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Semiprojectivity and the geometry of graphs

PhD thesis by

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Abstract

In this thesis semiprojectivity is investigated in three different settings; for commutative C^*-algebras, for Real C^*-algebras, and for C^*-algebras with a group action. In the setting of commutative C^*-algebras we give, in joint work with Thiel, a complete characterization of which algebras are semiprojective. This is used to answer open questions about semiprojectivity in the special case of commutative C^*-algebras. Semiprojectivity for Real C^*-algebras is used to study a perturbation problem for real valued matrices, concretely we prove, jointly with Loring, a real version of Lin's theorem. Finally, we study, in joint work with Phillips and Thiel, the properties of a version of semiprojectivity that respects group actions.

We also study so-called geometric classification of graph C^*-algebras. That is, classification of graph C^*-algebras in terms of their underlying graphs. In joint work with Eilers and Ruiz, we classify amplified graph C^*-algebras using geometric and K-theoretic methods. The techniques thus developed to deal with infinite emitters, are expanded upon and used to give a geometric classification of simple unital graph C^*-algebras.

The following is a Danish translation of the abstract as required by the rules of the University of Copenhagen.

Resumé

I denne afhandling undersøges semiprojektivitet inden for tre forskellige områder; for kommutative C^*-algebraer, for Reelle C^*-algebraer og for C^*-algebraer med en gruppevirkning. I sammenarbejde med Thiel giver vi en fuldstændig karakteristik af semiprojektivitet for kommutative C^*-algebraer. Det bruger vi til at besvare åbne spørgsmål om semiprojektivitet for C^*-algebraer i det specielle tilfælde, hvor C^*-algebraerne er kommutative. Semiprojektivitet for Reelle C^*-algebraer bliver brugt til, at undersøge et perturbations problem for matricer med reelle indgange. Mere præcist viser vi, i fællesskab med Loring, en reel version af Lins sætning. Endelig studerer vi, sammen med Phillips og Thiel, nogle egenskaber ved en form for semiprojektivitet, der respekterer gruppevirkninger.

Vi studerer desuden såkaldt geometrisk klassifikation af graf algebraer. Det vil sige klassifikation af graf C^*-algebraer ved hjælp af deres underliggende grafer. Fælles med Eilers og Ruiz klassificerer vi forstærkede graf C^*-algebraer, både ved brug af geometriske og K-teoretiske metoder. De metoder, der i den forbindelse blev udviklet til at håndterer uendelige udsendere, bliver videreudviklet og brugt til at give en geometrisk klassifikation af simple enhedsbærende graf C^*-algebraer.

Preface

This thesis is the result of research I have carried out as a PhD student at the Department of Mathematical Sciences at the University of Copenhagen from November 2009 to October 2012.

My PhD studies have primarily been focused on two areas: semiprojectivity and graph C^*-algebras. It was the initial hope that I would be able to combine the two topics. I have done this to some degree, as described in Chapter 3.

The main content of this thesis consists of five papers, they are attached as appendixes.

My work on semiprojectivity has lead to three papers. The first paper, "Almost commuting self-adjoint matrices — the real and self-dual cases" ([**LS10**]), is joint with Terry Loring. It is concerned with semiprojectivity for Real C^*-algebras, and is a direct outcome of my visit to Terry Loring at the University of New Mexico in the fall of 2010. The article "A characterization of semiprojectivity for commutative C^*-algebras" ([**ST**]) is joint with Hannes Thiel. It has been accepted for publication in the Proceedings of the London Mathematical Society, and appears in this thesis in the form it will be published in. In joint work with N. Christopher Phillips and Hannes Thiel, we investigate semiprojectivity for C^*-algebras with a group action in "Semiprojectivity with and without a group action" ([]). Finally there is the article "On a counterexample to a conjecture by Blackadar" ([**Sør12**]).

My work on graph C^*-algebras has lead to two papers. The paper "Amplified graph C^*-algebras" ([**ERS**]) is joint work with Søren Eilers and Efren Ruiz, it has been accepted for publication by the Münster Journal of Mathematics, but might still see small changes before it is published. Continuing that work, I wrote "Geometric classification of simple graph algebras" ([**Sør**]), which has been accepted for publication by Ergodic Theory and Dynamical Systems. It appears in this thesis in the form it will be published in.

In addition to these papers, which are all included as appendixes, the thesis contains five chapters. Chapters 1 through 4 provide the main results of the papers and put them in context. Additionally, in Chapter 2, I study semiprojectivity for C^*-algebras of real rank zero, and in Chapter 3, I explain how to use the classification of amplified graph C^* algebras to study semiprojectivity for them. In Chapter 5, I provide a real version of Lorings theory of generators and relations from [**Lor10**].

Acknowledgements

First and foremost I wish to thank my advisor Søren Eilers for his questions, his answers, and his support. Grateful thanks are extended to Terry Loring and his wife Rebecca (and their many dogs!) for welcoming me into their home, when I went to New Mexico in the fall of 2010.

I thank my fellow PhD students at the Department of Mathematical Sciences for being such good company, and I thank the Centre for Symmetry and Deformation for providing a fantastic research environment. Special thanks are due to Sara Arklint, Tim de Laat, Henrik Densing Petersen, and Hannes Thiel for many interesting conversations about mathematics and other things.

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Adam P W Sørensen
Frederiksberg, October 2012

Contents

CHAPTER 1

Introduction

1.1. On graph algebras

The definition of graph C^*-algebras is modeled on that of Cuntz-Krieger algebras. Indeed, if you start with a finite graph G with no sinks and no sources and denote by A_G its adjacency matrix, then the Cuntz-Krieger algebra of A_G, denoted $\mathcal{O}_{A_G}$, is isomorphic to the graph C^*-algebra $C^*(G)$ of G. Given a square, non-permutation, $\{0,1\}$-valued matrix A with no zero row or column there is a strong connection between the properties of A and the properties of $\mathcal{O}_A$. With that in mind, it is not surprising that we have many connections between properties of graphs and the properties of their associated C^*-algebras. We can, for instance, read off the ideal structure of a graph C^*-algebra from the underlying graph. We also get information about real rank, pure infiniteness, K-theory and semiprojectivity from the underlying graph.

While graph C^*-algebras have much in common with Cuntz-Krieger algebras, there are also many differences. For example, a simple Cuntz-Krieger algebra is always purely infinite, but the class of graph C^*-algebras contains, up to Morita equivalence, all AF algebras. The class of graph C^*-algebras also contains, up to Morita equivalence, all Kirchberg algebras (i.e. separable, simple, purely infinite, nuclear C^*-algebras that satisfy the universal coefficient theorem) with finitely generated K-theory and without torsion in K_1. So $\mathcal{O}_\infty$ is a graph algebra even though it is not a Cuntz-Krieger algebra.

In the case of a simple graph algebra, we have the dichotomy that it is either AF or purely infinite. Since graph algebras always are nuclear and satisfy the universal coefficient theorem, we get, from deep results, that simple graph C^*-algebras are classified by their K-groups. Combining this with the result that Cuntz-Krieger algebras with finitely many ideals are classified by K-theoretic data, one might hope that all graph C^*-algebras can be classified using (some form of) K-theory. There has been some progress towards this goal. Notably graph C^*-algebras with exactly one ideal are classified up to Morita equivalence by K-theoretic data. Other special ideal lattices have also been handled.

In this thesis, I attack the problem of classifying graph C^*-algebras by asking the following vague question:

Question 1.1.1. *What does it say about two graphs E and G that $C^*(E)$ is Morita equivalent to $C^*(E)$?*

Answering this question means classifying graph C^*-algebras in terms of their underlying graphs, we call that geometric classification. In the case where the graphs E and G both are so-called amplified graphs (meaning that if there is at least one edge between two vertices then there are infinitely many), and they both have only finitely many vertices, we give an answer to the question in [**ERS**]. Building on the ideas used to classify amplified graph algebras, and the ideas used by Franks to classify irreducible shifts of finite type, I answer question 1.1.1 in [**Sør**] under the assumption that $C^*(G)$ and $C^*(E)$ both are unital and simple. Both are examples of geometric classification.

1.2. On semiprojectivity

In [**Bla85**], Blackadar introduced semiprojectivity for C^*-algebras. It is a non-commutative analogue of the notion of a space being an absolute neighborhood retract. Blackadar showed that many C^*-algebras are semiprojective, such as the Cuntz-Krieger algebras, the Toeplitz algebra and the algebra of continuous functions on the circle. He also provided non-examples, such as the algebra of compact operators on infinite-dimensional Hilbert space and the algebra of continuous functions on the disc. Later Blackadar also showed that $\mathcal{O}_\infty$ is semiprojective ([**Bla04**]). Building on this result, Szymański showed in [**Szy02**] that all Krichberg algebras with finitely generated K-theory, no torsion in K_1, and with the rank of K_1 less than the rank of K_0, are semiprojective. Spielberg was able to remove the rank condition in [**Spi09**]. Thus all Kirchberg algebras with finitely generated K-theory and no torsion in K_1 are semiprojective.

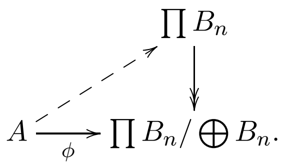
Semiprojectivity has been used successfully in the classification of C^*-algebras and has also found applications in problems concerning the structure of C^*-algebras. But there are still many open question about semiprojectivity. It is, for instance, not known if semiprojectivity of $M_2(A)$ implies that A is semiprojective. A recent example of Eilers and Katsura ([**EK12**]) shows that semiprojectivity does not pass from ideals to split extensions by $\mathbb{C}$. In [**Sør12**], the example of Eilers and Katsura is modified to prove the existence of a non-semiprojective non-split extension of a semiprojective ideal by $\mathbb{C}$. An important result by Enders shows that semiprojectivity passes to ideals of finite co-dimension ([**End11**]).

Even though semiprojectivity began as the non-commutative analog of a topological concept, there has not been a lot of study into the semiprojectivity of commutative C^*-algebras. In [**Lor96**], Loring shows that all finite one-dimensional CW complexes are semiprojective. This avenue of research was then left almost unexplored until Chigogidze and Dranishnikov gave a complete characterization of projectivity of commutative C^*-algebras in [**CD10**]. By extending their ideas, we were able to give a complete description of which commutative C^*-algebras are semiprojective in [**ST**]. Using this description we proved some permanence properties for commutative semiprojective C^*-algebras. For example, we show that for a commutative C^*-algebra A, the algebra $M_n(A)$ is semiprojective if and only if A is.

In [**Phi11**], Phillips takes a novel approach to the study of semiprojectivity by introducing a group equivariant form of semiprojectivity. Loosely speaking, he considers a topological group G and then ask that in the definition of semiprojectivity all algebras have a G-action and all maps are G-equivariant. The focus of [**Phi11**] is on providing examples of semiprojective

G-algebras, with an aim towards the classification of group actions on Kirchberg algebras. In [**PST12**], we study the connections between equivariant semiprojectivity and regular semiprojectivity. We also consider questions about semiprojectivity of crossed products and fixed point algebras.

An alternative formulation of semiprojectivity is through so-called stable relations. In this view, the fact that $\mathbb{C}$ is semiprojective takes the form that close to any element in a C^*-algebra that almost is a projection, there is an actual projection. Furthermore, if $x \in A$ is an almost-projection and we are given a surjection $\pi\colon A \to B$, we can pick the actual projection, p say, close to x and with $\pi(x) = \pi(p)$. If we drop the requirement about the surjective $*$-homomorphism, we get the notion of a weakly semiprojective C^*-algebra. Semiprojectivity is usually phrased as a lifting problem and we can also phrase weak semiprojectivity in that way. A C^*-algebra A is weakly semiprojective if given any sequence of C^*-algebras (B_n) we can solve all lifting problems of the form:



That these seemingly different notions coincide, is proved in [**EL99**].

