Risk Analyses of Financial Derivatives and Structured Products

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PhD Thesis
2009

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Preface

The work in this thesis was conducted in the period December 2006 to November 2009 under the supervision of Professor Rolf Poulsen. It is written in partial fulfilment of the requirements for achieving a PhD degree in finance at the Department of Mathematical Sciences under the Faculty of Science at University of Copenhagen.

The thesis consists of three papers, which study different risk aspects of pricing, issuing and investing in financial instruments. Two of the papers consider structured products: the constant proportion debt obligation, which is a structured credit derivative, and constant proportion portfolio insurance, which is a trading strategy ensuring investors a minimum return. These papers are of purely theoretical character and conduct numerical experiments to infer knowledge about the products. The third paper is an empirical study of the valuation of barrier options on a foreign exchange rate. The three papers can be read independently and are on track for submission to journals on quantitative finance.

Financial support from the Faculty of Science, which made this work possible, is gratefully acknowledged. In addition I would like to thank the Oticon Foundation who supported my attendance in several workshops.

I would like to take this opportunity to warmly thank Rolf Poulsen for his excellent supervision and valuable advice during the PhD programme, and for generously sharing his wide knowledge of the field of finance. Furthermore, I thank Rama Cont for his hospitality during my visit at Columbia University in the autumn 2007, as well as for inspiring and rewarding discussions. I wish to express my gratitude to my colleagues of the Department of Mathematical Sciences for the support and interest shown. I would like to thank friends and family for their support, especially Pernille: it has been a true privilege to have a sister doing a PhD within the same field as I. Finally, I thank Carsten for his love, confidence and constant encouragement.

Cathrine Jessen
Copenhagen, November 2009

List of papers

This thesis is based on the following three papers:

A. Cathrine Jessen & Rolf Poulsen (2009): *Empirical Performance of Models for Barrier Option Valuation.*

B. Cathrine Jessen (2009): *Constant Proportion Portfolio Insurance: Discrete-time Trading and Gap Risk Coverage.*

C. Rama Cont & Cathrine Jessen (2009): *Constant Proportion Debt Obligations: Modelling and Risk Analysis*; submitted for publication.

Abstracts

A. Empirical Performance of Models for Barrier Option Valuation

Joint with Rolf Poulsen, University of Copenhagen

In this paper the empirical performance of five different models for barrier option valuation is investigated: the Black-Scholes model, the constant elasticity of variance model, the Heston stochastic volatility model, the Merton jump-diffusion model, and the infinite activity Variance Gamma model. We use time-series data from the USD/EUR exchange rate market: standard put and call (plain vanilla) option prices and a unique set of observed market values of barrier options. The models are calibrated to plain vanilla option prices, and prediction errors at different horizons for plain vanilla and barrier option values are investigated. For plain vanilla options, the Heston and Merton models have similar and superior performance for prediction horizons up to one week. For barrier options, the continuous-path models (Black-Scholes, constant elasticity of variance, and Heston) do almost equally well, while both models with jumps (Merton and Variance Gamma) perform markedly worse.

B. Constant Proportion Portfolio Insurance: Discrete-time Trading and Gap Risk Coverage

This paper studies constant proportion portfolio insurance (CPPI) in a setup that accounts for market frictions. Trading costs, fees and borrowing restrictions are incorporated, and the assumption of continuous portfolio rebalancing is relaxed. The main goals are to cover issuer's gap risk and to maximize CPPI performance over possible multipliers; the proportionality

factor that determines the risky exposure of a CPPI. Investment objectives are described by the Sortino ratio and alternatively by a kinked constant relative risk aversion utility function. Investors with either objective will choose a lower multiplier than if CPPI performance is measured by the expected return. Discrete-time trading requires a portfolio rebalancing rule, which affects both performance and gap risk. Two commonly applied strategies, rebalancing at equidistant time steps and rebalancing based on market movements, are compared to a new rule, which takes trading costs into account. While the new and the market-based rules deliver similar CPPI performance, the new rebalancing rule achieves this by fewer trading interventions. Issuer's gap risk can be covered by a fee charge, hedging or by an artificial floor. A new approach for determining the artificial floor is introduced. Even though all three methods reduce losses from gap events effectively, the artificial floor and hedging are less costly to the investor.

C. Constant Proportion Debt Obligations (CPDOs): Modelling and Risk Analysis

Joint with Rama Cont, Columbia University

Constant proportion debt obligations (CPDOs) are structured credit derivatives indexed on a portfolio of investment grade debt, which generate high coupon payments by dynamically leveraging a position in an underlying portfolio of index default swaps. CPDO coupons and principal notes received high initial ratings from the major rating agencies, based on complex models for the joint transition of ratings and spreads for underlying names. We propose a parsimonious model for analysing the performance of CPDOs using a top-down approach which captures essential risk factors of the CPDO. Our analysis allows to compute default probabilities, loss distributions and other tail risk measures for the CPDO strategy and to analyse the dependence of these risk measures on parameters describing the risk factors. Though the probability of the CPDO defaulting on its coupon payments is found to be small, the ratings obtained strongly depend on the credit environment. CPDO loss distributions are found to be bimodal and our results point to a heterogeneous range of tail risk measures inside a given rating category, suggesting that credit ratings for such complex leveraged strategies should be complemented by other risk measures for the purpose of performance analysis. A worst-case scenario analysis indicates that CPDOs have a high exposure to persistent spread-widening scenarios. By calculating rating transition probabilities we find that ratings can be quite unstable during the lifetime of the CPDO.

1. Introduction

This thesis considers different risk aspects of trading in financial derivatives and structured products.

A financial derivative is an agreement to exchange cash or assets over time given some condition on the value of an underlying asset. In general, the purpose of derivatives trading is to transfer risk. Derivatives can be used for insurance purposes as provided by put options, for speculating in future market moves and for providing certainty in future cashflows as e.g. achieved by an interest rate swap. Derivatives are also used for hedging risky positions: by entering a derivative contract whose value moves in the opposite direction of the underlying portfolio, part of the risk can be mitigated.

Structured products are pre-specified trading strategies in an asset or a financial derivative. These strategies are typically designed to meet specific investment objectives that are not otherwise achievable by the financial instruments available in the market. In theory investors could simply follow the investment strategy themselves, but the costs and transaction volumes are beyond the scope of many individual investors. Dynamic trading strategies can provide principal protection, enhanced returns or reduced risk within an investment. A portfolio insurance strategy is studied in chapter 3.

The use of complex structured products, and especially credit products, has been criticized for playing a part in the financial crisis. Investors' demand for high return in the low spread environment prior to the crisis prompted financial institutions to develop complex structured products. Some of these involved structuring and repackaging of risky assets that made it (almost) impossible to track underlying risk factors. The constant proportion debt obligation studied in chapter 4 is a structured credit product that was created to generate high coupons at a low risk, however in hindsight the underlying risk factors were not fully understood.

Financial institutions, that issue structured products or act as intermediaries for financial derivatives trading, need efficient risk management tools. Institutions typically cover their risks by selling (part of) the risk in the market or by hedging, and require accurate methods for pricing the derivatives on their books. A range of models for derivatives pricing exist, nevertheless, they may produce quite different results. In chapter 2 this problem of model risk for barrier option valuation is considered.

