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Jonas B. Rasmussen

Higher congruences between modular forms

PhD Thesis

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Advisor	Ian Kiming University of Copenhagen, Denmark
Assessment Committee	Søren Jøndrup (chair) University of Copenhagen, Denmark
	Gabor Wiese Universität Duisburg-Essen, Germany
	Noriko Yui Queen's University, Ontario, Canada

Jonas B. Rasmussen
Department of Mathematical Sciences
University of Copenhagen
Universitetsparken 5
DK-2100 Copenhagen
Denmark
jonas@math.ku.dk
http://www.math.ku.dk/~jonas/

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Abstract

It is well-known that two modular forms on the same congruence subgroup and of the same weight, with coefficients in the integer ring of a number field, are congruent modulo a prime ideal in this integer ring, if the first B coefficients of the forms are congruent modulo this prime ideal, where B is an effective bound depending only on the congruence subgroup and the weight of the forms.

In this thesis, we generalize this result to congruences modulo powers of prime ideals and to modular forms of distinct weights. We also determine necessary conditions on the weights for there to be congruences between cusp forms modulo powers of prime ideals, with special emphasis on congruences between eigenforms because of their connection to Galois representations.

Additionally, we investigate the maximal congruences between newforms on $\Gamma_0(N)$, and also between newforms in case of level-lowering from $\Gamma_0(Np)$ to $\Gamma_0(N)$. This investigation leads to a very interesting set of conjectures, and we include all computed numerical evidence supporting these conjectures.

Resumé

Det er velkendt, at to modulformer på samme kongruensundergruppe og med samme vægt, og begge med koefficienter i heltalsringen for et tallegeme, er kongruente modulo et primideal i denne heltalsring, hvis de første B koefficienter for formerne er kongruente modulo dette primideal, hvor B er en effektiv grænse, der kun afhænger af kongruensundergruppen og formernes vægte.

I denne afhandling generaliserer vi dette resultat til kongruenser modulo potenser af primidealer og til modulformer med forskellige vægte. Vi bestemmer også nødvendige betingelser på vægtene for at der findes kongruenser mellem spidsformer modulo potenser af primidealer, med speciel fokus på kongruenser mellem egenformer på grund af deres forbindelse til Galoisrepræsentationer.

Derudover undersøger vi maksimale kongruenser mellem nyformer på $\Gamma_0(N)$, og også mellem nyformer i forbindelse med niveausænkning fra $\Gamma_0(Np)$ til $\Gamma_0(N)$. Dette fører til en meget interessant samling af formodninger, og vi inkluderer al beregnet numerisk data der understøtter disse formodninger.

Contents

Preface

This text constitutes my thesis for the PhD degree in mathematics at the University of Copenhagen.

My time as a PhD student, November 2005 to October 2009, has primarily been spent focusing on modular forms and Galois representations. I started out working on an analogue of Serre's conjecture modulo two different primes, and in the course of this work it became clear that it would be helpful to be able to compute many numerical examples involving modular forms.

This led me to look at an algorithm to compute cusp forms via cohomology, resulting in the paper *A detailed look at the complexity of computing cusp forms via Hecke action on cohomology* [Ras09].

While working with this algorithm, I was invited to look into higher congruences between modular forms, an area in which Imin Chen and Ian Kiming were already obtaining results, and our work led to the paper *On congruences mod* $\mathfrak{p}^m$ *between eigenforms and their attached Galois representations* [CKR08].

During the final months of my studies, I have systematically computed a database of higher congruences between newforms, which hints at a set of very interesting conjectures regarding maximal congruences between newforms.

I would like to thank, first and foremost, my advisor, Ian Kiming, who has been nothing but supportive and encouraging throughout the entire process. I have also appreciated the many discussions we have had during this time, both mathematical and otherwise.

I would also like to thank 'Rejselegat for Matematikere' for allowing me to travel the world for a year (including longer stays in Berkeley, Sydney and Oxford), meet wonderful people, and experience many things I look back on fondly.

Finally, I thank the Faculty of Science for my PhD stipend as well as my (former) office mate Anders Gaarde for copy-editing the thesis.

Copenhagen, October 2009
Jonas B. Rasmussen

Introduction

One can say that algebraic number theory is the study of the arithmetic properties of finite extensions of $\mathbb{Q}$, and this leads one to study the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$, as well as the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Representations of this group, i.e., (continuous) homomorphisms $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_n(R)$ for positive integers n and commutative rings R, are called *Galois representations.*

When $n = 1$ and $R = \mathbb{C}$, these representations lead to class field theory, and in case of general n, Langlands has given results and put forth conjectures concerning certain systems of n-dimensional p-adic Galois representations, and this is known as Langlands' program. This can be seen as an attempt to find a theory that generalizes class field theory to non-abelian extensions.

In this thesis, we consider the situation where $n = 2$ and R is of the form $\mathcal{O}/\mathfrak{p}^m$, where $\mathcal{O}$ is the ring of integers of a number field, $\mathfrak{p}$ is a prime ideal herein, and m is a positive integer. When $m > 1$, the ring $\mathcal{O}/\mathfrak{p}^m$ is no longer a field, and even contains zero-divisors and nilpotent elements (just consider the case of $R = \mathbb{Z}/p^m\mathbb{Z}$ for a prime p).

A way to generate such Galois representations is by a construction of Deligne, cf. Section 1.5, which to an eigenform (a cuspidal modular form that is an eigenvector for almost all Hecke operators) and a prime p gives representations $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathcal{O}/\mathfrak{p}^m)$ for all prime ideals $\mathfrak{p}$ over p in the integer ring $\mathcal{O}$ of a number field containing the Fourier coefficients of the eigenform.

Suppose that we are given two such Galois representations, and we know the eigenforms from which they are constructed. How can we determine if they are isomorphic?

Let us consider a fixed positive integer N and a fixed prime p, and suppose that we are given two (normalized) cusp forms f_1 and f_2 on $\Gamma_1(N)$ of weights k_1 and k_2, with coefficients in the integer ring $\mathcal{O}$ of a number field.

We say that f_1 and f_2 are *eigenforms outside* Np if they are eigenvectors for all Hecke operators T_ℓ for primes $\ell \nmid Np$. The corresponding eigenvalues for such T_ℓ acting on f_i are then exactly the coefficients $a_\ell(f_i)$.

Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}$ over p and let m be a positive integer. If f_i is an eigenform outside Np, we use the above-mentioned construction to obtain a Galois representation

$$\overline{\rho}_{f_i,\mathfrak{p}^m} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathcal{O}/\mathfrak{p}^m).$$

This representation is unramified outside Np, and we have

$$\operatorname{tr} \overline{\rho}_{f_i,\mathfrak{p}^m}(\mathrm{Frob}_\ell) = (a_\ell(f_i) \bmod \mathfrak{p}^m)$$

for all primes $\ell \nmid Np$. If we additionally suppose that the mod $\mathfrak{p}$ representation $\overline{\rho}_{f_i,\mathfrak{p}}$ is absolutely irreducible, $\overline{\rho}_{f_i,\mathfrak{p}^m}$ is determined (up to isomorphism) by this trace property.

As is obvious from the above, the key to determine whether $\overline{\rho}_{f_1,\mathfrak{p}^m}$ and $\overline{\rho}_{f_2,\mathfrak{p}^m}$ are isomorphic is to obtain a computationally decidable criterion for when we have $a_\ell(f_1) \equiv a_\ell(f_2) \pmod{\mathfrak{p}^m}$ for all primes $\ell \nmid Np$.

Now, for the case $m = 1$, and if the weights k_1 and k_2 are equal, there is a well-known theorem of Sturm that gives a necessary and sufficient condition for the forms to be congruent modulo $\mathfrak{p}$, in the sense that all their Fourier coefficients are congruent modulo $\mathfrak{p}$. It turns out to be very easy to generalize Sturm's theorem to the cases $m > 1$, provided that we still have $k_1 = k_2$. Then, still under the assumption that the weights are equal, a simple twisting argument allows us to discuss the case of eigenforms outside Np (or outside any finite set of primes containing the primes dividing Np).

In studying the case of distinct weights, we use two different approaches, and under favorable circumstances these approaches both result in computable necessary and sufficient conditions for the forms to be 'congruent modulo $\mathfrak{p}^m$ outside Np' in the above sense.

The first approach is to generalize a theorem of Katz-Serre on p-adic modular forms, cf. Theorem 2.4, which – under certain restrictions on the levels of the forms – gives a necessary congruence between the weights for the forms to be congruent modulo $\mathfrak{p}^m$. In Theorem 2.4, one needs to assume that $\mathfrak{p}$ is unramified over p, and we are able to generalize this to cases where $\mathfrak{p}$ is ramified over p.

Under certain technical restrictions, in particular that the ramification index relative to p of the Galois closure of the field of coefficients is not divisible by p, and that p is odd, Theorem 2.16 results in the desired computable necessary and sufficient conditions.

The second approach is via a study of the determinants of the attached mod $\mathfrak{p}^m$ representations. Again under certain technical restrictions, here notably a restriction on the characters of the forms, Theorem 2.20 leads to the desired computable necessary and sufficient conditions.

Even though we here only describe the results for eigenforms, many of the obtained results hold for general (cuspidal) modular forms on $\Gamma_1(N)$, and in some cases for modular forms on an arbitrary congruence subgroup (most notably the generalization of Sturm's theorem).

Consider the case where f_1 and f_2 are both newforms on $\Gamma_0(N)$ (still of weights k_1 and k_2 with coefficients in $\mathcal{O}$). Given a prime p, one can compute the *maximal congruence* between f_1 and f_2 modulo powers of prime ideals $\mathfrak{p}$ in $\mathcal{O}$ over p, where 'maximal congruence' means the highest power $\mathfrak{p}^m$ such that there is a congruence between $a_\ell(f_1) \equiv a_\ell(f_2) \pmod{\mathfrak{p}^m}$ for all primes $\ell \nmid Np$, taking into account the ramification of $\mathfrak{p}$ over p.

The results described above give upper bounds for such maximal congruences (depending on N, p and the weights k_1 and k_2), and in Section 2.6 we investigate whether these upper bounds are actually attained, cf. Conjecture 2.22.

We also consider these maximal congruences in case of level-lowering, i.e., we investigate maximal congruences between newforms on $\Gamma_0(Np)$ and $\Gamma_0(N)$. In this case two very different things seem to occur according to whether or not $p^2 \mid N$, cf. Conjecture 2.23 and Conjecture 2.24. Most notably, when $p^2 \mid N$, the upper bound for the maximal congruences appears to be independent of the weights.

All computational evidence supporting these conjectures are included in the appendices.

To even be able to consider such conjectures, we have to be able to compute spaces of modular forms. The standard way of doing this is via modular symbols, but another way of computing modular forms is via a cohomological approach, based on the Eichler-Shimura isomorphism (this approach is essentially the same as the modular symbols approach over the rationals).

In the final chapter, we describe an explicit implementation of an algorithm using this cohomological approach to compute spaces of cusp forms, and we determine its complexity.

Chapter 1

Modular forms

In this chapter we give a brief introduction to modular forms.

Most results are given without references since they are standard results that can be found in any textbook on modular forms, such as [DS05], [Lan95], [Miy06] or [Shi94]. For the more special results we give specific references.

1.1 Congruence subgroups and cusps

Let N be a positive integer. The subgroup of $\mathrm{SL}_2(\mathbb{Z})$ defined by

$$\Gamma(N) = \left\{ \gamma \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}$$

is called the *principal congruence subgroup of level* N. Any subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ containing such a subgroup is called a *congruence subgroup*, and the smallest N for which $\Gamma(N)$ is contained in Γ is called the *level* of Γ.

An example of a subgroup of level N is

$$\Gamma_H(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \pmod{N}, a \in H \right\},$$

for any subgroup H of $(\mathbb{Z}/N\mathbb{Z})^*$. We have $\Gamma_H(N) \subseteq \Gamma_H(M)$ if $M \mid N$.

The two trivial subgroups of $(\mathbb{Z}/N\mathbb{Z})^*$ give the most important congruence subgroups. In the case of $H = (\mathbb{Z}/N\mathbb{Z})^*$ we get

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \pmod{N} \right\},$$

while we for $H = \{1\}$ get

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \pmod{N}, a \equiv d \equiv 1 \pmod{N} \right\}.$$

Let Γ be a congruence subgroup. Γ acts on the set of *cusps*, $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$, by fractional linear transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} .s = \begin{cases} \frac{as+b}{cs+d}, & s \neq \infty, \\ \frac{a}{c}, & s = \infty, \end{cases}$$

and this action divides the cusps into finitely many cusp equivalence classes, $C(\Gamma)$, which we also call (abusing notation) the set of cusps.

We have $C\big(\mathrm{SL}_2(\mathbb{Z})\big) = \{\infty\}$, i.e., for any $s \in \mathbb{P}^1(\mathbb{Q})$, we have a $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma.\infty = s$.

The *width* of a cusp s of Γ is the smallest positive integer h such that

$$\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \gamma^{-1}\Gamma\gamma,$$

for any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ satisfying $\gamma.\infty = s$. We see that ∞ has width 1 for the groups $\Gamma_H(N)$, while it has width N for $\Gamma(N)$.

1.2 Modular forms

We denote by

$$\mathfrak{h} = \big\{ z \in \mathbb{C} \mid \mathrm{Im}\, z > 0 \big\}$$

the upper half-plane.

Let k be a positive integer and let Γ be a congruence subgroup. A *modular form* of weight k on Γ is a holomorphic function $f : \mathfrak{h} \to \mathbb{C}$ satisfying the transformation property

$$f(z) = (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right), \quad z \in \mathfrak{h}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,$$

and which is holomorphic at the cusps of Γ. Furthermore, f is called a *cusp form* if f vanishes at every cusp of Γ (see the references given above for what is meant by holomorphy and vanishing at the cusps).

The set of modular forms (resp. cusp forms) of weight k on Γ form a complex vector space $M_k(\Gamma)$ (resp. $S_k(\Gamma)$), and this space is always finite-dimensional. The modular forms that are not cusp forms are called *Eisenstein series*, and these form a subspace of $M_k(\Gamma)$, denoted $E_k(\Gamma)$, so that

$$M_k(\Gamma) = S_k(\Gamma) \oplus E_k(\Gamma).$$

At every cusp s (of width h) of Γ, the modular form f has a Fourier series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n(f) q_h^n, \quad q_h = e^{2\pi i z/h},$$

and we just write q for q_1. We call this Fourier series the *q-expansion of f* at s. If $s = \infty$, we just call this the q-expansion of f.

If f is a cusp form, we have $a_0(f) = 0$ (at every cusp).

For any subring R of $\mathbb{C}$, we denote by $M_k(\Gamma; R)$ (resp. $S_k(\Gamma; R)$) the modular forms (resp. cusp forms) in $M_k(\Gamma)$ (resp. $S_k(\Gamma)$) whose q-expansions (at ∞) have coefficients which all lie in R.

In keeping with the notation established above, we have $M_k(\Gamma) = M_k(\Gamma; \mathbb{C})$ and $S_k(\Gamma) = S_k(\Gamma; \mathbb{C})$.

1.3 The Hecke algebra

In this section we work with the congruence subgroup $\Gamma_1(N)$, and we write $M_k(N)$ (resp. $S_k(N)$) for the space of modular forms (resp. cusp forms) of weight k on $\Gamma_1(N)$.

Since $\Gamma_1(N)$ is the kernel of the surjetive homomorphism

$$\Gamma_0(N) \to (\mathbb{Z}/N\mathbb{Z})^*$$

given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (d \bmod N),$$

we get an isomorphism $\Gamma_0(N)/\Gamma_1(N) \cong (\mathbb{Z}/N\mathbb{Z})^*$. This gives rise to a set of operators on $M_k(N)$, called the *diamond operators*, and the subspace $S_k(N)$ is stable under the action of these operators. For $d \in (\mathbb{Z}/N\mathbb{Z})^*$, the diamond operator $\langle d \rangle$ is given by

$$(f|\langle d \rangle)(z) = (cz+d')^{-k} f\left(\frac{az+b}{cz+d}\right), \quad z \in \mathfrak{h}, \begin{pmatrix} a & b \\ c & d' \end{pmatrix} \in \Gamma_0(N), d \equiv d' \pmod{N},$$

When χ runs through the Dirichlet characters $(\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}^*$ satisfying $\chi(-1) = (-1)^k$ (otherwise $M_k(N, \chi)$ is trivial), the diamond operators decompose $M_k(N)$ into χ-eigenspaces

$$M_k(N, \chi) = \left\{ f \in M_k(N) \mid f|\langle d \rangle = \chi(d) f \text{ for all } d \in (\mathbb{Z}/N\mathbb{Z})^* \right\},$$

and similarly for $S_k(N)$.

We note that if χ_0 is the trivial character, we have $M_k(N, \chi_0) = M_k(\Gamma_0(N))$ (and similarly for cusp forms).

There is another family of commuting operators on $M_k(N)$ (resp. $S_k(N)$) called the *Hecke operators*. If the q-expansion of $f \in M_k(N, \chi)$ is

$$f(z) = \sum_{n=0}^{\infty} a_n(f) q^n,$$

then the m'th Hecke operator T_m acts on the q-expansion of f as

$$(T_m f)(z) = \sum_{n=0}^{\infty} c_n q^n,$$

where

$$c_n = \sum_{d \mid \gcd(m,n)} \chi(d) d^{k-1} a_{mn/d^2}(f),$$

with $\chi(d) = 0$ if $\gcd(d, N) > 1$. Note that $a_1(T_m f) = a_m(f)$.

The subspace $M_k(N, \chi)$ (resp. $S_k(N, \chi)$) is stable under the Hecke action.

Since our main focus is cusp forms, we now restrict to $S_k(N)$ (resp. $S_k(N, \chi)$). The *Hecke algebra* of $S_k(N)$ (resp. $S_k(N, \chi)$) is defined as

$$\mathbb{T} = \mathbb{Z}[T_1, T_2, \ldots],$$

and from the context it will be clear if we are talking about $\mathbb{T}$ as a subalgebra of $\operatorname{End} S_k(N)$ or $\operatorname{End} S_k(N, \chi)$. We denote by $\mathbb{T}'$ the subalgebra of $\mathbb{T}$ generated by the Hecke operators T_n with $\gcd(n, N) = 1$.

$\mathbb{T}$ is finitely generated, contains the diamond operators, and for any commutative ring R we put

$$\mathbb{T}_R = \mathbb{T} \otimes R = R[T_1, T_2, \ldots],$$

and similarly we put $\mathbb{T}'_R = \mathbb{T}' \otimes R$.

It can be shown that the Hecke operators T_n on $S_k(N)$ (resp. $S_k(N, \chi)$) for $\gcd(n, N) = 1$ are diagonizable, and hence are simultaneously diagonizable (since they commute). A non-zero form $f \in S_k(N)$ (resp. $S_k(N, \chi)$) satisfying

$$T_n f = \lambda_n f, \quad \lambda_n \in \mathbb{C},$$

for all positive integers n with $\gcd(n, N) = 1$ is called an *eigenform*. If $T_n f = \lambda_n f$ for all positive integers n, we call f a *true* eigenform.

If f is *normalized*, i.e., has $a_1(f) = 1$, then the Fourier coefficients $a_n(f)$ are precisely the eigenvalues λ_n. Throughout we always assume eigenforms to be normalized.

If f is an eigenform, the Fourier coefficients $a_n(f)$ are algebraic integers, and the field of coefficients of f is a number field of degree at most the rank of $\mathbb{T}$.

1.4 Newforms

Let M be a positive divisor of N. For each positive divisor $t \mid N/M$, there is an injection $\varphi_{M,t} : S_k(M) \to S_k(N)$ given by $f(z) \mapsto f(tz)$, and if $f(z)$ is a (true) eigenform of $S_k(M)$, then $f(tz)$ is a (true) eigenform of $S_k(N)$, with the same eigenvalues. There are also maps $\psi_{M,t}$ in the other direction.

The subspace of $S_k(N)$ generated by the image of all the $\varphi_{M,t}$ (with M a proper divisor of N) is called the *old* part of $S_k(N)$, and is denoted $S_k(N)_{\text{old}}$. The intersection of the kernel of all the $\psi_{M,t}$ (with M a proper divisor of N) is called the *new* part of $S_k(N)$, and is denoted $S_k(N)_{\text{new}}$.

By results of Atkin-Lehner [AL70] and Li [Li75], we have a decomposition

$$S_k(N,\chi) = \bigoplus_{M \mid N} \bigoplus_{t \mid N/M} \varphi_{M,t}\big(S_k(M)_{\text{new}}\big),$$

where each $S_k(M)_{\text{new}}$ is a direct sum of non-isomorphic simple $\mathbb{T}'_{\mathbb{C}}$-modules.

$S_k(N)_{\text{new}}$ could also be defined as the orthogoal complement of $S_k(N)_{\text{old}}$ in $S_k(N)$ with respect to the Petersson inner product. This point of view gives that

$$S_k(N) = S_k(N)_{\text{old}} \oplus S_k(N)_{\text{new}},$$

and both $S_k(N)_{\text{old}}$ and $S_k(N)_{\text{new}}$ are stable under the action of the Hecke algebra $\mathbb{T}$. If $f \in S_k(N)$ is an eigenform, we call f an *oldform* if $f \in S_k(N)_{\text{old}}$, and a *newform* if $f \in S_k(N)_{\text{new}}$. It is an important result about $S_k(N)_{\text{new}}$ that the ring of Hecke operators on $S_k(N)_{\text{new}}$ is generated by the Hecke operators T_n with $\gcd(n, N) = 1$, i.e., a newform is a true eigenform.

The results of this section also hold if we restrict to $S_k(N,\chi)$ (as long as we require that the conductor of χ divides M where necessary such that the spaces $S_k(M,\chi)$ makes sense).

1.5 Galois representations

Assume $k \geq 2$ and let $f \in S_k(N,\chi)$ be an eigenform. Let K be a number field containing the Fourier coefficients of f, and denote by $\mathcal{O}$ the ring of integers of K. We assume all representations to be continuous (where we equip $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ with the Krull topology, any p-adic field with the usual topology, and any finite set with the discrete topology).

By a theorem of Deligne, cf. [Del71], there is for every prime p exactly one Galois representation (up to isomorphism) $\rho_{f,p} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(K \otimes \mathbb{Q}_p)$

which is unramified at every prime $\ell \nmid Np$, and such that $\rho_{f,p}(\mathrm{Frob}_\ell)$ has the characteristic polynomial

$$X^2 - a_\ell(f)X + \ell^{k-1}\chi(\ell)$$

for all primes $\ell \nmid Np$, where Frob_ℓ is an arithmetic Frobenius element for ℓ in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

That the characteristic polynomial has this form is equivalent to the following statement about the trace and determinant:

$$\begin{aligned} \operatorname{tr} \rho_{f,p}(\mathrm{Frob}_\ell) &= a_\ell(f), \\ \det \rho_{f,p}(\mathrm{Frob}_\ell) &= \widetilde{\chi}_p^{k-1}(\ell)\chi(\ell). \end{aligned}$$

Here, $\widetilde{\chi}_p : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}_p^*$ is the p-adic cyclotomic character, and we in fact have $\det \rho_{f,p} = \widetilde{\chi}_p^{k-1}\chi$.

The representation $\rho_{f,p}$ is called the *p-adic Galois representation associated to f*.

Let $\mathfrak{p}$ be a prime ideal of K over the rational prime p. The canonical homomorphism $K \otimes \mathbb{Q}_p \to K_\mathfrak{p}$ gives rise to the representation $\rho_{f,\mathfrak{p}} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(K_\mathfrak{p})$, called the *$\mathfrak{p}$-adic Galois representation associated to f*. It can be shown that $\rho_{f,\mathfrak{p}}$ is always irreducible, and is characterized (up to isomorphism) by $\rho_{f,\mathfrak{p}}(\mathrm{Frob}_\ell)$ having the characteristic polynomial

$$X^2 - a_\ell(f)X + \ell^{k-1}\chi(\ell)$$

for all primes $\ell \nmid Np$.

Let $\mathcal{O}_\mathfrak{p}$ be the $\mathfrak{p}$-adic ring of integers of $K_\mathfrak{p}$. From the characteristic polynomial, we see that both the trace $\operatorname{tr} \rho_{f,\mathfrak{p}}(\mathrm{Frob}_\ell)$ and the determinant $\det \rho_{f,\mathfrak{p}}(\mathrm{Frob}_\ell)$ are in $\mathcal{O}_\mathfrak{p}$ for almost all primes ℓ (since $a_\ell(f)$ is an algebraic integer and $\chi(\ell)$ is a root of unity). Thus, we always can (and will) assume that $\rho_{f,\mathfrak{p}}$ takes values in $\mathrm{GL}_2(\mathcal{O}_\mathfrak{p})$.

Let π be a uniformizer of $K_\mathfrak{p}$. We obtain the *mod $\mathfrak{p}$ reduction of $\rho_{f,\mathfrak{p}}$* by first reducing $\rho_{f,\mathfrak{p}}$ modulo π, then taking its semisimplification, and finally using the isomorphism $\mathcal{O}/\mathfrak{p} \cong \mathcal{O}_\mathfrak{p}/(\pi)$ to get a representation $\overline{\rho}_{f,\mathfrak{p}} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathcal{O}/\mathfrak{p})$. Since $\mathcal{O}/\mathfrak{p}$ is a finite extension of $\mathbb{F}_p$, we often call such a representation a *mod p Galois representation.* Note that $\overline{\rho}_{f,\mathfrak{p}}$ is not necessarily irreducible, but is semisimple by construction.

Similarly, we obtain the *mod $\mathfrak{p}^m$ reduction of* $\rho_{f,\mathfrak{p}}$ by reducing $\rho_{f,\mathfrak{p}}$ modulo π^m, and we again end up with a representation $\overline{\rho}_{f,\mathfrak{p}^m} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathcal{O}/\mathfrak{p}^m)$ (note that $\mathcal{O}/\mathfrak{p}^m$ is not a field for $m > 1$!), which is unramified at every prime $\ell \nmid Np$. If we assume that the mod $\mathfrak{p}$ representation $\overline{\rho}_{f,\mathfrak{p}}$ is absolutely irreducible, [Car94, Thm. 1] shows that $\overline{\rho}_{f,\mathfrak{p}^m}$ is uniquely determined (up to isomorphism) by $\overline{\rho}_{f,\mathfrak{p}^m}(\mathrm{Frob}_\ell)$ having the characteristic polynomial

$$X^2 - (a_\ell(f) \bmod \mathfrak{p}^m)X + (\ell^{k-1}\chi(\ell) \bmod \mathfrak{p}^m)$$

for all primes $\ell \nmid Np$.

A Galois representation ρ is called *modular* (of level N, weight k and character χ) if it is isomorphic to one of the Galois representations considered in this section, that is if ρ is isomorphic to $\rho_{f,p}$, $\rho_{f,\mathfrak{p}}$ or $\overline{\rho}_{f,\mathfrak{p}^m}$ for an eigenform $f \in S_k(N,\chi)$. Any modular Galois representation ρ is *odd*, i.e., $\det \rho(c) = -1$ for a complex conjugation $c \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. This follows because

$$\det \rho_{f,p}(c) = \widetilde{\chi}_p^{k-1}(c)\chi(c) = (-1)^{k-1}\chi(-1) = (-1)^{2k-1} = -1,$$

since c acts as inversion on roots of unity and $\chi(-1) = (-1)^k$.

Serre's conjecture, cf. [Ser87, §3], states that the necessary condition of a mod p Galois representation ρ being odd, is also sufficient for ρ to be modular if it is irreducible (the reducible Galois representations can be seen as coming from Eisenstein series).

Theorem 1.1 ([KW06a, Thm. 1.2], [Kis07, Cor. 0.2])**.** *Any odd, irreducible mod p Galois representation is modular.*

Serre also specifies an optimal level N, weight k and character χ such that the eigenform giving rise to the Galois representation is in $S_k(N,\chi)$.

Theorem 1.1 was proven for odd p and odd conductor by Khare and Wintenberger in [KW06a] (modulo two theorems which are proved in [KW06b]), in which they also reduce the general case to a modularity statement about 2-adic lifts of modular mod 2 representations (see also [KW09]). This statement was then proven by Kisin in [Kis07].

Chapter 2

Higher congruences between modular forms

The first two sections of this chapter list some well-known results about congruences between modular forms and construction of Eisenstein series with certain congruence properties. The next three sections are based mainly on the paper [CKR08], but with some slight generalizations as well as additional results. The subsequent section discusses conjectures regarding maximal congruences between newforms, and the final section of this chapter describes some of the computational difficulties in actually determining whether higher congruences exist between given modular forms.

2.1 Congruence results

Let N and k be positive integers. The following result is a very useful computational criterion in determining when two modular forms are congruent modulo a prime ideal.