In the lifting picture of weak semiprojectivity, it is natural to consider only sequences (B_n) in which all the B_n have some nice property. For instance we could ask that all the B_n be matrix algebras. In that case we say that A is weakly semiprojective with respect to matrices. Solving the lifting problem then becomes equivalent to solving the perturbation problem for matrices. A deep theorem of Lin ([**Lin97**]) shows that the algebra $C(\mathbb{D})$ of continuous functions on the disc is weakly semiprojective with respect to matrices. In the perturbation picture, this says that an almost normal matrix is close to a normal matrix. A standard trick shows that this is equivalent to the fact that two almost commuting, hermitian, and contractive matrices are close to two exactly commuting, hermitian, and contractive matrices. Voiculescu has shown in [**Voi81**], that three almost commuting, hermitian, and contractive matrices are not always close to three exactly commuting, hermitian, and contractive matrices. For a more concrete example, see [**Dav85**, Theorem 2.3].

In [**LS10**] we prove a real version of Lin's theorem. That is, we prove that given two real, almost commuting, hermitian, and contractive matrices there are two real, exactly commuting, hermitian, and contractive matrices close by. Our proof follows the lines of the elegant proof of a generalization of Lin's theorem given by Friis and Rørdam in [**FR96**], which in turns builds on many results concerning semiprojectivity. To follow Friis and Rørdam, we move many of the standard techniques used in semiprojectivity into the realm of Real C^*-algebras. In the lifting picture of weak semiprojectivity, our result states that the Real C^*-algebra $C(\mathbb{D}, \mathrm{id})$ is weakly semiprojective with respect to matrices.

The question of whether Lin's theorem holds in the real case, is not only natural as a basic question about matrices, but it is also relevant in the physics of topological insulators. For details on this connection see [**HL10**].

1.3. Unanswered questions

The work done in connection with this thesis raises various questions. The main theorem of [**Sør**] gives a geometric classification of simple unital graph algebras. Therefore it is natural to ask if we can extend it to non-simple unital graph algebras. One could also ask if we can remove the unitality condition. This is probably more problematic, as none of the moves described in [**Sør**] deals with infinitely many vertices. Thus it is more reasonable to ask how to modify the moves in [**Sør**], or perhaps find entirely new moves, so that we can obtain a geometric classification for non-unital (simple) graph algebras.

Currently, the notion of nuclear dimension of a C^*-algebra, introduced in [**WZ10**], is receiving a lot of attention. We can recast the main theorem of [**ST**] as: A commutative C^*-algebra A is semiprojective if and only if the spectrum of A is an absolute neighborhood retract and the nuclear dimension of A is less than or equal to one. In this form, the result might raise the question whether all nuclear semiprojective C^*-algebras must have nuclear dimension at most one. That this should be the case, is, however, far from obvious. For example, it is still unknown if the nuclear dimension of the Toeplitz algebra is one or two.

In [**LS10**], a real version of Lin's theorem is proved. It is well know, that the complex version of Lin's theorem fails if we consider triples of almost commuting matrices. Boersema, Loring and Ruiz have shown in [**BLR12**, Corollary 7.11], that five almost commuting, real, self-adjoint, and contractive matrices cannot always be perturbed to a commuting five tuple. This leaves the question open for three and four real matrices.

CHAPTER 2

Semiprojectivity

In this chapter, I describe the main results of [**ST**] and [**Sør12**]. In Section 2.4, I discuss projectivity for real rank zero C^*-algebras.

2.1. Semiprojective C^*-algebras

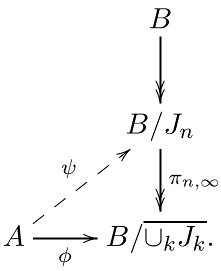
Semiprojectivity was introduced by Blackadar in [**Bla85**]. It is a lifting property for C^*-algebras modeled on the notion of absolute neighborhood retract for spaces. It can also be seen as a weakening of the notion of projectivity for C^*-algebra. Before Blackadar gave his definition of semiprojectivity Effros and Kaminker defined a similar concept, also called semiprojectivity, in [**EK86**]. We will only use Blackadar's definition in the thesis.

DEFINITION 2.1.1 ([**Bla85**, Definition 2.10]). A C^*-algebra A is *semiprojective* if for every C^*-algebra B, every increasing sequence of ideals $J_1 \subseteq J_2 \subseteq \cdots$ in B, and every $*$-homomorphism $\phi\colon A \to B/\overline{\cup_k J_k}$, there exists an $n \in \mathbb{N}$ and a $*$-homomorphism $\psi\colon A \to B/J_n$ such that

$$\pi_{n,\infty} \circ \psi = \phi,$$

where $\pi_{n,\infty}\colon B/J_n \to B/\overline{\cup_k J_k}$ is the natural quotient map.

The lifting problem is illustrated in the following figure, where the $*$-homomorphism associated with the solid arrows are given. Our task is then to find $n \in \mathbb{N}$ and a $*$-homomorphism that fits on the dashed arrow and makes the diagram commute.



Many C^*-algebras are known to be semiprojective, for instance finite-dimensional C^*-algebras ([**Bla85**]), $C(X)$ for finite one-dimensional CW complexes ([**Lor96**]), and Kirchberg algebras with finitely generated K-theory and no torsion in K_1 ([**Spi09**]). Examples of non-semiprojective C^*-algebras are the algebra $\mathbb{K}$ of compact operators, $C(\mathbb{D})$, and the irrational rotation algebras. The standard reference for material about semiprojectivity is the book [**Lor97**]. Also, the paper [**Bla04**] contains a very nice exposition of parts of the theory.

In [**Sør12**], I prove a result related to the following question. Consider an extension of C^*-algebras

$$0 \to I \to A \to F \to 0,$$

and assume that F is finite-dimensional, is A semiprojective if and only if I is? This question was raised in [**Lor97**, Chapter 16], a special case of the question appeared in [**Bla04**, Conjecture 4.5]. If A is unital and $F = \mathbb{C}$, then $A \cong \widetilde{I}$, and so the question has a positive answer. A more interesting partial result was obtained by Enders ([**End11**]), who proved that if A is semiprojective then I is. Recently, Eilers and Katsura gave a counterexample in [**EK12**] to the other direction. Concretely, they prove the following theorem.

THEOREM 2.1.2 (Eilers-Katsura). *There exists a split extension of C^*-algebras*

$$0 \to I \to A \to \mathbb{C} \to 0,$$

such that I is semiprojective, but A is not.

In [**Sør12**], I show how to modify the example given by Eilers and Katsura to get a *non-split* extension

$$0 \to J \to B \to \mathbb{C} \to 0,$$

where J is semiprojective but B is not. The algebra B is constructed as a pull-back over A and another C^*-algebra.

In [**ST**], we consider semiprojectivity for commutative C^*-algebras. We follow in the footsteps of Chigogidze and Dranishnikov, who gave a complete characterization of projectivity for commutative C^*-algebras in [**CD10**]. As mentioned above, it is known that $C(\mathbb{D})$ is not semiprojective. Therefore one expects that if $C(X)$ is semiprojective, then X must have low dimension. By [**Bla85**, Proposition 2.11], we have that if $C(X)$ is semiprojective, then X must be an absolute neighborhood retract. The main theorem of [**ST**] confirms that these are the only restrictions:

THEOREM 2.1.3 ([**ST**, Theorem 1.2]). *Let X be a compact, metric space. The following are equivalent:*

(1) $C(X)$ is semiprojective.
(2) X is an absolute neighborhood retract and $\dim(X) \leq 1$.

We use the result to answer questions about semiprojectivity in the commutative case. The two main applications are the following two results:

THEOREM 2.1.4 ([**ST**, Corollary 6.3]). *Let A be a separable, commutative C^*-algebra, and I an ideal in A. Assume A/I is finite-dimensional, i.e. $A/I \cong \mathbb{C}^k$ for some k. Then A is semiprojective if and only if I is semiprojective.*

THEOREM 2.1.5 ([**ST**, Corollary 6.9]). *Let A be a separable, commutative C^*-algebra, and let $k \in \mathbb{N}$. If $M_k(A)$ is semiprojective, then A is semiprojective.*

2.2. Group actions

Recently, Phillips has introduced a version of semiprojectivity that respects group actions in [**Phi11**]. Given a topological group G we call a triple (G, A, α) a G-algebra if A is a C^*-algebra and α is a strongly continuous action of G on A.

Definition 2.2.1 (see [**Phi11**, Definition 1.1]). Let G be a topological group. A G-algebra (G, A, α) is *equivariantly semiprojective* if for every G-algebra (G, B, β), every increasing sequence of invariant ideals $J_1 \subseteq J_2 \subseteq \cdots$ in B, and every G-equivariant $*$-homomorphism $\phi\colon A \to B/\overline{\cup_k J_k}$, there exists an $n \in \mathbb{N}$ and a G-equivariant $*$-homomorphism $\psi\colon A \to B/J_n$ such that

$$\pi_{n,\infty} \circ \psi = \phi,$$

where $\pi_{n,\infty}\colon B/J_n \to B/\overline{\cup_k J_k}$ is the natural quotient map.

Phillips goes on to prove that a number of actions are semiprojective, including all actions of compact groups on finite dimensional C^*-algebras and certain actions on the Cuntz algebras $\mathcal{O}_n$. In [**PST12**], we study various properties of equivariantly semiprojective C^*-algebras. One of our main results is, that if (G, A, α) is semiprojective and G is compact, then A must be semiprojective ([**PST12**, Corollary 3.11]). More generally we prove the following.

Theorem 2.2.2 ([**PST12**, Theorem 3.10]). *Let G be a locally compact group, and let $H \leqslant G$ be a closed subgroup such that G/H is compact. Let $\alpha\colon G \to Aut(A)$ be a strongly continuous action of G on a C^*-algebra A. If (G, A, α) is equivariantly semiprojective, then $\alpha|_H$ is equivariantly semiprojective.*

We also give an example ([**PST12**, Example 3.12]) of an equivariantly semiprojective $\mathbb{Z}$ algebra $(\mathbb{Z}, A, \alpha)$ such that A is not semiprojective.

2.3. Semiprojectivity in a subcategory

Let $\mathbf{C}^*$ denote the category of all C^*-algebras. We say that a subcategory $\mathcal{C}$ of $\mathbf{C}^*$ is closed under quotients, if for every $A \in \mathcal{C}_{\mathrm{obj}}$ and every surjective $*$-homomorphism $\pi\colon A \to B$, we have $B \in \mathcal{C}_{\mathrm{obj}}$ and $\pi \in \mathcal{C}_{\mathrm{mor}}$.

Definition 2.3.1. Let $\mathcal{C}$ be a subcategory of $\mathbf{C}^*$ that is closed under quotients.

A C^*-algebra $A \in \mathcal{C}_{\mathrm{obj}}$ is *projective in $\mathcal{C}$*, if for every C^*-algebras $B \in \mathcal{C}_{\mathrm{obj}}$, every ideal J in B, and every $*$-homomorphism $\phi\colon A \to B/J$ in $\mathcal{C}_{\mathrm{mor}}$, there exists a $*$-homomorphism $\psi\colon A \to B$ in $\mathcal{C}_{\mathrm{mor}}$ such that

$$\pi \circ \psi = \phi,$$

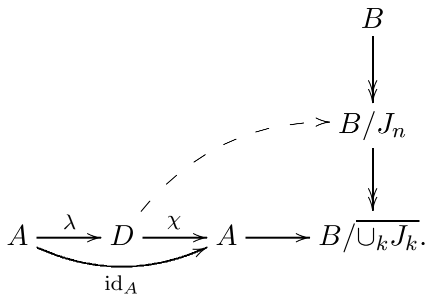
where $\pi\colon B \to B/J$ is the quotient map.

Let X be a locally compact metric space. By [**Bla85**, Proposition 2.7] $C_0(X)$ is projective in the category of all commutative C^*-algebras if and only if X is an absolute retract.

Loring has observed, that if $\mathbb{D}$ embeds into a compact Hausdorff space X, then $C(X)$ is not semiprojective. In [**ST**, Remark 3.3] we give a two step proof of this observation. The first step is to observe that $C(\mathbb{D})$ is not semiprojective (in $\mathbf{C}^*$) but that it is projective in the full subcategory of commutative C^*-algebras. The second step is to apply the following lemma.

