=N} C_{N_1} C_{N_2} C_{N_3} + \cdots , \qquad (2.29)$$

which, according to the rules given in Sec. 2.1.1, translates into the following functional equation for the generating function $C(x)$

$$C(x) - 1 = \sum_{k=1}^{\infty} (xC(x))^k = \frac{xC(x)}{1 - xC(x)}. \qquad (2.30)$$

We can apply the Lagrange inversion formula to find the coefficients C_N. First define $\tilde{C}(x) = xC(x)$, such that $\tilde{c}_N = C_N$, for $N \geq 1$. We have that $\tilde{C}(x)$ satisfies the equation

$$\tilde{C}(x) = x\varphi(\tilde{C}(x)), \qquad (2.31)$$

with $\varphi(y) = (1-y)^{-1}$. Hence, using the formula (2.24) with $H(x) = x$, we find

$$C_N = \tilde{c}_N = \frac{1}{N} \left[y^{N-1}\right] \frac{1}{(1-y)^N} = \frac{1}{N} \binom{2N-2}{N-1} = \frac{(2N-2)!}{N!(N-1)!}, \qquad (2.32)$$

where we used that

$$\frac{1}{(1-y)^N} = \sum_{k=0}^{\infty} \binom{k+N-1}{k} y^k. \qquad (2.33)$$

We get the same result noting that the solution of eq. (2.30), hence the generating function $C(x)$, is actually an analytic function, namely

$$C(x) = \frac{1 - \sqrt{1-4x}}{2x}, \tag{2.34}$$

which is analytic in the disc $|x| < \frac{1}{4}$.

Expanding the square root around $x = 0$ using Newton's generalized binomial formula

$$\sqrt{1+y} = \sum_{k=0}^{\infty} \binom{2k}{k} \frac{(-1)^{k+1}}{4^k(2k-1)} y^k, \tag{2.35}$$

we get

$$C(x) = \frac{1}{2} \sum_{k=1}^{\infty} \binom{2k}{k} \frac{x^{k-1}}{2k-1} \tag{2.36}$$

and finally

$$C_N = \binom{2N}{N} \frac{1}{2(2N-1)} = \frac{(2N-2)!}{N!(N-1)!}. \tag{2.37}$$

□

It is easy to see that the number D_N of rooted planar trees with N edges is related to C_N by

$$D_N = C_{N+1}, \qquad N \geq 0, \tag{2.38}$$

where $D_0 = 1$ refers to the tree with only one vertex. Thus we have the following corollary.

Corollary 2.1.5. *The number D_N of rooted planar trees with N edges is given by*

$$D_N = \frac{(2N)!}{(N+1)!N!}. \tag{2.39}$$

The asymptotic behaviour of C_N for large N can be computed using

Stirling's approximation formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n. \tag{2.40}$$

Thus we find

$$C_N \sim \frac{1}{\sqrt{\pi}} N^{-\frac{3}{2}} 4^N. \tag{2.41}$$

Note that we have a square root singularity in the generating function (2.34) and in the coefficients of its expansion a behaviour like $N^{-\frac{3}{2}} x_0^{-N}$, for N large, x_0 being the singularity point ($\frac{1}{4}$ in this case). This is exactly the same correspondence we found in Sec. 1.2.2, discussing the finite volume partition functions of the generic trees. As will be seen in the following, the relation between the behaviour of analytic functions near their singularity points and the asymptotic behaviour of their Taylor coefficients can be studied in a systematic way, called *singularity analysis*.

2.2 Asymptotic Analysis

In this section we consider generating functions as analytic functions. As pointed out at the beginning of section 2.1.1, a generating function $A(x) = \sum_{n\geq 0} a_n x^n$ represents an analytic function whenever the power series has a radius of convergence ρ, defined by (2.3), greater than 0. In this case $A(x)$ is analytic in the disc $D = \{x \mid |x| < \rho\}$ and we can apply the Cauchy's formula to recover the coefficients (a_n) of the series, i.e.

$$a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z^{n+1}}\, dz, \qquad n \geq 0, \tag{2.42}$$

where γ is a contour in D around 0 with winding number 1.

The first part of the section deals with the basics of the so-called *singularity analysis*, a technique developed by Flajolet and Odlyzko in [47]. It consists of a set of theorems that allow to *transfer* the singular behaviour of certain functions to the asymptotic behaviour of their Taylor coefficients, which basically relies on a systematic use of the Cauchy's formula.

At the end of the section we will see how to apply this analysis to the study of certain types of functional equations which are frequent in enumeration problems.

2.2.1 Singularity Analysis

In its original formulation, the singularity analysis process is more general than the version described in this section. On the other hand, all the cases considered in this thesis can be analyzed within the scheme described below. For a complete study we refer the reader to [48], Ch. VI.

The main result, stated in Thm. 2.2.4, relies essentially on two lemmas. The first one gives the asympotitc behaviour of the Taylor coefficients of the function $(1-z)^{-\alpha}$,

Lemma 2.2.1. *Given the function*

$$f(z) = (1-z)^{-\alpha}, \tag{2.43}$$

with $\alpha \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, the coefficient of z^n in $f(z)$, for large n, is given by

$$[z^n]\, f(z) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)} \left(1 + \sum_{k=1}^{\infty} \frac{p_k}{n^k}\right), \tag{2.44}$$

where p_k is a polynomial in α that is divisible by $\alpha(\alpha-1)\cdots(\alpha-k)$. In particular

$$[z^n]\, f(z) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)} \left(1 + O\left(\frac{1}{n}\right)\right), \tag{2.45}$$

as $n \to \infty$.

Sketch of proof. The proof of this result relies, basically, on the Cauchy formula

$$[z^n]\, f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{(1-z)^{-\alpha}}{z^{n+1}}\, dz \tag{2.46}$$

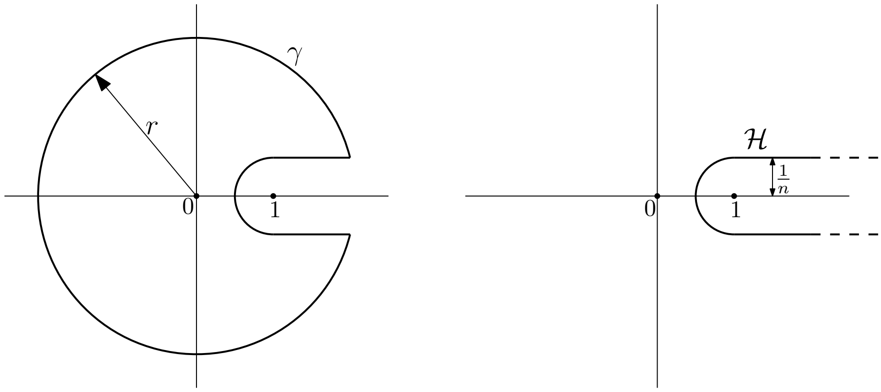

Figure 2.2: Contour of integration used in Lemma 2.2.1 and a Hankel contour.

where the contour γ is suitably chosen. In particular, consider γ as given in Fig. 2.2, where the external arc is assumed to have radius r. The integral along this arc is easily seen to vanish for $r \to \infty$. The remaining path $\mathcal{H}$, depicted in Fig. 2.2, is called a *Hankel contour*. It is chosen to pass at distance $\frac{1}{n}$ from the real half-line $\mathbb{R}_{\geq 1}$

Performing a change of variable

$$z = 1 + \frac{t}{n} \tag{2.47}$$

in eq. (2.46) gives

$$[z^n]\, f(z) = \frac{n^{\alpha-1}}{2\pi i} \int_\gamma (-t)^{-\alpha} \left(1 + \frac{t}{n}\right)^{-n-1} dt. \tag{2.48}$$

Formally, the result can be obtained by noting that

$$\left(1 + \frac{t}{n}\right)^{-n-1} = e^{-t} \left(1 + O\left(\frac{1}{n}\right)\right). \tag{2.49}$$

Then, one substitutes this expression in (2.48) and uses the Hankel's integral representation of the gamma function (see e.g. [53]):

$$\frac{1}{\Gamma(\alpha)} = -\frac{1}{2\pi i} \int_{\mathcal{H}} (-t)^{-\alpha} e^{-t} \, dt. \tag{2.50}$$

to get the desired result.

A justification of this formal argument, as well as details of the proof, can be found in [48]. □

In order to apply the singularity analysis scheme to a function $f(z)$, the essential condition that we need is the analyticity of $f(z)$ in a so-called Δ-*domain*. It is defined as follows.

Given two numbers ϑ, R with $R > 1$ and $0 < \vartheta < \frac{\pi}{2}$, the open domain $\Delta(\theta, R)$ is defined as (see Fig. 2.3)

$$\Delta(\theta, R) = \left\{ z \;\middle|\; |z| < R, z \neq 1, |\arg(z-1)| > \theta \right\}. \tag{2.51}$$

A domain is a Δ-domain if it is of the form $\Delta(\theta, R)$ for some R and ϑ. For a complex number z_0, we denote by $z_0\Delta$ the image by the mapping $z \mapsto z_0 z$ of a Δ-domain. We say that a function f is Δ-analytic, if f is analytic in a Δ-domain.

Lemma 2.2.2 (Error terms transfer)**.** *Let $a \in \mathbb{R}$ be an arbitrary real number and $f(z)$ a Δ-analytic function such that*

$$f(z) = O\left((1-z)^{-a}\right), \qquad \text{as } z \to 1,\ z \in \Delta, \tag{2.52}$$

then

$$[z^n]\, f(z) = O\left(n^{a-1}\right). \tag{2.53}$$

The same is true replacing $O(\cdot)$ with $o(\cdot)$.

Sketch of proof. We use the Cauchy formula

$$[z^n]\, f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z^{n+1}} \, dz \tag{2.54}$$

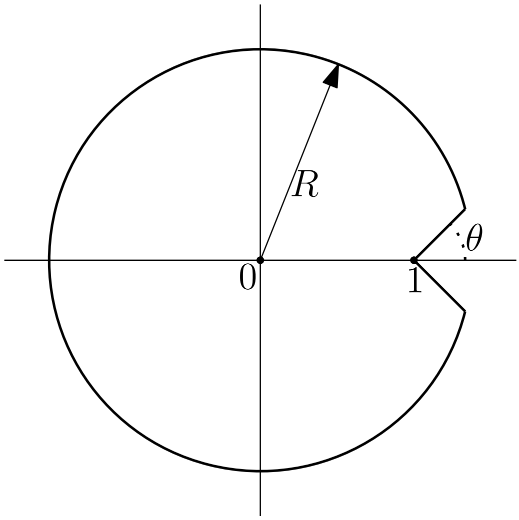

Figure 2.3: Δ-domain $\Delta(\theta, R)$

with path of integration $\gamma = \bigcup_{i=1}^{4} \gamma_i$ given by (see Fig. 2.4)

$$\gamma_1 = \left\{ z \mid |z-1| = \frac{1}{n}, \; |\arg(z-1)| \geq \phi \right\}, \tag{2.55}$$

$$\gamma_2 = \left\{ z \mid |z-1| \geq \frac{1}{n}, \; |z| \leq r, \; \arg(z-1) = \phi \right\}, \tag{2.56}$$

$$\gamma_3 = \left\{ z \mid |z| = r, \; |\arg(z-1)| \geq \phi \right\}, \tag{2.57}$$

$$\gamma_4 = \left\{ z \mid |z-1| \geq \frac{1}{n}, \; |z| \leq r, \; \arg(z-1) = -\phi \right\}. \tag{2.58}$$

If the Δ-domain is $\Delta(\theta, R)$, r and ϕ have to be chosen in such a way that $\gamma \in \Delta$, that is $1 < r < R$ and $\theta < \phi < \frac{\pi}{2}$. The result is then obtained using the bound $|f(z)| \leq C|1-z|^{-\alpha}$, for some positive constant C, to bound the absolute value of (2.54).

Details of the proof can be found in [48]. □

From these two lemmas a corollary immediately follows.

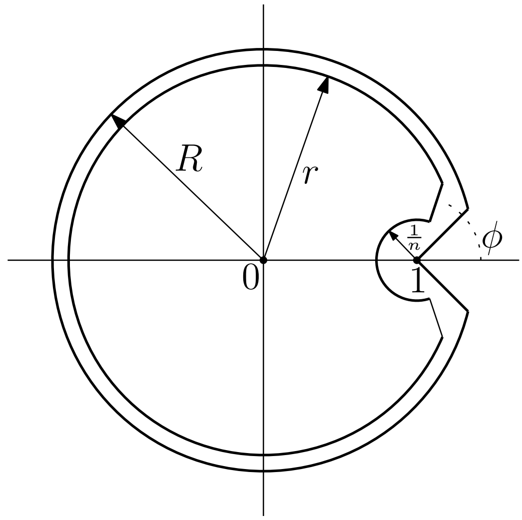

Figure 2.4: Delta domain and integration contour

Corollary 2.2.3. *Assume that $f(z)$ is Δ-analytic and*

$$f(z) \sim (1-z)^{-\alpha}, \qquad z \to 1,\ z \in \Delta \tag{2.59}$$

with $\alpha \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$. Then

$$[z^n]\, f(z) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}. \tag{2.60}$$

Proof. We have that $f(z) \sim (1-z)^{-\alpha}$ if and only if $f(z) = (1-z)^{-\alpha} + o\left((1-z)^{-\alpha}\right)$, then the result follows applying Lemma 2.2.1 to the first term and Lemma 2.2.2 to the error term. □

We are now ready to state the main result of this section.

Theorem 2.2.4 (Singularity analysis)**.** *Let $f(z)$ be a function with a singularity at z_0 and analytic in a Δ-domain $\Delta_0 = z_0\Delta$. Assume that, for some constant C,*

$f(z)$ admits an expansion of the form

$$f(z) = C\left(1 - \frac{z}{z_0}\right)^{-\alpha} + O\left(\left(1 - \frac{z}{z_0}\right)^{-a}\right), \tag{2.61}$$

as $z \to z_0$, $z \in \Delta_0$, with $a < \mathrm{Re}\,(\alpha)$, $\alpha \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$. Then

$$[z^n]\, f(z) = C\,\frac{n^{\alpha-1}}{\Gamma(\alpha)} z_0^{-n} + O\left(z_0^{-n} n^{\max\{\mathrm{Re}\,(\alpha)-2, a-1\}}\right) \tag{2.62}$$

Proof. The theorem is an immediate consequence of lemmas 2.2.1 and 2.2.2, once we note that $g(z) = f(z_0 z)$ is singular at 1, and by the scaling rule of the Taylor expansion

$$[z^n]\, f(z) = z_0^{-n}\, [z^n]\, f(z_0 z) = z_0^{-n}\, [z^n]\, g(z). \tag{2.63}$$

□

Remark 2.2.5. *As mentioned above, the singularity analysis process has a broader range of application than the functions described in Thm. 2.2.4. In particular, defining*

$$\mathcal{S} = \left\{(1-z)^{-\alpha}\lambda(z)^{\beta} \mid \alpha, \beta \in \mathbb{C}\right\}, \qquad \lambda(z) = \frac{1}{z}\log\frac{1}{1-z}, \tag{2.64}$$

the transfer theorems can be proved for any function $f(z)$ such that

$$f(z) = \sigma(z/z_0) + O\left(\tau(z/z_0)\right) \qquad \text{as } z \to z_0,\ z \in z_0\Delta, \tag{2.65}$$

where $\sigma(z)$ is a finite linear combination of functions in $\mathcal{S}$ and $\tau(z) \in \mathcal{S}$.

Further, the theorem can be extended to functions that have finitely many singularities on the boundary of their disc of convergence. Roughly speaking, in this case the asymptotic behaviour of the coefficients is given by adding up the contributions from each singularity, obtained by the basic singularity analysis process.

The interested reader should consult [48], where these cases are studied in detail.

2.2.2 Functional equations

Generating functions associated to graphs are often seen to be solutions of functional equations of the type $y = F(x, y)$. These equations arise naturally in combinatorics, e.g. enumeration problems, since they mirror the recursive nature of graphs.

As we will see in the following, even if an explicit solution of the functional equation is not known, under some rather general circumstances, it is possible to know the behaviour of the solution near its singularities. Early studies in this direction go back to Bender [13], Canfield [24] and Meir and Moon [67].

Single functional equation The next theorem shows that solutions of a functional equation of the type mentioned above usually possess a square-root singularity and can be analytically extended to a Δ-domain, hence they are amenable to the singularity analysis process . We refer the reader to [38] for a proof.

Theorem 2.2.6. *Let $F(x, y)$ be a function satisfying the following conditions.*

$(\mathbf{C_1})$ *$F(x, y)$ is analytic in x, y around $x = y = 0$ and*

$$F(0, y) = 0, \qquad [x^n y^m]\, F(x, y) \geq 0. \tag{2.66}$$

for $n, m \geq 0$.

$(\mathbf{C_2})$ *Within the domain of analyticity of $F(x, y)$ the system*

$$\begin{cases} y = F(x, y) \\ 1 = F_y(x, y) \end{cases} \tag{2.67}$$

admits a positive solution (x_0, y_0)*, with* $F_x(x_0, y_0) \neq 0$ *and* $F_{yy}(x_0, y_0) \neq 0$.

Then there exists a unique analytic solution $y = y(x)$ *of the functional equation*

$$y = F(x, y) \tag{2.68}$$

with $y(0) = 0$, $[x^n]\, y(x) \geq 0$ *and domain of analyticity* $|x| < x_0$.

Moreover, there exist functions $g(x)$ *and* $h(x)$ *that are analytic around* $x = x_0$*, such that* $y(x)$ *near* x_0 *is of the form*

$$y(x) = g(x) - h(x)\sqrt{1 - \frac{x}{x_0}} \tag{2.69}$$

with

$$g(x_0) = y(x_0), \qquad h(x_0) = \sqrt{\frac{2x_0 F_x(x_0, y_0)}{F_{yy}(x_0, y_0)}}. \tag{2.70}$$

The expression (2.69) provides an analytic continuation of $y(x)$ *for* $\arg(x - x_0) \neq 0$.

