Faculty of Science
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Ph.D. Thesis

Volatility-of-Volatility Perspectives: Variance Derivatives and Other Equity Exotics

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Contents

Preface

The present thesis has been prepared as part of the requirements of the Ph.D. degree at the Department of Mathematical Sciences, University of Copenhagen. The work has been carried out from August 2008 to July 2011, under the supervision of Prof. Rolf Poulsen.

Special thanks and sincere gratitude go first and foremost to my supervisor Prof. Rolf Poulsen for providing maximum support and ideal research conditions to make the work on this project enjoyable and productive. Certain parts or sections of this project have also benefited from the author's discussions or feedback kindly offered to the author by several people including Prof. Walter Farkas, Dr. Hans Buehler, Dr. Artur Sepp, Prof. Mark Broadie, Prof. Peter Carr and Dr. Lorenzo Bergomi.

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Summary

The principal theme of the thesis is the valuation of derivative securities sensitive to the volatility-of-volatility; important examples include forward-starting options, variance derivatives and cliquets. The thesis comprises five research papers and one short introductory note. All chapters closely follow the research manuscripts, which have been prepared in a format suitable for publication in international peer-reviewed journals on quantitative finance and financial mathematics. The chapters, along with a short description, are presented below.

0. **Volatility-of-Volatility: A simple model-free motivation**

In this introductory note, we aim to provide a simple, intuitive and model-free motivation for the importance of volatility-of-volatility in pricing certain kinds of exotic and structured products.

I. **Closed form convexity and cross-convexity adjustments for Heston prices**

We present a new and general technique for obtaining closed form expansions for prices of options in the Heston model, in terms of Black-Scholes prices and Black-Scholes greeks up to arbitrary orders. We then apply the technique to solve, in detail, the cases for the second order and third order expansions. In particular, such expansions show how the convexity in volatility, measured by the Black-Scholes volga, and the sensitivity of delta with respect to volatility, measured by the Black-Scholes vanna, impact option prices in the Heston model. The general method for obtaining the expansion rests on the construction of a set of new probability measures, equivalent to the original pricing measure, and which retain the affine structure of the Heston volatility diffusion. Finally, we extend our method to the pricing of forward-starting options in the Heston model.

This chapter is based on the research manuscript Drimus (2011a), accepted for publication in *Quantitative Finance*.

II. **Options on realized variance by transform methods: A non-affine stochastic volatility model**

We study the pricing and hedging of options on realized variance in the 3/2 non-affine stochastic volatility model, by developing efficient transform based pricing methods. This non-affine model gives prices of options on realized variance which

allow upward sloping implied volatility of variance smiles. Heston's model, the benchmark affine stochastic volatility model, leads to downward sloping volatility of variance smiles — in disagreement with variance markets in practice. Using control variates, we show a robust method to express the Laplace transform of the variance call function in terms of the Laplace transform of realized variance. The proposed method works in any model where the Laplace transform of realized variance is available in closed form. Additionally, we apply a new numerical Laplace inversion algorithm which gives fast and accurate prices for options on realized variance, simultaneously at a sequence of variance strikes. The method is also used to derive hedge ratios for options on variance with respect to variance swaps.

This chapter is based on the research manuscript Drimus (2011b), accepted for publication in *Quantitative Finance*.

III. **Options on realized variance in Log-OU models**

We consider the pricing of options on realized variance in a general class of Log-OU stochastic volatility models. The class includes several important models proposed in the literature. Having as common feature the log-normal law of instantaneous variance, the application of standard Fourier-Laplace transform methods is not feasible. By extending Asian pricing methods, we obtain bounds, in particular, a very tight lower bound for options on realized variance.

This chapter is based on the research manuscript Drimus (2010b), submitted and currently under review.

IV. **Options on discretely sampled variance: Discretization effect and Greeks**

The valuation of options on discretely sampled variance requires proper adjustment for the extra volatility-of-variance induced by discrete sampling. Under general stochastic volatility dynamics, we provide a detailed theoretical characterization of the discretization effect. In addition, we analyze several numerical methods which reduce the dimensionality of the required pricing scheme, while accounting for most of the discretization effect. The most important of these, named the conditional Black-Scholes scheme, leads to an explicit discretization adjustment term, easily computable by standard Fourier transform methods in any stochastic volatility model which admits a closed-form expression for the characteristic function of continuously sampled variance. In the second part of the chapter, we provide a practical analysis of the most important risk sensitivities ('*greeks*') of options on discretely sampled variance.

This chapter is based on the research manuscript Drimus, Farkas (2010), submitted and currently under review.

V. **A forward started jump-diffusion model and pricing of cliquet style exotics**

We present an alternative model for pricing exotic options and structured products with forward-starting components. The pricing of such exotic products (which consist primarily of different variations of locally / globally, capped / floored, arithmetic / geometric etc. cliquets) depends critically on the modeling of the *forward-return* distributions. Therefore, in our approach, we directly take up the modeling of forward variances corresponding to the tenor structure of the product to be priced. We propose a two factor forward variance market model with jumps in returns and volatility. It allows the model user to directly control the behavior of future smiles and hence properly price forward smile risk of cliquet-style exotic products. The key idea, in order to achieve consistency between the dynamics of forward variance swaps and the underlying stock, is to adopt a forward starting model for the stock dynamics over each reset period of the tenor structure. We also present in detail the calibration steps for our proposed model.

This chapter is based on the research manuscript Drimus (2010a), accepted for publication in *Review of Derivatives Research*.

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Volatility-of-Volatility: A simple model-free motivation

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Abstract

In this introductory note, we aim to provide a simple, intuitive and model-free motivation for the importance of volatility-of-volatility in pricing certain kinds of exotic and structured products.

KEYWORDS: volatility of volatility, variance derivatives, exotic options, structured products.

0.1 Introduction

It is intuitively clear that for exotic products that are strongly dependent on the *dynamics* of the volatility surface proper modeling of the volatility-of-volatility is critical. Several authors, including Schoutens et al. (2004), Gatheral (2006) and Bergomi (2005, 2008), have shown that the same exotic product can have significantly different valuations under different stochastic volatility models.

In this short note, we want to illustrate the importance of the volatility-of-volatility without referring to any of the standard models from the literature. We compare the pricing of a couple of fundamental payoffs *with* and *without* volatility-of-volatility.

0.2 A model free motivation

Let us begin by recalling the important payoff spanning formula, first observed in Breeden, Litzenberger (1978). A payoff function $H \in C^2(0, \infty)$ satisfies, for any

$x_0 > 0$:

$$H(x) = H(x_0) + \frac{\partial H}{\partial x}(x_0) \cdot (x - x_0) \quad + \quad \int_0^{x_0} \frac{\partial^2 H}{\partial x^2}(K) \cdot (K - x)_+ \, dK$$
$$+ \quad \int_{x_0}^{\infty} \frac{\partial^2 H}{\partial x^2}(K) \cdot (x - K)_+ \, dK \qquad (1)$$

This can be generalized to less smooth payoff functions H in several ways. For example, if $H \in C^2(0,\infty)\backslash\{x_0\}$, continuous at x_0 with left and right first derivatives $\frac{\partial H^-}{\partial x}(x_0)$, $\frac{\partial H^+}{\partial x}(x_0)$, the spanning formula becomes

$$\begin{aligned} H(x) &= H(x_0) - \frac{\partial H^-}{\partial x}(x_0) \cdot (x_0 - x)_+ + \frac{\partial H^+}{\partial x}(x_0) \cdot (x - x_0)_+ \\ &+ \int_0^{x_0} \frac{\partial^2 H}{\partial x^2}(K) \cdot (K - x)_+ \, dK + \int_{x_0}^{\infty} \frac{\partial^2 H}{\partial x^2}(K) \cdot (x - K)_+ \, dK \end{aligned} \qquad (2)$$

More generally, the spanning formula can be extended to convex H using generalized derivatives. For our purposes, in this section, statements (1) and (2) will suffice.

In what follows, we fix two future dates $0 < T_1 < T_2$. Suppose we want to value a contract whose payoff at time T_2 is

$$\frac{1}{T_2 - T_1} \cdot \log^2\left(\frac{S_{T_2}}{S_{T_1}}\right)$$

where we have denoted by S the price of some underlying asset. We first consider the value of this contract at the *future* time T_1. From the standpoint of time T_1, this payoff can be spanned into a portfolio of vanilla options. Specifically, if we take $H(x) = \frac{1}{T_2 - T_1} \cdot \log^2\left(\frac{x}{S_{T_1}}\right)$ and use

$$\begin{aligned} \frac{\partial H}{\partial x}(x) &= \frac{2}{x \cdot (T_2 - T_1)} \log\left(\frac{x}{S_{T_1}}\right) \\ \frac{\partial^2 H}{\partial x^2}(x) &= \frac{2}{x^2 \cdot (T_2 - T_1)} \left(1 - \log\left(\frac{x}{S_{T_1}}\right)\right) \end{aligned}$$

an application of the spanning formula (1) gives

$$\begin{aligned} \frac{1}{T_2 - T_1} \log^2\left(\frac{S_{T_2}}{S_{T_1}}\right) &= \int_0^{S_{T_1}} \frac{2}{K^2 \cdot (T_2 - T_1)} \left(1 - \log\left(\frac{K}{S_{T_1}}\right)\right) \cdot (K - x)_+ \, dK \\ &+ \int_{S_{T_1}}^{\infty} \frac{2}{K^2 \cdot (T_2 - T_1)} \left(1 - \log\left(\frac{K}{S_{T_1}}\right)\right) \cdot (x - K)_+ \, dK \end{aligned}$$

Assuming European Put and Call options, of all strikes $K > 0$, are tradeable in the market, we obtain that the value of the contract at time T_1 is given by

$$\begin{aligned} V_{T_1}^H &= \int_0^{S_{T_1}} \frac{2}{K^2 \cdot (T_2 - T_1)} \left(1 - \log\left(\frac{K}{S_{T_1}}\right)\right) \cdot P(S_{T_1}, K, T_2 - T_1) dK \\ &+ \int_{S_{T_1}}^{\infty} \frac{2}{K^2 \cdot (T_2 - T_1)} \left(1 - \log\left(\frac{K}{S_{T_1}}\right)\right) \cdot C(S_{T_1}, K, T_2 - T_1) dK \end{aligned}$$

where we assume the market option prices $P(S_{T_1}, K, T_2 - T_1)$ and $C(S_{T_1}, K, T_2 - T_1)$ are such that the two integrals converge. Making the change of variable $K = S_{T_1} \cdot x$ and using the Black-Scholes pricing function, we can write

$$\begin{aligned} P(S_{T_1}, K, T_2 - T_1) &= S_{T_1} \cdot P^{BS}\left(1, x; \hat{\sigma}(x), T_2 - T_1\right) \\ C(S_{T_1}, K, T_2 - T_1) &= S_{T_1} \cdot C^{BS}\left(1, x; \hat{\sigma}(x), T_2 - T_1\right) \end{aligned}$$

where we denoted by $\hat{\sigma}(x)$ the Black-Scholes implied volatility for moneyness $x = \frac{K}{S_{T_1}}$. We finally obtain the value, at time T_1, as

$$\begin{aligned} V_{T_1}^H &= \int_0^1 \frac{2}{x^2 \cdot (T_2 - T_1)} \left(1 - \log(x)\right) \cdot P^{BS}\left(1, x; \hat{\sigma}(x), T_2 - T_1\right) dx \\ &+ \int_1^{\infty} \frac{2}{x^2 \cdot (T_2 - T_1)} \left(1 - \log(x)\right) \cdot C^{BS}\left(1, x; \hat{\sigma}(x), T_2 - T_1\right) dx \qquad (3) \end{aligned}$$

Note that, for our contract, its future value at time T_1 depends *only* on the volatility-by-moneyness curve (i.e. the smile) $\hat{\sigma}(x)$ (of maturity $\Delta T = T_2 - T_1$) that will prevail in the market at time T_1. Of course, at present, we do not know what ΔT-smile will prevail in the market at time T_1. Therefore, the valuation of this product will depend entirely on the future smile scenarios assumed possible for time T_1.

Today's ΔT-smile, which is observable in the market, will be denoted by $\hat{\sigma}_0(x)$. If we make the assumption that the future ΔT-smile, which prevails in the market at time T_1, will be identical to today's smile (that is the case, for example, in any pure Levy model), we obtain the present value of the contract as

$$e^{-rT_1} \cdot V_{T_1}^H\left(\hat{\sigma}_0(x)\right) \qquad (4)$$

where we have used today's ΔT-smile $\hat{\sigma}_0(x)$ in formula (3).

Assume now that we recognize the uncertainty in the future smile and consider three possible scenarios: the smile moves up to $\hat{\sigma}_u(x)$, stays the same at $\hat{\sigma}_0(x)$ or moves down to $\hat{\sigma}_d(x)$ – with probabilities p_u, p_0 and p_d respectively. The value of the contract is now computed as

$$e^{-rT_1} \cdot \left[p_u \cdot V_{T_1}^H\left(\hat{\sigma}_u(x)\right) + p_0 \cdot V_{T_1}^H\left(\hat{\sigma}_0(x)\right) + p_d \cdot V_{T_1}^H\left(\hat{\sigma}_d(x)\right)\right]. \qquad (5)$$

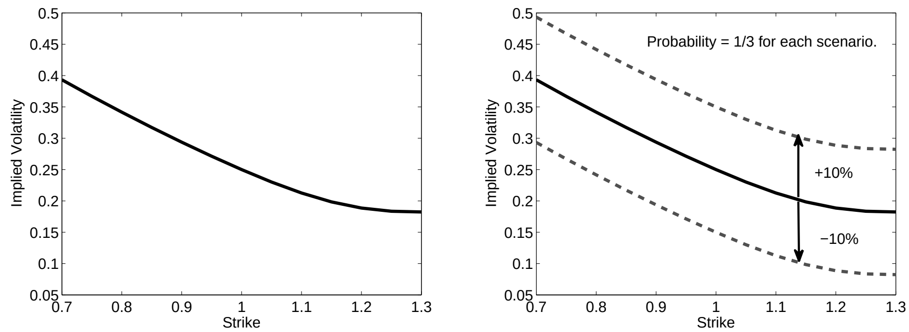

Figure 1: Comparison of two 3m-smile behaviors: (Left) the future 3m-smile assumed identical to today's 3m smile, (Right) the future 3m-smile assumed to take on 3 possible realizations (shifted up by 10 volatility points, remains the same and shifted down by 10 volatility points) with equal probabilities 1/3.

An interesting question is how the valuation without volatility-of-volatility in (4) compares to the valuation with volatility-of-volatility in (5).We next consider a simple numerical example. The left panel of Figure (1) shows the three-months, $\Delta T = 0.25$, S&P500 smile from July 31 2009; assume this is today's observed smile, denoted above by $\hat{\sigma}_0(x)$. With volatility-of-volatility, we assume three possible smile shifts: up by 10 volatility points ($\hat{\sigma}_u(x) = \hat{\sigma}_0(x) + 0.1$), constant and down 10 volatility points ($\hat{\sigma}_d(x) = \hat{\sigma}_0(x) - 0.1$) each with equal probability $\frac{1}{3}$. Remaining parameters are taken $T_1 = 0.25$, $T_2 = T_1 + \Delta T = 0.5$, interest rate $r = 0.4\%$ and dividend yield $\delta = 1.9\%$. We obtain the (undiscounted) contract value, without vol-of-vol, at 0.0863 and the value, with vol-of-vol, at $\frac{1}{3} \cdot (0.1727 + 0.0863 + 0.0313) = 0.0968$, for a relative difference of approximately 12.17%. We emphasize that, in both cases, the *expected smile is the same*; note that $\frac{1}{3} \cdot (\hat{\sigma}_u(x) + \hat{\sigma}_0(x) + \hat{\sigma}_d(x)) = \hat{\sigma}_0(x)$. Therefore, the significant valuation difference stems entirely from the volatility-of-volatility. We conclude that, a model which does not properly reflect the stochasticity of the future smile can severely misprice this product.

Let us now consider the valuation of a slightly more complicated contract, whose payoff at time T_2 is given by

$$\left(\frac{1}{T_2 - T_1} \cdot \log^2 \left(\frac{S_{T_2}}{S_{T_1}} \right) - \sigma_K^2 \right)_+$$

and which resembles (albeit remotely) an option on realized variance with volatility strike $\sigma_K > 0$. As before, we begin by determining the value of the contract at time T_1. This payoff can be decomposed as

$$\left(\frac{1}{T_2 - T_1} \cdot \log^2 \left(\frac{S_{T_2}}{S_{T_1}} \right) - \sigma_K^2 \right) \cdot \left(1_{S_{T_2} < S_{T_1} e^{-\sigma_K \sqrt{T_2 - T_1}}} + 1_{S_{T_2} > S_{T_1} e^{\sigma_K \sqrt{T_2 - T_1}}} \right)$$

and we let

$$\begin{aligned}
H_L(x) &= \left(\frac{1}{T_2-T_1}\cdot\log^2\left(\frac{x}{S_{T_1}}\right)-\sigma_K^2\right)\cdot 1_{x<S_{T_1}e^{-\sigma_K\sqrt{T_2-T_1}}}\\
H_R(x) &= \left(\frac{1}{T_2-T_1}\cdot\log^2\left(\frac{x}{S_{T_1}}\right)-\sigma_K^2\right)\cdot 1_{x>S_{T_1}e^{\sigma_K\sqrt{T_2-T_1}}}.
\end{aligned}$$

The function $H_L(x)$ is twice differentiable on $(0,\infty)\backslash\{S_{T_1}e^{-\sigma_K\sqrt{T_2-T_1}}\}$ with left and right derivatives at $S_{T_1}e^{-\sigma_K\sqrt{T_2-T_1}}$ given by

$$\begin{aligned}
\frac{\partial H_L^-}{\partial x}\left(S_{T_1}e^{-\sigma_K\sqrt{T_2-T_1}}\right) &= \frac{-2\sigma_K}{S_{T_1}\sqrt{T_2-T_1}e^{-\sigma_K\sqrt{T_2-T_1}}}\\
\frac{\partial H_L^+}{\partial x}\left(S_{T_1}e^{-\sigma_K\sqrt{T_2-T_1}}\right) &= 0.
\end{aligned}$$

Therefore, by applying to $H_L(x)$ the statement (2) of the spanning formula, we obtain

$$\begin{aligned}
H_L(x) &= \frac{2\sigma_K}{S_{T_1}\sqrt{T_2-T_1}e^{-\sigma_K\sqrt{T_2-T_1}}}\cdot\left(S_{T_1}e^{-\sigma_K\sqrt{T_2-T_1}}-x\right)_+\\
&\quad+\int_0^{S_{T_1}e^{-\sigma_K\sqrt{T_2-T_1}}}\frac{2}{K^2(T_2-T_1)}\left(1-\log\left(\frac{K}{S_{T_1}}\right)\right)\cdot(K-x)_+\,dK.
\end{aligned}$$

After proceeding analogously with the function $H_R(x)$, we finally obtain that the value of the contract at the future time T_1 will be given by

$$\begin{aligned}
V_{T_1}^H &= \frac{2\sigma_K}{S_{T_1}\sqrt{T_2-T_1}e^{-\sigma_K\sqrt{T_2-T_1}}}\cdot P(S_{T_1},S_{T_1}e^{-\sigma_K\sqrt{T_2-T_1}},T_2-T_1)\\
&\quad+\frac{2\sigma_K}{S_{T_1}\sqrt{T_2-T_1}e^{\sigma_K\sqrt{T_2-T_1}}}\cdot C(S_{T_1},S_{T_1}e^{\sigma_K\sqrt{T_2-T_1}},T_2-T_1)\\
&\quad+\int_0^{S_{T_1}e^{-\sigma_K\sqrt{T_2-T_1}}}\frac{2}{K^2(T_2-T_1)}\left(1-\log\left(\frac{K}{S_{T_1}}\right)\right)\cdot P(S_{T_1},K,T_2-T_1)dK\\
&\quad+\int_{S_{T_1}e^{\sigma_K\sqrt{T_2-T_1}}}^{\infty}\frac{2}{K^2(T_2-T_1)}\left(1-\log\left(\frac{K}{S_{T_1}}\right)\right)\cdot C(S_{T_1},K,T_2-T_1)dK.
\end{aligned}$$

As before, making the change of variable $K=x\cdot S_{T_1}$ and using the Black-Scholes implied volatility-by-moneyness smile $\hat{\sigma}(x)$ prevailing in the market at time T_1, we obtain

$$
\begin{aligned}
V_{T_1}^H &= \frac{2\sigma_K}{\sqrt{T_2-T_1}e^{-\sigma_K\sqrt{T_2-T_1}}}\cdot P^{BS}\left(1, e^{-\sigma_K\sqrt{T_2-T_1}}; \hat{\sigma}\left(e^{-\sigma_K\sqrt{T_2-T_1}}\right), T_2-T_1\right) \\
&\quad + \frac{2\sigma_K}{\sqrt{T_2-T_1}e^{\sigma_K\sqrt{T_2-T_1}}}\cdot C^{BS}\left(1, e^{\sigma_K\sqrt{T_2-T_1}}; \hat{\sigma}\left(e^{\sigma_K\sqrt{T_2-T_1}}\right), T_2-T_1\right) \\
&\quad + \int_0^{e^{-\sigma_K\sqrt{T_2-T_1}}} \frac{2}{x^2(T_2-T_1)}\left(1-\log(x)\right)\cdot P^{BS}(1, x; \hat{\sigma}(x), T_2-T_1)dx \\
&\quad + \int_{e^{\sigma_K\sqrt{T_2-T_1}}}^{\infty} \frac{2}{x^2(T_2-T_1)}\left(1-\log(x)\right)\cdot C^{BS}(1, x; \hat{\sigma}(x), T_2-T_1)dx.
\end{aligned}
$$

Again, we notice that the value of the contract at time T_1 depends *only* on the ΔT-smile which will prevail in the market at time T_1; in particular, note that the value does not depend on the future stock price S_{T_1}. Similar to our earlier comparison, we consider the two smile behaviors depicted in Figure (1): (a) the ΔT-smile remains identical to today's smile and (b) the smile can shift up/down by 10 volatility points around today's smile. The two valuations are then given by formulas (4) and (5) with $V_{T_1}^H$ as above. Using $\sigma_K^2 = 0.0968$ (the value of the previous contract), we obtain the (undiscounted) price, without vol-of-vol, at 0.044 and, with vol-of-vol, at $\frac{1}{3}(0.1161 + 0.044 + 0.0091) = 0.0564$ — for a relative difference of approximately 28.18%! As before, the expected smile is the same in both cases and, therefore, the pricing difference comes entirely from the volatility-of-volatility.

Both contracts considered so far had a substantially higher value *with* vol-of-vol than *without* vol-of-vol. This is explained by their positive convexity in volatility. Specifically, in our setting, the value $V_{T_1}^H(\hat{\sigma}(x))$ was convex in the level of the smile $\hat{\sigma}(x)$ and thus the average computed in equation (5) across the three possible smiles is larger than the value computed with the expected smile in equation (4). The importance of vol-of-vol is greater, the more volatility convexity a product has. In practice, this sensitivity is usually called Volga which, in turn, is just short-hand for Volatility Gamma.

As expected, different products can have vastly different Volgas. As another example, let us consider a contract whose payoff at time T_2 is

$$\left(\frac{S_{T_2}}{S_{T_1}} - 1\right)_+$$

i.e. a forward-started at-the-money call. It it straightforward to see that the value, at time T_1, of this contract is $C^{BS}(1, 1, \hat{\sigma}(1), T_2 - T_1)$, where $\hat{\sigma}(1)$ is the at-the-money implied Black-Scholes volatility of maturity ΔT prevailing in the market at time T_1. Proceeding as before, we compare the value without vol-of-vol

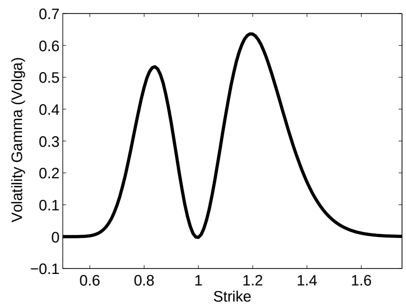

Figure 2: Volatility Gamma (Volga) of European vanilla options as a function of strike, for a Black-Scholes volatility of 25% and maturity 3-months.

$C^{BS}(1, 1, \hat{\sigma}_0(1), 0.25) = 4.782\%$ and the value with vol-of-vol

$$\begin{aligned} & \frac{1}{3} \cdot \left(C^{BS}(1, 1, \hat{\sigma}_u(1), 0.25) + C^{BS}(1, 1, \hat{\sigma}_0(1), 0.25) + C^{BS}(1, 1, \hat{\sigma}_d(1), 0.25)\right) \\ = \; & \frac{1}{3} \cdot (6.765\% + 4.782\% + 2.797\%) = 4.781\% \end{aligned}$$

and observe that the two valuations are essentially identical. This is explained by the fact that at-the-money options are almost *linear* in volatility i.e. have a Volga close to zero[1]. Figure (2) shows the Volga of European vanilla options across strikes. Indeed, we notice that ATM options have little Volga and that Volga peaks in a region OTM before dying off for far-OTM options. If we consider an OTM forward-started call with payoff

$$\left(\frac{S_{T_2}}{S_{T_1}} - 1.25\right)_+$$

by repeating the calculations above, we obtain a price without vol-of-vol of about 2.23 bps whereas the price with vol-of-vol is about 12.95 bps. Unlike the ATM case, vol-of-vol now has a substantial impact on valuation.

0.3 Conclusion

All the elementary payoffs that we have been considering in this short account appear, either explicitly or implicitly, in many types of exotic and structured products. Among these, we mention variance derivatives and the different variations of locally/globally, floored/capped, arithmetic/geometric cliquets. As noted in Eberlein, Madan (2009), the market for such products has been on an exponential

[1]We remark that it can, in fact, be slightly negative depending on the sign of $(r - \delta)^2 - \frac{\sigma^4}{4}$.

growth trend. Therefore, for dealers pricing these products proper modeling of the volatility-of-volatility is of major importance. Bergomi (2005, 2008) proposes a forward-started modeling approach which allows direct control of the future smiles; a version which includes jumps is also given in Drimus (2010). In addition to pricing, the monitoring and risk-management of the Volatility Gamma (or Volga) becomes critical for an exotics book, as it drives the Profit & Loss of the daily rebalancing of the Vega. A further discussion of the Volga and Vanna[2], in a stochastic volatility model, can be found in Drimus (2011).

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Closed form convexity and cross-convexity adjustments for Heston prices

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Abstract

We present a new and general technique for obtaining closed form expansions for prices of options in the Heston model, in terms of Black-Scholes prices and Black-Scholes greeks up to arbitrary orders. We then apply the technique to solve, in detail, the cases for the second order and third order expansions. In particular, such expansions show how the convexity in volatility, measured by the Black-Scholes volga, and the sensitivity of delta with respect to volatility, measured by the Black-Scholes vanna, impact option prices in the Heston model. The general method for obtaining the expansion rests on the construction of a set of new probability measures, equivalent to the original pricing measure, and which retain the affine structure of the Heston volatility diffusion. Finally, we extend our method to the pricing of forward-starting options in the Heston model.

KEYWORDS: stochastic volatility, Heston model, price approximation, forward starting options.

1.1 Introduction

In the area of pricing and hedging equity derivatives, an increasing body of literature has been focused on the problem of stochastic volatility. After the seminal work of Black, Scholes (1973), a number of alternative models have been proposed and designed to directly model the stochastic nature of volatility. With the exception of the special class of local volatility models, among which we mention Cox (1975) and Dupire (1994), the proposed models introduce a second stochastic factor to describe volatility movements. The idea of modeling volatility as a separate stochastic process leads, in turn, to several possible choices for its dynamics. Among the first contributions in this line of research we mention Scott (1987) and Chesney, Scott (1989), where the logarithm of the instantaneous volatility is assumed to follow a mean-reverting Ornstein-Uhlenbeck process, and Hull, White (1987a), where a geometric Brownian motion is used to model the instantaneous variance. However, these early models proved to be numerically cumbersome as they do not offer the possibility of fast valuation algorithms when large numbers of options have to be priced. As an alternative, came one of the most popular stochastic volatility models, proposed in Heston (1993), which employed the square root diffusion to model the evolution of instantaneous variance – dynamics which were first used by Cox, Ross (1985) in the area of interest rate modeling. Heston (1993) also introduced the technique of inversion of characteristic functions in order to compute option prices. This approach was subsequently refined and extended by Carr, Madan (1999) where the method of fast Fourier transforms, as introduced by Cooley, Tukey (1965), is employed and showed to provide superior results in terms of both accuracy and speed.

In this paper, we work under the stochastic volatility dynamics proposed in Heston (1993). The existence of fast numerical methods for pricing options in the Heston model, makes it a viable modeling tool in practice. However, these numerical recipes fail to reveal the inner logic and structure of the model by acting as 'black boxes', able to provide fast prices for the supplied sets of inputs. Specifically, such methods do not make explicit the connection between the prices obtained in a stochastic volatility model and the corresponding prices in the classical Black Scholes model. We believe that, for a financial engineer, it is of critical importance to have a deeper and more concrete understanding of the main features which make a stochastic volatility price different from the benchmark Black Scholes price. In this paper, we present a simple and general technique which allows one to expand the price of options in the Heston model in terms of Black-Scholes prices and higher order Black-Scholes greeks. In particular, this gives an explicit and exact means of quantifying the contribution of important features arising in stochastic volatility modeling, such as the convexity of option prices with respect to volatility (or, equivalently, the Volatility Gamma, also known in practice as the Volga) or the dependence of the delta hedge ratio on the level of volatility (measured by the Vanna). The method can also be applied beyond these second order effects, to compute the contribution of Black-Scholes greeks up to any order. We note that, under geometric Brownian motion dynamics for the instantaneous variance, Hull, White (1987a) provide a third order approximation to option prices under the assumption of zero correlation between stock and volatility movements. Our method does not impose such a restriction on correlation which, as revealed by other studies, for example, Bakshi et al. (1997), tends to be strongly negative.

Our approach rests on the construction of a set of new probability measures, equivalent to the original pricing measure, and which retain the affine structure of the square root diffusion. This enables us to make use, under the new probability measures, of the same results which have been derived in the literature for affine square root diffusions, such as the closed-form Laplace transform and the moments of integrated variance; see, for example, Cox et al. (1985) and Dufresne(2001). The method is applied to the pricing of European call and put options in the Heston model and shown that it can be extended to the case of forward starting options. Our paper is organized as follows. In the next section, we present the general results which apply to price expansions up to any order. In particular, we show how to construct the new probability measures which will be used in the rest of the paper and how the parameters of the square root diffusion change under the newly defined probability measures. In the subsequent two sections, we apply the general results to solve, in detail, the important cases of second order and third order expansions. In section five, we illustrate how our approach can be directly extended to forward starting options. The last section draws the main conclusions.

1.2 Heston expansions : The general case

In this paper we work under the stochastic volatility framework first proposed in Heston (1993). Let $(W_t, Z_t)_{0\leq t\leq T}$ be a standard two-dimensional Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})$ satisfying the usual conditions. We assume that the stock price and its instantaneous variance $(S_t, v_t)_{0\leq t\leq T}$ satisfy the following dynamics under the risk neutral measure $\mathbb{Q}$:

$$\begin{aligned} \frac{dS_t}{S_t} &= (r-\delta)dt + \sqrt{v_t}\left(\rho dW_t + \sqrt{1-\rho^2}dZ_t\right) \\ dv_t &= k(\theta - v_t)dt + \epsilon\sqrt{v_t}dW_t \end{aligned}$$

where the risk free interest rate and the dividend yield are assumed constant and denoted by r and δ, respectively. The instantaneous stochastic variance $(v_t)_{0\leq t\leq T}$ follows a square root diffusion, or Cox-Ingersoll-Ross (CIR) process, with speed of mean reversion k, long term mean variance θ and volatility of volatility ϵ. In the subsequent development of our method, we shall assume that the correlation parameter ρ is non-positive, i.e. $\rho \leq 0$. We note that this assumption on the correlation parameter is consistent with earlier studies, e.g. Bakshi et al. (1997), which show that the stock and the volatility are negatively correlated. In what follows, a central role will be played by the total integrated variance, denoted by V_T and defined as:

$$V_T = \int_0^T v_t dt.$$

Under CIR dynamics for the instantaneous variance, the Laplace transform of V_T is known in closed form; see, for example, Cox et al. (1985) or Dufresne (2001). In deriving the expansion terms, we will make repeated use of the transform function and we therefore recall its form below:

$$\mathcal{L}(s; k, \theta, \epsilon) = E\left(e^{-sV_T}\right) = A(s; k, \theta, \epsilon)\, e^{-v_0 B(s;k,\theta,\epsilon)}$$

where

$$\begin{aligned}
A(s; k, \theta, \epsilon) &= \left(\frac{e^{kT/2}}{\cosh(P(s)T/2) + \frac{k}{P(s)}\sinh(P(s)T/2)}\right)^{\frac{2k\theta}{\epsilon^2}} \\
B(s; k, \theta, \epsilon) &= \frac{s}{P(s)}\frac{2\sinh(P(s)T/2)}{\cosh(P(s)T/2) + \frac{k}{P(s)}\sinh(P(s)T/2)} \\
P(s) &= \sqrt{k^2 + 2\epsilon^2 s}.
\end{aligned}$$

In approximating the Heston prices with Black-Scholes prices and greeks, we will find it convenient to work with the Black-Scholes pricing function written in terms of the total variance, as opposed to the traditional form which uses the annualized volatility parameter. Specifically, we will use the following form for the Black-Scholes pricing function:

$$C^{BS}(S_0, V; r, \delta, K, T) = S_0 e^{-\delta T} N(d_1) - K e^{-rT} N(d_2)$$

where

$$\begin{aligned}
d_1 &= \frac{\log\frac{S_0}{K} + (r-\delta)T + \frac{V}{2}}{\sqrt{V}} \\
d_2 &= d_1 - \sqrt{V}.
\end{aligned}$$

Using this definition of the call pricing function, for a Black-Scholes volatility parameter $\sigma > 0$, we obtain the Black-Scholes call option price by computing $C^{BS}(S_0, \sigma^2 T)$. We begin by applying Itô's lemma to $\log(S_t)$, over the interval $[0, T]$, to obtain for S_T:

$$\begin{aligned}
S_T &= S_0 \exp\Big[(r-\delta)T - \frac{1-\rho^2}{2}\int_0^T v_t dt - \frac{\rho^2}{2}\int_0^T v_t dt + \\
&\quad + \sqrt{1-\rho^2}\int_0^T \sqrt{v_t} dZ_t + \rho\int_0^T \sqrt{v_t} dW_t\Big]
\end{aligned}$$

or

$$S_T = S_0 \cdot \xi_T \cdot \exp\Big[(r-\delta)\, T - \frac{1-\rho^2}{2} \int_0^T v_t dt + \sqrt{1-\rho^2} \int_0^T \sqrt{v_t} dZ_t \Big] \tag{1.1}$$

where we define the process $(\xi_t)_{0\leq t\leq T}$ as the stochastic exponential of $\int_0^t \rho\sqrt{v_u}dW_u$:

$$\xi_t = \mathcal{E}\left(\int_0^t \rho\sqrt{v_u}dW_u \right) = \exp\left(-\frac{\rho^2}{2} \int_0^t v_u du + \rho \int_0^t \sqrt{v_u} dW_u \right).$$

Our goal is to obtain an expansion for the Heston call price, of strike K and maturity T, which, using the pricing measure $\mathbb{Q}$, is given by:

$$C_{\text{HES}}\left(S_0, K, T, v_0, k, \theta, \epsilon\right) = e^{-rT} E^{\mathbb{Q}} \left(S_T - K\right)_+$$

Denoting by $(\mathcal{F}_t^W)_{0\leq t\leq T}$ the filtration generated by the standard Brownian motion $(W_t)_{0\leq t\leq T}$ driving the CIR diffusion $(v_t)_{0\leq t\leq T}$ and making use of (1.1), we can write the Heston call price as an expectation over Black-Scholes prices in the following way:

$$\begin{aligned} C_{\text{HES}}\left(S_0, K, T, v_0, k, \theta, \epsilon\right) &= e^{-rT} E^{\mathbb{Q}} \left(E^{\mathbb{Q}} \left(\left(S_T - K\right)_+ \middle| \mathcal{F}_T^W \right)\right) \\ &= E^{\mathbb{Q}} \left(C^{BS} \left(S_0 \cdot \xi_T, \left(1-\rho^2\right) \int_0^T v_t dt \right)\right) \end{aligned}$$

or, using our notation for the total integrated variance :

$$C_{\text{HES}}\left(S_0, K, T, v_0, k, \theta, \epsilon\right) = E^{\mathbb{Q}} \left(C^{BS} \left(S_0 \cdot \xi_T, \left(1-\rho^2\right) V_T \right)\right). \tag{1.2}$$

In other words, we will make use of the fact that, conditional on a realization of the instantaneous variance path, the Heston option price becomes a Black-Scholes option price with initial spot $S_0 \cdot \xi_T$ and total variance $(1-\rho^2)\int_0^T v_t dt$. It is important to remark that the conditioning argument leading to representation (1.2) works in exactly the same way for any European-style option whose payoff depends only on the final value S_T at maturity. The method no longer applies when path-dependent derivatives, such as barrier or Asian options, are considered. As first noted in Breeden, Litzenberger (1978), the payoff of any European-style option can be spanned into a portfolio of European vanilla payoffs. Therefore, we shall focus on the pricing of European vanilla call options which can be used as the fundamental building blocks of any European-style option. In a different context, Romano & Touzi (1997) used a similar representation in their study of market completeness in stochastic volatility models.

In our subsequent development, the key idea is to use ξ_T to define a set of new probability measures retaining the affine structure of the variance diffusion. Specifically, we note that, by Girsanov theorem

$$\xi_T = \exp\left(-\frac{\rho^2}{2}\int_0^T v_t dt + \rho\int_0^T \sqrt{v_t} dW_t\right)$$

can be used to define a new probability measure $\mathbb{Q}^1$ equivalent to $\mathbb{Q}$, with Radon-Nikodym derivative $\frac{d\mathbb{Q}^1}{d\mathbb{Q}} = \xi_T$, provided $(\xi_t)_{0\le t\le T}$ defines a true $\mathbb{Q}$-martingale. By Novikov's sufficient condition (see, for example, Revuz, Yor (1999)), we know that $(\xi_t)_{0\le t\le T}$ defines a martingale provided that :

$$E^{\mathbb{Q}}\left(\exp\left(\frac{1}{2}\rho^2\int_0^T v_t dt\right)\right) < \infty.$$

In terms of the Laplace transform of V_T this condition reads $\mathcal{L}(-\frac{1}{2}\rho^2) < \infty$. By inspecting the Laplace transform given earlier, this condition is satisfied if:

$$\rho^2 < \frac{k^2}{\epsilon^2}. \tag{1.3}$$

In practice, this sufficient condition usually holds and does not impose a severe limitation on the range of acceptable parameters. To see this, we note that the speed of mean reversion, k, tends to be higher than 1 while the volatility of volatility, ϵ, tends to be less than 1 (e.g. see the study of Bakshi et al. (1997)). In all of these cases the inequality will be automatically satisfied since the absolute value of correlation is less than one. Therefore our new probability measure $\mathbb{Q}^1$ will be well defined.

The sufficient condition (1.3) is, however, not necessary in this case. As mentioned in Cheridito, Filipovic, Kimmel (2005), under Heston (1993) dynamics, it can be shown that $(\xi_t)_{0\le t\le T}$ is a true martingale for any ρ, by applying a "local" version of the Novikov condition, as given in Karatzas, Shreve (1991). For later use, we make a note of this result below.

Lemma 1.2.1 *Under Heston (1993) $\mathbb{Q}$-dynamics for the instantaneous variance $(v_t)_{0\le t\le T}$, the local $\mathbb{Q}$-martingale process $(\xi_t)_{0\le t\le T}$ defined by*

$$\xi_t = \mathcal{E}\left(\alpha\cdot\int_0^t \sqrt{v_u} dW_u\right) = \exp\left(-\frac{\alpha^2}{2}\int_0^t v_u du + \alpha\int_0^t \sqrt{v_u} dW_u\right)$$

is a true $\mathbb{Q}$-martingale for any $\alpha \in \mathbb{R}$.

The key observation which enables our subsequent results is that, under the newly defined probability measure $\mathbb{Q}^1$, the instantaneous variance process $(v_t)_{0\leq t\leq T}$ remains CIR, with a set of new parameters easily computed from the original ones. Specifically, recall that $(v_t)_{0\leq t\leq T}$ is CIR(k,θ,ϵ) under $\mathbb{Q}$ i.e.

$$dv_t = k(\theta - v_t)dt + \epsilon\sqrt{v_t}dW_t$$

By Girsanov's theorem, under $\mathbb{Q}^1$, $W_t^1 = W_t - \int_0^t \rho\sqrt{v_u}du$ is a standard Brownian motion and hence we obtain:

$$dv_t = k(\theta - v_t)dt + \epsilon\sqrt{v_t}\left(dW_t^1 + \rho\sqrt{v_t}dt\right)$$

or

$$\begin{aligned} dv_t &= \left(k\left(\theta - v_t\right) + \epsilon\rho v_t\right)dt + \epsilon\sqrt{v_t}dW_t^1 \\ &= (k - \epsilon\rho)\left(\frac{\theta}{1-\frac{\epsilon\rho}{k}} - v_t\right)dt + \epsilon\sqrt{v_t}dW_t^1 \end{aligned}$$

Hence, we recognize that, under $\mathbb{Q}^1$, the dynamics of $(v_t)_{0\leq t\leq T}$ become CIR$(k - \epsilon\rho, \frac{\theta}{1-\frac{\epsilon\rho}{k}}, \epsilon)$. It is worth noting that this result relies intimately on the square root form of the CIR dynamics. In particular, this technique would not work for arbitrary stochastic volatility models. We will make use of the following straightforward generalization, which we state below.