LEMMA 2.3.2. *Let A, D be C^*-algebras, and suppose we are given two $*$-homomorphisms $\lambda\colon A \to D$, $\chi\colon D \to A$ such that $\chi \circ \lambda = \mathrm{id}_A$. If D is semiprojective, then so is A.*

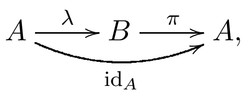
The proof of [**Thi11**, Lemma 5.1] can be modified slightly to give a proof of Lemma 2.3.2. The following diagram illustrates the proof. We are given $*$-homomorphisms corresponding to the solid arrows, and use semiprojectivity of D to find one that will fit on the dashed arrow.



The main point of the proof of Loring's observation is that $C(\mathbb{D})$ is projective in the category of commutative C^*-algebras but not semiprojective in the category of all C^*-algebras. However, there is nothing special about the category of commutative C^*-algebras, so we can use the same techniques in any category.

PROPOSITION 2.3.3. *Let $\mathcal{C}$ be a category of C^*-algebras that is closed under quotients. Let $A \in \mathcal{C}_{obj}$ be projective in $\mathcal{C}$ but not semiprojective (in $\mathbf{C}^*$). If $B \in \mathcal{C}_{obj}$ surjects onto A, then B is not semiprojective (in $\mathbf{C}^*$).*

PROOF. Let $\pi\colon B \to A$ be a surjection. Since $B \in \mathcal{C}_{\mathrm{obj}}$ and A is projective in $\mathcal{C}$, we can find a $*$-homomorphism $\lambda\colon A \to B$ (in $\mathcal{C}_{\mathrm{mor}}$) such that the diagram



commutes. Since A is not semiprojective (in $\mathbf{C}^*$), Lemma 2.3.2 tells us that B is not semiprojective (in $\mathbf{C}^*$). □

2.4. Semiprojectivity and real rank zero

To apply Proposition 2.3.3 in a different setting than commutative C^*-algebras, we need a category of C^*-algebras that is rich on objects that are internally projective, but not semiprojective. Since projections do not lift in general, but do lift between real rank zero C^*-algebras, we will consider the category of real rank zero C^*-algebras.

DEFINITION 2.4.1. Let $\mathcal{RR}_0$ denote the full subcategory of $\mathbf{C}^*$ were the objects are all the real rank zero C^*-algebras.

It follows from [**BP91**, Theorem 3.14] that $\mathcal{RR}_0$ is closed under quotients and that $\mathbb{C}$ is projective in $\mathcal{RR}_0$.

In this section, we will study projectivity in $\mathcal{RR}_0$. To get something that is projective in $\mathcal{RR}_0$, but not semiprojective in $\mathbf{C}^*$, we will use infinite direct sums.

LEMMA 2.4.2. *If (A_n) is a sequence of algebras in $\mathcal{RR}_0$, then $\bigoplus_n A_n$ is in $\mathcal{RR}_0$.*

PROOF. If A and B are in $\mathcal{RR}_0$ then $A \oplus B$ is in $\mathcal{RR}_0$ by [**BP91**, Theorem 3.14]. It now follows from [**BP91**, Proposition 3.1] that $\bigoplus_n A_n$ is in $\mathcal{RR}_0$. □

PROPOSITION 2.4.3. *If (A_n) is a sequence of σ-unital algebras in $\mathcal{RR}_0$ then $\bigoplus_n A_n$ is projective in $\mathcal{RR}_0$ if and only if each A_n is projective in $\mathcal{RR}_0$.*

PROOF. Suppose first that $\bigoplus_n A_n$ is projective in $\mathcal{RR}_0$. For every $k \in \mathbb{N}$ we have a commutative diagram

$$\underbrace{A_k \longrightarrow \bigoplus_n A_n \xrightarrow{\pi_k} A_k}_{\mathrm{id}_{A_k}} ,$$

where π_k is the projection onto the k'th summand. Reasoning as in the proof of Lemma 2.3.2, we see that A_k is projective in $\mathcal{RR}_0$.

Now suppose that each A_n is projective in $\mathcal{RR}_0$, and that we are given a $*$-homomorphism $\phi\colon \bigoplus_n A_n \to B/J$ for some B in $\mathcal{RR}_0$ and some ideal J in B. Let $\bar{h}_l$ be a strictly positive element in A_l, and let h_l be its image in $\bigoplus_n A_n$ under the natural inclusion. Denote by π the quotient map from B to B/J. By [**Lor97**, Lemma 10.1.12], we can find pairwise orthogonal positive elements $k_l \in B$ such that $\pi(k_l) = \phi(h_l)$ for all $l \in \mathbb{N}$.

For $l \in \mathbb{N}$, define

$$B_l = \overline{k_l B k_l} \quad \text{and} \quad D_l = \overline{\phi(h_l) B/J \phi(h_l)}.$$

By [**BP91**, Corollary 2.8], all the B_l are in $\mathcal{RR}_0$. Since π maps B_l onto D_l by [**Lor97**, Corollary 8.2.4], we can use the projectivity of the A_l one at a time, to define a $*$-homomorphism $\psi\colon \bigoplus_n A_n \to \bigoplus_n B_n$ such that $\pi \circ \psi = \phi$. Composing ψ with the inclusion of $\bigoplus_n B_n$ into B shows that $\bigoplus A_n$ is projective in $\mathcal{RR}_0$. □

Notice that the proof is just the proof of [**Lor97**, Theorem 10.1.13] combined with the fact that real rank zero passes to hereditary sub-C^*-algebras.

PROPOSITION 2.4.4. *Let (A_n) be a sequence of C^*-algebras that all contain a non-zero projection. Then $\bigoplus_n A_n$ is not semiprojective.*

PROOF. Let $B_k = \bigoplus_{n=1}^k A_n$. We have inclusion maps $\iota_{k,k+1}\colon B_k \to B_{k+1}$ and the direct limit of the associated inductive system is $\bigoplus_n A_n$. Let $\iota_{n,\infty}$ be the natural map from B_k to $\bigoplus_n A_n$. Suppose, to reach a contradiction, that $\bigoplus_n A_n$ is semiprojective. Then, by [**Bla04**, Proposition 3.9], we can can find some $k \in \mathbb{N}$ and a $*$-homomorphism $\psi\colon \bigoplus A_n \to B_k$ such that $\iota_{k,\infty} \circ \psi$ is homotopic to the identity map on $\bigoplus A_n$. Let $p \in A_{k+1}$ be a non-zero projection, and let $\rho_{k+1}\colon \bigoplus_n A_n \to A_{k+1}$ be the natural projection onto the $k+1$'th summand. Then

$$p = \rho_{k+1}(0,\ldots,0,p,0,0,\ldots) \sim_h \rho_{k+1}((\iota_{k,\infty}\circ\psi)(0,\ldots,0,p,0,0,\ldots)) = 0,$$

which implies that $p = 0$. This contradicts the choice of p, so we must conclude that $\bigoplus_n A_n$ is not semiprojective. □

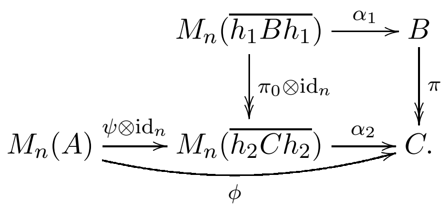
Combining Lemma 2.3.2 with Propositions 2.4.3 and 2.4.4, we get the following.

COROLLARY 2.4.5. *Let A be in $\mathcal{RR}_0$ and let (A_n) be a sequence of unital algebras that are projective in $\mathcal{RR}_0$. If A has a quotient that is isomorphic to $\bigoplus_n A_n$ then A is not semiprojective.*

Of course the utility of the corollary depends heavily on whether there are (many) examples of C^*-algebras that are projective in $\mathcal{RR}_0$. As we have already mentioned $\mathbb{C}$ is projective in $\mathcal{RR}_0$. We can also prove that all finite-dimensional C^*-algebras are projective in $\mathcal{RR}_0$.

PROPOSITION 2.4.6. *If A is projective in $\mathcal{RR}_0$ then $M_n(A)$ is projective in $\mathcal{RR}_0$ for all $n\in\mathbb{N}$.*

PROOF. First note that by [**BP91**, Theorem 2.10], we have that $M_n(A)$ is in $\mathcal{RR}_0$. Suppose we are given a surjective $*$-homomorphism $\pi\colon B\to C$ between C^*-algebras in $\mathcal{RR}_0$, and a $*$-homomorphism $\phi\colon M_n(A)\to C$. Following the proof of [**Lor93**, Theorem 3.3], we see that we can find positive elements $h_1\in B$, $h_2\in C$ and $*$-homomorphisms α_1,α_2,π_0 and ψ such that the following diagram commutes.



Here, id_n denotes the identity map on M_n. Since real rank zero passes to hereditary sub-C^*-algebras $\overline{h_1Bh_1}$ is in $\mathcal{RR}_0$. Hence we can use the projectivity of A to find a $*$-homomorphism λ that lifts ψ. Then $\alpha_1\circ(\lambda\otimes\mathrm{id}_n)$ is a lift of ϕ, and hence $M_n(A)$ is projective in $\mathcal{RR}_0$. □

THEOREM 2.4.7. *Every finite-dimensional C^*-algebra is projective in the category $\mathcal{RR}_0$. Every C^*-algebra of the form $\bigoplus_k M_{n_k}$, where $(n_k)\subseteq\mathbb{N}$, is projective in $\mathcal{RR}_0$.*

PROOF. Since $\mathbb{C}$ is projective in $\mathcal{RR}_0$, this follows from Proposition 2.4.6 and Proposition 2.4.3. □

We state a special case of Corollary 2.4.5.

COROLLARY 2.4.8. *Let A be in $\mathcal{RR}_0$. If there is a sequence $(n_k)\subseteq\mathbb{N}$ such that A has a quotient that is isomorphic to $\bigoplus_k M_{n_k}$ then A is not semiprojective.*

If a separable C^*-algebra P is projective (in $\mathbf{C}^*$), then P must be contractible ([**Lor97**, Lemma 10.1.6]) and residually finite-dimensional ([**Lor97**, Theorem 11.2.1]). Theorem 2.4.7 states that certain AF algebras are projective in $\mathcal{RR}_0$, they are certainly not contractible. But they are all residually finite-dimensional. We will now show that this is no coincidence, as projectivity in $\mathcal{RR}_0$ implies residually finite-dimensionality for separable C^*-algebras.

First we find a real rank zero C^*-algebra that surjects onto $B(H)$.

Lemma 2.4.9. *We will view M_n as sitting in the top left corner of $\mathbb{K}$. Let*

$$\mathcal{D} = \left\{ f \in C\left(\left\{ \frac{1}{n} \,\middle|\, n \in \mathbb{N} \right\} \cup \{0\}, \mathbb{K} \right) \,\middle|\, f\left(\frac{1}{n}\right) \in M_n \right\}.$$

Then the multiplier algebra $M(\mathcal{D})$ of $\mathcal{D}$ is in $\mathcal{RR}_0$, it is residually finite-dimensional, and it has a surjective $*$-homomorphism onto $B(H)$.

Proof. We have the following extension

$$0 \to \bigoplus M_n \to \mathcal{D} \to \mathbb{K} \to 0.$$

Since both the ideal and the quotient are AF algebras, it follows from [**Dav96**, Theorem III.6.3] that $\mathcal{D}$ is an AF algebra. Hence, by [**Lin93**, Corollary 12], the multiplier algebra $M(\mathcal{D})$ has real rank zero.

For every non-zero element in $\mathcal{D}$ at least one of the maps $\mathrm{ev}_{1/n}$ will map it to a non-zero element in some finite-dimensional algebra. That is, $\mathcal{D}$ is residually finite-dimensional. It now follows from [**BO08**, Proposition 10.3.1] that $M(\mathcal{D})$ is residually finite dimensional.

The map ev_0 is a surjection from $\mathcal{D}$ onto $\mathbb{K}$. By the non-commutative version of the Tietze extension theorem (see, e.g. [**Lor97**, Theorem 9.2.1]) there is a surjection from $M(\mathcal{D})$ onto $B(H)$. □

Theorem 2.4.10. *If A is separable and projective in $\mathcal{RR}_0$ then A is residually finite-dimensional.*

Proof. We can find an injective $*$-homomorphism $\phi\colon A \to B(H)$. By lemma 2.4.9, we can find a residually finite-dimensional, real rank zero C^*-algebra D and a surjective $*$-homomorphism $\pi\colon D \to B(H)$. Since D is in $\mathcal{RR}_0$ and A is projective in $\mathcal{RR}_0$, we can lift ϕ to a $*$-homomorphism $\psi\colon A \to D$. Since ϕ is injective ψ is injective. Thus, A is isomorphic to a sub-C^*-algebra of a residually finite-dimensional algebra, and therefore residually-finite dimensional. □

Using Theorem 2.4.10, we can show that many real rank zero C^*-algebras are not projective in $\mathcal{RR}_0$.