Furthermore, if there exists an n_0 *such that* $[x^n]\, y(x) > 0$ *for* $n \geq n_0$*, then* $y(x)$ *admits an analytic continuation in a* Δ*-domain* $x_0\Delta$*, and*

$$[x^n]\, y(x) = \sqrt{\frac{x_0 F_x(x_0, y_0)}{2\pi F_{yy}(x_0, y_0)}}\, x_0^{-n} n^{-3/2} \left(1 + O\left(\frac{1}{n}\right)\right). \tag{2.71}$$

The conditions $(\mathbf{C_1})$ ensure that the inversion problem is well defined around 0, that is we have a non trivial unique solution $y(x)$. Non-negativity of the coefficients is naturally satisfied in enumeration problems or, for instance, when they are partition functions, as in the cases studied in Sec. 1.2. However, this condition implies non-negativity of the coefficients of $y(x)$ around 0, therefore if $y(x)$ is analytic at a real positive x', then $y(x)$ is analytic for $|x| \leq x'$, by Pringsheim's Theorem (see e.g. [48], Thm. IV.6).

Note that, setting $F(x, y) = x\varphi(y)$ the theorem reduces to Thm. VI.6

(Singular Inversion) in [48], where the *aperiodicity* condition of φ is ensured in our case by $[x^n]\,y(x) > 0$, for $n > n_0$ (see [48], Definition IV.5 for the notion of periodicity of a generating function). In both cases, the requirement are needed to ensure that x_0 is the only singularity on the circle of convergence of $y(x)$.

Systems of functional equations The result of Thm. 2.2.6 can be generalized to systems of equations of the type (2.68). The assumptions that we will need are similar to the assumptions $(\mathbf{C_1})$, $(\mathbf{C_2})$ of the single equation case. We only need to impose a condition on the dependency graph for the system of functional equations. This is defined as follows.

Let $\mathbf{F}(x,\mathbf{y}) = (F_1(x,\mathbf{y}),\ldots,F_N(x,\mathbf{y}))$ be a vector of functions $F_i(x,\mathbf{y})$, $1 \le i \le N$, of complex variables x, $\mathbf{y} = (y_1,\ldots,y_N)$. The *dependency graph* $G_{\mathbf{F}}(V,E)$ for the system of equations $\mathbf{y} = \mathbf{F}(x,\mathbf{y})$ is a directed graph whose vertex set V is given by $V = \{y_1,\ldots,y_N\}$, whereas an oriented edge (y_i,y_j) is contained in E if and only if $F_i(x,\mathbf{y})$ really depends on y_j.

We say that a dependency graph is *strongly connected* if every pair of vertices is connected by an oriented path, or equivalently, if the jacobian matrix $\mathbf{F}_{\mathbf{y}}(x,\mathbf{y}) = \left(\frac{\partial F_i}{\partial x_j}\right)$ is irreducible (see e.g. [70]).

In the following we denote by $\mathbf{I}$ the $N \times N$ identity matrix and by $\mathbf{A}^t$ the transpose of a matrix $\mathbf{A}$.

Theorem 2.2.7. *Let $\mathbf{F}(x,\mathbf{y}) = (F_1(x,\mathbf{y}),\ldots,F_N(x,\mathbf{y}))$ be a vector of functions satisfying the following conditions.*

$(\mathbf{D_1})$ $\mathbf{F}(x,\mathbf{y})$ *is analytic around $x = 0$, $\mathbf{y} = \mathbf{0}$ and*

$$\mathbf{F}(0,\mathbf{y}) = \mathbf{0}, \qquad \mathbf{F}(x,\mathbf{0}) \neq \mathbf{0}, \quad x \neq 0 \tag{2.72}$$

and with non-negative Taylor coefficients.

($\mathbf{D_2}$) *Within the domain of analyticity of* $\mathbf{F}(x, \mathbf{y})$ *the system*

$$\begin{cases} \mathbf{y} = \mathbf{F}(x, \mathbf{y}) \\ 0 = \det\left(\mathbf{I} - \mathbf{F_y}(x, \mathbf{y})\right) \end{cases} \tag{2.73}$$

admits a positive solutions $(x_0, \mathbf{y}_0)$. *Further, assume that* $\mathbf{F}_x(x_0, \mathbf{y}_0) \neq 0$ *and that the dependency graph* $G_\mathbf{F}$ *is strongly connected.*

Then the system of equations

$$\mathbf{y} = \mathbf{F}(x, \mathbf{y}) \tag{2.74}$$

admits an analytic solution

$$\mathbf{y} = \mathbf{y}(x) = (y_1(x), \ldots, y_N(x)) \tag{2.75}$$

with $\mathbf{y}(0) = 0$ *and radius of convergence* x_0 *(common for all* $y_i(x)$*). Further, for each* $1 \leq i \leq N$, *there exist analytic functions* $g_i(x) \neq 0$, $h_i(x) \neq 0$, *such that* $y_i(x)$ *has a representation*

$$y_i(x) = g_i(x) - h_i(x)\sqrt{1 - \frac{x}{x_0}}, \tag{2.76}$$

locally near x_0 *but* $|\arg(x - x_0)| \neq 0$, *where* $g_i(x_0) = y_i(x_0) = y_0$.

Assuming that there exists an n_1 *such that* $[x^n]\, y_i(x) > 0$ *for* $n \geq n_1$, $1 \leq i \leq N$, *then the* $y_i(x)$*'s admit an analytic continuation in a* Δ*-domain* $x_0\Delta$ *and*

$$[x^n]\, y_i(x) = \frac{|a_i|}{\sqrt{4\pi}}\, x_0^{-n} n^{-\frac{3}{2}} \left(1 + O\left(\frac{1}{n}\right)\right). \tag{2.77}$$

where $\mathbf{a} = (a_1, \ldots, a_N)$, *is a solution of*

$$\left(\mathbf{I} - \mathbf{F_y}(x_0, \mathbf{y}_0)\right)\ \mathbf{a}^t = 0 \tag{2.78}$$

$$\mathbf{a}\ \mathbf{F_{yy}}(x_0, \mathbf{y}_0)\ \mathbf{a}^t = -2\mathbf{F}_x(x_0, \mathbf{y}_0) \tag{2.79}$$

This theorem relies on more general results due to Drmota [37, 38].

Possible remarks about the assumptions are similar to those previously made for the single equation case. We only want to stress that the assumption of strong connectivity is needed to avoid that, informally speaking, a subsystem of (2.74) is solved prior to the whole system of equations.

Infinite systems of functional equations For the sake of completeness, we conclude this chapter recording a result due to Morgenbesser [72], about the singular behaviour of the solutions of infinite systems of functional equations of the type (2.68).

Before we state the next theorem, we give some definitions from functional analysis. All the definitions not explicitly stated here can be found e.g. in [79]. We also refer the reader to [34], Sec. 7.7, for details about differentiability in Banach spaces.

Let B be a Banach space with norm $\|\cdot\|$, a function $F : B \to B$ is called *Fréchet differentiable* at x_0 if there exists a bounded linear operator $(\partial F/\partial x)(x_0)$ on B such that

$$\lim_{h\to 0} \frac{\|F(x_0+h) - F(x_0) - \frac{\partial F}{\partial x}(x_0)h\|}{\|h\|} = 0. \tag{2.80}$$

The operator $(\partial F/\partial x)(x_0)$ is called *Fréchet derivative* of F at x_0. If B is a complex vector space and (2.80) holds for all h, then F is said to be *analytic* at x_0. F is analytic in $V \subseteq B$, if it is analytic for all $x_0 \in V$.

When $B \equiv \ell^p$, where ℓ^p, $1 \le p < \infty$, is the Banach space of all complex valued sequences a_n, $n \in \mathbb{N}$, the Fréchet derivative is also called the *Jacobian operator*. We say that a function $\mathbf{F} : \mathbb{C} \times \ell^p \to \ell^p$ is *positive* in a domain $U \times V \in \mathbb{C} \times \ell^p$ if there exist non-negative real numbers $f_{i\mathbf{j},k}$ such that

$$F_k(x, \mathbf{y}) = \sum_{i,\mathbf{j}} f_{i\mathbf{j},k} x^i \mathbf{y}^{\mathbf{j}}, \quad \text{for all } (x, \mathbf{y}) \in U \times V, \qquad k \ge 1, \tag{2.81}$$

where $\mathbf{j} = (j_1, j_2, \ldots)$, $j_i \in \mathbb{N}$, with only finitely many nonzero components, and $\mathbf{y}^{\mathbf{j}} = y_1^{j_1} y_2^{j_2} \cdots$.

Recalling that any bounded linear operator A on an ℓ^p space is uniquely determined by an infinite dimensional matrix $(A_{ij})_{1\leq i,j<\infty}$, A is called *positive* if $A_{i,j} \geq 0$ for all $i, j \in \mathbf{N}$. A positive operator A is said to be *irreducible* if for every i, j there exist n such that $(A^n)_{ij} > 0$.

Theorem 2.2.8. *Let $1 \leq p < \infty$ and $\mathbf{F} : \mathbb{C} \times \ell^p \to \ell^p$, $(x, \mathbf{y}) \mapsto \mathbf{F}(x, \mathbf{y})$ a function satisfying the following conditions.*

($\mathbf{E_1}$) $\mathbf{F}(x, \mathbf{y})$ *is an analytic and positive function defined in an open neighborhood $U \times V$ of $(0, \mathbf{0})$ and*

$$\mathbf{F}(0, \mathbf{y}) = \mathbf{0} \quad \textit{for all } \mathbf{y} \in V, \qquad \mathbf{F}(x, \mathbf{0}) \neq \mathbf{0} \quad \textit{in } U \setminus \{0\}\,. \tag{2.82}$$

($\mathbf{E_2}$) $\mathbf{F_y}(x, \mathbf{y})$ *is a compact operator on ℓ^p for all $(x, \mathbf{y}) \in U \times V$ and irreducible for strictly positive $(x, \mathbf{y}) \in U \times V$. Further, the system*

$$\begin{cases} \mathbf{y} = \mathbf{F}(x, \mathbf{y}) \\ 1 = R\big(\mathbf{F_y}(x, \mathbf{y})\big)\,, \end{cases} \tag{2.83}$$

admits a positive solution $(x_0, \mathbf{y}_0) \in U \times V$, where $R(\mathbf{F_y}(x, \mathbf{y}))$ denotes the spectral radius of the Jacobian operator.

Then there exists an analytic solution $\mathbf{y}(x)$ of

$$\mathbf{y} = \mathbf{F}(x, \mathbf{y}), \tag{2.84}$$

with $\mathbf{y}(0) = 0$. Further, there exist analytic functions $\mathbf{g}(x)$ and $\mathbf{h}(x)$ such that such that $\mathbf{y}(x)$ has a representation

$$\mathbf{y}(x) = \mathbf{g}(x) - \mathbf{h}(x)\sqrt{1 - \frac{x}{x_0}}, \tag{2.85}$$

locally near x_0 but $|\arg(x - x_0)| \neq 0$.

Moreover, if there exist two integers n_2 and n_3 that are relatively prime such that $[x^{n_2}]\, y_1(x) > 0$ and $[x^{n_3}]\, y_1(x) > 0$, then x_0 is the only singularity of $\mathbf{y}(x)$

on the circle $|x| = x_0$ and we obtain for every $i \geq 1$

$$[x^n]\, y_i(x) \sim c_i\, x_0^{-n}\, n^{-\frac{3}{2}}, \qquad (2.86)$$

where c_i is a positive constant.

Chapter 3

Random graphs as Quantum Gravity models

In this chapter we give an account of two dimensional quantum gravity models described in terms of random graphs.

After a brief introduction to the path integral formalism in Sec. 3.1, we review the path-integral formulation of quantum gravity in Sec. 3.2. In Sec. 3.3 we restrict the discussion to the two dimensional case. In particular, we describe in Sec. 3.3.1 the discretization procedure known as Regge calculus, which will be the applied to the definition of the dynamical triangulation model in Sec 3.3.2, and the causal dynamical triangulation model in Sec. 3.3.3.

Most of the basic notions from quantum field theory and general relativity are omitted. We refer the reader to [55] for the former and [75] for the latter.

3.1 Introduction

The discussion in this section is purely formal and it is intended to give the reader an idea of the path-integral formalism.

The path integral approach as a quantization scheme of classical systems was first introduced by Feynman in his Ph.d. thesis [46], inspired by an early work by Dirac [35]. First applied in the context of non-relativistic quantum mechanics, later it became a powerful tool for the study of quantum field theories and provided an interpretation of quantum systems as statistical mechanical models.

For ordinary quantum mechanics, the central result of the path integral formalism can be summarized saying that the probability amplitude for a particle to travel from a point x_i at time t_i to a point x_f at time t_f can be obtained by integrating a phase factor $e^{iS[x(t)]}$, assigned to each path $x(t)$ connecting (x_i, t_i) and (x_f, t_f), over all such paths. Here $S[x(t)]$ denotes the functional action describing the evolution of a classical system, given by

$$S[x(t)] = \int_{t_i}^{t_f} \mathrm{d}t \mathcal{L}(x(t), \dot{x}(t)). \tag{3.1}$$

and we consider the case where the Lagrangian $\mathcal{L}(x(t), \dot{x}(t))$ is of the form

$$\mathcal{L}(x(t), \dot{x}(t)) = \frac{1}{2} m \dot{x}(t)^2 - V(x(t)), \tag{3.2}$$

with m denoting the mass of the particle and V the potential.

Thus, the probability amplitude, usually called *propagator*, or *Green's function*, is formally given by

$$G(x_i, t_i; x_f, t_f) = \int_{\mathcal{P}(x_i, x_f)} \mathscr{D}[x(t)]\, e^{iS[x(t)]}, \tag{3.3}$$

where $\mathcal{P}(x_i, x_f)$ indicates the space of all paths between (x_i, t_i) and (x_f, t_f) and $\mathscr{D}[x(t)]$ is a formal measure on such space.

The formal expression (3.3) of the non-relativistic propagator can be given a precise mathematical sense by the following procedure. First, subdivide the time interval $|t_f - t_i|$ in $n+1$ small intervals $|t_j - t_{j-1}|$, $j = 0, 1 \ldots, n+1$, of size $\epsilon = |t_f - t_i|/(n+1)$, with $t_0 = t_i$, $t_{n+1} = t_f$.

Then, defining $\mathbf{x} = (x_0, \ldots, x_{n+1})$, with $x_j = x(t_j)$, the amplitude is given by

$$G(x_i, t_i; x_f, t_f) = \lim_{n \to \infty} \int_{-\infty}^{+\infty} \prod_{j=1}^{n} \frac{\mathrm{d}x_j}{\sqrt{2\pi i \epsilon / m}} \, e^{iS_n[\mathbf{x}]} \tag{3.4}$$

with

$$S_n[\mathbf{x}] = \sum_{j=1}^{n+1} \epsilon \left[\frac{1}{2} m \left(\frac{x_j - x_{j-1}}{\epsilon} \right)^2 - V\left(\frac{x_j + x_{j-1}}{2} \right) \right]. \tag{3.5}$$

We refer the reader to [55] for further details.

Extending the integral in eq. (3.3) to the space of all *closed* paths, i.e. with $x_i = x(t_i) = x(t_f) = x_f$, and integrating over all x_i, one obtains a quantity, called *partition function*, which is formally written as

$$Z = \int \mathscr{D}[x(t)] \, e^{iS[x(t)]}. \tag{3.6}$$

The path-integral defining the propagator (3.3) and the partition function (3.6) can be generalized to the relativistic case, when one considers field configurations instead of particle positions and the integration is performed on suitable *spaces of fields*. For instance, consider a massive scalar field ϕ propagating in a 4-dimensional Minkowski spacetime (i.e. with metric $\eta_{\mu\nu}$ with signature $(-+++)$), with action $S[\phi]$ given by

$$S_{SF}[\phi] = \int \mathrm{d}^4 x \mathcal{L}(\phi, \dot{\phi}), \tag{3.7}$$

where the Lagrangian is

$$\mathcal{L}(\phi, \dot{\phi}) = -\frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m^2 \phi^2. \tag{3.8}$$

Here we used the sum over repeated index convention.

Then, the probability amplitude for ϕ to propagate from a configuration ϕ_i at time t_i to a configuration ϕ_f at time t_f, is formally given by the path integral

$$G_{SF}(\phi_i, t_i; \phi_f, t_f) = \int_{\mathcal{F}(\phi_i, \phi_f)} \mathscr{D}[\phi]\, e^{iS[\phi]}, \tag{3.9}$$

whereas the partition function is

$$Z_{SF} = \int \mathscr{D}[\phi]\, e^{iS[\phi]}. \tag{3.10}$$

Here $\mathcal{F}(\phi_i, \phi_f)$ is the space of fields with initial configuration ϕ_i at time t_i and final configuration ϕ_f at time t_f, whereas the integral in eq. (3.10) is performed over all fields with periodic boundary conditions in the time coordinate.

Generally, to make sense of the integral in eqs. (3.9) and (3.10) one can still perform a discretization analogous to the non-relativistic case and perform a Wick rotation, that is an analitic continuation of the action to imaginary time. This can be explained using the scalar field example. Indeed, the Wick rotation modifies the volume integration in the action (3.7) of a factor i, hence the partition function becomes

$$Z_{SF} = \int \mathscr{D}[\phi]\, e^{-\hat{S}[\phi]}, \tag{3.11}$$

where $\hat{S}[\phi] = -iS[\phi]$. For real fields ϕ, the Euclidean action $\hat{S}[\phi]$ is non-negative, therefore after a proper regularization the integral (3.11) should eventually converge.

Hence, the idea is to perform all the calculations in the Euclidean sector and only at the end rotate the time axis to restore the Lorentzian signature of the spacetime. This procedure can be made mathematically precise, for a certain class of field theories, by means of the Osterwalder-Schrader axioms [77].