Theorem 2.1 ([Stu87, Thm. 1]). *Let f_1 and f_2 be modular forms of weight k on a congruence subgroup Γ of level N and index $\mu = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma]$. Assume that f_1 and f_2 have coefficients in the ring of integers of a number field, and let $\mathfrak{p}$ be a prime ideal herein. If $a_n(f_1) \equiv a_n(f_2) \pmod{\mathfrak{p}}$ for all non-negative integers*

$$n \leq \begin{cases} k\mu/12 - (\mu - 1)/N, & f_1 - f_2 \in S_k(\Gamma), \\ k\mu/12, & \textit{otherwise}, \end{cases}$$

then $f_1 \equiv f_2 \pmod{\mathfrak{p}}$.

The integer $\lfloor k\mu/12 \rfloor$ is known as the *Sturm bound* for $M_k(\Gamma)$, and we will refer to $\lfloor k\mu/12 - (\mu-1)/N \rfloor$ as the Sturm bound for $S_k(\Gamma)$.

That one can improve the Sturm bound for cusp forms is mentioned in the proof of [Stu87, Thm. 1], and a proof that the bound given above works can be found in [Ste07, Sec. 9.4].

Theorem 2.1 only deals with the case where the forms have the same weight and where the power of the prime ideal is one. We will later generalize this result to distinct weights and powers of prime ideals, cf. Section 2.3.

It should be noted that by applying Theorem 2.1 to every prime ideal of the integer ring $\mathcal{O}$ of the field of coefficients, we obtain the fact that any $f \in M_k(\Gamma; \mathcal{O})$ is uniquely determined by its first B Fourier coefficients $a_1(f), \dots, a_B(f)$ (where B is the Sturm bound).

In the case of newforms of squarefree level we have the following improved bound.

Theorem 2.2 ([Stu87, Thm. 2]). *Let f_1 and f_2 be newforms in $S_k(N, \chi)$ with coefficients in the ring of integers of a number field, and let $\mathfrak{p}$ be a prime ideal herein. Assume that N is squarefree, let B be the Sturm bound, and let S be a subset of the prime divisors of N.*

If $a_\ell(f_1) = a_\ell(f_2)$ for all $\ell \in S$ and there is a congruence

$$a_\ell(f_1) \equiv a_\ell(f_2) \pmod{\mathfrak{p}}$$

for all primes $\ell \leq B/2^{\#S}$, then $f_1 \equiv f_2 \pmod{\mathfrak{p}}$.

There is a similar result about congruences between eigenforms in case of general level.

Theorem 2.3 ([BS02, Cor. 1.7]). *Let f_1 and f_2 be eigenforms in $S_k(N, \chi)$ with coefficients in the ring of integers of a number field, and let $\mathfrak{p}$ be a prime ideal herein. Assume that $N \geq 5$, let B be the Sturm bound, and let S be the set of prime divisors of N not dividing $N/\operatorname{cond}(\chi)$.*

If there is a congruence

$$a_\ell(f_1) \equiv a_\ell(f_2) \pmod{\mathfrak{p}}$$

for all primes $\ell \in S$ and all primes $\ell \leq B/2^{\#S}$, then $f_1 \equiv f_2 \pmod{\mathfrak{p}}$.

In the case of the prime p being unramified in the field of coefficients, the following theorem of Katz (and Serre) gives a necessary condition on the weights for the forms to be congruent modulo (powers of) p (see also [Kat73, Thm. 3.2]).

We first introduce some notation. Let p be a prime and let $\beta(n)$ denote the maximal element order in $(\mathbb{Z}/p^n\mathbb{Z})^*$ if n is a positive integer, and put $\beta(n) = 1$

otherwise. We define the function $\alpha : \mathbb{Z} \to \mathbb{N}_0$ by putting $\alpha(n) = \operatorname{ord}_p \beta(n)$. We see that, for odd p, we have

$$\alpha(n) = \max\{0, n-1\},$$

while we for $p = 2$ have

$$\alpha(n) = \begin{cases} 0, & n \leq 1, \\ 1, & n = 2, \\ n-2, & n \geq 3. \end{cases}$$

Theorem 2.4 ([Kat73, Cor. 4.4.2])**.** *Assume that $N \geq 3$ and let p be a prime not dividing N. Let f_1 and f_2 be cusp forms on $\Gamma_1(N) \cap \Gamma_0(p)$ of weights k_1 and k_2 with coefficients (when embedded into the p-adic completion of field of coefficients) in the ring of Witt vectors $W(\mathbb{F}_{p^f})$. Assume that not both q-expansions of f_1 and f_2 vanish modulo p.*

If $f_1 \equiv f_2 \pmod{p^m}$, then we have the congruence

$$k_1 \equiv k_2 \pmod{p^{\alpha(m)}(p-1)}$$

between the weights, where α is as above.

This theorem is one of the few results on higher congruences between modular forms, and we will later generalize this, cf. Theorem 2.16.

2.2 Construction of Eisenstein series

One of the ways of changing the weight of a modular form, while still keeping a congruence modulo (a power of) a prime ideal, is by multiplying the form with powers of certain Eisenstein series with nice congruence properties. In this section we give two well-known results on construction of such Eisenstein series.

Lemma 2.5. *Let p be a prime and let N be a positive integer satisfying $p \mid N$ if p is odd and either $3 \mid N$ or $4 \mid N$ if $p = 2$.*

Then, for any prime ideal $\mathfrak{p}$ over p in the $(p-1)$'st cyclotomic field there is an Eisenstein series E on $\Gamma_1(N)$ of weight 1 such that $E \equiv 1 \pmod{\mathfrak{p}}$.

The forms we construct will be on $\Gamma_1(p)$ for odd p, and on either $\Gamma_1(3)$ or $\Gamma_1(4)$ for $p = 2$.

Proof. We first consider the case of odd p.

Let ζ be a $(p-1)$'st root of unity and let $\mathcal{O}$ denote the ring of integers of $\mathbb{Q}(\zeta)$.

Let $\psi : (\mathbb{Z}/p\mathbb{Z})^* \to \mathcal{O}^*$ be a Dirichlet character satisfying $\psi(-1) = -1$. Using [DS05, Chap. 4.8], we get an Eisenstein series $E_{1,\psi}$ of weight 1 on $\Gamma_1(p)$ with q-expansion

$$E_{1,\psi} = 1 - \frac{2}{B_{1,\psi}} \sum_{n=1}^{\infty} \left(\sum_{d|n} \psi(d) \right) q^n,$$

where $B_{1,\psi}$ is the first Bernoulli number of ψ, and where the k'th Bernoulli number of ψ is defined by the equality of formal power series

$$\sum_{a=1}^{p-1} \frac{\psi(a) x e^{ax}}{e^{px} - 1} = \sum_{k=0}^{\infty} B_{k,\psi} \frac{x^k}{k!}.$$

Using generating functions of Bernoulli polynomials, cf. [Was97, Prop. 4.1], one finds that

$$B_{1,\psi} = \sum_{a=1}^{p-1} \psi(a) \Big(\frac{a}{p} - \frac{1}{2} \Big) = \frac{1}{p} \sum_{a=1}^{p-1} \psi(a) a,$$

where the last equality follows because ψ is non-trivial. We note that $pB_{1,\psi} \in \mathcal{O}$.

Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}$ over p. For an integer a not divisible by p, we define $\chi(a)$ to be the $(p-1)$'st root of unity such that

$$\chi(a) a \equiv 1 \pmod{\mathfrak{p}}.$$

This defines a Dirichlet character $\chi : (\mathbb{Z}/p\mathbb{Z})^* \to \mathcal{O}^*$, since $1, \zeta, \ldots, \zeta^{p-2}$ are all distinct modulo $\mathfrak{p}$ and $(\mathcal{O}/\mathfrak{p})^* \cong (\mathbb{Z}/p\mathbb{Z})^*$, and we also have $\chi(-1) = -1$.

The construction described above now gives an Eisenstein series $E_{1,\chi}$ of weight 1 on $\Gamma_1(p) \supseteq \Gamma_1(N)$ with q-expansion

$$E_{1,\chi} = 1 - \frac{2}{B_{1,\chi}} \sum_{n=1}^{\infty} \left(\sum_{d|n} \chi(d) \right) q^n.$$

By using the definition of χ, we find that

$$pB_{1,\chi} = \sum_{a=1}^{p-1} \chi(a) a \not\equiv 0 \pmod{\mathfrak{p}},$$

so that $E_{1,\chi}$ has coefficients in $\mathcal{O}$ and $E_{1,\chi} \equiv 1 \pmod{\mathfrak{p}}$.

We now turn to the case of $p = 2$, and consider the primitive Dirichlet character $\chi : (\mathbb{Z}/4\mathbb{Z})^* \to \{\pm 1\}$. We find that $B_{1,\chi} = -\frac{1}{2}$, so that the corresponding Eisenstein series $E_{1,\chi}$ on $\Gamma_1(4)$ has coefficients in $\mathbb{Z}$ and satisfies $E_{1,\chi} \equiv 1 \pmod 4$.

Similarly, we consider the primitive Dirichlet character $\chi : (\mathbb{Z}/3\mathbb{Z})^* \to \{\pm 1\}$. We find that $B_{1,\chi} = -\frac{1}{3}$, so that the Eisenstein series $E_{1,\chi}$ on $\Gamma_1(3)$ in this case also has coefficients in $\mathbb{Z}$ and satisfies $E_{1,\chi} \equiv 1 \pmod 6$, especially we have a congruence modulo 2. □

We will later use the fact that we for $p = 2$ have a congruence modulo 4 (and not just modulo 2) when $4 \mid N$.

Lemma 2.6. *Let p be a prime and let N be a positive integer satisfying $2 \mid N$ if $p = 3$ and either $3 \mid N$ or $4 \mid N$ if $p = 2$.*

Then, there is an Eisenstein series E on $\Gamma_1(N)$ of weight $p-1$ and a congruence $E \equiv 1 \pmod p$.

Proof. For even integers $k \geq 4$, there is a unique normalized Eisentein series E_k (by normalized we mean that the constant term is 1) of weight k on $\mathrm{SL}_2(\mathbb{Z})$, and [Ser73, Chap. VII, Prop. 4.8] gives the q-expansion of E_k as

$$E_k = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

where B_k is the k'th Bernoulli number defined by the equality of formal power series

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!},$$

and where we for any non-negative integer r let $\sigma_r(n)$ denote the sum of the r-powers of the positive divisors of n.

When $p \geq 5$, we conclude from [Lan95, Chap. X, Thm. 2.1] that $E = E_{p-1}$ has coefficients in $\mathbb{Z}_p$ and satisfies $E \equiv 1 \pmod p$.

If $p = 3$, there is a unique normalized Eisenstein series E of weight 2 on $\Gamma_1(2) \supseteq \Gamma_1(N)$, and the q-expansion of this Eisenstein series is

$$E = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n,$$

which clearly has coefficients in $\mathbb{Z}$ and reduces to 1 modulo 3.

Finally, if $p = 2$, we let E be the Eisenstein series $E_{1,\chi}$ from the proof of Lemma 2.5, with χ the primitive Dirichlet character of conductor 3 (resp. 4). Then, E is of weight 1 on $\Gamma_1(3) \supseteq \Gamma_1(N)$ (resp. $\Gamma_1(4) \supseteq \Gamma_1(N)$), has coefficients in $\mathbb{Z}$, and reduces to 1 modulo 2. □

2.3 Higher congruences

For a positive integer a, a commutative ring R, and a formal power series

$$h = \sum_{n=0}^{\infty} c_n q^n \in R[\![q]\!],$$

we define, for a prime ideal λ of R,

$$\operatorname{ord}_{\lambda^a} h = \inf \{ n \in \mathbb{N}_0 \mid \lambda^a \nmid (c_n) \},$$

with the convention that $\operatorname{ord}_{\lambda^a} h = \infty$ if $\lambda^a \mid (c_n)$ for all n.

We say that formal powers series h_1 and h_2 in $R[\![q]\!]$ are *congruent modulo* λ^a if $\operatorname{ord}_{\lambda^a}(h_1 - h_2) = \infty$, and we denote this by $h_1 \equiv h_2 \pmod{\lambda^a}$.

We consider a fixed integer ring $\mathcal{O}$ of a number field as well as a fixed prime ideal $\mathfrak{p}$ of $\mathcal{O}$. Let p be the rational prime under $\mathfrak{p}$, and put $e = e(\mathfrak{p}/p)$. We also consider a fixed positive integer m.

We have the following generalization of Theorem 2.1.

Proposition 2.7. *Let f_1 and f_2 be modular forms in $M_k(\Gamma; \mathcal{O})$ where Γ is a congruence subgroup of level N and index $\mu = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma]$.*

If there is a congruence

$$a_n(f_1) \equiv a_n(f_2) \pmod{\mathfrak{p}^m}$$

for all non-negative integers $n \leq B$, then $f_1 \equiv f_2 \pmod{\mathfrak{p}^m}$.

Here, B is the Sturm bound, cf. Section 2.1, that is $B = \lfloor k\mu/12 - (\mu - 1)/N \rfloor$ if $f_1 - f_2 \in S_k(\Gamma)$, and $B = \lfloor k\mu/12 \rfloor$ otherwise.

Proof. We prove this by induction on m. It will be convenient to prove a slightly more general statement, namely that the proposition holds for forms with coefficients in $\mathcal{O}_\mathfrak{p}$, the localization of $\mathcal{O}$ with respect to $\mathfrak{p}$. If h is such a form, we can define $\operatorname{ord}_{\mathfrak{p}^m} h$ in the same manner as above, and the claim is then that $\operatorname{ord}_{\mathfrak{p}^m} h > B$ implies $\operatorname{ord}_{\mathfrak{p}^m} h = \infty$ (where B is the appropriate Sturm bound).

This statement, for $m = 1$, follows immediately from Theorem 2.1: If h is a form on Γ of weight k with coefficients in $\mathcal{O}_\mathfrak{p}$, then there is a number $t \in \mathcal{O} \setminus \mathfrak{p}$ such that th has coefficients in $\mathcal{O}$; this follows from the 'bounded denominators' property for modular forms. Then, if $\operatorname{ord}_{\mathfrak{p}^m} h > B$, we have $\operatorname{ord}_{\mathfrak{p}^m}(th) > B$, and by Theorem 2.1 this implies $\operatorname{ord}_{\mathfrak{p}^m}(th) = \infty$, and hence also $\operatorname{ord}_{\mathfrak{p}^m} h = \infty$.

Assume now that $m > 1$, and that the proposition in the above slightly more general form is true for powers $\mathfrak{p}^a$ of $\mathfrak{p}$ with $a < m$. Consider then forms f_1

and f_2 on Γ of weight k with coefficients in $\mathcal{O}_{\mathfrak{p}}$ such that $\operatorname{ord}_{\mathfrak{p}^m}(f_1 - f_2) > B$. Let $\varphi = f_1 - f_2$. By assumption, we have $\operatorname{ord}_{\mathfrak{p}^m} \varphi > B$, and therefore also $\operatorname{ord}_{\mathfrak{p}^{m-1}} \varphi > B$, and hence the induction hypothesis gives $\operatorname{ord}_{\mathfrak{p}^{m-1}} \varphi = \infty$. Let π be a uniformizer for $\mathfrak{p}$, i.e., an element $\pi \in \mathfrak{p} \setminus \mathfrak{p}^2$.

We see that the form

$$\psi = \frac{1}{\pi^{m-1}}\varphi$$

is a form on Γ of weight k with coefficients in $\mathcal{O}_{\mathfrak{p}}$.

Since $\operatorname{ord}_{\mathfrak{p}^m} \varphi > B$, we must have $\operatorname{ord}_{\mathfrak{p}} \psi > B$, so that $\operatorname{ord}_{\mathfrak{p}} \psi = \infty$ by the induction hypothesis for $m = 1$. From this we conclude that $\operatorname{ord}_{\mathfrak{p}^m} \varphi = \infty$, as desired. $\square$

We now give a result in case of the forms having distinct weights (the $m = 1$ case is [Koh04, Thm. 1]).

Proposition 2.8. *Let f_1 and f_2 be modular forms of weights k_1 and k_2 on a congruence subgroup Γ of level N with coefficients in $\mathcal{O}$, and put $k = \max\{k_1, k_2\}$ and*

$$\widetilde{\mu} = \begin{cases} [\mathrm{SL}_2(\mathbb{Z}) : \Gamma \cap \Gamma_1(p)], & p \text{ odd}, \\ \min\{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma \cap \Gamma_1(3)], [\mathrm{SL}_2(\mathbb{Z}) : \Gamma \cap \Gamma_1(4)]\}, & p = 2. \end{cases}$$

Assume that $k_1 \equiv k_2 \pmod{p^s\kappa}$ for a non-negative integer s and a positive integer κ.

If $a_n(f_1) \equiv a_n(f_2) \pmod{\mathfrak{p}^m}$ for all non-negative integers

$$n \leq \widetilde{B} = \begin{cases} \frac{k\widetilde{\mu}}{12} - \frac{\widetilde{\mu}-1}{N}, & f_i \in S_{k_i}(\Gamma),\ i = 1, 2, \\ \frac{k\widetilde{\mu}}{12}, & \text{otherwise}, \end{cases}$$

then $f_1 \equiv f_2 \pmod{\mathfrak{p}^{\min\{e(s+1), m\}}}$.

Additionally, we obtain the congruence $f_1 \equiv f_2 \pmod{\mathfrak{p}^{\min\{2e(s+1), m\}}}$ if $p = 2$ and $\widetilde{\mu} = \max\{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma \cap \Gamma_1(3)], [\mathrm{SL}_2(\mathbb{Z}) : \Gamma \cap \Gamma_1(4)]\}$.

Proof. Assume without loss of generality that $k_2 \geq k_1$. Our hypotheses imply that we can then write

$$k_2 = k_1 + tp^s\kappa,$$

where t is a non-negative integer.

Let $\mathfrak{p}'$ be a prime ideal over p in the $(p-1)$'st cyclotomic field. By Lemma 2.5, there is an Eisenstein series of weight 1 on $\Gamma_1(p)$ if p is odd (resp. on $\Gamma_1(3)$ or $\Gamma_1(4)$ if $p = 2$) which reduces to 1 modulo $\mathfrak{p}'$, and we let E be the κ'th power of this Eisenstein series, so that E is of weight κ.

Now, view E as having coefficients in the compositum of the field of coefficients and the $(p-1)$'st cyclotomic field, and consider a prime ideal $\mathfrak{p}_1$ herein over $\mathfrak{p}$

and $\mathfrak{p}'$. Then, the ramification index of $\mathfrak{p}_1$ relative to p (and to $\mathfrak{p}'$) is e, since $e(\mathfrak{p}'/p) = 1$, and so the congruence $E \equiv 1 \pmod{\mathfrak{p}'}$ becomes $E \equiv 1 \pmod{\mathfrak{p}_1^e}$.

By induction on j, we see that $E^{p^j} \equiv 1 \pmod{\mathfrak{p}_1^{e(j+1)}}$ for all non-negative integers j, and hence also

$$E^{tp^s} \equiv 1 \pmod{\mathfrak{p}_1^{e(s+1)}}.$$

Consequently, the form

$$\widetilde{f} = E^{tp^s} \cdot f_1$$

satisfies $\widetilde{f} \equiv f_1 \pmod{\mathfrak{p}_1^{e(s+1)}}$. We now have that

$$a_n(\widetilde{f}) \equiv a_n(f_1) \pmod{\mathfrak{p}_1^{e(s+1)}},$$

and thus

$$a_n(\widetilde{f}) \equiv a_n(f_2) \pmod{\mathfrak{p}_1^{\min\{e(s+1),m\}}}$$

for all non-negative integers $n \leq \widetilde{B}$, because of our hypothesis on f_1 and f_2.

Now, $\widetilde{f}$ and f_2 are both forms of weight $k = k_2$ on $\Gamma \cap \Gamma_1(p)$ if p is odd, and on $\Gamma \cap \Gamma_1(3)$ or $\Gamma \cap \Gamma_1(4)$ if $p = 2$. Thus, Proposition 2.7 implies that

$$a_n(\widetilde{f}) \equiv a_n(f_2) \pmod{\mathfrak{p}_1^{\min\{e(s+1),m\}}},$$

and hence also

$$a_n(f_1) \equiv a_n(f_2) \pmod{\mathfrak{p}_1^{\min\{e(s+1),m\}}}$$

for all non-negative integers n.

Since both f_1 and f_2 have coefficients in $\mathcal{O}$ and $e(\mathfrak{p}_1/\mathfrak{p}) = 1$, we conclude that

$$a_n(f_1) \equiv a_n(f_2) \pmod{\mathfrak{p}^{\min\{e(s+1),m\}}}$$

for all non-negative integers n.

If $p = 2$ and $\widetilde{\mu} = \max\big\{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma \cap \Gamma_1(3)], [\mathrm{SL}_2(\mathbb{Z}) : \Gamma \cap \Gamma_1(4)]\big\}$, we use the Eisenstein series on $\Gamma_1(4)$, which according to the proof of Lemma 2.5 is congruent to 1 modulo 4, so that we get $a_n(\widetilde{f}) \equiv a_n(f_1) \pmod{\mathfrak{p}_1^{2e(s+1)}}$ for all non-negative integers $n \leq \widetilde{B}$. We finish the remaining part of the proof in the same way as above. $\square$

Since one of our main objectives in working with modular forms is to obtain results about Galois representations, we are especially interested in results on higher congruences involving eigenforms and newforms. In the formulation of Theorem 2.2 and Theorem 2.3 we used that it is sufficient to check coefficients indexed by primes when working with eigenforms and newforms. In the case of distinct weights the situation is more complicated.

We first give a useful lemma.

Lemma 2.9. *Let L/K be a finite extension of number fields. Let $\mathfrak{p}$ be a prime ideal of K and let $\mathfrak{P}$ be a prime ideal of L over $\mathfrak{p}$ of ramification index e.*

For every positive integer b we then have

$$\mathfrak{P}^b \cap K = \mathfrak{p}^{\lceil \frac{b}{e} \rceil}.$$

Proof. There is a non-negative integer a such that $ae < b \leq (a+1)e$, and then we have

$$\mathfrak{P}^{(a+1)e} \subseteq \mathfrak{P}^b \subseteq \mathfrak{P}^{ae}.$$

From this we get that

$$\mathfrak{p}^{a+1} = \mathfrak{P}^{(a+1)e} \cap K \subseteq \mathfrak{P}^b \cap K \subseteq \mathfrak{P}^{ae} \cap K = \mathfrak{p}^a,$$

and so $\mathfrak{P}^b \cap K$ is either $\mathfrak{p}^a$ or $\mathfrak{p}^{a+1}$.

Assume that $\mathfrak{P}^b \cap K = \mathfrak{p}^a$. Then $\mathfrak{p}^a \subseteq \mathfrak{P}^b$, i.e., $\mathfrak{P}^{ae} \subseteq \mathfrak{P}^b$, and so $ae \geq b$, a contradiction. We conclude that $\mathfrak{P}^b \cap K = \mathfrak{p}^{a+1}$, and since $a+1 = \lceil \frac{b}{e} \rceil$ by the definition of a, we are done. ☐

Proposition 2.10. *Let f_1 and f_2 be eigenforms of weights k_1 and k_2 on $\Gamma_1(N)$ with character χ and with coefficients in $\mathcal{O}$. Let ℓ be a prime and assume that k_1 and k_2 are congruent modulo the order of ℓ in $(\mathbb{Z}/p^{\lceil \frac{m}{e} \rceil}\mathbb{Z})^*$ if $\ell \nmid Np$, and that $\lceil \frac{m}{e} \rceil < \min\{k_1, k_2\}$ if $\ell = p$.*

Then, if $a_\ell(f_1) \equiv a_\ell(f_2) \pmod{\mathfrak{p}^m}$, we have $a_{\ell^n}(f_1) \equiv a_{\ell^n}(f_2) \pmod{\mathfrak{p}^m}$ for all positive integers n.

If we require $k_1 \equiv k_2 \pmod{p^{\alpha(\lceil \frac{m}{e} \rceil)}(p-1)}$ (where α is the function defined on p. 11), then we automatically have the congruence in the hypothesis satisfied for all $\ell \nmid Np$.

Proof. We prove this by induction on n. The $n = 1$ case holds because of our hypotheses, and we now consider the case of $n = 2$. We have

$$a_{\ell^2}(f_i) = a_\ell(f_i)^2 - \chi(\ell)\ell^{k_i - 1},$$

and so the claim is clearly true if $\ell \mid N$ since $\chi(\ell) = 0$ in this case.

Now assume that $\ell \nmid N$, so that $\chi(\ell) \notin \mathfrak{p}$. Then, $a_{\ell^2}(f_1) \equiv a_{\ell^2}(f_2) \pmod{\mathfrak{p}^m}$ if and only if

$$\chi(\ell)\ell^{k_1-1} \equiv \chi(\ell)\ell^{k_2-1} \pmod{\mathfrak{p}^m},$$

since $a_\ell(f_1)^2 \equiv a_\ell(f_2)^2 \pmod{\mathfrak{p}^m}$, and thus, if and only if

$$\ell^{k_1-1} \equiv \ell^{k_2-1} \pmod{p^{\lceil \frac{m}{e} \rceil}},$$

by Lemma 2.9. Assuming without loss of generality that $k_2 \geq k_1$, we can rewrite this as

$$\ell^{k_1-1}\big(1-\ell^{k_2-k_1}\big) \equiv 0 \pmod{p^{\lceil \frac{m}{e} \rceil}}.$$

By our hypotheses, we in all cases have this congruence satisfied, which completes the case of $n = 2$.

Finally, assume that $n > 2$, and that the proposition holds for all positive integers less than n. Using the identity

$$a_{\ell^n}(f_i) = a_\ell(f_i)a_{\ell^{n-1}}(f_i) - \chi(\ell)\ell^{k_i-1}a_{\ell^{n-2}}(f_i)$$

as well as the induction hypothesis, we conclude the desired. □

The following example shows that it is in general not sufficient to require congruences involving coefficients indexed by primes for newforms in order to conclude that there are congruences for all coefficients.

Example 2.11. Let f_1 be the newform in $S_6\big(\Gamma_0(3)\big)$ with q-expansion

$$f_1 = q - 6q^2 + 9q^3 + 4q^4 + \cdots$$

and let f_2 be the newform in $S_{20}\big(\Gamma_0(3)\big)$ with q-expansion

$$f_2 = q + \theta q^2 - 19683q^3 + (702\theta + 139840)q^4 + \cdots,$$

where θ is a root of the polynomial $x^2 - 702x - 664128$.

In the ring of integers $\mathcal{O}$ of $\mathbb{Q}(\theta)$, we have $5\mathcal{O} = \mathfrak{p}\mathfrak{p}_2$, and one finds that $a_2(f_1) \equiv a_2(f_2) \pmod{\mathfrak{p}}$.

However, $a_4(f_1) \not\equiv a_4(f_2) \pmod{\mathfrak{p}}$.

As mentioned previously, we work with modular forms in order to get results about Galois representations, and with this in mind it is too restrictive to work only with modular forms where we require congruences modulo $\mathfrak{p}^m$ for all non-negative integers n; we wish to be able to exclude indices divisible by a certain finite set of primes.

With this in mind, we consider a fixed finite set S of primes containing all prime divisors of Np. We define

$$N' = N \prod_{\ell \mid N} \ell \prod_{\substack{\ell \in S \\ \ell \nmid N}} \ell^2,$$

and note that $\mathrm{ord}_p(N') = 1 + \mathrm{ord}_p(N)$ if $p \mid N$, and that $\mathrm{ord}_p(N') = 2$ if $p \nmid N$.

We denote by B' the Sturm bound for $M_k(N')$.

Proposition 2.12. *Let f_1 and f_2 be modular forms in $M_k(N; \mathcal{O})$, and assume that there is a congruence*

$$a_\ell(f_1) \equiv a_\ell(f_2) \pmod{\mathfrak{p}^m}$$

for all primes $\ell \leq B'$ with $\ell \notin S$.

Then this congruence holds for all primes $\ell \notin S$.

Proof. We first apply [Miy06, Lem. 4.6.5]: By that lemma, we obtain from the f_i, forms f_i' in $M_k(N')$, by putting

$$f_i' = \sum_{\substack{\ell \nmid n \\ \ell \in S}} a_n(f_i) q^n.$$

Also, if the f_i are cusp forms, so are the f_i'.

The f_i' obviously still have coefficients in $\mathcal{O}$, and all Fourier coefficients, at any index n divisible by a prime in S, vanish. By our hypothesis, we can thus conclude that

$$a_n(f_1') \equiv a_n(f_2') \pmod{\mathfrak{p}^m}$$

for all non-negative integers $n \leq B'$.

From Proposition 2.7, we then get that $f_1' \equiv f_2' \pmod{\mathfrak{p}^m}$, and from this it follows that

$$a_\ell(f_1) \equiv a_\ell(f_2) \pmod{\mathfrak{p}^m}$$

for all primes $\ell \notin S$. □

That the increase of the bound is really necessary when we exclude a finite set of primes can be seen from the following example.

Example 2.13. Let f_1 and f_2 be the two newforms in $S_{14}(\Gamma_0(2))$. Both f_1 and f_2 have q-expansions with integral coefficients.