Proposition 1.2.2 *Under Heston (1993) $\mathbb{Q}$-dynamics for the instantaneous variance $(v_t)_{0\leq t\leq T}$, it is possible to define a sequence of equivalent probability measures $\mathbb{Q}^n$, $n = 1, 2, 3\ldots$, by $\frac{d\mathbb{Q}^n}{d\mathbb{Q}^{n-1}} = \xi_T^{(n-1)}$ where*

$$\xi_T^{(n-1)} = \exp\left(-\frac{\rho^2}{2}\int_0^T v_t dt + \rho\int_0^T \sqrt{v_t}dW_t^{n-1}\right)$$

where $W_t^n = W_t^{n-1} - \int_0^T \rho\sqrt{v_t}dt$ is a standard Brownian motion under $\mathbb{Q}^n$, $W_t^0 = W_t$ and $\mathbb{Q}^0 = \mathbb{Q}$ (the original pricing measure). Moreover, $(v_t)_{0\leq t\leq T}$ is CIR$(k - n\epsilon\rho, \frac{\theta}{1-\frac{n\epsilon\rho}{k}}, \epsilon)$ *under $\mathbb{Q}^n$.*

Proof The statement follows very easily by mathematical induction. It has already been shown that, for $n = 1$, the probability measure $\mathbb{Q}^1$ is well defined, by Lemma (1.2.1), and that $(v_t)_{0\leq t\leq T}$ is CIR$(k - \epsilon\rho, \frac{\theta}{1-\frac{\epsilon\rho}{k}}, \epsilon)$ under $\mathbb{Q}^1$. Suppose the statement holds for some $n \geq 1$. Firstly, Lemma (1.2.1) ensures that the probability measure $\mathbb{Q}^{n+1}$, defined by $\frac{d\mathbb{Q}^{n+1}}{d\mathbb{Q}^n} = \xi_T^{(n)}$, is well defined. Next, we check the form of $(v_t)_{0\leq t\leq T}$ under $\mathbb{Q}^{n+1}$. We have

$$dv_t = (k - n\epsilon\rho)\left(\frac{\theta}{1-\frac{n\epsilon\rho}{k}} - v_t\right)dt + \epsilon\sqrt{v_t}dW_t^n$$

under $\mathbb{Q}^n$. By Girsanov's theorem, $W_t^{n+1} = W_t^n - \int_0^t \rho\sqrt{v_u}du$ and hence we can write

$$dv_t = (k - n\epsilon\rho)\left(\frac{\theta}{1-\frac{n\epsilon\rho}{k}} - v_t\right)dt + \epsilon\sqrt{v_t}\left(dW_t^{n+1} + \rho\sqrt{v_t}dt\right)$$

Upon collecting terms we obtain

$$dv_t = (k - (n+1)\epsilon\rho)\left(\frac{\theta}{1-\frac{(n+1)\epsilon\rho}{k}} - v_t\right)dt + \epsilon\sqrt{v_t}dW_t^{n+1}$$

from where we conclude that, indeed, $(v_t)_{0\leq t\leq T}$ is CIR$(k-(n+1)\epsilon\rho, \frac{\theta}{1-\frac{(n+1)\epsilon\rho}{k}}, \epsilon)$ under $\mathbb{Q}^{n+1}$, as desired. □

Remark : We note that the following recursive relationship holds between the densities $\xi_T^{(n)}$, $n \geq 1$, defined in Proposition (1.2.2):

$$\xi_T^{(n)} = \xi_T^{(n-1)} e^{-\rho^2\int_0^T v_t dt}$$

which follows directly from the definition of $\xi_T^{(n)}$ and using the fact that $dW_t^n = dW_t^{n-1} - \rho\sqrt{v_t}dt$.

To obtain our approximation to Heston prices by Black-Scholes prices and Black-Scholes greeks, we proceed from relation (1.2) and use a Taylor expansion around the point $(S_0, (1-\rho^2)E^{\mathbb{Q}}(V_T))$. Using the fact that $E^{\mathbb{Q}}(\xi_T) = 1$, the Taylor polynomial of order $n \geq 2$, will have the form:

$$\begin{aligned} C_{\text{HES}}^{(n^{th})} &= C^{BS}\left(S_0, \left(1-\rho^2\right)E^{\mathbb{Q}}(V_T)\right) + \\ &+ \sum_{k=2}^{n}\sum_{l=0}^{k} \frac{1}{l!(k-l)!}\frac{\partial^k C^{BS}}{\partial S^l \partial V^{k-l}}\left(S_0, \left(1-\rho^2\right)E^{\mathbb{Q}}(V_T)\right)\cdot \\ &\cdot S_0^l \cdot \left(1-\rho^2\right)^{k-l} \cdot E^{\mathbb{Q}}\left((\xi_T - 1)^l\left(V_T - E^{\mathbb{Q}}(V_T)\right)^{k-l}\right) \end{aligned} \tag{1.4}$$

In order to obtain closed form expressions for the terms in the Taylor polynomial above, we notice that we require a technique to derive general moments of the form $E^{\mathbb{Q}}\left((\xi_T-1)^n\left(V_T - E^{\mathbb{Q}}(V_T)\right)^m\right)$ for n and m non-negative integers. As our subsequent results show, assuming $\rho \leq 0$, these moments will be finite for any n and m non-negative integers; the expansion can thus be constructed up to an arbitrary order. We remark that, in general, it is not possible to put a bound on the remainder term associated with the Taylor polynomial (1.4) because some greeks of our European call may be unbounded; the classical example is that of the Gamma which becomes unbounded as the total variance V goes to zero, as the call payoff has a kink located at the strike. In what follows, we will make use of the new probability measures $\mathbb{Q}^n$ and of Proposition (1.2.2). To illustrate our approach, we start with the simpler case of moments of the form $E^{\mathbb{Q}}(\xi_T^n)$. For $n = 2$, by the Radon-Nikodym theorem, we can write:

$$E^{\mathbb{Q}}\left(\xi_T^2\right) = E^{\mathbb{Q}}\left(\xi_T \cdot \xi_T\right) = E^{\mathbb{Q}^1}\left(\xi_T\right).$$

Using that $\xi_T = \xi_T^{(1)} \exp\left(\rho^2 \int_0^T v_t dt\right)$ and $\frac{d\mathbb{Q}^2}{d\mathbb{Q}^1} = \xi_T^{(1)}$ we obtain:

$$\begin{aligned} E^{\mathbb{Q}^1}\left(\xi_T\right) = E^{\mathbb{Q}^1}\left(\xi_T^{(1)} e^{\rho^2 \int_0^T v_t dt}\right) &= E^{\mathbb{Q}^2}\left(e^{\rho^2 \int_0^T v_t dt}\right) \\ &= \mathcal{L}\left(-\rho^2; k - 2\epsilon\rho, \frac{\theta}{1 - \frac{2\epsilon\rho}{k}}, \epsilon\right) \end{aligned}$$

where in the last step we use Proposition (1.2.2) by which we know that $(v_t)_{0 \le t \le T}$ is CIR$(k - 2\epsilon\rho, \frac{\theta}{1-\frac{2\epsilon\rho}{k}}, \epsilon)$ under $\mathbb{Q}^2$. The function $\mathcal{L}(\cdot)$ is just the Laplace transform of the integrated CIR diffusion and is known in closed form, as we have seen earlier. Applying the same steps, we can generalize and obtain the following result.

Lemma 1.2.3 *If $\rho \le 0$ the following holds for any non-negative integer $n \ge 1$:*

$$E^{\mathbb{Q}}\left(\xi_T^n\right) = \mathcal{L}\left(-\frac{n(n-1)}{2}\rho^2; k - n\epsilon\rho, \frac{\theta}{1 - \frac{n\epsilon\rho}{k}}, \epsilon\right).$$

Proof Recall the recursive relationship between the densities $\xi_T^{(n)}$, $n \ge 1$:

$$\xi_T^{(n)} = \xi_T^{(n-1)} e^{-\rho^2 \int_0^T v_t dt}$$

which gives that

$$\xi_T = \xi_T^{(0)} = \xi_T^{(n)} \cdot e^{n\rho^2 \int_0^T v_t dt}.$$

To compute the n-th moment of ξ_T, $E^{\mathbb{Q}}(\xi_T^n)$, we write

$$\xi_T^n = \prod_{k=1}^n \xi_T = \prod_{k=1}^n \xi_T^{(k-1)} \cdot e^{(k-1)\rho^2 \int_0^T v_t dt} = \left(\prod_{k=1}^n \xi_T^{(k-1)}\right) \cdot e^{\frac{n(n-1)}{2}\rho^2 \int_0^T v_t dt}$$

Using the fact that $\frac{d\mathbb{Q}^k}{d\mathbb{Q}^{k-1}} = \xi_T^{(k-1)}$, we obtain

$$E^{\mathbb{Q}}(\xi_T^n) = E^{\mathbb{Q}^n}\left(e^{\frac{n(n-1)}{2}\rho^2 \int_0^T v_t dt}\right) = \mathcal{L}\left(-\frac{n(n-1)}{2}\rho^2; k - n\epsilon\rho, \frac{\theta}{1 - \frac{n\epsilon\rho}{k}}, \epsilon\right)$$

where in the last step we have applied Proposition (1.2.2) which states that, under the probability measure $\mathbb{Q}^n$, $(v_t)_{0\le t\le T}$ is CIR$(k-n\epsilon\rho, \frac{\theta}{1-\frac{n\epsilon\rho}{k}}, \epsilon)$. Finally, to check that the moment obtained is finite, we can verify the sufficient condition:

$$n(n-1)\rho^2 < \frac{(k-n\epsilon\rho)^2}{\epsilon^2}$$

which simplifies to

$$\frac{k^2-2nk\epsilon\rho}{\epsilon^2}+n\rho^2>0.$$

This condition is clearly satisfied since $\rho\le 0$. □

Finally, in order to be able to compute general moments of the form $E^{\mathbb{Q}}\big(\,(\xi_T-1)^n \big(V_T-E^{\mathbb{Q}}(V_T)\big)^m\,\big)$, we can further generalize the previous lemma.

Lemma 1.2.4 *If $\rho\le 0$ the following holds for any non-negative integers $n\ge 1$ and $m\ge 1$:*

$$E^{\mathbb{Q}}\left(\xi_T^n\cdot V_T^m\right)=(-1)^m\frac{\partial^m}{\partial s^m}\mathcal{L}\left(-\frac{n(n-1)}{2}\rho^2;k-n\epsilon\rho,\frac{\theta}{1-\frac{n\epsilon\rho}{k}},\epsilon\right)$$

where V_T denotes the total integrated variance $V_T=\int_0^T v_t dt$.

Proof As in the proof of the previous lemma, we write

$$\xi_T^n=\left(\prod_{k=1}^n \xi_T^{(k-1)}\right)\cdot e^{\frac{n(n-1)}{2}\rho^2\int_0^T v_t dt}$$

and move from the original pricing measure $\mathbb{Q}$ to $\mathbb{Q}^n$:

$$E^{\mathbb{Q}}\left(\xi_T^n\cdot V_T^m\right)=E^{\mathbb{Q}^n}\left(e^{\frac{n(n-1)}{2}\rho^2 V_T}\cdot V_T^m\right)$$

Next, we check that this moment is finite. We recall from the proof of Lemma (1.2.3) that

$$n(n-1)\rho^2<\frac{(k-n\epsilon\rho)^2}{\epsilon^2}$$

and choose any $p\in\left(n(n-1)\rho^2,\frac{(k-n\epsilon\rho)^2}{\epsilon^2}\right)$. We can write

$$\begin{aligned}E^{\mathbb{Q}^n}\left(e^{\frac{n(n-1)}{2}\rho^2 V_T}\cdot V_T^m\right) &= E^{\mathbb{Q}^n}\left(e^{\frac{p}{2}V_T}\frac{V_T^m}{e^{\left(\frac{p}{2}-\frac{n(n-1)}{2}\rho^2\right)V_T}}\right)\\ &\le \frac{m!}{\left(\frac{p}{2}-\frac{n(n-1)}{2}\rho^2\right)^m}\cdot E^{\mathbb{Q}^n}\left(e^{\frac{p}{2}V_T}\right)<\infty\end{aligned}$$

where we use the fact that

$$e^{\left(\frac{p}{2}-\frac{n(n-1)}{2}\rho^2\right)V_T} \geq \frac{\left(\frac{p}{2}-\frac{n(n-1)}{2}\rho^2\right)^m V_T^m}{m!}.$$

Under $\mathbb{Q}^n$, $(v_t)_{0\leq t\leq T}$ is CIR $\left(k-n\epsilon\rho, \frac{\theta}{1-\frac{n\epsilon\rho}{k}}, \epsilon\right)$ and hence

$$E^{\mathbb{Q}^n}(e^{-sV_T}) = \mathcal{L}\left(s; k-n\epsilon\rho, \frac{\theta}{1-\frac{n\epsilon\rho}{k}}, \epsilon\right)$$

Taking the m-th derivative of both sides with respect to s and evaluating at $s = -\frac{n(n-1)}{2}\rho^2$ yields the desired result. $\square$

We now have the necessary results to derive, in closed form, all the terms in the Heston price expansion. Next, we apply this framework to work out the concrete details of the second and third order expansions.

1.3 A second order expansion

In this section we consider the Taylor polynomial (1.4) for the case $n = 2$ and explicitly compute all the convexity and cross-convexity adjustment terms that appear in the approximation. For brevity, we drop the common argument $\left(S_0, (1-\rho^2)\, E^{\mathbb{Q}}(V_T)\right)$ of the Black-Scholes function C^{BS} and obtain the following second order approximation to the Heston call price:

$$\begin{aligned} C^{\text{2nd}}_{\text{HES}} &= C^{BS} + \frac{1}{2}\frac{\partial^2 C^{BS}}{\partial S^2}\cdot S_0^2 \cdot E^{\mathbb{Q}}\left(\xi_T - 1\right)^2 + \\ &+ \frac{1}{2}\frac{\partial^2 C^{BS}}{\partial V^2}\cdot \left(1-\rho^2\right)^2 \cdot E^{\mathbb{Q}}\left(V_T - E^{\mathbb{Q}}\left(V_T\right)\right)^2 + \\ &+ \frac{\partial^2 C^{BS}}{\partial S \partial V}\cdot S_0 \cdot (1-\rho^2)\cdot E\left((\xi_T - 1)(V_T - E^{\mathbb{Q}}(V_T))\right). \end{aligned}$$

An important part in our expansions is played by the Black Scholes greeks. Their role in hedging and risk management is well recognized. The Black-Scholes greeks are discussed in many standard textbooks on quantitative finance and in a large number of scientific papers. Among these, we mention Hull, White (1987b) on the effects and role of vega and gamma hedging, Garman (1992) in which higher order Black Scholes partial derivatives are introduced and Carr (2000) who shows that the values of greeks of arbitrary order can be interpreted as the prices of certain contingent claims. The Black-Scholes greeks which appear in our second order approximation are:

$$\begin{aligned}
\text{Gamma} &= \frac{\partial^2 C^{BS}}{\partial S^2}(S,V) = \frac{e^{-\delta T}\varphi(d_1)}{S\sqrt{V}} \\
\text{Volga} &= \frac{\partial^2 C^{BS}}{\partial V^2}(S,V) = \frac{e^{-\delta T} S\varphi(d_1)}{4V^{3/2}}(d_1 d_2 - 1) \\
\text{Vanna} &= \frac{\partial^2 C^{BS}}{\partial S \partial V}(S,V) = -\frac{e^{-\delta T}\varphi(d_1) d_2}{2V}
\end{aligned}$$

where $\varphi(x)$ is the standard normal probability density function and d_1, d_2 as defined previously.

The moments of the total integrated variance can be computed in closed form by following the method in Dufresne (2001). In particular, we give below the expectation of total variance :

$$E^{\mathbb{Q}}(V_T) = E^{\mathbb{Q}}\left(\int_0^T v_t dt\right) = T\theta + \frac{1-e^{-kT}}{k}(v_0 - \theta).$$

For subsequent use, we will denote the expression above by $D_1(v_0, k, \theta, T)$. The closed form expression for the second central moment of the total integrated variance is given in the appendix. Applying Lemma (1.2.3) of the previous section we can compute:

$$E^{\mathbb{Q}}(\xi_T - 1)^2 = E^{\mathbb{Q}}(\xi_T^2) - 1 = \mathcal{L}\left(-\rho^2; k - 2\epsilon\rho, \frac{\theta}{1-\frac{2\epsilon\rho}{k}}, \epsilon\right) - 1.$$

Finally, applying Proposition (1.2.2) we can also determine the mixed moment below:

$$\begin{aligned}
E^{\mathbb{Q}}(\xi_T - 1)\left(V_T - E^{\mathbb{Q}}(V_T)\right) &= E^{\mathbb{Q}}(\xi_T V_T) - E^{\mathbb{Q}}(V_T) = E^{\mathbb{Q}^1}(V_T) - E^{\mathbb{Q}}(V_T) \\
&= D_1\left(v_0, k - \epsilon\rho, \frac{\theta}{1-\frac{\epsilon\rho}{k}}, \epsilon, T\right) - D_1(v_0, k, \theta, T)
\end{aligned}$$

where we have used the fact that, under $\mathbb{Q}^1$, the short variance process $(v_t)_{0\le t\le T}$ becomes $\mathrm{CIR}(k - \epsilon\rho, \frac{\theta}{1-\frac{\epsilon\rho}{k}}, \epsilon)$. This concludes the computation of all the terms in the second order Heston expansion.

We now present numerical tests to compare our second order approximation to the true Heston prices of plain vanilla options. To obtain the true prices of call options for a range of strikes and a fixed maturity T, we employ the Fast Fourier Transform (FFT) technique described by Carr, Madan (1999). In the original method of Carr, Madan (1999) a proper choice for the damping parameter must be made. In our computations, we apply a modification of the method, discussed

Parameter set	k	θ	ϵ	v_0
Heston (1993)	2	0.01	0.1	0.01
Bakshi et al. (1997)	1.15	0.0348	0.39	0.0348

Table 1.1: Parameter sets used for the CIR diffusion of instantaneous variance.

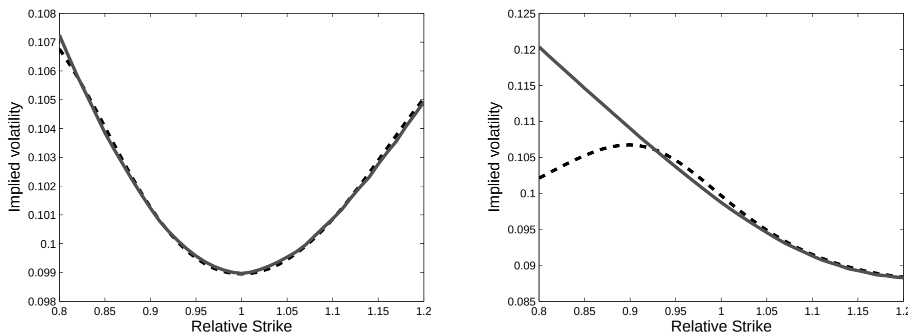

Figure 1.1: Comparison between second order expansion and true Heston prices, using original parameters from Heston (1993). *Solid gray:* True Heston prices, *Dashed black:* second order approximation (Left) Correlation parameter $\rho = 0$. (Right) Correlation parameter $\rho = -0.5$.

in Cont, Tankov (2004), which eliminates the need for choosing a damping parameter. Instead of directly inverting the Fourier transform of the (dampened) call price, one inverts the Fourier transform of the difference between the call price and a benchmark Black-Scholes price. For numerical illustrations and a discussion of the advantages of this modification we refer to Cont, Tankov (2004).

We illustrate our results using two different parameter sets for the CIR diffusion of instantaneous variance. Specifically, our parameter sets will consist of : (1) the original parameters used in Heston (1993) and (2) the parameters obtained in the study of Bakshi, Chen, Cao (1997). These parameters have been summarized in Table 1.1. For the correlation parameter, the study of Bakshi et al. (1997) found a value of $\rho = -0.64$; also, in Bakshi et al. (1997) the interest rate is $r = 3.4\%$ and $\delta = 0$ (as the authors work directly with dividend-adjusted asset prices). For the Heston (1993) parameter set, we use two choices for the correlation parameter, $\rho = 0$ and $\rho = -0.5$; in the parameter set of Heston (1993) we have $r = \delta = 0$. The impact on the approximation results of interest rate and dividend yield assumptions is small. This is so because the leading Black-Scholes price of the expansion will price-in the main influence of the assumed interest rate and dividend yield. The Black-Scholes greeks, which appear in the following terms, are only marginally affected by interest rate and dividend yield assumptions.

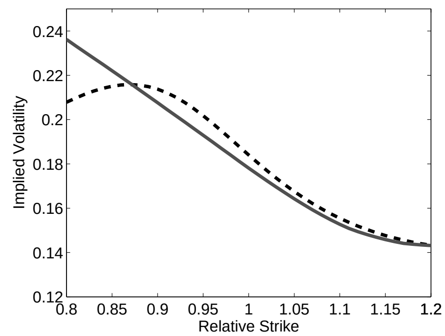

Figure 1.2: Comparison between second order expansion and true Heston prices, using parameters from Bakshi et al. (1997). *Solid gray:* True Heston prices, *Dashed black:* second order approximation.

We compare the Black-Scholes implied volatilities of the true Heston prices and the approximated prices for a maturity of six months, $T = 0.5$. In Figure (1.1) we can see the approximation results over a strike range of [80%-120%] using CIR parameters from the original paper of Heston (1993). In the left panel, for a correlation $\rho = 0$, we notice almost perfect agreement between the true prices and the second order approximation. The approximation holds remarkably well even for options which are 20% out-of-the money. In the right panel of Figure (1.1), we keep the same CIR parameters but change the correlation coefficient from $\rho = 0$ to $\rho = -0.5$. The right tail of the implied volatility curve remains well approximated. We note that for downside strikes, below 90% out-of-the money, the second order approximation will tend to underestimate the true value of options. In Figure (1.2), we run the same comparison using the CIR parameters from the study of Bakshi et al. (1997) and the correlation parameter found in their study, $\rho = -0.64$. The mean absolute error between the actual implied volatilities and the approximated implied volatilities, over the [80%,120%] strike range, was 0.61% (or 61bps).

1.4 A third order expansion

We present the calculations for the third order expansion of Heston prices in terms of Black-Scholes prices and Black-Scholes greeks. In addition to the terms we had to compute for the second order polynomial, we now have to consider also the terms below:

$$\begin{aligned}
C_{\text{HES}}^{3\text{rd}} &= C_{\text{HES}}^{2\text{nd}} + \frac{1}{3!}\frac{\partial^3 C^{BS}}{\partial S^3} \cdot S_0^3 \cdot E^{\mathbb{Q}}(\xi_T - 1)^3 + \\
&+ \frac{1}{3!}\frac{\partial^3 C^{BS}}{\partial V^3} \cdot (1-\rho^2)^3 \cdot E^{\mathbb{Q}}(V_T - E^{\mathbb{Q}}(V_T))^3 + \\
&+ \frac{1}{2}\frac{\partial^3 C^{BS}}{\partial S^2 \partial V} \cdot S_0^2 \cdot (1-\rho^2) \cdot E^{\mathbb{Q}}\left((\xi_T - 1)^2 (V_T - E^{\mathbb{Q}}(V_T))\right) + \\
&+ \frac{1}{2}\frac{\partial^3 C^{BS}}{\partial S \partial V^2} \cdot S_0 \cdot (1-\rho^2)^2 \cdot E^{\mathbb{Q}}\left((\xi_T - 1)(V_T - E^{\mathbb{Q}}(V_T))^2\right).
\end{aligned}$$

The Black-Scholes greeks which appear in this expansion are:

$$\begin{aligned}
\frac{\partial^3 C^{BS}}{\partial S^3}(S,V) &= -\frac{e^{-\delta T}\varphi(d_1)\left(d_1 + \sqrt{V}\right)}{VS^2} \\
\frac{\partial^3 C^{BS}}{\partial V^3}(S,V) &= \frac{e^{-\delta T} S \varphi(d_1)}{8V^{5/2}}\left((d_1 d_2 - 2)^2 - d_1^2 - d_2^2 - 1\right) \\
\frac{\partial^3 C^{BS}}{\partial S^2 \partial V}(S,V) &= \frac{e^{-\delta T}\varphi(d_1)}{2SV^{3/2}}(d_1 d_2 - 1) \\
\frac{\partial^3 C^{BS}}{\partial S \partial V^2}(S,V) &= \frac{e^{-\delta T}\varphi(d_1)}{4V^2}(d_1 + d_2 - d_2(d_1 d_2 - 1)).
\end{aligned}$$

As mentioned in the previous section, the moments of the total integrated variance can be computed in closed form as in Dufresne (2001). The expressions for the second and third central moments are provided in the appendix. Similar to the notation in the previous section, we will denote these moments as follows:

$$\begin{aligned}
E^{\mathbb{Q}}\left(V_T - E^{\mathbb{Q}}(V_T)\right)^2 &= D_2(v_0, k, \theta, \epsilon, T) \\
E^{\mathbb{Q}}\left(V_T - E^{\mathbb{Q}}(V_T)\right)^3 &= D_3(v_0, k, \theta, \epsilon, T)
\end{aligned}$$

Applying Lemma (1.2.3), we obtain :

$$\begin{aligned}
E^{\mathbb{Q}}(\xi_T - 1)^3 &= E^{\mathbb{Q}}(\xi_T^3) - 3E^{\mathbb{Q}}(\xi_T^2) + 2 = \mathcal{L}\left(-3\rho^2; k - 3\epsilon\rho, \frac{\theta}{1 - \frac{3\epsilon\rho}{k}}, \epsilon\right) - \\
&\quad -3\mathcal{L}\left(-\rho^2; k - 2\epsilon\rho, \frac{\theta}{1 - \frac{2\epsilon\rho}{k}}, \epsilon\right) + 2.
\end{aligned}$$

By applying Proposition (1.2.2) and Lemma (1.2.4), we next compute the remaining mixed moments. To determine :

$$E^{\mathbb{Q}}\left((\xi_T - 1)(V_T - E^{\mathbb{Q}}(V_T))^2\right)$$

write it as

$$E^{\mathbb{Q}}\left(\xi_T(V_T - E^{\mathbb{Q}}(V_T))^2\right) - E^{\mathbb{Q}}\left((V_T - E^{\mathbb{Q}}(V_T))^2\right)$$

which, by changing from $\mathbb{Q}$ to $\mathbb{Q}^1$, gives

$$\begin{aligned}
& E^{\mathbb{Q}^1}\left(V_T - E^{\mathbb{Q}}(V_T)\right)^2 - E^{\mathbb{Q}}\left(V_T - E^{\mathbb{Q}}(V_T)\right)^2 \\
= \; & E^{\mathbb{Q}^1}\left(V_T - E^{\mathbb{Q}^1}(V_T) + E^{\mathbb{Q}^1}(V_T) - E^{\mathbb{Q}}(V_T)\right)^2 - D_2(v_0, k, \theta, \epsilon, T) \\
= \; & D_2\left(v_0, k - \epsilon\rho, \frac{\theta}{1 - \frac{\epsilon\rho}{k}}, \epsilon, T\right) - D_2(v_0, k, \theta, \epsilon, T) + \\
& + \left(D_1\left(v_0, k - \epsilon\rho, \frac{\theta}{1 - \frac{\epsilon\rho}{k}}, T\right) - D_1(v_0, k, \theta, T)\right)^2.
\end{aligned}$$

Finally, to compute the last mixed moment :

$$E^{\mathbb{Q}}\left((\xi_T - 1)^2(V_T - E^{\mathbb{Q}}(V_T))\right)$$

multiply inner parentheses to obtain

$$\begin{aligned}
& E^{\mathbb{Q}}\Big(\xi_T^2 V_T - \xi_T^2 \cdot E^{\mathbb{Q}}\left(V_T\right) - 2\xi_T V_T + 2\xi_T E^{\mathbb{Q}}\left(V_T\right)\Big) \\
= \; & -\frac{\partial}{\partial s}\mathcal{L}\left(-\rho^2; k - 2\epsilon\rho, \frac{\theta}{1 - \frac{2\epsilon\rho}{k}}, \epsilon\right) - \\
& -D_1(v_0, k, \theta, T) \cdot \mathcal{L}\left(-\rho^2; k - 2\epsilon\rho, \frac{\theta}{1 - \frac{2\epsilon\rho}{k}}, \epsilon\right) - \\
& -2\left(D_1\left(v_0, k - \epsilon\rho, \frac{\theta}{1 - \frac{\epsilon\rho}{k}}, T\right) - D_1(v_0, k, \theta, T)\right).
\end{aligned}$$

This completes the computation of all the terms which appear in the third order expansion. Next, we apply our results to the numerical examples considered earlier.

Similar to the previous section, we graph the true implied volatilities against the implied volatilities calculated from the third order approximation. In Figure (1.3), we use the original parameters from Heston (1993). In the case of zero correlation, the results are almost indistinguishable from the second order approximation. This remains true also when we look at the right wing of the implied volatility curve, with the correlation set to $\rho = -0.5$. The third order approximation shows some improvement over the second order case for strikes below 90%. The mean absolute error between the true and the approximated implied volatilities was 0.30% (or 30bps) for the second order expansion and 0.20% (or 20bps) for the third order expansion. Figure (1.4) shows the comparison using the Bakshi et al. (1997) parameters. We notice that, at the money, the third order expansion can be slightly worse than its second order counterpart.

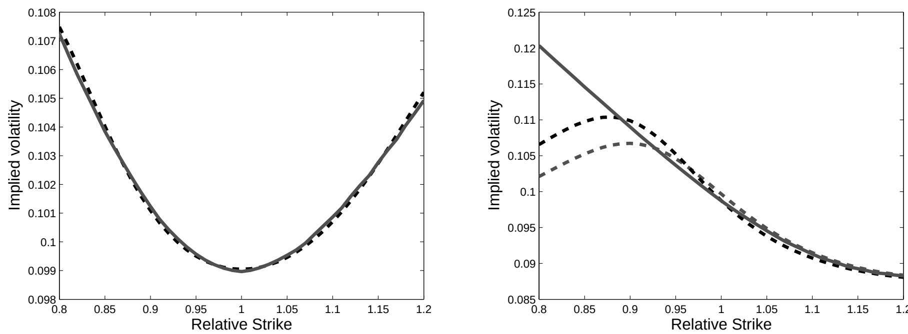

Figure 1.3: Comparison between third order expansion and true Heston prices, using original parameters from Heston (1993). *Solid gray:* true Heston prices, *Dashed black:* third order approximation (Left) Correlation parameter $\rho = 0$. (Right) Correlation parameter $\rho = -0.5$, *Dashed gray:* second order approximation.

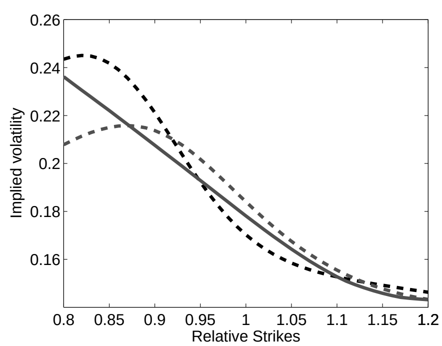

Figure 1.4: Comparison between second order expansion, third order expansion and true Heston prices, using parameters from Bakshi et al. (1997). *Solid gray:* true Heston prices, *Dashed black:* third order approximation, *Dashed gray:* second order approximation.

With regard to the use of higher order expansions, it is important to remark that it is not a priori guaranteed that they will provide more accurate approximations, since the Black-Scholes expansions may not converge. This follows by noting that, as shown in Carr (2000), Black-Scholes expansions in price and variance have a finite radius of convergence whereas inside the expectation of equation (1.4) we have to expand the Black-Scholes price over the entire domain $(S, V) \in (0, \infty) \times (0, \infty)$. This issue, however, is not specific to our expansion. The same situation arises, for example, in the well-known and popular convexity correction formula used for volatility swaps, as proposed in Brockhaus, Long (2000). Nevertheless, the second order convexity correction is widely used by practitioners as a first approximation for volatility swap prices.

1.5 Extension to forward starting options

In this section, we show how to extend the second order expansion proposed in section three to the case of forward starting options. To this end, let us consider a forward starting call option with forward start date $T > 0$, maturity $\Delta > 0$, and relative strike k. By definition, the payoff of this option, which occurs at time $T+\Delta$, will be :

$$\left(\frac{S_{T+\Delta}}{S_T} - k\right)_+ .$$

As in the case of spot started options, we seek to express the Heston price of the forward started call in terms of an integral over Black-Scholes prices. We have that

$$\begin{aligned} \frac{S_{T+\Delta}}{S_T} &= \exp\Bigg[(r-\delta)\,\Delta - \frac{1-\rho^2}{2}\int_T^{T+\Delta} v_t dt - \frac{\rho^2}{2}\int_T^{T+\Delta} v_t dt + \\ &\quad +\sqrt{1-\rho^2}\int_T^{T+\Delta} \sqrt{v_t} dZ_t + \rho\int_T^{T+\Delta} \sqrt{v_t} dW_t\Bigg]. \end{aligned}$$

It can be seen that this is similar to the case of spot started options except that, we now integrate the short variance process from its value at time T, v_T, which will be random, as opposed to the known v_0. Setting

$$\xi_T^{(\Delta)} = \exp\left[-\frac{\rho^2}{2}\int_T^{T+\Delta} v_t dt + \rho\int_T^{T+\Delta} \sqrt{v_t} dW_t\right]$$

we obtain

$$\frac{S_{T+\Delta}}{S_T} = \xi_T^{(\Delta)} \cdot \exp\left[(r-\delta)\,\Delta - \frac{1-\rho^2}{2}\int_T^{T+\Delta} v_t dt + \sqrt{1-\rho^2}\int_T^{T+\Delta} \sqrt{v_t} dZ_t\right].$$

By conditioning on the filtration generated by the Brownian motion W_t up to time $T+\Delta$, we can now write the value of the forward starting option as follows:

$$\begin{aligned} e^{-r(T+\Delta)} E^{\mathbb{Q}} \left(\frac{S_{T+\Delta}}{S_T} - k \right)_+ &= E^{\mathbb{Q}} \left(e^{-r(T+\Delta)} E^{\mathbb{Q}} \left[\left(\frac{S_{T+\Delta}}{S_T} - k \right)_+ \middle| \mathcal{F}^W_{T+\Delta} \right] \right) \\ &= E^{\mathbb{Q}} \left(C^{BS} \left(\xi_T^{(\Delta)}, \left(1-\rho^2\right) \int_T^{T+\Delta} v_t dt \right) \right). \end{aligned}$$

In the case of spot started options an important variable was the total integrated variance V_T. Its counterpart, in the forward starting case, will be the forward total integrated variance, which we denote as:

$$V_\Delta^T = \int_T^{T+\Delta} v_t dt.$$

Again, we notice that the key difference which arises in the pricing of forward starting options is that the value of the instantaneous variance at the forward start time T is not known to us at time zero, when we price the option. To make our approach similar to that of the previous sections, we will have to proceed by conditioning on the history of the variance process up to time T. Then, conditional on the value of v_T, the problem becomes identical to that which we have already solved in the first part of the paper. We will make use of the Laplace transform of v_T, whose expression we recall below (see for example Dufresne (2001)):

$$\mathcal{T}(s) = E^{\mathbb{Q}} \left(e^{-s v_T} \right) = \left(\frac{1}{1+\lambda s} \right)^{\frac{2k\theta}{\epsilon^2}} \exp \left(- \frac{s \cdot v_0 \cdot e^{-kT}}{1+\lambda s} \right)$$

where

$$\lambda = \frac{\epsilon^2}{2} \left(\frac{1-e^{-kT}}{k} \right).$$

Finally, in order to compute the terms in the expansion of the forward starting option price, we also need to recall the following two moments of v_T (see for example Andersen (2008) or Dufresne (2001)):

$$\begin{aligned} \beta_1(v_0,k,\theta,T) &= E^{\mathbb{Q}}\left(v_T\right) = \theta + (v_0-\theta)e^{-kT} \\ \beta_2(v_0,k,\theta,\epsilon,T) &= E^{\mathbb{Q}}\left(v_T - E^{\mathbb{Q}}(v_T)\right)^2 = \frac{v_0 \epsilon^2 e^{-kT}}{k}\left(1-e^{-kT}\right) + \\ &\quad + \frac{\theta\epsilon^2}{2k}\left(1-e^{-kT}\right)^2. \end{aligned}$$

We expand the forward call price in a Taylor polynomial around the point $\left(1, (1-\rho^2) \cdot E^{\mathbb{Q}}(V_\Delta^T)\right)$. Similar to the sections on plain vanilla options, the second order approximation of the forward-starting call option price will consist of the following terms:

$$\begin{aligned}C^{2\text{nd}}_{\text{FWD}} &= C^{BS} + \frac{1}{2}\frac{\partial^2 C^{BS}}{\partial S^2}\cdot E^{\mathbb{Q}}\left(\xi_T^{(\Delta)}-1\right)^2 + \\ &+\frac{1}{2}\frac{\partial^2 C^{BS}}{\partial V^2}\cdot\left(1-\rho^2\right)^2\cdot E^{\mathbb{Q}}\left(V_\Delta^T - E^{\mathbb{Q}}\left(V_\Delta^T\right)\right)^2 + \\ &+\frac{\partial^2 C^{BS}}{\partial S \partial V}\cdot(1-\rho^2)\cdot E^{\mathbb{Q}}\left((\xi_T^{(\Delta)}-1)(V_\Delta^T - E^{\mathbb{Q}}(V_\Delta^T))\right).\end{aligned}$$

The Black-Scholes greeks in this expansion have already been presented earlier. We proceed to compute the various moments which appear in the expansion and begin with the moments of the forward integrated variance.

Lemma 1.5.1 *The first two moments of the forward integrated variance V_Δ^T are given by:*

$$\begin{aligned}E^{\mathbb{Q}}(V_\Delta^T) &= D_1\left(\beta_1(v_0,k,\theta,T),k,\theta,\Delta\right) \\ E^{\mathbb{Q}}\left(V_\Delta^T - E^{\mathbb{Q}}\left(V_\Delta^T\right)\right)^2 &= D_2(\beta_1(v_0,k,\theta,T),k,\theta,\epsilon,\Delta) + \\ &+\left(\frac{1-e^{-k\Delta}}{k}\right)^2\cdot\beta_2(v_0,k,\theta,\epsilon,T).\end{aligned}$$

Proof Using the linearity of $D_1(v_0,k,\theta,\Delta)$ in v_0, we obtain for $E^{\mathbb{Q}}(V_\Delta^T)$:

$$\begin{aligned}E^{\mathbb{Q}}(V_\Delta^T) = E^{\mathbb{Q}}\left(E^{\mathbb{Q}}\left(V_\Delta^T|\mathcal{F}_T^W\right)\right) &= E^{\mathbb{Q}}\left(D_1(v_T,k,\theta,\Delta)\right) \\ &= D_1\left(\beta_1(v_0,k,\theta,T),k,\theta,\Delta\right).\end{aligned}$$

Similarly, using the linearity of $D_2(v_0,k,\theta,\epsilon,\Delta)$ in v_0 we can compute the second central moment of V_Δ^T. For brevity, in the calculation below we only show the first argument of $D_1(\cdot)$, $D_2(\cdot)$ and $\beta_1(\cdot)$:

$$\begin{aligned}E^{\mathbb{Q}}\left(V_\Delta^T - E^{\mathbb{Q}}\left(V_\Delta^T\right)\right)^2 &= E^{\mathbb{Q}}\left(V_\Delta^T - D_1(v_T) + D_1(v_T) - D_1(\beta_1(v_0))\right)^2 \\ &= E^{\mathbb{Q}}\left(V_\Delta^T - D_1(v_T)\right)^2 + E^{\mathbb{Q}}\left(D_1(v_T)-D_1(\beta_1(v_0)\right)^2.\end{aligned}$$

Recalling that

$$D_1(v_0) = \Delta\theta + \frac{1-e^{-k\Delta}}{k}\left(v_0-\theta\right)$$

we obtain

$$\begin{aligned}E^{\mathbb{Q}}\left(D_1(v_T)-D_1(\beta_1(v_0)\right)^2 &= \left(\frac{1-e^{-k\Delta}}{k}\right)^2 E^{\mathbb{Q}}\left(v_T-\beta_1(v_0)\right)^2 \\ &= \left(\frac{1-e^{-k\Delta}}{k}\right)^2\cdot\beta_2(v_0).\end{aligned}$$

Finally, conditioning on the history up to time T, gives:

$$\begin{aligned} E^{\mathbb{Q}}\left(V_\Delta^T - D_1(v_T)\right)^2 &= E^{\mathbb{Q}}\left[E^{\mathbb{Q}}\left(\left(V_\Delta^T - D_1(v_T)\right)^2 \middle| \mathcal{F}_T^W\right)\right] \\ &= E^{\mathbb{Q}}\left(D_2(v_T)\right) = D_2\left(\beta_1(v_0)\right). \end{aligned}$$

Therefore, we obtain:

$$\begin{aligned} E^{\mathbb{Q}}\left(V_\Delta^T - E^{\mathbb{Q}}\left(V_\Delta^T\right)\right)^2 &= D_2(\beta_1(v_0,k,\theta,T),k,\theta,\epsilon,\Delta) + \\ &+ \left(\frac{1-e^{-k\Delta}}{k}\right)^2 \cdot \beta_2(v_0,k,\theta,\epsilon,T). \end{aligned}$$

□

The remaining moments involve the density $\xi_T^{(\Delta)}$. Making use of Lemma (1.2.3) and the Laplace transform of v_T we obtain:

$$\begin{aligned} E^{\mathbb{Q}}\left(\xi_T^{(\Delta)} - 1\right)^2 &= E^{\mathbb{Q}}\left[E^{\mathbb{Q}}\left(\left(\xi_T^{(\Delta)}\right)^2 \middle| \mathcal{F}_T^W\right)\right] - 1 \\ &= E^{\mathbb{Q}}\left(\mathcal{L}\left(-\rho^2; v_T, k-2\epsilon\rho, \frac{\theta}{1-\frac{2\epsilon\rho}{k}}, \epsilon, \Delta\right)\right) - 1 \\ &= E^{\mathbb{Q}}\left(A(-\rho^2)e^{-v_T B(-\rho^2)}\right) - 1 \\ &= A\left(-\rho^2\right)\mathcal{T}\left(B\left(-\rho^2\right)\right) - 1. \end{aligned}$$

where

$$\begin{aligned} A\left(-\rho^2\right) &= A\left(-\rho^2; k-2\epsilon\rho, \frac{\theta}{1-\frac{2\epsilon\rho}{k}}, \epsilon\right) \\ B\left(-\rho^2\right) &= B\left(-\rho^2; k-2\epsilon\rho, \frac{\theta}{1-\frac{2\epsilon\rho}{k}}, \epsilon\right) \end{aligned}$$

with A and B as defined in section two. For the last mixed moment, we have to compute:

$$E^{\mathbb{Q}}\left((\xi_T^{(\Delta)} - 1)(V_\Delta^T - E^{\mathbb{Q}}(V_\Delta^T))\right).$$

We proceed in two steps. Firstly, rewrite the expectation above as:

$$E^{\mathbb{Q}}\left[(\xi_T^{(\Delta)} - 1)(V_\Delta^T - D_1(v_T) + D_1(v_T) - D_1\left(\beta_1(v_0)\right)\right]$$

and note that

$$E^{\mathbb{Q}}\left(\left(\xi_T^{(\Delta)} - 1\right)\left(D_1(v_T) - D_1\left(\beta_1(v_0)\right)\right)\right) = 0$$

as can be obtained by conditioning on $\mathcal{F}_T^W$:

$$E^{\mathbb{Q}}\left[\Big[D_1(v_T) - D_1\left(\beta_1(v_0)\right)\Big] E^{\mathbb{Q}}\left(\xi_T^{(\Delta)} - 1 \Big| \mathcal{F}_T^W\right)\right] = 0.$$

We thus have

$$E^{\mathbb{Q}}\left((\xi_T^{(\Delta)} - 1)(V_\Delta^T - E^{\mathbb{Q}}(V_\Delta^T))\right) = E^{\mathbb{Q}}\left(\left(\xi_T^{(\Delta)} - 1\right)\left(V_\Delta^T - D_1(v_T)\right)\right).$$

Secondly, recalling the following result derived in section 1.3 :

$$E^{\mathbb{Q}}\left(\xi_T - 1\right)\left(V_T - E^{\mathbb{Q}}(V_T)\right) = D_1\left(v_0, k - \epsilon\rho, \frac{\theta}{1 - \frac{\epsilon\rho}{k}}, \epsilon, T\right) - D_1(v_0, k, \theta, T)$$

and using the usual trick of conditioning on the history of the variance process up to time T, we obtain:

$$\begin{aligned}
&E^{\mathbb{Q}}\left(\xi_T^{(\Delta)} - 1\right)\left(V_\Delta^T - D_1(v_T)\right) = E^{\mathbb{Q}}\left[E^{\mathbb{Q}}\Big[\left(\xi_T^{(\Delta)} - 1\right)\left(V_\Delta^T - D_1(v_T)\right)\Big| \mathcal{F}_T^W\Big]\right] \\
&= E^{\mathbb{Q}}\left[D_1\left(v_T, k - \epsilon\rho, \frac{\theta}{1 - \frac{\epsilon\rho}{k}}, \epsilon, \Delta\right) - D_1(v_T, k, \theta, \Delta)\right] \\
&= D_1\left(\beta_1(v_0, k, \theta, T), k - \epsilon\rho, \frac{\theta}{1 - \frac{\epsilon\rho}{k}}, \epsilon, \Delta\right) - D_1(\beta_1(v_0, k, \theta, T), k, \theta, \Delta).
\end{aligned}$$

This completes the calculation of the terms in the expansion of the forward-starting call option price. This expansion will also allow us to explain the following phenomenon: as the forward-start date T increases, the forward smile becomes more pronounced with both wings of the curve steepening their slopes. The behavior of forward implied volatilities is particularly relevant in the valuation of exotic options and structured products (see, for example, Eberlein, Madan (2009)).