However, a detailed discussion of the path integral formulation goes far beyond the scope of this thesis. We refer the interested reader to e.g. [55] for further details. Our aim is to show how it eventually translates in a statistical mechanical problem in terms of random graphs, when it is applied as a quantization scheme for general relativity.

3.2 Lorentzian and Euclidean path-integral for gravity

In order to apply the path integral formalism to the gravitational field, the partition function $\mathcal{Z}$ for pure gravity (i.e. without matter fields) is formally obtained as follows. First, to a given metric g on a closed, compact d-dimensional Lorentzian manifold M we associate a phase factor $e^{iS_{EH}}$, where S_{EH} is the Einstein-Hilbert action defined as

$$S_{EH}(g; G, \Lambda) = \Lambda \int_M \mathrm{d}^d x \sqrt{|\det g|} - \frac{1}{2\pi G} \int_M \mathrm{d}^d x \sqrt{|\det g|}\, R, \qquad (3.12)$$

for $d \geq 2$. Here G denotes Newton's constant, Λ the cosmological constant and R the scalar curvature (or Ricci scalar) associated to g.

The gravitational Lorentzian partition function is formally defined as

$$\mathcal{Z}(G, \Lambda) = \sum_M \int_{\mathrm{Geom}(M)} \mathscr{D}[g]\; e^{iS_{EH}(g;G,\Lambda)}, \qquad (3.13)$$

where $\mathrm{Geom}(M) = \mathrm{Metric}(M)/\,\mathrm{Diff}(M)$ is the so-called *space of geometries* of the manifold M. Here $\mathrm{Metric}(M)$ is the space of Lorentzian metrics on M and $\mathrm{Diff}(M)$ the diffeomorphism group of M. The definition of the space of geometries keeps track of the diffeomorphism invariance of general relativity. The "sum" $\sum_M$ indicates a formal sum over diffeomorphism classes of manifolds and $[g] \in \mathrm{Geom}(M)$ the equivalence class of metrics which are isometric to g.

There are several problems when one tries to give a precise mathematical description of the gravitational path-integral (3.13). For instance, it is not known how to unambiguously define the measure $\mathscr{D}[g]$ (see e.g. [49]).

A different approach is to start directly from the Euclidean partition function

$$\hat{\mathcal{Z}}(G,\Lambda) = \sum_{\hat{M}} \int_{\text{Geom}((\hat{M})} \mathscr{D}[\hat{g}]\, e^{-S_{EH}(\hat{g};G,\Lambda)}. \tag{3.14}$$

where we use the notation ˆ to denote Euclidean quantities. Here, the action is defined by the same expression as in eq. (3.12), except for the fact that in this case the integration is over the manifold $\hat{M}$ equipped with Riemannian metric $\hat{g}$.

Even in this case some problems arise. To mention some of them:

i) In dimension $d \geq 3$ no classification of diffeomorphism classes of manifolds is known (in dimension $d = 2$, orientable compact manifolds are characterized by their Euler characteristic). Thus the sum $\sum_M$ is far from being tractable.

ii) There are no known analogs of the Osterwalder-Schrader axioms for quantum gravity.

iii) Unboundedness from below of the Einstein-Hilbert action (see e.g. [4]).

Further discussions about Euclidean quantum gravity can be found in [52, 4].

In the remainder of this chapter we will focus on the two-dimensional theory. In the next section we shall see how to deal with these problems introducing certain regularization procedures called *dynamical triangulation* (DT) for the Euclidean case and *causal dynamical triangulation* (CDT) for Lorentzian manifolds.

3.3 Two-dimensional quantum gravity

We start discussing the Euclidean partition function (3.14). In two dimensions the path-integral in eqs. (3.14) extremely simplifies, due to the Gauss-Bonnet theorem. This states that, for a compact closed oriented Riemannian manifold $\hat{M}_h$, with h handles, one has

$$\int_{\hat{M}_h} \mathrm{d}^2 x \sqrt{\det \hat{g}}\, \hat{R} = 4\pi\chi(h), \tag{3.15}$$

where the *Euler characteristic* $\chi(h)$ of $\hat{M}_h$ is given by $\chi(h) = 2 - 2h$,

Therefore, the two-dimensional Euclidean Einstein-Hilbert action reduces to

$$S_{EH}(\hat{g}; G, \Lambda) = \Lambda V_{\hat{g},h} - \frac{\chi(h)}{G}. \tag{3.16}$$

$V_{\hat{g},h}$ denotes the volume of $\hat{M}_h$,

$$V_{\hat{g},h} = \int_{\hat{M}_h} \mathrm{d}^2 x \sqrt{\det \hat{g}}. \tag{3.17}$$

Further, the diffeomorphism classes for closed oriented manifolds with genus h are completely determined by the Euler characteristic. Therefore, the sum over diffeomorphism classes in (3.14) reduces to a sum over the genus of the manifold. Hence from (3.14) and (3.16) we get

$$\hat{\mathcal{Z}}(\Lambda, G) = \sum_{h=0}^{\infty} e^{\chi(h)/G} \hat{\mathcal{Z}}_h(\Lambda) \tag{3.18}$$

where the partition function $\hat{\mathcal{Z}}_h(\Lambda)$ of fixed topology h is defined as

$$\hat{\mathcal{Z}}_h(\Lambda, G) = \int_{\mathrm{Geom}(\hat{M}_h)} \mathscr{D}[\hat{g}]\; e^{-\Lambda V_{\hat{g},h}}, \tag{3.19}$$

In the following we shall consider only manifolds with fixed topology of S^2 (possibly with boundaries), hence we will consider the partition function

$$\hat{\mathcal{Z}}(\Lambda) \equiv \hat{\mathcal{Z}}_0(\Lambda) = \int_{\mathrm{Geom}(\hat{S}^2)} \mathscr{D}[\hat{g}]\; e^{-S_{EH}(\hat{g};\Lambda)} \tag{3.20}$$

where

$$S_{EH}(\hat{g}; \Lambda) = \Lambda\, V_{\hat{g}}, \tag{3.21}$$

with $V_{\hat{g}} = V_{\hat{g},0}$.

When b boundary components are present, the action (3.21) get also a boundary term

$$S_{EH}(\hat{g}; \Lambda, \lambda_1, \ldots, \lambda_b) = \Lambda\, V_{\hat{g}} + \sum_{i=1}^{b} \lambda_i L_{\hat{g},i} \tag{3.22}$$

where $L_{\hat{g},i}$ denotes the length of the ith boundary component with respect to the metric $\hat{g}$ and λ_i the associated boundary cosmological constant.

The Gauss-Bonnet formula has an analog for Lorentzian manifolds [8, 27]. Also in this case the contribution from the curvature term in the action is a topological quantity and can be omitted fixing the topology (see also [63, 15, 66]). We also note that in two dimensions the vacuum Einstein equations are trivially satisfied and it is rather natural to drop the curvature term in the action, see [56].

Therefore, in the 2-dimensional Lorentzian case the partition function is defined as

$$\mathcal{Z}(\Lambda) = \int_{\mathrm{Geom}(M)} \mathscr{D}[g]\; e^{iS_{EH}(g;\Lambda)}, \tag{3.23}$$

where M is a compact 2-dimensional Lorentzian manifold and

$$S_{EH}(g; \Lambda) = \Lambda V_M. \tag{3.24}$$

Next step to properly define the path-integrals above is to introduce a regularization technique, which would make sense of the integration over metrics. Following the idea used for the non-relativistic case of integrating on piecewise linear segments, this procedure uses a suitable discretization of two-dimensional surfaces.

3.3.1 Triangulations and Regge action

The first attempt to find a discrete analog of the Einstein theory of general relativity is due to Regge [78]. The central idea of this approach is to approximate the 4-dimensional spacetime with piecewise flat building blocks, four-simplices, such that the metric tensor, the dynamical variable of the continuum theory, becomes a function of the edge lengths of the four-simplices.

In the following we outline Regge's construction for an Euclidean 2-dimensional surface. However, it can be equivalently applied to Lorentzian manifolds [78].

In two dimensions the spacetime is regarded as a piecewise linear surface, whose fundamental blocks are triangles. We consider such a surface embedded in $\mathbb{R}^D$, for sufficiently high D. The triangles are glued together to form a *triangulation* T satisfying the following constraints:

i) The identification of two edges implies identification of their endpoints.

ii) For any two triangles there exists a sequence of adjacent triangles connecting them.

iii) Closed paths of edges of length one and two are not allowed.

Two triangulations are said to be *isomorfic* if there is a bijective map between links and vertices which preserves the incidence relations.

Now we see how the geometry is encoded in a triangulation. To each vertex v one associates a *deficit angle* ε_v defined by

$$\varepsilon_v = 2\pi - \sum_{t \ni v} \theta_v(t), \tag{3.25}$$

where $\theta_v(t)$ denotes the angles at v of the triangle t to which v belongs, see Fig. 3.1. Note that on Lorentzian manifolds, the deficit angles might be imaginary, depending on the nature of the edges, i.e. whether they are timelike, spacelike or null, see [78, 81].

As will be seen in a moment the deficit angles ε_v encode the geometric properties of the curvature. First, recall [64] that on a smooth manifold, the

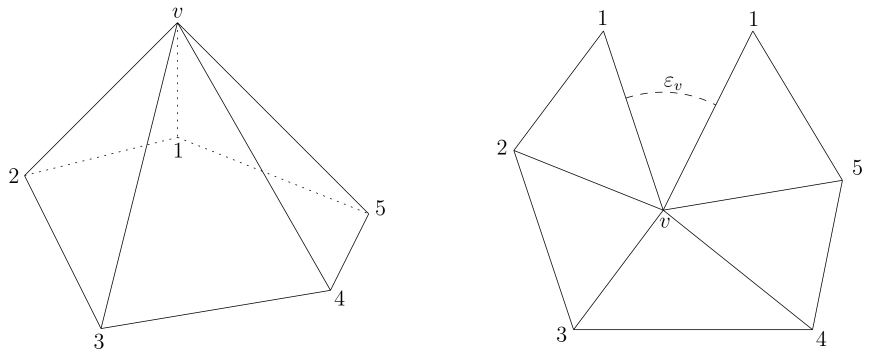

Figure 3.1: Deficit angle

easiest way to reveal the curvature of the manifold is to parallel transport a vector around a closed loop. For a curve enclosing an infinitesimal area $\mathrm{d}V$, a vector parallel transported along such curve undergoes a rotation of an angle $\mathrm{d}\theta = \frac{1}{2}R\,\mathrm{d}V$. On a piecewise linear 2-manifold parallel transporting a vector around a vertex v, rotates the vector of an angle ε_v. Hence, according to the continuum formalism we define the curvature R_v at the vertex v as

$$R_v = \frac{2\varepsilon_v}{V_v}, \tag{3.26}$$

where V_v denotes the barycentric area at the vertex v, i.e.

$$V_v = \frac{1}{3}\sum_{t \ni v} V_t, \tag{3.27}$$

with V_t being the area of the triangle to which v belongs.

With these definitions we have that the discretized counterpart of the curvature term in the two-dimensional Einstein-Hilbert action (3.12) is given by

$$\sum_{v \in T} \varepsilon_v, \tag{3.28}$$

and discrete volume term by

$$\sum_{v \in T} V_v \tag{3.29}$$

where T is a given triangulation. Hence the so-called *Regge action* $S_{Regge}(T)$ is defined as

$$S_{Regge}(T) = \sum_{v \in T} \left(\Lambda V_v - \frac{\varepsilon_v}{G} \right), \tag{3.30}$$

The above construction can be extended to higher dimension, see [78] and [71], Ch.42.

We stress that in this formulation edge lengths are allowed to vary, indeed they are the new dynamical variables, whereas the connectivity of the edges, i.e. the incidence matrix of the triangulation, is kept fixed.

As will be seen in the next section, this point of view will need to be reversed when applying Regge calculus to the Euclidean gravitational path integral.

3.3.2 Dynamical Triangulations

As discussed above, the triangulation contains all the informations about the geometry of the discretized manifold, via the edge lengths of the triangles. It is then a rather natural ansatz to define the discretized counterpart of the integral over the space of metrics with a sum over triangulation weighted by the Regge action.

A problem with this ansatz is that, as pointed out in the previous section, the edge lengths of the triangulation are allowed to vary continuously, hence the sum would include also equivalent triangulations, i.e. that can be continuously deformed into each other. Further, as in the quantum mechanics example discussed at the beginning of this chapter, we need a cut-off parameter (ε in that case) which might possibly be sent to 0 to recover the continuum theory.

A possible solution is to consider only triangulations made of equilateral triangles with same squared edge length $L_a^2 = a^2$. The study of this model, called *dynamical triangulation*, goes back to 1985 when it appeared in [2, 31, 60]. In particular, in [2, 60] it was used as a regularization technique for Polyakov string theory. Shortly after, the first numerical results appeared [3, 19]. We refer the reader to [4] for details.

It should be noted that even fixing the edges length new problems arise, such as the loss of the diffeomorphism invariance since deformations of the triangulations are not allowed (see [33] for a discussion).

Denoting by $\mathcal{T}$ the set of triangulations of S^2, we define the discretized counterpart of the partition function (3.20) for closed surfaces of genus $h = 0$ as

$$\hat{\mathcal{Z}}(\Lambda) = \sum_{T \in \mathcal{T}} \frac{1}{C_T} e^{-S(T,\Lambda)}, \tag{3.31}$$

where C_T is a symmetry factor equal to the order of the automorphism group of T. Here the action $S(T, \Lambda)$ is given by eq. (3.30) dropping the constant curvature term, that is

$$S(T, \Lambda) = \Lambda\, N_T, \tag{3.32}$$

with N_T denoting the number of triangles in the triangulation T. A factor $\frac{\sqrt{3}}{4}a^2$ coming from the area of a single triangle has been absorbed in Λ.

Defining the set

$$\mathcal{T}_k = \{ T \in \mathcal{T} \mid N_T = k \} \tag{3.33}$$

eq. (3.31) can be written as

$$\hat{\mathcal{Z}}(g) = \sum_{k=0}^{\infty} g^k \hat{\mathcal{Z}}(k) \tag{3.34}$$

where the *fugacity of triangles* g is defined by

$$g = e^{-\Lambda}, \tag{3.35}$$

and

$$\hat{\mathcal{Z}}(k) = \sum_{T \in \mathcal{T}_k} \frac{1}{C_T}. \tag{3.36}$$

Hence, the partition function (3.31) is the generating function for the number $\hat{\mathcal{Z}}(k)$ of non-isomorfic triangulations of S^2.

In analogy to the continuum theory (cf. eq. (3.22)), the discrete action in the presence of a number b of boundary components is given by

$$S(T, \Lambda, \lambda_1, \ldots, \lambda_b) = \Lambda\, N_T + \sum_{i=1}^{b} \lambda_i l_i, \tag{3.37}$$

where l_i denotes the numbers of edges in the ith boundary component. A factor a from the length of a single edge has been absorbed in λ_i, for $i = 1, \ldots, b$.

Denoting by $\mathcal{T}(l_1, \ldots, l_b)$ the set of triangulations with b boundary components of length $l_1, \ldots, l_b$, the partition function associated to the action (3.37) is defined by

$$\mathcal{W}(\Lambda, \lambda_1, \ldots, \lambda_b) = \sum_{l_1,\ldots,l_b} \sum_{T \in \mathcal{T}(l_1,\ldots,l_b)} e^{-S(T,\Lambda,\lambda_1,\ldots,\lambda_b)} \tag{3.38}$$

It follows from eq. (3.37) that this expression can be written as

$$\mathcal{W}(\Lambda, \lambda_1, \ldots, \lambda_b) = \sum_{l_1,\ldots,l_b} w(\Lambda, l_1, \ldots, l_b) \prod_{i=1}^{b} e^{-\lambda_i l_i} \tag{3.39}$$

where the *Hartle-Hawking wave functionals* (see [51, 4] for their continuum counterpart) $w(\Lambda, l_1, \ldots, l_b)$ are defined by

$$w(\Lambda, l_1, \ldots, l_b) = \sum_{T \in \mathcal{T}(l_1,\ldots,l_b)} e^{-S(T,\Lambda)}. \tag{3.40}$$

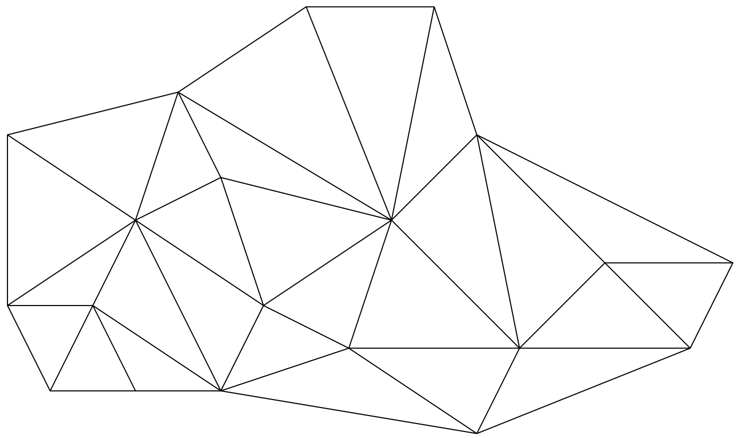

Figure 3.2: Regular triangulation

Setting

$$\mathcal{T}_k(l_1,\ldots,l_b) = \left\{ T \in \mathcal{T}(l_1,\ldots,l_b) \;\middle|\; N_T = k \right\}, \tag{3.41}$$

eq. (3.40) can be written as

$$w(\Lambda, l_1, \ldots, l_b) = \sum_{k=0}^{\infty} e^{-\Lambda k} w_{k,l_1,\ldots,l_b} \tag{3.42}$$

where

$$w_{k,l_1,\ldots,l_b} = \sharp\mathcal{T}_k(l_1,\ldots,l_b), \tag{3.43}$$

i.e. the number of triangulations with k triangles and boundary lengths $l_1,\ldots,l_b$.