Corollary 2.4.11. *The Cuntz algebras $\mathcal{O}_n$ are not projective in $\mathcal{RR}_0$.*

Proof. They are not residually finite-dimensional. □

Corollary 2.4.12. *The algebra of compact operators is not projective in $\mathcal{RR}_0$.*

Proof. It is not residually finite-dimensional. □

CHAPTER 3

Geometric classification

In this chapter, I describe the main results of [**ERS**] and [**Sør**]. I also use the results of [**ERS**] to study semiprojectivity for certain graph C^*-algebras.

3.1. Graphs and their algebras

There are, unfortunately, two definitions of graph C^*-algebras. In this thesis and in both [**ERS**] and [**Sør**], we use the convention used in, for instance, [**DT05**]. We define a graph G as a four-tuple (G^0, G^1, r, s), where G^0 is a set of vertices, G^1 a set of edges, r is a function from G^1 to G^0 that gives the range of the edges, and $s\colon G^1 \to G^0$ is a function giving the sources. We only consider graphs where G^0 and G^1 are countable. Graphs are usually denoted by G or E.

DEFINITION 3.1.1. Let $G = (G^0, G^1, r, s)$ be a graph. The *graph C^*-algebra of G* (sometimes simply called the graph algebra of G), denoted by $C^*(G)$, is the universal C^*-algebra generated by a set of mutually orthogonal projections $\{p_v \mid v \in G^0\}$ and a set $\{s_e \mid e \in G^1\}$ of partial isometries satisfying the following conditions:

- $s_e^* s_f = 0$ if $e, f \in G^1$ and $e \neq f$,
- $s_e^* s_e = p_{r_G(e)}$ for all $e \in G^1$,
- $s_e s_e^* \leq p_{s_G(e)}$ for all $e \in G^1$, and,
- $p_v = \sum_{e \in s_G^{-1}(v)} s_e s_e^*$ for all $v \in G^0$ with $0 < |s_G^{-1}(v)| < \infty$.

A good source for the theory of graph C^*-algebras is Raeburn's book [**Rae05**], which concisely covers a large part of the theory. It uses the other convention for graph C^*-algebras. Another source is the paper [**BPRS00**], which restricts its attention to row-finite graphs but gives a very well-written account of the ideal theory of graph algebras.

A path in a graph is a finite sequence of edges $e_1 e_2 \cdots e_n$ such that $r(e_i) = s(e_{i+1})$ for $1 \leq i \leq n-1$. The length of the path $\alpha = e_1 e_2 \cdots e_n$ is n, the source is $s(\alpha) = s(e_1)$ and the range is $r(\alpha) = r(e_n)$. Given two vertices u, v, we write $u \geq v$ if there is a path with source u and range v or $u = v$.

A path α with $s(\alpha) = r(\alpha)$ is called a loop. Thus a loop of length one is simply an edge e with $s(e) = r(e)$. We say that a vertex u supports a loop if there is a loop α with $s(\alpha) = u = r(\alpha)$.

3.2. Amplified graphs

In [**ERS**], we study a specialized class of graphs, namely the so-called amplified graphs. To define an amplified graph, it is convenient to speak of the amplification of a graph.

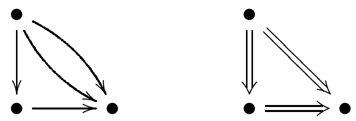
DEFINITION 3.2.1 ([**ERS**, Definition 2.4]). Let G be a graph. The *amplification of* G, denoted by $\overline{G}$, is defined by $\overline{G}^0 = G^0$,

$$\overline{G}^1 = \{e(v,w)^n \mid n \in \mathbb{N}, v, w \in G^0, \text{ and there exists an edge from } v \text{ to } w\},$$

and $s_{\overline{G}}(e(v,w)^n) = v$, and $r_{\overline{G}}(e(v,w)^n) = w$

If $E = \overline{G}$ for some graph G we say that E is an *amplified graph.*

The notion of an amplification is best understood through a picture. Below we have a graph E on the left, and its amplification $\overline{E}$ on the right. We write " $\Longrightarrow$ " to indicate infinitely many edges.



We note that there is very little connection between the C^*-algebras of E and $\overline{E}$. Consider:

$$E = \bullet \longrightarrow \bullet \longrightarrow \bullet, \qquad G = \bullet \rightrightarrows \bullet.$$

We have $C^*(E) \cong M_3 \cong C^*(G)$. The following table illustrates the lack of connection between the C^*-algebra of a graph and that of its amplification.

	non-trivial ideals	K_0 group
$C^*(G), C^*(E)$	0	$\mathbb{Z}$
$C^*(\overline{G})$	1	$\mathbb{Z}^2$
$C^*(\overline{E})$	2	$\mathbb{Z}^3$

The K-theory of an amplified graph algebra is easily computed using [**DT02**, Theorem 3.1 or Corollary 3.2]; one gets $K_0(C^*(E)) = \mathbb{Z}^{E^0}$ and $K_1(C^*(E)) = 0$. Hence, the K-theory remembers nothing about the edges in $\overline{E}$. Since amplified graphs satisfy the technical condition (K), their ideal structure is determined by the hereditary and saturated subsets of $\overline{E}^0$, see [**BHRS02**]. Since every vertex in $\overline{E}^0$ is either a sink or an infinite emitter, all subsets of $\overline{E}^0$ are saturated. Therefore the ideal structure of an amplified graph algebra is determined the hereditary subsets of $\overline{E}^0$, which in turn are completely determined by the path structure of $\overline{E}^0$. In other words, the K-theory and the ideal structure cannot see whether two vertices are connected by an edge or by a path. It has been conjectured that all graph C^*-algebras are classified by ideal related K-theory ([**ERR10**, Conjecture 1.1]). If we believe that conjecture, we must also believe that if we let the notions of path and edge coincide in an amplified graph the associated C^*-algebra will not change. That this is the case is expressed in the following theorem.

THEOREM 3.2.2 ([**ERS**, Theorem 3.8], Move (T)). *Let $\alpha = \alpha_1\alpha_2\cdots\alpha_n$ be a path in a graph G. Let E be the graph with vertex set G^0, edge set*

$$E^1 = G^1 \cup \{\alpha^m \mid m \in \mathbb{N}\},$$

and range and source maps that extend those of G and have $r_E(\alpha^m) = r_G(\alpha)$ and $s_E(\alpha^m) = s_G(\alpha)$. If

$$|s_G^{-1}(s_G(\alpha_1)) \cap r_G^{-1}(r_G(\alpha_1))| = \infty,$$

then $C^*(G) \cong C^*(E)$.

The condition on α states that there are infinitely many edges $e_1, e_2, \ldots$ such that $s(e_n) = s(\alpha_1)$ and $r(e_n) = r(\alpha_1)$ for all $n \in \mathbb{N}$.

We refer to adding the edges α^n in the theorem, as preforming move (T) on the graph. In an amplified graph, we can use move (T) to add edges between any two vertices that are connected by a path.

We note that there is nothing in Theorem 3.2.2 that requires the graphs to be amplified. Therefore, it can also be useful when studying other classes of graphs. It is for instance crucial in the way infinite emitters are handled in [**Sør**].

In order to state a version of the main theorem of [**ERS**], we introduce the notion of the transitive closure of a graph.

Definition 3.2.3 ([**ERS**, Definition 2.5]). Let $G = (G^0, G^1, r_G, s_G)$ be a graph. Define $\mathtt{t}G$ as follows:

$\mathtt{t}G^0 = G^0$,

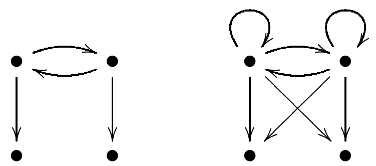
$\mathtt{t}G^1 = G^1 \cup \{e(v,w) \mid \text{there is a path but no edge from } v \text{ to } w\}$,

with range and source maps that extend those of G and satisfy

$$s_{\mathtt{t}G}(e(v,w)) = v, \quad \text{and} \quad r_{\mathtt{t}G}(e(v,w)) = w.$$

We call $\mathtt{t}G$ *the transitive closure of G.*

The idea of this definition is that in $\mathtt{t}G$ the relations "there is an edge from u to v" and "there is a path from u to v" coincide. The following figure shows a graph G on the left, and its transitive closure, $\mathtt{t}G$, on the right.



Ignoring, for a moment, the K-theory, we can state the main classification result for amplified graphs from [**ERS**] as follows.

Theorem 3.2.4 ([**ERS**, Theorem 5.7]). *Let E, G be finite graphs. The following are equivalent:*

(1) $\overline{\mathtt{t}G} \cong \overline{\mathtt{t}E}$.
(2) $C^*(\overline{\mathtt{t}G}) \cong C^*(\overline{\mathtt{t}G})$.
(3) $C^*(\overline{G}) \cong C^*(\overline{E})$.
(4) $C^*(\overline{G}) \otimes \mathbb{K} \cong C^*(\overline{E}) \otimes \mathbb{K}$.

Looking at (1) and (3), we get a clear classification in terms of graphs, i.e. a geometric classification. Our proof of Theorem 3.2.4 uses K-theory and the very explicit description of the ideal structure of a graph algebra in terms of the underlying graph.

In the paper, we introduce a K-theoretic invariant called the tempered ideal space, denoted $\mathrm{Prim}_\tau(-)$. This turns out to be a complete invariant for unital amplified graph algebras. We also define a class of C^*-algebras, which we call $\mathfrak{C}_{\text{free}}$, without any reference to graph C^*-algebras. We show that all amplified graph algebras are in $\mathfrak{C}_{\text{free}}$ and that $\mathrm{Prim}_\tau(-)$ is a complete invariant for C^*-algebras in $\mathfrak{C}_{\text{free}}$. Using that, we can obtain a nice closure property for the class of amplified graph algebras.

THEOREM 3.2.5 ([**ERS**, Corollary 7.5]). *Let G_1 and G_2 be finite graphs. If A is a unital C^*-algebra and A fits into the following exact sequence*

$$0 \to C^*(\overline{G_1}) \otimes \mathbb{K} \to A \to C^*(\overline{G_2}) \to 0,$$

then there exists a finite graph G such that $A \cong C^*(\overline{G})$.

3.3. Semiprojectivity for amplified graph algebras

In this section, we will discuss how to use move (T) to study semiprojectivity for amplified graph algebras. The work in this section is joint with Jack Spielberg; it was carried out while I visited him at Arizona State University in December 2010. Afterwards, we learned that Eilers and Katsura had given a complete characterization of semiprojectivity for unital graph algebras, so we decided not to try to publish our results.

Our approach to proving semiprojectivity is very similar to the approach used by Eilers and Katsura. Namely, provide a (far reaching) generalization of the techniques used by Blackadar in [**Bla04**], Szymański in [**Szy02**], and Spielberg in [**Spi09**]. Here we will not prove how to do this generalization. Instead we will import Eilers and Katsura's result and apply it to our setting. When proving non-semiprojectivity, we make strong use of the fact that we are dealing with a very special class of graphs.

As already remarked, the K_0-group of an amplified graph algebra counts the number of vertices in the graph. This means that an amplified graph algebra $C^*(\overline{E})$ has finitely generated K-theory if and only if $\overline{E}$ has finitely many vertices. Using subgraphs it is not hard to see that every amplified graph algebra is an inductive limit of amplified graph algebras with finitely many vertices. Therefore, we can apply [**Bla04**, Corollary 2.10] to deduce that if $C^*(\overline{E})$ is semiprojective then $\overline{E}$ has finitely many vertices. It is easy to see that this is not a sufficient condition, as $C^*(\bullet \Longrightarrow \bullet) \cong \widetilde{\mathbb{K}}$ is not semiprojective. We will show that if $\overline{E}$ contains a subgraph that, in a certain sense, looks like $\bullet \Longrightarrow \bullet$ then $C^*(\overline{E})$ cannot be semiprojective. First we need a small lemma about maps out of the stabilization of the unitized compacts.