We have $N = 2$ and $k = 14$, so that $B = 2$ (using the improved bound we get for cusp forms). Let p be any prime and m any positive integer. Putting $S = \{2, p\}$, we automatically have that $a_\ell(f_1) \equiv a_\ell(f_2) \pmod{p^m}$ for all primes $\ell \leq B$ with $\ell \notin S$, but clearly this is not true for all primes $\ell \notin S$.

If p is odd, we have $N' = 4p^2$, and so $\mu' = 6p(p+1)$, giving a B' of roughly $7p(p+1)$, which for just $p = 3$ means at least 20 additional primes would need to be checked in order to conclude that such congruences exist for all primes $\ell \notin S$.

It would be very useful to obtain a better bound than the B' used here, since this bound is very big compared to the original bound B.

Let us note that we do have certain congruences between f_1 and f_2. In fact, $a_\ell(f_1) \equiv a_\ell(f_2) \pmod{2^{10} \cdot 3}$ for all primes $\ell \neq 2$. There are no congruences for all primes $\ell \notin \{2, p\}$ for any other prime p.

We have the following application of Lemma 2.12 to Proposition 2.8.

Corollary 2.14. *Let f_1 and f_2 be modular forms of weights k_1 and k_2 on $\Gamma_1(N)$ with coefficients in $\mathcal{O}$, and assume that $k_1 \equiv k_2 \pmod{p^s \kappa}$ for a non-negative integer s and a positive integer κ. Assume additionally that either $3 \mid N$ or $4 \mid N$ if $p = 2$, and let B' be the Sturm bound for $M_k(N')$, where $k = \max\{k_1, k_2\}$.*

If $a_\ell(f_1) \equiv a_\ell(f_2) \pmod{\mathfrak{p}^m}$ for all primes $\ell \leq B'$ with $\ell \notin S$, then

$$a_\ell(f_1) \equiv a_\ell(f_2) \pmod{\mathfrak{p}^{\min\{e(s+1),m\}}}$$

for all primes $\ell \notin S$.

Additionally, $a_\ell(f_1) \equiv a_\ell(f_2) \pmod{\mathfrak{p}^{\min\{2e(s+1),m\}}}$ if $p = 2$ and $4 \mid N$.

Proof. Observe first the following (which we use in the case of odd p): If $p \nmid N$, then upon replacing N by Np, and re-calculating N', we end up with the same number N' as had we calculated it from N, since $p \in S$, cf. p. 18, and so the Sturm bound B' remains unchanged. And of course our forms on $\Gamma_1(N)$ are also forms on $\Gamma_1(Np)$.

This means that we may well assume that N is divisible by p; our hypotheses remain unchanged when N is replaced by Np if N is not divisible by p. We therefore have that $\Gamma_1(N)$ is contained in $\Gamma_1(p)$ for odd p and in either $\Gamma_1(3)$ or $\Gamma_1(4)$ for $p = 2$.

Following the proof of Proposition 2.8, we assume that $k_2 \geq k_1$, then look at the same Eisenstein series E of weight κ (which is now on $\Gamma_1(N)$) and the same form $\widetilde{f} = E^{tp^s} \cdot f_1$, so that we have

$$a_\ell(\widetilde{f}) \equiv a_\ell(f_1) \pmod{\mathfrak{p}_1^{e(s+1)}}$$

for all primes $\ell \leq B'$ with $\ell \notin S$ (recall that $\mathfrak{p}_1$ is a prime ideal over $\mathfrak{p}$ in the compositum of the field of coefficients and the $(p-1)$'st cyclotomic field).

From this we get that

$$a_\ell(\widetilde{f}) \equiv a_\ell(f_2) \pmod{\mathfrak{p}_1^{\min\{e(s+1),m\}}}$$

for all primes $\ell \leq B'$ with $\ell \notin S$.

Now, $\widetilde{f}$ and f_2 are both forms of weight $k = k_2$ on $\Gamma_1(N)$ because of our hypothesis on N. Thus, Lemma 2.12 implies that

$$a_\ell(\widetilde{f}) \equiv a_\ell(f_2) \pmod{\mathfrak{p}_1^{\min\{e(s+1),m\}}},$$

and hence also

$$a_\ell(f_1) \equiv a_\ell(f_2) \pmod{\mathfrak{p}_1^{\min\{e(s+1),m\}}}$$

for all primes $\ell \notin S$.

We finish the proof in the same way as for Proposition 2.8 $\square$

We note the following result, which says that the Sturm bounds for modular forms on $\Gamma_1(N)$ with character are the same as the Sturm bounds for $\Gamma_0(N)$.

Corollary 2.15 (Buzzard)**.** *Let f_1 and f_2 be modular forms in $M_k(N, \chi; \mathcal{O})$ and let B and B' be the Sturm bounds corresponding to weight-k modular forms on $\Gamma_0(N)$ and $\Gamma_0(N')$.*

(i) If there is a congruence

$$a_n(f_1) \equiv a_n(f_2) \pmod{\mathfrak{p}^m}$$

for all non-negative integers $n \leq B$, then $f_1 \equiv f_2 \pmod{\mathfrak{p}^m}$.

(ii) If there is a congruence

$$a_\ell(f_1) \equiv a_\ell(f_2) \pmod{\mathfrak{p}^m}$$

for all primes $\ell \leq B'$ with $\ell \notin S$, then this congruence holds for all primes $\ell \notin S$.

Corollary 2.15 is in its original form only stated and proved for $m = 1$ and congruences for all indices n, but because of Proposition 2.7 it is easily generalized.

Proof. Let $\varphi = f_1 - f_2$, and let d be the order of the Dirichlet character χ. Then φ^d is a modular form on $\Gamma_0(N)$ of weight dk, and

$$d \operatorname{ord}_{\mathfrak{p}^m} \varphi = \operatorname{ord}_{\mathfrak{p}^m}(\varphi^d) > dB,$$

showing that $\operatorname{ord}_{\mathfrak{p}^m} \varphi > B$, so that $\operatorname{ord}_{\mathfrak{p}^m} \varphi = \infty$ by Proposition 2.7, as desired.

Part *(ii)* follows from *(i)* by replacing the forms f_i on $\Gamma_1(N)$ with the forms f_i' on $\Gamma_1(N')$, as in the proof of Lemma 2.12. □

2.4 Distinct weights via Katz-Serre

We again consider a fixed integer ring $\mathcal{O}$ of a number field K, as well as a prime ideal $\mathfrak{p}$ of $\mathcal{O}$ over the prime p with ramification index $e = e(\mathfrak{p}/p)$. Let L denote the Galois closure of K, and let $\mathfrak{P}$ be a prime ideal of L over $\mathfrak{p}$. We write $e(L, p)$ for the ramification index $e(\mathfrak{P}/p)$, and put $r = \operatorname{ord}_p e(L, p)$.

Let f_1 and f_2 be normalized cusp forms of weights k_1 and k_2 with coefficients in $\mathcal{O}$, and put $k = \max\{k_1, k_2\}$. If f_1 and f_2 are forms on $\Gamma_1(N)$, we let B and B' be the Sturm bounds for $S_k(N)$ and $S_k(N')$, and we let S be a finite set of primes containing the prime divisors of Np.

Recall the function $\alpha : \mathbb{Z} \to \mathbb{N}_0$ introduced on p. 11, which can be seen as being defined by

$$\alpha(n) = \max\{0, n - 1\},$$

for odd p, and

$$\alpha(n) = \begin{cases} 0, & n \leq 1, \\ 1, & n = 2, \\ n-2, & n \geq 3. \end{cases}$$

for $p = 2$.

Theorem 2.16. *(i) Assume that $N \geq 3$ is not divisible by p, and that f_1 and f_2 are forms on $\Gamma_1(N) \cap \Gamma_0(p)$.*

If $f_1 \equiv f_2 \pmod{\mathfrak{p}^m}$, then we have the congruence

$$k_1 \equiv k_2 \pmod{p^{\alpha(\lceil \frac{m}{e} \rceil - r)}(p-1)}$$

between the weights.

(ii) Let N be arbitrary, and assume that f_1 and f_2 are forms on $\Gamma_1(N)$. Assume additionally that either $3 \mid N$ or $4 \mid N$ if $p = 2$.

Suppose that $k_1 \equiv k_2 \pmod{p^s(p-1)}$ for a non-negative integer s.

If $a_\ell(f_1) \equiv a_\ell(f_2) \pmod{\mathfrak{p}^m}$ for all primes $\ell \leq B'$ with $\ell \notin S$, then we have

$$a_\ell(f_1) \equiv a_\ell(f_2) \pmod{\mathfrak{p}^{\min\{e(s+1),m\}}}$$

for all primes $\ell \notin S$, and additionally $a_\ell(f_1) \equiv a_\ell(f_2) \pmod{\mathfrak{p}^{\min\{2e(s+1),m\}}}$ for all primes $\ell \notin S$ if $p = 2$ and $4 \mid N$.

In particular, if $s = \alpha(\lceil \frac{m}{e} \rceil - r)$, we have the congruence

$$a_\ell(f_1) \equiv a_\ell(f_2) \pmod{\mathfrak{p}^m}$$

for all primes $\ell \notin S$ if $m \leq e$ or (p is odd and $r = 0$) or ($p = 2$, $4 \mid N$ and $r = 0$) or ($p = 2$, $3 \mid N$, $r = 0$ and $m \leq 2e$).

Part *(i)* of Theorem 2.16 can be seen as a generalization of Theorem 2.4, and this theorem is also the main point of the proof. Theorem 2.4 takes care of the unramified case, and the strategy of the proof of *(i)* is to reduce the general case to the unramified case, and then apply the theorem.

Proof. *Part (i)*: Let L_0 be the subfield of L corresponding to the inertia group $I(\mathfrak{P}/p)$. Let $\mathfrak{p}_0$ be the prime ideal of L_0 under $\mathfrak{P}$.

We let $\sigma \in I(\mathfrak{P}/p)$ act on the f_i by acting on their Fourier coefficients. Since $f_1 \equiv f_2 \pmod{\mathfrak{p}^m}$, we have $\sigma(f_1) \equiv \sigma(f_2) \pmod{\mathfrak{P}^{me(\mathfrak{P}/\mathfrak{p})}}$ for all $\sigma \in I(\mathfrak{P}/p)$. Letting

$$F_1 = \sum_\sigma \sigma(f_1) \quad \text{and} \quad F_2 = \sum_\sigma \sigma(f_2),$$

with the sums taken over all $\sigma \in I(\mathfrak{P}/p)$, we therefore obtain

$$F_1 \equiv F_2 \pmod{\mathfrak{P}^{me(\mathfrak{P}/\mathfrak{p})}}.$$

Since F_1 and F_2 are invariant under the action of $I(\mathfrak{P}/p)$, they actually have coefficients in L_0, and we therefore have

$$F_1 \equiv F_2 \pmod{\mathfrak{p}_0^{\lceil \frac{m}{e} \rceil}},$$

since $\mathfrak{P}^b \cap L_0 = \mathfrak{p}_0^{\lceil \frac{b}{e(L,p)} \rceil}$ for non-negative integers b, cf. Lemma 2.9, and because

$$e(L,p) = e(\mathfrak{p}/p)e(\mathfrak{P}/\mathfrak{p}) = e \cdot e(\mathfrak{P}/\mathfrak{p}).$$

Now, the extension $(L_0)_{\mathfrak{p}_0}/\mathbb{Q}_p$ of local fields is unramified, and so $(L_0)_{\mathfrak{p}_0}$ is the field of fractions of the ring $W = W(\mathbb{F}_{p^f})$ of Witt vectors over $\mathbb{F}_{p^f}$, for some positive integer f. Since the F_i have integral coefficients in L_0, we can view them as having coefficients in W.

Let a be the largest non-negative integer such that all Fourier coefficients of F_1 and F_2 are divisible by p^a. Then the forms $p^{-a}F_1$ and $p^{-a}F_2$ are cusp forms on $\Gamma_1(N) \cap \Gamma_0(p)$ of weights k_1 and k_2, respectively, and with coefficients in W. At least one of these forms has a q-expansion that does not reduce to 0 identically modulo p. Their q-expansions are congruent modulo

$$\mathfrak{p}_0^{\max\{0, \lceil \frac{m}{e} \rceil - a\}},$$

and hence also modulo

$$\mathfrak{p}_0^{\max\{0, \lceil \frac{m}{e} \rceil - r\}},$$

since certainly $a \le r$, because the coefficient of q for each F_i equals $\#I(\mathfrak{P}/p)$, which is just $e(L,p)$.

By Theorem 2.4, and also [Kat73, Thm. 3.2], we then deduce that

$$k_1 \equiv k_2 \pmod{p^{\alpha(\lceil \frac{m}{e} \rceil - r)}(p-1)},$$

where α is the function given before the theorem. Notice that we need our hypothesis $N \ge 3$ because of this reference to [Kat73].

Part (ii): That we get a congruence

$$a_\ell(f_1) \equiv a_\ell(f_2) \pmod{\mathfrak{p}^{\min\{e(s+1), m\}}}$$

for all primes $\ell \notin S$, follows directly from Corollary 2.14 with $\kappa = p-1$.

Now, assume that $s = \alpha(\lceil \frac{m}{e} \rceil - r)$. We will show that $m \le e(s+1)$ (resp. $m \le 2e(s+1)$) in the cases mentioned in the theorem.

If $m \leq e$, we clearly always have $m \leq e(s+1)$.

If p is odd and $r = 0$, we have $s = \lceil \frac{m}{e} \rceil - 1$, and so

$$e(s+1) = e\lceil \tfrac{m}{e} \rceil \geq m.$$

Next, assume that $p = 2$, $r = 0$ and either $3 \mid N$ or $4 \mid N$. The case $m \leq e$ is already taken care of, so assume that $e < m \leq 2e$. Then $s = 1$, and so $e(s+1) = 2e \geq m$.

Finally, assume that $p = 2$, $r = 0$ and $4 \nmid N$. Write $\lceil \frac{m}{e} \rceil = n$ for an integer $n \geq 3$ (we already considered $n \leq 2$). Then $s = n - 2$ and we get

$$2e(s+1) = 2e(n-1) > en \geq m,$$

since $2(n-1) > n$ because $n \geq 3$.

In each case, we thus conclude that there is a congruence

$$a_\ell(f_1) \equiv a_\ell(f_2) \pmod{\mathfrak{p}^m}$$

for all primes $\ell \notin S$. □

We have the following corollary to Theorem 2.16 (in the case of odd p with $p \nmid e(L,p)$).

Corollary 2.17. *Retain the setup and notation of Theorem 2.16, and assume that p is odd, $r = 0$, that $N \geq 3$ is not divisible by p and $2 \mid N$ if $p = 3$, and that f_1 and f_2 are forms on $\Gamma_1(N)$.*

Then $f_1 \equiv f_2 \pmod{\mathfrak{p}^m}$ if and only if $a_n(f_1) \equiv a_n(f_2) \pmod{\mathfrak{p}^m}$ for all positive integers $n \leq B$ and we have the congruence

$$k_1 \equiv k_2 \pmod{p^{\lceil \frac{m}{e} \rceil - 1}(p-1)}$$

between the weights.

Proof. That the congruence $f_1 \equiv f_2 \pmod{\mathfrak{p}^m}$ implies the congruence between the first B coefficients, as well as between the weights, is a direct consequence of Theorem 2.16.

To prove the converse, we use the same type of argument as in the proof of Proposition 2.8, just with some other Eisenstein series.

We can assume that $k_2 \geq k_1$, and we write

$$k_2 = k_1 + tp^{\lceil \frac{m}{e} \rceil - 1}(p-1),$$

where t is a non-negative integer.

Letting E be the Eisenstein series of weight $p-1$ on $\Gamma_1(N)$ from Lemma 2.6 satisfying $E \equiv 1 \pmod{p}$, we define

$$\widetilde{f} = E^{tp^{\lceil \frac{m}{e} \rceil - 1}} \cdot f_1.$$

Then, $\widetilde{f}$ has weight k_2 and is congruent to f_1 modulo $p^{\lceil \frac{m}{e} \rceil}$, which we write as

$$\widetilde{f} \equiv f_1 \pmod{\mathfrak{p}^m},$$

using that $m \leq e\lceil \frac{m}{e} \rceil$.

Consequently, we get that

$$a_n(\widetilde{f}) \equiv a_n(f_2) \pmod{\mathfrak{p}^m}$$

for all positive integers $n \leq B$, and we conclude from Proposition 2.7 that we have $\widetilde{f} \equiv f_2 \pmod{\mathfrak{p}^m}$, and hence also $f_1 \equiv f_2 \pmod{\mathfrak{p}^m}$. □

Theorem 2.16 gives a necessary condition on the weights when one is looking for congruences modulo $\mathfrak{p}^m$. We apply this condition and use the mathematics software program Magma, cf. [BCP97], to give some examples of higher congruences between newforms where p is ramified in the field of coefficients.

Example 2.18. Consider the (normalized) cusp form f_1 on $\Gamma_0(9)$ of weight 4 with q-expansion

$$f_1 = q - 8q^4 + 20q^7 + \cdots.$$

We try to find congruences of the coefficients of f_1 and f_2 modulo powers of a prime ideal above 5, for a form f_2 of weight k_2 satisfying $k_2 \equiv 4 \pmod{20}$. We let $S = \{3, 5\}$.

The smallest possible choice of weight for f_2 is $k_2 = 24$, i.e., $s = 1$. There is a newform f_2 on $\Gamma_0(9)$ of weight 24 with coefficients in the number field $K = \mathbb{Q}(\theta)$, with θ a root of $x^4 - 29258x^2 + 97377280$. The prime 5 is ramified in K and has the decomposition $5\mathcal{O} = \mathfrak{p}^2\mathfrak{p}_2$, where $\mathcal{O}$ is the ring of integers of K.

We have $k = 24$, $N = 9$, $N' = 675$ and $\mu' = 1080$, from which we get the Sturm bound $B' = 2040$. We find that

$$a_\ell(f_1) \equiv a_\ell(f_2) \pmod{\mathfrak{p}^3}$$

for all primes $\ell \leq B'$ with $\ell \neq 3, 5$.

Since $[K : \mathbb{Q}] = 4$, the Galois closure L of K satisfies $[L : K] \mid 24$ (in fact $[L : K] = 8$ in this case). This shows that $5 \nmid e(L, 5)$, so that $r = 0$. Combined with the fact that 5 is odd, we conclude from Theorem 2.16 that

$$a_\ell(f_1) \equiv a_\ell(f_2) \pmod{\mathfrak{p}^3}$$

for all primes $\ell \neq 3, 5$.

Next we do an example where the weights are congruent modulo a higher power of p.

Example 2.19. We again start with the cusp form

$$f_1 = q - 8q^4 + 20q^7 + \cdots$$

on $\Gamma_0(9)$ of weight 4. Another possible choice of weight for f_2 is $k_2 = 104$, so that $k_1 \equiv k_2 \pmod{5^2}$. There is a newform f_2 on $\Gamma_0(9)$ of weight 104 with coefficients in a number field K of degree 18. We do not give the defining polynomial for K here, since the coefficients are huge (the constant term for instance has 279 digits). The prime 5 is ramified in K and has the decomposition

$$5\mathcal{O} = \mathfrak{p}^2\mathfrak{p}_2\mathfrak{p}_3\mathfrak{p}_4^2\mathfrak{p}_5^2\mathfrak{p}_6^2\mathfrak{p}_7^2\mathfrak{p}_8^2\mathfrak{p}_9^2,$$

where $\mathcal{O}$ is the ring of integers of K.

As N is the same is in the previous example, we again have $\mu' = 1080$, which for $k = 104$ gives the Sturm bound $B' = 9240$. We find that

$$a_\ell(f_1) \equiv a_\ell(f_2) \pmod{\mathfrak{p}^5}$$

for all primes $\ell \leq B'$ with $\ell \neq 3, 5$.

We can compute that the degree of the Galois closure L of K is $2^{16} \cdot 3^4 \cdot 5 \cdot 7$, which means that r is either 0 or 1. Since $s = 2$, $e = e(\mathfrak{p}/5) = 2$ and $m = 5$, we find that $\min\{e(s+1), m\} = 5$, so that we from Theorem 2.16 can conclude that

$$a_\ell(f_1) \equiv a_\ell(f_2) \pmod{\mathfrak{p}^5}$$

for all primes $\ell \leq B'$ with $\ell \neq 3, 5$.

The last example took over 9 hours to compute at the fastest server available at the Department of Mathematical Sciences at the University of Copenhagen. By far the most time-consuming part was computing the newforms on $\Gamma_0(9)$ of weight 104 (there are four). In Section 2.7, we discuss in more detail how we are able to compute prime decompositions in number fields of large degree.

2.5 Distinct weights via Galois representations

We still consider the fixed integer ring $\mathcal{O}$ of a number field, as well as a prime ideal $\mathfrak{p}$ of $\mathcal{O}$ over the prime p with ramification index $e = e(\mathfrak{p}/p)$.

Let f_1 and f_2 now be (normalized) cusp forms in $S_{k_1}(N, \chi_1)$ and $S_{k_2}(N, \chi_2)$ with coefficients in $\mathcal{O}$, and put $k = \max\{k_1, k_2\}$. As before, we let B and B' be the Sturm bounds for $S_k(N)$ and $S_k(N')$, and we again let S be a finite set of primes containing the prime divisors of Np.

We say that f_1 and f_2 are *eigenforms outside* S if they are eigenforms for all Hecke operators T_ℓ for primes $\ell \notin S$. The corresponding eigenvalue for such a T_ℓ acting on f_i is then exactly the coefficient $a_\ell(f_i)$.

We now give a result in the vein of Theorem 2.16, but where the strategy of the proof is to work with determinant characters of mod $\mathfrak{p}^m$ Galois representations associated to eigenforms outside S, cf. Section 1.5. Note that we allow p to divide N in this approach.

Theorem 2.20. *(i) Assume that f_1 and f_2 are eigenforms outside S, and that at least one of the mod $\mathfrak{p}$ Galois representations $\overline{\rho}_{f_1,\mathfrak{p}}$ or $\overline{\rho}_{f_2,\mathfrak{p}}$ is absolutely irreducible.*

View the characters χ_1 and χ_2 as finite order characters on $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and let the order of the character

$$\left(\chi_2\chi_1^{-1} \bmod \mathfrak{p}^m\right)|_{I_p},$$

where I_p is an inertia group at p, be $p^\delta d$ with d a positive divisor of $p-1$.

If we have $a_\ell(f_1) \equiv a_\ell(f_2) \pmod{\mathfrak{p}^m}$ for all primes $\ell \notin S$, then we have $\delta \le \alpha(\lceil \frac{m}{e} \rceil)$ and the congruence

$$k_1 \equiv k_2 \pmod{p^{\alpha(\lceil \frac{m}{e} \rceil)-\delta} \cdot (p-1)/d},$$

between the weights.

(ii) Let d be a positive divisor of $p-1$ and let s be a non-negative integer. Assume that there is a congruence

$$k_1 \equiv k_2 \pmod{p^s \cdot (p-1)/d}$$

between the weights, and assume additionally that either $3 \mid N$ or $4 \mid N$ if $p=2$.

If $a_\ell(f_1) \equiv a_\ell(f_2) \pmod{\mathfrak{p}^m}$ for all primes $\ell \le B'$ with $\ell \notin S$, then we have

$$a_\ell(f_1) \equiv a_\ell(f_2) \pmod{\mathfrak{p}^{\min\{e(s+1),m\}}}$$

for all primes $\ell \notin S$, and additionally $a_\ell(f_1) \equiv a_\ell(f_2) \pmod{\mathfrak{p}^{\min\{2e(s+1),m\}}}$ for all primes $\ell \notin S$ if $p=2$ and $4 \mid N$.

In particular, if $s = \alpha(\lceil \frac{m}{e} \rceil)$, we have the congruence

$$a_\ell(f_1) \equiv a_\ell(f_2) \pmod{\mathfrak{p}^m}$$

for all primes $\ell \notin S$ if p is odd or ($p=2$ and $4 \mid N$) or ($p=2$, $3 \mid N$ and $m \le 2e$).

Proof. Part (i): Consider the associated mod $\mathfrak{p}^m$ Galois representations $\overline{\rho}_{f_1,\mathfrak{p}^m}$ and $\overline{\rho}_{f_2,\mathfrak{p}^m}$ of f_1 and f_2.

Since $a_\ell(f_1) \equiv a_\ell(f_2) \pmod{\mathfrak{p}^m}$ for all primes $\ell \notin S$, we can conclude by Chebotarev's density theorem that the representations $\overline{\rho}_{f_1,\mathfrak{p}^m}$ and $\overline{\rho}_{f_2,\mathfrak{p}^m}$ have the same traces. As at least one of the mod $\mathfrak{p}$ representations is assumed absolutely irreducible, the mod $\mathfrak{p}^m$ representations $\overline{\rho}_{f_1,\mathfrak{p}^m}$ and $\overline{\rho}_{f_2,\mathfrak{p}^m}$ are isomorphic, cf. [Car94, Thm. 1]. Hence, the determinants of these representations are also isomorphic. These determinants are

$$\det \overline{\rho}_{f_i,\mathfrak{p}^m} = \big(\chi_i \widetilde{\chi}^{k_i-1} \bmod \mathfrak{p}^m\big),$$

where $\widetilde{\chi}$ denotes the p-adic cyclotomic character $\widetilde{\chi} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}_p^*$, and the characters χ_i are now seen as finite order characters on $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Observe that the characters χ_i take values in $\mathcal{O}$, so that it makes sense to reduce them modulo $\mathfrak{p}^m$. Also, reducing $\widetilde{\chi}$ modulo $\mathfrak{p}^m$ is to be taken in the obvious sense.

We can now deduce that

$$\big(\chi_2\chi_1^{-1} \bmod \mathfrak{p}^m\big)|_{I_p} = \big(\widetilde{\chi} \bmod \mathfrak{p}^m\big)^{k_1-k_2}|_{I_p}.$$

Now let us view, via local class field theory, the character $\big(\widetilde{\chi} \bmod \mathfrak{p}^m\big)|_{I_p}$ as a character on $\mathbb{Z}_p^*$. As such it factors through $(\mathbb{Z}/p^{\lceil \frac{m}{e} \rceil}\mathbb{Z})^*$ and has order

$$p^{\alpha(\lceil \frac{m}{e} \rceil)}(p-1),$$

cf. Lemma 2.9. By definition, the character $\big(\chi_2\chi_1^{-1} \bmod \mathfrak{p}^m\big)|_{I_p}$ has order $p^\delta d$, with d a divisor of $p-1$. Hence, we see that $p^\delta d$ is a divisor of $p^{\alpha(\lceil \frac{m}{e} \rceil)}(p-1)$, which implies that $\delta \le \alpha(\lceil \frac{m}{e} \rceil)$. We conclude from this, that $k_1 - k_2$ is divisible by $p^{\alpha(\lceil \frac{m}{e} \rceil)-\delta} \cdot (p-1)/d$, as desired.

Part (ii): That we get a congruence

$$a_\ell(f_1) \equiv a_\ell(f_2) \pmod{\mathfrak{p}^{\min\{e(s+1),m\}}}$$

for all primes $\ell \notin S$, follows directly from Corollary 2.14 with $\kappa = (p-1)/d$.

The remaining statement involving the case $s = \alpha(\lceil \frac{m}{e} \rceil)$ is shown just as in the proof of Theorem 2.16. □

The following corollary follows immediately from Theorem 2.20.

Corollary 2.21. *Retain the setup and notation of Theorem 2.20, and assume that $\delta = 0$ and p is odd.*

Then, $a_\ell(f_1) \equiv a_\ell(f_2) \pmod{\mathfrak{p}^m}$ for all primes $\ell \notin S$ if and only if this congruence holds for all primes $\ell \le B'$ with $\ell \notin S$ and we have the congruence

$$k_1 \equiv k_2 \pmod{p^{\lceil \frac{m}{e} \rceil - 1} \cdot (p-1)/d}$$

between the weights.

2.6 Maximal congruences between newforms

Assume that f_1 and f_2 are newforms on $\Gamma_0(N)$ of distinct weights k_1 and k_2, and let s be the largest non-negative integer such that $k_1 \equiv k_2 \pmod{p^s(p-1)}$. Let $\mathfrak{p}$ be a prime ideal over the prime p in the integer ring of the field of coefficients of f_1 and f_2, and let m be a positive integer such that there is a congruence

$$a_\ell(f_1) \equiv a_\ell(f_2) \pmod{\mathfrak{p}^m}$$

for all primes $\ell \nmid Np$.

Then it follows from Theorem 2.20 that $\lceil \frac{m}{e(\mathfrak{p}/p)} \rceil \le s+1$ if p is odd, and that $\lceil \frac{m}{e(\mathfrak{p}/p)} \rceil \le s+2$ if $p = 2$ (supposing the additional hypotheses of Theorem 2.20 to be satisfied).

It is interesting to ask whether these upper bounds for $\lceil \frac{m}{e(\mathfrak{p}/p)} \rceil$ are attained if we allow f_1 and f_2 to run through all newforms of $S_{k_1}\big(\Gamma_0(N)\big)$ and $S_{k_2}\big(\Gamma_0(N)\big)$, as well as letting $\mathfrak{p}$ run through all prime ideals over p in the field of coefficients.