Thus, also in this case the partition function (3.38) is the generating function for the number $w_{k,l_1,\ldots,l_b}$ of non-equivalent triangulations of S^2 with b boundaries.

Finally, the problem of evaluating the partition function of 2-dimensional quantum gravity is translated into the problem of counting the number of triangulations of the 2-sphere (possibly with boundaries).

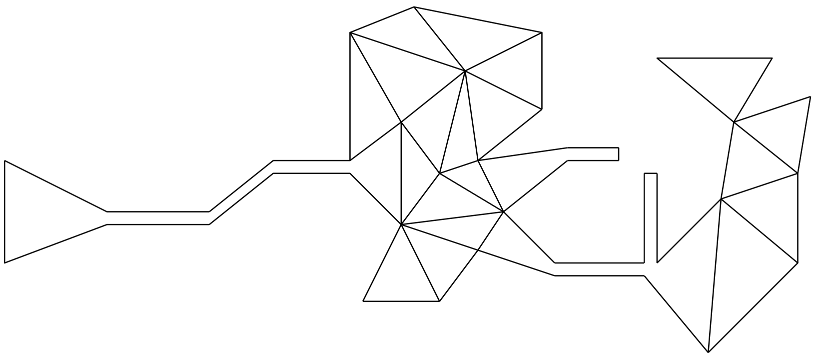

Figure 3.3: Unrestricted triangulation

Two classes of triangulations are usually considered (see [4]). The first class consists of triangulations obtained from the triangulations defined in the previous section further requiring that no two vertices in the same boundary component can be connected by an interior link, see Fig. 3.2. These triangulations are called *regular triangulations*.

Triangulations of the second class, called *unrestricted triangulations*, are constructed gluing together triangles and double edges, considered as infinitesimally narrow strips, see Fig. 3.3.

The combinatorial problem associated to the regular triangulations was first studied by Tutte in his seminal paper [85], and we refer the reader to that paper for details.

On the other hand, the unrestricted triangulation case can also be solved. However a complete discussion of the general case of a triangulation with an arbitrary number of boundaries is rather involved and we do not discuss it here. We refer the interested reader to [4].

Instead, in the following we study the simplest case of unrestricted triangulations with one boundary component and one marked edge on the boundary.

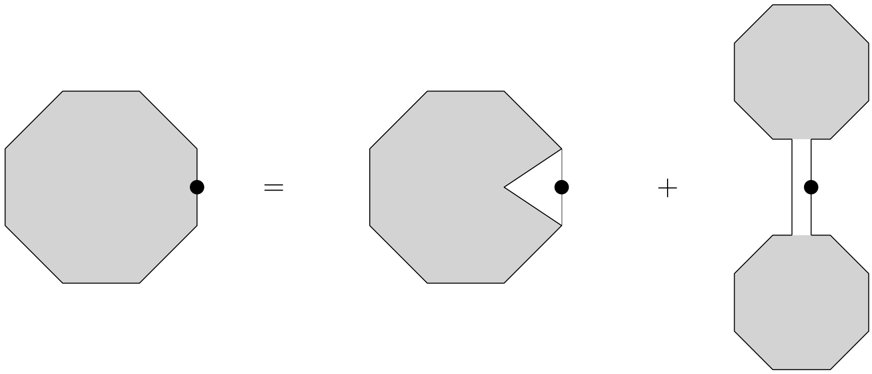

Figure 3.4: Unrestricted triangulation decomposition

We define the generating function for this type of triangulations as

$$W(g,z) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} w_{k,l} g^k z^{-(l+1)}, \tag{3.44}$$

where g is defined in (3.35) and z is defined by

$$z = e^{\lambda}. \tag{3.45}$$

Here z^{-1} is the *fugacity of a boundary edge*. The quantity $w_{k,l}$ denotes the number of unrestricted triangulations with k triangles and one boundary component with l edges. Note that eq. (3.44) includes the contribution of a triangulation with just one vertex, that is $w_{0,0} = 1$.

This function satisfies the equation

$$W(g,z) = gzW(g,z) + \frac{1}{z} W^2(g,z) - \frac{gW_1(g) + gz}{z} + \frac{1}{z} \tag{3.46}$$

where

$$W_l(g) = \sum_{k=0}^{\infty} w_{k,l} g^k, \qquad \text{for } l \geq 0. \tag{3.47}$$

In the literature, eq. (3.46) is usually called the *loop equation*. It can be obtained considering the decomposition of an unrestricted triangulation schematically depicted in Fig. 3.4. The meaning of the picture is the following. Removing a triangle with one boundary edge produces two new boundary edges, whereas removing a triangle with two boundary edges produces a double-edge. In both cases the number of triangles (the power of g in $W(g,z)$) decreases by one and the number of edges (the power of z) increases by one. This correspond to the first term in the rhs of eq. (3.46). If a double link is removed from the boundary, we get a decreasing of two in the length of the boundary and two triangulations. This corresponds to the second term. The remaining terms are added to cancel contribution coming from triangulations with boundaries of length 0 and 1 (third term), and in order to make the equation valid for the single vertex triangulation (fourth term).

Setting

$$A(g,z) = z - gz^2 \qquad B(z,g) = 1 - gW_1(g) - gz, \tag{3.48}$$

the second order equation (3.46) can be solved and we obtain

$$W(g,z) = \frac{1}{2}\left(A(g,z) - \sqrt{A(g,z)^2 - 4B(g,z)}\right), \tag{3.49}$$

where the sign in front of the square root is chosen in accordance with the expansion $W(g,z) = 1/z + O\left(1/z^2\right)$ for large z (since $w_{0,0} = 1$).

For $g = 0$ there are no triangles in the triangulation, that is we are counting unrestricted triangulations made only of double edges with one marked edge. These are usually called *rooted branched polymers*, see Fig. 3.5.

A rooted branched polymer with a marked edge can be seen as a rooted planar tree, whose number, for given finite size of the tree, as been calculated in 2.1.3 and is given by the Catalan numbers. Thus, we expect the same result also in this case.

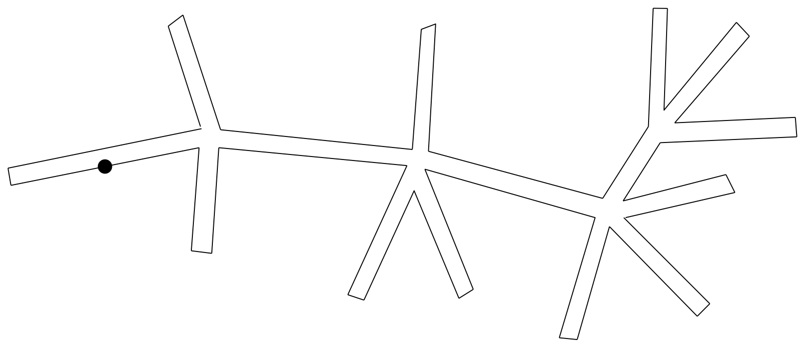

Figure 3.5: Rooted branched polymer

To make this argument precise, note that for $g = 0$ eq. (3.49) reduces to

$$W(0,z) = \frac{z - \sqrt{z^2 - 4}}{2}. \tag{3.50}$$

Expanding in power of $1/z$ we get

$$W(0,z) = \sum_{l=0}^{\infty} \frac{w_{2l}}{z^{2l+1}} \tag{3.51}$$

where

$$W_{2l} = \frac{(2l)!}{(l+1)!l!}, \tag{3.52}$$

as expected.

The general solution of (3.46) has been obtained in [4],

$$W(g,z) = \frac{1}{2}\left(z - gz^2 + (gz - c)\sqrt{(z - c_+)(z - c_-)}\right) \tag{3.53}$$

where c, c_+ and c_- can be obtained by requiring again that $W(g,z) = 1/z + O\left(1/z^2\right)$.

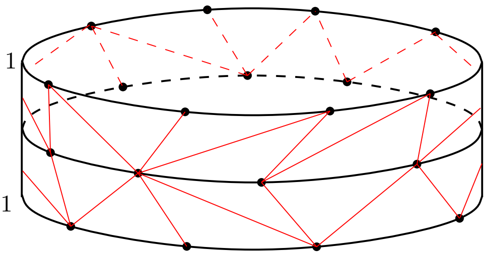

Figure 3.6: Causal triangulation

3.3.3 Causal Dynamical Triangulation

The causal dynamical triangulation (CDT) model was first studied in [6] to introduce the notion of time in the dynamical triangulation, discussed in the previous section. The basic idea is to approximate a Lorentzian manifold with a piecewise linear manifold, whose building blocks now include the notion of time orientation.

In the following for simplicity we discuss the CDT model for a two-dimensional Euclidean surface with fixed topology of $S^1 \times [0,1]$. For such a surface a causal triangulation is constructed as follows. We refer to [6] where the Lorentzian case id discussed.

First, assume that at each fixed time t the spatial slice is a loop, which is approximated by a piecewise linear curve with k_t edges (and vertices). As for dynamical triangulations, the squared edge length is fixed to $L_s^2 = a^2$. Likewise, the time direction is discretized, setting the time to assume discrete values $t \in \mathbb{N}$.

Finally, the vertices on a spatial slice at time t are connected to the vertices on the consecutive slice at $t+1$ by timelike edges of squared length $L_t^2 = a^2$, such that in the resulting graph all faces are triangles, see Fig. 3.6. We denote by $\mathcal{T}$ the set of all such triangulations.

Note that, according to the above construction, each triangle is made of one spacelike edge and two timelike edges. The volume of a triangle is given by $V = \frac{\sqrt{3}}{4}a^2$.

We denote by $\mathcal{T}_{l_1,l_2}$ the set of causal triangulations with boundaries (spatial slices) with l_1 and l_2 edges. Further, $\mathcal{T}_{l_1,l_2,n}$ denotes the subset of $\mathcal{T}_{l_1,l_2}$ of triangulations with n spatial slices between the boundaries.

We have that the volume of $T \in \mathcal{T}_{l_1,l_2,n}$ is given by

$$V_T = \frac{\sqrt{3}}{4} a^2 \sum_{i=1}^{n+1} N_T(i) \tag{3.54}$$

where the number $N_T(i)$ of triangles between two consecutive spatial slices at $t = i$ and $t = i + 1$ is given by

$$N_T(i) = k_i + k_{i+1}, \tag{3.55}$$

with $k_1 = l_1$ and $k_{n+2} = l_2$.

Thus, we find that, for $T \in \mathcal{T}_{l_1,l_2}$, the discrete two dimensional action can be written

$$S(\Lambda, \lambda_1, \lambda_2; T) = \Lambda V_T + \lambda_1 l_1 + \lambda_2 l_2, \tag{3.56}$$

where $\lambda_1 l_1$ and $\lambda_2 l_2$ are the boundary contributions and λ_1, λ_2 are the boundary cosmological constants. Here the constants coming from the area of the triangles and from the length of the edges have been absorbed in the cosmological constants.

The partition function for CDT reads

$$\mathcal{Z} = \sum_{T \in \mathcal{T}} e^{-S(\Lambda,\lambda_1,\lambda_2,T)}, \tag{3.57}$$

The sum over the set $\mathcal{T}$ can be decomposed into the sum over triangulations with fixed boundaries and a finite number of slices, that is

$$\mathcal{Z}(\Lambda, \lambda_1, \lambda_2) = \sum_{n=0}^{\infty} \mathcal{Z}(\Lambda, \lambda_1, \lambda_2, n) \tag{3.58}$$

where

$$\mathcal{Z}(\Lambda, \lambda_1, \lambda_2, n) = \sum_{l_1, l_2} \sum_{T \in \mathcal{T}_{l_1, l_2, n}} e^{-S(\Lambda, \lambda_1, \lambda_2, T)}. \tag{3.59}$$

As for the DT model, we define

$$g = e^{-\Lambda}, \qquad x = e^{-\lambda_1}, \qquad y = e^{-\lambda_2} \tag{3.60}$$

thus eq. (3.59) can be written as, using (3.56),

$$\mathcal{Z}(g, x, y, n) = \sum_{\substack{k_i \geq 1 \\ n+2 \geq i \geq 1}} \prod_{j=1}^{n+1} \binom{k_j + k_{j+1} - 1}{k_j - 1} (gx)^{k_1} (gy)^{k_{n+2}} g^{2(k_2 + \cdots + k_{n+1})}. \tag{3.61}$$

Here the binomial coefficient counts the number of ways of connecting k_j vertices on a spatial slice to k_{j+1} vertices on the consecutive slice. Moreover, one of the edges on the boundary has been marked, in order to cancel factors due to possibly rotational symmetries.

Summing repeatedly over $k_1, \ldots, k_{n+2}$ we find

$$\mathcal{Z}(g, x, y, n) = \left(\prod_{i=1}^{n} \frac{F_i(x)}{1 - F_i(x)} \right) \frac{g^2 xy}{(1 - gx)(1 - F_n(x) - gy)} \tag{3.62}$$

where

$$F_i(x) = \frac{g^2}{1 - F_{i-1}(x)}, \quad F_1(x) = gx. \tag{3.63}$$

One can use a standard fixed point technique to obtain the solution of eq. (3.63), which reads

$$F_i(x) = F \, \frac{g - xF + (F/g)^{2i-3}(gx - F)}{g - xF + (F/g)^{2i-1}(gx - F)}, \tag{3.64}$$

where F is the fixed point of eq. (3.63),

$$F(g) = \frac{1 - \sqrt{1 - 4g^2}}{2}, \quad |g| < \frac{1}{2}. \tag{3.65}$$

Finally, eq. (3.64) can be substituted in (3.62) to obtain the explicit expression of the partition function $\mathcal{Z}(g, x, y, n)$.

This conclude our study of the CDT model. The interested reader should consult ([6, 43]) for further details.

3.3.4 Gravity and matter

In this final section we discuss the idea of coupling matter fields to two-dimensional quantum gravity. At discrete level this coupling can be implemented defining a statistical mechanical model on a triangulated space-time [4]. From the point of view explained above, triangulations are regarded as dynamical variables, hence the coupling between the statistical mechanical system and the random triangulations is realized by defining a copy of the same statistical system for each different triangulation.

Let $\mathcal{S}$ be a statistical model and $\mathcal{T}$ an ensemble of triangulations. Let us denote by T_k a triangulation in $\mathcal{T}$ and by $\mathcal{S}_k$ a copy of the model $\mathcal{S}$ defined on T_k. Further, we denote by $\{\sigma_i^k\}$ the dynamical variables of $\mathcal{S}_k$ and by $S_k(\{\sigma_i^k\})$ the corresponding action. Then, the partition funcition of the coupled system is defined by

$$\mathcal{Z} = \sum_k w_k \mathcal{Z}_k, \tag{3.66}$$

where w_k denotes the weight of the statistical system $\mathcal{S}_k$ and Z_k is the partition function of the single model $\mathcal{S}_k$, that is

$$\mathcal{Z}_k = \sum_{\{\sigma_i^k\}} e^{-S_k(\{\sigma_i^k\})}. \tag{3.67}$$

A well-known example of such a model is the Ising model on planar random lattice. This was studied and exactly solved by Kazakov et al. in [59, 18, 21]. In particular, in [59] the author considered the Ising model on a random planar graph with all the vertices of degree 4.

Let us discuss this example in more detail. Let $\mathcal{G}_n$ be the set of planar lattices with n vertex, where each vertex has degree 4. The partition function of the Ising model at inverse temperature β on a graph $G \in \mathcal{G}_n$ reads

$$Z(G,\beta) = \sum_{\{s\}} \exp\left(\beta \sum_{\langle i,j \rangle} s_i s_j\right), \tag{3.68}$$

where $s_i = \pm 1$ are Ising spins and the notation $\langle i,j \rangle$ means that the sum is over neighbouring spins. Here, the first summation is over all the spin configurations on G. Therefore, according to the above discussion the partition function of the coupled system is given by

$$Z_n(\beta) = \sum_{G \in \mathcal{G}_n} \sum_{\{s\}} \exp\left(\beta \sum_{\langle i,j \rangle} s_i s_j\right). \tag{3.69}$$

The model was solved considering the generating function for $Z_n(\beta)$, that is

$$\mathscr{Z}(c,g) = \sum_{n=1}^{\infty} \left(\frac{-4gc}{(1-c^2)^2}\right)^n Z_n(\beta), \tag{3.70}$$

with $c = e^{-2\beta}$, and proving that it is equal to the free energy of an exactly solvable matrix model. In particular, the author gives an exact expression of the partition function $Z_n(\beta)$ in the thermodynamical limit $n \to \infty$. From this, it was found that the system undergoes a third-order phase transition at finite temperature $T_c = 1/\log 2$.

In the following chapter we study the Ising model with external field on a certain class of random trees. As will be seen, we are able to give a detailed description of this system in the infinite size limit, and of its magnetization properties.

Chapter 4

Generic Ising Trees

4.1 Introduction

Since its appearance, the Ising model has been considered in various geometrical backgrounds. Most familiar are the regular lattices, where it is well known that in dimension $d = 1$, originally considered by Ising and Lenz [54, 65], there is no phase transition as opposed to dimension $d \geq 2$, where spontaneous magnetization occurs at sufficiently low temperature [76, 82].

The Ising model on a Cayley tree turns out to be exactly solvable [36, 45, 73]. Despite the fact that the free energy, in this case, is an analytic function of the temperature at vanishing magnetic field, the model does have a phase transition and exhibits spontaneous magnetization at a central vertex. One may attribute this unusual behavior to the large size of the boundary of a ball in the tree as compared to its volume.

Studies of the Ising model on non-regular graphs are generally non-tractable from an analytic point of view. For numerical studies see e.g. [9]. See also [14], where the Ising model with external field coupled to the causal dynamical triangulation model is studied via high- and low-temperature expansion techniques. In [5] a grand canonical ensemble of

Ising models on random finite trees was considered, motivated by studies in two dimensional quantum gravity [4]. It was argued in [5] that the model does not exhibit spontaneous magnetization at values of the fugacity where the mean size of the trees diverges.