LEMMA 3.3.1. *Let $\lambda\colon \widetilde{\mathbb{K}} \otimes \mathbb{K} \to F \otimes \mathbb{K}$ be a $*$-homomorphism. If F is finite-dimensional then λ has non-zero kernel.*

PROOF. For any finite-dimensional C^*-algebra F we have that $F \otimes \mathbb{K} \cong \bigoplus_{i=1}^n \mathbb{K}$ for some $n \in \mathbb{N}$. We will suppress this isomorphism. Denote by $\rho_k\colon \bigoplus_{i=1}^n \mathbb{K} \to \mathbb{K}$ the projection map onto the k'th summand. Since $\ker \lambda = \bigcap_{i=1}^n \ker(\rho_i \circ \lambda)$ and the only possibilities for the kernel of λ are 0, $\mathbb{K} \otimes \mathbb{K}$, and $\widetilde{\mathbb{K}} \otimes \mathbb{K}$, it suffices to consider the case $n = 1$.

So suppose $\lambda\colon \widetilde{\mathbb{K}} \otimes \mathbb{K} \to \mathbb{K}$ is given. Let e_{ij} denote the standard matrix units of $\mathbb{K}$ and let $1_{\mathbb{K}}$ be the added unit. If $q = \lambda(1_{\mathbb{K}} \otimes e_{11})$ is zero, then $\ker \lambda = \widetilde{\mathbb{K}} \otimes \mathbb{K}$. Otherwise, $(\lambda(e_{nn} \otimes e_{11}))$ is a sequence of mutually orthogonal projections that are all dominated by q. Since $q \in \mathbb{K}$, this can only happen if at most finitely many of them are non-zero. Since they are all Murray-von Neumann equivalent, this implies that they are all zero, so $\ker \lambda = \mathbb{K} \otimes \mathbb{K}$. ☐

We will use the notion of a relative graph algebra in the proof of the next proposition. Relative graph algebras are defined and then discussed in some detail in section 3.1 of [**MT04**]. The definition is almost the same as that of a graph algebra (see Definition 3.2.3), the only difference being that we do not necessarily enforce the summation relation on all regular vertices. We will also need to know about the ideal structure of a graph C^*-algebra. For that and the notions of condition (K), and hereditary and saturated sets we refer to [**BHRS02**].

PROPOSITION 3.3.2. *Let $\overline{E}$ be an amplified graph with finitely many vertices. Suppose that there are two vertices $u, v \in E^0$ such that v is a sink and u is an infinite emitter that only emits to v. Then $C^*(\overline{E})$ is not semiprojective.*

PROOF. The set $\{u, v\}$ is hereditary. Let I be the ideal associated to $\{u, v\}$, and notice that by [**BHRS02**, Proposition 3.4], I is stably isomorphic to $\widetilde{\mathbb{K}}$.

Put $\{f_1, f_2, \ldots\} = s^{-1}(u)$. For each $n \in \mathbb{N}$ we define a graph E_n by letting $E_n^0 = \overline{E}^0$, $E_n^1 = \overline{E}^1 \setminus \{f_{n+1}, f_{n+2}, \ldots\}$, and restricting the range and source maps of $\overline{E}$. That is, E_n is just like $\overline{E}$, except there are only n edges from u to v. Let A_n be the relative graph algebra $C^*(E_n, \emptyset)$. Note that while u is regular in all the E_n we do not enforce the summation relation at u in any A_n

Let $\{s_e, p_v\}$ be the universal generators of $C^*(\overline{E})$, and denote the universal generators of A_n by $\{s_e^{(n)}, p_v^{(n)}\}$. Define for each $n \in \mathbb{N}$ a $*$-homomorphism $\iota_{n,n+1}\colon A_n \to A_{n+1}$ by sending $s_e^{(n)}$ to $s_e^{(n+1)}$ and p_v^n to $p_v^{(n+1)}$. Then $C^*(\overline{E})$ is the direct limit of the A_n with the $\iota_{n,n+1}$ as bounding maps. Denote by $\iota_{n,\infty}$ the map from A_n to $C^*(\overline{E})$, and note that it sends generators to generators.

By [**MT04**, Theorem 3.7], we have $A_n \cong C^*(G_n)$, where G_n is the graph with $G_n^0 = E_n^0 \cup \{u'\}$,

$$G_n^1 = E_n^1 \cup \{e' \mid e \in E_n^1, r_{E_n}(e) = u\},$$

the range and source maps extend those of E_n, and have $r_{G_n}(e') = u'$ and $s_{G_n}(e') = s_{E_n}(e)$. We observe that since $\overline{E}$ is amplified it satisfies the so-called condition (K), and that by construction, all the graphs E_n and thus all the graphs G_n satisfy condition (K). This means that they also satisfy the weaker condition that every loop in has an exit. Therefore, we see, by combining the concrete isomorphism from $A_n \cong C^*(G_n)$ with the Cuntz-Krieger uniqueness theorem (see [**FLR00**, Theorem 2]), that for all $n \in \mathbb{N}$ the map $\iota_{n,\infty}$ is injective.

Using the isomorphism $A_n \cong C^*(G_n)$ together with [**BHRS02**, Proposition 3.4], we see that $p_u^{(n)}$ and $p_v^{(n)}$ generate an ideal I_n in A_n that contains no other vertex projections in A_n. Furthermore $I_n \cong M_n \oplus \mathbb{C}$ and it is the unique ideal in A_n that contains $p_u^{(n)}$, $p_v^{(n)}$, and no other vertex projections.

We claim that $\iota_{n,\infty}(A_n) \cap I = \iota_{n,\infty}(I_n)$. To see this, we use that $\iota_{n,\infty}$ is injective, so we can think of $\iota_{n,\infty}(A_n) \cap I$ as an ideal in A_n. It contains the vertex projections $p_u^{(n)}$ and $p_v^{(n)}$, and, by construction of I, no others. Therefore, $\iota_{n,\infty}(A_n) \cap I = \iota_{n,\infty}(I_n)$.

Suppose, for the sake of reaching a contradiction, that $C^*(\overline{E})$ is semiprojective. By [**Bla04**, Proposition 2.9], we can find some $n \in \mathbb{N}$ and a $*$-homomorphism $\psi\colon C^*(\overline{E}) \to A_n$ such that $\iota_{n,\infty} \circ \psi$ is homotopic to $\mathrm{id}_{C^*(\overline{E})}$. For any projection $p \in I$, we then have that p is homotopic to $(\iota_{n,\infty} \circ \psi)(p)$. Since I is an ideal, this implies that $(\iota_{n,\infty} \circ \psi)(p) \in I$. Thus $\psi(p) \in I_n$. Because I is generated as an ideal by its projections, we can view $\psi|_I$ is a $*$-homomorphism from I to I_n. Call that $*$-homomorphism ϕ.

We recall that $I \otimes \mathbb{K} \cong \widetilde{\mathbb{K}} \otimes \mathbb{K}$ and that $I_n \otimes \mathbb{K}$ isomorphic to a stabilized finite-dimensional C^*-algebra, so Lemma 3.3.1 tells us that $\phi \otimes \mathrm{id}_{\mathbb{K}}$ is not injective. The only ideals in $I \otimes \mathbb{K}$ are stabilizations of ideals in I, so ϕ is not injective. In fact, the only non-trivial ideal of I contains the vertex projection p_v, so $\phi(p_v) = 0$. But then p_v is homotopic to $\iota(\phi(p_v)) = 0$, which contradicts that it is non-zero. Hence, we are forced to conclude that $C^*(\overline{E})$ is not semiprojective. $\square$

Loosely speaking the above proposition shows, that if $\widetilde{\mathbb{K}} \otimes \mathbb{K}$ sits as an ideal in an amplified graph C^*-algebra $C^*(\overline{E})$, then $C^*(\overline{E})$ cannot be semiprojective. On one hand this is not so surprising, since $\widetilde{\mathbb{K}} \otimes \mathbb{K}$ is not semiprojective, but on the other hand, it is surprising, since we know that semiprojective C^*-algebras can contain non-semiprojective ideals. For example, the Toeplitz algebra is semiprojective but has $\mathbb{K}$ as an ideal.

We now give a more general criterion for non-semiprojectivity of amplified graph algebras.

Proposition 3.3.3. *Let $\overline{E}$ be an amplified graph with finitely many vertices. If $\overline{E}^0$ contains two distinct vertices u, v such that*

(1) $u \geq v$, and,
(2) there is no vertex $x \in E^0$ such that $u \geq x \geq v$ and x supports a loop,

then $C^*(\overline{E})$ is not semiprojective.

Proof. Since move (T) (Theorem 3.2.2) preserves isomorphism, and the conditions are preserved under transitive closure, we may assume that $\overline{E}$ is transitively closed. Since $\overline{E}^0$ is finite, (2) implies that there is a path β from u to v of maximal length. Let $v' = r(\beta_1)$. Since u is not on a loop $v' \neq u$, and by (2) there is no loop based at v'.

Define $H = \{w \in \overline{E}^0 \mid u \geq w\} \setminus \{v'\}$. Let $w \in H$. If $w \geq v'$, then, by maximality of β, we must have that $w = r(\beta_i)$ for some i, thus $v' \geq w$. In particular, v' is on a loop, which is a contradiction. Therefore no vertex in H has a path to v', so H is hereditary. Since $\overline{E}$ is amplified H is also saturated. Let I_H be the ideal associated to H, and note that, since H is finite, it is generated by finitely many projections. We have the following short exact sequence, see [**BHRS02**, Proposition 3.4],

$$0 \to I_H \to C^*(\overline{E}) \to C^*(\overline{E}/H) \to 0.$$

Because I_H is generated by finitely many projections, semiprojectivity of $C^*(\overline{E})$ would imply semiprojectivity of $C^*(\overline{E}/H)$ (see for instance [**Sør12**, Proposition 2]). Thus it suffices to show that $C^*(\overline{E}/H)$ is not semiprojective. Observe that the graph $\overline{E}/H$ satisfies the hypotheses of Proposition 3.3.2, so by that proposition $C^*(\overline{E}/H)$ is not semiprojective. □

We now have our main tool for proving non-semiprojectivity. To prove semiprojectivity we rely on a theorem due to Eilers and Katsura. Denote by $\sim$ Murray-von Neumann equivalence of projections, and recall that a projection p is called properly infinite if it has orthogonal sub-projections $p_1, p_2 \leq p$, such that $p_1 \sim p \sim p_2$. By definition, the zero projection is properly infinite. If $\Lambda \subseteq E^0$ is a finite set we write p_Λ for the projection $\sum_{v\in\Lambda} p_v$.

THEOREM 3.3.4 ([**EK12**]). *Let E be a graph with finitely many vertices. Define for each $v \in E^0$ the set*

$$\Omega_v = \{w \in E^0 \mid v \text{ emits infinitely many edges to } w\}.$$

If p_{Ω_v} is properly infinite for all $v \in E^0$, then $C^*(E)$ is semiprojective.

To ease the use of the theorem we prove a lemma.