The following sections introduce the three financial products considered in the thesis: barrier options, constant proportion portfolio insurance and constant proportion debt obligations. Furthermore, motivation for studying different risk aspects of pricing, issuing and investing in these products are provided. Chapters 2–4 contain the papers in their full length.

1.1 Barrier options

Barrier options belong to a group of financial derivatives called exotic options. An exotic option is characterized by having a payoff that is contingent on the path of the underlying asset up to expiry. In contrast, the payoff of a plain vanilla (put and call) option depends only on the value of the underlying asset at expiry. A barrier option delivers a payoff of put or call type and is initiated (knocked-in) or exterminated (knocked-out) if the value of the underlying asset crosses a pre-specified barrier prior to expiry. For example, while an ordinary call option on an asset with value S_T at expiry T and strike K pays $[S_T - K]^+ := \max\{S_t - K, 0\}$, the corresponding down-and-out call option with barrier B pays

$$\begin{cases} [S_T - K]^+ & \text{if } \forall t < T : \ S_t > B \\ \quad 0 & \text{else} \end{cases} .$$

I.e. the option is knocked-out if the value of the underlying asset at some point prior to expiry falls below the barrier. More complex types of barrier options exist (double barrier options, options with time-dependent barriers, barrier options paying a rebate at knock-out, etc.), however, this thesis focuses on single barrier options with no rebate.

The barrier events characterizing single barrier options can be divided into four categories: up-and-in, down-and-in, up-and-out and down-and-out. Each of these can be combined with a payoff of put or call type resulting in a total of eight barrier option varieties. Barrier options whose plain vanilla counterpart is in-the-money, when the barrier event happens are called reverse barrier options. These are down-and-out puts, up-and-out calls and their knock-in counterparts.

The path dependency of barrier options makes analytical valuation complex and in some models even impossible. Therefore, numerical pricing methods are often used. Accurate approximations can be hard to obtain, though, especially for reverse barrier options, which have discontinuous payoff functions. Model risk is another challenge when valuing barrier options; it has been documented (e.g. by Schoutens, Simons & Tisteart (2004) and Hirsa, Courtadon & Madan (2002)), that while different models price plain vanilla options similarly, they may produce markedly different barrier option prices.

A barrier option can be an attractive alternative to its plain vanilla counterpart, because it has a lower price due to its payoff being conditioned on the occurrence of a barrier event. Barrier options are traded over the counter, which means that buyer and seller negotiate the terms of the contract directly. Therefore price information for barrier options is generally not accessible. In comparison, plain vanilla options are exchange traded derivatives, where price information is publicly available. For this reason empirical studies of barrier options are scarce.

The valuation of barrier options within different models has previously been considered by Maruhn, Nalholm & Fengler (2009) and An & Suo (2009) who study hedge portfolios for barrier options. Yet, actual market values of barrier options are absent from both studies. Few papers investigate actual market values of barrier options: Easton, Gerlach, Graham & Tuyl (2004) and Wilkens & Stoimenov (2007) work solely in realm of the Black-Scholes model, while the empirical application of Carr & Crosby (2009) is limited to two specific days.

Paper A *Empirical Performance of Models for Barrier Option Valuation* is an empirical investigation of how well different models work for barrier option valuation. The paper employs a unique data-set of values of barrier options on the USD/EUR exchange rate obtained from the risk-management department of the largest Danish bank.

We study the issue of model risk by comparing the prices produced by five different models: the Black-Scholes model, the constant elasticity of variance model, Heston's stochastic volatility model, Merton's jump-diffusion model, and the infinite jump activity Variance Gamma model. These models are commonly used in the literature, and are qualitatively different: one model with state dependent volatility versus one with stochastic volatility, and a model with low jump activity and large jumps versus one with high jump activity and small jumps.

We choose an experimental design that reflects market practice: the models are calibrated to liquid plain vanilla options and the calibrated models are then used to price barrier options. Closed-form formulas for barrier option prices exist in the Black-Scholes model, whereas prices in the other models are found by numerical methods. We test the predictive qualities of the models over a horizon h by comparing observed option values to model values using h days old parameters. Such conditional predictions are important for example in a context where plain vanilla options are used as hedge instruments to create portfolios that are immunized to changes in state variables.

For pricing plain vanilla options we find that the Heston and Merton models have comparable performance, and that this performance is superior to that of the three other models. With respect to barrier option valuation, the continuous-path models (Black-Scholes, constant elasticity of variance and Heston) perform similarly, while both models with jumps (Merton and Variance Gamma) are very inaccurate. General for all the models is that pricing errors for reverse barrier options are larger than for the non-reverse. The good behaviour of the Black-Scholes model could be explained as a self-fulfilling prophesy if market participants use (variations of) the Black-Scholes model for valuation, however, this cannot be the only reason, since the results are invariant to the prediction horizon.

1.2 Constant proportion portfolio insurance

Portfolio insurance is a portfolio management technique for ensuring a lower bound on the portfolio value at a given maturity. Theoretically, insurance against unfavourable market scenarios for some asset can be obtained by investing in a put option with strike equal to the desired lower bound. However, such a put option may not be available in the market, for example if the investment horizon is long. Still it is possible (if assuming complete markets) to replicate the payoff of the put option by trading in the underlying and a risk free asset. This portfolio insurance technique is referred to as option based portfolio insurance (OBPI) and dates back to Leland & Rubinstein (1976) and Brennan & Schwartz (1976).

Another popular portfolio insurance strategy is constant proportion portfolio insurance (CPPI) introduced by Perold (1986) and Black & Jones (1987). A CPPI provides a capital guarantee by a dynamical investment strategy in an underlying risky asset and the risk free asset. The risky exposure is a constant proportion, called the multiplier, of the excess of the portfolio value over the present value of the capital guarantee, referred to as the floor. If the portfolio value falls below the floor, all funds are invested in risk free assets in order not to jeopardize the guarantee any further. The main advantages of the CPPI over the OBPI strategy[1] are its simplicity, and its flexibility in the choice of multiplier, which can be specified to accommodate investor's risk appetite.

The CPPI strategy is widely used in the financial industry (see Pain & Rand (2008)). Typical buyers are large individual investors and institutional investors such as pension funds. A variety of underlying asset classes can be employed; stock indices, hedge funds, corporate bonds, property and credit default swaps. An implication of the CPPI trading rule is that in a falling market assets are sold at a lower price than they were bought. Because of this trend-chasing behaviour ("buy high, sell low"), portfolio insurance strategies were criticized for worsening the stock market decline in October 1987 (Shiller (2001)), and were also linked to the collapse of Long-Term Capital Management (LTCM) in 1998 (Pain & Rand (2008)).

The CPPI trading rule ensuring the capital guarantee relies on an assumption of frictionless markets. In a stylized setup analytical tractability of the strategy is preserved as demonstrated by Cont & Tankov (2009) in a general Lévy framework. However, when including trading costs in the setup, continuous portfolio rebalancing becomes too expensive and some discrete-time trading rule must be employed. This will typically compromise the analytical tractability. Furthermore, since the CPPI issuer has effectively sold a capital guarantee to the investor, this must be honoured even if the portfolio value should fall below the floor. The risk of such event is referred to as gap risk. A gap event may happen between

[1] Bertrand & Prigent (2005) provide a comparison of option based and constant proportion portfolio insurance techniques.

portfolio adjustments or due to jumps in the risky asset.