In the following numerical calculations, we use an option maturity of six months ($\Delta = 0.5$) forward-starting in six months ($T = 0.5$). In addition to the forward implied volatilities, we also show the spot-started implied volatilities (*dashed-gray* on the graph) to clearly see the more pronounced forward smile, in agreement with the effect mentioned earlier. This is particularly apparent in the left panel of Figure (1.5) where the correlation is set to zero; we notice that the spot-started smile is shallower than the forward smile. When the correlation is switched to -0.5, the effect is more pronounced on the right wing. As in the case of spot-started options, the accuracy of the second order approximation is nearly perfect in the [80%,120%] strike range when the correlation parameter is zero. For a correlation parameter of

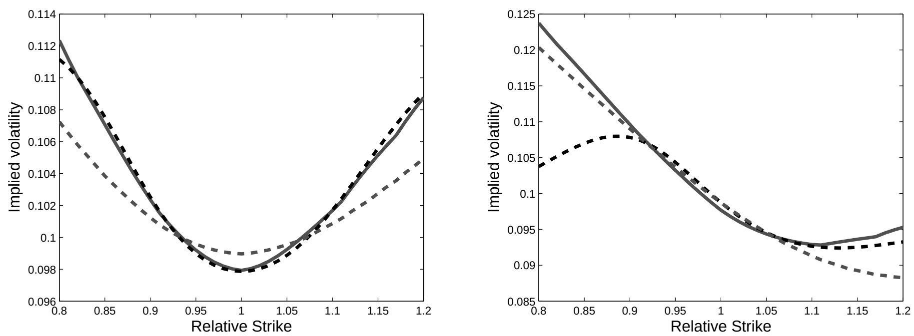

Figure 1.5: Comparison between second order expansion and true prices of forward-starting options in Heston model, using original parameters from Heston (1993). *Solid gray:* true Heston prices, *Dashed black:* second order approximation, *Dashed gray:* spot-started options. (Left) Correlation parameter $\rho = 0$. (Right) Correlation parameter $\rho = -0.5$. In both cases, $T = \Delta = 0.5$ years.

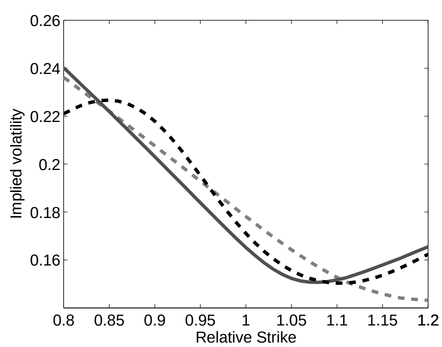

Figure 1.6: Comparison between second order expansion and true prices of forward-starting options, using parameters from Bakshi et al. (1997). *Solid gray:* true Heston prices, *Dashed black:* second order approximation, *Dashed gray:* spot-started options. In both cases, $T = \Delta = 0.5$ years.

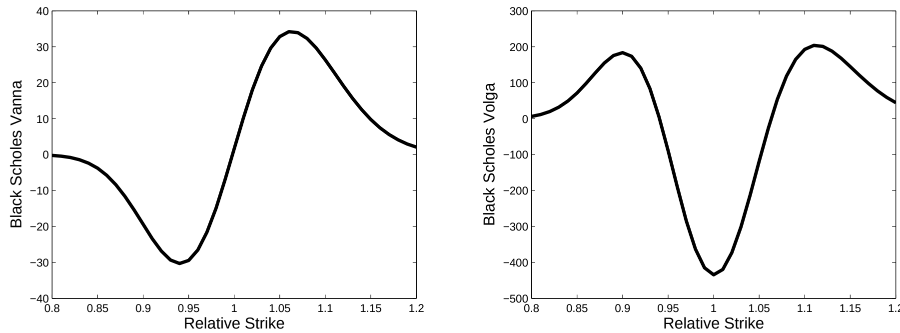

Figure 1.7: (Left) Black-Scholes Vanna as a function of strike. (Right) Black-Scholes Volga as a function of strike. In both cases, we have $r = \delta = 0$, $T = 0.5$, $\bar{V} = E^{\mathbb{Q}}(V_T) = 0.0038$ using Heston (1993).

-0.5, the mean absolute error was 0.33% (or 33bps). The same numerical test is also run for the parameter set of Bakshi et. al (1997) and shown in Figure(1.6).

To understand the forward smile effect, we plot the Black-Scholes greeks vanna and volga in Figure (1.7). When the correlation parameter is zero, our previous calculations show that the mixed moment involving $\xi_T^{(\Delta)}$ and V_Δ^T will vanish and hence there will be no vanna term in the second order expansion. In this case, the forward smile effect can be explained by inspecting the shape of the Black-Scholes volga. Specifically, its two local maxima reached out-of-the-money and its bottom at-the-money will tend to make the out-of-the-money options more expensive relative to at-the-money options thus leading to a steepening of both wings of the forward implied volatility curve. Therefore, we would expect the spot-started smile to be shallower than the forward-started smile if the variance of V_T is less than the variance of the forward integrated variance V_Δ^T. For our numerical example, we obtain the variance of V_T as $2.1 \cdot 10^{-6}$ and the variance of V_Δ^T as $4.26 \cdot 10^{-6}$. When the correlation parameter is different from zero, the vanna term also contributes to the forward smile effect and, as its shape from Figure (1.7) shows, it will make the effect more pronounced at the right wing and less pronounced at the left wing.

1.6 Conclusion

We have presented a new technique to decompose the price of options in the Heston stochastic volatility model in terms of Black-Scholes prices and Black-Scholes greeks. The square root form of the short variance dynamics is the key property which allows us to define a set of new probability measures which retain the affine form of the Heston dynamics. These structure preserving measure transformations give us the possibility to compute, in closed form, the expansion terms by reusing known results about the integrated variance process. The main role of these expansions is to make explicit the relation between Heston model prices and higher order risks which do not arise in the classical Black-Scholes setting, in particular, the convexity in volatility and the dependence of delta on the level of volatility. From a trading point of view, we note that for typical Heston parameters in the equity market, such as the Bakshi et al. (1997) parameter set, the error of the second order expansion in the [90%, 110%] at-the-money region is usually less than 1 volatility point (as can be seen in Figure (1.2)), which is about the size of the typical bid-offer spread, even for liquid equity underlyings. For other markets, such as foreign exchange, where the correlation parameter is (close to) zero the accuracy of the expansion almost matches the true model prices.

1.7 Appendix

For completeness, we provide in this appendix the formulas for the second and third central moments of the total integrated variance; these moments follow directly from the closed form results given in Dufresne (2001). As in the main paper, suppose the short variance $(v_t)_{t\in[0,T]}$ satisfies the SDE :

$$dv_t = k(\theta - v_t)dt + \epsilon\sqrt{v_t}dW_t$$

and define the total integrated variance V_T as:

$$V_T = \int_0^T v_t dt.$$

For the second and third central moments of V_T, denoted as follows:

$$\begin{aligned} D_2(v_0,k,\theta,\epsilon,T) &= E^{\mathbb{Q}}\left(V_T - E^{\mathbb{Q}}(V_T)\right)^2 \\ D_3(v_0,k,\theta,\epsilon,T) &= E^{\mathbb{Q}}\left(V_T - E^{\mathbb{Q}}(V_T)\right)^3 \end{aligned}$$

we have the closed form formulas below:

$$\begin{aligned} D_2(v_0,k,\theta,\epsilon,T) &= \frac{-5\theta\epsilon^2}{2k^3} + \frac{v_0\epsilon^2}{k^3} + \frac{T\theta\epsilon^2}{k^2} + \frac{2\theta\epsilon^2}{k^3}e^{-Tk} - \frac{v_0\epsilon^2}{k^3}e^{-2Tk} + \\ &\quad \frac{\theta\epsilon^2}{2k^3}e^{-2kT} - \frac{2Tv_0\epsilon^2}{k^2}e^{-Tk} + \frac{2T\theta\epsilon^2}{k^2}e^{-Tk} \end{aligned}$$

$$\begin{aligned} D_3(v_0, k, \theta, \epsilon, T) &= \frac{3v_0\epsilon^4}{k^5} - \frac{11\theta\epsilon^4}{k^5} + \frac{3T\theta\epsilon^4}{k^4} + \frac{3v_0\epsilon^4}{2k^5}e^{-Tk} + \frac{15\theta\epsilon^4}{2k^5}e^{-Tk} - \\ & \frac{3v_0\epsilon^4}{k^5}e^{-2Tk} + \frac{3\theta\epsilon^4}{k^5}e^{-2Tk} - \frac{3v_0\epsilon^4}{2k^5}e^{-3Tk} + \frac{\theta\epsilon^4}{2k^5}e^{-3Tk} - \\ & \frac{3Tv_0\epsilon^4}{k^4}e^{-Tk} + \frac{9T\theta\epsilon^4}{k^4}e^{-Tk} - \frac{3T^2v_0\epsilon^4}{k^3}e^{-Tk} + \frac{3T^2\theta\epsilon^4}{k^3}e^{-Tk} - \\ & \frac{6Tv_0\epsilon^4}{k^4}e^{-2Tk} + \frac{3T\theta\epsilon^4}{k^4}e^{-2Tk}. \end{aligned}$$

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II

Options on realized variance by transform methods: A non-affine stochastic volatility model

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Abstract

In this paper we study the pricing and hedging of options on realized variance in the 3/2 non-affine stochastic volatility model, by developing efficient transform based pricing methods. This non-affine model gives prices of options on realized variance which allow upward sloping implied volatility of variance smiles. Heston's (1993) model, the benchmark affine stochastic volatility model, leads to downward sloping volatility of variance smiles — in disagreement with variance markets in practice. Using control variates, we show a robust method to express the Laplace transform of the variance call function in terms of the Laplace transform of realized variance. The proposed method works in any model where the Laplace transform of realized variance is available in closed form. Additionally, we apply a new numerical Laplace inversion algorithm which gives fast and accurate prices for options on realized variance, simultaneously at a sequence of variance strikes. The method is also used to derive hedge ratios for options on variance with respect to variance swaps.

2.1 Introduction

The trading and risk management of variance and volatility derivatives requires models which both adequately describe the stochastic behavior of volatility as well as allow for fast and accurate numerical implementations. This is especially important for variance markets since their underlying asset, namely, variance, displays much more volatility than the corresponding stock or index, in the spot market. It is not uncommon for the volatility of variance to be several orders of magnitude higher than the volatility of the underlying stock or index. Many practical aspects relevant to variance and volatility markets are discussed in Bergomi (2005, 2008), Gatheral (2006) and Eberlein, Madan (2009).

Simple volatility derivatives, such as variance swaps, corridor variance swaps, gamma swaps and other similar variations, can be priced and hedged in a model free way and hence do not require the specification of a stochastic volatility model. Neuberger (1994) made a first contribution to this area by proposing the use of the log-contract as an instrument to hedge volatility risk. Due to their role in trading and hedging volatility, variance swaps have become liquidly traded instruments and have led to the development of other volatility derivatives. A comprehensive treatment of model free pricing and hedging of variance contracts can be found in Demeterfi, Derman, Kamal, Zou (1999), Carr, Madan (2002) and Friz, Gatheral (2005).

More complicated volatility derivatives, particularly, options on realized variance and volatility, require explicit modeling of the dynamics of volatility. Important early stochastic volatility models studied in the literature include Scott (1987), Hull, White (1987) and Chesney, Scott (1989). Since no fast numerical methods are available to compute large sets of European option prices in these models, calibration procedures can become difficult. Heston (1993) proposed the use of an affine square root diffusion process to model the dynamics of instantaneous variance. The model has become widely popular due to its tractability and existence of a closed form expression for the characteristic function of log returns. The important result of Carr, Madan (1999) shows how to apply fast Fourier inversion techniques to price European options when the characteristic function is available in closed form.

The problem of pricing options on realized variance received increasing attention in the recent literature. Broadie, Jain (2008a) and Sepp (2008) develop methods for pricing and hedging options on realized variance in the Heston model. Gatheral (2006) and Carr, Lee (2007) show how to use variance swap and volatility swap prices to fit a log-normal distribution to realized variance, thus arriving at Black-Scholes (1973) style formulas for prices and hedge ratios of options on variance. Several authors have considered the pricing of volatility derivatives in models with jumps. Carr, Geman, Madan, Yor (2005) price options on realized variance by assuming the underlying asset follows a pure jump Sato process; Albanese, Lo, Mijatovic (2009) develop spectral methods for models of constant elasticity of variance (CEV) mixtures and Variance-Gamma (VG) jumps; Sepp (2008) augments the Heston dynamics with simultaneous jumps in returns and volatility and also considers the pricing of options on forward variance.

In this paper we determine and compare the prices and hedge ratios of options on realized variance in the 3/2 non-affine stochastic volatility model versus the Heston (1993) model. The 3/2 model has been used previously by Ahn, Gao (1999) to model the evolution of short interest rates, by Andreasen (2003) as a default intensity model in pricing credit derivatives and by Lewis (2000) to price equity stock options. More recently, Carr, Sun (2007) discuss the 3/2 model in the context of a new framework in which variance swap prices are modeled instead of the short variance process. Besides its analytical tractability, the 3/2 diffusion specification enjoys empirical support in the equity market. Using S&P100 implied volatilities, studies by Jones (2003) and Bakshi, Ju, Yang (2006) estimate that the variance exponent should be around 1.3 which favors the 3/2 model over the 1/2 exponent in the Heston (1993) model. Additionally, as we show in this paper, the 3/2 and Heston models predict opposite dynamics for the short term equity skew as a function of the level of short variance and, more importantly, the Heston model wrongly generates downward sloping volatility of variance smiles, at odds with variance markets in practice.

We develop robust transform methods, based on control variates, to express the Laplace transform of the variance call function in terms of the Laplace transform of realized variance. Our approach works in any model where the Laplace transform of realized variance is available in closed form. We then apply a fast and accurate numerical Laplace inversion algorithm, recently proposed by Iseger (2006), which allows the use of the FFT technique of Cooley, Tukey (1965) to recover the variance call function at a sequence of strikes simultaneously. Finally, we show how these tools can be used to obtain hedge ratios for options on variance.

The paper is organized as follows. In section 2, we present general properties of the 3/2 and Heston models and compare them from the standpoint of short variance dynamics, equity skew dynamics and fitting to vanilla options. Section 3 is the main section, where we develop our transform methods and then apply them to pricing options on realized variance. In section 4, we discuss the derivation of hedge ratios with respect to variance swaps. Section 5 summarizes the main conclusions. All proofs not shown in the main text can be found in the appendix.

2.2 Model descriptions and properties

Two parametric stochastic volatility models are considered in this paper. Let $(B_t)_{t\geq 0}$ and $(W_t)_{t\geq 0}$ be standard Brownian motions, with correlation ρ, defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})$ satisfying the usual conditions. Under the well-known Heston (1993) model, we assume the stock price and its instantaneous variance $(S_t, v_t)_{t\geq 0}$ satisfy the following dynamics under the risk neutral measure $\mathbb{Q}$:

$$\begin{aligned} \frac{dS_t}{S_t} &= (r-\delta)dt + \sqrt{v_t}dB_t \\ dv_t &= k(\theta - v_t)dt + \epsilon\sqrt{v_t}dW_t \end{aligned}$$

where r denotes the risk-free rate in the economy and δ the dividend yield. The parameters of the instantaneous variance diffusion have the usual meaning: k is the speed of mean reversion, θ is the mean reversion level and ϵ is the volatility of volatility. The theoretical results in this paper allow the mean reversion level to be time dependent, but deterministic. This can be useful if the model user wants to interpolate the entire term structure of variance swaps. Therefore, we allow the short variance process to obey the following extended Heston dynamics:

$$dv_t = k(\theta(t) - v_t)dt + \epsilon\sqrt{v_t}dW_t \tag{2.1}$$

where $\theta(t)$ is a time-dependent and deterministic function of time. An alternative model, which forms the main focus of our study, is known in the literature as the *3/2-model*. It prescribes the following dynamics under the pricing measure $\mathbb{Q}$:

$$\begin{aligned} \frac{dS_t}{S_t} &= (r-\delta)dt + \sqrt{v_t}dB_t \\ dv_t &= kv_t(\theta(t) - v_t)dt + \epsilon v_t^{\frac{3}{2}}dW_t \end{aligned} \tag{2.2}$$

where, as in the case of the extended Heston model, $\theta(t)$ is the time-dependent mean reversion level. However, it is important to note that the parameters k and ϵ no longer have the same interpretation and scaling as in the Heston model. The speed of mean reversion is now given by the product $k \cdot v_t$, which is a stochastic quantity; in particular, we see that variance will mean revert more quickly when it is high. Also, we should expect the parameter k in the 3/2-process to scale as $1/v_t$ relative to the parameter k in the Heston model; the same scaling applies to the parameter ϵ. These scaling considerations are useful when interpreting the parameter values obtained from model calibration.

We next address a couple of technical conditions needed to have a well defined 3/2-model. An application of Itô's lemma to the process $1/v_t$ when v_t follows dynamics (2.2) gives:

$$d\left(\frac{1}{v_t}\right) = k\theta(t)\left(\frac{k+\epsilon^2}{k\theta(t)} - \frac{1}{v_t}\right)dt - \frac{\epsilon}{\sqrt{v_t}}dW_t$$

which reveals that the reciprocal of the 3/2 short variance process is, in fact, a Heston process of parameters $\left(k\theta(t), \frac{k+\epsilon^2}{k\theta(t)}, -\epsilon\right)$. Using Feller's boundary conditions, it is known that a time-homogeneous Heston process of dynamics (2.1), with $\theta(t) = \theta$, can reach the zero boundary with non-zero probability, unless:

$$2k\theta \geq \epsilon^2.$$

This result has been extended by Schlögl & Schlögl (2000) to the case of time-dependent piecewise-constant Heston parameters. We have seen that, if v_t is a 3/2-process, then $1/v_t$ is a Heston process. A non-zero probability of reaching zero for $1/v_t$ would imply a non-zero probability for the short variance process to reach infinity. For a piecewise constant $\theta(t)$, applying the result of Schlögl & Schlögl (2000) to the dynamics of $1/v_t$, we obtain the non-explosion condition for the 3/2-process as:

$$2k\theta(t) \cdot \frac{k + \epsilon^2}{k\theta(t)} \geq \epsilon^2$$

or

$$k \geq -\frac{\epsilon^2}{2}. \tag{2.3}$$

In what follows we assume that $k > 0$ which will automatically ensure that the non-explosion condition is satisfied. Another technical condition necessary in the 3/2 model refers to the martingale property of the process $S_t \cdot \exp(-(r - \delta)t)$; Lewis (2000) shows that for this process to be a true martingale, and not just a local martingale, the non-explosion test for $(v_t)_{t\geq 0}$ must be satisfied also under the measure which takes the asset price as numeraire. Applying the results in Lewis (2000), leads to the additional condition on the 3/2 model parameters:

$$k - \epsilon\rho \geq -\frac{\epsilon^2}{2}. \tag{2.4}$$

If we require that the correlation parameter ρ be non-positive, this condition will be automatically satisfied. In practice, imposing the restriction $\rho \leq 0$ does not raise problems since market behavior of prices and volatility usually displays negative correlation. To summarize, conditions (2.3) and (2.4) together ensure that we have a well-behaved 3/2-model. They are both satisfied if we impose the sufficient conditions $k > 0$ and $\rho \leq 0$.

Of importance to our subsequent analysis will be the joint Fourier-Laplace transform of the log-price $X_T = \log(S_T)$ and the annualized variance $V_T = \frac{1}{T}\int_0^T v_t dt$. In both models, it is possible to derive a closed form solution for this joint transform. In particular, using the characteristic function of $X_T = \log(S_T)$ it is possible to price European options by Fourier inversion using the method developed in Carr, Madan (1999). Also, in the next section, we develop fast transform methods to price options on realized variance using the Laplace transform of V_T. Propositions (2.2.1) and (2.2.2) below give the expression of the joint transforms in the two models. Below, we let $X_t = \log(S_t e^{(r-\delta)(T-t)})$ denote the log-forward price process.

Proposition 2.2.1 *In the Heston model with time-dependent mean-reversion level, the joint conditional Fourier-Laplace transform of X_T and the de-annualized realized variance $\int_t^T v_s ds$ is given by:*

$$E\left(e^{iuX_T-\lambda\int_t^T v_s ds}\middle| X_t, v_t\right) = \exp\left(iuX_t + a\left(t,T\right) - b\left(t,T\right)v_t\right)$$

where

$$\begin{aligned} a(t,T) &= -\int_t^T k\theta(s)b(s,T)ds \\ b(t,T) &= \frac{\left(iu+u^2+2\lambda\right)\left(e^{\gamma(T-t)}-1\right)}{\left(\gamma+k-i\epsilon\rho u\right)\left(e^{\gamma(T-t)}-1\right)+2\gamma} \\ \gamma &= \sqrt{\left(k-i\epsilon\rho u\right)^2+\epsilon^2\left(iu+u^2+2\lambda\right)}. \end{aligned}$$

For the case when the mean reversion level $\theta(t)$, $t \in [0,T]$, is a piecewise constant function it is possible to calculate explicitly the integral which defines $a(t,T)$ in Proposition (2.2.1). If we let $0 = t_0 < t_1 < \ldots < t_N = T$ be a partition of $[0,T]$ such that $\theta(t) = \theta_j$ on the interval (t_j, t_{j+1}), $j \in \{0,1,2,\ldots,N-1\}$, then the function $a(t,T)$ is given by:

$$\begin{aligned} a(t,T) &= \sum_{j=0}^{N-1}\frac{k\theta_j}{\epsilon^2}\Bigg[\left(k-\gamma-i\epsilon\rho u\right)\left(t_{j+1}-t_j\right)- \\ &\qquad -2\log\left(\frac{\alpha e^{\gamma(T-t_{j+1})}+\beta e^{-\gamma(t_{j+1}-t_j)}}{\alpha e^{\gamma(T-t_{j+1})}+\beta}\right)\Bigg] \end{aligned}$$

where

$$\begin{aligned} \alpha &= \gamma+k-i\epsilon\rho u \\ \beta &= \gamma-k+i\epsilon\rho u \end{aligned}$$

with γ as defined in Proposition (2.2.1). A similar result which gives the closed form expression of the joint Fourier-Laplace transform can be obtained in the 3/2-model. The result is due to Carr, Sun (2007).

Proposition 2.2.2 (Carr, Sun (2007)) *In the 3/2-model with time-dependent mean-reversion level, the joint conditional Fourier-Laplace transform of X_T and the de-annualized realized variance $\int_t^T v_s ds$ is given by:*

$$E\left(e^{iuX_T-\lambda\int_t^T v_s ds}\middle| X_t, v_t\right) = e^{iuX_t}\frac{\Gamma\left(\gamma-\alpha\right)}{\Gamma\left(\gamma\right)}\left(\frac{2}{\epsilon^2 y\left(t,v_t\right)}\right)^{\alpha} M\left(\alpha,\gamma,\frac{-2}{\epsilon^2 y\left(t,v_t\right)}\right)$$

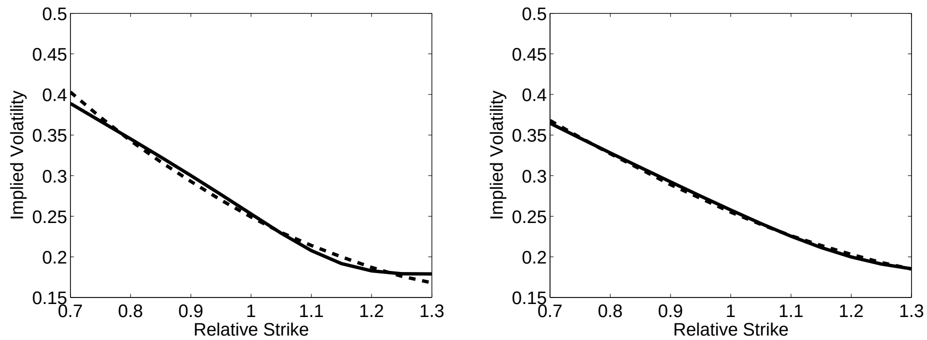

Figure 2.1: Simultaneous fit of Heston model to 3-months (left) and 6-months (right) S&P500 implied volatilities on July 31st 2009. *Solid:* Heston implied volatilities, *Dashed:* Market implied volatilities. Heston parameters obtained: $v_0 = 25.56\%^2$, $k = 3.8$, $\theta = 30.95\%^2$, $\epsilon = 92.88\%$ and $\rho = -78.29\%$.

where

$$\begin{aligned}
y(t, v_t) &= v_t \int_t^T e^{\int_t^u k\theta(s)ds} du \\
\alpha &= -\left(\frac{1}{2} - \frac{p}{\epsilon^2}\right) + \sqrt{\left(\frac{1}{2} - \frac{p}{\epsilon^2}\right)^2 + 2\frac{q}{\epsilon^2}} \\
\gamma &= 2\left(\alpha + 1 - \frac{p}{\epsilon^2}\right) \\
p &= -k + i\epsilon\rho u \\
q &= \lambda + \frac{iu}{2} + \frac{u^2}{2}
\end{aligned}$$

$M(\alpha, \gamma, z)$ is the confluent hypergeometric function defined as:

$$M(\alpha, \gamma, z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\gamma)_n} \frac{z^n}{n!}$$

and

$$(x)_n = x(x+1)(x+2)\cdots(x+n-1).$$

Proof We refer the reader to Carr, Sun (2007). □

Before looking at how the models differ in pricing exotic contracts, such as options on realized variance, we discuss in the rest of this section the pricing of European vanilla options. The calibration of the models to vanilla options is important because the prices of European options determine the values of variance swaps which, in turn, are the main hedging instruments for options on realized variance;

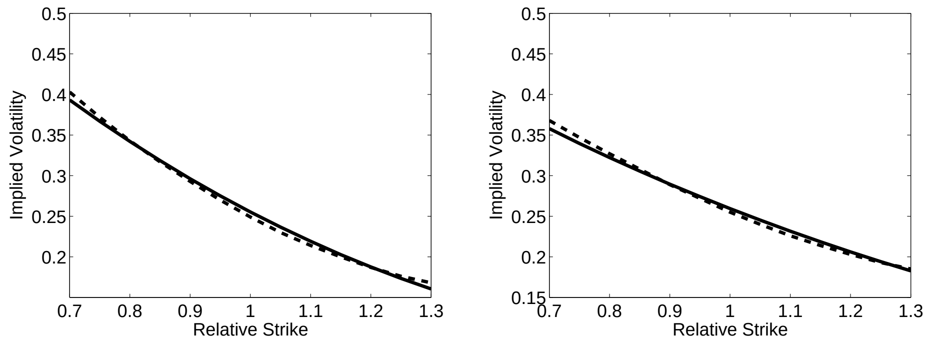

Figure 2.2: Simultaneous fit of 3/2 model to 3-months (left) and 6-months (right) S&P500 implied volatilities on July 31st 2009. *Solid:* 3/2 implied volatilities, *Dashed:* Market implied volatilities. 3/2 parameters obtained: $v_0 = 24.50\%^2$, $k = 22.84$, $\theta = 46.69\%^2$, $\epsilon = 8.56$ and $\rho = -99.0\%$.

for a broad introduction to variance swaps we refer to Carr, Madan (2002). Moreover, vanilla options are also often used to hedge the vega exposure of options on realized variance.

We begin by fitting both models to market prices of S&P500 European options; the fit is done simultaneously to two maturities: 3 months ($T = 0.25$) and 6 months ($T = 0.5$) on July 31 2009[1]. In performing the calibration, we employ the FFT algorithm for European options developed in Carr, Madan (1999). The results are shown in figures (2.1) and (2.2). The parameters obtained are $(v_0, k, \theta, \epsilon, \rho)$: $(25.56\%^2, 3.8, 30.95\%^2, 92.88\%, -78.29\%)$ in the Heston model and $(24.50\%^2, 22.84, 46.69\%^2, 8.56, -99.0\%)$ in the 3/2 model.

We remark that, while both models are able to fit the two maturities simultaneously, the Heston model parameters violate the non-zero boundary condition. This usually happens when calibrating the Heston model in the equity markets; the empirical study of Bakshi, Chao, Chen (1997) also finds Heston parameters which violate the non-zero boundary condition. This occurs because the Heston model requires a high volatility-of-volatility parameter ϵ to fit the steep skews in equity markets. On the other hand, the 3/2 parameters yield a well-behaved variance process which does not reach either zero or infinity. We notice, however, that the 3/2 model requires a more negative correlation parameter (-99.0%) compared to the Heston model (-78.29%). We explain this by the much 'wilder' dynamics (also called 'dynamite dynamics' by Andreasen (2003)) of short variance in the 3/2 model – as illustrated in figure (2.3) – which can cause a decorrelation with the spot process.

The modeling viewpoint of the 3/2 model as well as the modeling viewpoint of

[1]The data were kindly provided to us by an international investment bank.

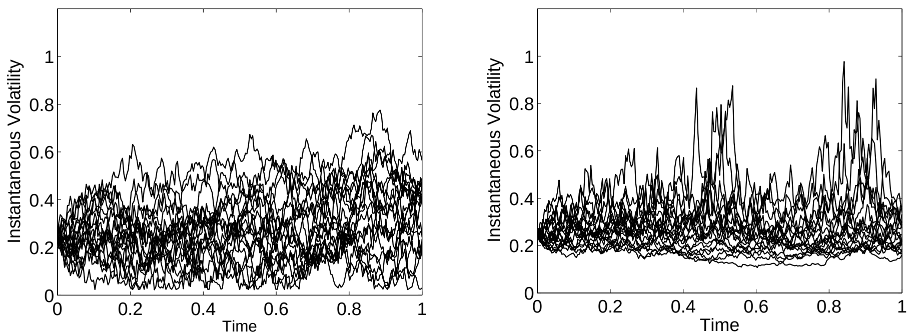

Figure 2.3: Instantaneous volatility (i.e. $\sqrt{v_t}$) paths in the Heston (left) and 3/2-model (right) using parameters calibrated to July 31st 2009 implied volatilities.

the Heston (1993) model can be both accommodated in the more general framework:

$$dv_t = \left(\alpha_0 + \alpha_1 v_t + \alpha_2 v_t^2 + \alpha_3 v_t^{-1}\right) dt + \beta_2 v_t^{\beta_3} dW_t$$

which was analyzed in the econometric study of Bakshi, Ju, Yang (2006). The authors use the square of the VIX index, sampled daily over a period of more than ten years, as a proxy for the instantaneous variance process. Among the results of their statistical tests, the authors emphasize that "*an overarching conclusion is that* $\beta_3 > 1$ *is needed to match the time-series properties of the VIX index*" and find strong evidence rejecting the null hypothesis $\beta_3 \leq 1$. Indeed, the authors estimate a value of approximately 1.28 for the exponent β_3 which lends more support for the 1.5 exponent of the 3/2-model than the 0.5 exponent in the Heston (1993) model. The estimation for β_3 obtained in Bakshi, Ju, Yang (2006) reinforces similar results from an earlier study by Jones (2003). Additionally, the study of Bakshi et al. (2006) finds evidence in favor of a non-linear drift for the instantaneous variance diffusion. In particular, they find that the role of the quadratic term $\alpha_2 \cdot v_t^2$ is statistically more important than the linear term $\alpha_1 \cdot v_t$, thus rejecting a linear drift specification as in the Heston (1993) model; the authors find a significant $\alpha_2 < 0$ as in the quadratic drift of the 3/2 process.

Having noted the statistical evidence above, Figure (2.3) also illustrates two qualitative differences between the evolution of instantaneous variance in the Heston model versus the 3/2 model : (a) the Heston variance paths spend much more time around the zero-boundary and (b) the 3/2 model allows for the occurrence of extreme paths with short-term spikes in instantaneous volatility. From a trading and risk management perspective, both of these observations favor the 3/2 model. It is hard to justify a vanishing variance process and the nonexistence of high (or, extreme) volatility scenarios. In the 3/2-model, as indicated by the reversion speed $k \cdot v_t$, the process will revert faster towards the mean after short-term spikes in instantaneous

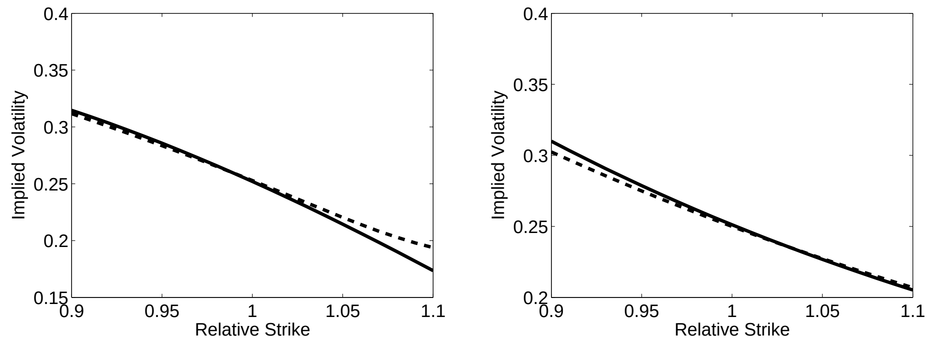

Figure 2.4: Comparison between true model implied volatilities and the Medvedev, Scaillet (2007) expansion, for a maturity of one month. Left: Heston model. Right: 3/2 model. *Solid*: Medvedev, Scaillet (2007) approximation. *Dashed*: true model implied volatilities.

variance. The right panel of Figure (2.3) illustrates this behavior graphically. We remark that short-term market volatility, as reflected for example by CBOE's VIX index, exhibits similar short-term spikes during periods of market stress. In the next section we see that these differences have a major effect on the prices of options on realized variance.

Another important difference is in the behavior of implied volatility smiles. Specifically, the steepness of the smile responds in opposite ways to changes in the level of short variance in the two models. To show this, we make use of the implied volatility expansion derived in Medvedev, Scailett (2007) for short times to expiration and near the money options. Letting $X = \log(K/S_0 e^{(r-\delta)T})$ denote the log-forward-moneyness corresponding to a European option with strike K and maturity T, we apply Proposition 1 in Medvedev, Scaillet (2007) to the case of the Heston and 3/2 models. One obtains the following expansions for implied volatility $I(X,T)$ in a neighborhood of $X = T = 0$. For the Heston model:

$$\begin{aligned} I(X,T) &= \sqrt{v_0} + \frac{\rho\epsilon X}{4\sqrt{v_0}} + \left(1 - \frac{5\rho^2}{2}\right)\frac{\epsilon^2 X^2}{24 v_0^{3/2}} + \\ &\quad + \left(\frac{k\,(\theta - v_0)}{4\sqrt{v_0}} + \frac{\rho\epsilon\sqrt{v_0}}{8} + \frac{\rho^2\epsilon^2}{96\sqrt{v_0}} - \frac{\epsilon^2}{24\sqrt{v_0}}\right) T + \ldots \end{aligned}$$

which gives the at-the-money-forward skew

$$\frac{\partial I}{\partial X}\,(X,T)\Big|_{X=0} = \frac{\rho\epsilon}{4\sqrt{v_0}}, \qquad T \to 0.$$

And for the 3/2 model:

$$
\begin{aligned}
I(X,T) \quad &= \quad \sqrt{v_0} + \frac{\rho\epsilon\sqrt{v_0}X}{4} + \left(1 - \frac{\rho^2}{2}\right)\frac{\epsilon^2\sqrt{v_0}X^2}{24} + \\
&\quad + \left(\frac{k(\theta - v_0)}{4} + \frac{\rho\epsilon v_0}{8} - \frac{7\rho^2\epsilon^2 v_0}{96} - \frac{\epsilon^2 v_0}{24}\right)\sqrt{v_0}T + ...
\end{aligned}
$$

which gives the at-the-money-forward skew

$$
\frac{\partial I}{\partial X}(X,T)\Big|_{X=0} = \frac{\rho\epsilon\sqrt{v_0}}{4}, \qquad T \to 0.
$$

In figure (2.4) we see a very good agreement between the Medvedev, Scaillet (2007) expansion and the true implied volatilities calculated by Fourier inversion. Since the expansions are valid only for short expirations and close to the money, we chose a one month maturity and a [90%, 110%] relative strike range. Therefore, in the Heston model, the short term skew flattens when the instantaneous variance increases whereas, in the 3/2 model, the short term skew steepens when the instantaneous variance increases. It is important to realize that this implies very different dynamics for the evolution of the implied volatility surface. The Heston model predicts that, in periods of market stress, when the instantaneous volatility increases, the skew will flatten. Under the same scenario, the skew will steepen in the 3/2 model. From a trading and risk management perspective, since the magnitude of the skew is itself a measure of market stress, the behavior predicted by the 3/2 model appears more credible.

2.3 Transform pricing of options on realized variance

The main quantity of interest in pricing options on realized variance is the annualized integrated variance, given by:

$$
V_T = \frac{1}{T}\int_0^T v_t dt.
$$

We study the prices of call options on realized variance; prices of put options follow by put-call parity. The payoff of a call option on realized variance with strike K and maturity T is defined as:

$$
\left(\frac{1}{T}\int_0^T v_t dt - K\right)_+ = (V_T - K)_+ .
$$

In a key result Carr, Madan (1999) showed that, starting from the characteristic function of the log stock price $\log(S_T)$, it is possible to derive a closed form expression for the Fourier transform of the (dampened) call price viewed as a function of the log strike $k = \log(K)$. Once the call price transform is known, fast inversion algorithms – such as the FFT method developed by Cooley, Tukey (1965)– can be applied to recover the call prices at a sequence of strikes simultaneously. This technique is now widely used in the literature to price stock options in non-Black-Scholes models which have closed form expressions for the characteristic function of $\log(S_T)$. We next develop a similar idea for the problem of pricing options on realized variance. Specifically, we show that, starting from the Laplace transform of integrated variance, it is possible to derive in closed form the Laplace transform of the variance call price viewed as a function of the variance strike. This idea was first suggested by Carr, Geman, Madan, Yor (2005). After providing a proof, we propose an important improvement of the result by the use of control variates. Additionally, we show the application of a new numerical Laplace inversion algorithm which gives prices of options on realized variance for a sequence of variance strikes simultaneously.

Proposition 2.3.1 *Let $\mathcal{L}(\cdot)$ denote the Laplace transform of the annualized realized variance over $[0,T]$:*

$$\mathcal{L}(\lambda) = E\left(e^{-\lambda \frac{1}{T}\int_0^T v_t dt}\right).$$

Then the undiscounted variance call function $C : [0,\infty) \to \mathbb{R}$ defined by

$$C(K) = E\left(\frac{1}{T}\int_0^T v_t dt - K\right)_+$$

has Laplace transform given by

$$L(\lambda) = \int_0^\infty e^{-\lambda K} C(K) dK = \frac{\mathcal{L}(\lambda) - 1}{\lambda^2} + \frac{C(0)}{\lambda}. \tag{2.5}$$

Proof Let $\mu(dx)$ denote the probability law of the annualized realized variance $\frac{1}{T}\int_0^T v_t dt$. We have to compute

$$L(\lambda) = \int_0^\infty e^{-\lambda K} \int_K^\infty (x - K)\, \mu(dx) dK.$$

Since the integrand in this double integral is non-negative, we can apply Fubini's theorem to change the order of integration and we obtain

$$\int_0^\infty \int_0^x e^{-\lambda K} (x - K)\, dK \mu(dx).$$

The inner integral now follows easily by integration by parts

$$\int_0^x e^{-\lambda K}(x-K)\,dK = \frac{e^{-\lambda x}-1}{\lambda^2} + \frac{x}{\lambda}.$$

Finally, we can compute the Laplace transform as follows

$$L(\lambda) = \int_0^\infty \left(\frac{e^{-\lambda x}-1}{\lambda^2} + \frac{x}{\lambda}\right)\mu(dx) = \frac{\mathcal{L}(\lambda)-1}{\lambda^2} + \frac{C(0)}{\lambda}$$

where we have used that

$$C(0) = E\left(\frac{1}{T}\int_0^T v_t dt\right).$$

$\square$

Relation (2.5) of Proposition (2.3.1) gives a closed form solution for the Laplace transform $L(\lambda)$ of the variance call function $C(K)$ in terms of the Laplace transform $\mathcal{L}(\lambda)$ of the annualized realized variance $V_T = \frac{1}{T}\int_0^T v_t dt$. The closed from expression for $\mathcal{L}(\lambda)$ is obtained from Proposition (2.2.1) and Proposition (2.2.2) by setting $t=0$, $u=0$ and $\lambda = \frac{\lambda}{T}$.