In the present chapter we study the Ising model on certain infinite random trees, constructed as "thermodynamic" limits of Ising systems on random finite trees. These are subject to a certain genericity condition for which reason we call them *generic Ising trees*. Using tools developed in [39, 42] we prove for such ensembles that spontaneous magnetization is absent. The basic reason is that the generic infinite tree has a certain one dimensional feature despite the fact that we prove its Hausdorff dimension to be 2. Furthermore, we obtain results on the spectral dimension of the generic Ising trees.

This chapter is organized as follows. In Section 4.2 we define the finite size systems whose infinite size limits are our main object of study. The remainder of Section 4.2 is devoted to an overview of the main results, including the existence and detailed description of the infinite size limit, the magnetization properties and the determination of the annealed Hausdorff and spectral dimensions of the generic Ising trees.

The next two sections provide detailed proofs and, in some cases, more precise statements of those results. Under the genericity assumption mentioned above we determine, in Section 4.3, the asymptotic behavior of the partition functions of ensembles of spin systems on finite trees of large size. This allows a construction of the limiting distribution on infinite trees and also leads to a precise description of the limit. In Section 4.4 we exploit the latter characterization to determine the annealed Hausdorff and spectral dimensions of the generic Ising trees, whereafter we establish absence of magnetization in Section 4.5.

This chapter is mainly based on a joint work with Bergfinnur Durhuus [44].

4.2 Definition of the models and main results

4.2.1 The models and the thermodynamic limit

The statistical mechanical models considered in this chapter are defined in terms of planar trees as follows. Let Λ_N be the set of rooted planar trees of size N decorated with Ising spin configurations,

$$\Lambda_N = \left\{ s : V(\tau) \to \{\pm 1\} \mid \tau \in \mathcal{T}_N \right\}, \tag{4.1}$$

and set

$$\Lambda = \left(\bigcup_{N=1}^{\infty} \Lambda_N \right) \cup \Lambda_\infty, \tag{4.2}$$

where Λ_∞ denotes the set of infinite decorated trees. In the following we will often denote by τ_s a generic element of Λ, in particular when stressing the underlying tree structure τ of the spin configuration s. Furthermore, we shall use both s_v and $s(v)$ to denote the value of the spin at vertex v.

The set Λ is a metric space with metric d defined by

$$d(\tau_s, \tau'_{s'}) = \inf \left\{ \frac{1}{R+1} \mid B_R(\tau) = B_R(\tau'), \, s|_{B_R(\tau)} = s'|_{B_R(\tau')} \right\}, \tag{4.3}$$

as a generalization of (1.6).

We define a probability measure μ_N on Λ_N by

$$\mu_N(\tau_s) = \frac{1}{Z_N} e^{-H(\tau_s)} \rho(\tau), \quad \tau_s \in \Lambda_N, \tag{4.4}$$

where the Hamiltonian $H(\tau_s)$, describing the interaction of each spin with its neighbors and with the constant external magnetic field h at inverse temperature β, is given by

$$H(\tau_s) = -\beta \sum_{(v_i,v_j)\in E(\tau)} s_{v_i} s_{v_j} - h \sum_{v_i \in V(\tau)\setminus r} s_{v_i}. \tag{4.5}$$

The weight function $\rho(\tau)$ is defined in terms of the *branching weights* $p_{\sigma_v - 1}$ associated to vertices $v \in V(\tau) \setminus r$, and is given by

$$\rho(\tau) = \prod_{v \in V(\tau) \setminus r} p_{\sigma_v - 1}. \tag{4.6}$$

Here $(p_n)_{n \geq 0}$ is a sequence of non-negative numbers such that $p_0 \neq 0$ and $p_n \neq 0$ for some $n \geq 2$ (otherwise only linear chains would contribute). We will further assume the branching weights to satisfy a *genericity condition* explained below in (4.28), and which defines the *generic Ising tree ensembles* considered in this chapter (see also [42]). Finally, the partition function Z_N in (4.4) is given by

$$Z_N(\beta, h) = \sum_{\tau \in \mathcal{T}_N} \sum_{s \in S_\tau} e^{-H(\tau_s)} \rho(\tau), \tag{4.7}$$

where $S_\tau = \{\pm 1\}^{V(\tau)}$.

Our first result (see Sec. 4.3) is to establish the existence of the thermodynamic limit of this model, in the sense that we prove the existence of a limiting probability measure $\mu = \mu^{(\beta,h)} = \lim_{N \to \infty} \mu_N$ defined on the set of trees of infinite size decorated with spin configurations. Here, the limit should be understood in the weak sense, that is

$$\int_\Lambda f(\tau_s)\, d\mu_N(\tau_s) \xrightarrow{N \to \infty} \int_\Lambda f(\tau_s)\, d\mu(\tau_s) \tag{4.8}$$

for all bounded continuous functions f on Λ (cf. Sec. 1.1.2). In particular, we find that, as in the cases discussed in Sec. 1.2, the measure μ is concentrated on the set of infinite trees with a single infinite path, the *spine*, starting at the root r, and with finite trees attached to the spine vertices, the *branches*.

As will be shown, the limiting distribution μ can be expressed in explicit terms in such a way that a number of its characteristics, such as the Hausdorff dimension, the spectral dimension, as well as the magnetization properties of the spins, can be analyzed in some detail. For the reader's convenience we now give a brief account of those results.

4.2.2 Magnetization properties

As a first result we show that the generic Ising tree exhibits no single site spontaneous magnetization at the root r or at any other spine vertex, i. e.

$$\lim_{h\to 0} \mu^{(\beta,h)}(\{\tau_s \mid s(v) = +1\}) = \frac{1}{2}, \tag{4.9}$$

for any vertex v on the spine and all $\beta \in \mathbb{R}$. Details of this result can be found in Theorem 4.5.2.

The fact that the measure μ is supported on trees with a single spine gives rise to an analogy with the one-dimensional Ising model. In fact, we show that the spin distribution on the spine equals that of the Ising model on the half-line at the same temperature but in a modified external magnetic field. As a consequence, we find that also the mean magnetization of the spine vanishes for $h \to 0$.

A different and perhaps more relevant result concerns the the total mean magnetization, which may be stated as follows. First, let us define the mean magnetization in the ball of radius R around the root by

$$M_R(\beta,h) = \langle |B_R(\tau)| \rangle^{-1}_{\beta,h} \left\langle \sum_{v\in B_R(\tau)} s_v \right\rangle_{\beta,h} \tag{4.10}$$

and the mean magnetization on the full infinite tree as

$$M(\beta,h) = \limsup_{R\to\infty} M_R(\beta,h). \tag{4.11}$$

For the generic Ising tree, we prove in Theorem 4.5.4 that this quantity satisfies

$$\lim_{h\to 0} M(\beta,h) = 0, \quad \beta \in \mathbb{R}. \tag{4.12}$$

4.2.3 Hausdorff and spectral dimension

We show in Theorem 4.4.1 that the annealed Hausdorff dimension of a generic Ising tree can be evaluated and equals that of generic random trees as introduced in [42], i.e.

$$\bar{d}_H = 2\,. \tag{4.13}$$

Furthermore, we show in Theorem 4.4.6 that the annealed spectral dimension of a generic Ising tree is

$$\bar{d}_s = \frac{4}{3}. \tag{4.14}$$

The values of the Hausdorff dimension and the spectral dimension of generic Ising trees are thus found to coincide with those of generic random trees [42] (see Sec. 1.2.2). This indicates that the geometric structure of the underlying trees is not significantly influenced by the coupling to the Ising model as long as the model is generic.

4.3 Ensembles of infinite trees

In this section we establish the existence of the measure $\mu^{(\beta,h)}$ on the set of infinite trees for values of β, h that will be specified below. Our starting point is the Ising model on finite but large trees. We first consider the dependence of its partition function on the size of trees.

4.3.1 Asymptotic behavior of partition functions

Let the branching weights $(p_n)_{n\geq 0}$ be given as above and consider the generating functions

$$\varphi(z) = \sum_{n=0}^{\infty} p_n z^n, \tag{4.15}$$

which we assume to have radius of convergence $\xi > 0$, and

$$Z(\beta, h, g) = \sum_{N=0}^{\infty} Z_N(\beta, h) g^N, \tag{4.16}$$

where Z_N is given by (4.7).

Decomposing the set S_τ into the two disjoint sets

$$S_\tau^{\pm} = \left\{ s \in S_\tau \;\middle|\; s(r) = \pm 1 \right\}, \tag{4.17}$$

gives rise to the decompositions

$$\Lambda_N = \Lambda_{N+} \cup \Lambda_{N-} \tag{4.18}$$

and

$$\Lambda = \Lambda_+ \cup \Lambda_-. \tag{4.19}$$

Correspondingly, we get

$$Z(\beta, h, g) = Z_+(\beta, h, g) + Z_-(\beta, h, g), \tag{4.20}$$

where the generating functions $Z_\pm(\beta, h, g)$ are given by

$$Z_\pm(\beta, h, g) = \sum_{N=0}^{\infty} Z_{N\pm}(\beta, h) g^N, \tag{4.21}$$

and $Z_{N\pm}$ are defined by restricting the second sum in (4.7) to $S_\tau^{\pm}$.

Decomposing the tree as in Fig.4.1, it is easy to see that the functions $Z_\pm(g)$ are determined by the system of equations

$$\begin{cases} Z_+ = g(a\,\varphi(Z_+) + a^{-1}\,\varphi(Z_-)) \\ Z_- = g(b\,\varphi(Z_+) + b^{-1}\,\varphi(Z_-)) \end{cases} \tag{4.22}$$

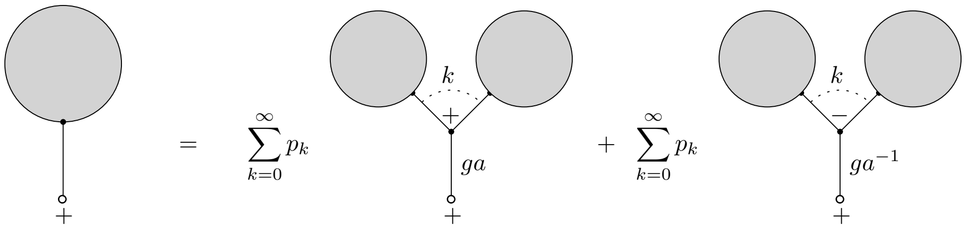

Figure 4.1: Decomposition of a tree of with $s(r) = +1$. The tree is decomposed according to the spin and the degree of the root's neighbor.

where

$$a = e^{\beta+h}, \quad b = e^{-\beta+h}. \tag{4.23}$$

Let us define $F : \{|z| < \xi\}^2 \times \mathbb{C} \to \mathbb{C}^2$ by

$$F(Z_+, Z_-, g) = \mathcal{Z} - g\,\Phi(Z_+, Z_-), \tag{4.24}$$

where,

$$\mathcal{Z} \equiv \begin{pmatrix} Z_+ \\ Z_- \end{pmatrix}, \qquad \Phi(Z_+, Z_-) \equiv \begin{pmatrix} a\,\varphi(Z_+) + a^{-1}\,\varphi(Z_-) \\ b\,\varphi(Z_+) + b^{-1}\,\varphi(Z_-) \end{pmatrix}. \tag{4.25}$$

With the assumption $\xi > 0$, we have

$$\frac{\partial F}{\partial \mathcal{Z}} = \mathbb{1} - g\,\frac{\partial \Phi}{\partial \mathcal{Z}} = \mathbb{1} - g \begin{pmatrix} a\,\varphi'(Z_+) & a^{-1}\,\varphi'(Z_-) \\ b\,\varphi'(Z_+) & b^{-1}\,\varphi'(Z_-) \end{pmatrix}, \tag{4.26}$$

and in particular, $F(0,0,0) = 0$ and $\frac{\partial F}{\partial \mathcal{Z}}(0,0,0) = \mathbb{1}$. The holomorphic implicit function theorem (see e.g. [48], Appendix B.5 and refs. therein) implies that the fixpoint equation (4.22) has a unique holomorphic solution $Z_\pm(g)$ in a neighborhood of $g = 0$. Let g_0 be the radius of convergence of the Taylor series of $Z_+(g)$. Since the Taylor coefficients of Z_+ are non-negative, $g = g_0$ is a singularity of $Z_+(g)$ by Pringsheim's Theorem ([48]

Thm.IV.6). Setting

$$Z_+(g_0) = \lim_{g \nearrow g_0} Z_+(g) \tag{4.27}$$

we have that $Z_+(g_0) < +\infty$. In fact, if $\xi = \infty$ this follows from (4.22), since $\varphi(Z_+)$ increases faster than linearly at $+\infty$, assuming that $p_n > 0$ for some $n \geq 2$. If $\xi < +\infty$ we must have $Z_\pm(g_0) \leq \xi$, because otherwise there would exist $0 < g_1 < g_0$ such that $Z_+(g_1) = \xi$ and $Z_-(g_1) \leq \xi$ (or vice versa), contradicting (4.22) (the LHS would be analytic at g_1 and the RHS not). In particular, we also have $g_0 < +\infty$ and that g_0 equals the radius of convergence for the Taylor series of $Z_-(g)$ by (4.22).

The *genericity assumption* mentioned above states that

$$Z_\pm(g_0) < \xi\,, \tag{4.28}$$

which we shall henceforth assume is valid.

Remark 4.3.1. *It should be noted that, in the absence of an external magnetic field, i. e. for $h = 0$, one has $Z_+(\beta, 0, g) = Z_-(\beta, 0, g) \equiv \bar{Z}(\beta, g)$ and the system (4.22) determining $Z_\pm$ reduces to the single equation $\bar{Z} = 2g \cosh\beta\, \varphi(\bar{Z})$. On the other hand, this equation characterizes the random tree models considered in [42] except for a rescaling of the coupling constant g by the factor $2\cosh\beta$. It follows that the condition* (4.28) *can be considered as a generalization of the genericity condition introduced in [42]. For this reason, the results on the Hausdorff dimension and the spectral dimension established in this chapter follow from [42] in case $h = 0$.*

Under the assumption (4.28), the implicit function theorem gives

$$\det\left(\mathbb{1} - g_0\,\Phi_0'\right) = 0, \tag{4.29}$$

where

$$\Phi_0' = \Phi'(Z_+^0, Z_-^0) = \begin{pmatrix} a\,\varphi'(Z_+^0) & a^{-1}\,\varphi'(Z_-^0) \\ b\,\varphi'(Z_+^0) & b^{-1}\,\varphi'(Z_-^0) \end{pmatrix}, \tag{4.30}$$

with $Z^0_\pm = Z_\pm(g_0)$. Expanding (4.22) around $Z^0_\pm$ we get

$$\Delta\mathcal{Z} = \Delta g\,\Phi_0 + g_0\,\Phi_0'\,\Delta\mathcal{Z} + \frac{g_0}{2}\,\Phi_0''\,\Delta\mathcal{Z}^2 + O(\Delta\mathcal{Z}^3, \Delta g\,\Delta\mathcal{Z})\,, \tag{4.31}$$

where

$$\Delta\mathcal{Z}^n = \begin{pmatrix} (\Delta Z_+)^n \\ (\Delta Z_-)^n \end{pmatrix} = \begin{pmatrix} (Z_+ - Z^0_+)^n \\ (Z_- - Z^0_-)^n \end{pmatrix}, \qquad \Delta g = g - g_0, \tag{4.32}$$

$$\Phi_0'' = \begin{pmatrix} a\,\varphi''(Z^0_+) & a^{-1}\,\varphi''(Z^0_-) \\ b\,\varphi''(Z^0_+) & b^{-1}\,\varphi''(Z^0_-) \end{pmatrix}. \tag{4.33}$$

By (4.29), we have

$$\begin{pmatrix} c_1 & c_2 \end{pmatrix} \left(\mathbb{1} - g_0\,\Phi_0'\right) = 0, \tag{4.34}$$

where

$$c_1 = g_0\,b\,\varphi'(Z^0_+), \quad c_2 = 1 - g_0\,a\,\varphi'(Z^0_+). \tag{4.35}$$

Hence, multiplying (4.31) on the left by $c = (c_1\ c_2)$ gives

$$\Delta g\,c\,\Phi_0 + \frac{g_0}{2}\,c\,\Phi_0''\,\Delta\mathcal{Z}^2 + O(\Delta\mathcal{Z}^3, \Delta g\,\Delta\mathcal{Z}) = 0. \tag{4.36}$$

This equation, together with (4.31), gives

$$(\Delta Z_\pm)^2 = -K_\pm\,\Delta g + o(\Delta g), \tag{4.37}$$

where the constants $K_\pm$ (depending only on β and h) are given by

$$K_+ = \alpha^2 K_- \tag{4.38}$$

with

$$\alpha \equiv \frac{g_0\,a^{-1}\,\varphi'(Z^0_-)}{1 - g_0\,a\,\varphi'(Z^0_+)} = \frac{1 - g_0\,b^{-1}\,\varphi'(Z^0_-)}{g_0\,b\,\varphi'(Z^0_+)}, \tag{4.39}$$

where the identity follows from (4.29), and

$$K_- \equiv \frac{2}{g_0} \frac{\alpha\, a\, \varphi(Z_+^0) + b^{-1}\, \varphi(Z_-^0)}{\alpha^3\, a\, \varphi''(Z_+^0) + b^{-1}\, \varphi''(Z_-^0)}\,. \tag{4.40}$$

This proves that $Z_\pm(g)$ has a square root branch point at $g = g_0$ in the disc $\{g \mid |g| \le g_0\}$.