LEMMA 3.3.5. *Let $\overline{E}$ be an amplified graph, and let $x \in \overline{E}^0$ be a vertex that supports a loop of length one. If $\Lambda \subseteq r(s^{-1}(x))$ is finite and contains x then p_Λ is properly infinite.*

PROOF. Denote by $e_1, e_2, \ldots$, the edges with both range and source x, and pick for each $w \in \Lambda$ an edge f_w from x to w. Choose numbers $m_w, k_w \in \mathbb{N}$ for each $w \in \Lambda$ that are all distinct. Define

$$p_1 = \sum_{w\in\Lambda} s_{e_{m_w}} s_{f_w} s_{f_w}^* s_{e_{m_w}}^*,$$

and

$$p_2 = \sum_{w\in\Lambda} s_{e_{k_w}} s_{f_w} s_{f_w}^* s_{e_{k_w}}^*.$$

Note that since all the k_w and m_w are distinct, the projections that appear in the sums are pairwise mutually orthogonal. Therefore, p_1 and p_2 are orthogonal projections. Furthermore, for each $w \in \Lambda$, we have $s_{e_{m_w}} s_{f_w} s_{f_w}^* s_{e_{m_w}}^* \leq p_x$ and

$$s_{e_{m_w}} s_{f_w} s_{f_w}^* s_{e_{m_w}}^* \sim s_{f_w}^* s_{e_{m_w}}^* s_{e_{m_w}} s_{f_w} = s_{f_w}^* p_x s_{f_w} = s_{f_w}^* s_{f_w} = p_w,$$

so $p_1 \leq p_x \leq p_\Lambda$ and $p_1 \sim p_\Lambda$. Likewise, $p_2 \leq p_x \leq p_\Lambda$ and $p_2 \sim p_\Lambda$. Thus, p_Λ is properly infinite. □

We can now give a complete characterization of semiprojectivity for amplified graph algebras.

THEOREM 3.3.6. *Let $\overline{E}$ be an amplified graph. Then $C^*(\overline{E})$ is semiprojective if and only if the following holds:*

(1) $\overline{E}^0$ *is finite, and*
(2) *for any $u, v \in \overline{E}^0$ with $u \geq v$ there exists some $x \in \overline{E}^0$ such that $u \geq x \geq v$ and x is on a loop.*

PROOF. Suppose first that $\overline{E}$ satisfies (1) and (2). As in the proof of Proposition 3.3.3, move (T) shows that there is no loss of generality in assuming that $\overline{E}$ is transitively closed. Let $v \in \overline{E}^0$ be given. We will show that p_{Ω_v} is properly infinite. There are two cases: $v \in \Omega_v$ and $v \notin \Omega_v$.

If $v \in \Omega_v$, then, since $\overline{E}$ is transitively closed, there is a loop of length one based at v. Thus it follows from Lemma 3.3.5 that p_{Ω_v} is properly infinite.

Assume now that $v \notin \Omega_v$. If Ω_v is empty then p_{Ω_v} is the zero projection, so it is properly infinite. Suppose that $\Omega_v \neq \emptyset$. Since $\overline{E}$ is transitively closed and satisfies (2), there is at least one vertex in Ω_v that supports a loop of length one. Let $x_1, x_2, \ldots x_n$ be the vertices in Ω_v that support a loop of length one, $n < \infty$ by (1). Using again that $\overline{E}$ is transitively closed, we see that $\Omega_{x_i} \subseteq \Omega_v$ for $1 \leq i \leq n$. Since $\overline{E}$ satisfies (2), we can pick pairwise disjoint subsets $\Lambda_i \subseteq \Omega_{x_i}$ such that $x_i \in \Lambda_i$, for $1 \leq i \leq n$, and $\cup_i \Lambda_i = \Omega_v$. By Lemma 3.3.5, all the p_{Λ_i} are properly infinite. A sum of orthogonal properly infinite projections is again properly infinite, so

$$p_{\Omega_v} = \sum_{i=1}^{n} p_{\Lambda_i}$$

is properly infinite. We have now shown that for all $v \in \overline{E}^0$ the projection p_{Ω_v} is properly infinite. Since $\overline{E}$ has finitely many vertices by (1), it follows from Theorem 3.3.4 that $C^*(\overline{E})$ is semiprojective.

Suppose now that $\overline{E}$ fails (1) or (2). If $\overline{E}$ fails (1) then $K_0(C^*(\overline{E}))$ is not finitely generated, in which case $C^*(\overline{E})$ is not semiprojective as remarked in the beginning of this section. If it fails (2) (but not (1)), then it follows from Proposition 3.3.3 that $C^*(\overline{E})$ is not semiprojective. □

3.4. Simple graph algebras

In [**Sør**] we consider moves one can use to change graphs without changing the Morita equivalence class of the associated C^*-algebras. The moves are described in detail in Section 3 of [**Sør**], stated shortly they are

(S) Remove a source, if it is a regular vertex,
(I) In-split the graph (as in [**BP04**]),
(O) Out-split the graph (as in [**BP04**]), and,
(R) Reduction (remove a regular vertex u with $s_G^{-1}(u)$ and $s_G(r_G^{-1}(u))$ one point sets).

All the moves were previously known to preserve Morita equivalence of graph algebras. We say that two graphs G and E are move equivalent if we can transform one into the other using the moves and their inverses. We write $G \sim_M E$. If $C^*(E)$ is Morita equivalent to $C^*(G)$ we say that G and E are C^*-equivalent and write $G \sim_{C^*} E$. The main result of [**Sør**] is:

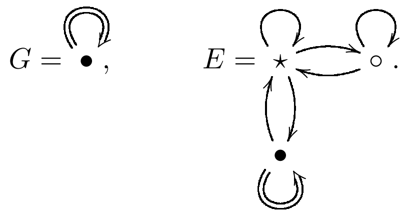
THEOREM 3.4.1 ([**Sør**, Theorem 4.4]). *Let G, E be graphs with simple unital C^*-algebras. If G has at least one infinite emitter or sink then*

$$G \sim_M E \iff G \sim_{C^*} E.$$

In the absence of infinite emitters, we add the so-called Cuntz splice to the list of moves, and get a similar result. The Cuntz splice is known, by the deep classification result of Kirchberg and Phillips, to preserve Morita equivalence for simple purely infinite graph algebras.

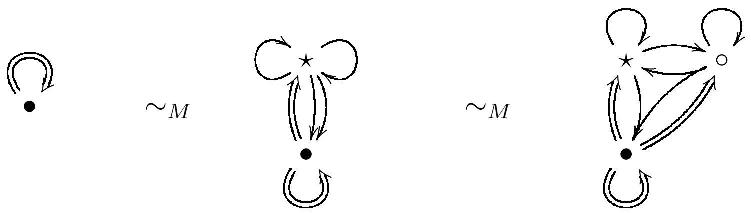
As a first step in [**Sør**], it is proved that two other moves, one is move (T) the other collapses a regular vertex that does not support a simple loop (see [**Sør**, Theorem 5.2]), are in $\sim_M$. This means that they can be achieved by preforming a sequence of the other moves. The result is then proved by adapting the techniques Franks used to classify irreducible shifts of finite type in [**Fra84**]. Therefore, the main technical results of [**Sør**] are concerned with under what conditions we can add rows and columns in the adjacency matrix of a graph without changing the associated algebra.

As an example of how to get by without the Cuntz splice in the presence of an infinite emitter, consider the following graphs:

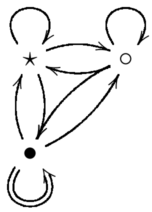


Here E arises from G by preforming a Cuntz splice. Both $C^*(E)$ and $C^*(G)$ are Kirchberg algebras. A K-theory computation shows that they are stably isomorphic, in fact they are both stably isomorphic to $\mathcal{O}_\infty$, but they are not isomorphic.

We will now describe how to go from G to E using only the allowed moves. First we do two out-splits leading to the following sequence of graphs.

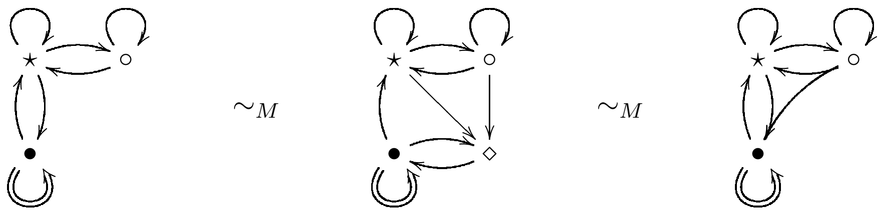


In the first, we divided the loops at $\bullet$ into two groups, one containing two of them and one containing the rest. In the next step, we partitioned the edges leaving $\star$ into two groups, each with one loop and one edge to $\bullet$. Using move (T), we see that the last graph is move equivalent to the following graph, that we will call E'.



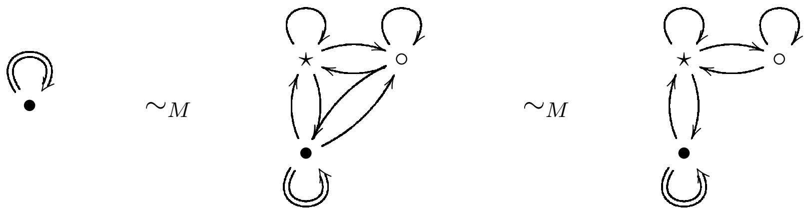
Looking at E' and E, we see that the only difference is that there are edges between $\circ$ and $\bullet$ in E', but not in E.

We will now work with E, and add those missing edges. First we show how to get the edge from $\circ$ to $\bullet$. The needed series of moves is described in the picture below.



We started by out-spliting at $\star$, dividing the outgoing edges into two groups, one containing the edge from $\star$ to $\bullet$, and one containing the remaining two edges $\star$ emits. Then we collapsed $\diamond$ using one of the moves derived from the original list ([**Sør**, Theorem 5.2]), this adds an edge for each path of length two that goes to $\diamond$ and to something else. This is similar to what is described in [**Sør**, Lemma 7.2].

Now one application of move (T) adds infinitely many edges from $\bullet$ to $\circ$, a second removes all but one of them. Therefore we have $G \sim_M E' \sim_M E$. The figure below shows this using pictures of graphs.



CHAPTER 4

Almost commuting matrices

In this chapter we describe the main results of [**LS10**]. The purpose of [**LS10**] is to provide a real version of Lin's theorem, but along the way we prove many things about semiprojectivity for Real C^*-algebras. Our proof follows the path laid out by Friis and Rørdam, when they proved a generalization of Lin's theorem. This, perhaps, leads to a longer proof of the real version of Lin's theorem than strictly necessary, but this way we get many auxiliary results that can be useful in other contexts.

4.1. Real C^*-algebras

The literature on Real C^*-algebras contains both real C^*-algebras and Real C^*-algebras. One is like a C^*-algebra, only with the scalar field being the real numbers, the other is a genuine C^*-algebra together with a conjugation operation, that lets you recognize the real elements. In [**LS10**] we abandon these names, as we find that it is a mistake to make a distinction between upper and lower case letters so important. Instead, we use R^*-algebra for the algebras with real scalars; for the others, we consider reflections on C^*-algebras. A map $\tau\colon A \to A$ is called a reflection if it is linear, anti-multiplicative, $*$-preserving, and satisfies $\tau(\tau(a)) = a$ for all $a \in A$. We then define a $C^{*,\tau}$-algebra as a pair (A, τ), where A is a C^*-algebra and τ is a reflection on A. The real elements of (A, τ) are those $a \in A$ with $a^* = \tau(a)$. We use Real C^*-algebras as a common term for the theory of both R^*-algebras and $C^{*,\tau}$-algebras.

The definition of a $C^{*,\tau}$-algebra is supposed to be reminiscent of M_n with the transpose map. Clearly the transpose map T is a reflection, and the real-valued matrices are exactly those $X \in M_n$ with $X^* = X^T$. For this reason we usually write $\tau(a)$ as a^τ. It turns out that when n is an even number, there is an additional reflection on M_n, namely the dual-operation $\sharp$. It is defined by

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^{\sharp} = \begin{pmatrix} A_{22}^T & -A_{12}^T \\ -A_{21}^T & A_{11}^T \end{pmatrix}.$$

Section 2 in [**LS10**] is devoted to explaining some of the basics of $C^{*,\tau}$-algebras, R^*-algebras and their relation to each other.

4.2. Semiprojectivity for $C^{*,\tau}$-algebras

A large amount of the work done in [**LS10**] concerns semiprojectivity for $C^{*,\tau}$-algebras. We give the obvious definition of semiprojectivity, and prove many of the basic results. To get to the more advanced topics in semiprojectivity, we also develop some of the theory of multiplier algebras and corona algebras for $C^{*,\tau}$-algebras. The most prominent theorem about semiprojectivity for $C^{*,\tau}$-algebras in [**LS10**] is:

THEOREM 4.2.1 ([**LS10**, Theorem 5.1]). *If X is a one-dimensional finite CW complex then $(C(X), \mathrm{id})$ is semiprojective.*

The complex version of this result is crucial in Friis and Rørdams proof of their generalization of Lin's theorem.