However, we notice that the following two, polynomially decaying terms, appear in expression (2.5):

$$\frac{-1}{\lambda^2} + \frac{C(0)}{\lambda}.$$

These vanish slowly as $|\lambda| \to \infty$ affecting the accuracy of numerical inversion algorithms. The term $\frac{C(0)}{\lambda}$ appears because the function has a discontinuity of size $C(0)$ at 0, while the term $\frac{-1}{\lambda^2}$ appears because the first derivative has a discontinuity of size -1 at 0 (as shown next). We propose to eliminate these slowly decaying terms by applying the idea of control variates. Specifically, we choose a proxy distribution for the realized variance which allows the calculation of the variance call function in closed form. Denote this control variate function by $\tilde{C}(\cdot)$. If we choose the proxy distribution such that it has the same mean as the true distribution of realized variance, we have $C(0) = \tilde{C}(0)$. Then, by the linearity of the Laplace transform we obtain:

$$L_{C-\tilde{C}}(\lambda) = L_C(\lambda) - L_{\tilde{C}}(\lambda) = \frac{\mathcal{L}(\lambda) - \tilde{\mathcal{L}}(\lambda)}{\lambda^2}.$$

Both power terms have been eliminated since the difference $C(\cdot) - \tilde{C}(\cdot)$ is now a function which is both continuous and differentiable at zero. Differentiability comes from the fact that both functions have a right derivative at 0 equal to -1. This is seen in the following simple lemma.

Lemma 2.3.2 *Let V be a random variable such that $V > 0$ a.s. and $E(V) < \infty$. Then the function $C : [0, \infty) \to \mathbb{R}$ defined by:*

$$C(K) = E(V - K)_+$$

satisfies

$$\lim_{K \downarrow 0} \frac{C(K) - C(0)}{K} = -1.$$

In summary, by making use of a control variate we can achieve smooth pasting at 0. In choosing the proxy distribution for realized variance, one appealing choice is the log-normal distribution. This would give Black-Scholes style formulas for the control variate function $\tilde{C}(\cdot)$. However, this choice does not work because the Laplace transform of the log-normal distribution is not available in closed form. Instead, we choose the Gamma distribution as our proxy distribution. The Laplace transform is known in closed form and the following lemma shows how to compute the control variate function $\tilde{C}(\cdot)$.

Lemma 2.3.3 *Let the (annualized) realized variance over $[0, T]$ follow a Gamma distribution of parameters (α, β). Specifically, assume the density of realized variance is:*

$$\frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}}, x > 0.$$

Then the control variate function $\tilde{C}(\cdot)$ is given by

$$\tilde{C}(K) = \alpha\beta \left(1 - F(K; \alpha + 1, \beta)\right) - K \left(1 - F(K; \alpha, \beta)\right)$$

where $F(x; \alpha, \beta)$ is the Gamma cumulative distribution function of parameters (α, β).

To ensure that $\tilde{C}(0) = C(0)$, the only necessary condition on α and β is that the mean of the Gamma distribution matches $C(0)$:

$$\alpha\beta = C(0).$$

Since we have two parameters, from a theoretical standpoint, we can choose one of them freely. Optionally, the extra parameter could be used to fix the second moment of the proxy distribution. For example, we can match the second moment of the model realized variance:

$$\alpha\beta^2 + (\alpha\beta)^2 = E\left(\frac{1}{T}\int_0^T v_t dt\right)^2 = \left.\frac{\partial^2 \mathcal{L}}{\partial \lambda^2}(\lambda)\right|_{\lambda=0}$$

where $\mathcal{L}(\cdot)$ is the Laplace transform of realized variance. In the Heston model, the second moment of realized variance is available in closed form; see Dufresne (2001). In the 3/2 model, however, we do not have a closed form formula for the second moment of realized variance; as shown later, even the calculation of the first moment requires the development of some additional results. In this case, the second moment could be approximated by using a finite difference for the second derivative of $\mathcal{L}(\cdot)$ at zero. Alternatively, an easier approach to choose a reasonable second moment for the control variate distribution is to match the second moment of a log-normal distribution of the form:

$$C(0)e^{\sigma\sqrt{T}N(0,1)-\frac{\sigma^2 T}{2}}$$

where $N(0,1)$ is a standard normal random variable and σ is a parameter of our choice – a sensible pick would have an order of magnitude that is representative for the implied volatility of variance. As the subsequent numerical results reveal, any choice for σ in the range, say, $[50\%, 150\%]$ would be reasonable. The second moment condition on α and β reads:

$$\alpha\beta^2 + (\alpha\beta)^2 = C(0)^2 e^{\sigma^2 T}$$

We obtain that a possible choice for the parameters of the proxy distribution is:

$$\begin{aligned}\alpha &= \frac{C(0)}{\beta}\\ \beta &= C(0)(e^{\sigma^2 T} - 1).\end{aligned}$$

To implement the above calculations one needs to be able to determine

$$E\left(\int_0^T v_t dt\right)$$

in both models – Heston and 3/2. The computation is straightforward in the Heston model but is more complicated in the 3/2 model. We first show the calculation for the Heston model with a piecewise constant mean reversion level $\theta(t)$, $t \in [0, T]$. If we let $\theta(t) = \theta_i$ on (t_i, t_{i+1}), $i \in \{0, 1, 2, \ldots, N-1\}$ we can write

$$E\left(\int_0^T v_t dt\right) = \sum_{i=0}^{N-1} E\left(\int_{t_i}^{t_{i+1}} v_t dt\right)$$

where

$$\begin{aligned}E\left(\int_{t_i}^{t_{i+1}} v_t dt\right) &= \frac{e^{-kt_i} - e^{-kt_{i+1}}}{k} \cdot \left(v_0 + \sum_{j=0}^{i-1} \theta_j \left(e^{kt_{j+1}} - e^{kt_j}\right)\right) + \\ &\quad + \frac{\theta_i}{k}\left(e^{-k(t_{i+1}-t_i)} - 1 + k\left(t_{i+1} - t_i\right)\right).\end{aligned}$$

In the case of the 3/2 model, Carr, Sun (2007) (see Theorem 4 therein) show that

$$E\left(\int_0^T v_t dt\right) = h\left(v_0 \int_0^T e^{k\int_0^t \theta(s)ds} dt\right)$$

where

$$h(y) = \int_0^y e^{-\frac{2}{\epsilon^2 z}} \cdot z^{\frac{2k}{\epsilon^2}} \cdot \int_z^\infty \frac{2}{\epsilon^2} e^{\frac{2}{\epsilon^2 u}} u^{\frac{-2k}{\epsilon^2}-2} du dz. \tag{2.6}$$

The integral appearing in the argument to the function $h(\cdot)$ is straightforward to compute for a piecewise constant $\theta(t)$:

$$\int_0^T e^{k\int_0^t \theta(s)ds} dt = \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} e^{k\int_0^t \theta(s)ds} dt$$

where

$$\int_{t_i}^{t_{i+1}} e^{k\int_0^t \theta(s)ds} dt = \frac{e^{k\theta_i(t_{i+1}-t_i)} - 1}{k\theta_i} \cdot \exp\left(k \sum_{j=0}^{i-1} \theta_j (t_{j+1} - t_j)\right).$$

However, the integral representation (2.6) of the function $h(\cdot)$ is hard to use for fast and accurate numerical implementations. We prove an alternative representation, based on a uniformly convergent series whose terms are easy to calculate and the total error can be controlled a priori. The result is formulated in Proposition (2.3.4) and Lemma (2.3.5).

Proposition 2.3.4 *The function $h(\cdot)$ admits the following uniformly convergent series representation*

$$h(y) = \alpha \cdot \left(\frac{E\left(\frac{\alpha}{y}\right)}{1-\beta} + \sum_{n=1}^{\infty} \frac{F_{\frac{\alpha}{y}}(n)}{n(n-\beta+1)}\right)$$

where

$$\begin{aligned} E(x) &= \int_x^\infty e^{-t} \cdot t^{-1} dt, x > 0 \\ F_\nu(n) &= P(Z \le n), \quad Z \sim \text{Poisson}(\nu) \\ \alpha &= \frac{2}{\epsilon^2} \\ \beta &= \frac{-2k}{\epsilon^2}. \end{aligned}$$

In Proposition (2.3.4) we recognize the special function $E(x)$ as the exponential integral which is readily accessible in any numerical package. The terms appearing in the infinite series are very fast and easy to compute. Moreover, as shown in Lemma (2.3.5) next, the total error arising from truncating the series can be controlled a priori.

Lemma 2.3.5 *The infinite series of Proposition (2.3.4) has a remainder term*

$$R_k = \sum_{n=k}^{\infty} \frac{F_{\frac{\alpha}{y}}(n)}{n(n-\beta+1)}$$

which is positive and satisfies the following bounds

$$\frac{F_{\frac{\alpha}{y}}(k)}{m+1}\left(\frac{1}{k}+\frac{1}{k+1}+\ldots+\frac{1}{k+m}\right) < R_k < \frac{1}{k}$$

where $m = \lceil -\beta \rceil$. If we let $\bar{R}$ the mid-point between the two bounds i.e.

$$\bar{R} = \frac{1}{2}\left(\frac{1}{k}+\frac{F_{\frac{\alpha}{y}}(k)}{m+1}\left(\frac{1}{k}+\frac{1}{k+1}+\ldots+\frac{1}{k+m}\right)\right).$$

then we also have

$$\left|R_k - \bar{R}\right| < \frac{m+\frac{\alpha}{y}}{4k^2}. \tag{2.7}$$

The application of Lemma (2.3.5) proceeds as follows. To compute $h(y)$, for a given y, use bound (2.7) to determine the number of terms needed to achieve the desired precision and then set

$$h(y) \approx \alpha \cdot \left(\frac{E\left(\frac{\alpha}{y}\right)}{1-\beta} + \sum_{n=1}^{k-1} \frac{F_{\frac{\alpha}{y}}(n)}{n(n-\beta+1)} + \bar{R}\right).$$

Having completed our discussion about the determination of the Laplace transform of the variance call function, we now turn to the problem of choosing a fast and accurate Laplace inversion algorithm. Many numerical Laplace inversion algorithms have been proposed in the literature; some important early contributions in this area include Weeks (1966), Dubner, Abate (1968), Stehfest (1970), Talbot (1979) and Abate, Whitt (1992). In what follows, we apply the very efficient algorithm recently proposed by Iseger (2006). Extensive analysis and numerical tests indicate that this algorithm is faster and more accurate than the other methods available. For a detailed treatment of the numerical and mathematical properties of this new method we refer the reader to Iseger (2006). We next outline the main steps of the method.

Suppose we want to recover the difference between the variance call functions $C - \tilde{C}$ at a sequence of variance strikes $k\Delta$, $k = 0, 1, \ldots, M-1$; let $g(k) = C(k\Delta) - \tilde{C}(k\Delta)$ and $\hat{g}$ the Laplace transform of g. The starting point of the method is the Poisson summation formula which states that, for any $v \in [0, 1)$, the following identity holds for the function g:

$$\sum_{k=-\infty}^{\infty} \hat{g}\left(a + 2\pi i(k+v)\right) = \sum_{k=0}^{\infty} e^{-ak} e^{-2\pi i k v} g(k) \tag{2.8}$$

where a is a positive damping factor. The Poisson summation formula applies to functions of bounded variation and in $L^1[0, \infty)$. To check these conditions for the function g, we derive the simple Lemma (2.3.6).

Lemma 2.3.6 *Let V be a random variable such that $V > 0$ a.s. and $E(V^2) < \infty$. Then the function $C : [0, \infty) \to \mathbb{R}$ defined by:*

$$C(K) = E(V - K)_+$$

belongs to $L^1[0, \infty)$ and is of bounded variation.

In both the Heston and the 3/2 model, the Laplace transform of integrated variance exists in a neighborhood of zero, which implies that all moments of integrated variance are finite. The same is true for our control variate distribution, Gamma. Applying Lemma (2.3.6), we conclude that functions $C(\cdot)$ and $\tilde{C}(\cdot)$ are in $L^1[0, \infty)$ and of bounded variation. It follows that the function g satisfies the same conditions.

Equation (2.8) relates an infinite sum of Laplace transform values (the LHS) to a dampened series of function values (the RHS). This result also forms the basis of the method developed by Abate, Whitt (1992). The series of Laplace transform values usually converges slowly and Abate, Whitt (1992) proposed a technique, known as Euler summation, to increase the rate of convergence for this series. Iseger (2006) proposes a completely different idea for handling the infinite series of Laplace transform values. It constructs a Gaussian quadrature rule for the series on the LHS of (2.8). Specifically, the infinite sum is approximated with a finite sum of the form

$$\sum_{k=1}^{n} \beta_k \cdot \hat{g}(a + i\lambda_k + 2\pi i v)$$

where β_k are the quadrature weights and λ_k are the quadrature points. The exact numbers β_k and λ_k can be found in Iseger (2006) (see Appendix A therein) for various values of n. It is found that a number of $n = 16$ quadrature points is sufficient for results attaining machine precision.

Having developed a fast and accurate approximation for the LHS of (2.8), we next turn to the dampened series of function values on the RHS. This series is much easier to handle. As shown in Iseger (2006) it is possible to choose the damping parameter a and a truncation rank M_2 to attain any desired level of truncation error. For double precision, the authors recommend truncating the series after $M_2 = 8M$ terms and using a dampening factor $a = 44/M_2$. Finally, applying the identity (2.8) repeatedly for all $v \in \{0, \frac{1}{M_2}, \frac{2}{M_2}, \dots, \frac{M_2-1}{M_2}\}$, we can recover each function value $g(k)$ by inverting the discrete Fourier series on the RHS as follows:

$$\frac{e^{ak}}{M_2} \cdot \sum_{j=0}^{M_2-1} \left[e^{2\pi i k \frac{j}{M_2}} \sum_{l=1}^{n} \beta_l \cdot \hat{g}\left(a + i\lambda_l + 2\pi i \frac{j}{M_2} \right) \right]. \tag{2.9}$$

An important advantage of this method is that these sums can all be calculated simultaneously for all $k \in \{0, 1, 2, \dots, M_2 - 1\}$ using the FFT algorithm of Cooley, Tukey (1965). In the end, we retain the first M values in which we are interested. For the FFT algorithm it is recommended that M be a power of 2. In the rest of the paper, we shall refer to the Iseger (2006) numerical inversion algorithm as the Gaussian-Quadrature-FFT algorithm or *GQ-FFT*, for short.

As an application of the tools developed so far, we now price options on realized variance in the Heston and 3/2 models. Similar to options on stocks, market practitioners express the prices of realized variance options in terms of Black-Scholes implied volatilities. Specifically, the undiscounted variance call price obtained from the model – Heston or 3/2, in our case – is matched to a Black-Scholes formula with zero rate and zero dividend yield:

$$C(K) = C(0)N(d_1) - KN(d_2)$$

where $C(0) = E\left(\frac{1}{T}\int_0^T v_t dt\right)$ is the fair variance as seen at time 0, and

$$\begin{aligned} d_1 &= \frac{\log\left(\frac{C(0)}{K}\right) + \frac{\xi^2 T}{2}}{\xi\sqrt{T}} \\ d_2 &= d_1 - \xi\sqrt{T}. \end{aligned}$$

are the usual Black-Scholes terms. The parameter ξ, ensuring the equality between the model price and the Black-Scholes price, will be called the implied *volatility of variance*, corresponding to strike K.

As a simple first numerical example, we take a standard choice for the Heston parameters from the existing literature; in an empirical investigation Bakshi, Cao,

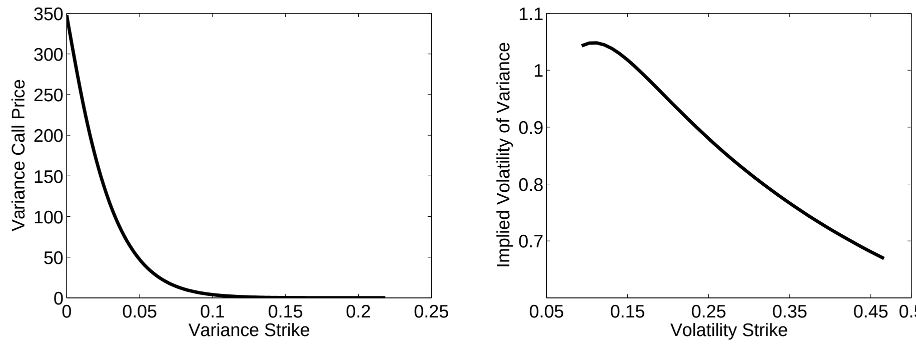

Figure 2.5: Left: Variance call prices as a function of variance strike, using Heston parameters from Bakshi et al. (1997) and a maturity of 6 months. Right: Implied volatility of variance as a function of volatility strike.

Chen (1997) estimated the following parameter set for the Heston model : $v_0 = 0.0348$, $k = 1.15$, $\theta = 0.0348$, and $\epsilon = 0.39$. Gatheral (2006) also uses the parameter set of Bakshi et al. (1997) to analyze prices of options on realized variance [2]. Here we apply our previous Laplace transform techniques to determine the prices of call options on 6-months realized variance. The left part of Figure (2.5) shows the (undiscounted) variance call function recovered over a wide strike range, from $K = 0$ to $K = 0.48^2$; the call prices have been scaled by a notional of $10,000$. A single run of the GQ-FFT algorithm computes the variance call prices for the entire sequence of strikes considered. As mentioned earlier, it is natural to convert these absolute prices to implied volatilities of variance. The right part of Figure (2.5) shows the implied volatilities of variance as a function of strike expressed as a volatility.

Compared to the spot market, we see that volatilities of variance can be several orders of magnitude higher than the volatility of the underlying stock or index. Depending on the volatility strike and maturity, it is common to see volatilities of variance in the range $[50\%, 150\%]$. For short maturities, the implied volatilities of variance increase very quickly; this makes trading sense, since, the shorter the period, the more uncertainty about the future realized variance. We refer the reader to Bergomi (2005, 2008) where many practical aspects of volatility markets are discussed. Figure (2.5) also reveals the main drawback of pricing volatility derivatives in the Heston model. We obtain a downward sloping smile for the volatility of variance whereas the slope is strongly positive in practice; see, for example, Bergomi (2008). From a trading and risk management perspective, it is clear that upside calls on variance should be more expensive, since, during periods of market stress when the volatility is high, the volatility of volatility is also very high. This behavior cannot be captured by the Heston model.

[2] Note that Gatheral (2006) use slightly modified values for v_0 and θ.

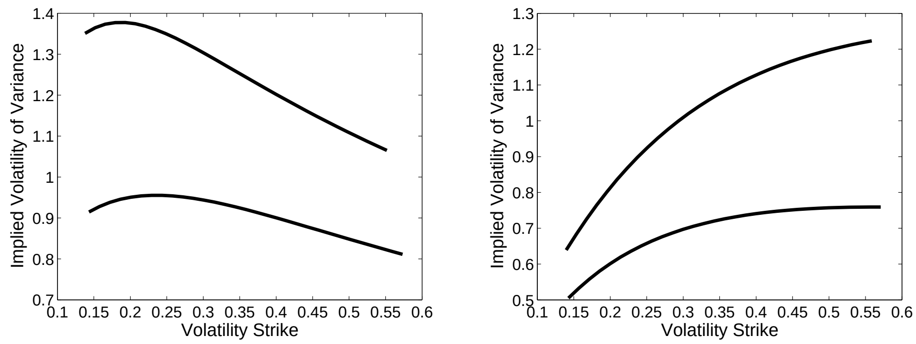

Figure 2.6: Implied volatility of variance as a function of volatility strike. Left: Heston model. Right: 3/2 model. Maturities of 3 months (higher curve) and 6 months (lower curve).

In Figure (2.6), we see the implied volatilities of variance in the Heston versus the 3/2 model, with the parameters calibrated in the previous section. We price 3-months and 6-months options on realized variance. Notice that, in both models, the volatilities are higher for 3-months variance than for 6-month variance; this is in line with our expectation. Most importantly, Figure (2.6) shows that, unlike the Heston model, the 3/2 model generates upward sloping volatility of variance smiles, thus capturing an important feature of the volatility derivatives market.

To further investigate the differences between the Heston and 3/2 model, we compare the densities of realized variance. Since the GQ-FFT algorithm gives us the variance call function for a sequence of strikes simultaneously, we can apply the well-known Breeden, Litzenberger (1978) formula to obtain the density of realized variance as follows:

$$\phi_{RV}(K) = \frac{\partial^2}{\partial K^2} C(K).$$

Applying this formula, we obtain in the left part of figure (2.7) the densities of the 3-months realized variance in the Heston and 3/2 models. We see that the well behaved and accurate prices obtained by our Laplace methods, allow us to get a smooth and positive density for a wide range of variance values (in this case, from $K = 0$ to $K = 55\%^2$). We notice that the 3/2 density is much more peaked and puts less weight for variance near zero. In the equity markets, the Heston fits often violate the zero boundary test, thus assigning significant probabilities to low variance scenarios. To better explain the downward / upward sloping volatility of variance smiles, we plot, in the right panel of figure (2.7), the density of the log realized variance return:

$$\log\left(\frac{\frac{1}{T}\int_0^T v_t dt}{C(0)}\right)$$

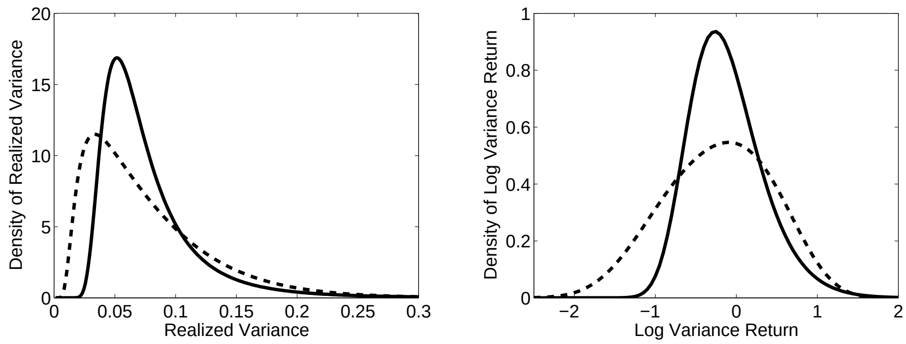

Figure 2.7: Left: Densities of 3 months realized variance. Right: Densities of 3 month log realized variance return. *Solid*: 3/2 model, *Dashed*: Heston model.

which we obtain from the density of the realized variance $\frac{1}{T}\int_0^T v_t dt$ as follows:

$$\phi_{LogRV}(k) = C(0) \cdot e^k \cdot \phi_{RV}(C(0)e^k).$$

We notice in the right panel of figure (2.7) that in the Heston model the variance log return is skewed to the left, while in the 3/2 model the variance log return is skewed to the right. This explains why the variance smile is downward sloping in the Heston model and upward sloping in the 3/2 model.

We conclude this section with a detailed explanation of the key advantages of our pricing method for options on realized variance, compared to a direct application of the standard Fourier methods from the literature. As seen next, recovering the variance call function $C(K)$ can, indeed, be easily written in terms of Fourier inversion. Specifically, since in both models — Heston (1993) and 3/2 — the Laplace transform $L(\lambda)$ of the variance call function $C(K)$ is well defined in the entire half-plane, for any $a \geq 0$ fixed, we can write:

$$L(a+iu) = \int_0^\infty e^{-iuK} \cdot e^{-aK} C(K) dK = \mathcal{F}\left\{e^{-aK} \cdot 1_{K\geq 0} \cdot C(K)\right\}(u)$$

where $u \in \mathbb{R}$ and $\mathcal{F}\{\cdot\}$ denotes the Fourier transform. Applying Fourier inversion, we can recover the variance call function at any strike $K \geq 0$ by

$$C(K) = \frac{e^{aK}}{2\pi} \int_{-\infty}^{\infty} e^{iuK} L(a+iu) du.$$

As in Carr, Madan (1999), since the function $C(K)$ is real, this can be written equivalently as

$$C(K) \quad = \quad \frac{e^{aK}}{\pi} Re\left\{ \int_0^\infty e^{iuK} L(a+iu) du \right\}. \tag{2.10}$$

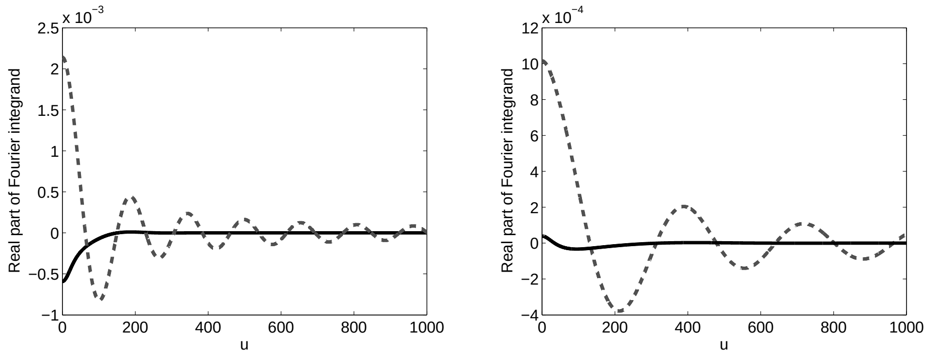

Figure 2.8: Real part of Fourier integrand without control variates (dashed gray) and with control variates (solid black) in the 3/2 model. Left: 6M maturity. Right: 3M maturity.

Hence, from a mathematical standpoint, the problem becomes identical to that of European vanilla options as treated in Carr, Madan (1999). The next step in the standard Fourier method is the truncation and then the discretization of the integral in (2.10). However, a key difference arising in our case is the singularity of the call function $C(K)$ at zero, which will cause the oscillations in the integrand $e^{iuK}L(a+iu)$ to decay slowly as $u \to \infty$ thus making the truncation of the integral a highly heuristic and unstable exercise. This singularity problem is circumvented in the case of European vanilla options by changing the variable from absolute strikes K to log-strikes $k = \log K$ and thus 'pushing' the singularity to $-\infty$. Suppose we tried to follow the same idea for variance call options and define:

$$c(k) = E\left(\frac{1}{T}\int_0^T v_t dt - e^k\right)_+$$

as in the standard method for European vanilla options. Taking $a \geq 0$ such that $e^{ak}c(k)$ is integrable on $\mathbb{R}$, the main result in Carr, Madan (1999) would give us a closed-form relation between the Fourier transform of $e^{ak}c(k)$ and the Fourier transform of log-realized variance $\log\left(\frac{1}{T}\int_0^T v_t dt\right)$. However, in both the Heston (1993) model and the 3/2 model, the Fourier transform of log-realized variance is not available in closed-form. We have closed-form expressions for the Fourier transform of realized variance, but not for the log-realized variance. Therefore a direct application of the standard Fourier methods of Carr, Madan (1999) will not be possible for options on realized variance.

If we, nevertheless, proceed by truncating the integral in (2.10), the truncation problem can be significantly alleviated by applying the idea of control variates. As can be seen in figure (2.8), the integrand will decay much faster making the truncation more stable. In our opinion, for options on realized variance, it is not

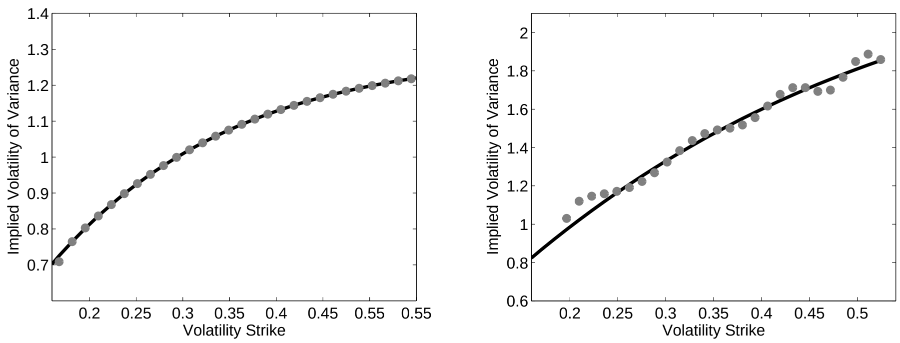

Figure 2.9: Left: 3M options on realized variance by GQ-FFT (solid black) and by standard Fourier inversion with Gamma control variates (gray dots) in the 3/2 model. Right: Same for 1M maturity.

possible to robustly truncate the integral in (2.10) without control variates. The idea of control variates, for European vanilla options, was first proposed in Andersen, Andreasen (2002) and further discussed in Cont, Tankov (2004). However, even with the help of control variates, the truncation bound for (2.10) will still retain some sensitivity to other factors such as the maturity of the option. Specifically, choosing a truncation bound which works well for a certain maturity will not be appropriate for another maturity; figure (2.9) illustrates this phenomenon. The sensitivity of the truncation bound is particularly severe for short maturities.

Similar to the seminal article of Abate, Whitt (1992), the new Laplace inversion algorithm in Iseger (2006) is a Fourier based inversion method. As seen on the LHS of equation (2.8), it involves Laplace transform values on the fixed line $a + iu$, $u \in \mathbb{R}$; as mentioned above, this is the same as looking at the Fourier transform of the dampened call price $e^{-aK} \cdot 1_{K \geq 0} \cdot C(K)$. However, the key advantage over the standard Fourier methods is that it completely eliminates the need to perform any (semi-heuristic) truncations. It achieves this by the simple, but powerful, idea of constructing a Gaussian quadrature for the series of Laplace transform (or Fourier transform) values. Numerical tests for an extensive list of functions can be found in Iseger (2006). It provides conclusive evidence that the new algorithm is faster and more accurate than previously developed methods. In the calculations reported in this section, we considered both the rather mild (at least, for equity markets) parameter set of Bakshi, Cao, Chen (1997) (with a volatility-of-volatility $\epsilon = 0.39$) as well as the more extreme parameter set (with a volatility-of-volatility $\epsilon = 0.9288$) fitted to the July, 31st, 2009 volatility smile. For all maturities and both models, we considered very wide ranges of volatility strikes from 50% to 200% of the ATM volatility. Moreover, the well behaved numerical results allowed us to differentiate twice the variance call prices, to yield the full smooth density of realized variance in both models. In all cases, we found the new pricing method based on the Gaussian Quadrature - Fast Fourier Transform (GQ-FFT) to be accurate and stable.

2.4 Hedge ratios for options on realized variance

The Laplace transform method developed in the previous section can also be applied to compute hedge ratios for options on realized variance. The natural hedging instruments for options on realized variance are their underlying variance swaps. Similar to options on stocks, we need to determine a delta, here with respect to variance swaps. As shown in Broadie, Jain (2008), one can derive that the correct amount of delta, for a variance call of strike K, is given by:

$$\Delta_{VS} = \frac{\frac{\partial}{\partial v_0} C(K)}{\frac{\partial}{\partial v_0} C(0)}$$

where v_0 is the current value of the short variance process and $C(0) = E\left(\frac{1}{T}\int_0^T v_t dt\right)$ is the current fair variance swap rate for maturity T. The formula is intuitively clear: to hedge against the randomness in the short variance process, one needs to look at the ratio of the sensitivities of the two instruments — option on variance and variance swap — with respect to the value of short variance. From the inversion equation (2.9) of the previous section, it can be seen that in order to derive $\frac{\partial}{\partial v_0} C(K)$, we need to have the expressions for $\frac{\partial}{\partial v_0}\mathcal{L}(\lambda)$ in both models, where $\mathcal{L}(\lambda)$ is the Laplace transform of annualized realized variance. From section (2), $\mathcal{L}(\lambda)$ is obtained from Propositions (2.2.1) and (2.2.2) by setting $t = 0$, $u = 0$ and $\lambda = \frac{\lambda}{T}$.

From Proposition (2.2.1), we obtain for the Heston model :

$$\frac{\partial}{\partial v_0}\mathcal{L}(\lambda) = -b(0,T) \cdot \exp\left(a\left(0,T\right) - b\left(0,T\right) v_0\right)$$

with $a(0,T)$ and $b(0,T)$ as defined in Proposition (2.2.1). Similarly, using Proposition (2.2.2) and the property of the confluent hypergeometric function

$$\frac{\partial}{\partial z} M(\alpha,\gamma,z) = \frac{\alpha}{\gamma} M(\alpha+1,\gamma+1,z)$$

we obtain for the 3/2-model :

$$\frac{\partial}{\partial v_0}\mathcal{L}(\lambda) = -\frac{\alpha z^{\alpha}}{v_0}\frac{\Gamma\left(\gamma-\alpha\right)}{\Gamma\left(\gamma\right)}\left[M\left(\alpha,\gamma,-z\right) - \frac{z}{\gamma} M\left(\alpha+1,\gamma+1,-z\right)\right]$$

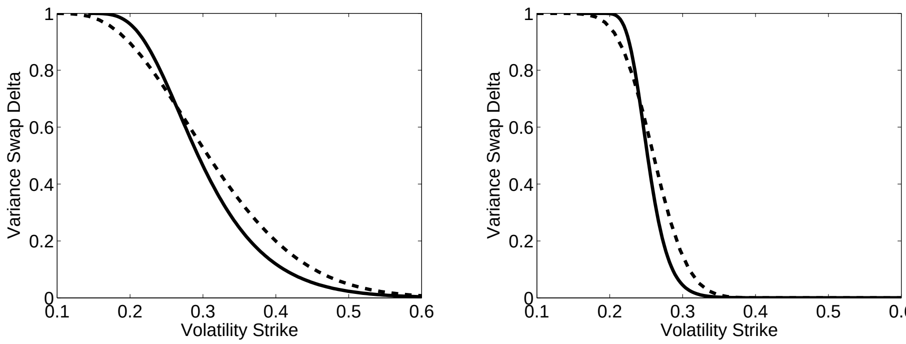

Figure 2.10: Variance swap delta for call options on realized variance with six months (left) and one week (right) to expiry, as a function of strike expressed as a volatility; *Solid*: 3/2 model, *Dashed*: Heston model.

with α, γ as defined in Proposition (2.2.2) and

$$z = \frac{2}{\epsilon^2 y(0,T)}.$$

Using these results, we can apply the inversion tools developed previously to obtain the variance swap deltas for call options on realized variance at a sequence of strikes simultaneously. In Figure (2.10) we show the results for the Heston and 3/2 models, calibrated in section 2.2. As expected, we notice a behavior of the variance swap delta similar to the delta of vanilla stock options. As we move from deep in the money calls to deep out of the money calls, the variance swap delta smoothly decreases from 1 to 0. Besides the six months maturity, we also show, in the right panel of Figure (2.10), a short maturity of one week and notice that delta approaches the expected digital behavior near expiry.

2.5 Conclusion

We have developed a fast and robust method for determining prices and hedge ratios for options on realized variance applicable in any model where the Laplace transform of realized variance is available in closed form. The method was used to price options on realized variance in the 3/2 stochastic volatility model and in the Heston (1993) model. It has been shown that the 3/2 model offers several advantages for trading and risk managing volatility derivatives. Unlike the 3/2 model, the Heston model assigns significant weight to very low and vanishing volatility scenarios and is unable to produce extreme paths with very high volatility of volatility. Most importantly, the 3/2 model generates upward sloping implied volatility of variance smiles — in agreement with the way variance options are traded in practice. Finally, we have shown that the transform methods can be efficiently used to obtain hedge ratios for options on realized variance.

2.6 Appendix

Proof of Proposition (2.2.1) We are in a special case of the general multi-factor affine framework introduced by Duffie, Pan, Singleton (2000). With $X_t = \log\left(S_t e^{(r-\delta)(T-t)}\right)$ denoting the log forward process, we have:

$$\begin{aligned} dX_t &= -\frac{v_t}{2}dt + \sqrt{v_t}dB_t \\ dv_t &= k\left(\theta(t) - v_t\right)dt + \epsilon\sqrt{v_t}dW_t. \end{aligned}$$

The state vector (X_t, v_t) is a two-dimensional affine process as defined in Duffie et. al (2000). Let

$$\psi\left(X_t, v_t, t\right) = E\left(e^{iuX_T - \lambda\int_t^T v_s ds}\middle| X_t, v_t\right)$$

denote the joint Fourier-Laplace transform of X_T and the de-annualized integrated variance $\int_t^T v_s ds$. Observing that the process

$$e^{-\lambda\int_0^t v_s ds} \cdot \psi\left(X_t, v_t, t\right) = E\left(e^{iuX_T - \lambda\int_0^T v_s ds}\middle| X_t, v_t\right)$$

is a martingale, an application of Ito's Lemma gives the following partial differential equation for $\psi(x, v, t)$:

$$\frac{1}{2}\epsilon^2 v\psi_{vv} + k\left(\theta(t) - v\right)\psi_v + \epsilon\rho v\psi_{xv} - \frac{1}{2}v\psi_x + \frac{1}{2}v\psi_{xx} - \lambda v\psi + \psi_t = 0$$

with terminal condition

$$\psi(x, v, T) = e^{iux}.$$

Looking for a solution of the form

$$\psi(x, v, t) = \exp\left(iux + a(t, T) - b(t, T)v\right)$$

leads to the ODEs for $a(\cdot, T)$ and $b(\cdot, T)$:

$$\begin{aligned} b' &= \frac{1}{2}\epsilon^2 b^2 + \left(k - i\epsilon\rho u\right)b - \left(\frac{1}{2}iu + \frac{1}{2}u^2 + \lambda\right) \\ a' &= k\theta(t)b \end{aligned}$$

with terminal condition $a(T,T) = b(T,T) = 0$, and $a^{'}$, $b^{'}$ denoting derivatives with respect to t. The complex valued ODE for $b(\cdot, T)$ can be solved in closed form; see Cox et. al (1985) or Heston (1993). In our case, the solution for $b(\cdot, T)$ is

$$b(t,T) = \frac{(iu + u^2 + 2\lambda)\left(e^{\gamma(T-t)} - 1\right)}{(\gamma + k - i\epsilon\rho u)\left(e^{\gamma(T-t)} - 1\right) + 2\gamma}$$

with

$$\gamma = \sqrt{(k - i\epsilon\rho u)^2 + \epsilon^2 (iu + u^2 + 2\lambda)}.$$

Once the function $b(\cdot, T)$ is known, the ODE for $a(\cdot, T)$ gives:

$$a(t,T) = -\int_t^T k\theta(s) b(s,T) ds.$$

$\square$

Proof of Lemma (2.3.2) Rewrite the limit as

$$\lim_{K \downarrow 0} \frac{C(K) - C(0)}{K} = \lim_{K \downarrow 0} \frac{E(V-K)_+ - E(V)}{K} = \lim_{K \downarrow 0} E\left(\frac{(V-K)_+ - V}{K}\right)$$

Since $V > 0$ a.s. we have

$$\lim_{K \downarrow 0} \frac{(V-K)_+ - V}{K} = \lim_{K \downarrow 0} \frac{V - K - V}{K} = -1 \qquad \text{a.s.}$$

Using the obvious bound

$$\left| \frac{(V-K)_+ - V}{K} \right| \leq 1$$

we can apply the dominated convergence theorem to interchange the order of limit and expectation

$$\lim_{K \downarrow 0} E\left(\frac{(V-K)_+ - V}{K}\right) = E\left(\lim_{K \downarrow 0} \frac{(V-K)_+ - V}{K}\right) = E(-1) = -1.$$

$\square$

Proof of Lemma (2.3.3) This result follows easily by direct integration. For $V \sim \text{Gamma}(\alpha, \beta)$, we have

$$\tilde{C}(K) = E(V-K)_+ = \int_K^\infty (x-K) \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}} dx$$

Computing separately the integral corresponding to each term in the parentheses, we obtain

$$\int_K^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} x^\alpha e^{-\frac{x}{\beta}} dx = \int_K^\infty \frac{\alpha\beta}{\Gamma(\alpha+1)\beta^{\alpha+1}} x^\alpha e^{-\frac{x}{\beta}} dx = \alpha\beta\left(1 - F(K; \alpha+1, \beta)\right)$$

and
$$\int_K^\infty \frac{K}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}} dx = K\left(1 - F(K;\alpha,\beta)\right)$$
where $F(\cdot,\alpha,\beta)$ is the CDF of the Gamma(α,β) distribution. $\square$

Proof of Proposition (2.3.4) In
$$h(y) = \int_0^y e^{-\frac{2}{\epsilon^2 z}} \cdot z^{\frac{2k}{\epsilon^2}} \cdot \int_z^\infty \frac{2}{\epsilon^2} \cdot e^{\frac{2}{\epsilon^2 u}} \cdot u^{\frac{-2k}{\epsilon^2}-2} du dz.$$
denote $\alpha = \frac{2}{\epsilon^2} > 0$, $\beta = \frac{-2k}{\epsilon^2} < 0$ and rewrite the integral as
$$\alpha \int_0^y e^{-\frac{\alpha}{z}} z^{-\beta} \int_z^\infty e^{\frac{\alpha}{u}} u^{\beta-2} du dz = \alpha \int_0^y e^{-\frac{\alpha}{z}} \int_z^\infty e^{\frac{\alpha}{u}} \left(\frac{z}{u}\right)^{-\beta} u^{-2} du dz.$$
In the inner integral, making the change of variable $\frac{z}{u} = t$ we obtain
$$\int_z^\infty e^{\frac{\alpha}{u}} \left(\frac{z}{u}\right)^{-\beta} u^{-2} du = \frac{1}{z} \int_0^1 e^{\frac{\alpha t}{z}} t^{-\beta} dt.$$
Expanding $e^{\frac{\alpha t}{z}}$ in a series gives
$$\frac{1}{z} \int_0^1 \sum_{n=0}^\infty \frac{1}{n!} \left(\frac{\alpha}{z}\right)^n t^{n-\beta} dt = \sum_{n=0}^\infty \frac{1}{n!} \frac{\alpha^n}{z^{n+1}} \frac{1}{n-\beta+1}$$
where we interchanged integration and summation as all terms are non-negative.
$$h(y) = \alpha \int_0^y e^{-\frac{\alpha}{z}} \sum_{n=0}^\infty \frac{1}{n!} \frac{\alpha^n}{z^{n+1}} \frac{1}{n-\beta+1} dz = \alpha \sum_{n=0}^\infty \frac{1}{n!(n-\beta+1)} \int_0^y \frac{\alpha^n e^{-\frac{\alpha}{z}}}{z^{n+1}} dz$$
Making the change of variable $\frac{\alpha}{z} = t$ we obtain
$$\int_0^y \frac{\alpha^n e^{-\frac{\alpha}{z}}}{z^{n+1}} dz = \int_{\frac{\alpha}{y}}^\infty e^{-t} \cdot t^{n-1} dt.$$
For $n = 0$ we recognize the exponential integral function
$$E\left(\frac{\alpha}{y}\right) = \int_{\frac{\alpha}{y}}^\infty e^{-t} \cdot t^{-1} dt$$
and for $n \geq 1$ we obtain the upper incomplete Gamma function which satisfies
$$\int_{\frac{\alpha}{y}}^\infty e^{-t} \cdot t^{n-1} dt = (n-1)! \cdot e^{-\frac{\alpha}{y}} \cdot \sum_{k=0}^n \frac{\left(\frac{\alpha}{y}\right)^k}{k!} = (n-1)! \cdot F_{\frac{\alpha}{y}}(n)$$

where $F_\nu(\cdot)$ is the CDF of a Poisson random variable of parameter ν. Collecting the previous results we obtain

$$h(y) = \alpha \cdot \left(\frac{E\left(\frac{\alpha}{y}\right)}{1-\beta} + \sum_{n=1}^{\infty} \frac{F_{\frac{\alpha}{y}}(n)}{n(n-\beta+1)} \right)$$

Uniform convergence follows from the bound

$$\sum_{n=k}^{\infty} \frac{F_{\frac{\alpha}{y}}(n)}{n(n-\beta+1)} < \sum_{n=k}^{\infty} \frac{1}{n(n-\beta+1)} < \sum_{n=k}^{\infty} \frac{1}{n(n+1)} = \frac{1}{k}.$$

□

Proof of Lemma (2.3.5) The upper bound has been derived in the proof of Proposition (2.3.4). If we let $m = \lceil -\beta \rceil$ (i.e. the smallest integer greater than or equal to $-\beta$), we can write

$$\begin{aligned} R_k = \sum_{n=k}^{\infty} \frac{F_{\frac{\alpha}{y}}(n)}{n(n-\beta+1)} &> F_{\frac{\alpha}{y}}(k) \sum_{n=k}^{\infty} \frac{1}{n(n-\beta+1)} \\ &\geq \frac{F_{\frac{\alpha}{y}}(k)}{m+1} \sum_{n=k}^{\infty} \frac{m+1}{n(n+m+1)} \\ &= \frac{F_{\frac{\alpha}{y}}(k)}{m+1} \left(\frac{1}{k} + \frac{1}{k+1} + \ldots + \frac{1}{k+m} \right). \end{aligned}$$

Let $\bar{R}$ denote the mid-point between the two bounds i.e.

$$\bar{R} = \frac{1}{2} \left(\frac{1}{k} + \frac{F_{\frac{\alpha}{y}}(k)}{m+1} \left(\frac{1}{k} + \frac{1}{k+1} + \ldots + \frac{1}{k+m} \right) \right).$$

and let $p = F_{\frac{\alpha}{y}}(k)$. We then have that $|R_k - \bar{R}|$ must be less than or equal to half the difference between the two bounds i.e.

$$\begin{aligned} |R_k - \bar{R}| &\leq \frac{1}{2} \left(\frac{1}{k} - \frac{p}{m+1} \left(\frac{1}{k} + \frac{1}{k+1} + \ldots + \frac{1}{k+m} \right) \right) \\ &= \frac{1}{2(m+1)} \sum_{j=0}^{m} \frac{j + (1-p)k}{k(k+j)} \\ &< \frac{1}{2k^2(m+1)} \left(\frac{m(m+1)}{2} + (1-p)k(m+1) \right) \\ &= \frac{m}{4k^2} + \frac{1-p}{2k}. \end{aligned}$$

Observing that

$$\begin{aligned}
1-p &= e^{-\frac{\alpha}{y}} \cdot \sum_{n=k+1}^{\infty} \frac{\left(\frac{\alpha}{y}\right)^n}{n!} \\
&= e^{-\frac{\alpha}{y}} \cdot \sum_{n=k+1}^{\infty} \frac{\frac{\alpha}{y}\left(\frac{\alpha}{y}\right)^{n-1}}{n \cdot (n-1)!} \\
&< \frac{\frac{\alpha}{y}}{k} e^{-\frac{\alpha}{y}} \cdot \sum_{n=k}^{\infty} \frac{\left(\frac{\alpha}{y}\right)^n}{n!} \\
&< \frac{\frac{\alpha}{y}}{k}
\end{aligned}$$

we finally obtain

$$|R_k - \bar{R}| < \frac{m + 2\frac{\alpha}{y}}{4k^2}.$$

□

Proof of Lemma (2.3.6) Bounded variation follows immediately by observing that $C(K) = E(V-K)_+$ is a monotone decreasing function of K. Next we check that $C(K) \in L^1[0,\infty)$ i.e.

$$\int_0^\infty C(K)dK = \int_0^\infty E(V-K)_+ dK < \infty.$$

Since the integrand is positive, we can apply Fubini to change the order of integration and expectation:

$$\begin{aligned}
\int_0^\infty E(V-K)_+ dK &= E\left(\int_0^\infty (V-K)_+ dK\right) \\
&= E\left(\int_0^V V - K dK\right) \\
&= E\left(\frac{V^2}{2}\right) < \infty.
\end{aligned}$$

□

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III

Options on realized variance in Log-OU models

Gabriel G. Drimus

Abstract

We consider the pricing of options on realized variance in a general class of Log-OU stochastic volatility models. The class includes several important models proposed in the literature. Having as common feature the log-normal law of instantaneous variance, the application of standard Fourier-Laplace transform methods is not feasible. By extending Asian pricing methods, we obtain bounds, in particular, a very tight lower bound for options on realized variance, similar to an idea first introduced in Rogers, Shi (1995).