Paper B *Constant Proportion Portfolio Insurance: Discrete-time Trading and Gap Risk Coverage* investigates the CPPI strategy in a setting including market frictions. Previous literature (e.g. Black & Perold (1992) and Paulot & Lacroze (2009)) study the CPPI strategy in a discrete-time setting, however, the choice of portfolio rebalancing rule has not been considered explicitly. Too rare rebalancing increases issuer's gap risk and too frequent rebalancing imposes high trading costs. In paper B, the choice of rebalancing rule is considered. I introduce a new rebalancing rule that depends on trading costs. This is found to be superior to rebalancing at equidistant time steps (e.g. once a week) and to deliver similar CPPI performance as rebalancing after pre-specified market moves. Yet, the new rebalancing rule achieves this by fewer trading interventions.

The new trading rule is inspired by results from the barrier option literature (Whalley & Wilmott (1997) and Poulsen & Siven (2008)). I establish a useful link to barrier options: future cashflows of a CPPI are equivalent to those from a position in down-and-out call options.[2]

Secondly, I compare different approaches for covering the CPPI issuer's gap risk. In the financial industry it is common practice to charge a (semi-) annual fee to cover losses incurred due to a gap event. Alternatively, gap risk can be hedged by a position in short maturity put options as suggested by Cont & Tankov (2009). A third approach is to introduce an artificial floor, which gives a buffer to absorb (part of) the losses. Paper B provides a theoretical foundation for choosing the buffer size by applying the results of Broadie, Glasserman & Kou (1997) for pricing discretely monitored barrier options. Both the artificial floor and hedging reduce gap risk effectively at a small cost to the investor, although the artificial floor approach has the advantage of not relying on availability of specific hedge instruments.

1.3 Constant proportion debt obligations

The benign credit environment in the period 2003–2006 induced financial institutions to develop structured credit products that offered higher re-

[2]Since the relation is based on cashflows, model risk as studied in paper A is not an issue in this context.

turns. One invention was the constant proportion debt obligation (CPDO) first offered by ABN AMRO in August 2006.

A CPDO is a dynamically leveraged credit trading strategy, which aims at generating high coupon payments by selling default protection on a portfolio of investment-grade debt. Exposure is taken in an unfunded format, which allows the CPDO to obtain high leverage without requiring additional capital. Main risk factors of this strategy are default risk in the underlying portfolio and spread risk, since the position in index default swaps is marked-to-market regularly.

CPDOs appeal not only to investors in search of yield, but also to investors who demand highly rated products due to the Basel II regulatory capital requirements. Both CPDO coupons and principal note were initially given top ratings by the major rating agencies. The rating caused controversy in the financial press, since CPDOs are exposed not only to credit risk, the assessment of which is considered the core competence of rating agencies, but also to market risk while operating with a high leverage factor. In hindsight, the market risk was greatly underestimated. Later, rating agencies have tightened their rating criteria for products that are highly sensitive to market risk of an underlying portfolio (Basel Committee on Banking Supervision (2008)).

The CPDO strategy is often compared to a credit CPPI with index credit default swaps as underlying. Unlike credit CPPIs, CPDOs do not offer principal protection, and investors are therefore potentially exposed to a full capital loss. Instead the CPDO delivers a high coupon, although at the cost of not having an upside potential. The main difference, though, is in the leverage mechanism; CPDO leverage is reduced when the transaction performs well and increased when losses occur. Thereby, the CPDO is a "buy low, sell high" strategy, as opposed to the CPPI.

Like with portfolio insurance, the CPDO strategy has been criticized for its impact on the market, in this case the index default swap market (Parker & Perzanowki (2008)). When CPDOs first appeared in the low spread environment, they sold large amounts of protection, and thus potentially drove index spreads further down. In contrast to the CPPI strategy, when CPDOs are first issued they tend to dampen volatility by buying back protection if spreads narrow, and vice versa. An important caveat, though, is that a spread widening may cause CPDOs to default and unwind all positions. The spread widening during the financial crisis caused the first CPDO to default in November 2007, see Wood (2007). By the end of 2008 the majority of CPDOs had defaulted or were bought back by issuers. The forced termination of massive amounts of default protection allegedly caused spreads to widening faster than they otherwise would have (Parker & Perzanowki (2008) and Basel Committee on Banking Supervision (2008)).

The CPDO strategy has mainly been studied by rating agencies (Wong, Chandler, Polizu, McCabe, Landschoot, Venus, Ding & Watson (2007), Linden, Neugebauer, Bund, Schiavetta, Zelter & Hardee (2007) and Jobst, Xuan, Zarya, Sandstrom & Gilkes (2007)) and by issuing entities (Varloot, Charpin & Charalampidou (2007)). The credit ratings given by the major rating agencies were based on complex models for the joint transition of ratings and spreads for all names in the underlying portfolio. In paper C *Constant Proportion Debt Obligations (CPDOs): Modelling and Risk Analysis* we present a parsimonious model for analysing the performance and risks of CPDO strategies. We model the index default intensity by a one factor top-down approach. This captures the risk features of a CPDO in a meaningful yet simple way, and allows us to study credit ratings, default probabilities, loss distributions and different tail risk measures.

Though the probability of the CPDO defaulting on its coupon payments is found to be small, the ratings obtained strongly depend on the credit environment. CPDOs are found to be less sensitive to default risk than to movements of spreads, and behave in this respect more like path-dependent derivatives on the index spread. Our scenario analysis clearly indicates that the worst case scenario for a CPDO manager is that of a sustained period of spread widening; the scenario that precisely happened during the financial crisis and resulted in the forced unwinding of many CPDOs as predicted by our analysis.

We find that over the lifetime of a CPDO its high parameter sensitivity leads to higher variability of the rating compared to standard top-rated products and is one of the main criticisms of CPDOs receiving a top rating. Furthermore, our analysis shows that within a given rating category a wide range of expected shortfalls may be observed. This leads us to conclude that for such complex products ratings tend to be misleading and cannot replace a detailed risk analysis.

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2. Empirical Performance of Models for Barrier Option Valuation[1]

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Abstract: In this paper the empirical performance of five different models for barrier option valuation is investigated: the Black-Scholes model, the constant elasticity of variance model, the Heston stochastic volatility model, the Merton jump-diffusion model, and the infinite activity Variance Gamma model. We use time-series data from the USD/EUR exchange rate market: standard put and call (plain vanilla) option prices and a unique set of observed market values of barrier options. The models are calibrated to plain vanilla option prices, and prediction errors at different horizons for plain vanilla and barrier option values are investigated. For plain vanilla options, the Heston and Merton models have similar and superior performance for prediction horizons up to one week. For barrier options, the continuous-path models (Black-Scholes, constant elasticity of variance, and Heston) do almost equally well, while both models with jumps (Merton and Variance Gamma) perform markedly worse.

Keywords: Barrier option valuation, empirical performance

[1]We are deeply indebted to Morten Nalholm for his help cleaning and organizing the barrier option data, and to Fiodar Kilin for his help calibrating the Bates and VG-CIR models.

2.1 Introduction

This paper is an empirical investigation of how well different models work for barrier option valuation. The study is performed using a unique dataset of exchange rate barrier option values.