During work on the results proven in the previous sections, we omputed many examples with newforms of various levels and weights. Based on these numerical examples it seems reasonable to conjecture the following.

Conjecture 2.22. *Let N be a positive integer and let p be a prime. Let k_1 and k_2 be distinct positive integers such that there are newforms on $\Gamma_0(N)$ of weights k_1 and k_2, and assume that $k_1 \equiv k_2 \pmod{p^s(p-1)}$ with the non-negative integer s as big as possible.*

Then there exist newforms f_1 and f_2 on $\Gamma_0(N)$ of weights k_1 and k_2, a prime ideal $\mathfrak{p}$ over p in the integer ring of a number field containing the coefficients of f_1 and f_2, and a positive integer m satisfying $s \le \lceil \frac{m}{e(\mathfrak{p}/p)} \rceil \le s+2$, such that

$$a_\ell(f_1) \equiv a_\ell(f_2) \pmod{\mathfrak{p}^m}$$

for all primes $\ell \nmid Np$.

Note that the conjecture does not say that such a 'maximal congruence' exists for any specific pair of newforms in $S_{k_1}\big(\Gamma_0(N)\big)$ and $S_{k_2}\big(\Gamma_0(N)\big)$, just that such a pair of newforms exist.

We denote by $L_{k_1,k_2}^{(N,p)}$ the maximally attained value for $\lceil \frac{m}{e(\mathfrak{p}/p)} \rceil$ between newforms on $\Gamma_0(N)$ of weights k_1 and k_2, and we write simply L_{k_1,k_2} if N and p are clear from the context.

When p is odd, we in our computations in general get $L_{k_1,k_2} = s+1$, but there are sporadic cases where we only get $L_{k_1,k_2} = s$. When $p = 2$, the evidence suggests that we get only $L_{k_1,k_2} = s+2$ when $5 \nmid N$ (except for a few sporadic cases where $L_{k_1,k_2} = s$), and that we get both $L_{k_1,k_2} = s + 1$ and $L_{k_1,k_2} = s + 2$ when $5 \mid N$ (plus a few sporadic cases with $L_{k_1,k_2} = s$). These phenomena could very well be due to the limited nature of the numerical data, and these limitations obviously also necessitate a caveat in connection with Conjecture 2.22.

In Appendix A, we give some computational evidence for Conjecture 2.22. These results show that the conjecture holds for all levels $N < 10$ and primes $p = 2, 3, 5$, with the weights k_1 and k_2 satisfying $2 \le k_1 < k_2 \le k_1 + 64$ and $k_1 \le k^{(N)}$, with the following choices of $k^{(N)}$: $k^{(1)} = 22$, $k^{(2)} = 24$, $k^{(3)} = 12$, $k^{(4)} = 14$, $k^{(5)} = 6$, $k^{(6)} = 22$, $k^{(7)} = 22$, $k^{(8)} = 10$ and $k^{(9)} = 6$.

We have also verified the conjecture for the levels $N = 10, 12, 15, 16, 18, 20, 24, 25, 27$ and primes $p = 2, 3, 5$, with k_1 and k_2 satisfying $2 \le k_1 < k_2 \le k_1 + 32$ and $k_1 \le k^{(N)}$, with the following choices of $k^{(N)}$: $k^{(10)} = 10$, $k^{(12)} = 14$, $k^{(15)} = 6$, $k^{(16)} = 10$, $k^{(18)} = 16$, $k^{(20)} = 6$, $k^{(24)} = 8$, $k^{(25)} = 4$ and $k^{(27)} = 2$.

When looking at the tables in Appendix A for fixed N and p, it is seen that when we fix k_1 and let k_2 vary, there is a distinct pattern in the values of L_{k_1,k_2}, and this pattern seems to depend only on s: When increasing k_2, we get (except for the sporadic cases) higher values for L_{k_1,k_2} exactly when we get higher values for s. The sporadic cases (which primarily occur when the weights are small and close to each other) are italicized in the tables of Appendix A.

We include as much numerical data as possible in Appendix A, since this is essentially all we have to support Conjecture 2.22 at this point. Hopefully, it can also serve as inspiration, or at least save time, for anyone interested in working with these questions.

We have also tested for maximal congruences when lowering the level from $\Gamma_0(Np)$ to $\Gamma_0(N)$, and the information gathered through these examples gives rise to a pair of conjectures (two distinct things happen according to whether or not $p^2 \mid N$).

Conjecture 2.23. *Let N be a positive integer and let p be a prime such that $p^2 \nmid N$. Let k_1 and k_2 be distinct positive integers such that there are newforms in $S_{k_1}\big(\Gamma_0(Np)\big)$ and $S_{k_2}\big(\Gamma_0(N)\big)$, and assume that $k_1 \equiv k_2 \pmod{p^s(p-1)}$ with the non-negative integer s as big as possible.*

Then there exist newforms $f_1 \in S_{k_1}\big(\Gamma_0(Np)\big)$ and $f_2 \in S_{k_2}\big(\Gamma_0(N)\big)$, a prime ideal $\mathfrak{p}$ over p in the integer ring of a number field containing the coefficients of f_1 and f_2, and a positive integer m satisfying $s \le \lceil \frac{m}{e(\mathfrak{p}/p)} \rceil \le s + 2$, such that

$$a_\ell(f_1) \equiv a_\ell(f_2) \pmod{\mathfrak{p}^m}$$

for all primes $\ell \nmid Np$.

We denote by $M_{k_1,k_2}^{(N,p)}$ the maximally attained value for $\lceil \frac{m}{e(\mathfrak{p}/p)} \rceil$ between newforms in $S_{k_1}\big(\Gamma_0(Np)\big)$ and $S_{k_2}\big(\Gamma_0(N)\big)$, and we write simply M_{k_1,k_2} if N and p are clear from the context.

We note that the conclusions of Conjecture 2.23 and those of Conjecture 2.22 are essentially identical.

The evidence suggests that we always get $M_{k_1,k_2} = s+1$ when p is odd, and that we generally get $M_{k_1,k_2} = s+1$ or $M_{k_1,k_2} = s+2$ when $p=2$ (plus a few sporadic cases with $M_{k_1,k_2} = s$). For $p=2$, it appears that we get $M_{k_1,k_2} = s+2$ when $5 \nmid N$ (except for the sporadic cases), and that both $M_{k_1,k_2} = s+1$ and $M_{k_1,k_2} = s+2$ can occur when $5 \mid N$. We again note that these phenomena could be due to the limited size of the numerical data.

In Appendix B, we give some computational evidence for Conjecture 2.23. These results show that the conjecture holds for all base levels $N < 10$ and primes $p = 2,3,5$ (the case $N=7$ and $p=5$ has not been computed), with the weights k_1 and k_2 satisfying $2 \le k_1 < k_2 \le k_1 + 64$ and $k_1 \le k^{(N,p)}$, with the following choices of $k^{(N,p)}$: $k^{(1,2)} = 24$, $k^{(1,3)} = 12$, $k^{(1,5)} = 6$, $k^{(2,2)} = 16$, $k^{(2,3)} = 22$, $k^{(2,5)} = 10$, $k^{(3,2)} = 22$, $k^{(3,3)} = 6$, $k^{(3,5)} = 6$, $k^{(4,3)} = 14$, $k^{(4,5)} = 6$, $k^{(5,2)} = 10$, $k^{(5,3)} = 6$, $k^{(5,5)} = 4$, $k^{(6,2)} = 14$, $k^{(6,3)} = 16$, $k^{(6,5)} = 12$, $k^{(7,2)} = 6$, $k^{(7,3)} = 2$, $k^{(8,2)} = 10$, $k^{(8,3)} = 8$, $k^{(8,5)} = 4$, $k^{(9,2)} = 16$, $k^{(9,3)} = 2$ and $k^{(9,5)} = 4$.

We have also verified the conjecture for the base levels $N = 10,12,15,16,18,20,24$ and primes $p = 2,3,5$, with k_1 and k_2 satisfying $2 \le k_1 < k_2 \le k_1 + 32$ and $k_1 \le k^{(N,p)}$, with the following choices of $k^{(N,p)}$: $k^{(10,2)} = 6$, $k^{(10,3)} = 12$, $k^{(10,5)} = 6$, $k^{(12,2)} = 8$, $k^{(12,3)} = 8$, $k^{(12,5)} = 8$, $k^{(15,2)} = 12$, $k^{(15,3)} = 4$, $k^{(15,5)} = 2$, $k^{(16,2)} = 4$, $k^{(16,3)} = 8$, $k^{(16,5)} = 4$, $k^{(18,2)} = 8$, $k^{(18,3)} = 6$, $k^{(18,5)} = 6$, $k^{(20,2)} = 4$, $k^{(20,3)} = 8$, $k^{(20,5)} = 2$, $k^{(24,2)} = 8$, $k^{(24,3)} = 6$ and $k^{(24,5)} = 4$.

When looking at the tables in Appendix B with $p^2 \nmid N$, we see the same patterns as in the tables of Appendix A (as before we have italicized the sporadic cases).

An interesting thing occurs when $p^2 \mid N$. The computational evidence suggests the following.

Conjecture 2.24. *Let N be a positive integer and let p be a prime with $p^2 \mid N$.*

There exists a positive integer M such that for all newforms $f_1 \in S_{k_1}\big(\Gamma_0(Np)\big)$ and $f_2 \in S_{k_2}\big(\Gamma_0(N)\big)$ with k_1 and k_2 distinct positive integers, and all prime ideals $\mathfrak{p}$ over p in a number field containing the coefficients of f_1 and f_2 the following holds: If there is a positive integer m and a congruence

$$a_\ell(f_1) \equiv a_\ell(f_2) \pmod{\mathfrak{p}^m}$$

for all primes $\ell \nmid Np$, then $\lceil \frac{m}{e(\mathfrak{p}/p)} \rceil \le M$.

In other words, Conjecture 2.24 says that there is an upper bound for $M_{k_1,k_2}^{(N,p)}$, which is independent of the weights k_1 and k_2, i.e., depending only on N and p, when $p^2 \mid N$. This is in stark contrast to the conjecture in the case of $p^2 \nmid N$, where we can attain arbitrarily high values by just increasing the weight, since we only need to increase s in the weight congruence $k_1 \equiv k_2 \pmod{p^s(p-1)}$.

We have not included all of the computed tables regarding Conjecture 2.24 in Appendix B, only those with base level $N < 25$, since these tables illustrate what happens in the case of $p^2 \mid N$. We have computed a few more examples with base level $N > 25$, and the tables below summarize all results in case of $p = 2$ and $p = 3$ (where we for each choice of k_1 have computed for all k_2 satisfying $k_1 < k_2 \leq k_1 + 32$, and also for $k_2 \leq k_1 + 64$ when $N < 10$):

$p = 2$		
N	k_1	M_{k_1,k_2}
4	$4, 6, 8, 10$	$3, 4, 5$
8	$4, 6, 8, 10$	3
12	$2, 4, 6, 8$	$3, 4, 5$
16	$2, 4$	2
20	$2, 4$	$1, 2, 3, 4, 5$
24	$2, 4, 6, 8$	3
32	$2, 4$	2

$p = 3$		
N	k_1	M_{k_1,k_2}
9	2	1
18	$2, 4, 6$	$1, 2$
36	2	$1, 2$

For the sake of clarity, we should mention that when we for (say) $N = 8$ and $p = 2$ state that the only value of M_{k_1,k_2} that occurs is 3, we mean that for each pair of weights k_1 and k_2, there exist newforms $f_1 \in S_{k_1}\big(\Gamma_0(16)\big)$ and $f_2 \in S_{k_2}\big(\Gamma_0(8)\big)$, as well as a prime ideal $\mathfrak{p}$ over p in the integer ring of the field of coefficients of f_1 and f_2, such that there is a congruence

$$a_\ell(f_1) \equiv a_\ell(f_2) \pmod{\mathfrak{p}^m}$$

for all primes $\ell \neq 2$ with $\lceil \frac{m}{e(\mathfrak{p}/p)} \rceil = 3$ (and no such congruence with $\lceil \frac{m}{e(\mathfrak{p}/p)} \rceil > 3$).

By looking at the above two tables, one naturally starts to consider if the bound M is actually independent of N, and only depends on p (with $M = 5$ if $p = 2$, and $M = 2$ if $p = 3$). In our computations, we have not encountered any congruences where these upper bounds were not correct, but there is a severe lack of examples due to the very time-consuming process of computing newforms when the level and/or weight increases, and so we have not chosen to include this in the formulation of Conjecture 2.24.

Another interesting question that immediately arises is if one can formulate similar conjectures for $\Gamma_1(N)$ (and $\Gamma_1(Np)$). In light of Theorem 2.20, one would need to adjust the attainable values to reflect the possible ramification of the characters involved.

One could also consider the case of level-lowering from $\Gamma_0(Nq)$ to $\Gamma_0(N)$ (or $\Gamma_1(Nq)$ to $\Gamma_1(N)$) modulo powers of p, where p and q are distinct primes.

Since work on these conjectures began only a few months before the due date of this thesis, we have not had the time to look much into either of these questions, but both would be very interesting to investigate.

2.7 Computational issues

Several computational problems arise when the level N (or Np) and weight $k = \max\{k_1, k_2\}$ become 'large'. The first complication is the computation of newforms, which takes a very long time for just something like $S_{100}\big(\Gamma_0(6)\big)$, and this limits how large N and k we can effectively test for. Note: It seems as if the command `Newforms(CuspidalSubspace(ModularForms(`N`,`k`)))` of the modular forms package of Magma is significantly faster than its modular symbols counterpart `SortDecomposition(NewformDecomposition(NewSubspace(CuspidalSubspace(ModularSymbols(`N`,`k`)))))` used with the `SystemOfEigenvalues` function. We use the `SortDecomposition` function as our way of numbering newforms of any given level and weight.

Another issue entirely is determining the decomposition of the prime p in the ring of integers $\mathcal{O}$ of the field of coefficients. The natural way to do this is to let Magma compute $\mathcal{O}$ via `IntegerRing` and then use `Decomposition(`$\mathcal{O}$`,`p`)`. The first step in determining $\mathcal{O}$ this way, is to compute the prime divisors of the discriminant of the number field, but this quickly turns into a very difficult problem since the discriminant becomes huge for relatively small N and k, and also contains very large prime divisors. Even for level $N = 1$ we cannot compute $\mathcal{O}$ for newforms of weight $k > 74$. This weight bound quickly decreases as the level increases, and for $N = 10$ we cannot go higher than $k = 26$.

There are other ways that one can compute the decomposition of p in $\mathcal{O}$, but what we need is the $\mathfrak{p}$-adic valuation of $a_\ell(f_1) - a_\ell(f_2)$ for certain primes ℓ and a prime ideal $\mathfrak{p}$ of $\mathcal{O}$ above p. This requires much more information about $\mathfrak{p}$, such as its generators expressed in terms of a primitive element of the field of coefficient.

This was a major concern, until we discovered the Montes package for Magma (available online since the summer of 2009, see also [GMN08a], [GMN08b] and [GMN08c]), which uses Newton polygons of higher order to compute many things related to integer rings of number fields. The authors of this package were kind enough to modify their existing package to include a function to compute exactly what we need:

First, given a rational prime p, compute the prime ideals of $\mathcal{O}$ over p (including the corresponding ramification indices, residue class degrees and generators, if so desired). Then, for an element θ of $\mathcal{O}$ and a prime ideal $\mathfrak{p}$ over p, return the $\mathfrak{p}$-adic valuation of θ.

The computation of prime ideals is extremely fast, even in cases where the degree of the field of coefficients is very large. Even though the computation of the $\mathfrak{p}$-adic valuation of $a_\ell(f_1) - a_\ell(f_2)$ is also fast, it can take a long time to compute this for all necessary primes ℓ, since B' becomes very big compared to B.

As an example, let us consider the case of $S_k\big(\Gamma_0(12)\big)$ and $p = 5$. We find $N' = 1800$ and $\mu' = 4320$, giving a Sturm bound of $B' = 360k - 361$, which is almost 180 times bigger than the original bound $B = 2k - 2$. If $k = 100$, this means that 3748 additional primes need to be checked.

Chapter 3

Complexity of computing cusp forms

This chapter is based on the paper [Ras09].

As we saw in the last part of the previous chapter, it is often extremely useful to be able to work explicitly with spaces of modular forms, for instance, in working with elliptic curves, testing conjectures, etc. The standard way of doing this is via modular symbols, and to our knowledge every available software package uses this approach to compute bases of spaces of modular forms. A good reference for the computational aspects of this is [Ste07].

Another way of computing modular forms is via a cohomological approach, based on the Eichler-Shimura isomorphism, as done by Wang in [Wan94]. This approach is essentially the same as the modular symbols approach over the rationals, due to the isomorphism (as Hecke modules) between modular symbols and certain comohology groups over $\mathbb{Q}$ (a detailed look into this is done in [Wie09]).

In this chapter we describe an explicit implementation of the algorithm suggested by Wang, and then analyze its complexity.

The main result of this complexity analysis is the following.

Theorem 3.1. *An upper bound on the theoretical complexity of determining a basis for $S_k(N, \chi)$ via the cohomological approach described below is*

$$\mathcal{O}\big(N^{3+\epsilon}k^{2+\epsilon}(N + k^4)\big),$$

for $\epsilon > 0$.

Finally, we give two examples, where we work through the main steps of the algorithm. The first of these examples is the easier case of trivial character, while the second example with non-trivial character showcases some other aspects of the algorithm.

3.1 Wang's work

In this section we sketch the theory behind Wang's approach (without proofs since these can be found in [Wan94]). Also, we introduce much of the notation of this chapter.

3.1.1 The Eichler-Shimura isomorphism and the Shapiro lemma

Let Δ denote all 2×2 matrices with coefficients in $\mathbb{Z}$, and let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ be the matrices herein with determinant 1. We also define

$$\Delta_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Delta \;\middle|\; c \equiv 0 \bmod N \right\},$$
$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \;\middle|\; c \equiv 0 \bmod N \right\}.$$

We denote by μ the index of $\Gamma_0(N)$ in Γ, and let $\gamma_1, \dots, \gamma_\mu$ denote the coset representatives of $\Gamma_0(N)$ (in Section 3.2.1 we describe the computation of these).

We have an operation of Δ on

$$M = \left\{ \sum_{j=0}^{k-2} a_j x^j y^{k-2-j} \;\middle|\; a_0, \dots, a_{k-2} \in \mathbb{Z} \right\}$$

given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} x^j y^{k-2-j} = (ax+by)^j (cx+dy)^{k-2-j}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Delta,$$

and we get an action of $\Delta_0(N)$ on $M_\chi = M \otimes R$, where $R = \mathbb{Z}[\frac{1}{6}, \chi]$, by setting $\delta_0.m = \chi(\delta_0)(\delta_0 m)$.

We wish to determine a basis for the space $S_k(N,\chi)$ of cusp forms of weight $k \geq 2$ with character χ satisfying $\chi(-1) = (-1)^k$.

By the Eichler-Shimura isomorphism, we have a canonical exact sequence (of complex vector spaces)

$$\begin{aligned} 0 \longrightarrow S_k(N,\chi) \oplus \overline{S_k(N,\chi)} &\longrightarrow H^1(\Gamma_0(N), M \otimes \mathbb{C}) \\ &\longrightarrow \bigoplus_{s \in C(\Gamma_0(N))} H^1(\Gamma_0(N)_s, M \otimes \mathbb{C}), \end{aligned}$$

where $\Gamma_0(N)_s = \{\gamma \in \Gamma_0(N) \mid \gamma.s = s\} = \langle T_s \rangle$ is an infinite cyclic group, cf. [Hab83, p. 284].

The dimension of

$$H^1(\Gamma_0(N)_s, M \otimes \mathbb{C}) \cong (M \otimes \mathbb{C})/(1 - T_s)(M \otimes \mathbb{C})$$

is 1, so that the dimension of $H^1(\Gamma_0(N), M \otimes \mathbb{C})$ is $\nu_\infty = \#C\big(\Gamma_0(N)\big)$, the number of inequivalent cusps of $\Gamma_0(N)$. In the case of $s = \infty$, we put $T = T_\infty = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.

We work with the coinduced module W_χ of M_χ on Γ, that is

$$W_\chi = \big\{ w : \Gamma \to M_\chi \mid w(\gamma_0\gamma) = \gamma_0.w(\gamma) \text{ for } \gamma_0 \in \Gamma_0(N) \big\},$$

and we get an action of $\delta \in \Delta$ on $w \in W_\chi$ by setting

$$(\delta.w)(\gamma) = \begin{cases} 0, & \gamma\delta \notin \Delta_0(N)\Gamma, \\ \delta_0.w(\gamma'), & \gamma\delta = \delta_0\gamma', \delta_0 \in \Delta_0(N), \gamma' \in \Gamma, \end{cases}$$

for $\gamma \in \Gamma$.

For a matrix $\delta \in \Delta$, we denote by W_χ^δ the submodule of W_χ invariant under the action of δ.

The Shapiro lemma gives a canonical isomorphism

$$H^1(\Gamma, W_\chi) \cong H^1(\Gamma_0(N), M_\chi)$$

as modules under the Hecke algebra, and we therefore study the cohomology of Γ with coefficients in W_χ.

3.1.2 A long exact sequence

Letting $H_c^n(G, W_\chi)$ denote the n'th cohomology of G with compact support, we obtain a long exact sequence

$$\begin{aligned} 0 \longrightarrow H^0(\Gamma, W_\chi) \longrightarrow H^0(\langle T\rangle, W_\chi) \longrightarrow H_c^1(\Gamma, W_\chi) \\ \longrightarrow H^1(\Gamma, W_\chi) \longrightarrow H^1(\langle T\rangle, W_\chi) \longrightarrow H_c^2(\Gamma, W_\chi) \longrightarrow 0, \end{aligned} \tag{$*$}$$

and by using the cup product along with a certain non-degenerate Hermitian pairing on W_χ, cf. [Wan94, pp. 97–99], we find that

$$\begin{aligned} \dim H_c^1(\Gamma, W_\chi) &= \dim H^1(\Gamma, W_\chi), \\ \dim H^1(\langle T\rangle, W_\chi) &= \dim H^0(\langle T\rangle, W_\chi), \\ \dim H_c^2(\Gamma, W_\chi) &= \dim H^0(\Gamma, W_\chi), \end{aligned}$$

as $\mathbb{Q}(\chi)$-vector spaces.

We denote by S and Q the following matrices:

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

Lemma 3.2 ([Wan94, Lem. 1 and Lem. 2])**.** *Letting χ_0 denote the trivial character, we have*

$$H^0(\Gamma, W_\chi) \cong \begin{cases} R, & (k,\chi) = (2,\chi_0), \\ 0, & \textit{otherwise}, \end{cases}$$

and there is an isomorphism

$$H^1(\Gamma, W_\chi) \cong W_\chi/(W_\chi^S + W_\chi^Q).$$

Additionally we have $H^0(\langle T\rangle, W_\chi) = W_\chi^T$ and $H^1(\langle T\rangle, W_\chi) \cong W_\chi/(1-T)W_\chi$, since $\langle T\rangle$ is an infinite cyclic group.

3.1.3 The connection to Manin symbols

We now describe the connection between W_χ and Manin symbols. Since $\Gamma_0(N)$ has finite index in Γ, the induced and coninduced modules of M_χ are isomorphic, so that

$$W_\chi \cong M_\chi \otimes_{\Gamma_0(N)} \Gamma,$$

where the operation of Γ on $M_\chi \otimes_{\Gamma_0(N)} \Gamma$ is given by $\gamma.(m \otimes \gamma') = m \otimes \gamma'\gamma^{-1}$.

By [Wan94, Lem. 3], we have an isomorphism

$$\mathrm{Hom}_{\Gamma_0(N)}(\Gamma, M_\chi) \xrightarrow{\cong} M_\chi \otimes_{\Gamma_0(N)} \Gamma$$

given by

$$f \mapsto \sum_{i=1}^{\mu} (f(\gamma_i) \otimes \gamma_i).$$

We let $\mathbb{P}^1_\chi(\mathbb{Z}/N\mathbb{Z})$ denote

$$\{(c,d) \in \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z} \mid \gcd(c,d,N) = 1\}$$

modulo the relation

$$(\lambda c, \lambda d) \sim \chi(\lambda)(c,d),\ \ \lambda \in (\mathbb{Z}/N\mathbb{Z})^*,$$

and we see that $\mathbb{P}^1_\chi(\mathbb{Z}/N\mathbb{Z})$ is just $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$ if χ is the trivial character.

For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, we put

$$\overline{\gamma} = (c \bmod N, d \bmod N) \in \mathbb{P}^1_\chi(\mathbb{Z}/N\mathbb{Z}),$$

and it is easily checked that $\overline{\gamma_0\gamma} = \chi(\gamma_0)\overline{\gamma}$ for $\gamma_0 \in \Gamma_0(N)$ and $\gamma \in \Gamma$.

For $\delta_0 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Delta_0(N)$, we define $\chi(\delta_0) = \chi(d)$.

Lemma 3.3 ([Wan94, Lem. 4]). *There is an isomorphism*

$$\mathrm{Hom}_{\Gamma_0(N)}(\Gamma, M_\chi) \xrightarrow{\cong} M \otimes_R \mathbb{P}^1_\chi(\mathbb{Z}/N\mathbb{Z})$$

given by

$$f \mapsto \sum_{i=1}^{\mu} (\gamma_i^{-1}.f(\gamma_i) \otimes \overline{\gamma}_i).$$

3.1.4 An involution

The matrix

$$\varepsilon = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

induces an involution on the cohomology groups $H^n(G, W_\chi)$ (resp. $H^n_c(G, W_\chi)$) by $(\varepsilon.\omega)(\gamma) = \varepsilon.\omega(\varepsilon^{-1}\gamma\varepsilon)$.

Letting $H^n(G, W_\chi)_\pm = \{\omega \in H^n(G, W_\chi) \mid \varepsilon.\omega = \pm\omega\}$ (and similarly for $H^n_c(G, W_\chi)$), we get from (∗), long exact sequences

$$\begin{aligned} 0 \longrightarrow H^0(\Gamma, W_\chi)_\pm \longrightarrow H^0(\langle T\rangle, W_\chi)_\pm \longrightarrow H^1_c(\Gamma, W_\chi)_\pm \\ \longrightarrow H^1(\Gamma, W_\chi)_\pm \xrightarrow{r^*_\pm} H^1(\langle T\rangle, W_\chi)_\pm \longrightarrow H^2_c(\Gamma, W_\chi)_\pm \longrightarrow 0, \end{aligned}$$

and a variant of the Eichler-Shimura isomorphism along with [Wan94, Lem. 5] now gives an exact sequence (of complex vector spaces)

$$0 \longrightarrow S_k(N, \chi) \longrightarrow H^1(\Gamma, W_\chi \otimes \mathbb{C})_+ \xrightarrow{r^*_+} H^1(\langle T\rangle, W_\chi \otimes \mathbb{C})_+ \longrightarrow 0.$$

We note that we from this exact sequence get

$$\dim S_k(N, \chi) = \dim H^1(\Gamma, W_\chi \otimes \mathbb{C})_+ - \dim H^1(\langle T\rangle, W_\chi \otimes \mathbb{C})_+.$$

We have already seen that $H^1(\langle T\rangle, W_\chi) \cong W_\chi/(1-T)W_\chi$ has dimension ν_∞, and that $W_\chi \cong M \otimes_R \mathbb{P}^1_\chi(\mathbb{Z}/N\mathbb{Z})$. Using these facts, it is shown in [Wan94, Lem. 6], that there is an isomorphism (of $\mathbb{Q}$-vector spaces)

$$H^1(\langle T\rangle, W_\chi \otimes \mathbb{Q}) \cong \bigoplus_{s \in C(\Gamma_0(N))} \mathbb{Q},$$

induced by the map

$$M \otimes_R \mathbb{P}^1_\chi(\mathbb{Z}/N\mathbb{Z}) \to \bigoplus_{s \in C(\Gamma_0(N))} \mathbb{Q}$$

given by

$$m \otimes (c, d) \mapsto m(0,1)\{\gamma^{-1}.\infty\} = m(0,1)\{-\tfrac{d}{c}\},\ \overline{\gamma} = (c, d).$$

The action of ε on $\bigoplus_{s\in C(\Gamma_0(N))}\mathbb{Q}$ is shown to be given by

$$\varepsilon.\{s\} = -\chi(\delta_0)\{s'\},\ \gamma_j\varepsilon = \delta_0\gamma_i,\ s = \gamma_i^{-1}\infty, s' = \gamma_j^{-1}\infty,$$

with $i, j \in \{1, \ldots, \mu\}$, and using this, cf. Section 3.2.3, one can determine the dimension of $H^1(\langle T\rangle, W_\chi \otimes \mathbb{C})_+$.

Since ε is an involution, we get from Lemma 3.2 that

$$H^1(\Gamma, W_\chi)_+ \cong W_\chi/(W_\chi^S + W_\chi^Q + W_\chi^\varepsilon).$$

3.1.5 Hecke action and basis computation

As shown in [Wan94, pp. 105–106], the action of the Hecke operator T_n on the space $H^1(\Gamma, W_\chi) \cong W_\chi/(W_\chi^S + W_\chi^Q)$, is given by

$$T_n x = \sum_{A\in X_n} c_A A.x,\quad c_A \in R,$$

where X_n is a set of matrices in Δ of determinant n (see also [Mer94]). We determine these matrices in Section 3.3.4.