3.1 Introduction

We introduce a general class of Log-OU stochastic volatility models and study the pricing of options on realized variance under such models. This framework includes, for example, Scott's (1987) model and the continuous version of Bergomi's (2005) model. A desirable property of this class of models is that the distribution of realized variance will be approximately log-normal. Many authors including Ahmad, Wilmott (2005), Bergomi (2005) and Gatheral (2006) have found that, in practice, the distribution of realized variance is close to log-normal.

Despite its numerical tractability, the popular Heston (1993) model has the drawback that it generates a downward sloping volatility-of-volatility skew, a feature which is at odds with variance markets in practice. The class of Log-OU models discussed in this paper will generate an approximately flat, in fact mildly upward sloping, volatility-of-volatility skew thus providing an improvement over the traditional Heston (1993) model.

The class of models considered have the common property that the marginal distribution of instantaneous variance is *log-normal*. By the well-known fact that a sum of log-normal variables does not remain log-normal, the realized variance — defined as an integral over the instantaneous variances — will not be exactly log-normal and, more importantly, will not have a moment generating function (or Laplace transform) in closed-form. Therefore familiar Fourier-Laplace transform methods cannot be applied to value options on realized variance. Motivated by the similarities between options on realized variance and Asian options, we extend the classical Asian bounds of Rogers, Shi (1995) and Thompson (1999) to Log-OU processes. In particular, we obtain a very tight lower bound, which can essentially be used as the true price.

The remaining of this paper is organized as follows. In the next section we introduce the class of Log-OU models and discuss some of their general properties. Section three is the main section, which develops the bounds for options on realized variance and illustrates their numerical performance. The final section summarizes the conclusions.

3.2 A class of Log-OU Models

We start by specifying the Log-OU stochastic volatility dynamics considered throughout the paper. Let $(B_t, W_t)_{t\geq 0}$ be a standard two-dimensional Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})$ satisfying the usual conditions. We assume that the stock price and its instantaneous variance $(S_t, v_t)_{t\geq 0}$ satisfy the following dynamics under the risk neutral measure $\mathbb{Q}$:

$$\frac{dS_t}{S_t} = (r-\delta)dt + \sqrt{v_t}\left(\rho dW_t + \sqrt{1-\rho^2}dB_t\right) \tag{3.1}$$

$$d\log(v_t) = \Big[k\left(\log(\theta(t)) - \log(v_t)\right) + \chi(t)\Big]dt + \epsilon dW_t \tag{3.2}$$

where $\theta(t)$ and $\chi(t)$ are arbitrary deterministic functions. The risk free interest rate, dividend yield and correlation parameters are denoted r, δ and ρ respectively. We recognize that the logarithm of the instantaneous variance $y_t = \log(v_t)$ follows a Gaussian Ornstein-Ühlenbeck process with extra drift term $\chi(t)$.

The simplest special case of (3.2) is obtained for $\theta(t) = \theta$ (a constant) and $\chi(t) = 0$, which yields the classical model of Scott (1987). More recently, we find in Bergomi (2005) an interesting variation where one starts by specifying the dynamics of the instantaneous forward variances $\xi_t^T = E^{\mathbb{Q}}\left(v_T|\mathcal{F}_t\right)$, similar to HJM forward interest rate modeling, as follows:

$$d\xi_t^T = \epsilon \cdot \xi_t^T \cdot e^{-k(T-t)}dW_t \tag{3.3}$$

along with an initial instantaneous forward variance curve ξ_0^T, for all $T > 0$. We now show that (3.3) is also a Log-OU model. Assuming the initial forward variance curve is sufficiently smooth (more precisely, differentiable in T), we start by defining the function $\theta(\cdot)$ as:

$$\log\left(\theta(T)\right) = \log \xi_0^T + \frac{1}{k} \cdot \frac{\partial \log \xi_0^T}{\partial T}.$$

A straightforward application of Itô's lemma to the logarithm of the instantaneous variance $y_t = \log(\xi_t^t)$ reveals the dynamics:

$$dy_t = \left[k\left(\log(\theta(t)) - y_t\right) - \frac{\epsilon^2}{4}(1 + e^{-2kt}) \right] dt + \epsilon dW_t \tag{3.4}$$

which we recognize as a special case of (3.2) with $\chi(t) = -\frac{\epsilon^2}{4}(1 + e^{-2kt})$. This parametrization may appear less intuitive, but has the advantage that it provides a perfect fit to any initial (forward) variance swap curve and it makes the dynamics of forward variances (3.3) simpler and cleaner.

In what follows, we will find it convenient to write the logarithm of the instantaneous variance y_t as $y_t = \bar{y}_t + Z_t$ where $\bar{y}_t$ is the (time dependent) mean of y_t and Z_t is a centered Gaussian-OU process given by

$$Z_t = \epsilon \cdot e^{-kt} \cdot \int_0^t e^{ks} dW_s. \tag{3.5}$$

Applying Itô's lemma to $e^{kt} \cdot y_t$ and using the Log-OU dynamics (3.2) we obtain, for the deterministic component $\bar{y}_t$:

$$\bar{y}_t = y_0 e^{-kt} + k e^{-kt} \int_0^t e^{ku} \log\left(\theta(u)\right) du + e^{-kt} \int_0^t e^{ku} \chi(u) du. \tag{3.6}$$

For Scott's (1987) model, the expression (3.6) simplifies to

$$\bar{y}_t = \log(\theta) + \left(y_0 - \log(\theta)\right) e^{-kt}$$

and for Bergomi's (2005) model we have

$$\bar{y}_t = y_0 e^{-kt} + k e^{-kt} \int_0^t e^{ku} \log\left(\theta(u)\right) du - \frac{\epsilon^2}{4k}\left(1 - e^{-2kt}\right).$$

In Log-OU stochastic volatility models of the form (3.1)-(3.2), there is no fast analytical method to compute prices of European vanilla options. Nevertheless, efficient one-dimensional Monte Carlo pricing can be obtained by using the so-called "mixing" approach (see Romano, Touzi (1997) or Lewis (2000)). We start with a simple application of Itô's lemma to $\log(S_t)$ on the interval $[0, T]$ to obtain:

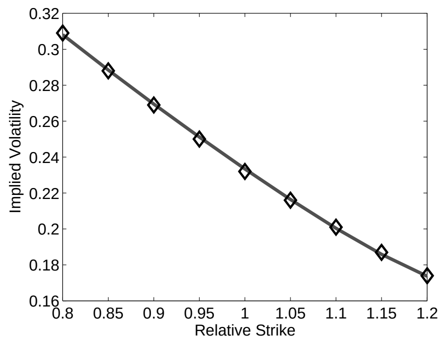

Figure 3.1: Fit of Scott (1987) model to 6M implied volatilities of S&P500 index options on December 14, 2009. *Diamond Black*: Market implied volatilities. *Solid grey*: Scott's model implied volatility curve. Parameters obtained are: $v_0 = 19.85\%^2$, $k = 1.786$, $\theta = 26.32\%^2$, $\epsilon = 2.19$, $\rho = -0.84$.

$$
\begin{aligned}
S_T &= S_0 \cdot \exp\left(-\frac{1}{2}\rho^2 \int_0^T e^{y_t} dt + \rho \int_0^T \sqrt{e^{y_t}} dW_t\right) \\
&\quad \cdot \exp\left((r-\delta)T - \frac{1}{2}(1-\rho^2) \int_0^T e^{y_t} dt + \sqrt{1-\rho^2} \int_0^T \sqrt{e^{y_t}} dB_t\right).
\end{aligned}
$$

By conditioning on the path of the Brownian Motion W_t, driving the instantaneous variance, we obtain the price of any European option as an expectation over Black-Scholes prices. If we let $C^{BS}(S, \sigma)$ denote the Black-Scholes price of a European vanilla option, for initial spot S and constant volatility σ, we arrive at the following mixing representation for option prices in our stochastic volatility models:

$$
C^{SV} = E^{\mathbb{Q}}\left(C^{BS}\left(S_0 \cdot \xi_T, \sqrt{1-\rho^2}\sqrt{\frac{1}{T}\int_0^T e^{y_t} dt}\right)\right)
$$

where

$$
\xi_T = \exp\left(-\frac{1}{2}\rho^2 \int_0^T e^{y_t} dt + \rho \int_0^T \sqrt{e^{y_t}} dW_t\right).
$$

To obtain a set of parameter values for our subsequent numerical examples, we end this section with a simple fit of the Scott (1987) model to market prices of 6-months S&P500 index options on Dec 14, 2009; figure (3.1) displays the fit. The parameters obtained are : $v_0 = 19.85\%^2$, $k = 1.786$, $\theta = 26.32\%^2$, $\epsilon = 2.19$, $\rho = -0.84$. We shall use this example parameter set to illustrate the numerical results in the next section.

3.3 Options on realized variance in Log-OU models

We now consider the problem of pricing options on realized variance in the general class of Log-OU stochastic volatility models (3.1)-(3.2). The payoff of a call option on realized variance, with strike K and maturity T, is given by:

$$\left(\frac{1}{T}\int_0^T v_t dt - K\right)_+ . \tag{3.7}$$

In stochastic volatility models, such as Heston (1993), where the Fourier-Laplace transform

$$E^{\mathbb{Q}}\left(\exp\left(-s \cdot \frac{1}{T}\int_0^T v_t dt\right)\right)$$

with $s \in \mathbb{C}$ and $\mathrm{Re}(s) \geq s^* \geq 0$ is known in closed form, semi-analytical transform techniques can be applied to value options on realized variance (see, for example, Sepp (2008)). In the class of Log-OU models the transform is not available in closed form.

In what follows, we shall regard the payoff (3.7) as the payoff of an Asian option written on the instantaneous variance process v_t. Of course, a direct valuation approach is to construct a Monte Carlo scheme. Similar to arithmetic average Asian stock options in Kemma, Vorst (1990), we can implement a Monte Carlo approach, enhanced with geometric average control variates. Specifically, recalling that $v_t = \exp(\bar{y}_t + Z_t)$, in order to value

$$C_{var}(K) = e^{-rT} E^{\mathbb{Q}}\left(\frac{1}{T}\int_0^T e^{\bar{y}_t + Z_t} dt - K\right)_+$$

we can use the control variate

$$C_{var}^{Geom.}(K) = e^{-rT} E^{\mathbb{Q}}\left(\exp\left(\frac{1}{T}\int_0^T (\bar{y}_t + Z_t) dt\right) - K\right)_+ .$$

The latter can be valued with a Black-Scholes style formula as provided in Appendix A.1.

Subsequently, Rogers, Shi (1995) and then Thompson (1999), developed elegant methods to bound the prices of arithmetic average Asian options. In particular, they discovered an extremely tight lower bound, which follows from a simple conditioning argument. Throughout the Asian pricing literature, the log-stock price was assumed in the classical Black-Scholes model:

$$\log(S_t) = \log(S_0) + \left(r - \delta - \frac{\sigma^2}{2}\right) t + \sigma Z_t$$

where Z_t is a standard Brownian motion. Note that, for our "Asian" option with payoff (3.7), $\log(v_t)$ has a similar form, namely $\log(v_t) = \bar{y}_t + Z_t$, except in this case, Z_t is a mean-reverting Ornstein-Ühlenbeck process. It is not clear that the classical Asian methods can be extended to this setting.

We next show how to extend the Asian bounds to a mean-reverting process Z_t and obtain, in particular, a very tight lower bound for options on realized variance in the general class of Log-OU stochastic volatility models (3.1)-(3.2). The lower bound is so accurate that, in practice, it could be used as a substitute for the true price.

The basic idea rests on the following straightforward inequalities: for any measurable event $A \in \mathcal{F}_T$ we have

$$\begin{aligned} E^{\mathbb{Q}}\left(\frac{1}{T}\int_0^T v_t dt - K\right)_+ &\geq E^{\mathbb{Q}}\left[\left(\frac{1}{T}\int_0^T v_t dt - K\right)_+ \cdot 1_A\right] \\ &\geq E^{\mathbb{Q}}\left[\left(\frac{1}{T}\int_0^T v_t dt - K\right) \cdot 1_A\right] \\ &= \frac{1}{T}\int_0^T E^{\mathbb{Q}}\left[(v_t - K)\cdot 1_A\right] dt. \end{aligned} \tag{3.8}$$

Next, we take events $A \in \mathcal{F}_T$ of the form $A = \{Z > \zeta\}$ with $\zeta \in \mathbb{R}$ a constant and

$$Z = \frac{\int_0^T Z_t dt}{\sqrt{\operatorname{Var}\int_0^T Z_t dt}} \sim N(0,1)$$

where Z_t is the zero-mean Gaussian-OU process defined by equation (3.5). To search for $\zeta \in \mathbb{R}$ which yields the highest lower bound in (3.8), we have to solve the optimization problem

$$\max_{\zeta\in\mathbb{R}} \; f(\zeta) = \frac{1}{T}\int_0^T E^{\mathbb{Q}}\Big[(v_t - K)\cdot 1_{\{Z>\zeta\}}\Big] dt.$$

Letting $g(t,\zeta) = E^{\mathbb{Q}}\Big[(v_t - K)\cdot 1_{\{Z>\zeta\}}\Big]$ denote the integrand in the expression of $f(\cdot)$ above, we begin by noting the identity below, which follows from the law of total probability:

$$g(t,\zeta) = \int_\zeta^\infty E^{\mathbb{Q}}\left(v_t - K \middle| Z = z\right)\cdot \phi(z) dz$$

where $\phi(\cdot)$ is the standard normal density. This allows us to easily see that

$$\frac{\partial g}{\partial \zeta}(t,\zeta) = -E^{\mathbb{Q}}\left(v_t - K \middle| Z = \zeta\right)\cdot \phi(\zeta).$$

It follows that the necessary maximum condition on ζ becomes

$$f^{'}(\zeta) = \frac{1}{T}\int_0^T \frac{\partial g}{\partial \zeta}(t,\zeta)dt = -\frac{\phi(\zeta)}{T}\int_0^T E^{\mathbb{Q}}\left(v_t - K\Big|Z=\zeta\right)dt = 0$$

or, equivalently

$$\int_0^T E^{\mathbb{Q}}\left(v_t\Big|Z=\zeta\right)dt = KT \tag{3.9}$$

where we recall that $v_t = \exp(\bar{y}_t + Z_t)$. Next, we want to rewrite equation (3.9) for ζ in a more explicit way. Computing the conditional expectations $E^{\mathbb{Q}}\left(\exp(\bar{y}_t + Z_t)\Big|Z=\zeta\right)$ is straightforward using the properties of the Gaussian-OU process Z_t. In particular, we have that, for each $t \in [0,T]$, the pair (Z_t, Z) is jointly normally distributed. Their covariance can be calculated explicitly to give

$$\gamma(t) = \mathrm{Cov}\left(Z_t, Z\right) = \frac{\frac{\epsilon}{2k}\left(2 - e^{-k(T-t)} + e^{-k(T+t)} - 2e^{-kt}\right)}{\sqrt{T - \frac{3-4e^{-kT}+e^{-2kT}}{2k}}}. \tag{3.10}$$

Denoting the variance of Z_t by

$$\nu(t) = \mathrm{Var}(Z_t) = \frac{\epsilon^2}{2k}\left(1 - e^{-2kt}\right) \tag{3.11}$$

we have that the conditional distribution of Z_t given $\{Z=\zeta\}$ is normal, with parameters

$$Z_t\Big|\{Z=\zeta\} \sim N\left(\gamma(t)\cdot\zeta, \nu(t) - \gamma(t)^2\right)$$

which allows us to write (3.9) more explicitly and arrive at the necessary maximum condition for ζ as

$$\int_0^T \exp\left(\bar{y}_t + \gamma(t)\cdot\zeta + \frac{1}{2}\left(\nu(t) - \gamma(t)^2\right)\right) = KT. \tag{3.12}$$

Since the left-hand side of (3.12) strictly increases from 0 to ∞ as ζ goes from $-\infty$ to ∞, we see that this equation has a unique solution $\zeta^* \in \mathbb{R}$. It is easy to check that $f^{''}(\zeta) < 0$, $\zeta \in \mathbb{R}$, and thus we conclude that ζ^* is our desired global maximum; in practice, the solution ζ^* is easily determined numerically, for example by applying Newton's root search algorithm to (3.12). Finally, to obtain the lower bound we have to compute the original integral from (3.8)

$$LB = \frac{e^{-rT}}{T}\int_0^T E^{\mathbb{Q}}\left(\left(e^{\bar{y}_t + Z_t} - K\right)\cdot 1_{Z-\zeta^*>0}\right)dt.$$

After carrying out the remaining algebraic calculations (included in Appendix A.2), we can summarize our lower bound result in the following proposition.

Proposition 3.3.1 *In the general Log-OU stochastic volatility model (3.1)-(3.2), a call option on realized variance with variance strike K and maturity T satisfies the lower bound*

$$C_{var}(K) \geq LB = \frac{e^{-rT}}{T}\left[\int_0^T e^{\bar{y}_t+\frac{1}{2}\nu(t)} \cdot N\left(-\zeta^* + \gamma(t)\right) dt - K \cdot N\left(-\zeta^*\right)\right]$$

where ζ^* is the unique solution to the equation

$$\int_0^T \exp\left(\bar{y}_t + \gamma(t)\cdot\zeta + \frac{1}{2}\left(\nu(t) - \gamma(t)^2\right)\right) = KT$$

with $\gamma(t)$ and $\nu(t)$ as defined in (3.10) and (3.11).

Before illustrating the numerical performance of this lower bound, we turn to the problem of deriving an upper bound. It will turn out that the upper bound is less sharp than the lower bound and hence less useful in practice. Nevertheless, the argument is interesting and we include it in our treatment for completeness.

Let f_t be any integrable stochastic process such that $\int_0^T f_t dt = 1$. We start with the following simple inequality:

$$\begin{aligned} E^{\mathbb{Q}}\left(\frac{1}{T}\int_0^T v_t dt - K\right)_+ &= \frac{1}{T}E^{\mathbb{Q}}\left(\int_0^T \left(v_t - KT\cdot f_t\right) dt\right)_+ \\ &\leq \frac{1}{T}\int_0^T E^{\mathbb{Q}}\left(v_t - KT\cdot f_t\right)_+ dt. \end{aligned} \tag{3.13}$$

Next, similar to the idea in Thompson (1999), we take f_t to have the particular form

$$f_t = u_t + Z_t - \frac{1}{T}\int_0^T Z_t dt \tag{3.14}$$

where $u \in C[0,T]$ is a deterministic, continuous function on $[0,T]$ such that $\int_0^T u_t dt = 1$ and Z_t the zero-mean Gaussian-OU process defined by (3.5). (Note that this choice of f_t clearly satisfies the condition $\int_0^T f_t dt = 1$.) To find the deterministic function $u \in C[0,T]$ which yields the lowest upper bound in (3.13), we want to solve the problem

$$\begin{cases} \min_{u\in C[0,T]} \frac{1}{T}\int_0^T E^{\mathbb{Q}}\left(v_t - KT\cdot(u_t + X_t)\right)_+ dt \\ \\ \text{with} \int_0^T u_t dt = 1 \end{cases} \tag{3.15}$$

where we denoted $X_t \triangleq Z_t - \frac{1}{T}\int_0^T Z_t dt$. This problem can be formulated and solved as a simple problem in the calculus of variations. Let us first define $\psi(t,x)$: $[0,T]\times\mathbb{R}\to\mathbb{R}$ as

$$\psi(t,x) = E^{\mathbb{Q}}\left(v_t - KT\cdot(x + X_t)\right)_+$$

and observe that we have

$$\begin{aligned} \frac{\partial \psi}{\partial x}(t,x) &= -KT \cdot P\left(v_t - KT \cdot X_t \geq KT \cdot x\right) \\ \frac{\partial^2 \psi}{\partial x^2}(t,x) &\geq 0. \end{aligned} \tag{3.16}$$

We now introduce the functional $F : C[0,T] \to \mathbb{R}$ associated with our constrained minimization problem (3.15):

$$F(u) = \frac{1}{T}\int_0^T E^{\mathbb{Q}}\left(v_t - KT \cdot (u_t + X_t)\right)_+ dt + \lambda\left(\int_0^T u_t dt - 1\right)$$

with $\lambda \in \mathbb{R}$ a Lagrange multiplier. It can be written more compactly in terms of $\psi(t,x)$ defined earlier

$$\begin{aligned} F(u) &= \frac{1}{T}\int_0^T \psi(t,u_t)dt + \lambda\left(\int_0^T u_t dt - 1\right) \\ &= \frac{1}{T}\left[\int_0^T \left(\psi(t,u_t) + \lambda T \cdot u_t - \lambda\right) dt\right]. \end{aligned}$$

We calculate the first and second variations of the functional F at u to obtain

$$\begin{aligned} \delta F(u)(h) &= \frac{1}{T}\int_0^T \left(\frac{\partial \psi}{\partial x}(t,u_t) + \lambda T\right) \cdot h_t dt \\ \delta^2 F(u)(h) &= \frac{1}{T}\int_0^T \frac{\partial^2 \psi}{\partial x^2}(t,u_t) \cdot h_t^2 dt \end{aligned}$$

where $h \in C[0,T]$ is any test function. Noting that the second variation is non-negative, we set the first variation equal to zero. From $\delta F(u)(h) = 0$ for all $h \in C[0,T]$ we obtain the optimality condition, namely, for each $t \in [0,T]$ we must have

$$\frac{\partial \psi}{\partial x}(t,u_t) + \lambda T = 0$$

or, from (3.16)

$$K \cdot P\left(v_t - KT \cdot X_t \geq KT \cdot u_t\right) = \lambda.$$

Upon writing out v_t and X_t explicitly, this optimality condition reads

$$K \cdot P\left(e^{\bar{y}_t + Z_t} - KT \cdot \left(Z_t - \frac{1}{T}\int_0^T Z_t dt\right) \geq KT \cdot u_t\right) = \lambda.$$

Because this expression combines log-normal and normal distributions, writing out an explicit solution for the optimal u_t is not possible. Following Thompson (1999),

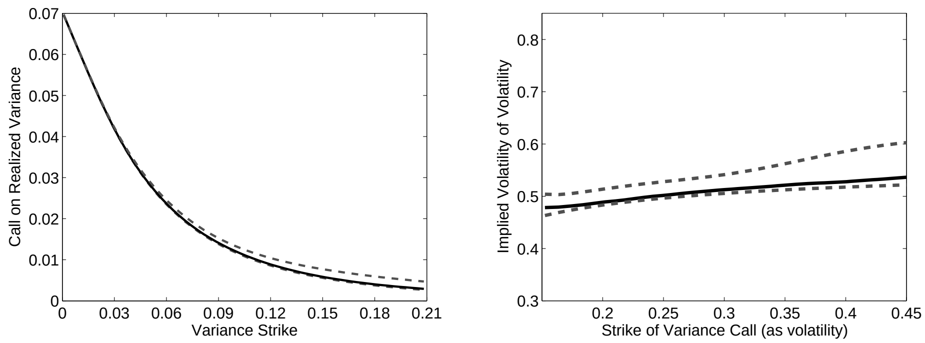

Figure 3.2: (Left) *Solid black*: 6M variance call prices in Scott's (1987) model (obtained by Monte-Carlo with geometric control variates), *Dashed gray*: Lower and Upper bounds from Propositions (3.3.1) and (3.3.2). (Right) *Solid black*: 6M variance call prices in Scott's (1987) model expressed as implied log-normal volatilities of volatility, *Dashed gray*: Lower and Upper bounds expressed as log-normal implied volatilities of volatility.

by replacing e^{Z_t} with $1+Z_t$ it becomes possible to determine a solution u_t explicitly. Note from inequality (3.13) that this, too, will provide us with an upper bound for the variance call option, although not the tightest one. Its performance will be illustrated numerically. The problem for $u \in C[0,T]$ now becomes

$$K \cdot P\left(e^{\bar{y}_t}\left(1+Z_t\right) - KT \cdot \left(Z_t - \frac{1}{T}\int_0^T Z_t dt\right) \geq KT \cdot u_t\right) = \lambda$$

for all $t \in [0,T]$ and such that $\int_0^T u_t dt = 1$. Once u_t has been determined (and, hence, also f_t in (3.13), (3.14)), the upper bound requires the computation of

$$UB = \frac{e^{-rT}}{T} \cdot \int_0^T E^{\mathbb{Q}}\left(v_t - KT \cdot f_t\right)_+ dt.$$

Leaving the remaining algebraic calculations for the Appendix A.3, we finally obtain the upper bound as stated in the following proposition.

Proposition 3.3.2 *In the general Log-OU stochastic volatility model (3.1)-(3.2), a call option on realized variance with variance strike K and maturity T satisfies the upper bound*

$$\begin{aligned} C_{var}(K) \leq UB \quad &= \quad \frac{e^{-rT}}{T}\int_0^T \int_{-\infty}^{\infty} \phi(x)\cdot\left[\alpha\left(t, x\sqrt{\nu(t)}\right) N\left(\frac{\alpha\left(t, x\sqrt{\nu(t)}\right)}{\beta(t)}\right) + \right. \\ &\left. \qquad + \beta(t)\phi\left(\frac{\alpha\left(t, x\sqrt{\nu(t)}\right)}{\beta(t)}\right)\right] dx dt \end{aligned}$$

where

$$\nu(t) = \mathrm{Var}\left(Z_t\right) = \frac{\epsilon^2}{2k}\left(1 - e^{-2kt}\right)$$

and $\alpha(t,x)$, $\beta(t)$ *as given in Appendix A.3.*

Figure (3.2) shows the performance of the lower and upper bounds of Propositions (3.3.1) and (3.3.2), against the actual variance call prices computed by Monte Carlo, with geometric control variates. We plot both the absolute prices (in the left panel) and the corresponding log-normal implied volatilities of volatility (in the right panel). Across a wide range of volatility strikes (from 0.15 to 0.45), we notice that, the lower bound remains very sharp. More exactly, the difference between the implied volatility-of-volatility corresponding to the Monte Carlo price and the one corresponding to the lower bound is roughly of the order of one volatility-of-volatility point — which is well within any reasonable bid-offer spread for such volatility products. The upper bound is less sharp and, hence, will be of less practical interest. Finally, we notice that the implied volatility-of-volatility skew is approximately flat, or more precisely, mildly upward sloping; the latter is a desirable feature which is not possessed by, for example, the traditional Heston (1993) model.

3.4 Conclusions

We have introduced a general class of Log-OU stochastic volatility models which, compared to the traditional Heston (1993) model, have the advantage of generating a more acceptable implied volatility-of-volatility skew. In this context, the valuation problem for options on realized variance bears certain analogies to Asian options. The main difference is that the driving process is now a mean-reverting OU process, as opposed to a standard Brownian motion. We show how to extend a couple of effective Asian option methods to the pricing of options on realized variance and obtain, in particular, a very tight lower bound. In practice, the error from using the lower bound should be smaller than the usual bid-offer spread for such volatility products. For completeness, we also derived an upper bound.

3.5 Appendix

A.1. A Black-Scholes style formula is obtained for the geometric control variate of a call option on realized variance as follows. Specifically, to value

$$C_{var}^{Geom.}(K) = e^{-rT} E^{\mathbb{Q}}\left(\exp\left(\frac{1}{T}\int_0^T (\bar{y}_t + Z_t)dt\right) - K\right)_+$$

we use that

$$\frac{1}{T}\int_0^T (\bar{y}_t + Z_t)dt \sim N\left(\frac{1}{T}\int_0^T \bar{y}_t dt, \frac{\epsilon^2}{T^2k^2}\left(T - \frac{3-4e^{-kT}+e^{-2kT}}{2k}\right)\right).$$

For example, in Scott's (1987), and respectively Bergomi's (2005), models we have

$$\begin{aligned}
\frac{1}{T}\int_0^T \bar{y}_t dt &= \log(\theta) + (y_0 - \log(\theta)) \cdot \frac{1-e^{-kT}}{kT} \\
\frac{1}{T}\int_0^T \bar{y}_t dt &= y_0 \cdot \frac{1-e^{-kT}}{kT} + \frac{1}{T}\int_0^T \log(\theta_u) \cdot \left(1 - e^{-k(T-u)}\right) du \\
&\quad -\frac{\epsilon^2}{4k}\left(1 - \frac{1-e^{-2kT}}{2kT}\right).
\end{aligned}$$

Denoting $m = \frac{1}{T}\int_0^T \bar{y}_t dt$ and $v^2 = \mathrm{Var}\left(\frac{1}{T}\int_0^T (\bar{y}_t + Z_t)dt\right)$ we obtain the Black-Scholes style formula for the control variate

$$C_{var}^{Geom.}(K) = e^{-rT}\left[e^{m+\frac{v^2}{2}} \cdot N(d_1) - K \cdot N(d_2)\right]$$

with

$$\begin{aligned}
d_1 &= \frac{\log\left(\frac{e^{m+\frac{v^2}{2}}}{K}\right) + \frac{v^2}{2}}{v} \\
d_2 &= d_1 - v.
\end{aligned}$$

A.2. To calculate the inner expectation appearing in the expression of the lower bound

$$LB = \frac{e^{-rT}}{T}\int_0^T E\left(\left(e^{\bar{y}_t+Z_t} - K\right) \cdot 1_{Z-\zeta^*>0}\right) dt$$

we use the following simple identity given in Thompson (1999):

$$E\left(\left(e^X - K\right) \cdot 1_{Y>0}\right) = e^{\mu_X + \frac{1}{2}\sigma_X^2} \cdot N\left(\frac{\mu_Y + \sigma_{XY}}{\sigma_Y}\right) - K \cdot N\left(\frac{\mu_Y}{\sigma_Y}\right)$$

which holds for any bivariate normal vector (X, Y) with mean (μ_X, μ_Y) and covariance matrix

$$\begin{pmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{XY} & \sigma_Y^2 \end{pmatrix}.$$

Setting $X = \bar{y}_t + Z_t$ and $Y = Z - \zeta^*$ we have $\mu_X = \bar{y}_t$, $\mu_Y = -\zeta^*$, $\sigma_X^2 = \nu(t) = \frac{\epsilon^2}{2k}\left(1 - e^{-2kt}\right)$, $\sigma_Y^2 = 1$ and covariance

$$\sigma_{XY} = \frac{\frac{\epsilon}{2k}\left(2 - e^{-k(T-t)} + e^{-k(T+t)} - 2e^{-kt}\right)}{\sqrt{T - \frac{3-4e^{-kT}+e^{-2kT}}{2k}}}.$$

as was given in the main text and denoted there $\gamma(t)$. Finally, plugging these terms in the identity above we obtain

$$LB = \frac{e^{-rT}}{T} \int_0^T \left[e^{\bar{y}_t + \frac{1}{2}\nu(t)} \cdot N\left(-\zeta^* + \gamma(t)\right) - K \cdot N\left(-\zeta^*\right) \right] dt.$$

A.3. Recall that we need to determine $u \in C[0,T]$ such that

$$K \cdot P\left(e^{\bar{y}_t}\left(1 + Z_t\right) - KT \cdot \left(Z_t - \frac{1}{T} \int_0^T Z_t dt \right) \geq KT \cdot u_t \right) = \lambda$$

for all $t \in [0,T]$, or equivalently

$$K \cdot P\left(\left(e^{\bar{y}_t} - KT\right) \cdot Z_t + K \int_0^T Z_t dt + e^{\bar{y}_t} \geq KT \cdot u_t \right) = \lambda.$$

Letting

$$N_t = \left(e^{\bar{y}_t} - KT\right) \cdot Z_t + K \int_0^T Z_t dt + e^{\bar{y}_t}$$

we first observe that, since $\left(Z_t, \int_0^T Z_t dt\right)$ are jointly normally distributed, it follows that N_t is also normal with

$$N_t \sim N\left(e^{\bar{y}_t}, \omega(t)\right)$$

where we denote $\mathrm{Var}(N_t) = \omega(t)$. Note that

$$\begin{aligned} \omega(t) &= \left(e^{\bar{y}_t} - KT\right)^2 \cdot \mathrm{Var}\left(Z_t\right) + K^2 \cdot \mathrm{Var}\left(\int_0^T Z_t dt \right) + \\ &\quad + 2K\left(e^{\bar{y}_t} - KT\right) \cdot \mathrm{Cov}\left(Z_t, \int_0^T Z_t dt \right) \end{aligned}$$

with

$$\mathrm{Var}\left(Z_t\right) = \frac{\epsilon^2}{2k}\left(1 - e^{-2kt}\right) \tag{3.17}$$

$$\mathrm{Var}\left(\int_0^T Z_t dt \right) = \frac{\epsilon^2}{k^2}\left(T - \frac{3 - 4e^{-kT} + e^{-2kT}}{2k} \right) \tag{3.18}$$

$$\mathrm{Cov}\left(Z_t, \int_0^T Z_t dt \right) = \frac{\epsilon^2}{2k^2}\left(2 - e^{-k(T-t)} + e^{-k(T+t)} - 2e^{-kt}\right). \tag{3.19}$$

Written in terms of N_t the condition for u_t becomes

$$P\left(N_t \geq KT \cdot u_t\right) = \frac{\lambda}{K}$$

which is equivalent to

$$\frac{KT \cdot u_t - e^{\bar{y}_t}}{\sqrt{\omega(t)}} = \gamma$$

where we denote $N^{-1}(1 - \frac{\lambda}{K}) = \gamma$. Solving for u_t gives

$$u_t = \frac{1}{KT} \cdot \left(e^{\bar{y}_t} + \gamma \sqrt{\omega(t)} \right).$$

To find the constant γ we impose the condition $\int_0^T u_t dt = 1$ which gives

$$\gamma = \frac{KT - \int_0^T e^{\bar{y}_t} dt}{\int_0^T \sqrt{\omega(t)}}.$$

Finally, having determined u_t, to calculate the upper bound we must compute

$$UB = \frac{e^{-rT}}{T} \int_0^T E \left(e^{\bar{y}_t + Z_t} - KT \cdot \left(u_t + Z_t - \frac{1}{T} \int_0^T Z_t dt \right) \right)_+ dt.$$

For the inner expectation, we proceed by conditioning on Z_t and using that

$$\int_0^T Z_t dt \bigg| \{Z_t = z\} \sim N \left(\frac{\mathrm{Cov}\left(Z_t, \int_0^T Z_t dt \right)}{\mathrm{Var}\,(Z_t)} \cdot z, \mathrm{Var} \left(\int_0^T Z_t dt \Big| Z_t \right) \right)$$

where

$$\mathrm{Var} \left(\int_0^T Z_t dt \Big| Z_t \right) = \mathrm{Var} \int_0^T Z_t dt - \frac{\mathrm{Cov}^2 \left(Z_t, \int_0^T Z_t dt \right)}{\mathrm{Var} Z_t}$$

and where the expressions of the variances and covariances are given explicitly in (3.17),(3.18) and (3.19). Integrating out with respect to the density of Z_t, we write the inner expectation as

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{\nu(t)}} \phi \left(\frac{z}{\sqrt{\nu(t)}} \right) \cdot E \left(\beta(t) \cdot N(0,1) + \alpha(t,z) \right)_+ dz$$

where

$$\beta(t) = K \sqrt{\mathrm{Var} \left(\int_0^T Z_t dt \Big| Z_t \right)}$$

$$\alpha(t,z) = e^{\bar{y}_t + z} - KT \cdot (u_t + z) + K \cdot \frac{\mathrm{Cov}\left(Z_t, \int_0^T Z_t dt \right)}{\mathrm{Var} Z_t} \cdot z.$$

Finally, using the basic identity from Thomson (1999)

$$E\left(b \cdot N(0,1) + a\right)_{+} = a \cdot N\left(\frac{a}{b}\right) + b \cdot \phi\left(\frac{a}{b}\right)$$

$a \in \mathbb{R}$, $b \in \mathbb{R}$, we obtain the expression for the upper bound

$$\frac{e^{-rT}}{T}\int_0^T \int_{-\infty}^{\infty} \frac{1}{\sqrt{\nu(t)}}\phi\left(\frac{z}{\sqrt{\nu(t)}}\right) \cdot \left[\alpha(t,z) \cdot N\left(\frac{\alpha(t,z)}{\beta(t)}\right) + \beta(t) \cdot \phi\left(\frac{\alpha(t,z)}{\beta(t)}\right)\right] dzdt.$$

Making the change of variable $z = x \cdot \sqrt{\nu(t)}$ we arrive at the final form

$$\begin{aligned} C_{var}(K) \leq UB \quad = \quad & \frac{e^{-rT}}{T}\int_0^T \int_{-\infty}^{\infty} \phi(x) \cdot \left[\alpha\left(t, x\sqrt{\nu(t)}\right) N\left(\frac{\alpha\left(t, x\sqrt{\nu(t)}\right)}{\beta(t)}\right) + \right. \\ & \left. + \beta(t)\phi\left(\frac{\alpha\left(t, x\sqrt{\nu(t)}\right)}{\beta(t)}\right)\right] dxdt \end{aligned}$$

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IV

Options on discretely sampled variance: Discretization effect and Greeks

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Abstract

The valuation of options on discretely sampled variance requires proper adjustment for the extra volatility-of-variance induced by discrete sampling. Under general stochastic volatility dynamics, we provide a detailed theoretical characterization of the discretization effect. In addition, we analyze several numerical methods which reduce the dimensionality of the required pricing scheme, while accounting for most of the discretization effect. The most important of these, named the conditional Black-Scholes scheme, leads to an explicit discretization adjustment term, easily computable by standard Fourier transform methods in any stochastic volatility model which admits a closed-form expression for the characteristic function of continuously sampled variance. In the second part of the chapter, we provide a practical analysis of the most important risk sensitivities ('*greeks*') of options on discretely sampled variance.

4.1 Introduction

Early literature on variance derivatives assumed continuous sampling of the realized variance. The first contribution belongs to Neuberger (1994) who introduced the concept of the log-contract and argued that delta hedging this contract leads to the replication of the continuously sampled variance; the result holds under general continuous semi-martingale dynamics. Carr, Madan (1998) subsequently extended

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the results and showed how variance swaps (as well as corridor variance swaps) can be replicated by a static position in a continuum of vanilla options, dynamically delta-hedged with the underlying asset. Broadie, Jain (2008a) consider the problem of approximating this static portfolio using a finite number of European vanilla options. The consistent joint-modeling of a term structure of variance swaps and an underlying spot price is treated in Buehler (2006a, 2006b). For a detailed overview of the various theoretical developments, we refer the reader to Carr, Lee (2009). Additionally, important aspects of volatility derivatives in practice are discussed in Gatheral (2006), Bergomi (2005, 2008) and Overhaus et al. (2007).