If we were bold, we would add the qualifiers "first" and "truly" to the word "empirical" in the opening paragraph. That, however, would be pushing the envelope as empirical studies of barrier options are not completely absent from the literature. The performance of "held-until-expiry" hedge portfolios for barrier options on the German DAX index is tested by Maruhn et al. (2009), and An & Suo (2009) investigate hedge portfolios for USD/EUR exchange rate barrier options by "marking-to-model". Actual market values of barrier options are, though, absent from both studies. We know of three previous papers that look at market values of barrier options. Easton et al. (2004) investigate Australian exchange traded index barrier options, and Wilkens & Stoimenov (2007) study the embedded barrier option in the German Turbo Certificates. But these both work solely in realm of the Black-Scholes model. Carr & Crosby (2009) offer an ingenious model construction that allows for efficient pricing of barrier options but their empirical application is limited to illustrative calibrations for two specific days.

A variety of empirical designs can be used when investigating model performance across time and markets (underlying, *plain vanilla* across strikes and expiry-dates, and exotics). We use one that resembles how the models are used by market participants without violating the basic premise of what constitutes a model. Parametric models are calibrated to liquid plain vanilla options and then used to value exotic options. While this (re-)calibration practice is almost impossible to justify theoretically, a model that does not get the basic contracts about right does not come across as trustworthy when it comes to valuing more advanced products. More specifically, our experimental design is this: on any given day in the sample, say t, each model's parameters are chosen to obtain the best fit of that day's plain vanilla option prices across strikes and expiry-dates. We calculate within-that-day ("horizon-0") pricing errors by comparing observed option values to model values. This is done separately for plain vanilla ("in-sample") and for barrier options ("out-of-sample" or more tellingly "out-of-market"). We then test the predictive qualities of the models over the horizon h by keeping the time-t calibrated parameters fixed, updating state variables (underlying and possibly volatility) and options (plain vanilla and barrier) to their time-$(t + h)$ values, and registering the discrepancies between model and market values. Some may frown at our use of the word "prediction" and say that we should at least add "conditionally on state variables", or better yet say that we test "parameter stability". That is a matter of taste, but what is not is that such conditional predictions are exactly what matter in a context where plain vanilla options are used as hedge instruments where the idea is to create portfolios that are immunized to changes in state variables.

We work with five popular, yet qualitatively different parametric models: the Black-Scholes model, the constant elasticity of variance model, the Heston stochastic volatility model, the Merton jump-diffusion model, and the infinite activity Variance Gamma model.

For the plain vanilla options we find that the Heston and Merton models have similar performance, and that this performance is superior to the three other models' at horizons of up to five days.

For the barrier options, the performance of the continuous-path models (Black-Scholes, constant elasticity of variance and Heston) is quite similar, and better than those in the few reported previous studies, which all deal with equity markets. And as a general rule, the performance is "half an order of magnitude" worse than for plain vanilla options; more for barrier options whose plain vanilla counterpart is in-the-money when the barrier event happens, less in the opposite case. Both models with jumps (Merton and Variance Gamma) fail miserably for barrier options. These results hold not only at horizon-0, which could be seen as a self-fulfilling prophesy if market participants use Black-Scholes'ish models for valuation, but for predictions at all horizons.

The rest of the paper is organized as follows. Section 2.2 describes the data-sets in detail, section 2.3 reviews the different models and option pricing techniques, section 2.4 reports the results of the empirical analysis, and section 2.5 briefly concludes and outlines topics for future research.

2.2 Data

Our study combines data from two independent sources.[2] Plain vanilla option prices on the USD/EUR exchange rate come from British Bankers' Association.[3]

For each day, we have observations of options with expiries in 1 week, 1 month, 3 months, 6 months, 1 year and 2 years; for the 1 month, 3 months and 1 year expiries we further have prices of options with strikes (roughly) 5% under and 5% over the current exchange rate. In all, 12 plain vanilla option prices are observed each day.

Figure 2.1 shows the data, with option prices being expressed through their implied volatilities. Implied volatility is not constant across time (it decreases throughout our sample), expiry (it increases with time to expiry), or strike (it increases as strike moves away from spot). Compared to equity options, these implied volatilities display a fairly symmetric smile across strikes on average, but there is a randomly varying asymmetry as measured by the skew, i.e. the difference between high- and low-strike implied volatilities.[4] The British Bankers' Association data does not give information about bid/ask-spreads, but according to Wystup (2007) a multiplicative spread on volatilities of 1-2% is (or: was at that time) common for at-the-money options in the Interbank market. Or in numerical terms: a typical at-the-money option is sold at 0.101, bought at 0.099.

The exchange rate barrier option data-set stems from the risk-management department of Danske Bank; the largest Danish bank. Every day the department calls (or: sends a spreadsheet to) the bank's foreign exchange trading desk asking for valuations of all the exchange rate barrier options that the bank currently has on its books. We see no indications in the data that the trading desk is not "doing its job properly"[5] – such as stale

[2] To be entirely precise: three independent sources. To further enhance the data quality, we cross-checked exchange and interest rates against the FED Release H.15.

[3] Historic data on so-called benchmark exchange rate option volatility can be found at `http://www.bba.org.uk/bba/jsp/polopoly.jsp?d=129&a=799`. This admirable free service was discontinued in early 2008 – possibly not completely surprising as the data quality had deteriorated noticeably throughout 2007.

[4] Others have noticed this and proposed stochastic skew models; Carr & Wu (2007) do it in a Levy setting and in unpublished work Nicole Branger and co-authors use a diffusion framework. We leave the investigation of these models to future work.

[5] The fact that the life-span of the barrier options is much shorter than the "bonus

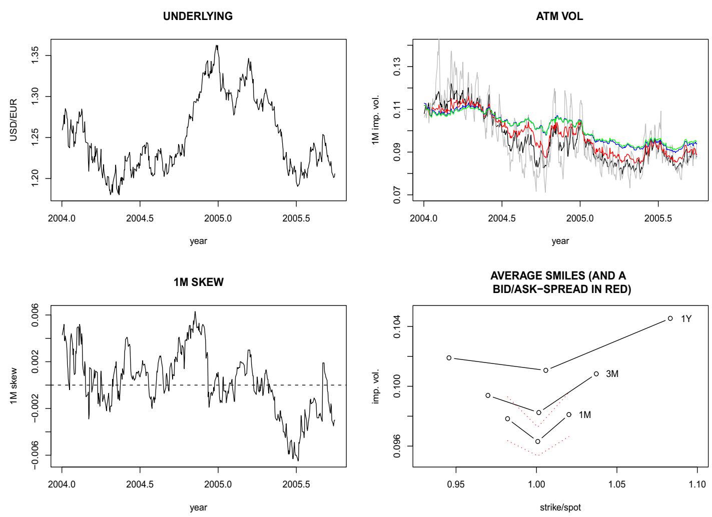

Figure 2.1: The plain vanilla data from British Bankers' Association. The top left graph is the USD/EUR exchange rate, i.e. the number of US dollars one has to pay to get 1 Euro. The top right panel is the implied volatility of at-the-money options with one week (gray), one month (black), three months (red), one year (green) and two years (blue) to expiry. The "smiles" in the bottom right panel are the time-averages of implied volatilities (connected) across strikes for expiries of 1 month (lower), 3 months (middle) and 1 year (upper). The dotted red curves around the 1 month smile indicate typical Interbank market bid/ask-spreads. The "skew" depicted in the bottom left graph is the time-series behaviour of the difference between the right and left end-points of the 1 month implied volatility curve.

quotes or suspiciously consistent over- or undervaluations of particular trades. The data-set contains values and characteristics for USD/EUR barrier options, but all proprietary information such as counter-party, size and direction of position, and initial price at which the option was sold to (or bought from, but it seems a safe bet that the bank is mostly short in barrier options) the counter-party has been removed. This would of course make for interesting reading and research, but on the plus-side the lack of sensitive information means that the data has been released for research

horizon" should alleviate moral hazard.

without "strings attached". More specifically, the data-set consists of daily observations of continuously monitored zero-rebate barrier contracts covering the period January 2, 2004 to September 27, 2005. We consider only single barrier options and disregard options with values lower than 10^{-5} and/or less than 7 days to expiry (thus staying within the expiry-range of the plain vanilla calibration instruments). This leaves us with a total of 3,108 observations on 156 individual contracts. These are broken down by characteristics in table 2.1.