Let φ be any $\mathbb{C}$-linear map $S_k(N,\chi) \to \mathbb{C}$ and let $f \in S_k(N,\chi)$. By [Wan94, Lem. 7], we get that the formal power series

$$\sum_{n=1}^{\infty} \varphi(T_n f) q^n,$$

actually is the q-expansion of a form in $S_k(N,\chi)$.

This way we can generate q-expansions, and it is a central point of the algorithm (suggested by Merel), which can be summarized as follows:

(1) Compute a basis B for $H^1(\Gamma, W_\chi)_+ \cong W_\chi/(W_\chi^S + W_\chi^Q + W_\chi^\varepsilon)$.

(2) Compute the dimension of $S_k(N,\chi)$, i.e., $\#B - \dim H^1(\langle T\rangle, W_\chi \otimes \mathbb{C})_+$.

(3) Choose an element $x \in \ker r_+^*$, and compute $T_n x = \sum_{b\in B} \lambda_n(b) b$ for all positive integers n up to at least the Sturm bound $\lfloor \frac{k\mu}{12} - \frac{\mu-1}{N} \rfloor$.

(4) Compute the dimension of the vector space generated by the q-expansions $\sum \lambda_n(b) q^n$ for all $b \in B$, and if this dimension is less than the dimension of $S_k(N,\chi)$, try (3) again with another x.

3.2 Implementation of the algorithm

In this section we work through the algorithm point by point, in each case determining how to explicitly compute the desired objects.

3.2.1 Coset representatives

We start out by getting coset representatives for $\Gamma_0(N)$ in Γ. Besides the N representatives $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \dots, \left(\begin{smallmatrix} 0 & -1 \\ 1 & N-1 \end{smallmatrix}\right)$, we also have $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ (if N is prime, these are all of them).

By [Cre97, Prop. 2.2.2], there is a bijection between coset representatives $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of $\Gamma_0(N)$ in Γ and elements $(c, d) \in \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$. To get the remaining representatives, we simply take the remaining elements (c, d) of $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$, and lift these to matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in Γ via the Euclidean algorithm.

It is described in [Cre97, Sec. 2.2] how to efficiently determine $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$. One starts out with the obvious elements $(0, 1)$, $(1, 0), \dots (1, N-1)$, and then look at elements (c, d), where $c \mid N$ and $d = 1, \dots, N-1$, adding (c, d) to the list if it is not equivalent to an element already on the list (one uses that two elements (c_1, d_1) and (c_2, d_2) are equivalent if and only if $c_1 d_2 \equiv c_2 d_1 \pmod N$). See also [Ste07, Sec. 8.7].

Recall that we denote the coset representatives by $\gamma_1, \dots, \gamma_\mu$.

3.2.2 Action of $\boldsymbol{\Delta}$ on $\boldsymbol{W_\chi}$ and relations matrix

We want to determine a basis for $H^1(\Gamma, W_\chi)_+ \cong W_\chi/(W_\chi^S + W_\chi^Q + W_\chi^\varepsilon)$, which is W_χ modulo the relations

$$w + S.w = w + Q.w + Q^2.w = w + \varepsilon.w = 0.$$

Therefore we need to be able to determine a matrix representation of the action of a matrix in Δ on W_χ.

An element of W_χ is determined by its values on the μ coset representatives, and since M_χ is generated by the $k-1$ homogeneous monomials of degree $k-2$, the space W_χ has $\mu(k-1)$ generators.

As a basis for W_χ, we thus have the elements

$$w_{ij} : \begin{cases} \gamma_r \mapsto x^j y^{k-2-j}, & r = i, \\ \gamma_r \mapsto 0, & r \neq i, \end{cases}$$

with $i = 1, \dots, \mu$ and $j = 0, \dots, k-2$.

To determine the action of $\delta \in \Delta$ on W_χ, we only need the action of $\delta.w_{ij}$ on the coset representatives. If $\gamma_r\delta \notin \Delta_0(N)\Gamma$, the action is 0, so we now assume that we can write $\gamma_r\delta = \delta_0\gamma$ for some $\delta_0 \in \Delta_0(N)$ and $\gamma \in \Gamma$. Since we have $\gamma = \gamma_0\gamma_s$ for some $\gamma_0 \in \Gamma_0(N)$ and a coset representative γ_s, we replace δ_0 with $\delta_0\gamma_0 \in \Delta_0(N)$, so that the action is given by

$$\delta_0.w_{ij}(\gamma_s) = \begin{cases} \delta_0.x^j y^{k-2-j}, & s = i, \\ 0, & s \neq i. \end{cases}$$

To get the action of δ on w_{ij}, we need to run through the coset representatives γ_r, get the corresponding $\delta_0 \in \Delta_0(N)$ such that $\gamma_r\delta = \delta_0\gamma_s$, and then compute the coefficients $a_0, \ldots, a_{k-2}$ of the polynomial

$$\delta_0.x^j y^{k-2-j} = \sum_{t=0}^{k-2} a_t x^t y^{k-2-t},$$

for $j = 0, \ldots, k-2$. These coefficients are then placed in the $(s + j\mu)$'th row and the $(i + t\mu)$'th columns $(t = 0, \ldots, k-2)$ of a $\mu(k-1) \times \mu(k-1)$-matrix.

This way, we obtain matrix representations of the actions of $I + S$, $I + Q + Q^2$ and $I + \varepsilon$ on W_χ (here I is the identity matrix), and $W_\chi/(W_\chi^S + W_\chi^Q + W_\chi^\varepsilon)$ is then the nullspace of the resulting relations matrix (the relations matrix is the above three matrix representations stacked on top of one another).

3.2.3 Dimension

To determine the dimension of $S_k(N, \chi)$, we need to compute the dimension of $H^1(\langle T\rangle, W_\chi \otimes \mathbb{C})_+$.

As we have already seen, there is an isomorphism (of $\mathbb{Q}$-vector spaces)

$$H^1(\langle T\rangle, W_\chi \otimes \mathbb{Q}) \cong \bigoplus_{s \in C(\Gamma_0(N))} \mathbb{Q},$$

and the action of ε on $\bigoplus_{s \in C(\Gamma_0(N))} \mathbb{Q}$ is shown to be given by

$$\varepsilon.\{s\} = -\chi(\delta_0)\{s'\},\ \gamma_j\varepsilon = \delta_0\gamma_i,\ s = \gamma_i^{-1}\infty, s' = \gamma_j^{-1}\infty,$$

with $i, j \in \{1, \ldots, \mu\}$.

Let us write $\varepsilon.\{s\} = c\{s'\}$. Since ε is an involution, we have $\varepsilon.\{s'\} = c^{-1}\{s\}$. If $\{s\} = \{s'\}$, we have $\varepsilon.\{s\} = \pm\{s\}$, showing that $\{s\}$ is in the corresponding $\pm$-space. If $\{s\} \neq \{s'\}$, we have $\{s\} \pm c\{s'\}$ in the corresponding $\pm$-space.

Thus, if $\{s\} \neq \{s'\}$, we get elements of both $\pm$-spaces, but when $\{s\} = \{s'\}$, we get an element of only one of these spaces – the space corresponding to the sign of $-\chi(\delta_0)$, i.e., we get an element of the $+$-space if and only if $\chi(\delta_0) = -1$, when $\{s\} = \{s'\}$.

We therefore need to determine when two cusps $\{\gamma_m^{-1}\infty\}$ and $\{\gamma_n^{-1}\infty\}$ are equivalent, and this happens exactly when $\gamma_m T^u = \gamma_0\gamma_n$ for some $\gamma_0 \in \Gamma_0(N)$ and $u \in \mathbb{Z}$. This groups the coset representatives into ν_∞ classes (one for each cusp equivalence class).

Thus, ε maps a cusp $\{s\}$ to $\pm\{s\}$ (with $s = \gamma_i^{-1}\infty$) if and only if the coset representatives γ_i and γ_j, satisfying $\gamma_j\varepsilon = \delta_0\gamma_i$, are in the same equivalence class, which is exactly when $\gamma_i T^u = \gamma_0\gamma_j$ for some $\gamma_0 \in \Gamma_0(N)$ and some $u \in \mathbb{Z}$.

Since we only need one representative for each equivalence class, we choose for a given γ_i, the unique representative γ_j satisfying $\gamma_i T = \gamma_0 \gamma_j$ for some $\gamma_0 \in \Gamma_0(N)$, and write $\gamma_j \varepsilon = \delta_0 \gamma_i$, checking the sign of $\chi(\delta_0)$.

This way, we find the part of the $\pm$-spaces coming from the case where ε maps $\{s\}$ to $\pm\{s\}$. When this does not happen, we get elements of both $\pm$-spaces, and since the sum of the dimensions is ν_∞, we get the dimension of the $+$-space.

The dimension of $S_k(N, \chi)$ is the difference between the dimension of the nullspace of the relations matrix and the dimension just found.

3.2.4 Hecke action and basis

Since $S_k(N, \chi)$ is the kernel of the homomorphism

$$r_+^* : H^1(\Gamma, W_\chi)_+ \to H^1(\langle T \rangle, W_\chi)_+,$$

we take elements in the kernel of this map, and compute the corresponding q-expansions, until we have enough forms to generate $S_k(N, \chi)$.

It is described in [Wan94, pp.107–108] how to choose elements in the kernel, and we will briefly recount this here.

Using that W_χ is isomorphic to $M_\chi \otimes_{\Gamma_0(N)} \Gamma$, we get from Lemma 3.3 that r_+^* becomes a homomorphism

$$W_\chi / (W_\chi^S + W_\chi^Q) \to \bigoplus_{s \in C(\Gamma_0(N))} \mathbb{Q},$$

and with the above isomorphisms, this map is on $W_\chi \cong M \otimes \mathbb{P}^1_\chi(\mathbb{Z}/N\mathbb{Z})$ given by

$$m \otimes (c, d) \mapsto m(0, 1)\{-\tfrac{d}{c}\} - m(1, 0)\{\tfrac{c}{d}\}. \tag{†}$$

For $m \otimes (c, d)$ to be in the kernel, we can use any $(c, d) \in \mathbb{P}^1_\chi(\mathbb{Z}/N\mathbb{Z})$ with m any (non-empty) linear combination of the monomials $xy^{k-3}, \ldots, yx^{k-3}$ if $k > 2$, and $m = 1$ if $k = 2$ (in the weight-2 case we have to have $\chi(c) = \chi(d)$ as well).

Since elements $(c, d) \in \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$ correspond bijectively to coset representatives $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of $\Gamma_0(N)$ in Γ, we can therefore represent elements of the kernel as $m \otimes \gamma_r$ with $r = 1, \ldots, \mu$ and m as above.

We use the Heilbronn-Merel matrices (cf. [Mer94, Prop. 20])

$$H_n = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Delta \;\middle|\; ad - bc = n, a > b \geq 0, d > c \geq 0 \right\},$$

to determine the action of the Hecke operator T_n on the elements of the kernel found above.

The action of the Hecke operator T_n on $W_\chi/(W_\chi^S + W_\chi^Q)$ is given by

$$T_n x = \sum_{A \in H_n} A.x.$$

Translating this through the isomorphisms above, the action on a kernel element $m \otimes \gamma_r$ is

$$T_n(m \otimes \gamma_r) = \sum_{A \in H_n} \chi(\delta_{0,A})(A.m) \otimes \gamma_{r_A},$$

where we for each $A \in H_n$ write $\gamma_r A = \delta_{0,A}\gamma_{r_A}$ with $\delta_{0,A} \in \Delta_0(N)$ (terms where it is not possible to write $\gamma_r A$ in this way are ignored).

The coefficients of the polynomial $\chi(\delta_{0,A})(A.m)$ are then saved in a vector t_n, where the coefficient of $x^j y^{k-2-j}$ is added to the $r_A(k-1) - (k-2-j)$'th entry, as A runs through H_n.

If we wish to compute the basis up to exponent q^M, we compute t_n for all $n = 1, \ldots, M$, where we require that $M \geq \lfloor k\mu/12 - (\mu - 1)/N \rfloor$ (the Sturm bound), and we let t be the matrix whose n'th column is t_n.

We then multiply the nullspace matrix of the relations matrix found earlier with the matrix t, and denote the resulting matrix by B. If B has rank equal to the dimension of $S_k(N, \chi)$, we have found a basis (the leading rows of B).

If B has rank less than the dimension, we choose another element $m \otimes \gamma_r$ in the kernel, compute the Hecke action $t_1, \ldots, t_M$ on this, multiply the nullspace matrix of the relations matrix with the resulting matrix, and get a matrix whose rows are concatenated to B, and we again compute the rank of B. This procedure continues until we get as many linearly independent rows in B as the dimension of $S_k(N, \chi)$.

Experimentation indicates that if we choose $m = xy^{k-3} + \cdots + x^{k-3}y$ (for $k > 2$), this procedure is likely to give a basis using just the first few γ_r's.

3.2.5 Determining the kernel

In the implementation described, we choose certain kernel elements, and generate Fourier coefficients from these, until we have enough forms to generate the space of cusp forms.

However, there is no certainty that this will work, i.e., there is no guarantee that this approach will give enough linearly independent forms (even though we have yet to see an example of this).

Another approach (which is certain to work every time) is the following. From the nullspace of the relations matrix, we get a basis for $W_\chi/(W_\chi^S + W_\chi^Q + W_\chi^\varepsilon)$. By using the isomorphism $W_\chi \cong M \otimes_R \mathbb{P}^1_\chi(\mathbb{Z}/N\mathbb{Z})$, we can get the image of this basis on a quotient of $M \otimes_R \mathbb{P}^1_\chi(\mathbb{Z}/N\mathbb{Z})$, expressed in terms of the standard basis of $M \otimes \mathbb{P}^1_\chi(\mathbb{Z}/N\mathbb{Z})$.

We use this basis to write up a matrix representation of the map (†) on this quotient, and determine its nullspace (which has dimension equal to the dimension of $S_k(N,\chi)$). From the nullspace matrix we read off a basis, and we then write the kernel elements as linear combinations of the $m \otimes \gamma_r$. We now compute the Hecke action on these elements, which we know will generate enough forms to give a basis for $S_k(N,\chi)$.

3.3 Complexity of implementation

We always assume that the level is given via its prime factorization, i.e., no work is needed to find the divisors of N.

We also use that a Dirichlet character on $(\mathbb{Z}/N\mathbb{Z})^*$ is defined via a lookup table, which takes $\mathcal{O}(N)$ to create, but we do not then need to worry about the cost of evaluating the character.

3.3.1 Coset representatives

The number of coset representatives is

$$\mu = N \prod_{p \mid N} \left(1 + \tfrac{1}{p}\right).$$

Let n be the number of prime divisors of N, and let $p_1, \ldots, p_n$ be the first n primes. By using [Lan99, p. 139], we find that

$$\begin{aligned}\prod_{p\mid N}\left(1+\tfrac{1}{p}\right) \le \prod_{p\mid N}\left(1+\tfrac{1}{p}+\tfrac{1}{p^2}+\cdots\right) = \prod_{p\mid N}\left(1-\tfrac{1}{p}\right)^{-1} \le \prod_{i=1}^{n}\left(1-\tfrac{1}{p_i}\right)^{-1} \\ = \mathcal{O}(\log p_n) = \mathcal{O}(\log n) = \mathcal{O}(\log\log N),\end{aligned}$$

since $N \ge 2^n$. We therefore have $\mu = \mathcal{O}(N \log\log N)$.

We determine the coset representatives by looking at elements (c,d), where $c \mid N$ and $d = 1, \ldots, N-1$. By [HW79, Thm. 315], the number of divisors of N is $\mathcal{O}(N^\delta)$ for $\delta > 0$, and we therefore look at $\mathcal{O}(N^{1+\delta})$ elements (c,d).

Every time we look at an element, we check if it is equivalent to something already found, and this takes $\mathcal{O}(\log^2 N)$ each time. Whenever we find a new element, we lift it to a matrix in Γ via the Euclidean algorithm, which also takes $\mathcal{O}(\log^2 N)$. This gives a total complexity of $\mathcal{O}(N^{1+\delta}\log^2 N)$.

3.3.2 Action of Δ on W_χ and relations matrix

All it takes to get $\gamma_r\delta$ on the form $\delta_0\gamma_s$, is to compute some greatest common divisors, and run through the coset representatives to see which one works. All in all the complexity for this is $\mathcal{O}(\mu \log^2 N)$.

The hardest part of computing the action of a matrix on a polynomial, is the binomial coefficients that show up when computing polynomial coefficients. The cost of computing $\binom{n}{m}$ is $\mathcal{O}(m^2 \log^2 n)$, and so a rough estimate for computing the action of δ_0 on the $k-1$ monomials $x^j y^{k-2-j}$ is $\mathcal{O}(k^4 \log^2 k)$. This needs to be done for every γ_r, i.e., μ times, giving a total complexity of $\mathcal{O}(\mu^2 \log^2 N + \mu k^4 \log^2 k)$ for determining the matrix representation of δ on W_χ.

We note that if we work over the finite field $\mathbb{F}_p$ instead of the integers, one can use a congruence first proved by Lucas [Luc78] to obtain the complexity $\mathcal{O}(p^2 \log^2 p \log k)$ for the binomial coefficient computations. A more modern reference for the Lucas congruence is [Sta97, p. 44].

The relations matrix consists of the matrix representations of the actions of $I + S$, $I + Q + Q^2$ and $I + \varepsilon$, and so is a matrix of size $3\mu(k-1) \times \mu(k-1)$. To compute the nullspace therefore takes $\mathcal{O}(\mu^3 k^3)$, and this is really the time-consuming function of this part.

3.3.3 Dimension

We first build an array whose i'th entry is the index j of the coset representative satisfying $\gamma_i T = \gamma_0 \gamma_j$, as well as a similar array giving the index j of the coset representative satisfying $\gamma_i \varepsilon = \delta_0 \gamma_j$. Doing this takes $\mathcal{O}(\mu^2 \log^2 N)$.

Next, we determine to which cusp equivalence class each coset representative belongs, computing ν_∞ along the way. The work needed is already done in the first array we created.

We then choose a representative γ_i of a cusp equivalence class, and use the second array to find an equivalent representative γ_j satisfying $\gamma_j \varepsilon = \delta_0 \gamma_i$ for some $\delta_0 \in \Delta_0(N)$. We then compute $\chi(\delta_0)$ and add 1 to the count of the corresponding $\pm$-variable $d_\pm$.

The dimension of the $+$-space is then $d_+ + (\nu_\infty - d_+ - d_-)/2$, and the dimension of $S_k(N, \chi)$ is the difference between the dimension of the nullspace of the relations matrix and the dimension of the $+$-space.

The work in creating the arrays is by far the most work in this, so the complexity of this algorithm is $\mathcal{O}(\mu^2 \log^2 N)$.

3.3.4 Hecke action and basis

Even though we use the Heilbronn-Merel matrices in the implementation, we turn to [Mer94] for the complexity analysis, since this paper gives another class of matrices, which can be used instead of the Heilbronn-Merel matrices, and we have estimates on the size of these classes.

In [Mer94, Sec. 3], Merel defines a set $\mathcal{S}_n$, where a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Delta$ is in $\mathcal{S}_n$ if it has determinant n and at least one of the following conditions is satisfied:

- $a > |b|$, $d > |c|$, $bc > 0$,
- $b = 0$, $|c| < d/2$,
- $c = 0$, $|b| < a/2$.

Merel also defines a set $\mathcal{S}'_n$, where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Delta$ is in $\mathcal{S}'_n$ if it has determinant n and one of the two following conditions is satisfied:

- $b = 0$, $|c| = d/2$,
- $c = 0$, $|b| = a/2$.

It is easily seen that an upper bound for $|\mathcal{S}'_n|$ is $2\sigma_1(n)$, where $\sigma_1(n)$ is the sum of the positive divisors of n, and from [Mer94, p. 85], we have, as $n \to \infty$,

$$|\mathcal{S}_n| \sim \frac{12 \log 2}{\pi^2} \sigma_1(n) \log n.$$

By [HW79, Thm. 322], we have $\sigma_1(n) = \mathcal{O}(n^{1+\delta})$ for $\delta > 0$, so that we have

$$|\mathcal{S}_n \cup \mathcal{S}'_n| = \mathcal{O}(n^{1+\delta} \log n) = \mathcal{O}(n^{1+\epsilon}),$$

for $\epsilon > 0$, since $\log n = \mathcal{O}(n^{\delta'})$ for $\delta' > 0$.

We need to compute $\mathcal{S}_n$ and $\mathcal{S}'_n$ for $n = 1, \dots, \lfloor k\mu/12 - (\mu - 1)/N \rfloor$ (or more n if we want higher precision), so $\mathcal{O}(\mu^{2+\epsilon} k^{2+\epsilon})$ matrices are needed to compute the Hecke action.

We want to compute the action of T_n on $m \otimes \gamma_r$, with m a linear combination of the monomials $xy^{k-3}, \dots, x^{k-3}y$ if $k > 2$, and $m = 1$ if $k = 2$. To do this we write, for each $A \in \mathcal{S}_n \cup \mathcal{S}'_n$, $\gamma_r A = \delta_{0,A} \gamma_{r_A}$, with $\delta_{0,A} \in \Delta_0(N)$, and this can be done in $\mathcal{O}(\mu \log^2 N)$ for each A. We then have

$$T_n(m \otimes \gamma_r) = \sum_{A \in \mathcal{S}_n} \chi(\delta_{0,A})(A.m) \otimes \gamma_{r_A} + \frac{1}{2} \sum_{A \in \mathcal{S}'_n} \chi(\delta_{0,A})(A.m) \otimes \gamma_{r_A},$$

and as mentioned earlier, the complexity of determining the action of A on a linear combination of all possible monomials is $\mathcal{O}(k^4 \log^2 k)$.

The coefficients of $\chi(\delta_{0,A})(A.m)$ are added for each A, and saved in a vector t_n, with indices depending on the monomial $x^j y^{k-2-j}$ and the index r_A. Determining all necessary t_n are done in $\mathcal{O}(\mu^{2+\epsilon} k^{2+\epsilon}(\mu \log^2 N + k^4 \log^2 k))$.

Multiplying the nullspace matrix of the relations matrix with the matrix whose n'th column is t_n, takes $\mathcal{O}(\mu^3 k^3)$, which is less than the complexity of determining the Hecke action.

The resulting matrix is the basis matrix, if it has rank equal to the dimension of $S_k(N, \chi)$. We therefore do Gaussian elimination, and compute the rank to see if we are done. If not, we choose another coset representative γ_r, get the resulting t_n's of the Hecke action on $m \otimes \gamma_r$, multiply the nullspace matrix of the relations matrix with the resulting matrix, and get a matrix whose rows are concatenated to B. We again do Gaussian elimination and compute the rank, and this is repeated until we get the right rank. If we run through all the coset representatives without getting the right rank, we can try with another m.

Gaussian elimination is done in $\mathcal{O}(\mu^3 k^3)$, and is therefore insignificant compared to the computation of the Hecke action.

For a given m, this procedure is repeated at most μ times, but experimentation indicates that it is likely to finish much sooner if m is chosen to be the sum of all possible monomials (in the case of $k > 2$). We see that the computation of the Hecke action is by far the hardest part of this basis determination, and we therefore get a total theoretical complexity of $\mathcal{O}\big(\mu^{3+\epsilon} k^{2+\epsilon}(\mu \log^2 N + k^4 \log^2 k)\big)$.

Since $\mu = \mathcal{O}(N \log\log N)$ and $\log\log N = \mathcal{O}(N^{\epsilon'})$ for an $\epsilon' > 0$, we have $\mu^\epsilon = \mathcal{O}(N^{\epsilon+\epsilon\epsilon'})$ for $\epsilon > 0$, and we find that

$$\begin{aligned}\mathcal{O}\big(\mu^{3+\epsilon} k^{2+\epsilon}(\mu \log^2 N + k^4 \log^2 k)\big) &= \mathcal{O}\big(\mu^{4+\epsilon} k^{2+\epsilon} \log^2 N + \mu^{3+\epsilon} k^{6+\epsilon} \log^2 k\big) \\ &= \mathcal{O}\big(\mu^{4+\epsilon+\epsilon''} k^{2+\epsilon} + \mu^{3+\epsilon} k^{6+\epsilon+\epsilon''}\big) \\ &= \mathcal{O}\big(N^{3+\epsilon+\epsilon\epsilon'+\epsilon''} k^{2+\epsilon+\epsilon''}(N + k^4)\big),\end{aligned}$$

using that $\log^2 \alpha = \mathcal{O}(\alpha^{\epsilon''})$ for $\epsilon'' > 0$ (and α either N or k), and this is Theorem 3.1, after replacing $\epsilon + \epsilon\epsilon' + \epsilon''$ with ϵ.

3.4 Examples

We use this implementation to determine bases for two spaces of cusp forms, one with trivial character and one with non-trivial character.

The first example gives more detail, while the second highlights an aspect which only occurs in the case of non-trivial character.

3.4.1 $S_4(\Gamma_0(25))$

We start out by getting the coset representatives, and we do this by determining $\mathbb{P}^1(\mathbb{Z}/25\mathbb{Z})$ in the way we described earlier. Thus, we get the 30 elements

$$(1,0),\ldots,(1,24),(0,1),(5,1),(5,2),(5,3),(5,4),$$

and these are lifted to matrices in Γ via the Euclidean algorithm:

$$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},\ldots,\begin{pmatrix}0 & -1\\ 1 & 24\end{pmatrix},\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},$$
$$\begin{pmatrix}1 & 0\\ 5 & 1\end{pmatrix},\begin{pmatrix}-2 & -1\\ 5 & 2\end{pmatrix},\begin{pmatrix}2 & 1\\ 5 & 3\end{pmatrix},\begin{pmatrix}-1 & -1\\ 5 & 4\end{pmatrix}.$$

We denote these representatives by $\gamma_1,\ldots,\gamma_{30}$, so that, for instance, $\gamma_{26}=I$.

Next, we determine the nullspace of the relations matrix. Since the matrix representations of $I+S$, $I+Q+Q^2$ and $I+\varepsilon$ all are 90×90-matrices, we do not write these up here. After bringing the matrix on echelon form and deleting zero rows, the nullspace matrix of the relations matrix is a 7×90-matrix.

Next, we compute the dimension of the +-space. We therefore build an array whose i'th entry is the index j of the coset representative γ_j satisfying $\gamma_i T=\gamma_0\gamma_j$ for some $\gamma_0\in\Gamma_0(25)$. This becomes:

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
j	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

i	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
j	17	18	19	20	21	22	23	24	25	1	26	27	28	29	30

This array shows that $\gamma_1,\ldots,\gamma_{25}$ are all in the same cusp equivalence class, and the rest are each in their own class. All in all, we see that $\Gamma_0(25)$ has 6 cusps, represented by γ_1, γ_{26}, γ_{27}, γ_{28}, γ_{29} and γ_{30}.

We also build an array whose i'th entry is the index j of the coset representative γ_j satisfying $\gamma_j\varepsilon=\delta_0\gamma_i$ for some $\delta_0\in\Delta_0(N)$. This becomes:

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
j	1	25	24	23	22	21	20	19	18	17	16	15	14	13	12

i	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
j	11	10	9	8	7	6	5	4	3	2	26	30	29	28	27

We now take each of the 6 representatives γ_i, match them with the corresponding γ_j from this table (checking that γ_j is in the same equivalence class).

We do not need to compute the δ_0's, since we have trivial character, and therefore always will get elements of the $-$-space.

From the table, we see that only γ_1 and γ_{26} give rise to γ_j's in the same cusp equivalence class, and we therefore get 2 elements of the $-$-space.

Since there are 6 cusps, this means that the remaining 4 are split evenly between the $+$- and $-$-spaces, giving that the dimension of the $+$-space is 2.

From this we get that the dimension of $S_4\big(\Gamma_0(25)\big)$ is $7-2=5$ (nullspace dimension minus $+$-space dimension).

Finally, we compute the Hecke action $t_1, \dots, t_M$, with $M = \lfloor \frac{4\cdot 30}{12} - \frac{29}{25} \rfloor = 8$.

Each coset representative γ_r gives rise to a kernel element $xy \otimes \gamma_r$, and to compute t_n for $xy \otimes \gamma_r$, we compute $A.xy$ for all $A \in H_n$, keeping track of the index r_A of $\gamma_r A = \delta_{0,A}\gamma_{r_A}$.

In the case of $n = 3$ we have

$$H_3 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix} \right\},$$

and we denote these $A_1, \dots, A_7$.