Remark 4.3.2. *The transpose of the matrix $g_0\Phi_0'$ has positive entries and eigenvalues 1 and λ, with*

$$\lambda = \det g_0\Phi'(Z_+^0, Z_-^0) = g_0^2(ab^{-1} - a^{-1}b)\varphi'(Z_+^0)\varphi'(Z_-^0). \tag{4.41}$$

In particular, we have $\lambda < 1$ by construction and $\lambda > -1$ since

$$1 + \lambda = g_0(a\varphi'(Z_+^0) + b^{-1}\varphi'(Z_-^0)) > 0. \tag{4.42}$$

Hence 1 is the Perron-Frobenius eigenvalue of the transpose of $g_0\Phi_0'$ (cf. [48] and refs. therein) and we have $c_1, c_2 > 0$ and accordingly $\alpha > 0$.

Making further use of the implicit function theorem we next show that $Z_\pm(g)$ have extensions to a so-called Δ-domain (cf. [48]), as described by the following proposition.

Proposition 4.3.3. *Suppose the greatest common divisor of $\{n \mid p_n > 0\}$ is 1. Then the functions $Z_\pm(g)$ can be analytically extended to a domain*

$$D_{\epsilon,\vartheta} = \{z \mid |z| < g_0 + \epsilon,\, z \neq g_0,\, |\arg(z - g_0)| > \vartheta\} \tag{4.43}$$

and (4.37) holds in $D_{\epsilon,\vartheta}$, for some $\epsilon > 0$ and $0 \le \vartheta < \frac{\pi}{2}$.

Proof. From $\det(\mathbb{1} - g\Phi'(Z_+, Z_-))|_{g=g_0} = 0$ and $\det(\mathbb{1} - g\Phi'(Z_+, Z_-))|_{g=0} = 1$, we have

$$\det(\mathbb{1} - g\Phi'(Z_+, Z_-)) > 0, \qquad 0 \le g < g_0. \tag{4.44}$$

Hence

$$|\det(\mathbb{1} - g\Phi'(Z_+, Z_-))| \geq \det\left(\mathbb{1} - |g|\Phi'(Z_+(|g|), Z_-(|g|))\right) > 0 \quad (4.45)$$

for $|g| < g_0$, where we have used that φ and $Z_\pm$ have positive Taylor coefficients. Moreover, in the limiting case $|g| = g_0$ we get that $\det(\mathbb{1} - g\Phi'(Z_+, Z_-)) = 0$ if and only if

$$g\varphi'(Z_\pm(g)) = g_0\varphi'(Z_\pm(g_0)). \quad (4.46)$$

In particular, $|\varphi'(Z_\pm(g))| = \varphi'(Z_\pm(g_0))$ which implies

$$|Z_\pm(g)| = Z_\pm(g_0). \quad (4.47)$$

By the definition of $Z_{N\pm}(\beta, h)$ we have that $Z_{N\pm}(\beta, h) > 0$ for all N of the form

$$N = 1 + n_1 + n_2 + \cdots + n_s, \quad (4.48)$$

where n_i are such that $p_{n_i} > 0$, $i = 1, \ldots, s$. Hence, eq. (4.47) implies

$$g^N = e^{i\theta} g_0^N \quad (4.49)$$

for some fixed $\theta \in \mathbb{R}$ and all such N. By the assumption on (p_n) this implies $g = g_0$. This proves that the functions $Z_\pm(g)$ can be analytically extended beyond the boundary of the disc $\{g \mid |g| \leq g_0\}$, except at g_0.

It remains to show that

$$\det(\mathbb{1} - g\Phi'(Z_+, Z_-)) \neq 0 \quad (4.50)$$

for $0 < |g - g_0| < \epsilon$ for some $\epsilon > 0$, since this together with the implicit function theorem proves the claim with $\vartheta = 0$. By (4.37) it suffices to show

$$\begin{aligned}&\frac{\partial}{\partial Z_+}\det(\mathbb{1} - g\Phi'(Z_+, Z_-))\Big|_{Z^0_\pm} \sqrt{K_+} \\ &+ \frac{\partial}{\partial Z_-}\det(\mathbb{1} - g\Phi'(Z_+, Z_-))\Big|_{Z^0_\pm} \sqrt{K_-} \neq 0.\end{aligned} \tag{4.51}$$

The LHS equals

$$\begin{aligned}&\Big[- g_0 a\varphi''(Z^0_+)\,(1 - g_0 b^{-1}\varphi'(Z^0_-)) - g_0^2 a^{-1} b\varphi''(Z^0_+)\varphi'(Z^0_-)\Big]\sqrt{K_+} \\ &+ \Big[- g_0 b^{-1}\varphi''(Z^0_-)\,(1 - g_0 a\varphi'(Z^0_+)) - g_0^2 a^{-1} b\varphi'(Z^0_+)\varphi''(Z^0_-)\Big]\sqrt{K_-}\end{aligned} \tag{4.52}$$

which obviously is < 0. The reader may also consult [38] for a general theorem on the asymptotic behavior of solutions to systems of functional equations of the type considered here (see 2.2.2). □

The above result allows us to use a standard transfer theorem [48], discussed in Sec. 2.2.1, to determine the asymptotic behavior of $Z_{N\pm}(\beta, h)$ for $N \to \infty$. We state it as follows.

Corollary 4.3.4. *Under the assumptions of Proposition 4.3.3, we have*

$$Z_{N\pm}(\beta, h) = \frac{1}{2}\sqrt{\frac{g_0 K_\pm}{\pi}}\, g_0^{-N}\, N^{-3/2}(1 + o(1)) \tag{4.53}$$

for $N \to \infty$, where g_0, $K_\pm > 0$ are determined by (4.22), (4.29), and (4.37)-(4.40).

4.3.2 The limiting measure

For $1 \leq N < \infty$ and fixed $\beta, h \in \mathbb{R}$ we define the probability distributions $\mu_{N\pm}$ on $\Lambda_{N\pm} \subset \Lambda$ by

$$\mu_{N\pm}(\tau_s) = \frac{1}{Z_{N\pm}} e^{-H(\tau_s)} \rho(\tau), \tag{4.54}$$

such that

$$\mu_N = \frac{Z_{N+}}{Z_N} \mu_{N+} + \frac{Z_{N-}}{Z_N} \mu_{N-}. \tag{4.55}$$

We shall need the following proposition, that can be obtained by a slight modification of the proof of Proposition 3.2 in [39], and whose details we omit.

Proposition 4.3.5. *Let K_R, $R \in \mathbb{N}$, be a sequence of positive numbers. Then the subset*

$$C = \bigcap_{r=1}^{\infty} \left\{ \tau_s \in \Lambda \;\middle|\; |B_R(\tau)| \leq K_R \right\} \tag{4.56}$$

of Λ is compact.

We are now ready to prove the following main result of this section.

Theorem 4.3.6. *Let $\beta, h \in \mathbb{R}$ and assume that the genericity condition (4.28) holds and that the greatest common divisor of $\{n \mid p_n > 0\}$ is 1. Then the weak limits*

$$\mu_\pm = \lim_{N\to\infty} \mu_{N\pm} \quad \textit{and} \quad \mu = \lim_{N\to\infty} \mu_N \tag{4.57}$$

exist as probability measures on Λ and

$$\mu = \frac{\alpha}{1+\alpha}\, \mu_+ + \frac{1}{1+\alpha}\, \mu_-, \tag{4.58}$$

where α is given by (4.39).

Proof. The identity (4.58) follows immediately from (4.55), Corollary 4.3.4 and (4.38), provided $\mu_\pm$ exist. Hence it suffices to show that μ_+ exists (since existence of μ_- follows by identical arguments).

According to [39] (see Sec. 1.1.2), it is sufficient to prove that the sequence (μ_{N+}) satisfies a certain *tightness condition* (see e.g. [17] for a definition) and that the sequence

$$\mu_{N+}(\left\{\tau_s \mid B_R(\tau) = \hat{\tau}, s|_{V(\hat{\tau})} = \hat{s}\right\}) \tag{4.59}$$

is convergent in $\mathbb{R}$ as $N \to \infty$, for each finite tree $\hat{\tau} \in \mathcal{T}$ and fixed spin configuration $\hat{s}$.

Tightness of (μ_{N+}): As a consequence of Proposition 4.3.5, this condition holds if we show that for each $\epsilon > 0$ and $R \in \mathbb{N}$ there exists $K_R > 0$ such that

$$\mu_{N+}(\{\tau_s \mid |B_R(\tau)| > K_R\}) < \epsilon, \quad N \in \mathbb{N}. \tag{4.60}$$

For $R = 1$ this is trivial. For $R = 2$, $k \geq 1$ we have

$$\begin{aligned}
&\mu_{N+}(\{\tau_s \mid |B_2(\tau)| = k+1\}) \\
&\quad = Z_{N+}^{-1} \sum_{N_1+\cdots+N_k=N-1} \left[a \prod_{i=1}^{k} Z_{N_i+} + a^{-1} \prod_{i=1}^{k} Z_{N_i-} \right] p_k \\
&\quad \leq k \sum_{\substack{N_1+\cdots+N_k=N-1 \\ N_1 \geq (N-1)/k}} Z_{N+}^{-1} \left[a \prod_{i=1}^{k} Z_{N_i+} + a^{-1} \prod_{i=1}^{k} Z_{N_i-} \right] p_k \\
&\quad \leq cst.\, k^{5/2} \left[Z_+(g_0)^{k-1} + Z_-(g_0)^{k-1} \right] p_k \,,
\end{aligned} \tag{4.61}$$

where we have used (4.53). The last expression tends to zero for $k \to \infty$ as a consequence of (4.28). This proves (4.60) for $R = 2$.

For $R > 2$ it is sufficient to show

$$\mu_{N+}(\left\{\tau_s \mid |B_{R+1}(\tau)| > K, B_R(\tau) = \hat{\tau}, s|_{V(\hat{\tau})} = \hat{s}\right\}) \to 0 \tag{4.62}$$

uniformly in N for $k \to \infty$, for fixed $\hat{\tau}$ of height R and fixed $\hat{s} \in \{\pm 1\}^{V(\hat{\tau})}$, as well as fixed $K > 0$. Let L denote the number of vertices in $\hat{\tau}$ at maximal height R. Any $\tau \in \Lambda$ with $B_R(\tau) = \hat{\tau}$ is obtained by attaching a sequence of trees $\tau_1, \ldots, \tau_S$ in Λ such that the root vertex of τ_i is identified with a vertex at maximal height in $\hat{\tau}$. We must then have

$$|\tau_1| + \cdots + |\tau_S| = |\tau| - |\hat{\tau}| \tag{4.63}$$

and

$$k_1 + \cdots + k_L = S, \tag{4.64}$$

where $k_i \geq 0$ denotes the number of trees attached to vertex v_i in $\hat{\tau}$, $i = 1, \ldots, L$. For fixed $k_1, \ldots, k_L$ we get a contribution to (4.62) equal to

$$\begin{aligned} Z_{N+}^{-1} \sum_{N_1+\cdots+N_S=N-|\hat{\tau}|} &\left(\prod_{i=1}^{L} (Z_{N_i \hat{s}_{v_i}})^{k_i} p_{k_i} \right) e^{-H(\hat{\tau}_{\hat{s}})} \prod_{v \in V(\hat{\tau}) \setminus \{r, v_1, \ldots, v_L\}} p_{\sigma_v - 1} \\ &\leq cst. \prod_{i=1}^{L} (\max Z_{\pm}^0)^{k_i} p_{k_i} \, (k_i+1)^{5/2} \end{aligned} \tag{4.65}$$

where the inequality is obtained as above for $R = 2$ and the constant is independent of $k_1, \ldots, k_L$.

Since

$$|B_{R+1}(\tau)| = |\hat{\tau}| + k_1 + \cdots + k_L > K \tag{4.66}$$

and the number of choices of $k_1, \ldots, k_L \geq 0$ for fixed $k = k_1 + \cdots + k_L$ equals

$$\binom{k+L-1}{L-1} \leq \frac{k^{L-1}}{(L-1)!} \tag{4.67}$$

the claim (4.62) follows from (4.28) and (4.65).

Convergence of $\mu_{N+}(\left\{ \tau_s \mid B_R(\tau) = \hat{\tau}, s|_{V(\hat{\tau})} = \hat{s} \right\})$: Using the decomposition of τ into $\hat{\tau}$ with branches described above and using the arguments in the last part of the proof of Theorem 3.3 in [39] we get, with notation as above, that

$$\mu_{N\pm}(\left\{\tau_s \mid B_R(\tau) = \hat{\tau}, s|_{V(\hat{\tau})} = \hat{s}\right\}) \xrightarrow{N\to\infty} \frac{g_0^{|\hat{\tau}|}}{\sqrt{K_\pm}} e^{-H(\hat{\tau}_{\hat{s}})} \sum_{i=1}^{L} \sqrt{K_{\hat{s}(v_i)}} \varphi'(Z^0_{\hat{s}(v_i)}) \prod_{j\neq i} \varphi(Z^0_{\hat{s}(v_j)}), \tag{4.68}$$

provided $\hat{s}(r) = \pm 1$ (if $\hat{s}(r) = \mp 1$ the limit is trivially 0). □

Introducing the notation

$$A(\hat{s}) = \left\{\tau_s \mid B_R(\tau) = \hat{\tau},\, s|_{V(\hat{\tau})} = \hat{s}\right\}, \tag{4.69}$$

where $\hat{\tau}$ is a finite tree of height R with spin configuration $\hat{s}$, and using (4.38), it follows from (4.68) that the $\mu_\pm$-volumes of this set are given by

$$\mu_\pm(A(\hat{s})) = g_0^{|\hat{\tau}|} e^{-H(\hat{\tau}_{\hat{s}})} \sum_{i=1}^{L} \alpha^{(\hat{s}(v_i)\mp 1)/2} \varphi'(Z^0_{\hat{s}(v_i)}) \prod_{j\neq i} \varphi(Z^0_{\hat{s}(v_j)}), \tag{4.70}$$

if $\hat{s}(r) = \pm 1$ and where $v_1, \ldots, v_L$ are the vertices at maximal distance from the root r in $\hat{\tau}$.

The above calculations show, by similar arguments as in [39, 25], that the limiting measures $\mu_\pm$ are concentrated on trees with a single infinite path starting at r, called the *spine*, and attached to each spine vertex u_i, $i = 1, 2, 3 \ldots$, is a finite number k_i of finite trees, called *branches*, some of which are attached to the left and some to the right as seen from the root, cf. Fig.4.2.

The following corollary provides a complete description of the limiting measures $\mu_\pm$.

Corollary 4.3.7. *The measures $\mu_\pm$ are concentrated on the sets*

$$\bar{\Lambda}_\pm = \left\{\tau_s \in \Lambda_\pm \mid \tau \textit{ has a single spine}\right\}, \tag{4.71}$$

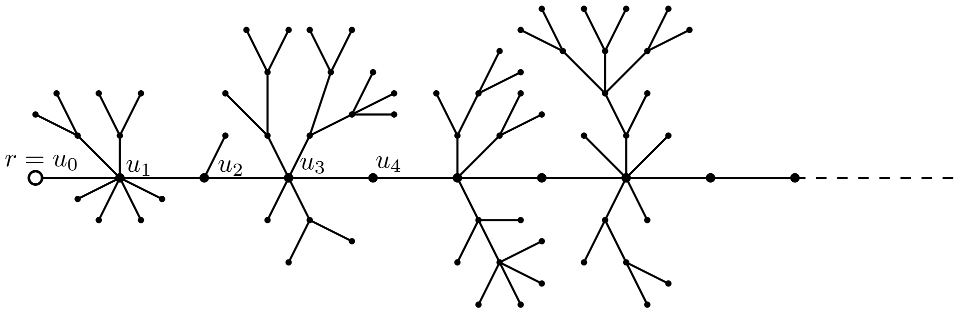

Figure 4.2: Example of an infinite tree, consisting of a spine and left and right branches.

respectively, and can be described as follows:

i) *The probability that the spine vertices* $u_0 = r, u_1, u_2, \ldots, u_N$ *have* $k'_1, \ldots, k'_N$ *left branches and* $k''_1, \ldots, k''_N$ *right branches and spin values* $s_0 = \pm 1$, s_1, s_2, $\ldots, s_N$, *respectively, equals*

$$\rho^{s_0}_{k'_1,\ldots,k'_N,k''_1,\ldots,k''_N}(s_0,\ldots,s_N) = g_0^N e^{-H_N}\left(\prod_{i=1}^{N}(Z^0_{s_i})^{k'_i+k''_i} p_{k'_i+k''_i+1}\right)\alpha^{(s_N-s_0)/2}, \tag{4.72}$$

with

$$H_N = -\beta\sum_{i=1}^{N} s_{i-i}s_i - h\sum_{i=1}^{N} s_i. \tag{4.73}$$

ii) *The conditional probability distribution of any finite branch* τ_s *at a fixed* u_i, $1 \le i \le N$, *given* $k'_1, \ldots, k'_N$, $k''_1, \ldots, k''_N$, $s_0, \ldots, s_N$ *as above, is given by*

$$\nu_{s_i}(\tau_s) = (Z^0_{s_i})^{-1}\, g_0^{|\tau|}\, e^{-H(\tau_s)} \prod_{v\in V(\tau)\setminus u_i} p_{\sigma_v - 1} \tag{4.74}$$

for $s(u_i) = s_i$, *and 0 otherwise.*

iii) *The conditional distribution of the infinite branch at* u_N, *given* $k'_1, \ldots, k'_N$,

$k''_1, \dots, k''_N$, $s_0, \dots, s_N$, equals μ_{s_N}.

4.4 Hausdorff and spectral dimensions

In this section we determine the values of the Hausdorff and spectral dimensions of the ensemble of trees $(\mathcal{T}, \bar{\mu})$ obtained from (Λ, μ) by integrating over the spin degrees of freedom, that is

$$\bar{\mu}(A) = \mu(\{\tau_s \mid \tau \in A\}) \tag{4.75}$$

for $A \subseteq \mathcal{T}$. Note that the mapping $\tau_s \to \tau$ from Λ to $\mathcal{T}$ is a contraction w. r. t. the metric (4.3) and the metric on $\mathcal{T}$ defined by (1.6).

Most of the arguments in this section are based on the methods of [42], and we shall mainly focus on the novel ingredients that are needed and otherwise refer to [42] for additional details.