	#contracts	#obs	$\overline{\tau_0}$	$\overline{\tau_i}$	$\overline{B/X_0}$	$\overline{K/X_0}$
Total	156	3,108	69	70	–	–
Reverse						
up-and-out call	38	535	44	40	1.041	1.003
up-and-in call	17	309	85	94	1.051	0.997
down-and-out put	16	105	31	25	0.964	1.000
down-and-in put	42	909	69	65	0.954	1.009
Straight						
down-and-out call	33	1,059	93	76	0.969	1.019
down-and-in call	5	52	29	21	0.978	0.991
up-and-out put	5	139	210	177	1.020	0.992
up-and-in put	0	0	–	–	–	–

Table 2.1: Descriptive statistics of the barrier option data-set; option type, frequency, (calendar) time to expiry, (relative) strike and barrier. A '0' indicates 'at initiation of a particular contract', so the 4th, 6th, and 7th column are averages of contracts at their initiation days of, respectively, time to expiry (measured in calendar days), barrier level (B) relative to spot (X_0), and strike (K) relative to spot. The $\overline{\tau_i}$'s denote times to expiry averaged over all observations (so having $\overline{\tau_i} > \overline{\tau_0}$ is not an error).

We see that the strikes are mostly set very close to the exchange-rate at contract initiation (i.e. the corresponding plain vanilla option is at-the-money), that the typical time to expiry is 70 (calendar) days, and that it is common to have the barrier 2-5% away from initial spot. This means that the barrier options fit nicely into the range (expiry and moneyness-wise) of our plain vanilla calibration instruments. Another important feature of a barrier option is whether it is *reverse* or not (which we term *straight*; non-reverse is too awkward). A barrier option is of reverse (a.k.a. live-out) type if the corresponding plain vanilla option is in-the-money when the barrier event (knock-in or knock-out) happens. This means that values of reverse barrier options change very rapidly in the vicinity of the barrier; there is a big difference between just crossing, and not crossing; exploding Greeks and gap risk are other terms used to describe this phenomenon. This makes them hard to hedge – be that statically or dynamically, see Nalholm & Poulsen (2006, Table 2) for instance. The reverse barrier options are the down-and-out put and the up-and-out call and their knock-in counterparts. From table 2.1 we see that the data-set is fairly balanced; in general reverse-type options are more common (73% of contracts, 59% of observations), but the single-most observed contract is the (straight) down-and-out call.

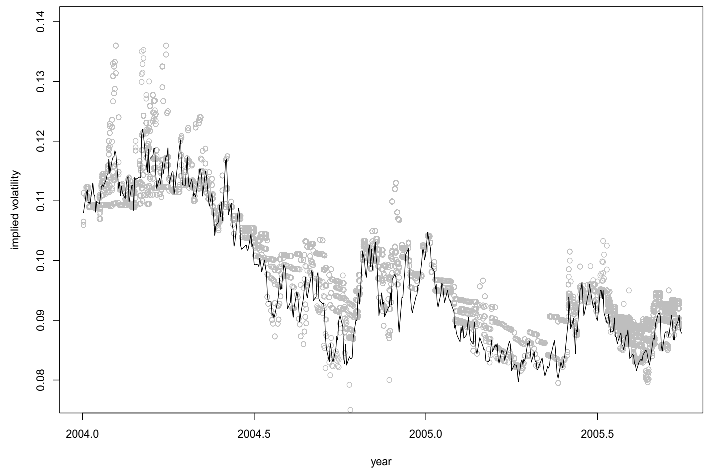

Figure 2.2: The barrier option data from Danske Bank. Each gray circle represents a barrier option data point in terms of an "implied" volatility. The fully drawn curve is the implied volatility of the 1 month, at-the-money plain vanilla option.

A final sanity check of the barrier option data is given in figure 2.2. It shows (all) the barrier options' "implied" volatilities as expressed by the bank.[6] We see that barrier option volatilities line up reasonably closely to the implied volatilities of the plain vanilla options.

2.3 Model selection and pricing methods

We consider five alternative models for the exchange rate. The model selection aims at including models with different features: a model with state dependent volatility versus one with stochastic volatility and a model with low jump activity and large jumps versus one with high jump activity and small jumps. One requirement though is the existence of reasonable methods for pricing barrier options either analytically or numerically. Under these criteria we have chosen the following models, which are frequently encountered in the literature: the constant elasticity of variance model (CEV), the stochastic volatility model of Heston, the Merton jump-diffusion model and the infinite activity Variance Gamma model (VG). Our benchmark model is that of Black-Scholes.

All models have the Black-Scholes model as a special or limiting case, but apart from that they are as non-nested as can be, thus covering a large range of qualitatively different (and popular) models. The models and pricing methods "at a glance" are shown in table 2.2; more detailed descriptions are given in the following subsections.

[6]The reason for the quotes and the disclaimer is that even in the Black-Scholes model barrier option values are not monotone functions of volatility. Hence, given an observed barrier option value there may be multiple sensible input volatilities that match the observation, and thus implied volatility is not uniquely defined. The bank's data-set contains both "implied" volatilities and actual prices; we use the former only for graphical purposes.

Model	Plain Vanilla	Barriers
Black-Scholes	Closed-form.	Closed-form portmanteau formulation in Rubinstein & Reiner (1991).
Constant elasticity of variance	Closed-form ala Schroder (1989) with Ding's algorithm for the non-central χ^2-distribution.	Collocation ala Nalhom & Poulsen (2006). Inversion ala Davydov & Linetsky (2001).
Heston stochastic volatility	Fourier inversion using the formulation in Lipton (2002).	Simulation ala Andersen (2008) with bridge and control. PDE solution ala Kluge (2002). Perturbation expansion ala Wong & Chan (2008).
Merton jump-diffusion	The original formula by Merton (1976) is found more efficient than Fourier inversion.	Simulation ala Joshi & Leung (2007) and Metwally & Atiya (2002).
Variance Gamma	Fourier inversion with tricks from Lee (2004) and Jaimungal (2004).	Simulation ala Glasserman (2004) with tricks from Avramidis (2004).

Table 2.2: Annotated taxonomy of pricing.