We need to compute up to $r = 4$, since the forms generated by using only $r \leq 3$ only spans a 4-dimensional space. As a matter of fact, the forms generated by using just $xy \otimes \gamma_4$ span the whole space, and we write a table of indices r_{A_i} with respect to γ_4, as well as the action of A_i on xy:

i	1	2	3	4
r_{A_i}	10	22	13	28
$A_i.xy$	$3xy$	$3xy + x^2$	$3xy + 2x^2$	$2y^2 + 5xy + 2x^2$

i	5	6	7
r_{A_i}	2	19	11
$A_i.xy$	$3xy$	$y^2 + 3xy$	$2y^2 + 3xy$

Thus, we get that

$$T_3(xy \otimes \gamma_4) = \sum_{i=1}^{7} A_i.xy \otimes \gamma_{r_{A_i}},$$

and we put the coefficients into a vector t_3, where $(t_3)_{3(r_{A_i}-1)+j}$ is the coefficient of $x^j y^{2-j}$ in $A_i.xy$.

We do this for all $n = 1, \dots, 8$, and build a matrix with the t_n as columns. We then multiply the nullspace matrix with this matrix, and get (after removing zero rows and bringing the matrix on echelon form)

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

which has rank 5, and the rows therefore form a basis for $S_4\big(\Gamma_0(25)\big)$. Thus, the standard basis (up to q^8) of this space is

$$q,\ q^2 - q^7 - q^8,\ q^3 + q^7 - 2q^8,\ q^4 - q^6,\ q^5.$$

We note that Schoen has described a basis for this space in [Sch86], in terms of the η-function:

$$\begin{aligned} f_0 &= \eta(z)^4\eta(5z)^4, \\ f_1 &= \eta(z)^3\eta(5z)^4\eta(25z), \\ f_2 &= \eta(z)^2\eta(5z)^4\eta(25z)^2, \\ f_3 &= \eta(z)\eta(5z)^4\eta(25z)^3, \\ f_4 &= \eta(5z)^4\eta(25z)^4. \end{aligned}$$

We can compute this (up to q^8) as

$$\begin{aligned} f_0 &= q - 4q^2 + 2q^3 + 8q^4 - 5q^5 - 8q^6 + 6q^7, \\ f_1 &= q^2 - 3q^3 + 5q^5 - 4q^7 + 5q^8, \\ f_2 &= q^3 - 2q^4 - q^5 + 2q^6 + q^7 - 2q^8, \\ f_3 &= q^4 - q^5 - q^6, \\ f_4 &= q^5, \end{aligned}$$

and since we can write

$$\begin{aligned} q &= f_0 + 4f_1 + 10f_2 + 12f_3 + 7f_4, \\ q^2 - q^7 - q^8 &= f_1 + 3f_2 + 6f_3 + 4f_4, \\ q^3 + q^7 - 2q^8 &= f_2 + 2f_3 + 3f_4, \\ q^4 - q^6 &= f_3 + f_4, \\ q^5 &= f_4, \end{aligned}$$

both sets of forms span the same space, namely $S_4\big(\Gamma_0(25)\big)$.

3.4.2 $S_5\big(12, (\frac{\cdot}{12})\big)$

We start out by getting the coset representatives, and we again do this by determining $\mathbb{P}^1(\mathbb{Z}/12\mathbb{Z})$. We get the 24 elements

$$(1,0),\ldots,(1,11),(0,1),(2,1),(2,3),(2,5),(3,1),$$
$$(3,2),(3,4),(3,7),(4,1),(4,3),(4,5),(6,1),$$

and these are lifted to matrices in Γ via the Euclidean algorithm:

$$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},\ldots,\begin{pmatrix}0 & -1\\ 1 & 11\end{pmatrix},\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix},\begin{pmatrix}1 & 1\\ 2 & 3\end{pmatrix},\begin{pmatrix}1 & 2\\ 2 & 5\end{pmatrix},\begin{pmatrix}1 & 0\\ 3 & 1\end{pmatrix},$$
$$\begin{pmatrix}-1 & -1\\ 3 & 2\end{pmatrix},\begin{pmatrix}1 & 1\\ 3 & 4\end{pmatrix},\begin{pmatrix}1 & 2\\ 3 & 7\end{pmatrix},\begin{pmatrix}1 & 0\\ 4 & 1\end{pmatrix},\begin{pmatrix}-1 & -1\\ 4 & 3\end{pmatrix},\begin{pmatrix}1 & 1\\ 4 & 5\end{pmatrix},\begin{pmatrix}1 & 0\\ 6 & 1\end{pmatrix}.$$

We denote these representatives by $\gamma_1,\ldots,\gamma_{24}$.

We again determine the relations matrix (of size 184×96) and its nullspace, which in this case has dimension 8.

Just as in the last example, we choose a coset representative from each cusp equivalence class, and we get 6 cusps, represented by γ_1, γ_{13}, γ_{14}, γ_{17}, γ_{21} and γ_{24}.

We now take each of these γ_i, match them with the corresponding γ_j satisfying $\gamma_j\varepsilon = \delta_0\gamma_i$ (checking that γ_j is in the same equivalence class as γ_i, which they all are in this case), and compute $\chi(\delta_0)$ to see to which of the $\pm$-spaces they correspond. We summarize the results in the following table:

i	1	13	14	17	21	24
j	1	13	16	20	23	24
δ_0	$\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}-5 & 2\\ -12 & 5\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}-7 & 2\\ -24 & 7\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}-5 & 1\\ -24 & 5\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}-1 & 0\\ -12 & 1\end{smallmatrix}\right)$
$\chi(\delta_0)$	-1	1	-1	1	-1	1
Space	$+$	$-$	$+$	$-$	$+$	$-$

From this we get that the dimension of $S_5\big(12, (\frac{\cdot}{12})\big)$ is $8 - 3 = 5$ (nullspace dimension minus $+$-space dimension).

As before, we compute the Hecke action $t_1,\ldots,t_M$, with $M = \lfloor\frac{5\cdot 24}{12} - \frac{23}{12}\rfloor = 8$, and in this case it is enough to use the kernel element $(xy^2 + xy^2)\otimes\gamma_1$ to generate a basis.

The matrix we get, after multiplying the nullspace matrix with the matrix having the t_n as columns (and removing zero rows and bringing it on echelon form) is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & 1 & 0 & 0 & 0 & -3 & 0 & -8 \\ 0 & 0 & 1 & 0 & 0 & 0 & -10 & 0 \\ 0 & 0 & 0 & 1 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -5 & 0 \end{pmatrix}.$$

Thus, the standard basis (up to q^8) of $S_5\big(12, (\frac{\cdot}{12})\big)$ is

$$q - 4q^7,\ q^2 - 3q^6 - 8q^8,\ q^3 - 10q^7,\ q^4 - 3q^6,\ q^5 - 5q^7.$$

Appendix A

Computational evidence for Conjecture 2.22

The tables in this appendix are computational results supporting Conjecture 2.22.

We denote by $f_{N,k,i}$ the i'th newform in $S_k\big(\Gamma_0(N)\big)$, with the ordering used by `SortDecomposition` in Magma. If $S_k\big(\Gamma_0(N)\big)$ contains only one newform, we simply denote this by $f_{N,k}$.

Looking at all newforms on $\Gamma_0(N)$ of weights k_1 and k_2, the tables give the maximal positive integer $L = \lceil \frac{m}{e(\mathfrak{p}/p)} \rceil$ such that

$$a_\ell(f_{N,k_1,j}) \equiv a_\ell(f_{N,k_2,i}) \pmod{\mathfrak{p}^m}$$

for all $\ell \leq B'$ (the largest of the extended Sturm bounds for $S_{k_i}\big(\Gamma_0(N)\big)$) with $\ell \nmid Np$, where $\mathfrak{p}$ runs through all prime ideals over p in the coefficient field of $f_{N,k_1,j}$ and $f_{N,k_2,i}$. If L is attained for several i's or j's, these are all listed. The tables also list the maximal integer s such that $k_1 \equiv k_2 \pmod{p^s(p-1)}$. The rows written in bold indicates the lowest weight k_2 for which there is an increase in L compared to the lower weights.

We call $S_{k_1}\big(\Gamma_0(N)\big)$ the *initial space*, and for most of the computations we have tried to choose initial spaces where there is only one newform (with integral coefficients), since this means less congruences to be checked, and we omit the index j if this is the case. We list only one table for each level, since the tables for the initial spaces $S_{k_1}\big(\Gamma_0(N)\big)$ and $S_{k_1+n}\big(\Gamma_0(N)\big)$ (for an even positive integer n) are very similar.

For levels $N < 10$ and initial weight k_1, we compute congruences for newforms on $\Gamma_0(N)$ of weights k_2 satisfying $k_1 < k_2 \leq k_1 + 64$, and for $N \geq 10$ we compute for newforms of weights k_2 satisfying $k_1 < k_2 \leq k_1 + 32$.

Level 1

The space $S_{12}\big(\Gamma_0(1)\big)$ contains the single newform

$$f_{1,12} = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 - 16744q^7 + 84480q^8 + \cdots,$$

and we get the following table (we omit the index i since there is at most one newform on $\Gamma_0(1)$ of any weight):

$p=2$			$p=3$			$p=5$		
k_2	L	s	k_2	L	s	k_2	L	s
16	**4**	**2**	**16**	**1**	**0**	**16**	**1**	**0**
18	3	1	**18**	**2**	**1**			
20	**5**	**3**	20	1	0	20	1	0
22	3	1	22	1	0			
24	4	2	24	2	1	24	1	0
26	3	1	26	1	0			
28	**6**	**4**	28	1	0	28	1	0
30	3	1	**30**	**3**	**2**			
32	4	2	32	1	0	**32**	**2**	**1**
34	3	1	34	1	0			
36	5	3	36	2	1	36	1	0
38	3	1	38	1	0			
40	4	2	40	1	0	40	1	0
42	3	1	42	2	1			
44	**7**	**5**	44	1	0	44	1	0
46	3	1	46	1	0			
48	4	2	48	3	2	48	1	0
50	3	1	50	1	0			
52	5	3	52	1	0	52	2	1
54	3	1	54	2	1			
56	4	2	56	1	0	56	1	0
58	3	1	58	1	0			
60	6	4	60	2	1	60	1	0
62	3	1	62	1	0			
64	4	2	64	1	0	64	1	0
66	3	1	**66**	**4**	**3**			
68	5	3	68	1	0	68	1	0
70	3	1	70	1	0			
72	4	2	72	2	1	72	2	1
74	3	1	74	1	0			
76	**8**	**6**	76	1	0	76	1	0

Table A.1: $N = 1$, $k_1 = 12$, $p = 2, 3, 5$ and $14 \leq k_2 \leq 76$.

We have computed similar tables for all $k_1 \leq 22$ (and $k_2 \leq 308$).

Level 2

The space $S_8\big(\Gamma_0(2)\big)$ contains the single newform

$$f_{2,8} = q - 8q^2 + 12q^3 + 64q^4 - 210q^5 - 96q^6 + 1016q^7 - 512q^8 - 2043q^9 + \cdots ,$$

and we get the following table:

$p=2$				$p=3$				$p=5$			
k_2	i	L	s	k_2	i	L	s	k_2	i	L	s
10	**1**	**3**	**1**	**10**	**1**	**1**	**0**				
14	1, 2	3	1	**14**	**1, 2**	**2**	**1**				
16	**1**	**5**	**3**	16	1	1	0	**16**	**1**	**1**	**0**
18	1	3	1	18	1	1	0				
20	1, 2	4	2	20	2	2	1	20	1	1	0
22	1, 2	3	1	22	1, 2	1	0				
24	**1**	**6**	**4**	24	1	1	0	24	1	1	0
26	1, 2	3	1	**26**	**2**	**3**	**2**				
28	1, 2	4	2	28	1, 2	1	0	**28**	**1**	**2**	**1**
30	1, 2	3	1	30	1, 2	1	0				
32	1, 2	5	3	32	2	2	1	32	2	1	0
34	1, 2	3	1	34	1, 2	1	0				
36	1, 2	4	2	36	1, 2	1	0	36	1	1	0
38	1, 2	3	1	38	1	2	1				
40	**1, 2**	**7**	**5**	40	1, 2	1	0	40	2	1	0
42	1, 2	3	1	42	1, 2	1	0				
44	1, 2	4	2	44	2	3	2	44	1	1	0
46	1, 2	3	1	46	1, 2	1	0				
48	1, 2	5	3	48	1, 2	1	0	48	2	2	1
50	1, 2	3	1	50	2	2	1				
52	1, 2	4	2	52	1, 2	1	0	52	1	1	0
54	1, 2	3	1	54	1, 2	1	0				
56	1, 2	6	4	56	2	2	1	56	2	1	0
58	1, 2	3	1	58	1, 2	1	0				
60	1, 2	4	2	60	1, 2	1	0	60	1	1	0
62	1, 2	3	1	**62**	**1**	**4**	**3**				
64	1, 2	5	3	64	1, 2	1	0	64	2	1	0
66	1, 2	3	1	66	1, 2	1	0				
68	1, 2	4	2	68	2	2	1	68	1	2	1
70	1, 2	3	1	70	1, 2	1	0				
72	**1, 2**	**8**	**6**	72	1, 2	1	0	72	2	1	0

Table A.2: $N = 2$, $k_1 = 8$, $p = 2, 3, 5$ and $10 \le k_2 \le 72$.

We have computed similar tables for all $k_1 \le 24$ (and $k_2 \le 176$).

Level 3

The space $S_6\big(\Gamma_0(3)\big)$ contains the single newform

$$f_{3,6} = q - 6q^2 + 9q^3 + 4q^4 + 6q^5 - 54q^6 - 40q^7 + 168q^8 + 81q^9 \cdots ,$$

and we get the following table:

$p=2$				$p=3$				$p=5$			
k_2	i	L	s	k_2	i	L	s	k_2	i	L	s
8	**1**	**3**	**1**	**8**	**1**	**1**	**0**				
10	**1**	**4**	**2**	10	1, 2	1	0	**10**	**2**	**1**	**0**
12	1	3	1	*12*	*1*	*1*	*1*				
14	**2**	**5**	**3**	14	1, 2	1	0	14	2	1	0
16	1, 2	3	1	16	1, 2	1	0				
18	2	4	2	**18**	**2**	**2**	**1**	18	1	1	0
20	1, 2	3	1	20	1, 2	1	0				
22	**3**	**6**	**4**	22	1–3	1	0	22	3	1	0
24	1, 2	3	1	**24**	**1, 2**	**3**	**2**				
26	2	4	2	26	1, 2	1	0	**26**	**1**	**2**	**1**
28	1, 2	3	1	28	1, 2	1	0				
30	2	5	3	30	1, 2	2	1	30	2	1	0
32	1, 2	3	1	32	1, 2	1	0				
34	1	4	2	34	1, 2	1	0	34	2	1	0
36	1, 2	3	1	36	1, 2	2	1				
38	**2**	**7**	**5**	38	1, 2	1	0	38	2	1	0
40	1, 2	3	1	40	1, 2	1	0				
42	2	4	2	42	1, 2	3	2	42	1	1	0
44	1, 2	3	1	44	1, 2	1	0				
46	1	5	3	46	1, 2	1	0	46	1	2	1
48	1, 2	3	1	48	1, 2	2	1				
50	2	4	2	50	1, 2	1	0	50	1	1	0
52	1, 2	3	1	52	1, 2	1	0				
54	2	6	4	54	1, 2	2	1	54	2	1	0
56	1, 2	3	1	56	1, 2	1	0				
58	2	4	2	58	1, 2	1	0	58	1	1	0
60	1, 2	3	1	**60**	**1, 2**	**4**	**3**				
62	2	5	3	62	1, 2	1	0	62	2	1	0
64	1, 2	3	1	64	1, 2	1	0				
66	2	4	2	66	1, 2	2	1	66	1	2	1
68	1, 2	3	1	68	1, 2	1	0				
70	**2**	**8**	**6**	70	1, 2	1	0	70	2	1	0

Table A.3: $N = 3$, $k_1 = 6$, $p = 2, 3, 5$ and $8 \leq k_2 \leq 70$.

We have computed similar tables for all $k_1 \leq 12$ (and $k_2 \leq 132$).

Level 4

The space $S_6\big(\Gamma_0(4)\big)$ contains the single newform

$$f_{4,6} = q - 12q^3 + 54q^5 - 88q^7 - 99q^9 + \cdots ,$$

and we get the following table (we can again omit the index i):

$p=2$			$p=3$			$p=5$		
k_2	L	s	k_2	L	s	k_2	L	s
10	**4**	**2**	**10**	**1**	**0**	**10**	**1**	**0**
12	3	1	**12**	**2**	**1**			
14	**5**	**3**	14	1	0	14	1	0
16	3	1	16	1	0			
18	4	2	18	2	1	18	1	0
20	3	1	20	1	0			
22	**6**	**4**	22	1	0	22	1	0
24	3	1	**24**	**3**	**2**			
26	4	2	26	1	0	**26**	**2**	**1**
28	3	1	28	1	0			
30	5	3	30	2	1	30	1	0
32	3	1	32	1	0			
34	4	2	34	1	0	34	1	0
36	3	1	36	2	1			
38	**7**	**5**	38	1	0	38	1	0
40	3	1	40	1	0			
42	4	2	42	3	2	42	1	0
44	3	1	44	1	0			
46	5	3	46	1	0	46	2	1
48	3	1	48	2	1			
50	4	2	50	1	0	50	1	0
52	3	1	52	1	0			
54	6	4	54	2	1	54	1	0
56	3	1	56	1	0			
58	4	2	58	1	0	58	1	0
60	3	1	**60**	**4**	**3**			
62	5	3	62	1	0	62	1	0
64	3	1	64	1	0			
66	4	2	66	2	1	66	2	1
68	3	1	68	1	0			
70	**8**	**6**	70	1	0	70	1	0

Table A.4: $N = 4$, $k_1 = 6$, $p = 2, 3, 5$ and $8 \leq k_2 \leq 70$.

We have computed similar tables for all $k_1 \leq 16$.

Level 5

The space $S_4\big(\Gamma_0(5)\big)$ contains the single newform

$$f_{5,4} = q - 4q^2 + 2q^3 + 8q^4 - 5q^5 - 8q^6 + 6q^7 - 23q^9 + \cdots ,$$

and we get the following table:

$p=2$				$p=3$				$p=5$			
k_2	i	L	s	k_2	i	L	s	k_2	i	L	s
6	***1***	***1***	***1***	**6**	**1**	**1**	**0**				
8	**2**	**3**	**2**	8	2	1	0	**8**	**1, 2**	**1**	**0**
10	1	2	1	**10**	**2**	**2**	**1**				
12	**2**	**4**	**3**	12	2	1	0	12	2	1	0
14	1	2	1	14	2	1	0				
16	2	3	2	16	2	2	1	16	1, 2	1	0
18	1	2	1	18	2	1	0				
20	**2**	**5**	**4**	20	2	1	0	20	1, 2	1	0
22	1	2	1	**22**	**2**	**3**	**2**				
24	2	3	2	24	2	1	0	*24*	*1, 2*	*1*	*1*
26	1	2	1	26	2	1	0				
28	2	5	3	28	2	2	1	28	1, 2	1	0
30	1	2	1	30	2	1	0				
32	2	3	2	32	2	1	0	32	1, 2	1	0
34	1	2	1	34	2	2	1				
36	**2**	**6**	**5**	36	2	1	0	36	1, 2	1	0
38	1	2	1	38	2	1	0				
40	2	3	2	40	2	3	2	40	1, 2	1	0
42	1	2	1	42	2	1	0				
44	2	5	3	44	2	1	0	**44**	**2**	**2**	**1**
46	1	2	1	46	2	2	1				
48	2	3	2	48	2	1	0	48	1, 2	1	0
50	1	2	1	50	2	1	0				
52	2	6	4	52	2	2	1	52	1, 2	1	0
54	1	2	1	54	2	1	0				
56	2	3	2	56	2	1	0	56	1, 2	1	0
58	1	2	1	**58**	**2**	**4**	**3**				
60	2	5	3	60	2	1	0	60	1, 2	1	0
62	1	2	1	62	2	1	0				
64	2	3	2	64	2	2	1	64	1, 2	2	1
66	1	2	1	66	2	1	0				
68	**2**	**7**	**6**	68	2	1	0	68	1, 2	1	0

Table A.5: $N = 5$, $k_1 = 4$, $p = 2, 3, 5$ and $6 \leq k_2 \leq 68$.

We have computed a similar table for $k_1 = 6$.

Level 6

The space $S_4\big(\Gamma_0(6)\big)$ contains the single newform

$$f_{6,4} = q - 2q^2 - 3q^3 + 4q^4 + 6q^5 + 6q^6 - 16q^7 - 8q^8 + 9q^9 + \cdots ,$$

and we get the following table:

$p=2$				$p=3$				$p=5$			
k_2	i	L	s	k_2	i	L	s	k_2	i	L	s
6	**1**	**3**	**1**	**6**	**1**	**1**	**0**				
8	1	3	2	8	1	1	0	**8**	**1**	**1**	**0**
10	1	3	1	10	1	1	1				
12	**3**	**5**	**3**	12	1–3	1	0	12	3	1	0
14	1	3	1	14	1	1	0				
16	2, 3	4	2	**16**	**3**	**2**	**1**	16	1	1	0
18	1–3	3	1	18	1–3	1	0				
20	**3**	**6**	**4**	20	1–3	1	0	20	3	1	0
22	1–3	3	1	**22**	**1, 2**	**3**	**2**				
24	1, 3	4	2	24	1–4	1	0	**24**	**4**	**2**	**1**
26	1–3	3	1	26	1–3	1	0				
28	2, 4	5	3	28	3, 4	2	1	28	4	1	0
30	1–4	3	1	30	1–4	1	0				
32	1, 3	4	2	32	1–4	1	0	32	4	1	0
34	1–4	3	1	34	1, 2	2	1				
36	**1, 4**	**7**	**5**	36	1–4	1	0	36	4	1	0
38	1–4	3	1	38	1–4	1	0				
40	3, 4	4	2	40	1, 4	3	2	40	2	1	0
42	1–4	3	1	42	1–4	1	0				
44	1, 4	5	3	44	1–4	1	0	44	4	2	1
46	1–4	3	1	46	2, 3	2	1				
48	1, 3	4	2	48	1–4	1	0	48	4	1	0
50	1–4	3	1	50	1–4	1	0				
52	2, 4	6	4	52	3, 4	2	1	52	4	1	0
54	1–4	3	1	54	1–4	1	0				
56	1, 3	4	2	56	1–4	1	0	56	4	1	0
58	1–4	3	1	**58**	**1, 2**	**4**	**3**				
60	1, 4	5	3	60	1–4	1	0	60	4	1	0
62	1–4	3	1	62	1–4	1	0				
64	3, 4	4	2	64	1, 4	2	1	64	2	2	1
66	1–4	3	1	66	1–4	1	0				
68	**1, 4**	**8**	**6**	68	1–4	1	0	68	4	1	0

Table A.6: $N=6$, $k_1=4$, $p=2,3,5$ and $6 \le k_2 \le 68$.

We have computed similar tables for all $k_1 \le 22$.

Level 7

The space $S_4\big(\Gamma_0(7)\big)$ contains the single newform

$$f_{7,4} = q - q^2 - 2q^3 - 7q^4 + 16q^5 + 2q^6 - 7q^7 + 15q^8 - 23q^9 + \cdots ,$$

and we get the following table:

$p=2$				$p=3$				$p=5$			
k_2	i	L	s	k_2	i	L	s	k_2	i	L	s
6	**2**	**3**	**1**	**6**	**1**	**1**	**0**				
8	**2**	**4**	**2**	8	2	1	0	**8**	**1**	**1**	**0**
10	2	3	1	**10**	**1**	**2**	**1**				
12	**2**	**5**	**3**	12	2	1	0	12	2	1	0
14	2	3	1	14	1	1	0				
16	2	4	2	16	2	2	1	16	1	1	0
18	2	3	1	18	1	1	0				
20	**2**	**6**	**4**	20	2	1	0	20	2	1	0
22	2	3	1	**22**	**1**	**3**	**2**				
24	2	4	2	24	2	1	0	**24**	**1**	**2**	**1**
26	2	3	1	26	1	1	0				
28	2	5	3	28	2	2	1	28	2	1	0
30	2	3	1	30	1	1	0				
32	2	4	2	32	2	1	0	32	1	1	0
34	2	3	1	34	1	2	1				
36	**2**	**7**	**5**	36	2	1	0	36	2	1	0
38	2	3	1	38	1	1	0				
40	2	4	2	40	2	3	2	40	1	1	0
42	2	3	1	42	1	1	0				
44	2	5	3	44	2	1	0	44	2	2	1
46	2	3	1	46	1	2	1				
48	2	4	2	48	2	1	0	48	1	1	0
50	2	3	1	50	1	1	0				
52	2	6	4	52	2	2	1	52	2	1	0
54	2	3	1	54	1	1	0				
56	2	4	2	56	2	1	0	56	1	1	0
58	2	3	1	**58**	**1**	**4**	**3**				
60	2	5	3	60	2	1	0	60	2	1	0
62	2	3	1	62	1	1	0				
64	2	4	2	64	2	2	1	64	1	2	1
66	2	3	1	66	1	1	0				
68	**2**	**8**	**6**	68	2	1	0	68	2	1	0

Table A.7: $N=7$, $k_1=4$, $p=2,3,5$ and $6 \le k_2 \le 68$.

Level 8

The space $S_4(\Gamma_0(8))$ contains the single newform

$$f_{8,4} = q - 4q^3 - 2q^5 + 24q^7 - 11q^9 + \cdots ,$$

and we get the following table:

$p=2$				$p=3$				$p=5$			
k_2	i	L	s	k_2	i	L	s	k_2	i	L	s
6	**1**	**3**	**1**	**6**	**1**	**1**	**0**				
8	**1, 2**	**4**	**2**	8	1	1	0	**8**	**2**	**1**	**0**
10	1, 2	3	1	**10**	**2**	**2**	**1**				
12	**1, 2**	**5**	**3**	12	2	1	0	12	2	1	0
14	1, 2	3	1	14	2	1	0				
16	1–3	4	2	16	3	2	1	16	2	1	0
18	1, 2	3	1	18	1	1	0				
20	**1, 2**	**6**	**4**	20	2	1	0	20	2	1	0
22	1, 2	3	1	**22**	**2**	**3**	**2**				
24	1, 2	4	2	24	1	1	0	**24**	**2**	**2**	**1**
26	1, 2	3	1	26	2	1	0				
28	1, 2	5	3	28	2	2	1	28	2	1	0
30	1, 2	3	1	30	2	1	0				
32	1, 2	4	2	32	1	1	0	32	2	1	0
34	1, 2	3	1	34	2	2	1				
36	**1, 2**	**7**	**5**	36	2	1	0	36	2	1	0
38	1, 2	3	1	38	2	1	0				
40	1, 2	4	2	40	1	3	2	40	2	1	0
42	1, 2	3	1	42	1	1	0				
44	1, 2	5	3	44	2	1	0	44	2	2	1
46	1, 2	3	1	46	2	2	1				
48	1, 2	4	2	48	1	1	0	48	2	1	0
50	1, 2	3	1	50	1	1	0				
52	1, 2	6	4	52	2	2	1	52	2	1	0
54	1, 2	3	1	54	2	1	0				
56	1, 2	4	2	56	1	1	0	56	2	1	0
58	1, 2	3	1	**58**	**1**	**4**	**3**				
60	1, 2	5	3	60	2	1	0	60	2	1	0
62	1, 2	3	1	62	2	1	0				
64	1, 2	4	2	64	2	2	1	64	1	2	1
66	1, 2	3	1	66	2	1	0				
68	**2**	**8**	**6**	68	2	1	0	68	2	1	0

Table A.8: $N = 8$, $k_1 = 4$, $p = 2, 3, 5$ and $6 \leq k_2 \leq 68$.

We have computed similar tables for all $k_1 \leq 8$.

Level 9

The space $S_4\big(\Gamma_0(9)\big)$ contains the single newform

$$f_{9,4} = q - 8q^4 + 20q^7 + \cdots ,$$

and we get the following table:

$p=2$				$p=3$				$p=5$			
k_2	i	L	s	k_2	i	L	s	k_2	i	L	s
6	***1***	***1***	***1***	**6**	**1**	**1**	**0**				
8	**2**	**4**	**2**	8	1, 2	1	0	**8**	**2**	**1**	**0**
10	1	3	1	**10**	**1–3**	**2**	**1**				
12	**3**	**5**	**3**	12	1–3	1	0	12	3	1	0
14	2	3	1	14	1–3	1	0				
16	1, 5	4	2	16	1–5	2	1	16	1, 5	1	0
18	3	3	1	18	1–4	1	0				
20	**4**	**6**	**4**	20	1–4	1	0	20	4	1	0
22	1, 5	3	1	**22**	**1, 3, 6**	**3**	**2**				
24	4	4	2	24	1–4	1	0	**24**	**4**	**2**	**1**
26	4	3	1	26	1–4	1	0				
28	1, 5	5	3	28	1–5	2	1	28	1, 5	1	0
30	4	3	1	30	1–4	1	0				
32	4	4	2	32	1–4	1	0	32	4	1	0
34	1, 5	3	1	34	1–5	2	1				
36	**4**	**7**	**5**	36	1–4	1	0	36	4	1	0
38	4	3	1	38	1–4	1	0				
40	1, 5	4	2	40	1–5	3	2	40	1, 5	1	0
42	4	3	1	42	1–4	1	0				
44	4	5	3	44	1–4	1	0	44	4	2	1
46	1, 5	3	1	46	1–5	2	1				
48	4	4	2	48	1–4	1	0	48	4	1	0
50	4	3	1	50	1–4	1	0				
52	1, 5	6	4	52	1–5	2	1	52	1, 5	1	0
54	4	3	1	54	1–4	1	0				
56	4	4	2	56	1–4	1	0	56	4	1	0
58	1, 5	3	1	**58**	**1, 3, 4**	**4**	**3**				
60	4	5	3	60	1–4	1	0	60	4	1	0
62	4	3	1	62	1–4	1	0				
64	1, 5	4	2	64	1–5	2	1	64	5	2	1
66	4	3	1	66	1–4	1	0				
68	**4**	**8**	**6**	68	1–4	1	0	68	4	1	0

Table A.9: $N = 9$, $k_1 = 4$, $p = 2, 3, 5$ and $6 \leq k_2 \leq 68$.