In derivative markets, variance contracts are specified with discrete sampling; in particular, daily sampling is the most common convention. For linear contracts on realized variance (such as variance swaps), the discretization effect is usually small, as found in Buehler (2006a) and Broadie, Jain (2008b). The explanation follows from the fact that linear contracts on variance do not depend on the volatility of variance. Itkin, Carr (2010) show how to price discrete variance swaps in general time-changed Lévy models, by the method of forward characteristic functions. Under the assumption of a stochastic clock independent of the driving Lévy process, section 6 in Carr, Lee and Wu (2011) shows that discrete sampling increases the value of variance swaps.

For non-linear contracts on variance (such as options on variance) the discretization effect becomes substantial, especially for shorter maturities. The short-time limit of the discretization gap, under general semi-martingale dynamics, has been derived recently in Keller-Ressel, Muhle-Karbe (2011); the authors also develop Fourier pricing methods for options on discrete variance under exponential Lévy dynamics. In the context of the Heston (1993) model, Sepp (2011) proposes an approximation by combining the distribution of quadratic variation in the Heston (1993) model with that of discrete variance in an independent Black, Scholes (1973) model. The approach leads to a tractable characteristic function for the discretely sampled variance, thus allowing the application of the semi-analytical methods treated in the earlier work Sepp (2008) and provides good accuracy near the at-the-money region, across maturities.

The aim of this paper is to provide a comprehensive treatment of the discretization effect, arising in the valuation of variance derivatives, under general stochastic volatility dynamics. Additionally, we do not restrict attention to particular strike ranges, such as at-the-money strikes, nor to particular maturity ranges. We prove that, conditional on the realization of the instantaneous variance process, the (properly scaled) residual randomness arising from discrete sampling can be well approximated with a normally distributed random variable. In the financial econometrics literature, related results were obtained and used in the analysis of high frequency data and the estimation of stochastic volatility models; in particular, we remark the results in Barndorff-Nielsen, Shephard (2002), and more generally, in Barndorff-Nielsen et al. (2006). In contrast, we adopt a conditional approach, which was pioneered in the study of stochastic volatility models by Hull, White (1987), in the case of zero correlation between volatility and the underlying asset, and by Romano, Touzi (1997) and Willard (1997) in the case of non-zero correlation. Additionally, as made precise in the next section, we consider a different limiting sequence, which accounts for the correlation induced terms in the discrete variance. In the first step, we reduce the dimensionality of the Monte-Carlo scheme by eliminating the need to simulate the path of the log-returns. A further simplification, makes it possible to (even) avoid the simulation of the instantaneous variance path, by simulating directly from the distribution of the integrated continuous variance. Most importantly, a variation on the latter approach, termed hereafter the conditional Black-Scholes scheme, leads to an explicit discretization adjustment term — which is easily computable by standard Fourier transform methods, in any stochastic volatility model with a closed-form expression for the characteristic function of continuously sampled variance (e.g. Heston (1993) or the 3/2 model in Lewis (2000) and Carr, Sun (2007)).

An important area which appears to be lacking in the current literature concerns the risk-sensitivities (also known as greeks) of options on discretely sampled variance. In the context of continuous sampling and the Heston (1993) model, risk-sensitivities are discussed in Broadie, Jain (2008a). Unlike the case of continuous sampling, options on discrete variance display sensitivities with respect to the spot price process (e.g. Delta, Gamma) in the periods between resets. They also display cross-sensitivities with respect to spot and volatility. In the second part of the paper, we define and illustrate the behavior of the most important risk sensitivities in the context of discrete sampling.

The remaining of the paper is divided into two parts. The next section is the main part which develops the theory underlying our proposed numerical schemes and then provides several numerical examples. The second part analyzes the greeks of options on discrete variance, with an emphasis on their practical use and interpretation. Proofs not given in the main text, can be found in the appendix.

4.2 Options on discretely sampled variance

In this section we study the magnitude of the discretization effect in valuing options on realized variance. We start by setting a general stochastic volatility framework. Let $(B_t, W_t)_{t\geq 0}$ be a standard two-dimensional Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{Q})$ satisfying the usual conditions. Assume that the stock price and its instantaneous variance $(S_t, v_t)_{t\geq 0}$ satisfy the following risk-neutral dynamics under $\mathbb{Q}$:

$$\begin{aligned} \frac{dS_t}{S_t} &= (r-\delta)dt + \sqrt{v_t}\left(\rho dW_t + \sqrt{1-\rho^2}dB_t\right) &(4.1)\\ dv_t &= a(v_t)dt + b(v_t)dW_t &(4.2)\end{aligned}$$

where $a : \mathbb{R}_+ \to \mathbb{R}$ and $b : \mathbb{R}_+ \to \mathbb{R}$ are Borel measurable functions such that the two-dimensional SDE (4.1), (4.2), admits a unique and non-exploding solution $(S_t, v_t)_{t\geq 0}$. The risk free interest rate, dividend yield and correlation parameters are denoted by r, δ and ρ respectively. As shown in Buehler (2006a), to ensure that the discounted, and dividend adjusted, process $S_t \cdot \exp(-(r-\delta)t)$ is a true $\mathbb{Q}$-martingale (and not just a local martingale) we must also assume that the variance diffusion v_t is non-explosive under the measure which takes the asset S_t as numeraire. A large number of stochastic volatility models proposed in the literature belong to this framework. Important examples include Scott (1987), Heston (1993) and the 3/2 model discussed in Lewis (2000) and Carr, Sun (2007).

Consider a finite maturity $T > 0$ and let $0 = t_0 < t_1 < t_2 < \ldots < t_n = T$ be an equally spaced partition of $[0, T]$ with step-size $\Delta = \frac{T}{n}$. The (annualized) discretely sampled variance of $\log(S_t)$ over $[0, T]$ is defined as

$$RV_n = \frac{1}{T} \cdot \sum_{i=1}^{n} \log^2\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right). \tag{4.3}$$

A standard result in stochastic calculus (see, for example, Revuz, Yor (1999)) establishes that, as $n \to \infty$, RV_n converges in probability to the continuously sampled variance (or quadratic variation) of $\log(S_t)$. Specifically, in our setup we have $\operatorname{plim}_{n\to\infty} RV_n = \frac{1}{T}[\log(S_t)]_T = \frac{1}{T}\int_0^T v_t dt$, where plim denotes the limit in probability. In practice, variance contracts must be specified by using discrete sampling. For example, a call option on realized variance with maturity T and volatility strike σ_K delivers, at time T, the payoff:

$$VN \cdot \left(\frac{1}{T} \cdot \sum_{i=1}^{n} \log^2\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right) - \sigma_K^2\right)_+$$

where VN is a constant known as the variance notional; throughout this section, we set $VN = 1$. Our goal here is to compare the prices of options on discretely sampled variance to those of options on quadratic variation. An application of Itô's lemma to $\log(S_t)$ gives

$$d\log(S_t) = (r - \delta - \frac{v_t}{2})dt + \sqrt{v_t}\left(\rho dW_t + \sqrt{1-\rho^2}dB_t\right)$$

from where, each discrete log-return can be written as

$$\log\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right) = (r-\delta)\cdot\frac{T}{n} - \frac{1}{2}\int_{t_{i-1}}^{t_i} v_s ds + \rho\int_{t_{i-1}}^{t_i}\sqrt{v_s}dW_s + \sqrt{1-\rho^2}\int_{t_{i-1}}^{t_i}\sqrt{v_s}dB_s$$

for all $i \in \{1,2,\ldots,n\}$. In what follows we denote by $\mathcal{F}_t^W$ the filtration generated by the Brownian motion W_t driving the variance diffusion v_t; we recall that the process v_t serves to model the (stochastic) instantaneous variance of the asset price. A key observation is that, conditional on $\mathcal{F}_T^W$, the log-returns $\log\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)$, $i = \{1,2,\ldots,n\}$, form a sequence of independent normally distributed (but not identically) random variables with means and variances given by:

$$\log\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)\Bigg|_{\mathcal{F}_T^W} \sim \quad N\Bigg((r-\delta)\cdot\frac{T}{n} - \frac{1}{2}\int_{t_{i-1}}^{t_i} v_s ds + \rho\int_{t_{i-1}}^{t_i}\sqrt{v_s}dW_s,$$
$$(1-\rho^2)\int_{t_{i-1}}^{t_i} v_s ds\Bigg). \tag{4.4}$$

The result in (4.4) follows immediately from the property that the Brownian integral of any deterministic, locally-bounded function is a Gaussian process (see, for example, Revuz , Yor (1999)). We note that, conditional on $\mathcal{F}_T^W$, the continuously sampled variance $\frac{1}{T}\int_0^T v_t dt$ is just a constant, whereas the discretely sampled variance RV_n still has a residual randomness driven by B_t. In practice, for typical parameter values, this residual randomness is not negligible and can lead to substantially higher prices for options on discretely sampled variance, especially for maturities less than one year. It is the properties of this residual randomness that we investigate next.

In what follows, a key result (formulated in Theorem 4.2.2) shows that the *conditional* distribution of RV_n is asymptotically normal. To establish this result, we use a generalized version of the central limit theorem (CLT) for triangular arrays of unequal components. To see why this is necessary, note that for each $n \geq 1$, in the expression of RV_n, we have a different sequence of squared log-returns and the components of each sequence have different variances. Specifically, we shall use the *Lindeberg-Feller* generalized CLT (see, for example, Ferguson (1996)) as formulated in Theorem 4.2.1.

Theorem 4.2.1 (Generalized CLT: Lindeberg-Feller) *Let $Z_{n,i}$, $n = 1,2,\ldots$ $i = 1,2,\ldots,n$ be a triangular sequence of random variables such that $E\left(Z_{n,i}\right) = 0$, $E\left(Z_{n,i}^2\right) < \infty$ and for each fixed $n = 1,2,\ldots$ the random variables $Z_{n,1}, Z_{n,2},\ldots,Z_{n,n}$*

are independent. If the Lindeberg condition *is satisfied i.e. for all* $\epsilon > 0$

$$\lim_{n\to\infty} \frac{1}{s_n^2} \sum_{i=1}^{n} E\left(|Z_{n,i}|^2; |Z_{n,i}| > \epsilon \cdot s_n\right) = 0 \tag{4.5}$$

where $s_n^2 = \sum_{i=1}^n E\left(Z_{n,i}^2\right)$*, then we have the convergence in distribution*

$$\frac{Z_{n,1} + Z_{n,2} + \ldots + Z_{n,n}}{s_n} \xrightarrow{d} N(0,1). \tag{4.6}$$

To verify the Lindeberg condition (4.5), it is usually easier to check the sufficient condition of *Lyapnupov* (see, for example, Petrov (1995)). Specifically, if there exits $\delta > 0$ such that

$$\lim_{n\to\infty} \frac{1}{s_n^{2+\delta}} \sum_{i=1}^{n} E\left(|Z_{n,i}|^{2+\delta}\right) = 0 \tag{4.7}$$

then the conclusion of the Lindeberg-Feller theorem (4.6) holds. Before we proceed to Theorem 4.2.2, we remark that related results were derived in Barndorff-Nielsen, Shephard (2002), and more generally, in Barndorff et al. (2006). Our results differ in several ways. Firstly, as noted in the introduction, we adopt a conditional approach (along the lines of Hull, White (1987) and Romano, Touzi (1997)). Secondly, unlike our version, the elements of the CLT sequence, in both Barndorff-Nielsen, Shephard (2002) and Barndorff et al. (2006), do not include the correlation induced terms – specifically, as in $\mu_{n,i}$, $\sigma_{n,i}^2$ (and then M_n, Σ_n) defined below. For further reference, we denote the conditional means and variances of the log-returns in (4.4) by:

$$\begin{aligned} \mu_{n,i} &= (r-\delta)\cdot\frac{T}{n} - \frac{1}{2}\int_{t_{i-1}}^{t_i} v_s ds + \rho \int_{t_{i-1}}^{t_i} \sqrt{v_s} dW_s \\ \sigma_{n,i}^2 &= (1-\rho^2)\int_{t_{i-1}}^{t_i} v_s ds. \end{aligned}$$

for all $n \geq 1$ and $i \in \{1, 2, \ldots, n\}$.

Theorem 4.2.2 *Conditional on* $\mathcal{F}_T^W$ *, the discretely sampled realized variance*

$$\mathrm{RV}_n = \frac{1}{\Delta \cdot n} \cdot \sum_{i=1}^{n} \log^2\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)$$

converges in distribution to a normal random variable. More precisely, as $n \to \infty$*, we have*

$$\frac{n\cdot\Delta}{s_n}\left(\mathrm{RV}_n - \frac{\sum_{i=1}^n \mu_{n,i}^2 + \sigma_{n,i}^2}{n\cdot\Delta}\right) \xrightarrow{d} N(0,1) \tag{4.8}$$

where

$$s_n^2 = \sum_{i=1}^{n} 2\sigma_{n,i}^4 + 4\mu_{n,i}^2 \cdot \sigma_{n,i}^2.$$

Proof Take any $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$. By the local properties of Brownian paths (see, for example, Revuz, Yor (1999)), the function $h(t) = \int_0^t \sqrt{v_s} dW_s$ is Hölder continuous with index α on $[0, T]$. We conclude that there exists a positive constant $K_1 > 0$, independent of n, such that

$$\left| \int_{t_{i-1}}^{t_i} \sqrt{v_s} dW_s \right| = \left| h\left(t_i\right) - h\left(t_{i-1}\right) \right| \leq K_1 \cdot \left| t_i - t_{i-1} \right|^{\alpha} = K_1 \cdot \frac{T^\alpha}{n^\alpha}$$

for all $n \geq 1$. Similarly, since the function $g(t) = \int_0^t v_s ds$ is Lipschitz continuous on $[0, T]$, there exists a positive constant $K_2 > 0$, independent of n, such that

$$\left| \int_{t_{i-1}}^{t_i} v_s ds \right| = \left| g\left(t_i\right) - g\left(t_{i-1}\right) \right| \leq K_2 \cdot \left| t_i - t_{i-1} \right| = K_2 \cdot \frac{T}{n}$$

for all $n \geq 1$. From the definition of $\mu_{n,i}$, we see that for all positive integers $n \geq T$, the following bound holds:

$$\begin{aligned} \left|\mu_{n,i}\right| &\leq \left|r - \delta\right| \cdot \frac{T}{n} + \frac{1}{2} K_2 \cdot \frac{T}{n} + |\rho| K_1 \cdot \frac{T^\alpha}{n^\alpha} \\ &\leq \left(\left|r - \delta\right| + \frac{1}{2} K_2 + |\rho| K_1 \right) \cdot \frac{T^\alpha}{n^\alpha} = C_1 \cdot \frac{T^\alpha}{n^\alpha}. \end{aligned} \tag{4.9}$$

Similarly, for $\sigma_{n,i}^2$ we obtain

$$\sigma_{n,i}^2 \leq (1 - \rho^2) \cdot K_2 \cdot \frac{T}{n} = C_2 \cdot \frac{T}{n}. \tag{4.10}$$

Define the triangular sequence $Y_{n,i} = X_{n,i}^2 - \left(\mu_{n,i}^2 + \sigma_{n,i}^2\right)$, where $X_{n,i} = \log\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)$ is the triangular sequence of log-returns. Making use of the conditional normality of the log-returns $X_{n,i}$ we obtain (all expectations are conditional on $\mathcal{F}_T^W$):

$$\begin{aligned} E\left(Y_{n,i}^2\right) &= 2\sigma_{n,i}^4 + 4\mu_{n,i}^2 \cdot \sigma_{n,i}^2 \\ E\left(Y_{n,i}^4\right) &= 60\sigma_{n,i}^8 + 240\sigma_{n,i}^6 \cdot \mu_{n,i}^2 + 48\sigma_{n,i}^4 \cdot \mu_{n,i}^4 + 4\sigma_{n,i}^2 \cdot \mu_{n,i}^6. \end{aligned}$$

The computation of these two expectations follows by the straightforward (but tedious) use of the higher moments of the normal distribution. We seek to apply the Lindeberg-Feller Theorem 4.2.1 to the sequence $Y_{n,i}$ by verifying the sufficient condition of Lyapnupov (4.7) for $\delta = 2$. Specifically, we show that

$$\lim_{n\to\infty} \frac{1}{s_n^4} \sum_{i=1}^{n} E\left(Y_{n,i}^4\right) = 0$$

where

$$s_n^2 = \sum_{i=1}^{n} E\left(Y_{n,i}^2\right) = \sum_{i=1}^{n} 2\sigma_{n,i}^4 + 4\mu_{n,i}^2 \cdot \sigma_{n,i}^2.$$

Using inequalities (4.9), (4.10) we have for all positive integers $n \geq T$

$$\begin{aligned} E\left(Y_{n,i}^4\right) \leq \;& 60 \cdot C_2^4 \cdot \frac{T^4}{n^4} + 240 \cdot C_2^3 \cdot C_1^2 \cdot \frac{T^{3+2\alpha}}{n^{3+2\alpha}} + 48 \cdot C_2^2 \cdot C_1^4 \cdot \frac{T^{2+4\alpha}}{n^{2+4\alpha}} + \\ & 4 \cdot C_2 \cdot C_1^6 \cdot \frac{T^{1+6\alpha}}{n^{1+6\alpha}} \leq C_3 \cdot \frac{T^{1+6\alpha}}{n^{1+6\alpha}}. \end{aligned}$$

where we have used the fact that $1 + 6\alpha < 2 + 4\alpha < 3 + 2\alpha < 4$ and $\frac{T}{n} \leq 1$. By a simple application of the classic Cauchy-Schwarz (or, alternatively, Jensen's) inequality, we also obtain:

$$s_n^2 \geq 2\sum_{i=1}^{n} \sigma_{n,i}^4 \geq \frac{2}{n}\left(\sum_{i=1}^{n} \sigma_{n,i}^2\right)^2 = \frac{2}{n}\left(1-\rho^2\right)^2 \left(\int_0^T v_t dt\right)^2 = \frac{C_4}{n}$$

where $C_4 > 0$ does not depend on n. Finally, this gives

$$\frac{1}{s_n^4} \sum_{i=1}^{n} E\left(Y_{n,i}^4\right) \leq \frac{n \cdot C_3 \cdot \frac{T^{1+6\alpha}}{n^{1+6\alpha}}}{\frac{C_4^2}{n^2}} = \frac{C_3 \cdot T^{1+6\alpha}}{C_4^2} \cdot \frac{1}{n^{6\left(\alpha - \frac{1}{3}\right)}} \to 0 \text{ as } n \to \infty$$

where we have used that $\alpha > \frac{1}{3}$. By the Lindeberg-Feller Theorem 4.2.1 we conclude

$$\frac{n \cdot \Delta}{s_n}\left(\mathrm{RV}_n - \frac{\sum_{i=1}^{n} \mu_{n,i}^2 + \sigma_{n,i}^2}{n \cdot \Delta}\right) = \frac{\sum_{i=1}^{n} Y_{n,i}}{s_n} \xrightarrow{d} N(0,1).$$

$\square$

In practical terms, Theorem 4.2.2 implies that, conditional on the realization of the instantaneous variance path v_t, $t \in [0, T]$, the distribution of the discretely sampled variance RV_n can be approximated with a normal $N\left(M_n, \Sigma_n^2\right)$ where

$$\begin{aligned} M_n &= \frac{\sum_{i=1}^{n} \mu_{n,i}^2 + \sigma_{n,i}^2}{n \cdot \Delta} \\ \Sigma_n^2 &= \frac{s_n^2}{n^2 \cdot \Delta^2}. \end{aligned}$$

Basic properties of normal random variables give that (conditional on the variance path v_t, $t \in [0,T]$), the (undiscounted) value of the discrete variance call can be approximated by

$$C^n(\sigma_K) = \Sigma_n \cdot \phi\left(\frac{M_n - \sigma_K^2}{\Sigma_n}\right) + \left(M_n - \sigma_k^2\right) \cdot N\left(\frac{M_n - \sigma_K^2}{\Sigma_n}\right) \tag{4.11}$$

where $\phi(\cdot)$ and $N(\cdot)$ denote the density and distribution functions of the standard normal law. The results derived so far indicate the following method to price options on discrete variance by eliminating the need to simulate paths of the spot price S_t: for each simulated instantaneous variance path, compute the conditional option price by (4.11) and, finally, average over these conditional prices. Later in the section, we explore ways to further simplify this conditional scheme by eliminating the need to compute the quantities $\mu_{n,i}$ and $\sigma_{n,i}^2$ for each variance path. The performance of the simplified schemes will be illustrated numerically.

Before continuing the analysis of the conditional pricing schemes, we remark that, on the purely theoretical front, it is possible to draw further on the tools of the generalized CLT to establish bounds on the approximation (4.11). In particular, we can use the following generalization of the *Berry-Essen* inequalities due to Bikelis (1966), and which can also be found in Petrov (2007).

Theorem 4.2.3 (Bikelis) *Assume $Z_1, Z_2, \ldots, Z_n$ are independent random variables with mean zero and $E\left(|Z_i|^3\right) < \infty$. Let $s_n^2 = \sum_{i=1}^n E\left(Z_i^2\right)$ and $L_n = s_n^{-3} \cdot \sum_{i=1}^n E\left(|Z_i|^3\right)$. If $F_n(x)$ denotes the distribution function defined as*

$$F_n(x) = P\left(\frac{\sum_{i=1}^n Z_i}{s_n} \leq x\right)$$

then, for any $x \in \mathbb{R}$, we have

$$\left|F_n(x) - N(x)\right| \leq \frac{A \cdot L_n}{(1+|x|)^3}$$

where $N(\cdot)$ is the standard normal CDF *and A is an absolute positive constant.*

By making use of Theorem 4.2.3, we can provide a bound on the difference between the conditional variance call price $C(\sigma_K) = E\Big[(RV_n - \sigma_K^2)_+ \,\Big|\mathcal{F}_T^W\Big]$ and the conditional normal approximation $C^n(\sigma_K)$ in (4.11).

Proposition 4.2.4 *If $\sigma_K^2 \leq M_n$, then*

$$\Big|C(\sigma_K) - C^n(\sigma_K)\Big| \leq \frac{A \cdot \Sigma_n \cdot L_n}{2\left(1 + \frac{M_n - \sigma_K^2}{\Sigma_n}\right)^2}.$$

If $\sigma_K^2 > M_n$, *then*

$$\left|C(\sigma_K) - C^n(\sigma_K)\right| \leq A \cdot \Sigma_n \cdot L_n \cdot \left(1 - \frac{1}{2\left(1 + \frac{\sigma_K^2 - M_n}{\Sigma_n}\right)^2}\right).$$

where $L_n = s_n^{-3} \cdot \sum_{i=1}^n E\left(|Y_{n,i}|^3\right)$ *with* $Y_{n,i}$ *as in Lemma 5.1 in Appendix.*

Proof See Appendix. □

The standard and most basic model which fits in our framework is the Black-Scholes (1973) model, which will prove useful in our simplified conditional pricing schemes. In the standard Black-Scholes framework, we set $v_t = \sigma^2$ (a positive constant) and the log-returns now become i.i.d. normal:

$$\log\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right) \sim N\left(\left(r - \delta - \frac{\sigma^2}{2}\right) \cdot \frac{T}{n}, \sigma^2 \cdot \frac{T}{n}\right)$$

which gives that the distribution of RV_n satisfies

$$RV_n \overset{d}{=} \frac{\sigma^2}{n} \cdot \sum_{i=1}^n \left(\frac{r - \delta - \frac{\sigma^2}{2}}{\sigma} \cdot \sqrt{\frac{T}{n}} + Z_i\right)^2$$

where Z_i, with $1 \leq i \leq n$, here denotes a sequence of independent standard normal variables. We obtain that $RV_n \overset{d}{=} \frac{\sigma^2}{n} \cdot \chi'(n, \lambda)$ where $\chi'(n, \lambda)$ denotes the non-central chi-square distribution with n degrees of freedom and non-centrality parameter λ given by:

$$\lambda = \frac{\left(r - \delta - \frac{\sigma^2}{2}\right)^2 T}{\sigma^2}. \tag{4.12}$$

A well-known result in mathematical statistics (see, for example, the classic treatment in Muirhead (2005)) establishes the following convergence in distribution to a standard normal:

$$\frac{\chi'(n, \lambda) - (n + \lambda)}{\sqrt{2(n + 2\lambda)}} \xrightarrow{d} N(0, 1)$$

as the number of degrees of freedom $n \to \infty$. Using the value of λ from (4.12), simple algebraic computations show that the distribution of RV_n converges to a normal with mean and variance given by:

$$N\left(\sigma^2 + \frac{\left(r - \delta - \frac{\sigma^2}{2}\right)^2 T}{n}, \frac{2\sigma^4}{n} + \frac{4\left(r - \delta - \frac{\sigma^2}{2}\right)^2 \sigma^2 T}{n^2}\right). \tag{4.13}$$

This is a special case of our more general result in Theorem 4.2.2 and is obtained by using that, in the standard Black-Scholes model, we have $\mu_{n,i} = \left(r - \delta - \frac{\sigma^2}{2}\right)\frac{T}{n}$ and $\sigma_{n,i}^2 = \sigma^2 \cdot \frac{T}{n}$ which give:

$$
\begin{aligned}
M_n &= \sigma^2 + \frac{\left(r-\delta-\frac{\sigma^2}{2}\right)^2 T}{n} \\
\Sigma_n &= \sqrt{\frac{2\sigma^4}{n} + \frac{4\left(r-\delta-\frac{\sigma^2}{2}\right)^2 \sigma^2 T}{n^2}}.
\end{aligned}
$$

In the Black-Scholes model it is possible to derive an exact closed-form formula for the price of options on discrete variance. We formulate this result in Lemma 4.2.5 and note that it will be used in our conditional Black-Scholes scheme introduced later.

Lemma 4.2.5 *In the Black-Scholes model with constant volatility σ, we have*

$$
\begin{aligned}
E^{\mathbb{Q}}\left(RV_n - \sigma_K^2\right)_+ &= \sigma^2 \cdot \left(1 - F_{\chi'}\left(\frac{\sigma_K^2 \cdot n}{\sigma^2}; \lambda, n+2\right)\right) \\
&\quad + \frac{\sigma^2 \cdot \lambda}{n} \cdot \left(1 - F_{\chi'}\left(\frac{\sigma_K^2 \cdot n}{\sigma^2}; \lambda, n+4\right)\right) \\
&\quad - \sigma_K^2 \cdot \left(1 - F_{\chi'}\left(\frac{\sigma_K^2 \cdot n}{\sigma^2}; \lambda, n\right)\right) \qquad (4.14)
\end{aligned}
$$

where $F_{\chi'}(\cdot;\lambda,n)$ denotes the non-central chi-square CDF with n degrees of freedom and non-centrality parameter λ; the value of λ is given by (4.12).

Proof See Appendix. $\square$.

We now return to the general stochastic volatility case and seek to derive further simplified versions of the conditional pricing scheme implied by Theorem 4.2.2 and approximation (4.11). We note that, in a separate study, Sepp (2011) explores a different approach to adjusting for the discretization effect. Specifically, the discretization effect is treated independently by making the following approximation, in distribution:

$$
RV_n \overset{d}{\simeq} \frac{1}{T}\int_0^T v_t dt - E^{\mathbb{Q}}\left(\frac{1}{T}\int_0^T v_t dt\right) + RV_n^{BS} \qquad (4.15)
$$

where RV_n^{BS} is the discretely sampled variance in an independent Black-Scholes model with time-dependent volatility $\sigma(t)^2 = E^{\mathbb{Q}}(v_t)$. Given that, in a Black-Scholes model, the Fourier-Laplace transform of discretely sampled variance is easily obtained in closed-form and using the independence assumption, we can approximate the transform of RV_n:

$$L(\lambda) = E^{\mathbb{Q}}\left(e^{-\lambda \cdot RV_n}\right) \simeq e^{\lambda \cdot M} \cdot L_{QV}(\lambda) \cdot L_{RV_n^{BS}}(\lambda)$$

where $M = E^{\mathbb{Q}}\left(\frac{1}{T}\int_0^T v_t dt\right)$ and $L_{QV}(\lambda)$ is the transform of continuously sampled variance. As in several stochastic volatility models $L_{QV}(\lambda)$ is known in closed-form, this approach is attractive from the standpoint of using semi-analytical transform techniques and provides good accuracy for near the at-the-money region.

We will propose an alternative, transform-based approach, which does not rely on an independence assumption. The advantage will be that it leads to improved accuracy for out-of-the-money options. The magnitude of the discretization effect depends on the realization of continuously sampled variance; the smaller (larger) the latter, the smaller (larger) the discretization effect. Ignoring this dependence, will tend to overprice downside variance puts and underprice upside variance calls.

The following lemma will prove useful in our calculations. It was given in Barndorff-Nielsen, Shephard (2002), under the assumption of a variance process of finite variation. This would be unsuitable for our dynamics (4.2) of v_t but, fortunately, the assumption can be removed and, hence, we modify its statement accordingly.

Lemma 4.2.6 *Let $(v_t)_{t\geq 0}$ be a process which is a.s. locally bounded and has at most a countable number of discontinuity points on every finite interval. Then, for any fixed $T > 0$ and positive integer $k \in \mathbb{N}\backslash\{0\}$ we have*

$$\frac{n^{k-1}}{T^{k-1}} \cdot \sum_{i=1}^{n} \left(\int_{\frac{(i-1)T}{n}}^{\frac{iT}{n}} v_t dt\right)^k \to \int_0^T v_t^k dt \qquad a.s.$$

as $n \to \infty$.

Proof The proof is identical to Barndorff-Nielsen, Shephard (2002) except that we do not require the process $(v_t)_{t\geq 0}$ to be of locally bounded variation. The argument only requires that v_t^k be (a.s.) Riemann integrable on $[0, T]$, a condition which is satisfied under our assumptions for Lemma 4.2.6 (by, for example, Theorem 6.10 in Rudin (1976)).

□

Recall that, by Theorem 4.2.2, the conditional randomness of RV_n satisfies

$\left(RV_n \middle| \mathcal{F}_T^W - M_n\right)/\Sigma_n \xrightarrow{d} N(0,1)$ where

$$\begin{aligned} M_n &= \frac{\sum_{i=1}^n \mu_{n,i}^2 + \sigma_{n,i}^2}{n \cdot \Delta} \\ \Sigma_n^2 &= \frac{s_n^2}{n^2 \cdot \Delta^2} = \frac{\sum_{i=1}^n 2\sigma_{n,i}^4 + 4\mu_{n,i}^2 \cdot \sigma_{n,i}^2}{n^2 \cdot \Delta^2}. \end{aligned}$$

The simplified conditional schemes will set the variance-spot correlation ρ to zero. It turns out that setting $\rho = 0$ has little material impact on the prices of discrete variance options. To see this intuitively, note that, in the limit of continuous sampling, the correlation parameter plays no role in the price of variance options. In the numerical examples, we will see that for typical market parameters with strongly negative correlation, the simplified schemes perform very well.

We next want to apply Lemma 4.2.6 to obtain an approximation for M_n and Σ_n^2. Fix the following notations for real sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$: write $a_n = o(b_n)$ iff $\lim_{n\to\infty} |a_n/b_n| = 0$ and $a_n = O(b_n)$ iff $\limsup_{n\to\infty} |a_n/b_n| < \infty$. Firstly, observe that our instantaneous variance process $(v_t)_{t\in[0,T]}$, having a.s. continuous paths, clearly satisfies the assumptions of Lemma 4.2.6. Hence, letting $I_k = \frac{1}{T}\int_0^T v_t^k dt$ and recalling that, for $\rho = 0$, $\sigma_{n,i}^2 = \int_{t_{i-1}}^{t_i} v_t dt$, we obtain by Lemma 4.2.6:

$$\frac{1}{T}\sum_{i=1}^n \sigma_{n,i}^4 = \frac{T}{n} \cdot I_2 + o\left(\frac{1}{n}\right) \tag{4.16}$$

$$\frac{1}{T}\sum_{i=1}^n \sigma_{n,i}^6 = \frac{T^2}{n^2} \cdot I_3 + o\left(\frac{1}{n^2}\right). \tag{4.17}$$

Also, we note that $\frac{1}{T}\sum_{i=1}^n \sigma_{n,i}^2 = I_1$. Expanding $\mu_{n,i}^2$, we have:

$$\begin{aligned} \mu_{n,i}^2 &= (r-\delta)^2 \cdot \frac{T^2}{n^2} - (r-\delta)\frac{T}{n} \cdot \int_{t_{i-1}}^{t_i} v_t dt + \frac{1}{4}\left(\int_{t_{i-1}}^{t_i} v_t dt\right)^2 \\ &= (r-\delta)^2 \cdot \frac{T^2}{n^2} - (r-\delta)\frac{T}{n} \cdot \sigma_{n,i}^2 + \frac{1}{4}\sigma_{n,i}^4 \end{aligned}$$

and, writing T for $n \cdot \Delta$, we obtain

$$\begin{aligned} M_n &= \frac{1}{T}\sum_{i=1}^n \mu_{n,i}^2 + \frac{1}{T}\sum_{i=1}^n \sigma_{n,i}^2 \\ &= (r-\delta)^2 \frac{T}{n} - (r-\delta)\frac{T}{n} I_1 + \frac{1}{4}\frac{T}{n} I_2 + o\left(\frac{1}{n}\right) + I_1 \\ &= I_1 + O\left(\frac{1}{n}\right) \end{aligned} \tag{4.18}$$

We next apply a similar approach to Σ_n^2. Using again the relations (4.16) , (4.17) obtained by Lemma 4.2.6, simple algebra gives:

$$\begin{aligned}
\Sigma_n^2 &= \frac{2}{T^2}\sum_{i=1}^{n}\sigma_{n,i}^4 + \frac{4}{T^2}\sum_{i=1}^{n}\mu_{n,i}^2\cdot\sigma_{n,i}^2 \\
&= \frac{2}{n}\cdot I_2 + o\left(\frac{1}{n}\right) + 4\left((r-\delta)^2\cdot\frac{T}{n^2}I_1 - (r-\delta)\frac{T}{n^2}I_2 + \frac{1}{4}\cdot\frac{T}{n^2}I_3 + o\left(\frac{1}{n^2}\right)\right) \\
&= \frac{2}{n}\cdot I_2 + o\left(\frac{1}{n}\right) \qquad (4.19)
\end{aligned}$$

From (4.18) and (4.19), we have obtained the following result for the conditional distribution of RV_n:

$$RV_n\Big|\mathcal{F}_T^W \sim N\left(I_1 + O\left(\frac{1}{n}\right), \frac{2}{n}\cdot I_2 + o\left(\frac{1}{n}\right)\right) \qquad (4.20)$$

The Simplified Conditional Normal Scheme. Keeping the leading order terms in (4.20), we obtain the approximation, in distribution:

$$RV_n\Big|\mathcal{F}_T^W \sim N\left(\frac{1}{T}\int_0^T v_t dt, \frac{2}{n}\cdot\frac{1}{T}\int_0^T v_t^2 dt\right). \qquad (4.21)$$

By virtue of relation (4.21), we formulate the *simplified conditional normal* (SCN) pricing scheme as follows: (a) simulate a variance path v_t, $t\in[0,T]$ and compute I_1 and I_2, (b) price the conditional variance call by setting $M_n = I_1$ and $\Sigma_n = \sqrt{\frac{2}{n}\cdot I_2}$ in formula (4.11) and (c) average conditional prices by repeating (a), (b). Note that this approach, while no longer requiring to compute the quantities $\mu_{n,i}$ and $\sigma_{n,i}^2$, still requires the simulation of the entire variance path v_t on $[0,T]$ in order to allow us to extract both I_1 and I_2.

The Simplified Conditional Black-Scholes Scheme. Next, we present a further simplification and link it to discrete variance in the Black-Scholes model. By Jensen's inequality $I_2 \geq I_1^2$ and hence we expect that by using the approximation

$$RV_n\Big|\mathcal{F}_T^W \sim N\left(\frac{1}{T}\int_0^T v_t dt, \frac{2}{n}\left(\frac{1}{T}\int_0^T v_t dt\right)^2\right) \qquad (4.22)$$

may cause some underpricing of options on realized variance (at least, relative to the SCN scheme). On the other hand, this approximation will make it possible to simulate just from the law of integrated continuous variance $I_1 = \frac{1}{T}\int_0^t v_t dt$, without the need to generate the entire variance path v_t on $[0,T]$. We shall observe, in the numerical examples, that such underpricing is usually small.

In fact, it can be shown that approximation (4.22) is asymptotically equivalent to assuming that the *conditional* pricing model is Black-Scholes. Specifically, conditional on a realization of the integrated continuous variance $\frac{1}{T}\int_0^T v_t dt$, suppose the model for the underlying price is Black-Scholes with variance parameter:

$$\sigma^2 = \frac{1}{T}\int_0^T v_t dt. \tag{4.23}$$

We have seen that, in a Black-Scholes model, the discretely sampled variance is non-central chi-square distributed $\chi^{'}(n,\lambda)$. In turn, keeping only the leading order terms in (4.13), we have that $\chi^{'}(n,\lambda)$ is approximately $N\left(\sigma^2, \frac{2}{n}\sigma^4\right)$. Replacing the value of σ^2 from (4.23), we see that the conditional Black-Scholes approach leads, in fact, to approximation (4.22).

By virtue of relation (4.22), we formulate the *simplified conditional Black-Scholes* (SCBS) scheme as follows: (a) simulate from the law of integrated continuous variance $\frac{1}{T}\int_0^T v_t dt$, (b) price the conditional variance call by setting $\sigma^2 = I_1$ in the *exact* Black-Scholes formula of Lemma 4.2.5, (c) average conditional prices by repeating (a), (b). We note that the SCBS scheme does not require to simulate the entire path of v_t, $t \in [0,T]$.

Pursuing further the SCBS scheme, it is possible to derive a simple discretization adjustment requiring *no* Monte-Carlo simulation. Specifically, under the assumption (4.22) and provided the continuously sampled variance I_1 posses a Fourier transform in closed-form, we next derive a leading-order discretization adjustment based on a simple Fourier inversion[2]. In the following, we regard the (undiscounted) prices of options on realized variance as functions of the variance strike $\mathcal{V} = \sigma_K^2$ and define $C_n, C : \mathbb{R} \to \mathbb{R}_{\geq 0}$ by

$$\begin{aligned} C_n(\mathcal{V}) &= 1_{\mathcal{V}\geq 0} \cdot \mathbb{E}\Big(RV_n - \mathcal{V}\Big)_+ \\ C(\mathcal{V}) &= 1_{\mathcal{V}\geq 0} \cdot \mathbb{E}\Big(I_1 - \mathcal{V}\Big)_+ \end{aligned}$$

where $I_1 = \frac{1}{T}\int_0^T v_t dt$ and, under the SCBS scheme (4.22), $RV_n\big|\mathcal{F}_T^W \sim N\left(I_1, \frac{2\cdot I_1^2}{n}\right)$. Assuming $\mathbb{E}(RV_n^2) < \infty$ and $\mathbb{E}(I_1^2) < \infty$, we first check that both functions $C_n(\cdot)$ and $C(\cdot) \in L^1(\mathbb{R})$, i.e. are integrable on $\mathbb{R}$. For example, we have for $C_n(\mathcal{V})$:

$$\begin{aligned} \int_{-\infty}^{\infty} \big|C_n(\mathcal{V})\big| d\mathcal{V} &= \int_0^{\infty} \mathbb{E}\left(RV_n - \mathcal{V}\right)_+ d\mathcal{V} \\ &= \mathbb{E}\int_0^{RV_n} \left(RV_n - \mathcal{V}\right) d\mathcal{V} = \frac{1}{2}\cdot\mathbb{E}\left(RV_n^2\right) < \infty \end{aligned}$$

[2]We are grateful to an anonymous referee for explicitly steering us in this direction.

where we interchanged integration and expectation as the integrand is non-negative; an identical argument holds for $C(\mathcal{V})$. Therefore, both functions will have well defined Fourier transforms, hereafter denoted by $\widehat{C}_n(u)$ and $\widehat{C}(u)$, respectively. The following formula, which first appeared in Carr et al. (2005), can be established for the Fourier transforms:

$$\begin{aligned}\widehat{C}_n(u) &= \int_{-\infty}^{\infty} e^{iu\mathcal{V}} \cdot C_n(\mathcal{V})d\mathcal{V} = \frac{1-\varphi_n(u)}{u^2} - i \cdot \frac{\mathbb{E}(RV_n)}{u} \\ \widehat{C}(u) &= \int_{-\infty}^{\infty} e^{iu\mathcal{V}} \cdot C(\mathcal{V})d\mathcal{V} = \frac{1-\varphi(u)}{u^2} - i \cdot \frac{\mathbb{E}(I_1)}{u}\end{aligned}$$

where $\varphi_n(u) = \mathbb{E}\left(e^{iuRV_n}\right)$ and $\varphi(u) = \mathbb{E}\left(e^{iuI_1}\right)$ denote the Fourier transforms of RV_n and I_1, respectively.