2.3.1 Black-Scholes model

In the Black-Scholes model the foreign exchange rate X follows a geometric Brownian motion under the risk-neutral pricing measure:[7]

$$dX_t = (r_d - r_f)X_t dt + \sigma X_t dW_t,$$

where r_d and r_f denote the assumed-constant domestic (US) and foreign (Euro) interest rates. In this setup, closed-form formulas for both plain vanilla and barrier option prices exist and will be used for pricing. It is well-known that the one-parameter Black-Scholes model is not the best model to describe observed option prices – especially not for a wider range of strikes and maturities simultaneously. However, it may still turn out to be the preferred model choice for pricing barrier options due to its fast, stable and easily implementable pricing procedure.

[7]By construction the price calibration estimates the pricing measure used by the market, thus all parameters are under the/a risk-neutral pricing measure; say Q. For our price analysis, this is not a restriction; if we were to study construction and performance of hedge portfolios both the risk-neutral pricing measure and the real-world measure would matter – though possibly less so in practice than in theory, see Poulsen, Schenk-Hoppé & Ewald (2009) and Siven & Poulsen (2009, Table 4).

2.3.2 Constant elasticity of variance model

A minimal extension of the Black-Scholes model is the constant elasticity of variance model, in which the foreign exchange rate has the risk neutral dynamics

$$dX_t = (r_d - r_f)X_t dt + \sigma X_t^{\alpha} dW_t,$$

where α denotes the so-called elasticity of variance. In this model there are two parameters, α and σ, to be estimated. For $\alpha < 1$ volatility increases as the exchange rate falls; vice versa for $\alpha > 1$.

For pricing plain vanilla options we use the closed-form formula of Schroder (1989). Several methods for pricing barrier options exist. Numerical techniques such as the finite difference method and Monte Carlo simulation can be applied. Alternatively, as we have chosen to do here, barrier option prices may be found via collocation as demonstrated in Nalhom & Poulsen (2006). Analytical formulas for barrier option prices based on inversion techniques do exist (see Davydov & Linetsky (2001)), however, these are rather involved and in our experience there is no real gain with respect to computation time compared to the direct numerical approaches.

2.3.3 Heston's stochastic volatility model

For the stochastic volatility model we have chosen the Heston model, where the exchange rate and its instantaneous variance follow

$$\begin{aligned} dX_t &= (r_d - r_f)X_t dt + \sqrt{v_t} X_t dW_t^1 \\ dv_t &= \kappa(\theta - v_t)dt + \eta\sqrt{v_t} dW_t^2. \end{aligned}$$

θ is the long term level of variance, κ is the speed of mean reversion, η is referred to as the volatility of volatility, and the driving Brownian motions have correlation ρ, leading to a skew in implied volatilities. In the Heston model there are four parameters, κ, θ, η and ρ, plus one state variable, v, to be estimated. The (conditional) characteristic function of X can be found in closed form (this was first done in Heston (1993)), and means that plain vanilla option pricing becomes a question of one-dimensional numerical integration; inverting a transform. There is a sizeable literature on this, see Lee (2004). We prefer the quadratic denominator formula of Lipton

(2002). Closed-form solutions for barrier options exist (see Lipton (2001)) in the case where domestic and foreign short rates are equal and correlation is zero, but Faulhaber (2002) shows that there is no simple way to relax those assumptions, which are unrealistic to impose on our data. During the sample period the US short rate r_d decreases from approximately 4% to 1%, while the European short rate r_f is more or less constant at 2.1%, and since we do see an implied skew in our data, fixing $\rho = 0$ is also too restrictive. Alternatives are the PDE method described by Kluge (2002) or Monte Carlo simulation using a quadratic exponential discretization scheme for the volatility process developed by Andersen (2008). We use the latter and combine it with a Black-Scholes model control variate.

2.3.4 Merton's jump-diffusion model

The Black-Scholes model can also be extended to include jumps in the exchange rate as done by Merton:

$$dX_t = (r_d - r_f - \lambda \mathbf{E}^Q(Z_t - 1))X_t dt + \sigma X_t dW_t + X_t(Z_t - 1)dN_t,$$

where N is a Poisson process with intensity λ, and $\log Z_t \sim \mathcal{N}(\mu_Z, \sigma_Z)$ describes the relative jump size as being normally distributed with mean μ_Z and variance σ_Z. The Merton model has four parameters to be estimated: σ, μ_Z, σ_Z and λ.

Pricing plain vanilla options in this model can be done by Fourier inversion techniques or, in our experience more efficiently, by using the formula provided by Merton (1976). Barrier option prices are found by Monte Carlo simulation methods as suggested by Metwally & Atiya (2002) and Joshi & Leung (2007) via the use of importance sampling.

2.3.5 Variance Gamma model

Another class of jump models are models exhibiting infinite jump activity as e.g. the Variance Gamma (VG) model proposed by Madan & Seneta (1990). The foreign exchange rate under the risk neutral measure in the VG model is of the form

$$X_t = X_0 \exp\{(r_d - r_f)t + Y_t^{\mathrm{VG}} + \omega t\},$$

where $Y_t^{\mathrm{VG}} = \theta G_t^\nu + \sigma W(G_t^\nu)$ is a variance gamma process; a time changed Brownian motion with drift, $\theta t + \sigma W_t$, using a gamma process G^ν with volatility ν as the stochastic clock. The martingale correction term $\omega = \frac{1}{\nu}\ln\left(1 - \theta\nu - \frac{1}{2}\sigma^2\nu\right)$ ensures that the expected rate of return on assets equals the risk-neutral rate $r_d - r_f$. The parameter ν controls for excess kurtosis and θ for skewness. The limit when $\nu \to 0$ (in which case the influence from θ also disappears) is the Black-Scholes model. Figure 2.3 shows some simulated paths of VG processes, and illustrates that for small ν-values, the process looks diffusion'ish, while high ν gives a more Poisson-jump-like appearance.

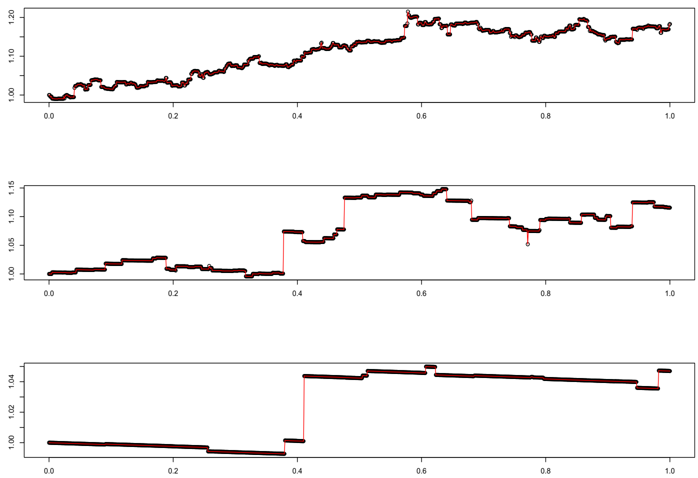

Figure 2.3: Simulated paths of Variance Gamma processes. The parameter values are $\sigma = 0.1$, $\theta = 0.0085$, and $\nu = 0.01$ (top), 0.1 (middle), 1 (bottom).

Plain vanilla option prices can be found by Fourier inversion, as done in Jaimungal (2004) or Lee (2004). Barrier option prices can be found by simulation methods as presented in Glasserman (2004) or the double-gamma bridge sampling algorithm by Avramidis (2004).