We have computed a similar table for $k_1 = 6$.

Level 10

The space $S_4\big(\Gamma_0(10)\big)$ contains the single newform

$$f_{10,4} = q + 2q^2 - 8q^3 + 4q^4 + 5q^5 - 16q^6 - 4q^7 + 8q^8 + 37q^9 + \cdots ,$$

and we get the following table:

$p=2$				$p=3$				$p=5$			
k_2	i	L	s	k_2	i	L	s	k_2	i	L	s
6	**2**	**3**	**1**	**6**	**1–3**	**1**	**0**				
8	*1*	*2*	*2*	8	1	1	0				
10	3	2	1	**10**	**2**	**2**	**1**				
12	**4**	**4**	**3**	12	1–4	1	0	**12**	**1, 4**	**1**	**0**
14	3	3	1	14	1–3	1	0				
16	3, 4	3	2	16	4	2	1	16	2, 3	1	0
18	1, 3	3	1	18	1–4	1	0				
20	**4**	**5**	**4**	20	1–4	1	0	20	1, 4	1	0
22	3	3	1	**22**	**4**	**3**	**2**				
24	2, 4	3	2	24	1–4	1	0	*24*	*2, 3*	*1*	*1*
26	1, 3	3	1	26	1–4	1	0				
28	3, 4	4	3	28	4	2	1	28	1, 4	1	0
30	1, 4	3	1	30	1–4	1	0				
32	2, 4	3	2	32	1–4	1	0	32	2, 3	1	0
34	1, 4	3	1	34	3	2	1				
36	**3, 4**	**6**	**5**	36	1–4	1	0	36	1, 4	1	0

Table A.10: $N = 10$, $k_1 = 4$, $p = 2, 3, 5$ and $6 \leq k_2 \leq 36$.

We have computed similar tables for all $k_1 \leq 10$.

Level 12

The space $S_4\big(\Gamma_0(12)\big)$ contains the single newform

$$f_{12,4} = q + 3q^3 - 18q^5 + 8q^7 + 9q^9 + \cdots ,$$

and we get the following table:

$p=2$				$p=3$				$p=5$			
k_2	i	L	s	k_2	i	L	s	k_2	i	L	s
8	**1**	**4**	**2**	**8**	**1, 2**	**1**	**0**	**8**	**2**	**1**	**0**
10	1	3	1	**10**	**1**	**2**	**1**				
12	**1**	**5**	**3**	12	1, 2	1	0	12	1	1	0
14	1, 2	3	1	14	1, 2	1	0				
16	2	4	2	16	1, 2	2	1	16	1	1	0
18	1, 2	3	1	18	1, 2	1	0				
20	**1**	**6**	**4**	20	1, 2	1	0	20	1	1	0
22	1, 2	3	1	**22**	**1, 2**	**3**	**2**				
24	1	4	2	24	1, 2	1	0	**24**	**2**	**2**	**1**
26	1, 2	3	1	26	1, 2	1	0				
28	2	5	3	28	1, 2	2	1	28	2	1	0
30	1, 2	3	1	30	1, 2	1	0				
32	1	4	2	32	1, 2	1	0	32	2	1	0
34	1, 2	3	1	34	1, 2	2	1				
36	**1**	**7**	**5**	36	1, 2	1	0	36	1	1	0

Table A.11: $N = 12$, $k_1 = 4$, $p = 2, 3, 5$ and $6 \leq k_2 \leq 36$.

We have computed similar tables for all $k_1 \leq 14$.

Level 15

The space $S_2\big(\Gamma_0(15)\big)$ contains the single newform

$$f_{15,2} = q - q^2 - q^3 - q^4 + q^5 + q^6 + 3q^8 + q^9 + \cdots ,$$

and we get the following table:

$p = 2$				$p = 3$				$p = 5$			
k_2	i	L	s	k_2	i	L	s	k_2	i	L	s
4	**1**	**3**	**1**								
6	**3**	**4**	**2**	**6**	**3**	**1**	**0**				
8	3	3	1	*8*	*1, 2*	*1*	*1*				
10	**4**	**5**	**3**	10	1, 4	1	0	**10**	**4**	**1**	**0**
12	1, 4	3	1	12	2, 3	1	0				
14	4	4	2	**14**	**4**	**2**	**1**	14	1, 3	1	0
16	2, 4	3	1	16	1, 3	1	0				
18	**4**	**6**	**4**	18	1, 4	1	0	18	4, 4	1	0
20	1, 4	3	1	**20**	**2, 3**	**3**	**2**				
22	5	4	2	22	4, 5	1	0	*22*	*1, 3, 4*	*1*	*1*
24	1, 4	3	1	24	2, 3	1	0				
26	4	5	3	26	1, 4	2	1	26	2, 4	1	0
28	1, 3	3	1	28	2, 4	1	0				
30	4	4	2	30	1, 4	1	0	30	1, 2	1	0
32	1, 4	3	1	32	2, 3	2	1				
34	**4**	**7**	**5**	34	2, 4	1	0	34	1, 4	1	0

Table A.12: $N = 15$, $k_1 = 2$, $p = 2, 3, 5$ and $4 \leq k_2 \leq 34$.

We have computed a similar table for $k_1 = 4$.

Level 16

The space $S_4\big(\Gamma_0(16)\big)$ contains the single newform

$$f_{16,4} = q + 4q^3 - 2q^5 - 24q^7 - 11q^9 + \cdots ,$$

and we get the following table:

$p=2$				$p=3$				$p=5$			
k_2	i	L	s	k_2	i	L	s	k_2	i	L	s
6	**1, 2**	**3**	**1**	**6**	**2**	**1**	**0**				
8	**1–3**	**4**	**2**	8	2	1	0	**8**	**3**	**1**	**0**
10	1–4	3	1	**10**	**2**	**2**	**1**				
12	**1–4**	**5**	**3**	12	4	1	0	12	4	1	0
14	1–5	3	1	14	5	1	0				
16	1–5	4	2	16	6	2	1	16	4	1	0
18	1–5	3	1	18	3	1	0				
20	**5–6**	**6**	**4**	20	6	1	0	20	6	1	0
22	1–6	3	1	**22**	**6**	**3**	**2**				
24	1–5	4	2	24	4	1	0	**24**	**5**	**2**	**1**
26	1–6	3	1	26	6	1	0				
28	1–6	5	3	28	6	2	1	28	6	1	0
30	1–6	3	1	30	6	1	0				
32	1–6	4	2	32	5	1	0	32	6	1	0
34	1–6	3	1	34	6	2	1				
36	**5–6**	**7**	**5**	36	6	1	0	36	6	1	0

Table A.13: $N = 16$, $k_1 = 4$, $p = 2, 3, 5$ and $6 \leq k_2 \leq 36$.

We have computed similar tables for all $k_1 \leq 10$.

Level 18

The space $S_4\big(\Gamma_0(18)\big)$ contains the single newform

$$f_{18,4} = q + 2q^2 + 4q^4 - 6q^5 - 16q^7 + 8q^8 + \cdots ,$$

and we get the following table:

$p=2$				$p=3$				$p=5$			
k_2	i	L	s	k_2	i	L	s	k_2	i	L	s
6	**2**	**3**	**1**	**6**	**1–3**	**1**	**0**				
8	1, 2	3	2	8	1, 2	1	0	**8**	**2**	**1**	**0**
10	2, 4	3	1	**10**	**3, 4**	**2**	**1**				
12	**2**	**5**	**3**	12	1–5	1	0	12	2	1	0
14	1, 4, 5	3	1	14	1–5	1	0				
16	3, 5	4	2	16	1–3	2	1	16	4	1	0
18	1–4	3	1	18	1–6	1	0				
20	**2**	**6**	**4**	20	1–7	1	0	20	2	1	0
22	1–5	3	1	**22**	**3–5, 7**	**3**	**2**				
24	3, 4	4	2	24	1–7	1	0	**24**	**6**	**2**	**1**
26	1–4, 7	3	1	26	1–7	1	0				
28	4, 6	5	3	28	1, 2, 6, 7	2	1	28	2, 6	1	0
30	1–6	3	1	30	1–8	1	0				
32	1, 3	4	2	32	1–8	1	0	32	4, 7	1	0
34	1–6	3	1	34	3, 4, 6, 8	2	1				
36	**2, 5**	**7**	**5**	36	1–8	1	0	36	1, 5	1	0

Table A.14: $N = 18$, $k_1 = 4$, $p = 2, 3, 5$ and $6 \leq k_2 \leq 36$.

We have computed similar tables for all $k_1 \leq 16$.

Level 20

The space $S_2\big(\Gamma_0(20)\big)$ contains the single newform

$$f_{20,2} = q - 2q^3 - q^5 + 2q^7 + q^9 + \cdots \,,$$

and we get the following table:

$p = 2$				$p = 3$				$p = 5$			
k_2	i	L	s	k_2	i	L	s	k_2	i	L	s
4	***1***	***1***	***1***	**4**	**1**	**1**	**0**				
6	**1**	**3**	**2**	6	1	1	0				
8	1	2	1	**8**	**2**	**2**	**1**				
10	**2**	**4**	**3**	10	1,2	1	0	**10**	**2**	**1**	**0**
12	1	2	1	12	1,2	1	0				
14	2	3	2	14	2	2	1	14	1,2	1	0
16	1	2	1	16	1,2	1	0				
18	**2**	**5**	**4**	18	1,2	1	0	18	1,2	1	0
20	1	2	1	**20**	**2**	**3**	**2**				
22	2	3	2	22	1,2	1	0	*22*	*1,2*	*1*	*1*
24	1	2	1	24	1,2	1	0				
26	2	4	3	26	2	2	1	26	1,2	1	0
28	1	2	1	28	1,2	1	0				
30	2	3	2	30	1,2	1	0	30	1,2	1	0
32	1	2	1	32	2	2	1				
34	**2**	**7**	**5**	34	1,2	1	0	34	1,2	1	0

Table A.15: $N = 20$, $k_1 = 2$, $p = 2, 3, 5$ and $4 \leq k_2 \leq 34$.

We have computed similar tables for all $k_1 \leq 6$.

Level 24

The space $S_2\big(\Gamma_0(24)\big)$ contains the single newform

$$f_{24,2} = q - q^3 - 2q^5 + q^9 + \cdots ,$$

and we get the following table:

$p=2$				$p=3$				$p=5$			
k_2	i	L	s	k_2	i	L	s	k_2	i	L	s
4	**1**	**3**	**1**								
6	**2, 3**	**4**	**2**	**6**	**3**	**1**	**0**	**6**	**1**	**1**	**0**
8	1–3	3	1	*8*	*2, 3*	*1*	*1*				
10	**3, 4**	**5**	**3**	10	2, 4	1	0	10	4	1	0
12	1–4	3	1	12	1, 3	1	0				
14	3, 4	4	2	**14**	**4**	**2**	**1**	14	2	1	0
16	1–4	3	1	16	2, 4	1	0				
18	**3, 4**	**6**	**4**	18	2, 4	1	0	18	4	1	0
20	1–4	3	1	**20**	**1, 3**	**3**	**2**				
22	3, 4	4	2	22	1, 4	1	0	**22**	**2**	**2**	**1**
24	1–4	3	1	24	3, 4	1	0				
26	3, 4	5	3	26	2, 4	2	1	26	4	1	0
28	1–4	3	1	28	1, 2	1	0				
30	3, 4	4	2	30	1, 3	1	0	30	2	1	0
32	1–4	3	1	32	3, 4	2	1				
34	**3, 4**	**7**	**5**	34	1, 4	1	0	34	4	1	0
36	1–4	3	1	36	1, 2	1	0				

Table A.16: $N = 24$, $k_1 = 2$, $p = 2, 3, 5$ and $4 \leq k_2 \leq 34$.

We have computed similar tables for all $k_1 \leq 8$.

Level 25

The space $S_4\big(\Gamma_0(25)\big)$ contains the three newforms

$$f_{25,4,1} = q + q^2 + 7q^3 - 7q^4 + 7q^6 + 6q^7 - 15q^8 + 22q^9 + \cdots ,$$
$$f_{25,4,2} = q - q^2 - 7q^3 - 7q^4 + 7q^6 - 6q^7 + 15q^8 + 22q^9 + \cdots ,$$
$$f_{25,4,3} = q + 4q^2 - 2q^3 + 8q^4 - 8q^6 - 6q^7 - 23q^9 + \cdots ,$$

and we get the following table:

$p=2$					$p=3$					$p=5$				
k_2	i	j	L	s	k_2	i	j	L	s	k_2	i	j	L	s
6	**3**	**2**	**3**	**1**	**6**	**1, 4**	**1, 3**	**1**	**0**					
6	**4**	**1**	**3**	**1**	**6**	**3**	**2**	**1**	**0**					
8	**3**	**1**	**4**	**2**	8	3, 5	1, 3	1	0	**8**	**1**	**2, 3**	**1**	**0**
8	**4**	**2**	**4**	**2**	8	4	2	1	0	**8**	**2–5**	**1–3**	**1**	**0**
10	3	2	3	1	**10**	**2**	**3**	**2**	**1**					
10	4	1	3	1	**10**	**3**	**2**	**2**	**1**					
					10	**4**	**1**	**2**	**1**					
12	**5**	**1**	**5**	**3**	12	3, 5	1, 3	1	0	12	1	2, 3	1	0
12	**6**	**2**	**5**	**3**	12	6	2	1	0	12	2	1	1	0
14	3	2	3	1	14	2, 4	1, 3	1	0	12	3–6	1–3	1	0
14	4	1	3	1	14	3	2	1	0					
16	4	1	4	2	16	3	3	2	1	16	1	2, 3	1	0
16	5	2	4	2	16	4	1	2	1	16	2–6	1–3	1	0
					16	5	2	2	1					
18	4	1	3	1	18	3, 4	1, 3	1	0					
18	5	2	3	1	18	5	2	1	0					
20	**4**	**1**	**6**	**4**	20	3, 4	1, 3	1	0	20	1	2, 3	1	0
20	**5**	**2**	**6**	**4**	20	5	2	1	0	20	2–6	1–3	1	0
22	4	2	3	1	**22**	**3**	**3**	**3**	**2**					
22	5	1	3	1	**22**	**4**	**2**	**3**	**2**					
					22	**5**	**1**	**3**	**2**					
24	4	1	4	2	24	3, 4	1, 3	1	0	**24**	**1**	**3**	**2**	**1**
24	5	2	4	2	24	5	2	1	0	**24**	**2, 3**	**1, 2**	**2**	**1**
26	4	2	3	1	26	3, 5	1, 3	1	0	**24**	**4–6**	**1–3**	**2**	**1**
26	5	1	3	1	26	4	2	1	0					
28	3	3	5	3	28	3	3	2	1	28	1–6	1–3	1	0
28	4	1	5	3	28	4	1	2	1					
28	5	2	5	3	28	5	2	2	1					
30	4	2	3	1	30	3, 5	1, 3	1	0					
30	5	1	3	1	30	4	2	1	0					
32	4	1	4	2	32	3, 4	1, 3	1	0	32	1–6	1–3	1	0
32	5	2	4	2	32	5	2	1	0					
34	4	2	3	1	34	3	3	2	1					
34	5	1	3	1	34	4	2	2	1					
					34	5	1	2	1					
36	**4**	**2**	**7**	**5**	36	3, 5	1, 3	1	0	36	1–6	1–3	1	0
36	**5**	**1**	**7**	**5**	36	4	2	1	0					

Table A.17: $N=25$, $k_1=4$, $p=2,3,5$ and $6 \le k_2 \le 36$.

Level 27

The space $S_2(\Gamma_0(27))$ contains the single newform

$$f_{27,2} = q - 2q^4 - q^7 + \cdots ,$$

and we get the following table:

$p=2$				$p=3$				$p=5$			
k_2	i	L	s	k_2	i	L	s	k_2	i	L	s
4	**3**	**2**	**1**	**4**	**1–3**	**1**	**0**				
6	1	1	2	6	1–4	1	0	**6**	**1**	**1**	**0**
6	**2**	**3**	**2**	**8**	**1–3**	**2**	**1**				
10	**4**	**4**	**3**	10	1–4	1	0	10	4	1	0
12	3	2	1	12	1–5	1	0				
14	1, 3	3	2	14	1–3	2	1	14	2	1	0
16	4	2	1	16	1–4	1	0				
18	**3**	**5**	**4**	18	1–5	1	0	18	3	1	0
20	1, 3	2	1	**20**	**1**	**3**	**2**				
22	4	3	2	22	1–4	1	0	**22**	**1**	**2**	**1**
24	3	2	1	24	1–5	1	0				
26	1, 3	4	3	26	1–3	2	1	26	1, 3	1	0
28	4	2	1	28	1–4	1	0				
30	3	3	2	30	1–5	1	0	30	1, 2	1	0
32	1, 3	2	1	32	1–3	2	1				
34	**4**	**6**	**5**	34	1–4	1	0	34	4	1	0

Table A.18: $N = 27$, $k_1 = 2$, $p = 2, 3, 5$ and $4 \leq k_2 \leq 34$.

Appendix B

Computational evidence for Conjectures 2.23 and 2.24

The tables in this appendix are computational results supporting Conjecture 2.23 and Conjecture 2.24.

We denote by $f_{N,k,i}$ the i'th newform in $S_k\big(\Gamma_0(N)\big)$, with the ordering used by `SortDecomposition` in Magma. If $S_k\big(\Gamma_0(N)\big)$ contains only one newform, we simply denote this by $f_{N,k}$.

Looking at all newforms in $S_{k_1}\big(\Gamma_0(Np)\big)$ and $S_{k_2}\big(\Gamma_0(N)\big)$, the tables give the maximal positive integer $M = \lceil \frac{m}{e(\mathfrak{p}/p)} \rceil$ such that

$$a_\ell(f_{Np,k_1,j}) \equiv a_\ell(f_{N,k_2,i}) \pmod{\mathfrak{p}^m}$$

for all $\ell \leq B'$ (the largest of the extended Sturm bounds for $S_{k_i}\big(\Gamma_0(Np)\big)$) with $\ell \nmid Np$, where $\mathfrak{p}$ runs through all prime ideals over p in the coefficient field of $f_{Np,k_1,j}$ and $f_{N,k_2,i}$. If M is attained for several i's or j's, these are all listed. The tables also list the maximal integer s such that $k_1 \equiv k_2 \pmod{p^s(p-1)}$. The rows written in bold indicates the lowest weight k_2 for which there is an increase in M compared to the lower weights.

We call $S_{k_1}\big(\Gamma_0(Np)\big)$ the *initial space*, and for most of the computations we have tried to choose initial spaces where there is only one newform (with integral coefficients), since this means less congruences to be checked, and we omit the index j if this is the case. We list only one table for each base level, since the tables for the initial spaces $S_{k_1}\big(\Gamma_0(Np)\big)$ and $S_{k_1+n}\big(\Gamma_0(Np)\big)$ (for an even positive integer n) are very similar.

For base levels $N < 10$ and initial weight $k_1^{(p)}$, we compute congruences for newforms on $\Gamma_0(N)$ of weights k_2 satisfying $k_1^{(p)} < k_2 \leq k_1^{(p)} + 64$, and for $N \geq 10$ we compute for newforms of weights k_2 satisfying $k_1^{(p)} < k_2 \leq k_1^{(p)} + 32$.

Base level 1

Each of the spaces $S_8\big(\Gamma_0(2)\big)$, $S_6\big(\Gamma_0(3)\big)$ and $S_4\big(\Gamma_0(5)\big)$ contains a single newform, and we get the following table (we omit the index i since there is at most one newform on $\Gamma_0(1)$ of any weight):

$p=2$			$p=3$			$p=5$		
k_2	M	s	k_2	M	s	k_2	M	s
12	**4**	**2**	**12**	**2**	**1**	**12**	**1**	**0**
16	**5**	**3**	16	1	0	16	1	0
18	3	1	18	2	1			
20	4	2	20	1	0	20	1	0
22	3	1	22	1	0			
24	**6**	**4**	**24**	**3**	**2**	**24**	**2**	**1**
26	3	1	26	1	0			
28	4	2	28	1	0	28	1	0
30	3	1	30	2	1			
32	5	3	32	1	0	32	1	0
34	3	1	34	1	0			
36	4	2	36	2	1	36	1	0
38	3	1	38	1	0			
40	**7**	**5**	40	1	0	40	1	0
42	3	1	42	3	2			
44	4	2	44	1	0	44	2	1
46	3	1	46	1	0			
48	5	3	48	2	1	48	1	0
50	3	1	50	1	0			
52	4	2	52	1	0	52	1	0
54	3	1	54	2	1			
56	6	4	56	1	0	56	1	0
58	3	1	58	1	0			
60	4	2	**60**	**4**	**3**	60	1	0
62	3	1	62	1	0			
64	5	3	64	1	0	64	2	1
66	3	1	66	2	1			
68	4	2	68	1	0	68	1	0
70	3	1	70	1	0			
72	**8**	**6**	72	2	1	72	1	0

Table B.1: $N=1$, $p=2,3,5$, $k_1^{(2)}=8$, $k_1^{(3)}=6$, $k_1^{(5)}=4$ and $k_2 \leq 72$.

We have computed similar tables for all $k_1^{(2)} \leq 24$, $k_1^{(3)} \leq 12$ and $k_1^{(5)} \leq 6$.

Base level 2

Each of the spaces $S_6(\Gamma_0(4))$, $S_4(\Gamma_0(6))$ and $S_4(\Gamma_0(10))$ contains a single newform, and we get the following table:

$p=2$				$p=3$				$p=5$			
k_2	i	M	s	k_2	i	M	s	k_2	i	M	s
8	**1**	**3**	**1**	**8**	**1**	**1**	**0**	**8**	**1**	**1**	**0**
10	**1**	**4**	**2**	**10**	**1**	**2**	**1**				
14	**1, 2**	**5**	**3**	14	1,2	1	0				
16	1	3	1	16	1	2	1	16	1	1	0
18	1	4	2	18	1	1	0				
20	1, 2	3	1	20	1,2	1	0	20	1	1	0
22	**1, 2**	**6**	**4**	**22**	**1**	**3**	**2**				
24	1	3	1	24	1	1	0	**24**	**1**	**2**	**1**
26	1, 2	4	2	26	1,2	1	0				
28	1, 2	3	1	28	1,2	2	1	28	1	1	0
30	1, 2	5	3	30	1,2	1	0				
32	1, 2	3	1	32	1,2	1	0	32	2	1	0
34	1, 2	4	2	34	2	2	1				
36	1, 2	3	1	36	1,2	1	0	36	1	1	0
38	**1, 2**	**7**	**5**	38	1,2	1	0				
40	1, 2	3	1	40	2	3	2	40	2	1	0
42	1, 2	4	2	42	1,2	1	0				
44	1, 2	3	1	44	1,2	1	0	44	1	2	1
46	1, 2	5	3	46	1	2	1				
48	1, 2	3	1	48	1,2	1	0	48	2	1	0
50	1, 2	4	2	50	1,2	1	0				
52	1, 2	3	1	52	2	2	1	52	1	1	0
54	1, 2	6	4	54	1,2	1	0				
56	1, 2	3	1	56	1,2	1	0	56	2	1	0
58	1, 2	4	2	**58**	**2**	**4**	**3**				
60	1, 2	3	1	60	1,2	1	0	60	1	1	0
62	1, 2	5	3	62	1,2	1	0				
64	1, 2	3	1	64	2	2	1	64	2	2	1
66	1, 2	4	2	66	1,2	1	0				
68	1, 2	3	1	68	1,2	1	0	68	1	1	0
70	**1, 2**	**8**	**6**	70	1	2	1				

Table B.2: $N=2$, $p=2,3,5$, $k_1^{(2)}=6$, $k_1^{(3)}=4$, $k_1^{(5)}=4$ and $k_2 \leq 70$.

We have computed similar tables for all $k_1^{(2)} \leq 16$, $k_1^{(3)} \leq 22$ and $k_1^{(5)} \leq 10$.

Base level 3

Each of the spaces $S_4\big(\Gamma_0(6)\big)$, $S_4\big(\Gamma_0(9)\big)$ and $S_2\big(\Gamma_0(15)\big)$ contains a single newform, and we get the following table:

$p=2$				$p=3$				$p=5$			
k_2	i	M	s	k_2	i	M	s	k_2	i	M	s
6	**1**	**3**	**1**	**6**	**1**	**1**	**0**	**6**	**1**	**1**	**0**
8	**1**	**4**	**2**	8	1	1	0				
10	1, 2	3	1	**10**	**1, 2**	**2**	**1**	10	2	1	0
12	**1**	**5**	**3**	12	1	1	0				
14	1, 2	3	1	14	1, 2	1	0	14	2	1	0
16	2	4	2	16	1, 2	2	1				
18	1, 2	3	1	18	1, 2	1	0	18	1	1	0
20	**2**	**6**	**4**	20	1, 2	1	0				
22	1–3	3	1	**22**	**1, 3**	**3**	**2**	**22**	**3**	**2**	**1**
24	2	4	2	24	1, 2	1	0				
26	1, 2	3	1	26	1, 2	1	0	26	1	1	0
28	2	5	3	28	1, 2	2	1				
30	1, 2	3	1	30	1, 2	1	0	30	2	1	0
32	2	4	2	32	1, 2	1	0				
34	1, 2	3	1	34	1, 2	2	1	34	2	1	0
36	**2**	**7**	**5**	36	1, 2	1	0				
38	1, 2	3	1	38	1, 2	1	0	38	2	1	0
40	1	4	2	40	1, 2	3	2				
42	1, 2	3	1	42	1, 2	1	0	42	1	2	1
44	2	5	3	44	1, 2	1	0				
46	1, 2	3	1	46	1, 2	2	1	46	1	1	0
48	2	4	2	48	1, 2	1	0				
50	1, 2	3	1	50	1, 2	1	0	50	1	1	0
52	2	6	4	52	1, 2	2	1				
54	1, 2	3	1	54	1, 2	1	0	54	2	1	0
56	2	4	2	56	1, 2	1	0				
58	1, 2	3	1	**58**	**1, 2**	**4**	**3**	58	1	1	0
60	2	5	3	60	1, 2	1	0				
62	1, 2	3	1	62	1, 2	1	0	62	2	2	1
64	1	4	2	64	1, 2	2	1				
66	1, 2	3	1	66	1, 2	1	0	66	1	1	0
68	**2**	**8**	**6**	68	1, 2	1	0				

Table B.3: $N=3$, $p=2,3,5$, $k_1^{(2)}=4$, $k_1^{(3)}=4$, $k_1^{(5)}=2$ and $k_2\leq 68$.

We have computed similar tables for all $k_1^{(2)}\leq 22$, $k_1^{(3)}\leq 6$ and $k_1^{(5)}\leq 4$.

Base level 4

Each of the spaces $S_4\big(\Gamma_0(8)\big)$, $S_4\big(\Gamma_0(12)\big)$ and $S_2\big(\Gamma_0(20)\big)$ contains a single newform, and we get the following table (we can again omit the index i):

$p=2$			$p=3$			$p=5$		
k_2	M	s	k_2	M	s	k_2	M	s
6	**3**	**1**	**6**	**1**	**0**	**6**	**1**	**0**
10	3	1	**10**	**2**	**1**	10	1	0
12	**5**	**3**	12	1	0			
14	3	1	14	1	0	14	1	0
16	4	2	16	2	1			
18	3	1	18	1	0	18	1	0
20	5	4	20	1	0			
22	3	1	**22**	**3**	**2**	**22**	**2**	**1**
24	4	2	24	1	0			
26	3	1	26	1	0	26	1	0
28	5	3	28	2	1			
30	3	1	30	1	0	30	1	0
32	4	2	32	1	0			
34	3	1	34	2	1	34	1	0
36	5	5	36	1	0			
38	3	1	38	1	0	38	1	0
40	4	2	40	3	2			
42	3	1	42	1	0	42	2	1
44	5	3	44	1	0			
46	3	1	46	2	1	46	1	0
48	4	2	48	1	0			
50	3	1	50	1	0	50	1	0
52	5	4	52	2	1			
54	3	1	54	1	0	54	1	0
56	4	2	56	1	0			
58	3	1	**58**	**4**	**3**	58	1	0
60	5	3	60	1	0			
62	3	1	62	1	0	62	2	1
64	4	2	64	2	1			
66	3	1	66	1	0	66	1	0
68	5	6	68	1	0			

Table B.4: $N=4$, $p=2,3,5$, $k_1^{(2)}=4$, $k_1^{(3)}=4$, $k_1^{(5)}=2$ and $k_2 \leq 68$.