The key idea is to now consider the difference between the price of options on discrete variance and the price of options on continuous variance by defining the new function $\Lambda(\mathcal{V}) = C_n(\mathcal{V}) - C(\mathcal{V}) \in L^1(\mathbb{R})$. Using that $\mathbb{E}(RV_n) = \mathbb{E}\left(\mathbb{E}\left(RV_n \middle| \mathcal{F}_T^W\right)\right) = \mathbb{E}(I_1)$ and by the linearity of the Fourier transform, we obtain

$$\widehat{\Lambda}(u) = \int_{-\infty}^{\infty} e^{iu\mathcal{V}} \cdot \Lambda(\mathcal{V})d\mathcal{V} = \widehat{C}_n(u) - \widehat{C}(u) = \frac{\varphi(u) - \varphi_n(u)}{u^2}.$$

Using that

$$\varphi_n(u) = \mathbb{E}\left(e^{iuRV_n}\right) = \mathbb{E}\left(\mathbb{E}\left(e^{iuRV_n}\middle|\mathcal{F}_T^W\right)\right) = \mathbb{E}\left(e^{iuI_1 - \frac{u^2 I_1^2}{n}}\right)$$

we obtain

$$\widehat{\Lambda}(u) = \mathbb{E}\left(e^{iuI_1} \cdot \frac{1 - e^{-\frac{u^2 I_1^2}{n}}}{u^2}\right).$$

Expanding the second term of the product under the expectation and keeping the term of order $O\left(\frac{1}{n}\right)$, we can write

$$\widehat{\Lambda}(u) = \mathbb{E}\left(e^{iuI_1} \cdot \frac{I_1^2}{n}\right) + O\left(\frac{1}{n^2}\right) := \widehat{\Lambda}_1(u) + O\left(\frac{1}{n^2}\right). \tag{4.24}$$

Making use again of the assumption $\mathbb{E}(I_1^2) < \infty$, we have that $\varphi(u)$, the characteristic function of I_1, is twice continuously differentiable with respect to u and $\frac{\partial^2\varphi(u)}{\partial u^2} = -\mathbb{E}\left(e^{iuI_1} \cdot I_1^2\right)$. This gives that the leading term $\widehat{\Lambda}_1(u)$ in (4.24) can be written as

$$\widehat{\Lambda}_1(u) = -\frac{1}{n} \cdot \frac{\partial^2\varphi(u)}{\partial u^2}.$$

We now proceed to determine the discretization adjustment term that results from considering only the term $\widehat{\Lambda}_1(u)$ in the expansion (4.24). More precisely, we compute

$$\begin{aligned}
\Lambda_1(\mathcal{V}) \;&:=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iu\mathcal{V}}\cdot\widehat{\Lambda}_1(u)du = \frac{-1}{2\pi}\int_{-\infty}^{\infty} e^{-iu\mathcal{V}}\cdot\frac{1}{n}\cdot\frac{\partial^2\varphi(u)}{\partial u^2}du \\
&= \frac{-1}{2\pi n}\cdot\left[e^{-iu\mathcal{V}}\cdot\frac{\partial\varphi(u)}{\partial u}\bigg|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} i\mathcal{V}\cdot e^{-iu\mathcal{V}}\cdot\frac{\partial\varphi(u)}{\partial u}du\right] \\
&= \frac{-1}{2\pi n}\cdot\left[e^{-iu\mathcal{V}}\cdot\frac{\partial\varphi(u)}{\partial u}\bigg|_{-\infty}^{\infty} + i\mathcal{V}\cdot e^{-iu\mathcal{V}}\cdot\varphi(u)\bigg|_{-\infty}^{\infty}\right. \\
&\qquad\left. - \int_{-\infty}^{\infty}\mathcal{V}^2\cdot e^{-iu\mathcal{V}}\cdot\varphi(u)du\right] \\
&= \frac{\mathcal{V}^2}{n}\cdot\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iu\mathcal{V}}\cdot\varphi(u)du
\end{aligned}$$

where both boundary terms, resulting from the integration by parts, will vanish by a simple application of the classical Riemann-Lebesgue lemma (see, for example, Feller (1991)). In conclusion, we have obtained

$$\Lambda_1(\mathcal{V}) = \frac{\mathcal{V}^2}{n}\cdot\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iu\mathcal{V}}\cdot\varphi(u)du. \tag{4.25}$$

We notice that the computation of the discretization adjustment term $\Lambda_1(\mathcal{V})$ involves a simple Fourier inversion of $\varphi(u)$. Alternatively, the discretization adjustment can be written more compactly as

$$\Lambda_1(\mathcal{V}) = \frac{\mathcal{V}^2}{n}\cdot q\left(\mathcal{V}\right) \geq 0 \tag{4.26}$$

where $q\left(\mathcal{V}\right)$ denotes the density of the continuously sampled variance $I_1 = \frac{1}{T}\int_0^T v_t dt$. Both representations (4.25) and (4.26) provide a remarkably simple formula for the leading order discretization adjustment term.

We note that, as expected, the discretization adjustment is non-negative reflecting that options on discrete variance are more expensive than options on continuous variance. Therefore, if we work in a stochastic volatility model which admits a closed-form solution for $\varphi(u)$ (e.g. Heston (1993) or the 3/2 model in Carr, Sun (2007)), we first price options on continuously sampled variance using standard Fourier methods from the literature (e.g. Sepp (2008)) and then add the positive adjustment term (4.25), which is also computable by simple Fourier inversion.

In the following numerical examples, we consider a standard Heston (1993) parameter set from the literature, namely, the one estimated in the study of Bakshi, Cao, Chen (1997) for the S&P500 index. The same set is also used in Gatheral (2006) in pricing options on realized variance. The estimated parameters are $(v_0, k, \theta, \epsilon, \rho) = (18.65\%^2, 1.15, 18.65\%^2, 0.39, -0.64)$.

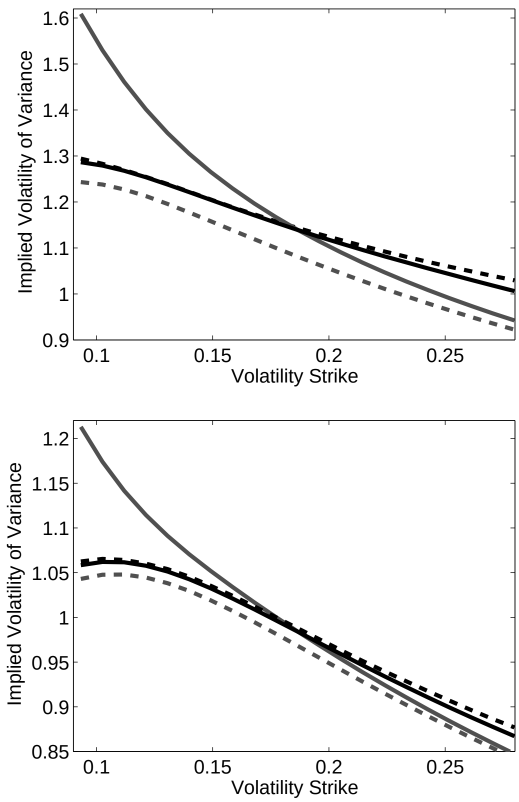
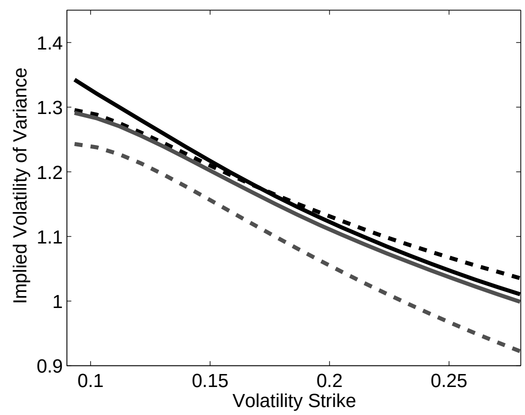


Figure 4.1: Comparison of the independent composition approach (*solid gray*) and the simple Fourier adjustment (4.25) (*solid black*) for discrete variance options with 3 months (top) and 6 months (bottom) time to maturity. Bakshi, Cao, Chen (1997) Heston parameters $(v_0, k, \theta, \epsilon, \rho) = (18.65\%^2, 1.15, 18.65\%^2, 0.39, -0.64)$.

In all the figures of this section, the prices of options on variance are expressed as *implied volatilities-of-variance* (VoV) across a wide range of volatility strikes and the following conventions apply: (1) the *dashed-gray* curve depicts the prices corresponding to continuous sampling, computed by semi-analytical transform methods (see Carr, Madan (1999), Lee (2004) and Sepp (2008)), (2) the *dashed-black* curve depicts the prices corresponding to (daily) discrete sampling, computed by full Monte-Carlo simulation of the SDE (4.1), (4.2) by the technique in Andersen (2008); the simulation technique in Andersen (2008) combines the exact simulation scheme of Broadie, Kaya (2006) with a numerically efficient local moment-matching method, and (3) the *solid* curves depict the various approximations developed in



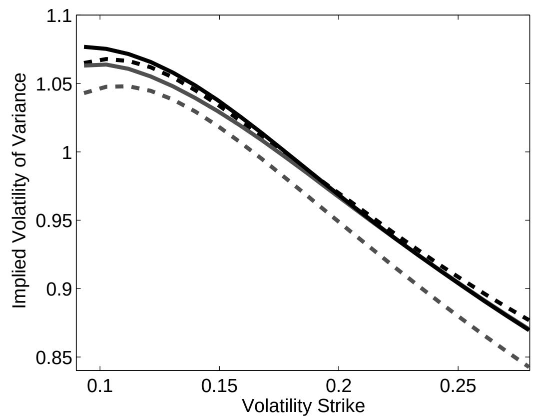


Figure 4.2: Comparison of the conditional Black-Scholes (*solid gray*) and simplified conditional normal (*solid black*) methods for discrete variance options with 3 months (top) and 6 months (bottom) time to maturity. Bakshi, Cao, Chen (1997) Heston parameters $(v_0, k, \theta, \epsilon, \rho) = (18.65\%^2, 1.15, 18.65\%^2, 0.39, -0.64)$.

the paper.

Figure (4.1) compares two Fourier based methods, namely, the independent composition (IC) method (4.15) (*solid gray*) and the simple Fourier adjustment method given by (4.25). The top panel of Figure (4.1) shows the results for a maturity of 3 months. The average difference between the discrete and continuous VoV smiles, across the strike range, was 7.4%. As expected, we observe that the independence assumption, on which the IC method is based, performs well in the at-the-money region but will tend to overprice downside puts and underprice upside calls. The alternative Fourier method, derived in the previous section, behaves well across a wide strike range, with an average error of 1.34% . A similar pattern can be observed for the 6 months maturity (bottom panel).

Next, we compare the simplified conditional normal (SCN) and conditional Black-Scholes (SCBS) methods. The results are shown in Figure (4.2), where the *solid black* curves depict the SCN method and the *solid gray* curves depict the SCBS method. As anticipated in the main text, the conditional Black-Scholes method will tend to provide lower prices of discrete variance options than the SCN scheme. Indeed, the SCBS volatility-of-variance smile was an average of 1.8% points below the actual discrete variance smile; this compares well to the average error of 1.6% of the SCN method. We observe a similar behavior also for the 6-months maturity, shown in the bottom panel of Figure (4.2).

4.3 Greeks of options on realized variance

We found that present literature provides limited treatment of the risk management sensitivities, commonly known as '*greeks*', for options on realized variance, in general, and options on discrete variance, in particular. In this section, we aim to identify and define the relevant greeks for options on variance, with a focus on their practical use and interpretation. We begin with the more familiar (and simpler) case of continuous sampling. A contract on continuously sampled variance with arbitrary, integrable payoff $\Psi\left(\int_0^T v_u du\right)$ at time T, has value at time $t \in [0,T]$ given by

$$V\left(t, v_t, \int_0^t v_u du\right) = e^{-r(T-t)} \cdot E^{\mathbb{Q}}\left(\Psi\left(\int_0^T v_u du\right) \Big| \mathcal{F}_t\right)$$

where the value function $V(t,v,q)$ satisfies the PDE

$$V_t + a(v) \cdot V_v + v \cdot V_q + \frac{1}{2} b^2(v) \cdot V_{vv} - r \cdot V = 0 \tag{4.27}$$

with terminal condition

$$V\left(T, v, q\right) = \Psi\left(q\right). \tag{4.28}$$

We note that, with continuous sampling, the value of any variance derivative depends only on the dynamics of v_t and has *no* connection to S_t. Therefore, hedging such a contract requires to hedge only against movements in the instantaneous variance v_t. However, the process v_t is not the price of a traded instrument and hence the hedging of, say, an option on variance would be accomplished by trading another, simpler, variance contract such as a variance swap. Let the value function of the chosen hedge instrument be denoted V^h.

To understand the relevant greeks from a practical risk-management perspective, let us assume the true (and unknown) dynamics of v_t, under the real-world measure $\mathbb{P}$, are given by

$$dv_t = \alpha_t dt + \beta_t dW_t^{\mathbb{P}}$$

where α_t, β_t are *unknown*, integrable processes and $W_t^{\mathbb{P}}$ is a standard $\mathbb{P}$-Brownian motion. Consider a portfolio Π consisting of 1 unit of V and $-\Delta$ units of V^h. Using Itô's lemma and the PDE (4.27) satisfied by both V and V^h, we obtain:

$$\begin{aligned} d\Pi = dV - \Delta \cdot dV^h &= r\left(V - \Delta \cdot V^h\right) dt + \frac{1}{2}\left(\beta_t^2 - b^2(v_t)\right) \cdot \left(V_{vv} - \Delta \cdot V_{vv}^h\right) dt \\ &\quad + \left(V_v - \Delta \cdot V_v^h\right) \cdot \left(dv_t - a(v_t)dt\right). \end{aligned}$$

To make the portfolio Π instantaneously risk-free, we observe that the correct hedge ratio Δ^h is

$$\Delta^h = \frac{V_v\left(t, v_t, \int_0^t v_u du\right)}{V_v^h\left(t, v_t, \int_0^t v_u du\right)} \tag{4.29}$$

as was obtained also by Broadie, Jain (2008a); for a more general version, see Buehler (2006). To the extent that the actual volatility of variance β_t differs from the model $b(v_t)$, we see that the portfolio Π will also experience the so-called 'bleed' term:

$$\frac{1}{2}\left(\beta_t^2 - b^2(v_t)\right) \cdot \left(V_{vv} - \Delta^h \cdot V_{vv}^h\right) dt.$$

Therefore, the dealer hedging a position in variance options must also monitor the variance gamma V_{vv}, known in practice as the Volga[3].

From a practical perspective, expressing the greeks with respect to instantaneous variance is unnatural, since v_t is not a traded quantity. In our opinion, a better alternative is to report greeks with respect to the square root of fair variance. Specifically, let $K^2(t, v_t; T)$ denote the fair variance of maturity T:

$$K^2(t, v_t; T) = \frac{1}{T-t} E^{\mathbb{Q}}\left(\int_t^T v_u du \Big| \mathcal{F}_t\right).$$

The major advantage of $K(t, v_t; T)$ is that, at any time t, it is an observable model-free quantity, computable from the prices of vanilla options. In what follows, we call $K(t, v_t; T)$ the *fair volatility* (although it should not be mistaken for

[3] The name is a shorthand from 'Volatility Gamma', measuring the second order sensitivity w.r.t. volatility.

$\frac{1}{T-t}E^{\mathbb{Q}}\left(\sqrt{\int_t^T v_u du}\middle|\mathcal{F}_t\right)$, which is a model-dependent quantity). For any variance contract V in the maturity bucket T, we propose to define Vega as:

$$\text{Vega} \overset{d}{=} \frac{V_v\left(t, v_t, \int_0^t v_u du\right)}{K_v(t, v_t)}. \tag{4.30}$$

This has the intuitive interpretation as the change in the value of the contract per unit change in fair volatility. A dealer looking to hedge with instrument V^h, simply takes the ratio of the two contracts' Vegas, in order to arrive at the correct hedge ratio:

$$\Delta^h = \frac{\text{Vega}}{\text{Vega}^{\text{h}}} = \frac{V_v\left(t, v_t, \int_0^t v_u du\right)}{V_v^h\left(t, v_t, \int_0^t v_u du\right)}$$

as obtained in (4.29). Similarly, we propose the definition of Volga as

$$\text{Volga} \overset{d}{=} \frac{\text{Vega}_v\left(t, v_t, \int_0^t v_u du\right)}{K_v(t, v_t)} \tag{4.31}$$

which will measure the change in Vega per unit change in fair volatility of maturity T.

In the case of contracts on discretely sampled variance, in addition to Vega and Volga, intra-fixing[4] sensitivities with respect to spot price S_t arise. Let the current time $t \in [t_k, t_{k+1})$ with $k = 0, 1, \ldots, n-1$. The realized variance RV_n can be decomposed as:

$$\begin{aligned} \text{RV}_n &= \frac{1}{\Delta \cdot n} \cdot \sum_{i=1}^{n} \log^2\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right) = \frac{k}{n} \cdot \frac{1}{\Delta \cdot k} \cdot \sum_{i=1}^{k} \log^2\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right) \\ &+ \frac{1}{\Delta \cdot n} \cdot \left(\log\left(\frac{S_t}{S_{t_k}}\right) + \log\left(\frac{S_{t_{k+1}}}{S_t}\right)\right)^2 \\ &+ \frac{n-k-1}{n} \cdot \frac{1}{\Delta \cdot (n-k-1)} \cdot \sum_{i=k+2}^{n} \log^2\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right). \end{aligned}$$

If we let RV_k denote the variance realized up to, and including, fixing k :

$$\text{RV}_k = \frac{1}{\Delta \cdot k} \cdot \sum_{i=1}^{k} \log^2\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)$$

[4] In practice, the discrete price observations are usually referred to as *resets* or *fixings*.

and, similarly, FV_{k+1} the future realized variance over $[t_{k+1}, t_n]$:

$$\mathrm{FV}_{k+1} = \frac{1}{\Delta \cdot (n-k-1)} \cdot \sum_{i=k+2}^{n} \log^2 \left(\frac{S_{t_i}}{S_{t_{i-1}}} \right)$$

we can rewrite the realized variance RV_n as

$$RV_n = \frac{k}{n} \cdot \mathrm{RV}_k + \frac{1}{\Delta \cdot n} \cdot \left(\log\left(\frac{S_t}{S_{t_k}} \right) + \log\left(\frac{S_{t_{k+1}}}{S_t} \right) \right)^2 + \frac{n-k-1}{n} \cdot FV_{k+1}.$$

We note the presence of the intra-fixing term

$$\frac{1}{\Delta \cdot n} \cdot \left(\log\left(\frac{S_t}{S_{t_k}} \right) + \log\left(\frac{S_{t_{k+1}}}{S_t} \right) \right)^2$$

which will give rise to intra-fixing Delta and Gamma with respect to S_t. To see this, consider a contract on realized variance, with integrable payoff $\Psi\left(\mathrm{RV}_n\right)$, whose value function on $[t_k, t_{k+1})$ will be denoted by V_k:

$$\begin{aligned} V_k(t, v_t, S_t) &= e^{-r(T-t)} \cdot E^{\mathbb{Q}}\left[\Psi\left(\frac{k}{n} \cdot \mathrm{RV}_k + \frac{1}{\Delta \cdot n} \cdot \left(\log\left(\frac{S_t}{S_{t_k}} \right) + \log\left(\frac{S_{t_{k+1}}}{S_t} \right) \right)^2 \right.\right. \\ &\quad \left.\left. + \frac{n-k-1}{n} \cdot FV_{k+1} \right) \middle| \mathcal{F}_t \right]. \end{aligned}$$

Note that, in our stochastic volatility framework (4.1)-(4.2), the distributions of $\log\left(\frac{S_{t_{k+1}}}{S_t}\right)$ and FV_{k+1} do *not* depend on the current spot price S_t. Their distribution depends only on the current value of the pair (t, v_t). The variable S_t arises in the function V_k only through the presence of the term $\log\left(\frac{S_t}{S_{t_k}}\right)$ inside $\Psi(\cdot)$. Hence, unlike the case of continuously sampled variance, the dealer will have an explicit Delta and Gamma exposure with respect to S_t during each intra-fixing period.

A simple, but illuminating, example is that of a variance swap. Without loss of generality, consider a zero-strike variance swap contract which pays RV_n at time T, hence $\Psi(RV_n) = RV_n$. Its value function at time $t \in [t_k, t_{k+1})$ is

$$\begin{aligned} V_k(t, v_t, S_t) &= e^{-r(T-t)} \cdot E^{\mathbb{Q}}\left[\frac{k}{n} \cdot \mathrm{RV}_k + \frac{1}{\Delta \cdot n} \cdot \left(\log\left(\frac{S_t}{S_{t_k}} \right) + \log\left(\frac{S_{t_{k+1}}}{S_t} \right) \right)^2 \right. \\ &\quad \left. + \frac{n-k-1}{n} \cdot FV_{k+1} \middle| \mathcal{F}_t \right] \end{aligned}$$

or

$$
\begin{aligned}
V_k(t, v_t, S_t) &= e^{-r(T-t)} \cdot \left[\frac{k}{n} \cdot \mathrm{RV}_k + \frac{1}{\Delta \cdot n} \cdot \log^2 \left(\frac{S_t}{S_{t_k}} \right) + \right. \\
&\quad + \frac{2}{\Delta \cdot n} \cdot \log \left(\frac{S_t}{S_{t_k}} \right) \cdot \left((r-\delta)(t_{k+1}-t) - \frac{E^{\mathbb{Q}} \left(\int_t^{t_{k+1}} v_u du \middle| \mathcal{F}_t \right)}{2} \right) + \\
&\quad \left. + \frac{1}{\Delta \cdot n} \cdot E^{\mathbb{Q}} \left(\log^2 \left(\frac{S_{t_{k+1}}}{S_t} \right) \middle| \mathcal{F}_t \right) + \frac{n-k-1}{n} \cdot E^{\mathbb{Q}} \left(\mathrm{FV}_{k+1} \middle| \mathcal{F}_t \right) \right].
\end{aligned}
$$

Recalling that in our stochastic volatility framework (4.1)-(4.2), the distributions of $\log \left(\frac{S_{t_{k+1}}}{S_t} \right)$ and FV_{k+1} do not depend on the current spot price S_t, we have

$$
\begin{aligned}
\frac{\partial V_k}{\partial S}(t, v_t, S_t) &= e^{-r(T-t)} \cdot \left[\frac{2}{\Delta \cdot n} \cdot \log \left(\frac{S_t}{S_{t_k}} \right) \cdot \frac{1}{S_t} + \right. \\
&\quad \left. + \frac{2}{\Delta \cdot n} \cdot \frac{1}{S_t} \cdot \left((r-\delta)(t_{k+1}-t) - \frac{E^{\mathbb{Q}} \left(\int_t^{t_{k+1}} v_u du \middle| \mathcal{F}_t \right)}{2} \right) \right]
\end{aligned}
\tag{4.32}
$$

and

$$
\begin{aligned}
\frac{\partial^2 V_k}{\partial S^2}(t, v_t, S_t) &= e^{-r(T-t)} \cdot \left[\frac{2}{\Delta \cdot n} \cdot \left(\frac{1}{S_t^2} - \frac{1}{S_t^2} \cdot \log \left(\frac{S_t}{S_k} \right) \right) - \right. \\
&\quad \left. - \frac{2}{\Delta \cdot n} \cdot \frac{1}{S_t^2} \cdot \left((r-\delta)(t_{k+1}-t) - \frac{E^{\mathbb{Q}} \left(\int_t^{t_{k+1}} v_u du \middle| \mathcal{F}_t \right)}{2} \right) \right].
\end{aligned}
\tag{4.33}
$$

Finally, before proceeding to the numerical examples, we note that in the case of discrete sampling, a cross-sensitivity with respect to spot and volatility arises. The Vega of the product will change as the spot price S_t changes, a sensitivity known as Vanna. Specifically, the Vanna will be defined as:

$$
\mathrm{Vanna} = \frac{\partial \mathrm{Vega}}{\partial S}(t, v_t, S_t) \tag{4.34}
$$

measuring the change in Vega per unit change in spot price.

We next illustrate how the greeks behave for an option on discrete variance and for a variance swap. Specifically, we consider a 3-months at-the-money call on variance, with daily sampling ($\Delta = \frac{1}{252}$), and its corresponding variance swap. The

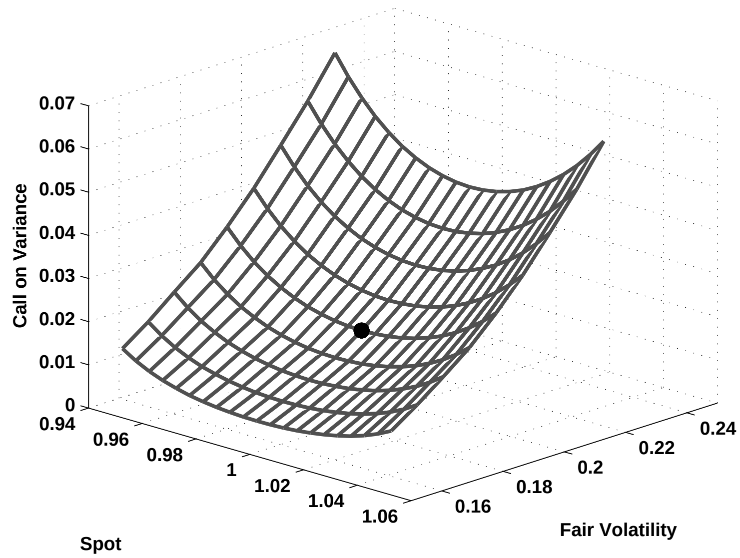


Figure 4.3: Value of 3-months call option on discrete variance with volatility strike $\sigma_K = 18.65\%$ as a function of spot price S_t and 3-months fair volatility $K(t, v_t; T)$. *Black-dot*: Value of the option of 2.04% at spot $S_t = 1$ and fair volatility 18.65%.

stochastic volatility model used is Heston (1993) with Bakshi, Cao, Chen (1997) parameters $(v_0, k, \theta, \epsilon, \rho) = (18.65\%^2, 1.15, 18.65\%^2,\ 0.39, -0.64)$. We denote the initial fixing $S_0 = S_{t_0} = 1$ and set the current time $t = \frac{\Delta}{2}$ i.e the middle of the first intra-fixing period. The fair variance for the maturity $T = 0.25$ is $18.65\%^2$ and our volatility strike is $\sigma_K = 18.65\%$. To aid in the interpretation of the numerical results, we take a vega notional $\mathcal{V} = 1$ which is equivalent to taking the variance notional $VN = \frac{1}{2 \cdot 18.65\%} = 2.68$.

As a first step, we plot in Figure (4.3) the value function for our call on variance as a function of current spot and current fair volatility. For a spot price $S_t = 1$ and fair volatility 18.65%, the value of the option is 2.04% (*black dot*). We notice that, indeed, the value of the option depends both on spot and fair volatility and that it is convex in both variables. In fact, the shape and behavior of most of the following greeks can be easily traced back to the value function plot in Figure (4.3).

Figure (4.4) plots the spot-price sensitivities (Delta and Gamma) of the variance call and its underlying variance swap. It is very important to carefully focus on the logic of the results shown in Figure (4.4). As intuitively expected, when the current spot S_t is at the value of the previous fixing $S_{t_0} = 1$, the Delta of both the variance call and the swap should be (close to) zero, as seen in the top panel of Figure (4.4). We remark that, while Delta is indeed very close to zero, it is not

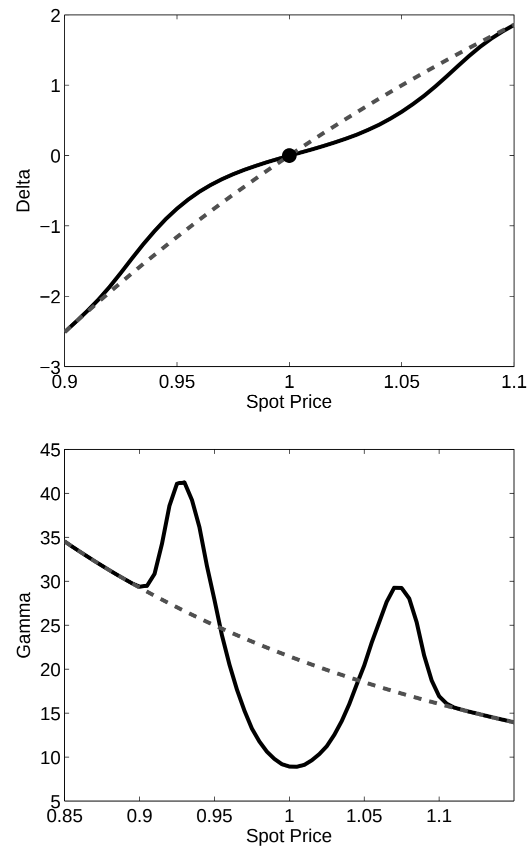


Figure 4.4: (Top) : Delta of variance call (*solid black*) and variance swap (*dashed gray*). (Bottom): Gamma of variance call (*solid black*) and variance swap (*dashed gray*).

necessarily exactly zero. To see this, recall the earlier example of the variance swap with Delta given by (4.32). The main term in the delta, namely

$$e^{-r(T-t)} \cdot \frac{2}{\Delta \cdot n} \cdot \log\left(\frac{S_t}{S_{t_k}}\right) \cdot \frac{1}{S_t}$$

will be exactly zero, for $S_t = S_{t_0}$ ($k = 0$, in our case) but, strictly mathematically, there is also the (usually very small) non-zero residual

$$e^{-r(T-t)} \cdot \frac{2}{\Delta \cdot n} \cdot \frac{1}{S_t} \cdot \left((r-\delta)(t_{k+1}-t) - \frac{E^{\mathbb{Q}}\left(\int_t^{t_{k+1}} v_u du \middle| \mathcal{F}_t\right)}{2}\right).$$

Note that the delta of both instruments is positive when $S_t > S_{t_0}$ and negative when $S_t < S_{t_0}$. This agrees with intuition : if spot has moved higher from the previous fixing, both contracts gain if spot increases still higher and, if spot has moved lower from the previous fixing, both contracts gain if spot decreases further. The Delta of the call changes more slowly than that of the swap near S_{t_0} but, if the move in spot is large enough, it will converge to the delta of the swap. This occurs because, with a large enough move in spot, the variance call becomes likely to finish in the money and, hence, it must resemble its underlying variance swap.

We can see, in the bottom panel of Figure (4.4), the Gamma of the swap decreasing with the level of spot, reflecting the fact that, the higher the spot price S_t, the same absolute move in S_t generates a smaller (in absolute value) log-return and hence less Gamma profits for the swap. In fact, in the neighborhood around S_{t_0}, the Gamma of the swap will decrease like $\frac{1}{S_t^2}$ as can be seen from the main term in (4.33), with $k = 0$:

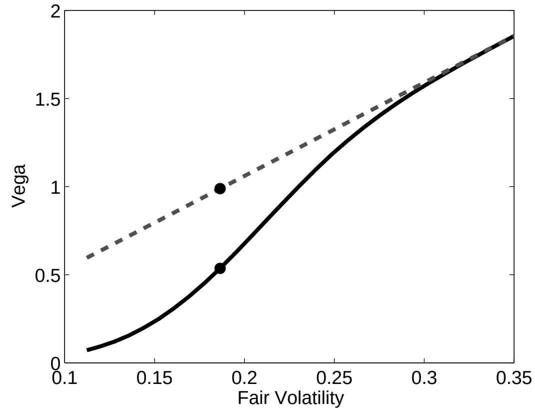
$$e^{-r(T-t)} \cdot \frac{2}{\Delta \cdot n} \cdot \frac{1}{S_t^2} \cdot \left(1 - \log\left(\frac{S_t}{S_{t_0}}\right)\right).$$

Close to the previous fixing, the Gamma of the variance call is below that of its underlying variance swap. In our case, with the call at-the-money, its Gamma is just below half of that of the swap. It would be still lower if the option was out-of-the-money, and closer to that of the swap if in-the-money. As in the case of the Delta, we notice that, if the spot price S_t makes a large swing from the previous fixing S_{t_0}, the Gamma of the call will approach that of the swap. Interestingly, in the interim, the Gamma of the call will spike above that of the swap, as the Delta of the call (see top panel of Figure (4.4)) catches up to that of the swap.

Figure (4.5) plots the Vega and Volatility Gamma (Volga) of the variance call and its underlying variance swap. As expected, the Vega of the swap is initially at 1 (by choice of the Vega notional) and is (approximately) linear in fair volatility. To see the latter point, recall that the value of the swap is driven by the square of fair volatility (and, hence, upon differentiation we obtain a Vega linear in volatility). Also, we note that the Vega of the call resembles the familiar S-shape of the Delta of vanilla stock options. This is not surprising given that Vega can be interpreted as a Delta with respect to fair volatility. The current Vega of our variance call is 0.54, hence, a dealer looking to hedge a long position in this variance call would have to sell about 0.54 units of its underlying variance swap. We now see the benefit of having defined Vega as in (4.30) (as opposed to the derivative w.r.t. 'instantaneous variance'); it allows interpretations similar to the case of vanilla stock options. Finally, if volatility were to increase substantially (making the call very likely to finish in the money) the Vega of the call will converge to that of the swap.

In, the bottom panel of Figure (4.5), we see the Volga of the swap is (approximately) constant at about 5.3 [5]. A quick sanity check of the swap Volga of 5.3,

[5] It would be perfectly constant if the variance swap had continuous sampling; our swap has



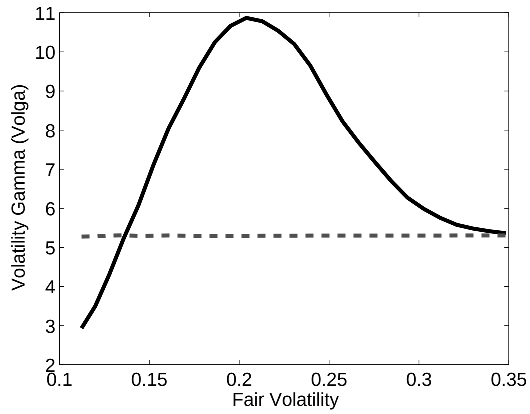


Figure 4.5: (Top) : Vega of variance call (*solid black*) and variance swap (*dashed gray*). (Bottom): Volatility Gamma (Volga) of variance call (*solid black*) and variance swap (*dashed gray*).

follows by observing that it should approximately equal $2 \cdot VN \cdot \frac{T-t}{T}$ or, in our case, roughly $2 \cdot 2.68$. Reassuringly, we notice that the Volatility Gamma of the variance call retains the bell-shape familiar from vanilla stock options Gamma. One notable difference is that its right tail no longer goes to zero but rather to the flat Volga of the underlying swap.

Finally, to illustrate the cross-sensitivity with respect to spot price and fair volatility, Figure (4.6) plots the Vanna of the variance call for different values of the spot price S_t and with fair volatility set at the current value of 18.65%. We recall that Vanna measures the sensitivity of Vega with respect to spot price. To understand the behavior of Vanna, note that, for a call on realized variance, whether spot has moved up or down from the previous fixing its Vega will increase if spot

discrete sampling.

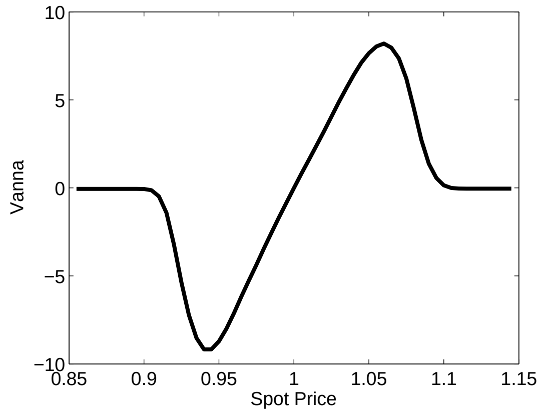


Figure 4.6: Vanna of variance call as a function of spot price S_t with fair volatility fixed at current value.

continues to move in the *same* direction; this is seen most easily in Figure (4.3). This observation helps to explain why, in Figure (4.6), Vanna is positive on the upside and negative on the downside.

Note that Figure (4.6) does not report the Vanna of the variance swap, as it is approximately zero. To see this, recall that the main term in the variance swap delta (4.32) does not depend on fair volatility. Note that, strictly mathematically, the second term in (4.32) is a cross spot-volatility term, but its contribution is very small. This can also be confirmed from Figure (4.6) where we see that, for a large enough move in spot, the Vanna of the variance call will go to zero, as it resembles its underlying variance swap.

We end the section with a plot of the value function of the combined Vega hedged position, consisting of long 1 unit of our variance call and short 0.54 units of the underlying variance swap. The interested reader can draw many interesting conclusions from Figure (4.7). Of particular importance is to notice that the Vega hedged position will be : (a) long Volatility Gamma (Volga) and (b) short spot Gamma. To see (a), note that keeping spot fixed at $S_t = 1$ the value of the combined position increases whether fair volatility moves up or down. Similarly, to see (b), note that keeping volatility fixed at 18.65% the value of the combined position decreases whether spot moves up or down.

4.4 Conclusions

Discrete sampling has significant impact on the valuation of options on realized variance. Conditional on a realization of the instantaneous variance process, while

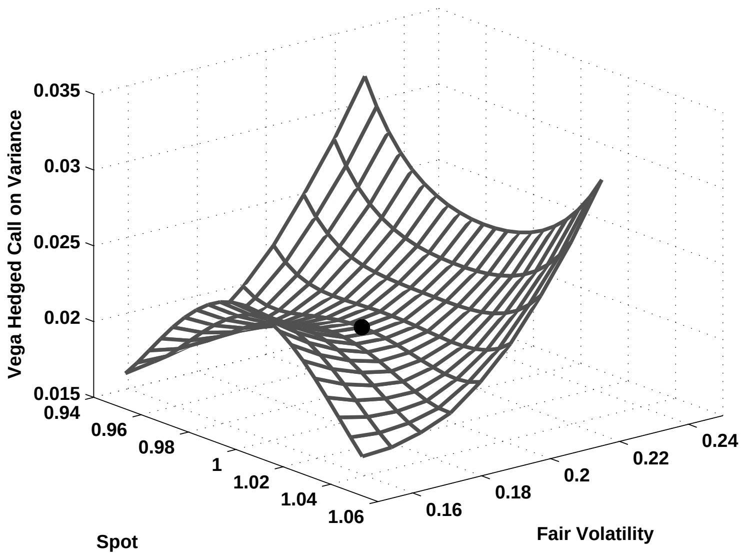


Figure 4.7: Value of Vega hedged 3-months call option on discrete variance with volatility strike $\sigma_K = 18.65\%$ as a function of spot price S_t and 3-months fair volatility $K(t, v_t; T)$. Vega hedged position: 1 x Variance call - 0.54 x Variance Swap.

the continuously sampled variance is simply a constant, the discretely sampled variance retains substantial randomness. We start by providing a characterization of this additional randomness, in Theorem 4.2.2, under general stochastic volatility dynamics and then construct several methods for pricing options on discrete variance without the need to simulate the paths of the log-returns.

Most importantly, we remark among the various methods, the conditional Black-Scholes scheme which leads to a remarkably simple and tractable leading-order discretization adjustment term, as obtained in equation (4.25). The result can be implemented in any stochastic volatility model which admits a closed-form expression for the Fourier transform of continuously sampled variance; important examples of models where our result is directly applicable include both affine stochastic volatility models (e.g. Heston (1993)) as well as tractable non-affine models, such as the 3/2 model in Lewis (2000) and Carr, Sun (2007).

4.5 Appendix

Proof of Proposition 4.2.4 Integration by parts shows that for any random variable X with $E|X| < \infty$ and distribution function $F(x)$ we have

$$E\left(X - \sigma_K^2\right)_+ = E(X) - \sigma_K^2 + \int_{-\infty}^{\sigma_K^2} F(x)dx. \tag{4.35}$$

For a fixed $n \geq 1$, we consider the sequence of independent random variables $Y_{n,1}, Y_{n,2}, \ldots, Y_{n,n}$ defined in the proof of Theorem 4.2.2. We have $E\left(Y_{n,i}\right) = 0$ and $E\left(|Y_{n,i}|^3\right) < \infty$; the exact formula for $E\left(|Y_{n,i}|^3\right)$ can be found in Lemma 4.5.1. Using that

$$\frac{\sum_{i=1}^n Y_{n,i}}{s_n} = \frac{1}{\Sigma_n}\left(\mathrm{RV}_n - \frac{\sum_{i=0}^n \mu_{n,i}^2 + \sigma_{n,i}^2}{n \cdot \Delta}\right)$$

the Theorem 4.2.3 of Bikelis implies

$$\left| F_{RV_n}\left(\Sigma_n \cdot x + M_n\right) - N(x) \right| \leq \frac{A \cdot L_n}{\left(1 + |x|\right)^3}$$

or, equivalently

$$\left| F_{RV_n}(x) - N\left(\frac{x - M_n}{\Sigma_n}\right) \right| \leq \frac{A \cdot L_n}{\left(1 + \left|\frac{x - M_n}{\Sigma_n}\right|\right)^3} \tag{4.36}$$

where F_{RV_n} denotes the conditional distribution of the discretely sampled variance and $L_n = s_n^{-3} \cdot \sum_{i=1}^n E\left(|Y_{n,i}|^3\right)$. By (4.35) and (4.36) we have

$$\left| C(\sigma_K) - C^n(\sigma_K) \right| \leq \int_{-\infty}^{\sigma_K^2} \left| F_{RV_n}(x) - N\left(\frac{x - M_n}{\Sigma_n}\right) \right| dx = \int_{-\infty}^{\sigma_K^2} \frac{A \cdot L_n}{\left(1 + \left|\frac{x - M_n}{\Sigma_n}\right|\right)^3} dx.$$

Making the change of variable $\frac{x - M_n}{\Sigma_n} = y$ the remaining calculations are straightforward. For example, in the case $\sigma_K^2 \leq M_n$ the integral becomes

$$\int_{-\infty}^{\frac{\sigma_K^2 - M_n}{\Sigma_n}} \frac{A \cdot \Sigma_n \cdot L_n}{\left(1 - y\right)^3} dy = \frac{A \cdot \Sigma_n \cdot L_n}{2\left(1 + \frac{M_n - \sigma_K^2}{\Sigma_n}\right)^2}.$$

The case $\sigma_K^2 > M_n$ is solved similarly. $\square$

Proof of Lemma 4.2.5 In the main text, we showed that, in the Black-Scholes model, $RV_n \overset{d}{=} \frac{\sigma^2}{n} \cdot \chi'\left(n, \lambda\right)$ where

$$\lambda = \frac{\left(r - \delta - \frac{\sigma^2}{2}\right)^2 T}{\sigma^2}$$

and $\chi^{'}(n,\lambda)$ denotes the non-central chi-square distribution with n degrees of freedom and non-centrality parameter λ. We recall the density of a $\chi^{'}(n,\lambda)$ random variable:

$$f_{\chi'}(x;\lambda,n) = \sum_{i=0}^{\infty} \frac{e^{-\frac{\lambda}{2}} \cdot \left(\frac{\lambda}{2}\right)^i}{i!} \cdot f_{\chi}(x;n+2i)$$

where $f_{\chi}(x;n)$ denotes the PDF of a chi-square random variable with n degrees of freedom :

$$f_{\chi}(x;n) = \frac{1}{2^{n/2}\Gamma(n/2)} x^{n/2-1} e^{-x/2} \cdot 1_{x>0}.$$

It is straightforward to show that $x \cdot f_{\chi}(x;n) = n \cdot f_{\chi}(x;n+2)$, which in turn allows us to write

$$\begin{aligned} x \cdot f_{\chi'}(x;\lambda,n) &= \sum_{i=0}^{\infty} \frac{e^{-\frac{\lambda}{2}} \cdot \left(\frac{\lambda}{2}\right)^i}{i!} \cdot (n+2i) \cdot f_{\chi}(x;n+2+2i) \\ &= n \cdot f_{\chi'}(x;\lambda,n+2) + \lambda \cdot f_{\chi'}(x;\lambda,n+4). \end{aligned} \tag{4.37}$$

The expectation to compute becomes

$$E^{\mathbb{Q}}\left(\frac{\sigma^2}{n} \cdot \chi^{'}(n,\lambda) - \sigma_K^2\right)_+ = \int_{\frac{\sigma_K^2 \cdot n}{\sigma^2}}^{\infty} \left(\frac{\sigma^2}{n} \cdot x - \sigma_K^2\right) \cdot f_{\chi'}(x;\lambda,n)\,dx$$

Using property (4.37) derived above, the result follows immediately. □

Lemma 4.5.1 *If we let*

$$Y_{n,i} = \log^2\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right) - \left(\mu_{n,i}^2 + \sigma_{n,i}^2\right)$$

we have

$$\begin{aligned} E\left|Y_{n,i}\right|^3 &= \sigma_{n,i}^6 \cdot \Big(\Psi(d_+) \cdot \phi(d_+) - \Psi(d_-) \cdot \phi(d_-) + (48\alpha^2+16) \cdot (N(d_-) - N(d_+)) + \\ &\quad 24\alpha^2 + 9\Big) \end{aligned}$$

where

$$d_{\pm} = \frac{-\mu_{n,i} \pm \sqrt{\mu_{n,i}^2 + \sigma_{n,i}^2}}{\sigma_{n,i}}.$$

and

$$\Psi(d) = 2d^5 + 12\alpha d^4 + (24\alpha^2+4)d^3 + (16\alpha^3+24\alpha)d^2 + (48\alpha^2+18)d + 32\alpha^3 + 60\alpha$$

with $\alpha = \frac{\mu_{n,i}}{\sigma_{n,i}}$.