2.4 Empirical results

2.4.1 Calibration and plain vanilla option valuation

On any specific date (t) and for any model j (naturally indexed by {BS, CEV, H, M, VG}), we estimate the parameter, $\vartheta_j(t)$, by minimizing the sum of absolute differences between the observed implied volatilities (IV) and the model's implied volatilities. Or with symbols:

$$\widehat{\vartheta}_j(t) = \arg\min_{\vartheta} \sum_{i|t(i)=t} |IV^{\text{obs}}(i) - IV^{\text{model } j}(i;\vartheta)|,$$

where the notational philosophy is that i denotes observations, and $t(\cdot)$ maps an observation to its date.

Implied volatilities put option prices in a comparable scale across strikes and expiries. Minimizing differences to raw prices does not alter our results but makes the numbers harder to relate to. One could also minimize differences to relative prices but in our experience that tends to put too much weight on out-of-the-money options.[8]

Sample characteristics of estimators are given in table 2.3. The calibrated parameters are not constant over the sample period, but they are more stable than the meta-analysis for S&P500 that is reported in Gatheral (2006, Table 5.4) indicating that exchange rate markets are more benign than equity markets. Only the dangerously naive observer would claim that a price outside the bid/ask-spread is an arbitrage opportunity, but it is nonetheless a sensible yardstick. On that count the models separate into two categories; Merton and Heston hit about two-thirds of bid/ask-spreads, and the rest about 40%. Other remarks:

- The difference between the average instantaneous variance, v_0, and the Heston model's (risk-adjusted parameter) θ reflects the typically increasing term structure of volatility.

[8]To illustrate: An implied volatility difference of 0.001 gives a 1% relative price difference for the at-the-money option, but 2% for the out-of-the-money option. A variety of weighting schemes have been suggested in the literature; *to every man his own*. An interesting point is made by Cont & Tankov (2004, p. 349) that calibration to squared price differences corresponds (to a first-order approximation) to using a vega-weighted average of implied volatility differences.

- In the Merton model most (75%) of the variance of daily returns is caused by the diffusion component.
- The sample mean of the VG estimates were used to generate the middle path in figure 2.3; the paths have a visible, but not extreme, non-diffusive character.

Model	Black-Scholes	CEV		Heston				
	σ	σ	α	v_0	θ	κ	η	ρ
mean	0.0998	0.0891	1.50	0.00964	0.0120	2.22	0.183	0.0634
std.dev	0.0088	0.0147	0.597	0.00244	0.00139	0.525	0.0633	0.117
median	0.0996	0.0899	1.24	0.00889	0.0125	2.12	0.184	0.0731
min	0.0877	0.0451	-0.091	0.00615	0.00894	0.879	0.00186	-0.465
max	0.116	0.118	3.39	0.0161	0.0150	7.91	0.412	0.470
within bid/ask	40.3 %	38.8%		69.2%				

Model	Merton				VG		
	σ	λ	σ_Z	μ_Z	σ	ν	θ
mean	0.0851	1.24	0.0443	0.00339	0.105	0.102	0.00847
std.dev	0.0157	0.104	0.0208	0.0116	0.00387	0.0851	0.0666
median	0.0806	1.26	0.0510	0.00280	0.105	0.0898	0.00929
min	0.0593	0.587	$5.00\cdot10^{-5}$	-0.0325	0.0921	$1.08\cdot10^{-4}$	-0.310
max	0.118	1.54	0.077	0.0524	0.113	0.357	0.294
within bid/ask	65.6%				44.3%		

Table 2.3: Sample characteristics of parameter estimates. The "within bid/ask" rows give the percentage of the calibrated model prices that fall within a bid/ask spread of ± 0.0015 on implied volatility.

A particular model may perform well on the data that it is calibrated to, but have poor predictive qualities (think of fitting a high-order polynomial to "a straight line with noise"). To investigate this, figure 2.4 shows the five models' prediction errors at different (business day) horizons. The fully drawn, differently coloured curves show the models' average absolute implied volatility differences for increasing horizons, i.e.

$$\frac{1}{\#\text{obs. dates} - h} \sum_t \frac{1}{\# i | t(i) = t} \sum_{i | t(i) = t} |IV^{\text{obs}}(i) - IV^{\text{model } j}(i; \hat{\vartheta}_{t(i)-h})|.$$

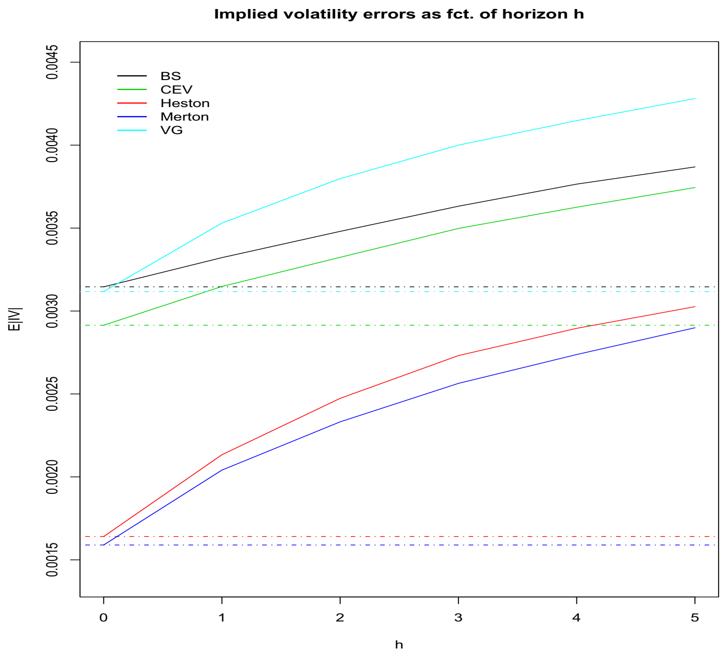

Figure 2.4: Prediction errors for plain vanilla options at different horizons.

It is only for the Variance Gamma model that the ordering is changed when we look at predictions; it is (slightly) better than Black-Scholes at horizon 0, but worse at longer horizons. The errors of the Merton model are marginally lower than those of the Heston model at all horizons, and the differences are statistically significant (at a 5% level) at horizons of two days or more. As a rule-of-thumb-quantification of how much better the Heston and Merton models perform, we can look at the horizons where their prediction errors match the horizon-0 errors of the others models, i.e. the points, say h_j, on the abscissa where the red and blue curves cross the dash-dotted horizontal lines. A way to interpret these numbers is to say: "Using model j with 'freshly estimated' parameters is (on average) as good as using a Heston or Merton model with h_j day old parameters". We see that the Heston and Merton models are caught up with by the other models after about one week.

Combining stochastic volatility and jumps; the Bates model

One may suspect a combination of the Heston and Merton models to perform even better. A stochastic volatility model including Poisson jumps in the exchange rate (also known as a Bates model following Bates (1996)) has the dynamics

$$\begin{aligned} dX_t &= (r_d - r_f - \lambda \mathbf{E}(Z_t - 1))X_t dt + \sqrt{v_t} X_t dW_t^1 + X_t(Z_t - 1)dN_t \\ dv_t &= \kappa(\theta - v_t)dt + \eta\sqrt{v_t}dW_t^2, \end{aligned}$$

with $\rho dt = \mathrm{cor}(dW_t^1, dW_t^2)$. The Bates model has a total of seven parameters, κ, θ, η, ρ, λ, μ_Z and σ_Z, plus one state variable, v_0. Calibration of this model is a numerically delicate matter but can be carried out as suggested by Kilin (2007).