We have computed similar tables for all $k_1^{(2)} \leq 10$, $k_1^{(3)} \leq 14$ and $k_1^{(5)} \leq 6$.

Base level 5

Each of the spaces $S_4\big(\Gamma_0(10)\big)$ and $S_2\big(\Gamma_0(15)\big)$ contains a single newform, while $S_4\big(\Gamma_0(25)\big)$ contains three, and we get the following table:

$p=2$				$p=3$				$p=5$				
k_2	i	M	s	k_2	i	M	s	k_2	i	j	M	s
				4	**1**	**1**	**0**					
6	**1**	**2**	**1**	6	1	1	0					
8	**1**	**3**	**2**	**8**	**2**	**2**	**1**	**8**	**1**	**1**	**1**	**0**
10	2	3	1	10	2	1	0	**8**	**2**	**1–3**	**1**	**0**
12	**1**	**4**	**3**	12	2	1	0	12	1	2, 3	1	0
14	2	3	1	14	2	2	1	12	2	1–3	1	0
16	1	3	2	16	2	1	0	16	1, 2	1–3	1	0
18	2	3	1	18	2	1	0					
20	**1**	**5**	**4**	**20**	**2**	**3**	**2**	20	1, 2	1–3	1	0
22	2	3	1	22	2	1	0					
24	1	3	2	24	2	1	0	**24**	**1**	**1,2**	**2**	**1**
26	2	3	1	26	2	2	1	**24**	**2**	**1–3**	**2**	**1**
28	1	4	3	28	2	1	0	28	1, 2	1–3	1	0
30	2	3	1	30	2	1	0					
32	1	3	2	32	2	2	1	32	1, 2	1–3	1	0
34	2	3	1	34	2	1	0					
36	**1**	**6**	**5**	36	2	1	0	36	1, 2	1–3	1	0
38	2	3	1	38	2	3	2					
40	1	3	2	40	2	1	0	40	1, 2	1–3	1	0
42	2	3	1	42	2	1	0					
44	1	4	3	44	2	2	1	44	1, 2	1–3	2	1
46	2	3	1	46	2	1	0					
48	1	3	2	48	2	1	0	48	1, 2	1–3	1	0
50	2	3	1	50	2	2	1					
52	1	5	4	52	2	1	0	52	1, 2	1–3	1	0
54	2	3	1	54	2	1	0					
56	1	3	2	**56**	**2**	**4**	**3**	56	1, 2	1–3	1	0
58	2	3	1	58	2	1	0					
60	1	4	3	60	2	1	0	60	1, 2	1–3	1	0
62	2	3	1	62	2	2	1					
64	1	3	2	64	2	1	0	64	1, 2	1–3	2	1
66	2	3	1	66	2	1	0					
68	**1**	**7**	**6**	68	2	1	0	68	1, 2	1–3	1	0

Table B.5: $N=5$, $p=2,3,5$, $k_1^{(2)}=4$, $k_1^{(3)}=2$, $k_1^{(5)}=4$ and $k_2 \le 68$.

We have computed similar tables for all $k_1^{(2)} \le 10$ and $k_1^{(3)} \le 6$.

Base level 6

Each of the spaces $S_4\big(\Gamma_0(12)\big)$, $S_4\big(\Gamma_0(18)\big)$ and $S_2\big(\Gamma_0(30)\big)$ contains a single newform, and we get the following table:

$p=2$				$p=3$				$p=5$			
k_2	i	M	s	k_2	i	M	s	k_2	i	M	s
6	**1**	**3**	**1**	**6**	**1**	**1**	**0**	**6**	**1**	**1**	**0**
8	**1**	**4**	**2**	8	1	1	0				
10	1	3	1	**10**	**1**	**2**	**1**	10	1	1	0
12	**1, 2**	**5**	**3**	12	1–3	1	0				
14	1	3	1	14	1	1	0	14	1	1	0
16	1	4	2	16	1, 2	2	1				
18	1–3	3	1	18	1–3	1	0	18	2	1	0
20	**1, 2**	**6**	**4**	20	1–3	1	0				
22	1–3	3	1	**22**	**3**	**3**	**2**	**22**	**2**	**2**	**1**
24	2, 4	4	2	24	1–4	1	0				
26	1–3	3	1	26	1–3	1	0	26	2	1	0
28	1, 3	5	3	28	1, 2	2	1				
30	1–4	3	1	30	1–4	1	0	30	4	1	0
32	2, 4	4	2	32	1–4	1	0				
34	1–4	3	1	34	3–4	2	1	34	4	1	0
36	**2, 3**	**7**	**5**	36	1–4	1	0				
38	1–4	3	1	38	1–4	1	0	38	4	1	0
40	1, 2	4	2	40	2, 3	3	2				
42	1–4	3	1	42	1–4	1	0	42	3	2	1
44	2, 3	5	3	44	1–4	1	0				
46	1–4	3	1	46	1, 4	2	1	46	3	1	0
48	2, 4	4	2	48	1–4	1	0				
50	1–4	3	1	50	1–4	1	0	50	3	1	0
52	1, 3	6	4	52	1, 2	2	1				
54	1–4	3	1	54	1–4	1	0	54	4	1	0
56	2, 4	4	2	56	1–4	1	0				
58	4–4	3	1	**58**	**3, 4**	**4**	**3**	58	4	1	0
60	2, 3	5	3	60	1–4	1	0				
62	1–4	3	1	62	1–4	1	0	62	4	2	1
64	1, 2	4	2	64	2, 3	2	1				
66	1–4	3	1	66	1–4	1	0	66	3	1	0
68	**2, 3**	**8**	**6**	68	1–4	1	0				

Table B.6: $N=6$, $p=2,3,5$, $k_1^{(2)}=4$, $k_1^{(3)}=4$, $k_1^{(5)}=2$ and $6 \leq k_2 \leq 68$.

We have computed similar tables for all $k_1^{(2)} \leq 14$, $k_1^{(3)} \leq 16$ and $k_1^{(5)} \leq 12$.

Base level 7

Each of the spaces $S_2\big(\Gamma_0(14)\big)$ and $S_2\big(\Gamma_0(21)\big)$ contains a single newform, and we get the following table:

$p = 2$				$p = 3$			
k_2	i	M	s	k_2	i	M	s
4	**1**	**3**	**1**	**4**	**1**	**1**	**0**
6	**2**	**4**	**2**	6	1	1	0
8	2	3	1	**8**	**2**	**2**	**1**
10	**2**	**5**	**3**	10	1	1	0
12	2	3	1	12	2	1	0
14	2	4	2	14	1	2	1
16	2	3	1	16	2	1	0
18	**2**	**6**	**4**	18	1	1	0
20	2	3	1	**20**	**2**	**3**	**2**
22	2	4	2	22	1	1	0
24	2	3	1	24	2	1	0
26	2	5	3	26	1	2	1
28	2	3	1	28	2	1	0
30	2	4	2	30	1	1	0
32	2	3	1	32	2	2	1
34	**2**	**7**	**5**	34	1	1	0
36	2	3	1	36	2	1	0
38	2	4	2	38	1	3	2
40	2	3	1	40	2	1	0
42	2	5	3	42	1	1	0
44	2	3	1	44	2	2	1
46	2	4	2	46	1	1	0
48	2	3	1	48	2	1	0
50	2	6	4	50	1	2	1
52	2	3	1	52	2	1	0
54	2	4	2	54	1	1	0
56	2	3	1	**56**	**2**	**4**	**3**
58	2	5	3	58	1	1	0
60	2	3	1	60	2	1	0
62	2	4	2	62	1	2	1
64	2	3	1	64	2	1	0
66	**2**	**8**	**6**	66	1	1	0

Table B.7: $N = 7$, $p = 2, 3$, $k_1^{(2)} = 2$, $k_1^{(3)} = 2$ and $4 \leq k_2 \leq 66$.

We have computed similar tables for all $k_1^{(2)} \leq 6$.

Base level 8

Each of the spaces $S_4\big(\Gamma_0(16)\big)$, $S_2\big(\Gamma_0(24)\big)$ and $S_2\big(\Gamma_0(40)\big)$ contains a single newform, and we get the following table:

$p=2$				$p=3$				$p=5$			
k_2	i	M	s	k_2	i	M	s	k_2	i	M	s
				4	**1**	**1**	**0**				
6	**1**	**3**	**1**	6	1	1	0	**6**	**1**	**1**	**0**
8	1, 2	3	2	**8**	**1**	**2**	**1**				
10	1, 2	3	1	10	2	1	0	10	1	1	0
12	1, 2	3	3	12	2	1	0				
14	1, 2	3	1	14	2	2	1	14	2	1	0
16	1–3	3	2	16	3	1	0				
18	1, 2	3	1	18	1	1	0	18	2	1	0
20	1, 2	3	4	**20**	**2**	**3**	**2**				
22	1, 2	3	1	22	2	1	0	**22**	**2**	**2**	**1**
24	1, 2	3	2	24	1	1	0				
26	1, 2	3	1	26	2	2	1	26	1	1	0
28	1, 2	3	3	28	2	1	0				
30	1, 2	3	1	30	2	1	0	30	2	1	0
32	1, 2	3	2	32	1	2	1				
34	1, 2	3	1	34	2	1	0	34	1	1	0
36	1, 2	3	5	36	2	1	0				
38	1, 2	3	1	38	2	3	2	38	2	1	0
40	1, 2	3	2	40	1	1	0				
42	1, 2	3	1	42	1	1	0	42	2	2	1
44	1, 2	3	3	44	2	2	1				
46	1, 2	3	1	46	2	1	0	46	2	1	0
48	1, 2	3	2	48	1	1	0				
50	1, 2	3	1	50	1	2	1	50	2	1	0
52	1, 2	3	4	52	2	1	0				
54	1, 2	3	1	54	2	1	0	54	2	1	0
56	1, 2	3	2	**56**	**1**	**4**	**3**				
58	1, 2	3	1	58	1	1	0	58	2	1	0
60	1, 2	3	3	60	2	1	0				
62	1, 2	3	1	62	2	2	1	62	2	2	1
64	1, 2	3	2	64	2	1	0				
66	1, 2	3	1	66	2	1	0	66	1	1	0
68	1, 2	3	6	68	2	2	1				

Table B.8: $N=8$, $p=2,3,5$, $k_1^{(2)}=4$, $k_1^{(3)}=2$, $k_1^{(5)}=2$ and $k_2 \le 68$.

We have computed similar tables for all $k_1^{(2)} \le 10$, $k_1^{(3)} \le 8$ and $k_1^{(5)} \le 4$.

Base level 9

Each of the spaces $S_4\big(\Gamma_0(18)\big)$, $S_2\big(\Gamma_0(27)\big)$ and $S_2\big(\Gamma_0(45)\big)$ contains a single newform, and we get the following table:

$p=2$				$p=3$				$p=5$			
k_2	i	M	s	k_2	i	M	s	k_2	i	M	s
				4	**1**	**1**	**0**				
6	**1**	**3**	**1**	6	1	1	0	**6**	**1**	**1**	**0**
8	**1**	**4**	**2**	8	1, 2	1	1				
10	2, 3	3	1	10	1–3	1	0	10	3	1	0
12	**2**	**5**	**3**	12	1–3	1	0				
14	1, 3	3	1	14	1–3	1	1	14	3	1	0
16	4	4	2	16	1–5	1	0				
18	1, 2, 4	3	1	18	1–4	1	0	18	1	1	0
20	**3**	**6**	**4**	20	1–4	1	2				
22	2–4, 6	3	1	22	1–6	1	0	**22**	**6**	**2**	**1**
24	3	4	2	24	1–4	1	0				
26	1–3	3	1	26	1–4	1	1	26	2	1	0
28	4	5	3	28	1–5	1	0				
30	1–3	3	1	30	1–4	1	0	30	3	1	0
32	3	4	2	32	1–4	1	1				
34	2–4	3	1	34	1–5	1	0	34	4	1	0
36	**2**	**7**	**5**	36	1–4	1	0				
38	1–3	3	1	38	1–4	1	2	38	3	1	0
40	2	4	2	40	1–5	1	0				
42	1–3	3	1	42	1–4	1	0	42	1	2	1
44	3	5	3	44	1–4	1	1				
46	2–4	3	1	46	1–5	1	0	46	3	1	0
48	3	4	2	48	1–4	1	0				
50	1–3	3	1	50	1–4	1	1	50	2	1	0
52	3	6	4	52	1–5	1	0				
54	1–3	3	1	54	1–4	1	0	54	3	1	0
56	3	4	2	56	1–4	1	3				
58	2–4	3	1	58	1–5	1	0	58	3	1	0
60	3	5	3	60	1–4	1	0				
62	1–3	3	1	62	1–4	1	1	62	3	2	1
64	2	4	2	64	1–5	1	0				
66	1–3	3	1	66	1–4	1	0	66	1	1	0
68	**3**	**8**	**6**	68	1–4	1	1				

Table B.9: $N = 9$, $p = 2, 3, 5$, $k_1^{(2)} = 4$, $k_1^{(3)} = 2$, $k_1^{(5)} = 2$ and $k_2 \leq 68$.

We have computed similar tables for all $k_1^{(2)} \leq 16$ and $k_1^{(5)} \leq 4$.

Base level 10

Each of the spaces $S_2\big(\Gamma_0(20)\big)$ and $S_2\big(\Gamma_0(30)\big)$ contains a single newform, while $S_2\big(\Gamma_0(50)\big)$ contains two, and we get the following table:

$p = 2$

k_2	i	M	s
4	***1***	***1***	***1***
6	**1, 3**	**3**	**2**
8	*1*	*1*	*1*
10	**1, 2**	**4**	**3**
12	1, 3	2	1
14	1, 2	3	2
16	1, 2	2	1
18	**2, 4**	**5**	**4**
20	1, 2	2	1
22	2, 4	3	2
24	1, 3	2	1
26	2, 4	4	3
28	1, 2	2	1
30	2, 3	3	2
32	1, 3	2	1
34	**2, 3**	**6**	**5**

$p = 3$

k_2	i	M	s
4	**1**	**1**	**0**
6	1–3	1	0
8	**1**	**2**	**1**
10	1–3	1	0
12	1–4	1	0
14	2	2	1
16	1–4	1	0
18	1–4	1	0
20	**4**	**3**	**2**
22	1–4	1	0
24	1–4	1	0
26	4	2	1
28	1–4	1	0
30	1–4	1	0
32	4	2	1
34	1–4	1	0

$p = 5$

k_2	i	j	M	s
6	**1**	**1**	**1**	**0**
6	**2, 3**	**2**	**1**	**0**
10	1	2	1	0
10	2, 3	1	1	0
14	1	1	1	0
14	2, 3	2	1	0
18	1, 2	2	1	0
18	3, 4	1	1	0
22	**1, 2**	**1**	**2**	**1**
22	**3, 4**	**2**	**2**	**1**
26	1, 2	2	1	0
26	3, 4	1	1	0
30	1, 2	1	1	0
30	3, 4	2	1	0
34	1, 2	2	1	0
34	3, 4	1	1	0

Table B.10: $N = 10$, $p = 2, 3, 5$, $k_1^{(2)} = 2$, $k_1^{(3)} = 2$, $k_1^{(5)} = 2$ and $k_2 \leq 34$.

We have computed similar tables for all $k_1^{(2)} \leq 6$, $k_1^{(3)} \leq 12$ and $k_1^{(5)} \leq 6$.

Base level 12

Each of the spaces $S_2\big(\Gamma_0(24)\big)$ and $S_2\big(\Gamma_0(36)\big)$ contains a single newform, while $S_4\big(\Gamma_0(60)\big)$ contains two, and we get the following table:

$p=2$				$p=3$				$p=5$				
k_2	i	M	s	k_2	i	M	s	k_2	i	j	M	s
4	**1**	**3**	**1**	**4**	**1**	**1**	**0**					
8	1, 2	3	1	**8**	**1, 2**	**2**	**1**	**8**	**1**	**1, 2**	**1**	**0**
10	**1**	**5**	**3**	10	1	1	0					
12	1, 2	3	1	12	1, 2	1	0	12	2	1, 2	1	0
14	2	4	2	14	1, 2	2	1					
16	1, 2	3	1	16	1, 2	1	0	16	2	1, 2	1	0
18	2	5	4	18	1, 2	1	0					
20	1, 2	3	1	**20**	**1, 2**	**3**	**2**	20	2	1, 2	1	0
22	2	4	2	22	1, 2	1	0					
24	1, 2	3	1	24	1, 2	1	0	**24**	**1**	**1, 2**	**2**	**1**
26	2	5	3	26	1, 2	2	1					
28	1, 2	3	1	28	1, 2	1	0	28	1	1, 2	1	0
30	2	4	2	30	1, 2	1	0					
32	1, 2	3	1	32	1, 2	2	1	32	1	1, 2	1	0
34	2	5	5	34	1, 2	1	0					
36	1, 2	3	1	36	1, 2	1	0	36	2	1, 2	1	0

Table B.11: $N=12$, $p=2,3,5$, $k_1^{(2)}=2$, $k_1^{(3)}=2$, $k_1^{(5)}=4$ and $k_2 \leq 36$.

We have computed similar tables for all $k_1^{(2)} \leq 8$, $k_1^{(3)} \leq 8$ and $k_1^{(5)} \leq 8$.

Base level 15

Each of the spaces $S_2\big(\Gamma_0(30)\big)$ and $S_2\big(\Gamma_0(45)\big)$ contains a single newform, while $S_2\big(\Gamma_0(75)\big)$ contains three, and we get the following table:

$p=2$				$p=3$				$p=5$				
k_2	i	M	s	k_2	i	M	s	k_2	i	j	M	s
4	**2**	**3**	**1**	**4**	**1**	**1**	**0**					
6	**2**	**4**	**2**	6	$1,2$	1	0	**6**	**1**	**3**	**1**	**0**
								6	**2**	**1**	**1**	**0**
								6	**3**	**2–3**	**1**	**0**
8	$1,2$	3	1	**8**	**3**	**2**	**1**					
10	**3**	**5**	**3**	10	$2,3$	1	0	10	1	2	1	0
								10	$2,4$	1	1	0
12	$2,3$	3	1	12	$1,4$	1	0	10	3	$2,3$	1	0
14	3	4	2	14	$2,3$	2	1	14	$2,4$	$2,3$	1	0
16	$1,3$	3	1	16	$2,4$	1	0	14	3	1	1	0
18	**2**	**6**	**4**	18	$2,3$	1	0	18	$1,2$	$2,3$	1	0
20	$2,3$	3	1	**20**	**4**	**3**	**2**	18	$3,4$	1	1	0
22	$1,3$	4	2	22	1–3	1	0	**22**	**2**	**3**	**2**	**1**
								22	**3, 4**	**1**	**2**	**1**
24	$2,3$	3	1	24	$1,4$	1	0	**22**	**5**	**2, 3**	**2**	**1**
26	3	5	3	26	$2,3$	2	1	26	$1,3$	$2,3$	1	0
28	$2,4$	3	1	28	$1,3$	1	0	26	$2,4$	1	1	0
30	2	4	2	30	$2,3$	1	0	30	$1,2$	1	1	0
32	$2,3$	3	1	32	$1,4$	2	1	30	$3,4$	$2,3$	1	0
34	**3**	**7**	**5**	34	$1,3$	1	0	34	$1,4$	1	1	0
								34	$2,3$	$2,3$	1	0

Table B.12: $N=15$, $p=2,3,5$, $k_1^{(2)}=2$, $k_1^{(3)}=2$, $k_1^{(5)}=2$ and $k_2\leq 34$.

We have computed similar tables for all $k_1^{(2)}\leq 12$ and $k_1^{(3)}\leq 4$.

Base level 16

Each of the spaces $S_2\big(\Gamma_0(32)\big)$ and $S_2\big(\Gamma_0(48)\big)$ contains a single newform, while $S_2\big(\Gamma_0(80)\big)$ contains two, and we get the following table:

$p = 2$

k_2	i	M	s
4	**1**	**2**	**1**
6	$1, 2$	2	2
8	1–3	2	1
10	1–4	2	3
12	1–4	2	1
14	1–5	2	2
16	1–6	2	1
18	1–5	2	4
20	1–6	2	1
22	1–6	2	2
24	1–5	2	1
26	1–6	2	3
28	1–6	2	1
30	1–6	2	2
32	1–6	2	1
34	1–6	2	5

$p = 3$

k_2	i	M	s
4	**1**	**1**	**0**
6	2	1	0
8	**2**	**2**	**1**
10	2	1	0
12	4	1	0
14	5	2	1
16	6	1	0
18	3	1	0
20	**6**	**3**	**2**
22	6	1	0
24	4	1	0
26	6	2	1
28	6	1	0
30	6	1	0
32	5	2	1
34	6	1	0

$p = 5$

k_2	i	j	M	s
6	**1**	**2**	**1**	**0**
6	**2**	**1**	**1**	**0**
10	1	1	1	0
10	4	2	1	0
14	2	2	1	0
14	5	1	1	0
18	4	2	1	0
18	5	1	1	0
22	**4**	**2**	**2**	**1**
22	**6**	**1**	**2**	**1**
26	4	2	1	0
26	5	1	1	0
30	4	2	1	0
30	6	1	1	0
34	4	2	1	0
34	5	1	1	0

Table B.13: $N = 16$, $p = 2, 3, 5$, $k_1^{(2)} = 2$, $k_1^{(3)} = 2$, $k_1^{(5)} = 2$ and $k_2 \leq 34$.

We have computed similar tables for all $k_1^{(2)} \leq 4$, $k_1^{(3)} \leq 8$ and $k_1^{(5)} \leq 4$.

Base level 18

The space $S_2\big(\Gamma_0(36)\big)$ contains a single newform, $S_2\big(\Gamma_0(54)\big)$ contains two, while $S_2\big(\Gamma_0(90)\big)$ contains three, and we get the following table:

$p = 2$				$p = 3$					$p = 5$				
k_2	i	M	s	k_2	i	j	M	s	k_2	i	j	M	s
4	***1***	***1***	***1***	**4**	**1**	**1, 2**	**1**	**0**					
6	**1, 3**	**4**	**2**	6	1–3	1, 2	1	0	**6**	**1**	**1**	**1**	**0**
8	*1, 2*	*1*	*1*	**8**	**1**	**1**	**2**	**1**	**6**	**2**	**3**	**1**	**0**
				8	**2**	**2**	**2**	**1**	**6**	**3**	**2**	**1**	**0**
10	**1, 3**	**5**	**3**	10	1–4	1, 2	1	0	10	1	2	1	0
									10	2	3	1	0
12	1, 5	3	1	12	1–5	1, 2	1	0	10	3	1	1	0
14	2, 3	4	2	14	1, 2	2	2	1	14	2	1	1	0
				14	3–5	1	2	1	14	3	2	1	0
16	2, 6	3	1	16	1–6	1, 2	1	0	14	4	3	1	0
18	**5, 6**	**6**	**4**	18	1–6	1, 2	1	0	18	1	3	1	0
20	1, 5	3	1	20	1–4	1	2	2	18	5	2	1	0
				20	5–7	2	2	2	18	6	1	1	0
22	6, 7	4	2	22	1–7	1, 2	1	0	**22**	**4**	**3**	**2**	**1**
									22	**6**	**1**	**2**	**1**
24	5, 7	3	1	24	1–7	1, 2	1	0	**22**	**7**	**2**	**2**	**1**
26	5, 6	5	3	26	1–3, 5	2	2	1	26	3	3	1	0
				26	4, 6, 7	1	2	1	26	5	2	1	0
28	7, 8	3	1	28	1–8	1, 2	1	0	26	6	1	1	0
30	7, 8	4	2	30	1–8	1, 2	1	0	30	6	3	1	0
32	6, 8	3	1	32	1, 2, 5, 6	1	2	1	30	7	1	1	0
				32	3, 4, 7, 8	2	2	1	30	8	2	1	0
34	**7, 8**	**7**	**5**	34	1–8	1, 2	1	0	34	5	3	1	0
									34	7	2	1	0
									34	8	1	1	0

Table B.14: $N = 18$, $p = 2, 3, 5$, $k_1^{(2)} = 2$, $k_1^{(3)} = 2$, $k_1^{(5)} = 2$ and $k_2 \leq 34$.

We have computed similar tables for all $k_1^{(2)} \leq 8$, $k_1^{(3)} \leq 6$ and $k_1^{(5)} \leq 6$.

Base level 20

Each of the spaces $S_2\big(\Gamma_0(40)\big)$ and $S_2\big(\Gamma_0(100)\big)$ contains a single newform, while $S_2\big(\Gamma_0(60)\big)$ contains two, and we get the following table:

$p=2$				$p=3$					$p=5$			
k_2	i	M	s	k_2	i	j	M	s	k_2	i	M	s
4	**1**	**2**	**1**									
6	1	1	2	**6**	**1**	**1, 2**	**1**	**0**	**6**	**1**	**1**	**0**
8	**2**	**3**	**1**	8	1, 2	1, 2	1	0				
10	**1**	**4**	**3**	**10**	**1**	**2**	**2**	**1**	10	1, 2	1	0
				10	**2**	**1**	**2**	**1**				
12	2	3	1	12	1, 2	1, 2	1	0				
14	1	3	2	14	1, 2	1, 2	1	0	14	1, 2	1	0
16	2	3	1	16	1	2	2	1				
				16	2	1	2	1				
18	**1**	**5**	**4**	18	1, 2	1, 2	1	0	18	1, 2	1	0
20	2	3	1	20	1, 2	1, 2	1	0				
22	1	3	2	**22**	**1**	**2**	**3**	**2**	**22**	**2**	**2**	**1**
				22	**2**	**1**	**3**	**2**				
24	2	3	1	24	1, 2	1, 2	1	0				
26	1	4	3	26	1, 2	1, 2	1	0	26	1, 2	1	0
28	2	3	1	28	1	2	2	1				
				28	2	1	2	1				
30	1	3	2	30	1, 2	1, 2	1	0	30	1, 2	1	0
32	2	3	1	32	1, 2	1, 2	1	0				
34	1	5	5	34	1	2	2	1	34	1, 2	1	0
				34	2	1	2	1				
36	2	3	1	36	1, 2	1, 2	1	0				

Table B.15: $N = 20$, $p = 2, 3, 5$, $k_1^{(2)} = 2$, $k_1^{(3)} = 4$, $k_1^{(5)} = 2$ and $k_2 \leq 36$.

We have computed similar tables for all $k_1^{(2)} \leq 4$ and $k_1^{(3)} \leq 8$..

Base level 24

Each of the spaces $S_2\big(\Gamma_0(48)\big)$ and $S_2\big(\Gamma_0(72)\big)$ contains a single newform, while $S_2\big(\Gamma_0(120)\big)$ contains two, and we get the following table:

$p=2$				$p=3$				$p=5$				
k_2	i	M	s	k_2	i	M	s	k_2	i	j	M	s
4	**1**	**3**	**1**	**4**	**1**	**1**	**0**					
6	1–3	3	2	6	1, 2	1	0	**6**	**2**	**2**	**1**	**0**
8	1–3	3	1	**8**	**1**	**2**	**1**	**6**	**3**	**1**	**1**	**0**
10	1–4	3	3	10	1, 3	1	0	10	1	1	1	0
12	1–4	3	1	12	2, 4	1	0	10	2	2	1	0
14	1–4	3	2	14	2, 3	2	1	14	3	2	1	0
16	1–4	3	1	16	1, 3	1	0	14	4	1	1	0
18	1–4	3	4	18	1, 3	1	0	18	1	1	1	0
20	1–4	3	1	**20**	**4**	**3**	**2**	18	2	2	1	0
22	1–4	3	2	22	2, 3	1	0	**22**	**3**	**2**	**2**	**1**
24	1–4	3	1	24	1, 2	1	0	**22**	**4**	**1**	**2**	**1**
26	1–4	3	3	26	1, 3	2	1	26	1	1	1	0
28	1–4	3	1	28	3, 4	1	0	26	2	2	1	0
30	1–4	3	2	30	2, 4	1	0	30	3	1	1	0
32	1–4	3	1	32	1, 2	2	1	30	4	2	1	0
34	1–4	3	5	34	2, 3	1	0	34	1	2	1	0
								34	2	1	1	0

Table B.16: $N=24$, $p=2,3,5$, $k_1^{(2)}=2$, $k_1^{(3)}=2$, $k_1^{(5)}=2$ and $k_2\leq 34$.

We have computed similar tables for all $k_1^{(2)}\leq 8$, $k_1^{(3)}\leq 6$ and $k_1^{(5)}\leq 4$.

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DEPARTMENT OF MATHEMATICAL SCIENCES
UNIVERSITY OF COPENHAGEN
UNIVERSITETSPARKEN 5
DK-2100 COPENHAGEN
DENMARK
EMAIL IMF@MATH.KU.DK
WWW.MATH.KU.DK

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