4.4.1 The annealed Hausdorff dimension

Theorem 4.4.1. *Under the assumptions of Theorem 4.3.6 the annealed Hausdorff dimension of $\bar{\mu}$ is 2 for all β, h:*

$$\bar{d}_H = \lim_{R \to \infty} \frac{\ln \langle |B_R| \rangle_{\bar{\mu}}}{\ln R} = 2\,. \tag{4.76}$$

Proof. Consider the probability distribution $\nu_\pm$ on $\{\tau_s \mid \tau \text{ is finite}\}$ given by (4.74) and denote by $D_R(\tau)$ the set of vertices at distance R from the root in τ. For a fixed branch T, we set

$$f_R^\pm = \langle |D_R| \rangle_{\nu_\pm} Z_\pm^0. \tag{4.77}$$

where $\langle \cdot \rangle_{\nu_\pm}$ denotes the expectation value w.r.t. $\nu_\pm$. Arguing as in the derivation of (4.22), we find

$$\begin{cases} f_R^+ = g_0 \left(a\, \varphi'(Z_+^0)\, f_{R-1}^+ + a^{-1}\, \varphi'(Z_-^0)\, f_{R-1}^- \right) \\ f_R^- = g_0 \left(b\, \varphi'(Z_+^0)\, f_{R-1}^+ + b^{-1}\, \varphi'(Z_-^0)\, f_{R-1}^- \right) , \end{cases} \tag{4.78}$$

for $R \geq 2$, and $f_1^\pm = Z_\pm^0$. Using that c, given by (4.35), is a left eigenvector of $g_0 \Phi_0'$ with eigenvalue 1, this implies

$$\begin{aligned} c_1\, f_R^+ + c_2\, f_R^- &= c_1\, f_{R-1}^+ + c_2\, f_{R-1}^- = \dots \\ &= c_1\, f_1^+ + c_2\, f_1^- = c_1\, Z_+^0 + c_2\, Z_-^0 \,. \end{aligned} \tag{4.79}$$

Since c_1, c_2, $Z_\pm^0$, $f_R^\pm > 0$, we conclude that

$$k_1 \leq \langle |D_R| \rangle_{\nu_\pm} \leq k_2, \quad R \geq 1 \,, \tag{4.80}$$

where k_1, k_2 are positive constants (depending on β, h). Using

$$\langle |B_R| \rangle_{\nu_\pm} = \sum_{R'=0}^{R} \langle |D_{R'}| \rangle_{\nu_\pm} \tag{4.81}$$

we then obtain

$$1 + k_1\, R \leq \langle |B_R| \rangle_{\nu_\pm} \leq 1 + k_2\, R, \tag{4.82}$$

Finally, it follows from (4.72) that

$$1 + R + k_1 \frac{1}{2} R(R+1) \leq \langle |B_R| \rangle_\mu \leq 1 + R + k_2 \frac{1}{2} R(R+1) \,, \tag{4.83}$$

which proves the claim. □

Remark 4.4.2. *By a more elaborate argument, using the methods of [42, 43], one can show that the Hausdorff dimension d_H defined by* (1.11) *exists and equals* 2 *almost surely, that is for all trees $\tau \in \mathcal{T}$ except for a set of vanishing $\bar{\mu}$-measure. We shall not make use of this result below and refrain from giving further details.*

4.4.2 The annealed spectral dimension

In this section we first establish two results needed for determining the spectral dimension. The first one is a version of a classical result, proven by Kolmogorov for Galton-Watson trees [50], on survival probabilities for $\nu_\pm$.

Proposition 4.4.3. *The measures $\nu_\pm$ defined by (4.74) fulfill*

$$\frac{k_-}{R} \leq \nu_\pm(\{\tau_s \in \Lambda \mid D_R(\tau) \neq \emptyset\}) \leq \frac{k_+}{R}, \quad R \geq 1, \tag{4.84}$$

where $k_\pm > 0$ are constants depending on β, h.

Proof. Let $H_R^\pm(w)$ be the generating function for the distribution of $|D_R|$ w.r.t. $\nu_\pm$,

$$H_R^\pm(w) = Z_\pm^0 \sum_{n=0}^{\infty} \nu_\pm(\{\tau_s \mid |D_R(\tau)| = n\})\, w^n. \tag{4.85}$$

Arguing as in the proof of (4.22), we have

$$\begin{cases} H_R^+ = g_0 \left(a\, \varphi(H_{R-1}^+) + a^{-1}\, \varphi(H_{R-1}^-) \right) \\ H_R^- = g_0 \left(b\, \varphi(H_{R-1}^+) + b^{-1}\, \varphi(H_{R-1}^-) \right) \end{cases}, \tag{4.86}$$

for $R \geq 2$, and $H_1^\pm = Z_\pm^0\, w$.

Note that

$$Z_\pm^0\, \nu_\pm(\{\tau_s \in \Lambda \mid D_R(\tau) \neq \emptyset\}) = Z_\pm^0 - H_R^\pm(0), \tag{4.87}$$

and that the radius of convergence for $H_R^\pm$ is ≥ 1. Also, $(H_R^\pm(0))_{R\geq 1}$ is an increasing sequence. In fact, $H_1^\pm(0) = 0$ and so $H_2^\pm(0) > 0$ by (4.86). Since φ is positive and increasing on $[0, \xi)$, it then follows by induction from (4.86) that $(H_R^\pm(0))_{R\geq 1}$ is increasing. Hence, we conclude from (4.86) and (4.22) that

$$H_R^\pm(0) \nearrow Z_\pm^0 \quad \text{for} \quad R \to \infty. \tag{4.88}$$

Taking R large enough and expanding $\varphi(H_R^{\pm}(0))$ around $Z_{\pm}^0$ we obtain, in matrix form,

$$\Delta_R = g_0\, \Phi_0'\, \Delta_{R-1} - \frac{g_0}{2}\, \Phi_0''\, \Delta_{R-1}^2 + O(\Delta_{R-1}^3)\,, \tag{4.89}$$

where

$$\Delta_R^n = \begin{pmatrix} (\Delta_R^+)^n \\ (\Delta_R^-)^n \end{pmatrix} = \begin{pmatrix} (Z_+^0 - H_R^+(0))^n \\ (Z_+^0 - H_R^+(0))^n \end{pmatrix}, \tag{4.90}$$

and where Φ_0', Φ_0'' are given by (4.30) and (4.33). Setting $L_R = c\,\Delta_R$, eq. (4.89) gives

$$L_R = L_{R-1} - \frac{g_0}{2}\, c\, \Phi_0''\, \Delta_{R-1}^2 + O(\Delta_{R-1}^3)\,. \tag{4.91}$$

From this we deduce that there exists $R_0 > 0$ such that

$$L_{R-1} - A_- L_{R-1}^2 \leq L_R \leq L_{R-1} - A_+ L_{R-1}^2, \qquad R \geq R_0, \tag{4.92}$$

where $A_{\pm} = A_{\pm}(\beta, h)$ are constants. Hence, it follows that

$$\frac{1}{L_{R-1}} + B_- \leq \frac{1}{L_{R-1}} \frac{1}{1 - A_- L_{R-1}} \leq \frac{1}{L_R} \leq \frac{1}{L_{R-1}} \frac{1}{1 - A_+ L_{R-1}} \leq \frac{1}{L_{R-1}} + B_+, \tag{4.93}$$

for $R \geq R_0$, where $B_{\pm} > 0$ are constants. This implies

$$B_- R + C_- \leq \frac{1}{L_R} \leq B_+ R + C_+ \tag{4.94}$$

for suitable constants $C_{\pm}$. Evidently, this proves that

$$\frac{D_-}{R} \leq Z_{\pm}^0 - H_R^{\pm}(0) \leq \frac{D_+}{R}, \quad R \geq 1, \tag{4.95}$$

where $D_{\pm} > 0$ are constants, which together with (4.87) proves the claim. □

We also note the following generalization of Lemma 4 in [42].

Lemma 4.4.4. *Suppose $u : \Lambda \to \mathbb{C}$ is a bounded function depending only on $\tau_s \in \Lambda$ through the ball $B_R(\tau)$ and the spins in $B_R(\tau)$, except those on its boundary, for some $R \geq 1$. Moreover, define the function $E_R : \Lambda \to \mathbb{R}$ by*

$$E_R(\tau_s) = \sum_{v \in D_R(\tau)} \frac{\sqrt{K_{s_v}}}{Z^0_{s_v}}, \tag{4.96}$$

with the convention $E_R(\tau_s) = 0$ if $D_R(\tau) = \emptyset$. Then

$$\int_\Lambda u(\tau_s) d\mu_\pm(\tau_s) = \frac{Z^0_\pm}{\sqrt{K_\pm}} \int_\Lambda u(\tau_s) E_R(\tau_s) d\nu_\pm(\tau_s). \tag{4.97}$$

Proof. Using (4.72-4.74) we may evaluate the LHS of (4.97) and get

$$\sum_{\tau_s \in \Lambda(R)} u(\tau_s)\, g_0^{|\tau|}\, e^{-H(\tau_s)}\, \alpha^{(s(v_R)-s_0)/2} \prod_{v \in V(\tau)\setminus r} p_{\sigma_v - 1}\,, \tag{4.98}$$

where $\Lambda(R)$ denotes the set of finite rooted trees in Λ with one marked vertex w_R of degree 1 at distance R from the root, and v_R is the neighbor of w_R.

On the other hand, the integral on the RHS can be written as

$$\frac{1}{Z^0_\pm} \sum_{\tau_s \in \Lambda(R)} u(\tau_s)\, g_0^{|\tau|} e^{-H(\tau_s)} \frac{\sqrt{K_{s(v_R)}}}{Z^0_{s(v_R)}} Z^0_{s(v_R)} \prod_{v \in V(\tau)\setminus r} p_{\sigma_v - 1}. \tag{4.99}$$

By comparing the two expressions the identity (4.97) follows. □

As a consequence of this result we have the following lemma.

Lemma 4.4.5. *There exist constants $c_\pm > 0$ such that*

$$\left\langle |B_R|^{-1} \right\rangle_{\mu_\pm} \leq c_\pm R^{-2} \tag{4.100}$$

Proof. Define, for fixed $R \geq 1$, the function

$$u(\tau) = \begin{cases} |D_R(\tau)|^{-1} & \text{if } D_R(\tau) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases} \tag{4.101}$$

Then $u(\tau)$ fulfills the assumptions of Lemma 4.4.4 for this value of R. Hence

$$\begin{aligned} \left\langle |D_R(\tau)|^{-1} \right\rangle_{\mu_\pm} &= \frac{Z^0_\pm}{\sqrt{K_\pm}} \sum_{\tau_s : D_R(\tau) \neq \emptyset} |D_R(\tau)|^{-1} \, E(\tau_s) \, e^{-H(\tau_s)} \prod_{v \in V(\tau) \setminus r} p_{\sigma_v - 1} \\ &\leq c'_\pm \sum_{\tau_s : D_R(\tau) \neq \emptyset} e^{-H(\tau_s)} \prod_{v \in V(\tau) \setminus r} p_{\sigma_v - 1} \leq \frac{c''_\pm}{R}, \end{aligned} \tag{4.102}$$

where Proposition 4.4.3 is used in the last step. Combining this fact with Jensen's inequality, we obtain

$$\begin{aligned} \left\langle |B_R|^{-1} \right\rangle_{\mu_\pm} &= \left\langle \frac{1}{|D_1| + \cdots + |D_R|} \right\rangle_{\mu_\pm} \\ &\leq R^{-1} \left\langle \left(|D_1||D_2| \cdots |D_R| \right)^{-1/R} \right\rangle_{\mu_\pm} \\ &\leq R^{-1} \prod_{i=1}^{R} \left\langle |D_i|^{-1} \right\rangle_{\mu_\pm}^{1/R} \\ &\leq c''_\pm (R!)^{-1/R} \leq c_\pm R^{-2}. \end{aligned} \tag{4.103}$$

□

Returning to the spectral dimension, let us define, with the notation of subsection 1.1.4, the generating function for return probabilities of the simple random walk on a tree τ by

$$Q_\tau(x) = \sum_{t=0}^{\infty} (1-x)^{\frac{t}{2}} \pi_t(\tau, r) \,, \tag{4.104}$$

and set

$$Q(x) = \langle Q_\tau(x) \rangle_{\bar{\mu}} \,. \tag{4.105}$$

The annealed spectral dimension as defined by (1.14) is related to the singular behavior of the function $Q(x)$ as follows. First, note that if $\bar{d}_s$ exists, we have

$$\langle \pi_t(\tau, r) \rangle_{\bar{\mu}} \sim t^{-\frac{\bar{d}_s}{2}}, \quad t \to \infty \,. \tag{4.106}$$

For $\bar{d}_s < 2$, this implies that $Q(x)$ diverges as

$$Q(x) \sim x^{-\gamma}, \quad \text{as} \quad x \to 0, \tag{4.107}$$

where

$$\gamma = 1 - \frac{\bar{d}_s}{2} \,. \tag{4.108}$$

We shall take (4.107) and (4.108) as the definition of $\bar{d}_s$ and prove (4.107) with $\gamma = \frac{1}{3}$ by establishing the estimates

$$\underline{c}\, x^{-1/3} \leq Q(x) \leq \bar{c}\, x^{-1/3} \tag{4.109}$$

for x sufficiently small, where $\underline{c}$ and $\bar{c}$ are positive constants, that may depend on β, h.

Theorem 4.4.6. *Under the assumptions of Theorem 4.3.6, the annealed spectral dimension of* $(\mathcal{T}, \bar{\mu})$ *is*

$$\bar{d}_s = \frac{4}{3} \,. \tag{4.110}$$

Proof. We first prove the lower bound in (4.109).

Let $R \geq 1$ be fixed and consider the spine vertices $u_0, u_1, \ldots, u_R$ with given spin values $s_0, \ldots, s_R$ and branching numbers $k'_1, \ldots, k'_R, k''_1, \ldots, k''_R \geq 0$ as in Corollary 4.3.7. The conditional probability that a given branch at u_j has length $\geq R$ is bounded by $\frac{c}{R}$ by Proposition 4.4.3. Hence, the conditional probability that at least one of the $k'_j + k''_j$ branches at u_j has height $\geq R$ is bounded by $(k'_j + k''_j)\frac{c}{R}$. Using Corollary 4.3.7 and summing over $k'_1, \ldots, k'_R, k''_1, \ldots, k''_R$, we get that the conditional probability q_R that at least one branch at u_j is of height $\geq R$, given $s_0, \ldots, s_R$, is bounded by

$$\frac{1}{1+\alpha} g_0^R e^{-H_R} \prod_{\substack{i=1 \\ i \neq j}}^{R} \varphi'(Z^0_{s_i}) \, \varphi''(Z^0_{s_j}) \alpha^{(s_R+1)/2} \frac{c}{R} \leq \frac{c'}{R}. \tag{4.111}$$

Using that the distributions of the branches at different spine vertices are independent for given $s_0, \ldots, s_R$, it follows that the conditional probability that no branch at $u_1, \ldots, u_R$ has length $\geq R$, for given $s_0, \ldots, s_R$, is bounded from below by

$$(1 - q_R)^R \geq \left(1 - \frac{c'}{R}\right)^R \geq e^{-c' + O(R^{-1})}. \tag{4.112}$$

Denoting this conditioned event by $\mathcal{A}_R$, it follows from Lemmas 6 and 7 in [42] that the conditional expectation of $Q_\tau(x)$, given $s_0, s_1, \ldots, s_R$, is

$$\begin{aligned} &\geq e^{c' + O(R^{-1})} \left\langle \left(\frac{1}{R} + Rx + \sum_{T \subset \tau}^{R} x \, |T| \right)^{-1} \right\rangle_R \\ &\geq e^{c' + O(R^{-1})} \left(\frac{1}{R} + Rx + x \left\langle \sum_{T \subset \tau}^{R} |T| \right\rangle_R \right)^{-1}. \end{aligned} \tag{4.113}$$

Here $\langle \cdot \rangle_R$ denotes the conditional expectation value w.r.t. μ on $\mathcal{A}_R$ and $\sum^R_{T \subset \tau}$ the sum over all branches T of τ attached to vertices on the spine at distance $\leq R$ from the root. We have

$$\begin{aligned} \left\langle \sum_{T \subset \tau}^{R} |T| \right\rangle_R &= \sum_{i=1}^{R} \left\langle |B^i_R(\tau)| \right\rangle_R \\ &\leq \sum_{i=1}^{R} \mu(\mathcal{A}_R \mid s_0, \ldots, s_R)^{-1} \left\langle |B^i_R| \right\rangle_\mu \\ &\leq e^{c' + O(R^{-1})} \sum_{i=1}^{R} \langle |B_R| \rangle_{\nu_{s_i}} \leq C \, R^2, \end{aligned} \tag{4.114}$$

where (4.82) is used in the last step.

This bound being independent of $s_0, \ldots, s_R$ we have proven that

$$Q(x) \geq cst. \left(\frac{1}{R} + Rx + CR^2 x \right)^{-1} \tag{4.115}$$

and consequently, choosing $R \sim x^{-\frac{1}{3}}$, it follows that

$$Q(x) \geq \underline{c}\, x^{-\frac{1}{3}}. \tag{4.116}$$

As concerns the upper bound in (4.109), it follows by an argument identical to the one in [42] on p.1245–50 by using Lemma 4.4.5. □

4.5 Absence of spontaneous magnetization

Using the characterization of the measure $\mu^{(\beta,h)}$ established in Section 4.3 and that $\bar{d}_H = 2$, we are now in a position to discuss the magnetization properties of generic Ising trees in some detail. In view of the fact that the trees have a single spine, we distinguish between the magnetization on the spine and the bulk magnetization. In subsection 4.5.1 we show that the former can be expressed in terms of an effective Ising model on the half-line $\{0, 1, 2, \ldots\}$. The bulk magnetization is discussed in subsection 4.5.2

4.5.1 Magnetization on the spine

The following result is crucial for the subsequent discussion.