4.3. Almost commuting (real) matrices

A slightly simplified version of the main Theorem in [**LS10**] is:

THEOREM 4.3.1 ([**LS10**, Theorem 1]). *For all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $n \in \mathbb{N}$ the following holds: whenever A, B are two n-by-n, contractive, self-adjoint, and real matrices such that $\|AB - BA\| < \delta$, there exist n-by-n, contractive, self-adjoint, and real matrices A', B' such that $A'B' = B'A'$ and*

$$\|A - A'\|, \|B - B'\| < \varepsilon.$$

The proof begins by observing that if A and B almost commute then $X = A + iB$ is almost normal. Note that X is not a real matrix. However, in addition to being almost normal, it is symmetric, i.e. $X^T = X$. So while the complex version of Lin's theorem states that almost normal matrices are close to normal matrices, our version of Lin's theorem says that symmetric, almost normal matrices are close to symmetric, normal matrices. We do not prove whether real, almost normal matrices are close to real, normal matrices. Therefore it is more accurate to say that we prove *a* real version of Lin's theorem rather than *the* real version of Lin's theorem.

Friis and Rørdam have generalized Lin's theorem ([**FR96**, Theorem 4.4]), showing that two almost commuting self-adjoint contractions in any stable rank one C^*-algebra are close to two exactly commuting self-adjoint contractions. We provide a similar generalization in [**LS10**, Theorem 7.10]. However, where Friis and Rørdam use C^*-algebras where the invertibles are dense, i.e. stable rank one, we use C^*-algebras where the self-τ invertible elements are dense in all the self-τ elements.

CHAPTER 5

Generators and relations for Real C^*-algebras

5.1. $C^{*,\tau}$-relations

The work done in [**LS10**] has shown the need for a theory of generators and relations in the category of $C^{*,\tau}$-algebras. A nice and general theory for C^*-algebra relations is presented in [**Lor10**]. We will follow it almost verbatim. Hence, we give the following definitions of (compact) $C^{*,\tau}$-relations (see [**Lor10**, Definition 2.1, Definition 2.2, and Definition 2.3]). For definitions and notions concerning $C^{*,\tau}$-algebras and R^*-algebras we refer to [**LS10**].

DEFINITION 5.1.1. Let $\mathcal{X}$ be a set. Define a category $\mathcal{N}_{\mathcal{X}}$, where the *objects* are pairs $(j,(A,\tau))$, where (A,τ) is a $C^{*,\tau}$-algebra and $j\colon \mathcal{X} \to (A,\tau)$ is a map. The *morphisms* from $(j,(A,\tau))$ to $(k,(B,\tau))$ are all $*$-τ-homomorphisms $\phi\colon (A,\tau) \to (B,\tau)$ such that $\phi\circ j = k$.

We call $\mathcal{N}_{\mathcal{X}}$ the *null $C^{*,\tau}$-relation on $\mathcal{X}$*. Sometimes we will, with slight abuse of notation, refer to $j\colon \mathcal{X} \to (A,\tau)$ as an object in the category.

DEFINITION 5.1.2. Given a set $\mathcal{X}$, a *$C^{*,\tau}$-relation on $\mathcal{X}$* is a full subcategory $\mathcal{R}$ of the null $C^{*,\tau}$-relation $\mathcal{N}_{\mathcal{X}}$ such that:

CT1: The unique map $\mathcal{X} \to \{0\}$ is an object in $\mathcal{R}$.

CT2: For any injective $*$-τ-homomorphism $\phi\colon (A,\tau) \hookrightarrow (B,\tau)$ and any function $f\colon \mathcal{X} \to (A,\tau)$, we have

$$f \text{ is an object} \quad \Longleftarrow \quad \phi\circ f \text{ is an object.}$$

CT3: For any $*$-τ-homomorphism $\phi\colon (A,\tau) \to (B,\tau)$ and any function $f\colon \mathcal{X} \to (A,\tau)$, we have

$$f \text{ is an object} \quad \Longrightarrow \quad \phi\circ f \text{ is an object.}$$

CT4f: If $f_j\colon \mathcal{X} \to (A_j,\tau)$ is an object for $1 \le j \le n$ then

$$\prod_{j=1}^{n} f_j\colon \mathcal{X} \to \prod_{j=1}^{n} (A_j,\tau)$$

is an object.

DEFINITION 5.1.3. A $C^{*,\tau}$-relation $\mathcal{R}$ on $\mathcal{X}$ is called *compact* if it satisfies:

CT4: For any non-empty set Λ and any family $f_\lambda\colon \mathcal{X} \to (A_\lambda,\tau)$ of objects indexed by Λ, the function

$$\prod_{\lambda\in\Lambda} f_\lambda\colon \mathcal{X} \to \prod_{\lambda\in\Lambda} (A_\lambda,\tau),$$

is an object in $\mathcal{R}$.

The definition of a universal object in [**Lor10**] is set up to allow an easy transition into a situation where we consider not only universal C^*-algebras but also universal pro-C^*-algebras (compare [**Lor10**, Definition 2.9 and Definition 3.12]). Since we at present have no desire to study pro-$C^{*,\tau}$-algebras (which have yet to be defined), we give a slightly different definition of a universal object than Loring does.

DEFINITION 5.1.4. If $\mathcal{R}$ is a $C^{*,\tau}$-relation on $\mathcal{X}$ then an object $((U,\tau),\iota)$ in $\mathcal{R}$ will be called *universal for* $\mathcal{R}$ if U is generated by $\iota(\mathcal{X})$ and whenever we are given any object $(f,(A,\tau))$ then there exists a unique $*$-τ-homomorphism $\phi\colon (U,\tau)\to(A,\tau)$ such that $f=\phi\circ\iota$.

From the definition one easily sees that there is at most one universal object.

The importance of the compact relations is their relationship with universal objects. The following is [**Lor10**, Theorem 2.10] modified to fit our category.

THEOREM 5.1.5. *Let $\mathcal{R}$ be a $C^{*,\tau}$-relation on $\mathcal{X}$. There exists a universal object for $\mathcal{R}$ if and only if $\mathcal{R}$ is compact.*

PROOF. Suppose that $((U,\tau),\iota)$ is a universal object for $\mathcal{R}$. We will show that $\mathcal{R}$ satisfies **CT4**. Let a set Λ and a family of objects $f_\lambda\colon \mathcal{X}\to(A_\lambda,\tau)$ be given, and define $f\colon\mathcal{X}\to\prod A_\lambda$ by $f(x)=(f_\lambda(x))$. We aim to show that $(\prod(A_\lambda,\tau),f)$ is an object.

Since $((U,\tau),\iota)$ is a universal object, we can, for each $\lambda\in\Lambda$, find a $*$-τ-homomorphism $\phi_\lambda\colon (U,\tau)\to(A_\lambda,\tau)$ such that $f_\lambda=\phi_\lambda\circ\iota$. Define a $*$-τ-homomorphism $\phi\colon(U,\tau)\to\prod(A_\lambda,\tau)$ by $\phi(a)=(\phi_\lambda(a))$. Observe that for all $x\in\mathcal{X}$, we have

$$f(x)=(f_\lambda(x))=((\phi_\lambda\circ\iota)(x))=(\phi_\lambda(\iota(x)))=(\phi\circ\iota)(x).$$

Thus, by **CT3**, **CT4** holds, and therefore $\mathcal{R}$ is compact.

Suppose now that $\mathcal{R}$ is compact. We will define four sets.

(i) Let S_1 be a set such that every $C^{*,\tau}$-algebra generated by a set no larger than $\mathcal{X}$ has cardinality at most that of S_1.
(ii) Let S_2 be the set of all $C^{*,\tau}$-algebras whose underlying set is a subset of S_2.
(iii) Let S_3 be the set of all functions from $\mathcal{X}$ to an element in S_2.
(iv) Let S_4 be the set of all functions $f\colon\mathcal{X}\to(A,\tau)$ in S_3 such that $f(\mathcal{X})$ generates (A,τ) and $(f,(A,\tau))\in\mathcal{R}$.

Index the elements in S_4 as $f_\lambda\colon\mathcal{X}\to(A_\lambda,\tau)$ for λ in some set Λ. By **CT1**, the set Λ is not empty.

We will now define (U,τ). By **CT4**, the function

$$h=\prod_\lambda f_\lambda\colon\mathcal{X}\to\prod_\lambda(A_\lambda,\tau),$$

is an object in $\mathcal{R}$. Let (U,τ) be the $C^{*,\tau}$-algebra generated by $h(\mathcal{X})$, let $\iota\colon\mathcal{X}\to(U,\tau)$ be the co-restriction of h, and let η be the $C^{*,\tau}$-inclusion of (U,τ) in the product $\prod(A_\lambda,\tau)$. Since $h=\eta\circ\iota$, **CT2** tells us that $(\iota,(U,\tau))$ is an object in $\mathcal{R}$.

To see that $(\iota, (U, \tau))$ has the desired universal property, let $(f, (A, \tau))$ be an object in $\mathcal{R}$. Let A_0 be the sub-$C^{*,\tau}$-algebra of A generated by $f(\mathcal{X})$, let f_0 be the co-restriction of f to A_0, and let α_0 be the inclusion of A_0 in A. By **CT2**, $(f_0, (A_0, \tau))$ is an object in $\mathcal{R}$. There is a $*$-τ-isomorphism $\psi\colon (A_0, \tau) \to (A_1, \tau)$ for some $(A_1, \tau) \in S_2$. By **CT3**, the map $g\colon \mathcal{X} \to (A_1, \tau)$ given by $g = \psi \circ f_0$ is an object in $\mathcal{R}$. Hence, by construction of S_4, there is some $\mu \in \Lambda$ such that $(g, (A_1, \tau)) = (f_\mu, (A_\mu, \tau))$. If we let π_μ be the quotient map from $\prod_\lambda (A_\lambda, \tau)$ onto (A_μ, τ), we can define a $*$-τ-homomorphism $\phi\colon (U, \tau) \to (A, \tau)$ by

$$\phi = \alpha_0 \circ \psi^{-1} \circ \pi_\mu \circ \eta.$$

Finally, we observe that

$$\begin{aligned} \phi \circ \iota &= \alpha_0 \circ \psi^{-1} \circ \pi_\mu \circ \eta \circ \iota \\ &= \alpha_0 \circ \psi^{-1} \circ \pi_\mu \circ h \\ &= \alpha_0 \circ \psi^{-1} \circ f_\mu \\ &= \alpha_0 \circ f_0 = f. \end{aligned}$$

Since $h(\mathcal{X})$ generates U, the map ϕ is unique, and, therefore, $(\iota, (U, \tau))$ is a universal object for $\mathcal{R}$. □

Just as it is the case for C^*-algebras, the proof that universal $C^{*,\tau}$-algebras exist, gives no insight into what the universal algebras are. However, inspired by the complex case, we can often find universal algebras.

Example 5.1.6. Let $n \in \mathbb{N}$ be given and let $X = [0,1]^n$. Then $C(X, \mathrm{id})$ is the universal $C^{*,\tau}$-algebra generated by n self-adjoint commuting contractions $h_1, h_2, \dots, h_n$ such that $h_i^\tau = h_i$ for $i = 1, 2, \dots, n$. The universal generators are the coordinate functions.

To see this, suppose we are given such a set of contractions h_i in some $C^{*,\tau}$-algebra (B, τ). Denote by f_i the coordinate projections in $C(X)$. By the universal property of $C(X)$, there exist a unique $*$-homomorphism $\psi\colon C(X) \to B$ such that $\psi(f_i) = h_i$. It remains to see that ψ respects the τ-operation. This follows, since the f_i and h_i are self-τ, and by the uniqueness of ψ.

5.2. R^*-relations

We can also give versions of Loring's definitions that fit with R^*-algebras.

Definition 5.2.1. Let $\mathcal{X}$ be a set. Define a category $\mathcal{M}_\mathcal{X}$, where the *objects* are pairs (j, A), where A is an R^*-algebra and $j\colon \mathcal{X} \to A$ is a map. The *morphisms* from (j, A) to (k, B) are all $*$-homomorphisms $\phi\colon A \to B$ such that $\phi \circ j = k$.