Proof To simplify notation we drop the subscripts and let $\mu_{n,i} = \mu$ and $\sigma_{n,i} = \sigma$. We note that

$$Y_{n,i} \stackrel{d}{=} (\mu + \sigma \cdot N(0,1))^2 - \mu^2 - \sigma^2$$

where $N(0,1)$ denotes a standard normal variable. We thus have to compute

$$\int_{-\infty}^{\infty} \left| (\mu + \sigma \cdot x)^2 - \mu^2 - \sigma^2 \right|^3 \cdot \phi(x) dx$$

where $\phi(x)$ denotes the standard normal density. By solving

$$\begin{aligned} (\mu + \sigma \cdot x)^2 - \mu^2 - \sigma^2 &= 0 \Leftrightarrow \\ \sigma^2 \cdot x^2 + 2\mu\sigma \cdot x - \sigma^2 = 0 \end{aligned}$$

we obtain the roots

$$d_{\pm} = \frac{-\mu \pm \sqrt{\mu^2 + \sigma^2}}{\sigma}.$$

Separating the positive and negative regions of the modulus, the integration becomes

$$\sigma^6 \cdot \Bigg[\int_{-\infty}^{d_-} \left(x^2 + 2\alpha \cdot x - 1\right)^3 \cdot \phi(x) dx - \int_{d_-}^{d_+} \left(x^2 + 2\alpha \cdot x - 1\right)^3 \cdot \phi(x) dx \\ + \int_{d_+}^{\infty} \left(x^2 + 2\alpha \cdot x - 1\right)^3 \cdot \phi(x) dx \Bigg]$$

where we put $\alpha = \frac{\mu}{\sigma}$. To compute these integrals, we note that by letting $u_n(x) = \int x^n \cdot \phi(x) dx$ and making use of the relationship

$$u_n(x) = -x^{n-1} \cdot \phi(x) + (n-1) \cdot u_{n-2}(x)$$

with $u_0(x) = N(x)$ and $u_1(x) = -\phi(x)$, we obtain

$$\begin{aligned} u_2(x) &= -x\phi(x) + N(x) \\ u_3(x) &= -(x^2 + 2) \cdot \phi(x) \\ u_4(x) &= -(x^3 + 3x) \cdot \phi(x) + 3 \cdot N(x) \\ u_5(x) &= -(x^4 + 4x^2 + 8) \cdot \phi(x) \\ u_6(x) &= -(x^5 + 5x^3 + 15x) \cdot \phi(x) + 15 \cdot N(x). \end{aligned}$$

Finally, the remaining steps involve only algebraic calculations to arrive at the result stated in the lemma. □

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V

A forward started jump-diffusion model and pricing of cliquet style exotics

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Abstract

We present an alternative model for pricing exotic options and structured products with forward-starting components. As presented in the recent study by Eberlein, Madan (2009), the pricing of such exotic products (which consist primarily of different variations of locally / globally, capped / floored, arithmetic / geometric etc. cliquets) depends critically on the modeling of the *forward-return* distributions. Therefore, in our approach, we directly take up the modeling of forward variances corresponding to the tenor structure of the product to be priced. We propose a two factor forward variance market model with jumps in returns and volatility. It allows the model user to directly control the behavior of future smiles and hence properly price forward smile risk of cliquet-style exotic products. The key idea, in order to achieve consistency between the dynamics of forward variance swaps and the underlying stock, is to adopt a forward starting model for the stock dynamics over each reset period of the tenor structure. We also present in detail the calibration steps for our proposed model.

5.1 Introduction

A critical component of trading and hedging exotic equity products is represented by the vega hedging costs. Among the more standard exotics, such as plain barrier options, vega hedging is concentrated primarily at time zero, or at trade inception, without major rebalancing required during the life of the trade, as such products tend to have small exposure to the volatility of volatility. This fact is intimately related to the success of static hedging approaches that have been proposed in the literature for such products; we refer the reader to Carr, Ellis, Gupta (1998) and Poulsen (2006). Therefore, the hedger is primarily exposed to the value of options today and requires a model that adequately reprices today's implied volatility surface. Several models have been proposed in the literature which are capable of fitting the volatility surface both across strikes and maturities. To achieve this flexibility such models typically combine stochastic volatility with jumps in the underlying. We note two main approaches that have been developed and proposed in the literature: diffusion based stochastic volatility models with Poisson arrival process for the jumps (Bates (1996), Bakshi, Cao, Chen (1997)) and general Levy processes run on a stochastic clock (Carr, Geman, Madan, Yor (2003)). A comprehensive survey which shows the calibration capabilities of these models is provided by Schoutens et. al (2004).

However, as soon as we consider the more sophisticated equity structured products which are now popular in the market[1], in particular, the many variations of cliquet-style products including Napoleons, accumulators, swing cliquets, reverse cliquets[2] etc. we face a difficult choice among the classical models. Such products, due to their forward starting components, require the trader to renew the vega hedge at the beginning of each reset period. It is thus very important that the chosen model prices-in dynamics for the future smiles which are consistent with the observed dynamics of volatility smiles in reality. Bergomi (2004) shows how the standard models developed in the literature impose constraints on the dynamics of forward skew and do not allow the model user to control this critical feature of the model. In a subsequent paper, Bergomi (2005) proposes a forward starting constant-elasticity of variance (or CEV) model for the asset, along with consistent log-normal two-factor dynamics for the forward variances term structure; the model enables its user to directly control the forward skew and the volatility of volatility term structure.

In this paper we propose an extension of the Bergomi (2005) model to include jumps in the asset return and in the dynamics of forward variances. Additionally, we replace the forward-starting CEV model with a forward starting Merton jump diffusion model for the asset. The advantages are threefold. Firstly, the cliquet-style products in question, normally, have reset periods of 1 to 6 months in length and, at such short expiries, the market implied volatilities are best matched with a model which incorporates jumps. Secondly, it is common to observe in the market that downward jumps in the asset price lead to almost instantaneous upward jumps

[1]see Eberlein, Madan (2009) for how the market volumes of equity structured products have been increasing in the United States and Europe.

[2]examples of payoffs of such products will be provided subsequently in the paper.

in implied volatilities, a phenomenon which directly impacts vega rehedging costs. In our model the forward variances term structure is allowed to experience upward jumps when the asset price jumps. Thirdly, the Merton jump diffusion model is easier to handle for Monte Carlo simulations and also allows us to derive closed form solutions for the forward-starting parameters in terms of the realization of fair variances.

The literature has provided substantial evidence on the phenomenon of volatility jumps. Using both options market data and historical return data, studies by Bakshi, Cao, Chen (1997) and Bates (2000) analyze the performance of stochastic volatility models with jumps in the underlying asset returns. The authors indicate that, while such models improve significantly over the misspecification of the standard Black-Scholes model (1973), they still require very high volatility-of-volatility and correlation parameters to match the pronounced negative skewness and excess kurtosis observed in the market. It is suggested that an additional improvement could be obtained from allowing random jumps in the volatility process. This suggestion is then formulated in the double-jump model of Duffie, Pan, Singleton (2000), in the context of the general affine-jump diffusion class. They analyze the impact of volatility jumps on the shape of the volatility smile, in the case of simultaneous jumps in log-returns and volatility. By calibrating their double-jump model to S&P500 options market data, they find evidence that, indeed, the volatility-of-volatility parameter is substantially reduced when allowing for jumps in the volatility process; this confirms the observation made in Bakshi et al. (1997) and Bates (2000). Similar to the case considered by Duffie et al. (2000), our model assumes that jumps in the underlying are contemporaneous with the jumps in volatility. Barndorff-Nielsen, Shephard (2001) propose a Levy-driven positive Ornstein-Uhlenbeck process for the short variance with simultaneous jumps in the log-return process. The use of Levy processes to construct general classes of stochastic volatility models is studied in Carr, Geman, Madan, Yor (2003). Our modeling approach remains different from that of the previously cited works, in that we directly model the evolution of the entire term structure of forward variances.

Other authors have also acknowledged the importance of constructing alternative models for the pricing of forward starting exotic options. Eberlein, Madan (2009) note that there is a need to build models in which the forward-return distributions do not depart too much from spot-return distributions. They propose the use of Sato processes to address this need. Overhaus et al. (2007) discuss the importance of skew propagation in models for pricing cliquet style exotics and they propose the use of a forward started Heston model. In our approach, in addition to providing direct control over the forward skew, we model the term structure of forward variance swaps which now become natural calibration and hedging instruments.

The paper is organized as follows. In section two, we introduce the model dynamics for forward variances and the stock. In particular, we show the consistency and smile behavior conditions that must be satisfied by the forward-starting model parameters. Unlike in the CEV-based model proposed by Bergomi (2005), we are able to derive closed form solutions for the forward-starting parameters. In section three, we give a detailed step-by-step description of the model calibration. We also price three different exotic contracts with the parameters calibrated to actual market data. The last section provides the main conclusions.

5.2 Model description

To motivate the subsequent development of our model, we begin with a definition of the types of exotic products which form the focus of this study. Cliquets form a broad class of exotic options whose payoff is defined in terms of a sequence of forward-started returns, with a given reset-period frequency. Specifically, if we consider the equally spaced tenor structure $0 = T_0 < T_1 < T_2 < \ldots < T_N$ and let $R_i = \frac{S(T_{i+1})}{S(T_i)} - 1$ denote the asset return corresponding to the reset period $[T_i, T_{i+1}]$ then, in the general case, the payoff of a cliquet option can be written as:

$$\text{Payoff} = h\left(\sum_{i=0}^{N-1} g\left(R_i\right)\right)$$

where $g(\cdot)$ is a local function applied to each forward-return and $h(\cdot)$ is a global function applied to the accumulated (transformed) returns. All the cliquet products considered in this paper can be accommodated in this definition. For example, the payoff of an "accumulator" is obtained by setting $g(x) = \max(\min(x, C), F)$ and $h(x) = \max(x, 0)$, where F and C denote return floor and cap levels specified in the contract. The basic building blocks of a cliquet product are a series of forward starting options. Therefore, as shown in Eberlein, Madan (2009), the forward return distribution plays the key role in pricing these products; in particular, in our approach we model directly the forward variance term structure and the behavior of future implied volatility skews.

We start by specifying the model for the forward variances. Similar to the idea of LIBOR market models, we will not model the instantaneous forward variances but rather a finite set of discrete forward variances corresponding to a given tenor structure. Let us consider the equally spaced tenor structure $0 = T_0 < T_1 < T_2 < \ldots < T_N$ where $T_{i+1} - T_i = \Delta$ for all $i = 0, 1, \ldots, N-1$. We assume that variance swaps of maturities T_i are tradable in the market (for a comprehensive introduction to variance swaps we refer the interested reader to Carr, Madan (2002)). Without loss of generality, we will work with zero-strike variance swaps. The forward variance over the interval $[T_i, T_{i+1}]$, as seen at time $t \in [T_0, T_i]$, will be denoted by ξ_t^i. It is straightforward to show that the forward variance swap rate ξ_t^i is given by the future value of a simple portfolio consisting of variance swaps of maturities T_i and T_{i+1}. Specifically:

$$\xi_t^i = \frac{e^{r(T_{i+1}-t)}}{\Delta}\left(T_{i+1}V_t^{i+1} - T_i e^{-r\Delta} V_t^i\right)$$

where r is the assumed risk-free rate in the economy, V_t^i and V_t^{i+1} denote the values at time t of the variance swaps of maturities T_i and T_{i+1} respectively. To see that ξ_t^i is indeed the fair price to pay at time T_{i+1} for the variance realized over the interval $[T_i, T_{i+1}]$, we note that $T_{i+1}V_t^{i+1}$ is the fair price to pay *now* at time t to receive, at time T_{i+1}, the deannualized realized variance over $[T_0, T_{i+1}]$; similarly, $T_i e^{-r\Delta} V_t^i$ is the fair price now of the variance realized over $[T_0, T_i]$ and received at time T_{i+1} (notice that, in this case, we have the payoff of V_t^i delayed until T_{i+1} i.e. Δ units of time) . Finally, multiplying by the future value factor will give the fair price to be paid at time T_{i+1}; Δ annualizes the result.

Since ξ_t^i is given by the future value of a tradable portfolio, by no arbitrage, is must be driftless under the pricing measure. In what follows, we propose the following driftless jump-diffusion dynamics:

$$\frac{d\xi_t^i}{\xi_{t-}^i} = \omega\left(e^{-k_1(T_i-t)}dW_1(t) + \theta e^{-k_2(T_i-t)}dW_2(t) + e^{-k_j(T_i-t)}\left(dZ(t) - mdt\right)\right) \tag{5.1}$$

where $W_1(t)$, $W_2(t)$ are standard Brownian motions with correlation ρ, i.e. $dW_1(t)dW_2(t) = \rho dt$, $Z_t = \sum_{l=1}^{N_t} J_l^\xi$ is an independent compound Poisson process with intensity λ, J_l^ξ independent and identically distributed non-negative random variables which model the jump sizes for variance and $m = E(Z(1))$. In this paper, we will take J_l^ξ to be exponentially distributed with parameter $1/\eta$, hence $m = \lambda\eta$, although other assumptions for the distribution of variance jumps can be easily incorporated in our framework. By applying Ito's lemma for jump-diffusions to the process $\log(\xi_t^i)$ over an interval $[0, t]$ with $t \leq T_i$, we obtain the following expression for ξ_t^i :

$$\begin{aligned}
\xi_t^i \;=\; & \xi_0^i \exp\Bigg[\omega\left(\int_0^t e^{-k_1(T_i-s)}dW_1(s) + \theta\int_0^t e^{-k_2(T_i-s)}dW_2(s)\right) - \\
& -\frac{\omega^2}{2}\Big(\int_0^t e^{-2k_1(T_i-s)}ds + \theta^2\int_0^t e^{-2k_2(T_i-s)}ds + 2\rho\theta\int_0^t e^{-(k_1+k_2)(T_i-s)}ds\Big) \\
& -m\omega\int_0^t e^{-k_j(T_i-s)}ds\Bigg] \cdot \prod_{\Delta Z_s \neq 0, s\in[0,t]}\left(1 + \omega e^{-k_j(T_i-s)}\Delta Z_s\right)
\end{aligned} \tag{5.2}$$

Similar to Bergomi (2005) , we introduce (X_t) and (Y_t) Gaussian Ornstein-Uhlenbeck processes satisfying:

$$\begin{aligned} dX(t) &= -k_1X(t)dt + dW_1(t), \quad X(0) = 0 \\ dY(t) &= -k_2Y(t)dt + dW_2(t), \quad Y(0) = 0 \end{aligned}$$

or, in integrated form:

$$\begin{aligned} X(t) &= \int_0^t e^{-k_1(t-s)}dW_1(s) \\ Y(t) &= \int_0^t e^{-k_2(t-s)}dW_2(s). \end{aligned}$$

Using X_t and Y_t, we rewrite the equation for ξ_t^i to obtain the final form:

$$\begin{aligned} \xi_t^i &= \xi_0^i \exp\Bigg[\omega\left(e^{-k_1(T_i-t)}X_t + \theta e^{-k_2(T_i-t)}Y_t\right) - \frac{\omega^2}{2}\Big(e^{-2k_1(T_i-t)}\mathrm{Var}(X_t) \\ &\quad +\theta^2 e^{-2k_2(T_i-t)}\mathrm{Var}(Y_t) + 2\theta e^{-(k_1+k_2)(T_i-t)}\mathrm{Cov}(X_t,Y_t)\Big) \\ &\quad -m\omega e^{-k_j(T_i-t)}\frac{1-e^{-k_jt}}{k_j}\Bigg] \cdot \prod_{\Delta Z_s \neq 0, s\in[0,t]} \left(1+\omega e^{-k_j(T_i-s)}\Delta Z_s\right) \end{aligned} \tag{5.3}$$

where

$$\begin{aligned} \mathrm{Var}(X_t) &= \frac{1-e^{-2k_1t}}{2k_1} \\ \mathrm{Var}(Y_t) &= \frac{1-e^{-2k_2t}}{2k_2} \\ \mathrm{Cov}(X_t,Y_t) &= \rho\frac{1-e^{-(k_1+k_2)t}}{k_1+k_2}. \end{aligned}$$

The parameters of the dynamics of forward variances will be calibrated to an input term structure of volatility of volatility. Specifically, we have to consider the pricing of options on forward variances of different maturities. A call option on the $[T_1, T_{n+1}]$, $n \leq N-1$, forward variance with expiry T_1 will have payoff:

$$\left(\frac{\sum_{i=1}^n \xi_{T_1}^i}{n} - K\right)_+ \tag{5.4}$$

For $K = (\sum_{i=1}^n \xi_0^i)/n$, the Black Scholes implied volatility of this option gives us the at-the-money (ATM) volatility of a forward variance swap of maturity $n\Delta$ starting at T_1. Since, in practice, it is more common to speak of volatilities of volatility, in what follows we work with the volatilities of variance divided by two [3]. As we vary n from 1 to $N-1$, we obtain the term structure of volatility of volatility[4] for expiry $T_1 = \Delta$. Under the dynamics (5.1), we do not have a closed form formula for options on forward variances (we note that this is the case even without jumps). Nevertheless, it is not difficult to price options of the form (5.4) by Monte Carlo. We notice that conditional on the realization of the jumps, all variables $\xi^i_{T_1}$ are log-normal. Similar to Kemma, Vorst (1990), in order to price options on $(\sum_{i=1}^n \xi^i_{T_1})/n$ by Monte Carlo, we can use the options on the geometric average, with payoff:

$$\left(\left(\prod_{i=1}^{n} \xi^i_{T_1}\right)^{1/n} - K\right)_+$$

as control variates; they can be priced with a Black-Scholes formula. Alternatively, the conditional moments of $(\sum_{i=1}^n \xi^i_{T_1})/n$ can be obtained in closed form and moment matching techniques can be applied. Such techniques for sums of log-normal random variables have been discussed widely in the literature; see, for example, Milevsky et al. (1998) or Mehta et al. (2006). In the last step, we only have to integrate over the jumps. For the calculations in this paper, we use the Monte Carlo approach with geometric control variates.

To illustrate the workings of the forward variance dynamics in (5.1) we show in figure (5.1) the effects of the main parameters on the term structure of Black ATM volatilities of volatility. For clarity, we take the particular case of $\theta = 0$ and $\lambda = 0$. We notice that the ratio ω/k_1 controls the long-term level of vol-of-vol, while k_1 controls the steepness of the term structure. Already with just two parameters, namely ω and k_1, we obtain a flexible vol-of-vol curve. When allowing for a second factor (i.e. $\theta \neq 0$) with $k_2 < k_1$, we gain more control over the long end part of the curve and thus an even richer set of term structure shapes become possible.

The stock will follow a series of forward starting Merton (1976) Jump Diffusion (MJD) models. Specifically, over each reset period $[T_i, T_{i+1}]$, with $i \in \{0, 1, 2, \ldots, N-1\}$, the stock will follow dynamics:

$$\frac{dS^i_t}{S^i_{t-}} = (r - \delta - \gamma_i)\, dt + \sigma_i dB_t + \left(e^{J^{S_i}} - 1\right) dN_t \qquad (5.5)$$

where $S^i_{T_i} = S^{i-1}_{T_i}$ for $i \geq 1$, $S^0_{T_0} = S_0$, r is the risk free rate, δ is the dividend yield, $dB(t)dW_1(t) = \rho_{SX} dt$ and $dB(t)dW_2(t) = \rho_{SY} dt$ are correlations between stock and

[3]Note that if X is a r.v. with B-S volatility σ, i.e. $X = X_0 e^{-\frac{\sigma^2}{2}T + \sigma\sqrt{T}Z}$ where $Z \sim N(0,1)$, then X^2 will have B-S volatility 2σ, since $X^2 = X_0^2 e^{\sigma^2} e^{-\frac{(2\sigma)^2}{2}T + 2\sigma\sqrt{T}Z}$.

[4]for brevity, we will sometimes use the abbreviation *vol-of-vol*.

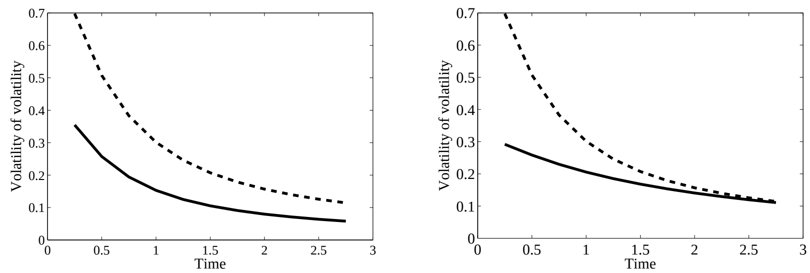

Figure 5.1: Term structures of Black ATM volatilities of volatility. (Left) Dashed: $\omega = 2$, $k_1 = 3$, Solid: $\omega = 1$, $k_1 = 3$. (Right) Dashed: $\omega = 2$, $k_1 = 3$, Solid : $\omega = 2/3$, $k_1 = 1$. Notice that in this case we keep $\omega/k_1 = 2/3$. We see that k_1 controls the speed of the decay of the vol-of-vol term structure and the ratio ω/k_1 controls the overall level of the curve.

the short-end (and long-end respectively) of the term structure of variance; (N_t) is the independent Poisson process which is also driving the variance dynamics, $J^{S_i} \sim N(a_i, b^2)$ and γ_i is the MJD drift compensator $\gamma_i = \lambda(\exp(a_i + b^2/2) - 1)$. We note that the correlations ρ_{SX}, ρ_{SY} and ρ between $B(t)$, $W_1(t)$ and $W_2(t)$ cannot be chosen arbitrarily as the correlation matrix has to be positive semidefinite; we adopt the parametrisation from Bergomi (2005) and use: $\rho_{SY} = \rho_{SX}\rho + \chi\sqrt{1-\rho_{SX}^2}\sqrt{1-\rho^2}$ with $\chi \in [-1, 1]$.

The key idea of the proposed model is now the following: both a_i and the diffusive volatility σ_i are determined at time T_i to achieve (a) consistency between the dynamics of variance and stock and (b) the desired behavior of future smiles. Note that the parameters b and λ are kept the same over all reset periods.

In our setting, consistency means that the expected quadratic variation of log returns over the interval $[T_i, T_{i+1}]$, as seen at time T_i, must agree with the realization of the fair variance $\xi^i_{T_i}$. This leads to the following *consistency* condition on the pair (a_i, σ_i); we note that, below, $[\cdot]$ denotes the quadratic variation process:

$$E_{T_i}\left(\left[\log S^i\right]_{T_{i+1}} - \left[\log S^i\right]_{T_i}\right) = E_{T_i}\left(\sigma_i^2(T_{i+1} - T_i) + \sum_{k=1}^{N_{T_{i+1}} - N_{T_i}} \left(J_k^{S_i}\right)^2\right) = \xi^i_{T_i}\Delta$$

or, after simplification and using $E_{T_i}\left((J_k^{S_i})^2\right) = a_i^2 + b^2$, we obtain:

$$\sigma_i^2 + \lambda(a_i^2 + b^2) = \xi^i_{T_i} \tag{5.6}$$

The second condition we propose on the pair (a_i, σ_i) is designed to give the model user control over the behavior of forward skew. We recall here that the distribution skew of the log-return at some time horizon Δ in an MJD model of parameters

$(\sigma_i, \lambda, a_i, b)$ is given by (see Cont & Tankov (2004)):

$$\frac{(3b^2 + a_i^2)a_i\lambda}{\left(\sigma_i^2 + \lambda(a_i^2 + b^2)\right)^{3/2}} \frac{1}{\sqrt{\Delta}}.$$

As the shape of the implied volatility skew is directly related to the distribution skew (Backus et al. (2004)[5] show that the at-the-money implied volatility skew is proportional to the distribution skew of the log return divided by the square root of time), we formulate the *smile behavior* condition as:

$$\frac{(3b^2 + a_i^2)a_i\lambda}{\left(\sigma_i^2 + \lambda(a_i^2 + b^2)\right)^{3/2}} \frac{1}{\sqrt{\Delta}} = \alpha\sqrt{\Delta} \tag{5.7}$$

where α is a parameter chosen by the trader or model user; we will call this the future smile shape parameter. As we show in the next section, α can also depend on $\xi^i_{T_i}$, in which case the trader can choose future smile shapes which depend on the level of fair variance. Together, conditions (5.6) and (5.7) give the following simple system of two equations for the pair (a_i, σ_i):

$$\begin{cases} \sigma_i^2 + \lambda a_i^2 &= \xi^i_{T_i} - \lambda b^2 \\ 3b^2 a_i + a_i^3 &= \frac{\alpha\Delta(\xi^i_{T_i})^{3/2}}{\lambda} \end{cases} \tag{5.8}$$

To address the nonlinear equation in a_i we make the substitution $a_i = b(Q_i - 1/Q_i)$ which reduces the cubic equation to a solvable quadratic equation. Retaining the solution which makes $a_i \leq 0$ (i.e. downward sloping skew in MJD) we obtain in the end:

$$\begin{aligned} a_i &= b\left(Q_i - \frac{1}{Q_i}\right) \qquad (5.9) \\ Q_i &= \sqrt[3]{\frac{\frac{R_i}{b^3} + \sqrt{\frac{R_i^2}{b^6} + 4}}{2}} \\ R_i &= \frac{\alpha\Delta(\xi^i_{T_i})^{3/2}}{\lambda}. \end{aligned}$$

Once the mean-jump parameter a_i has been calculated by equation (5.9), we get the diffusive volatility parameter as:

$$\sigma_i = \sqrt{\xi^i_{T_i} - \lambda b^2 - \lambda a_i^2}. \tag{5.10}$$

[5] unpublished paper Backus, D., Foresi, S., Wu, L., Accounting for Biases in Black-Scholes(August 31, 2004). Available at SSRN: http://ssrn.com/abstract=585623.

In figure (5.2) we show how the mean jump size a_i and the diffusive volatility σ_i depend on the level of fair variance $\xi^i_{T_i}$. In line with intuition, we notice that the higher the level of fair variance observed at the beginning of the reset period, the higher will be (in absolute value) the mean size of the jump and the diffusive volatility.

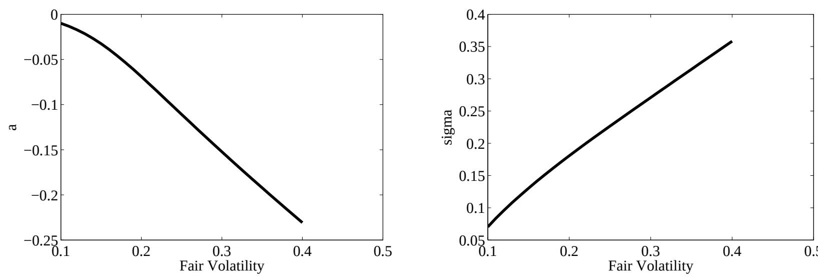

Figure 5.2: (Left) Mean-jump parameter a_i as a function of the fair variance ξ (here expressed in volatility terms i.e. $\sqrt{\xi}$). The higher the fair volatility, the higher (in absolute value) the expected downward jump in the stock. (Right) Same for diffusive volatility parameter σ_i as a function of the fair volatility.

We now summarize the main components of the proposed model. Firstly, given a fixed tenor structure, we start by specifying the dynamics of discrete forward variances using two Gaussian Ornstein-Uhlenbeck processes and exponentially distributed jumps with Poisson arrivals, according to SDE (5.1). As explained, in this setup it is possible to develop fast Monte Carlo pricing for options on forward variances; this will be important in the next section when parameters are calibrated to match a given term structure of implied volatilities of volatility. Secondly, the stock follows forward-starting Merton jump-diffusion dynamics over each reset period, according to SDE (5.5). Specifically, at the beginning of each period $[T_i, T_{i+1}]$ two of the MJD parameters – namely, the mean jump size (denoted by a_i) and the diffusive volatility parameter (denoted by σ_i) – are recalculated to achieve consistency with the realization of fair variance (equation (5.6)) and to match the desired skew for the forward-return distribution (equation (5.7)). The two conditions yield closed form solutions for the mean-jump and diffusive volatility parameters; they are given in equations (5.9) and (5.10).

5.3 Model implementation, pricing and numerical examples

In this section we present the steps for model calibration using actual market data. We then use the model to price cliquet-style exotic products. Indeed, in practice one can encounter many variations of such products. For many examples of products relevant in the market at present, we refer the reader to Eberlein, Madan (2009) and Bergomi (2004, 2005). In this paper, we will focus on the following three types of exotics: (a) swing cliquet, (b) accumulator and (c) reverse cliquet. Their payoffs are given below:

Swing Cliquet

$$SC(\omega) = \min\left(\sum_{i=0}^{N-1} \max\left(\left|\frac{S_{T_{i+1}}}{S_{T_i}} - 1\right| - K, 0\right), C\right)$$

Accumulator

$$AC(\omega) = \max\left(\sum_{i=0}^{N-1} \max\left(\min\left(\frac{S_{T_{i+1}}}{S_{T_i}} - 1, C\right), F\right), 0\right)$$

Reverse Cliquet

$$RC(\omega) = \max\left(C + \sum_{i=0}^{N-1} \min\left(\frac{S_{T_{i+1}}}{S_{T_i}} - 1, 0\right), 0\right)$$

where C, F, K are contract specifications that we will choose subsequently.

Before we examine the calibration procedure in detail, we list the three main steps of the process:

[1] Calibrate the initial MJD parameters $(\sigma_0, a_0, b, \lambda)$ to the market values of vanilla options with expiry $T_1 = \Delta$.

[2] Calibrate forward variance dynamics parameters to match a given input vol-of-vol curve.

[3] Calibrate the correlations between stock and forward variances to match skew term structure observed in the vanilla market at time T_0.

In the calibration procedure, the first reset period $[T_0, T_1]$ plays an important role. This is because over this reset period the stock follows usual spot-started MJD dynamics, as opposed to forward-started MJD dynamics for the rest of the reset periods. We start by calibrating the MJD parameters $(\sigma_0, a_0, b, \lambda)$ to the market value of options of expiry T_1. It is important to note that the parameters (b, λ) will remain fixed for the rest of the reset periods. Only (σ_i, a_i) will change according to the evolution of forward variances and the desired future smile behavior. In this section we use S&P500 options market data[6] from June 26th 2008 to illustrate the calculations. We use a quarterly tenor structure with total length of 3 years, i.e. $\Delta = 0.25$ and $N = 12$. Figure (5.3) shows the MJD fit on 06.26.08 for expiry $T_1 = 0.25$. We obtain the parameters $(\sigma_0, a_0, b, \lambda) = (10.35\%, -12.27\%, 0.06052, 2.6)$.

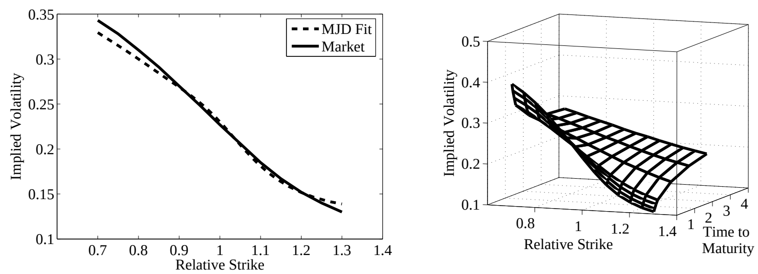

Figure 5.3: (Left) Dashed: MJD fit, Solid: 3-month market implied volatility curve on 06.26.08. Resulting parameters are $(\sigma_0, a_0, b, \lambda) = (10.35\%, -12.27\%, 0.06052, 2.6)$. (Right) The S&P500 implied volatility surface on 06.26.08.

We illustrate next the key feature of our forward-starting MJD model, namely the possibility to control the behavior and shape of the future smile through the use of the parameter α. As noted also in Bergomi (2005), one can consider at least two fundamental types of smile movement: one in which the smile preserves its shape but can float upwards or downwards depending on the size of the fair variance and one in which the smile also steepens for higher values of fair variance. Both of these behaviors can be obtained in our model by using the following two prescriptions for the parameter α:

parallel shift regime:

$$\alpha = \alpha_0$$

or, proportional volatility skew regime:

$$\alpha\left(\xi^i_{T_i}\right) = \alpha_0 \cdot \left(q + (1-q)\frac{\sqrt{\xi^i_{T_i}}}{\sqrt{\xi_0}}\right)$$

where

$$\alpha_0 = \frac{1}{\Delta}\frac{(3b^2 + a_0^2)a_0\lambda}{\left(\sigma_0^2 + \lambda(a_0^2 + b^2)\right)^{3/2}}$$

and

$$\xi_0 = \sigma_0^2 + \lambda(a_0^2 + b^2)$$

[6]The data were kindly provided to us by an international investment bank.

are the smile shape factor and fair variance corresponding to the first reset period; $q \in [0,1]$ is a blending factor between a parallel-shift regime and a proportional-skew regime. Figure (5.4) illustrates the shapes of future smiles obtained under the two regimes for values of fair variance in the interval $[0.2^2, 0.3^2]$ and $q = 0.5$; the fair variance for the first reset period is $\xi_0 = 0.2435^2$ and the smile shape factor $\alpha_0 = -2.2973$. We notice in figure (4) that, indeed, the smile displays the desired behavior.

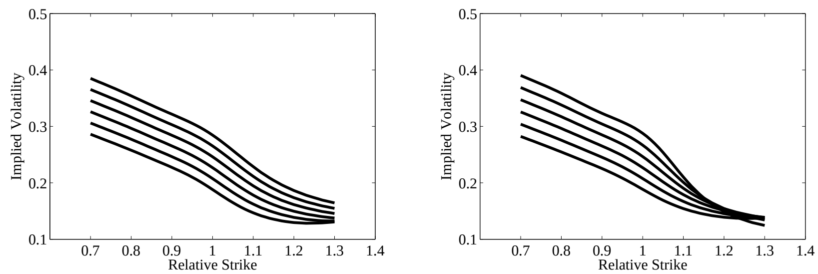

Figure 5.4: (Left) Future smile shapes for parallel shift regime $\alpha = -2.2973$. (Right) Future smile shapes for proportional skew regime with $q = 0.5$. In both cases the fair variance takes values in the interval $[0.2^2, 0.3^2]$.

With regard to the forward variance term structure, we first determine the variance swap rates, denoted by $VR(T_i)$ with $i \in \{1, 2, \dots, N\}$, from the S&P500 volatility surface, using the well known variance swap pricing formula; we refer the reader to Carr, Madan (2002) for a comprehensive presentation of variance swap pricing as well as the pricing of other contracts on realized variance. We then obtain the forward variances ξ_0^i as follows[7]:

$$\xi_0^i = \frac{T_{i+1} \cdot VR\left(T_{i+1}\right) - T_i \cdot VR\left(T_i\right)}{T_{i+1} - T_i}$$

The next step is to price options on forward variances in order to match an input Black ATM vol-of-vol term structure. If liquid market quotes for options on forward variance swaps of various maturities are available, the trader can use these to get the input vol-of-vol curve. Alternatively, the trader can rely on historic estimates for the volatility of variances and apply an appropriate risk premium. For our example, we take the input vol-of-vol curve of figure (5.5) which is obtained for the choice of parameters $\omega = 4$, $k_1 = 4$, $k_2 = 0.25$, $k_j = 3$, $\theta = 0.3$, $\eta = 0.25$ and $\rho = 0$. We recall that $\lambda = 2.6$. The figure also includes the variance rate curve calculated on 06.26.08.

[7]We note that, in section two, we denoted by V_t^i the value of a variance swap of maturity T_i; the relationship to the variance swap rate, at time zero, is $VR(T_i) = V_0^i e^{rT_i}$.

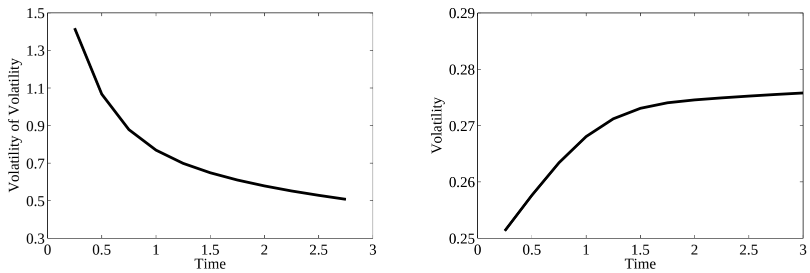

Figure 5.5: (Left) Input volatility of volatility term structure for parameters $\omega = 4$, $k_1 = 4$, $k_2 = 0.25$, $k_j = 3$, $\theta = 0.3$, $\eta = 0.25$ and $\rho = 0$. (Right) Fair variance swap rates calculated from S&P500 option market data on 06.26.08.

Finally, we are left with two more parameters: ρ_{SX} and ρ_{SY}, the correlations between the stock and the short-end / long-end of the forward variance curve. Proceeding as in Bergomi (2005), we fit these parameters to the 95%-105% skew[8] term structure observed in the vanilla market on the date of the model calibration. The motivation behind this approach is that the higher the correlation between the stock returns and the stochastic forward variances, the more persistent the term structure of skew at long maturities. We illustrate this on the right side of figure (5.6) where we see that, when the forward variances are held fixed, the skew falls off very quickly compared to the skew observed in the market. On the other hand, even without skew at the horizon Δ, i.e. using a *forward started* Black-Scholes model, when the forward variances are allowed to evolve stochastically, the model generates long term skew due to the correlation between the forward variances and the log returns. For S&P500 options data from 06.26.08, we found that $\rho_{SX} = -0.7$ and $\rho_{SY} = -0.4$ leads to very good agreement between market skew and model skew; see left side of figure (5.6).

Having calibrated all parameters of our model, we now turn to the pricing of the exotic products introduced at the beginning of this section. Similar to the approach in Bergomi (2005), we carry out four types of pricings: (1) Black Scholes: we use the Black Scholes model with deterministic time dependent volatility fitted to the original forward variance curve on the date of calibration, (2) deterministic forward skew : the stock follows MJD over each reset period but forward variances are held fixed at their starting values, (3) vol-of-vol, no skew: forward variances allowed to evolve stochastically but forward starting MJD is now replaced with forward starting Black-Scholes over each reset period (4) full model: both stochastic variances and future smiles as described in section two.

[8]By this we mean the difference between the Black-Scholes implied volatilities corresponding to a relative strike of 95%, and 105% respectively.

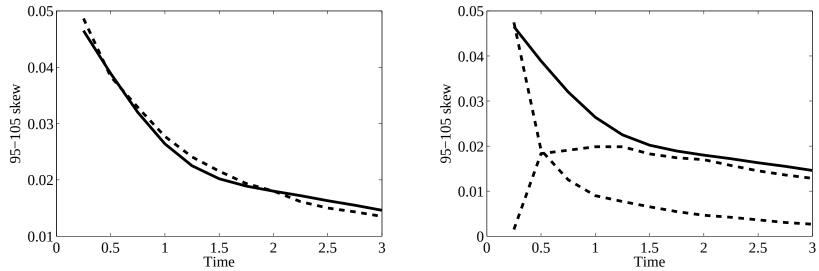

Figure 5.6: (Left) Dashed: model generated 95%-105% skew term structure, Solid: market 95%-105% skew term structure on 06.26.08. (Right) Dashed: model generated skew in two cases (a) forward variances held fixed: notice how quickly the skew decays compared to market observed skew, (b) no Δ-skew, i.e. we use a forward started Black-Scholes model; even though we start with no skew at horizon T_1, the model generates skew for maturities T_i, $i \geq 2$, due to correlation between stock returns and the variance term structure.

Model	Swing Cliquet	Acummulator	Rev. Cliquet
Black-Scholes	0.18%	1.96%	1.45%
Det. Fwd Skew	1.47%	4.41%	2.63%
VoV, No Skew	34.37%	1.83%	3.17%
Full model	32.45%	3.51%	3.47%

Table 5.1: Exotic option prices corresponding to the full MJD-Forward-Variance-Model and three additional variations. See text for details.

For the exotic options introduced earlier we choose the following specifications: (a) swing cliquet: $K = 0.4$, $C = 1$, (b) accumulator: $F = -0.02$, $C = 0.02$ and (c) reverse cliquet: $C = 0.4$. The prices are shown in table 5.1. We notice some marked differences between the full model price and its three variations across the products. These are best explained by considering the basic building blocks of each exotic. For example, the swing cliquet is a sequence of 3-months forward starting 60%-140% strangles on the underlying index. In line with intuition, the price table shows that this product is primarily a vol-of-vol product and, when the stochastic movement of variances is switched off, its value almost vanishes. In addition, we notice that the introduction of skew can even decrease its value. At the opposite end, we have the accumulator. This product is a sequence of 3-months forward starting 98%-102% call spreads and hence, unlike the swing cliquet, will be primarily a skew product. In fact, in our example we see that vol-of-vol can hurt its value; table 5.1 shows that, for the model calibrated on 06.26.08, the highest price is reached when vol-of-vol is turned off. Finally, the building blocks of the reverse cliquet are forward starting at-the-money puts and we see that vol-of-vol accounts for the most part of the value of this product.

We end this section with a review of the main insights of our numerical implementations. Firstly, we have seen – in detail – that our proposed model can be naturally calibrated to a vanilla options surface and a term structure of implied volatilities of volatility. Additionally, we have illustrated the key feature of the model, namely, the ability of the user to specify the desired behavior of future smiles; figure (5.4) displays two possible future smile behaviors: parallel smile shifts and steepening smiles. Secondly, after completing the calibration steps, the examination of the prices of different cliquet structures revealed that exotic options with forward started components draw most of their value from two sources: the volatility of volatility and the forward skew. Depending on the product, if either of these features is switched off the product can be grossly mis-priced. Some products (for example, the swing cliquet and the reverse cliquet) are primarily "vol-of-vol products" while others (for example, the accumulator) are primarily "skew products". Therefore, the key is to identify those risks which are relevant to each type of product and then carry out the pricing in a model which gives its user (i.e. the trader) direct access to those risks.

5.4 Conclusion

The paper shows that the key idea of forward-starting a classical model is a flexible and powerful tool to build a new class of models which allow the model user to directly control the behavior of future smiles. We model the term structure of forward variances consistently with the dynamics of the underlying asset by forward starting the asset dynamics at the beginning of each reset period. Our choice, of a forward started Merton jump diffusion process, provides good fits for market implied volatilities at short time horizons and allows us to derive closed form expressions for the forward starting parameters. The model incorporates positive jumps in the term structure of forward variance swaps thus capturing an important feature of the dynamics of volatility. We have shown that our framework is particularly important for products with high volatility-of-volatility and forward-skew sensitivity such as cliquets and their variations.

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