Proposition 4.5.1. *Under the assumptions of Theorem 4.3.6, the functions $Z^0_\pm$ are smooth functions of β, h.*

Proof. In Section 4.3.1 we have shown that $Z_\pm(\beta, h, g)$ is a solution to the equation

$$F(Z_+, Z_-, g) = 0$$

where F is defined in (4.24), and that

$$Z^0_\pm(\beta,h) = Z_\pm(g_0(\beta,h),\beta,h) \tag{4.117}$$

is a solution to

$$\begin{cases} F(Z^0_+,Z^0_-,g_0) = 0 \\ \det(\mathbb{1} - g_0\Phi'(Z^0_+,Z^0_-)) = 0\,, \end{cases} \tag{4.118}$$

considered as three equations determining (Z^0_+,Z^0_-,g_0) implicitly as functions of (β,h). Hence, defining $G:\ (-R,R)^2\times\mathbb{R}^3\to\mathbb{R}^3$ by

$$G(Z^0_+,Z^0_-,g_0,\beta,h) = \begin{pmatrix} F(Z^0_+,Z^0_-,g_0) \\ \det(\mathbb{1} - g_0\Phi'(Z^0_+,Z^0_-)) \end{pmatrix}$$

it suffices to show that its Jacobian J with respect to (Z^0_+,Z^0_-,g_0) is regular at $(Z^0_+(\beta,h),Z^0_-(\beta,h),g_0(\beta,h))$. We have

$$J = \begin{pmatrix} \mathbb{1} - g_0\Phi'(Z^0_+,Z^0_-) & & -\Phi(Z^0_+,Z^0_-) \\ A_+ & A_- & B \end{pmatrix},$$

where

$$A_\pm = \frac{\partial}{\partial Z^0_\pm}\det(\mathbb{1} - g_0\Phi'(Z^0_+,Z^0_-))\,, \qquad B = \frac{\partial}{\partial g_0}\det(\mathbb{1} - g_0\Phi'(Z^0_+,Z^0_-))$$

are readily calculated and equal

$$A_+ = -g_0\,a\,\varphi''(Z^0_+)\,(1 - g_0\,b^{-1}\varphi'(Z^0_-)) - g_0^2\,a^{-1}\,b\,\varphi''(Z^0_+)\varphi'(Z^0_-)\,, \tag{4.119}$$

$$A_- = -g_0\,b^{-1}\,\varphi''(Z^0_-)\,(1 - g_0\,a\varphi'(Z^0_+)) - g_0^2\,a^{-1}\,b\,\varphi'(Z^0_+)\varphi''(Z^0_-)\,, \tag{4.120}$$

and

$$B = -a\varphi'(Z^0_+) - b^{-1}\varphi'(Z^0_-) + 2g_0\,(ab^{-1} - a^{-1}b)\,\varphi'(Z^0_+)\varphi'(Z^0_-)\,. \tag{4.121}$$

Using eqs. (4.118) and (4.39), we get

$$\det J = (Z_+^0\, b\, \varphi'(Z_+^0) + g_0^{-1}\, Z_-^0\, (1 - g_0 a \varphi'(Z_+^0))) \begin{vmatrix} 1 & -\alpha \\ A_+ & A_- \end{vmatrix} < 0\,,$$

since clearly $A_\pm < 0$ and $\alpha > 0$ by Remark 4.3.2. This proves the claim. □

We can now establish the following result for the single site magnetization on the spine.

Theorem 4.5.2. *Under the assumptions of Theorem 4.3.6, the probability* $\mu^{(\beta,h)}(\{s_v = +1\})$ *is a smooth function of* β, h *for any spine vertex* v*. In particular, there is no spontaneous magnetization in the sense that*

$$\lim_{h\to 0} \mu^{(\beta,h)}(\{s_v = +1\}) = \frac{1}{2}\,. \tag{4.122}$$

Proof. For the root vertex r, we have by eq. (4.58) that

$$\mu^{(\beta,h)}(\{s(r) = +1\}) = \frac{\alpha(\beta,h)}{1+\alpha(\beta,h)}\,, \tag{4.123}$$

where $\alpha(\beta,h)$ is given by (4.39) and is a smooth function of β,h by Proposition 4.5.1. Hence, to verify (4.122) for $v = r$ it suffices to note that $\alpha(\beta,0) = 1$, since $a = b^{-1}$ and $Z_+^0 = Z_-^0$ for $h = 0$.

Now, assume $v = u_N$ is at distance N from the root, and define

$$p_{ij} = \mu_i(\{s_v = j\})\, \frac{\alpha^{\frac{1+i}{2}}}{1+\alpha}\,, \tag{4.124}$$

for i, $j \in \{\pm 1\}$, where we use ± 1 and $\pm$ interchangeably. From eq. (4.72) follows that

$$\begin{aligned}
\mu_{s_0}(\{s_v = s_N\}) &= \sum_{\substack{k_i',k_i''\geq 0\\ s_1,\dots,s_{N-1}}} \rho^{s_0}_{k_1',\dots,k_N',k_1'',\dots,k_N''}(s_0,\dots,s_N) \\
&= \sum_{s_1,\dots,s_{N-1}} \prod_{i=1}^{N} g_0[\Phi'(Z^0_+,Z^0_-)]_{s_{i-1}s_i}\,\alpha^{\frac{s_N-s_0}{2}} \\
&= [(g_0\,\Phi'(Z^0_+,Z^0_-))^N]_{s_0s_N}\,\alpha^{\frac{s_N-s_0}{2}},
\end{aligned} \tag{4.125}$$

where we have used that the matrix elements of $\Phi'(Z^0_+,Z^0_-)$ are given by

$$[\Phi'(Z^0_+,Z^0_-)]_{s_{i-1}s_i} = e^{\beta s_{i-i}s_i + hs_i}\varphi'(Z^0_{s_i})\,. \tag{4.126}$$

Hence, substituting into (4.124) we have

$$p_{ij} = \Big[(g_0\Phi'(Z^0_+,Z^0_-))^N\Big]_{ij}\,\frac{\alpha^{\frac{1+j}{2}}}{1+\alpha}. \tag{4.127}$$

By Proposition 4.5.1, all factors on the RHS of (4.127) are smooth functions of β, h, and by (4.58) we have

$$\mu^{(\beta,h)}(\{s_v = j\}) = p_{+j} + p_{-j}\,. \tag{4.128}$$

Eq. (4.122) is now obtained from (4.127) by noting again that for $h=0$ we have $\alpha = 1$ and hence $c_1 = c_2$, which by (4.34) gives

$$\begin{aligned}
p_{+j} + p_{-j} &= \Big[(1 \quad 1)(g_0\,\Phi'(Z^0,Z^0))^N\Big]_j\,\frac{1}{2} \\
&= (1 \quad 1)_j\,\frac{1}{2} = \frac{1}{2}\,.
\end{aligned} \tag{4.129}$$

□

The preceding proof together with (4.72) shows that the distribution of spin variables $s_0, \dots, s_N$ on the spine can be written in the form

$$\rho(s_0, \dots, s_N) = e^{-H'_N(s_0,\dots,s_N)} \left(g_0^2\, \varphi'(Z_+^0)\varphi'(Z_-^0)\right)^{N/2} \frac{\sqrt{\alpha}}{1+\alpha} \tag{4.130}$$

where

$$H'_N(s_0, \dots, s_N) = -\beta \sum_{i=1}^{N} s_{i-1}s_i - h' \sum_{i=1}^{N} s_i - \frac{s_N}{2} \log \alpha \tag{4.131}$$

and

$$h' = h + \frac{1}{2} \ln \frac{\varphi'(Z_+^0)}{\varphi'(Z_-^0)}\,. \tag{4.132}$$

Since $\rho(s_0, \dots, s_N)$ is normalized, the expectation value w.r.t. μ of a function $f(s_0, \dots, s_{N-1})$ hence coincides with the expectation value w. r. t. the Gibbs measure of the Ising chain on $[0, N]$, with Hamiltonian given by (4.131) and (4.132). In particular, we have that the mean magnetization on the spine vanishes in the absence of an external magnetic field, since h' is a smooth function of h, by Proposition 4.5.1, and vanishes for $h = 0$ (see e.g. [12] for details about the 1d Ising model).

We state this result as follows.

Corollary 4.5.3. *Under the assumptions of Theorem 4.3.6, the mean magnetization on the spine vanishes as $h \to 0$, i.e.*

$$\lim_{h\to 0} \lim_{N\to\infty} \left\langle \frac{s_0 + \cdots + s_{N-1}}{N} \right\rangle_{\beta,h} = 0. \tag{4.133}$$

4.5.2 Mean magnetization

For the mean magnetization on the full infinite tree, defined in Sec. 4.2.2, we have the following result, which requires some additional estimates in combination with Proposition 4.5.1.

Theorem 4.5.4. *Under the assumptions of Theorem 4.3.6, the mean magnetization vanishes for $h \to 0$, i.e.*

$$\lim_{h\to 0} M(\beta,h) = 0\,, \quad \beta \in \mathbb{R}\,, \tag{4.134}$$

where $M(\beta,h)$ is defined by (4.10)-(4.11).

Proof. Consider the measure $\nu_\pm$ given by (4.74) and, for a given finite branch T, let $S_R(T)$ denote the sum of spins at distance R from the root of T. Setting

$$m_R^\pm = Z_\pm^0 \, \langle S_R \rangle_{\nu_\pm} \tag{4.135}$$

it follows, by decomposing T according to the spin and the degree of the vertex closest to the root, that

$$\begin{cases} m_R^+ = g_0 \, \left(a\, \varphi'(Z_+^0)\, m_{R-1}^+ + a^{-1}\, \varphi'(Z_-^0)\, m_{R-1}^-\right) \\ m_R^- = g_0 \, \left(b\, \varphi'(Z_+^0)\, m_{R-1}^+ + b^{-1}\, \varphi'(Z_-^0)\, m_{R-1}^-\right), \end{cases} \tag{4.136}$$

for $R \geq 1$, and $m_0^\pm = \pm Z_\pm^0$. In matrix notation these recursion relations read

$$m_R = g_0 \Phi_0' \, m_{R-1}, \tag{4.137}$$

which, upon multiplication by the left eigenvector c of $g_0\Phi_0'$, leads to

$$c\, m_R = g_0\, c\, \Phi_0'\, m_{R-1} = c\, m_{R-1}, \tag{4.138}$$

and hence

$$c_1\, m_R^+ + c_2\, m_R^- = c_1\, Z_+^0 - c_2\, Z_-^0, \qquad R \geq 0\,. \tag{4.139}$$

Now, fix $N \geq 1$ and let $U_{R,N}$ denote the sum of all spins at distance $R \geq 1$ from the N'th spine vertex u_N in the branches attached to u_N. The conditional expectation of $U_{R,N}$, given $s_0, s_1, \ldots, s_N$, then only depends on s_N, and its value is obtained from Corollary 4.3.7 by summing over

$k'_N, k''_N \geq 0$, which yields

$$\left(\sum_{k=0}^{\infty}(Z^0_{s_N})^k(k+1)p_{k+1}\right)^{-1}\sum_{k=0}^{\infty}(Z^0_{s_N})^{k-1}p_{k+1}k(k+1)\,m_R^{s_N} \\ = \varphi'(Z^0_{s_N})^{-1}\varphi''(Z^0_{s_N})\,m_R^{s_N} \equiv d_R^{s_N}\,. \tag{4.140}$$

Using the matrix representation (4.127) for p_{ij}, this gives

$$\langle U_{R,N}\rangle_{\beta,h} = \frac{1}{1+\alpha}\,(1 \quad 1)\,(g_0\Phi'(Z^0_+, Z^0_-))^N \begin{pmatrix}\alpha\, d_R^+\\ d_R^-\end{pmatrix}. \tag{4.141}$$

As pointed out in Remark 4.3.2, the matrix $g_0\Phi'_0$ has a second left eigenvalue λ such that $|\lambda| < 1$. Let (e_1, e_2) be a smooth choice of eigenvectors corresponding to λ as a function of (β, h), e.g.

$$\begin{pmatrix}e_1\\ e_2\end{pmatrix} = \begin{pmatrix}g_0 b\varphi'(Z^0_-)\\ g_0 b^{-1}\varphi'(Z^0_-) - 1\end{pmatrix}, \tag{4.142}$$

and write

$$\begin{pmatrix}1\\ 1\end{pmatrix} = A\begin{pmatrix}c_1\\ c_2\end{pmatrix} + B\begin{pmatrix}e_1\\ e_2\end{pmatrix}. \tag{4.143}$$

From (4.141) we then have

$$\langle U_{R,N}\rangle_{\beta,h} = \frac{1}{1+\alpha}\left(A(c_1 \quad c_2) + B\lambda^N(e_1 \quad e_2)\right)\begin{pmatrix}\alpha\, d_R^+\\ d_R^-\end{pmatrix} \\ = \frac{A}{1+\alpha}(c_1\alpha d_R^+ + c_2 d_R^-) + \lambda^N\frac{B}{1+\alpha}(e_1\alpha d_R^+ + e_2 d_R^-)\,, \tag{4.144}$$

and from (the proof of) Theorem 4.5.2 it follows that $A \to \tilde{c}^{-1}$ and $B \to 0$ for $h \to 0$, where $\tilde{c} = c_1(\beta, 0) = c_2(\beta, 0)$.

Next, note that $|d_R^{\pm}|, R \geq 1$, are bounded by a constant $C_1 = C_1(\beta, h)$ as a consequence of (4.80), and that

$$\begin{aligned}\langle M_R(\beta,h)\rangle_{\beta,h} &\leq \langle |B_R|\rangle_{\beta,h}^{-1} \sum_{R',N\leq R} \left|\langle U_{R',N}\rangle_{\beta,h}\right| \\ &\leq C_2 R^{-2} \sum_{R',N\leq R} \left|\langle U_{R',N}\rangle_{\beta,h}\right|\end{aligned} \tag{4.145}$$

for some constant $C_2 = C_2(\beta,h)$ by (4.83). It now follows from (4.144) that

$$\left|\langle M_R(\beta,h)\rangle_{\beta,h}\right| \leq \frac{A\,C_2}{R(1+\alpha)} \sum_{R'=1}^{R} (c_1\alpha d_R^+ + c_2 d_R^-) + R^{-1}\,B\,C_1\,C_2\,\max\{e_1,e_2\}. \tag{4.146}$$

Obviously, the second term on the RHS vanishes in the limit $R \to \infty$. Rewriting the summand in the first term on the RHS as

$$\begin{aligned}c_1\alpha d_R^+ + c_2 d_R^- &= c_1\,\alpha\,m_R^+\varphi'(Z_+^0)^{-1}\varphi''(Z_+^0) + c_2\,m_R^-\varphi'(Z_-^0)^{-1}\varphi''(Z_-^0) \\ &= (c_1 m_R^+ + c_2 m_R^-)\,\varphi'(Z^0)^{-1}\varphi''(Z^0) \\ &+ c_1 m_R^+\left[\alpha\varphi'(Z_+^0)^{-1}\varphi''(Z_+^0) - \varphi'(Z^0)^{-1}\varphi''(Z^0)\right] \\ &+ c_2 m_R^-\left[\varphi'(Z_-^0)^{-1}\varphi''(Z_-^0) - \varphi'(Z^0)^{-1}\varphi''(Z^0)\right],\end{aligned} \tag{4.147}$$

we see the last two terms in this expression tend to 0 uniformly in R as $h \to 0$ by continuity of $Z_\pm^0$, g_0 and boundedness of $|m_R^\pm|$, and the same holds for the first term as a consequence of (4.139) and continuity of c_1, c_2, $Z_\pm^0$ and g_0. In conclusion, given $\epsilon > 0$ there exists $\delta > 0$ such that

$$\left|\langle M_R(\beta,h)\rangle_\mu\right| \leq \epsilon\frac{A\,C_2}{1+\alpha} + C'\,R^{-1}, \tag{4.148}$$

if $|h| < \delta$, where C' is a constant. This completes the proof of the theorem. □

Remark 4.5.5. *A natural alternative to the mean magnetization as defined by*

(4.10)-(4.11) *is the quantity*

$$\bar{M}(\beta, h) = \limsup_{R\to\infty} \bar{M}_R(\beta, h)\,, \tag{4.149}$$

where

$$M_R(\beta, h) = \left\langle |B_R(\tau)|^{-1} \sum_{v\in B_R(\tau)} s_v \right\rangle_{\beta,h} . \tag{4.150}$$

It is natural to conjecture that $\lim_{h\to 0} \bar{M}(\beta, h) = 0$ *holds for generic Ising trees.*

Conclusions

The statistical mechanical models on random graphs considered in Ch. 4 chapter possess two simplifying features, beyond being Ising models, the first being that the graphs are restricted to be trees and the second that they are generic, in the sense of (4.28). Relaxing the latter condition might be a way of producing models with different magnetization properties from the ones considered here. Infinite non-generic trees having a single vertex of infinite degree have been investigated in [57, 58], but it is unclear whether a non-trivial coupling to the Ising model is possible. A different question is whether validity of the genericity condition (4.28) for $h = 0$ implies its validity for all $h \in \mathbb{R}$. The arguments presented in Section 4.3.1 only show that the domain of genericity in the (β, h)-plane is an open subset containing the β-axis, and thus leaves open the possibility of a transition to non-generic behavior at the boundary of this set.

Coupling the Ising model to other ensembles of infinite graphs represents a natural object of future study. In particular, models of planar graphs may be tractable. The so-called uniform infinite causal triangulations of the plane are known to be closely related to planar trees [43, 62], and a quenched version of this model coupled to the Ising model without external field has been considered in [62], and found to have a phase transition. Analysis of the non-quenched version, analogous to the models considered in the present chapter, seem to require developing new techniques. Surely, this is also the case for other planar graph models such as the uniform infinite planar triangulation [7] or quadrangulation [25].

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