We call $\mathcal{M}_\mathcal{X}$ the *null R^*-relation on $\mathcal{X}$*. Sometimes we will, with slight abuse of notation, refer to $j\colon \mathcal{X} \to A$ as an object in the category.

Definition 5.2.2. Given a set $\mathcal{X}$, an *R^*-relation on $\mathcal{X}$* is a full subcategory $\mathcal{R}$ of the null R^*-relation $\mathcal{M}_\mathcal{X}$ such that:

R1: The unique map $\mathcal{X} \to \{0\}$ is an object in $\mathcal{R}$.

R2: For any injective $*$-homomorphism $\phi\colon A \hookrightarrow B$ and any function $f\colon \mathcal{X} \to A$, we have

$$f \text{ is an object} \iff \phi \circ f \text{ is an object.}$$

R3: For any $*$-homomorphism $\phi\colon A \to B$ and any function $f\colon \mathcal{X} \to A$, we have

$$f \text{ is an object} \implies \phi \circ f \text{ is an object.}$$

R4f: If $f_j\colon \mathcal{X} \to A_j$ is an object for $1 \leq j \leq n$ then

$$\prod_{j=1}^{n} f_j\colon \mathcal{X} \to \prod_{j=1}^{n} A_j$$

is an object.

Definition 5.2.3. An R^*-relation $\mathcal{R}$ on $\mathcal{X}$ is called *compact* if it satisfies:

R4: For any non-empty set Λ and any family $f_\lambda\colon \mathcal{X} \to A_\lambda$ of objects indexed by Λ, the function

$$\prod_{\lambda\in\Lambda} f_\lambda\colon \mathcal{X} \to \prod_{\lambda\in\Lambda} A_\lambda,$$

is an object in $\mathcal{R}$.

Definition 5.2.4. If $\mathcal{R}$ is an R^*-relation on $\mathcal{X}$ then an object (U, ι) in $\mathcal{R}$ will be called *universal for* $\mathcal{R}$ if U is generated by $\iota(\mathcal{X})$ and whenever we are given any object (f, A) then there exists a unique $*$-homomorphism $\phi\colon U \to A$ such that $f = \phi \circ \iota$.

Recall from [**LS10**, Section 2.2] that we have a functor $\Re$ from the category of $C^{*,\tau}$-algebras to the category of R^*-algebras, and that

$$\Re(A, \tau) = \{a \in A \mid a^* = a^\tau\}.$$

We also have a functor $\overline{\bigstar}$ from the category of R^*-algebras to the category of $C^{*,\tau}$-algebras. The functors are almost inverses in the sense that $\Re(\overline{\bigstar}(A)) \cong A$ and $\overline{\bigstar}(\Re(B, \tau)) \cong (B, \tau)$.

We will deduce the R^*-algebra version of Theorem 5.1.5 from the $C^{*,\tau}$-algebra version.

Proposition 5.2.5. *Let $\mathcal{R}$ be an R^*-relation on $\mathcal{X}$. Define a full subcategory $\mathcal{R}_\tau$ of $\mathcal{N}_\mathcal{X}$ by letting its objects be pairs $(f, (A, \tau))$, where $f(\mathcal{X}) \subseteq \Re(A, \tau)$ and, if we let $\tilde{f}$ be the co-restriction of f to $\Re(A, \tau)$, the pair $(\tilde{f}, \Re(A, \tau))$ is an object in $\mathcal{R}$. The category $\mathcal{R}_\tau$ is a $C^{*,\tau}$-relation.*

Moreover, $\mathcal{R}_\tau$ is compact if $\mathcal{R}$ is compact.

Proof. The real part of the zero $C^{*,\tau}$-algebra is the zero $C^{*,\tau}$-algebra, so because $\mathcal{R}$ satisfies **R1**, $\mathcal{R}_\tau$ satisfies **CT1**. Any $*$-τ-homomorphism $\phi\colon (A, \tau) \to (B, \tau)$ maps $\Re(A, \tau)$ into $\Re(B, \tau)$. If ϕ is injective, then for any $a \in A$, the fact that $\phi(a) \in \Re(B, \tau)$ implies $a \in \Re(A, \tau)$. Combined with the fact that $\mathcal{R}$ satisfies **R2** and **R3**, this shows that $\mathcal{R}_\tau$ satisfies **CT2** and **CT3**.

To see that $\mathcal{R}_\tau$ satisfies **CT4f** (and **CT4** when $\mathcal{R}$ satisfies **R4**), simply notice that for any non-empty set Λ and any family (A_λ, τ) of $C^{*,\tau}$-algebras indexed by Λ, we have

$$\Re\left(\prod_\lambda (A_\lambda, \tau)\right) = \prod_\lambda \Re(A_\lambda, \tau). \qquad \square$$

We can now prove that universal R^*-algebras exist.

THEOREM 5.2.6. *Let $\mathcal{R}$ be an R^*-relation on $\mathcal{X}$. There exists a universal object for $\mathcal{R}$ if and only if $\mathcal{R}$ is compact.*

Moreover, if $((U,\tau),\iota)$ is the universal object for $\mathcal{R}_\tau$ then $(\Re(A,\tau),\kappa)$, where κ is the co-restrict of ι to $\Re(A,\tau)$, is the universal object for $\mathcal{R}$.

PROOF. The proof that $\mathcal{R}$ must be compact if a universal object exists, is exactly as in the proof of Theorem 5.1.5, so we omit it.

Suppose that $\mathcal{R}$ is compact, and define $\mathcal{R}_\tau$ as in Proposition 5.2.5. Since $\mathcal{R}_\tau$ is compact, there exists a universal object $((U,\tau),\iota)$ for it. Let κ be the co-restriction of ι to $\Re(U,\tau)$ and put $V = \Re(U,\tau)$. We claim that (V,κ) is universal for $\mathcal{R}$. First note that since $((U,\tau),\iota)$ is an object of $\mathcal{R}_\tau$, the pair (V,κ) is an object in $\mathcal{R}$. To see that V is universal, let (f, A) be an object in $\mathcal{R}$. Let $(B,\tau) = \overline{\bigstar}(A)$, and let $\bar{f}\colon \mathcal{X} \to B$ be defined by $\bar{f}(x) = f(x)$. Then $(\bar{f}, (B,\tau))$ is an object in $\mathcal{R}_\tau$. Thus there is a $*$-τ-homomorphism $\phi\colon (U,\tau) \to (B,\tau)$ such that $\bar{f} = \phi \circ \iota$. Since the image of ι is in $\Re(U,\tau)$ and the image of $\bar{f}$ is in $\Re(B,\tau)$, we also have that $f = \Re(\phi) \circ \kappa$. Since $\kappa(\mathcal{X})$ generates V, we see that $\Re(\phi)$ is the unique $*$-homomorphism with that property, so (V,κ) is universal for $\mathcal{R}$. $\square$

EXAMPLE 5.2.7. Let $n \in \mathbb{N}$ be given and let $X = [0,1]^n$. Then $C(X,\mathbb{R})$ is the universal R^*-algebra generated by n self-adjoint commuting contractions $h_1, h_2, \ldots, h_n$. The universal generators are the coordinate functions. To see this let $\mathcal{R}$ be the category associated with the relation. Then the category associated with the relation in Example 5.1.6 is exactly $\mathcal{R}_\tau$.

5.3. An application of $C^{*,\tau}$-relations

We will use the existence of universal $C^{*,\tau}$-algebras to prove the following:

THEOREM 5.3.1. *If (P,τ) is a separable projective $C^{*,\tau}$-algebra then P is a projective C^*-algebra.*

To prove the Theorem 5.3.1 we need two lemmas. Firstly, we have a simple application of Theorem 5.1.5.

LEMMA 5.3.2. *There exists a universal $C^{*,\tau}$-algebra generated by a sequence of self-τ contractions and a sequence of skew-τ contractions.*

We also need the following special case of Theorem 5.3.1.

LEMMA 5.3.3. *Let (Q,τ) be the universal $C^{*,\tau}$-algebra generated by a sequence of self-τ contractions and a sequence of skew-τ contractions. Then (Q,τ) is a projective $C^{*,\tau}$-algebra and Q is a projective C^*-algebra.*

PROOF. Contractions always lift between C^*-algebras. If x is a contractive lift of a self-τ contraction then $(x+x^\tau)/2$ is a contractive self-τ lift. We can lift the skew-τ elements in a similar fashion. With that in mind we see that (Q,τ) is projective by virtue of its universal property.

Let (x_n) be the universal self-τ generators of (Q,τ) and let (y_n) be the universal skew-τ generators. To see that Q is projective as a C^*-algebra, suppose we are given a surjective $*$-homomorphism $\pi\colon B \twoheadrightarrow Q$. Since contractions lift, we can find contractions $(\bar{x}_n),(\bar{y}_n) \subseteq B$ such that $\pi(\bar{x}_n) = x_n$ and $\pi(\bar{y}_n) = y_n$ for all $n \in \mathbb{N}$. Let $D = B \oplus B^{\mathrm{op}}$, and define a reflection $\tau\colon D \to D$ by $\tau(a,b) = (b,a)$. Put $a_n = (\bar{x}_n, \bar{x}_n)$ and $b_n = (\bar{y}_n, -\bar{y}_n)$. Note that (a_n) is a sequence of self-τ contractions in (D,τ) and (b_n) is a sequence of skew-τ contractions. By the universal property of (Q,τ), there is a $*$-τ-homomorphism $\phi\colon (Q,\tau) \to (D,\tau)$ such that $\phi(x_n) = a_n$ and $\phi(y_n) = b_n$ for all $n \in \mathbb{N}$. Let $\rho\colon D \twoheadrightarrow B$ be the $*$-homomorphism that projects onto the first summand of D, and put $\psi = \rho \circ \phi$. Then ψ is a $*$-homomorphism from Q to B such that

$$(\pi \circ \psi)(x_n) = \pi(\rho((\bar{x}_n, \bar{x}_n))) = \pi(\bar{x}_n) = x_n,$$

for all $n \in \mathbb{N}$. Likewise, $(\pi \circ \psi)(y_n) = y_n$ for all $n \in \mathbb{N}$. Since Q is generated by the x_n and y_n not only as a $C^{*,\tau}$-algebra, but also as a C^*-algebra (since $x_n^\tau = x_n$ and $y_n^\tau = -y_n$), we have $\pi \circ \psi = \mathrm{id}_Q$. That is, Q is a projective C^*-algebra. □

Finally, we recall that to prove projectivity of a C^*-algebra A, it suffices to find a projective C^*-algebra P that has a split surjection onto A (see for instance [**Thi11**, Lemma 5.1]).

PROOF OF THEOREM 5.3.1. Let (Q,τ) be the universal $C^{*,\tau}$-algebra generated by a sequence of self-τ contractions and a sequence of skew-τ contractions. Since (P,τ) is separable, there is a sequence of contractions z_n that generate P as a $C^{*,\tau}$-algebra. Put

$$x_n = \frac{z_n + z_n^\tau}{2} \quad \text{and} \quad y_n = \frac{z_n - z_n^\tau}{2},$$

for all $n \in \mathbb{N}$. Then the x_n and y_n generate P, and by the universal property of (Q,τ) we have a surjective $*$-τ-homomorphism $\rho\colon (Q,\tau) \twoheadrightarrow (P,\tau)$. Using the projectivity of (P,τ), we get a $*$-τ-homomorphism $\lambda\colon (P,\tau) \to (Q,\tau)$ such that $\rho \circ \lambda = \mathrm{id}_P$.

Changing category, we have a surjective $*$-homomorphism $\rho\colon Q \twoheadrightarrow P$ and a $*$-homomorphism $\lambda\colon P \to Q$ such that $\rho \circ \lambda = \mathrm{id}_A$. By Lemma 5.3.3, Q is projective and therefore P is projective. □

APPENDIX A

A characterization of semiprojectivity for commutative C^*-algebras

APPENDIX B

On a counterexample to a conjecture by Blackadar

APPENDIX C

Semiprojectivity with and without a group action

APPENDIX D

Amplified graph C^*-algebras

APPENDIX E

Geometric classification of simple graph algebras

APPENDIX F

Almost commuting self-adjoint matrices — the real and self-dual cases

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