For the plain vanilla data the Bates model's average absolute implied volatility error is up to two significant digits (and no statistical significance) identical to the Merton and Heston models'.

Combining Levy-models and stochastic volatility; the VG-CIR model

A way to introduce stochastic volatility into pure jump models such as the Variance Gamma is to subject the driving process to a random time-change, i.e. to work with $Y_{Z_t}^{\mathrm{VG}}$ where Z_t is an increasing stochastic process. Carr, Geman, Madan & Yor (2003) show how characteristic functions in some cases can be expressed by composition of the Laplace-transform of the time change process and the characteristic function of the original model. A convenient choice of time-change process is an integrated Cox-Ingersoll-Ross process (independent of the original Y^{VG}-process), whose Laplace-transform is part of the interest rate theory vocabulary. Again, calibration of this six-parameter (and one more or less latent state variable; the current value of the subordinator) VG-CIR model is a delicate matter for which we refer to Kilin (2007).

The cross-sectional average implied volatility error for the model is 0.23%; lower than the VG model's error, but not as good as the Heston and Merton models.

%		Black-Scholes		CEV		Heston	
h	split	$\overline{e^{\mathrm{BS}}}$	$\overline{\lvert e^{\mathrm{BS}}\rvert}$	$\overline{e^{\mathrm{CEV}}}$	$\overline{\lvert e^{\mathrm{CEV}}\rvert}$	$\overline{e^{\mathrm{H}}}$	$\overline{\lvert e^{\mathrm{H}}\rvert}$
0	all	$0.102^{\ddagger}$	4.93	$-0.0028^{\ddagger}$	4.49	0.53	2.21
(ATM , OTM)		(1.15 , -0.495)	(3.48 , 6.39)	(1.03 , -1.03)	(3.38 , 5.60)	(1.95 , -0.89)	(2.62 , 1.80)
1	all	$0.184^{\ddagger}$	5.19	$0.081^{\ddagger}$	4.86	0.618	3.02
(ATM , OTM)		(1.20 , -0.831)	(3.67 , 6.71)	(1.08 , -0.916)	(3.59 , 6.14)	(2.01 , -0.769)	(3.03 , 3.02)
5	all	0.537	6.06	0.479	5.84	0.977	4.47
(ATM , OTM)		(1.42 , $-0.345^{\ddagger}$)	(4.26 , 7.85)	(1.33 , $-0.369^{\ddagger}$)	(4.18 , 7.50)	(2.23 , $-0.275^{\dagger}$)	(3.87 , 5.07)

%		Merton		VG	
h	split	$\overline{e^{\mathrm{M}}}$	$\overline{\lvert e^{\mathrm{M}}\rvert}$	$\overline{e^{\mathrm{VG}}}$	$\overline{\lvert e^{\mathrm{VG}}\rvert}$
0	all	-0.348	2.31^{**}	0.58	4.79
(ATM , OTM)		(-0.36 , -0.34)	(1.80 , 2.83)	(1.07 , $0.085^{\ddagger}$)	(2.95 , 6.64)
1	all	-0.257	3.00^{**}	0.590	5.40
(ATM , OTM)		(-0.30 , $-0.21^{\dagger}$)	(2.31 , 3.69)	(1.08 , $0.0998^{\ddagger}$)	(3.41 , 7.39)
5	all	$0.118^{\ddagger}$	4.30	0.675	6.51
(ATM , OTM)		($-0.064^{\ddagger}$, $0.30^{\dagger}$)	(3.35 , 5.25^{*})	(1.15 , $0.201^{\ddagger}$)	(4.35 , 8.68)

Table 2.4: Relative pricing errors for plain vanilla options in %.

%		Black-Scholes		CEV		Heston	
h	split	$\overline{\xi^{\mathrm{BS}}}$	$\overline{\lvert\xi^{\mathrm{BS}}\rvert}$	$\overline{\xi^{\mathrm{CEV}}}$	$\overline{\lvert\xi^{\mathrm{CEV}}\rvert}$	$\overline{\xi^{\mathrm{H}}}$	$\overline{\lvert\xi^{\mathrm{H}}\rvert}$
0	all	2.71	8.27^{**}	$0.13^{\ddagger}$	8.50	-3.48	7.82
	(rev. , str.)	(4.12 , $0.61^{\dagger}$)	(11.1^{**} , 4.0^{**})	($-0.37^{\ddagger}$, 0.88)	(11.1^{**} , 4.68)	(-4.61 , -1.80)	(10.4 , 3.89)
	(up , down)	($-0.37^{\ddagger}$, 4.13)	(6.35 , 9.16)	(-1.47 , $0.87^{\dagger}$)	(7.66^{**} , 8.94)	(-4.09 , -3.20)	(7.83 , 7.81)
1	all	2.75	8.33^{**}	$0.075^{\ddagger}$	8.69^{**}	-3.09	8.22
	(rev. , str.)	(4.18 , $0.61^{\dagger}$)	(11.2^{**} , 4.02^{**})	($-0.55^{\ddagger}$, 1.00)	(11.4^{*} , 4.59)	(-4.61 , -1.68)	(10.5 ,3.78)
	(up , down)	($-0.50^{\ddagger}$, 4.25)	(6.33 , 9.26^{*})	(-1.75 , $0.92^{\ddagger}$)	(7.63^{**} , 9.19^{**})	(-4.21 , -2.57)	(7.84 , 8.39)
5	all	3.60	10.6^{**}	0.91	10.3^{**}	-2.31	10.1
	(rev. , str.)	(5.16 , 1.23)	(14.0^{**} , 5.49)	($0.30^{\ddagger}$, 1.83)	(13.2^{**} , 6.02)	(-3.03 , -1.23)	(13.6 , 4.83)
	(up , down)	($-0.010^{\ddagger}$, 5.26)	(8.38 , 11.6)	(-1.39 , 1.98)	(9.36^{**} , 10.8^{**})	(-3.87 , -1.59)	(9.50 , 10.3)

%		Merton		VG	
h	spilt	$\overline{\xi^{\mathrm{M}}}$	$\overline{\lvert\xi^{\mathrm{M}}\rvert}$	$\overline{\xi^{\mathrm{VG}}}$	$\overline{\lvert\xi^{\mathrm{VG}}\rvert}$
0	all	-22.5	29.3	66.7	79.5
	(rev. , str.)	(-21.1 , -24.7)	(31.7 , 25.7)	(106 , 8.39)	(123 , 14.4)
	(up , down)	(-8.91 , -28.8)	(26.3 , 30.7)	(49.6 , 74.7)	(65.5 , 86.0)
1	all	-22.5	29.3	63.4	76.0
	(rev. , str.)	(-21.1 , -24.7)	(31.7 , 25.6)	(100 , 8.39)	(118 , 14.1)
	(up , down)	(-9.01 , -28.7)	(27.0 , 30.8)	(48.1 , 70.5)	(63.6 , 81.7)
5	all	-21.3	30.3	68.0	78.1
	(rev. , str.)	(-19.6 , -23.9)	(33.5 , 25.6)	(102 , 10.1)	(119 , 15.9)
	(up , down)	(-7.82 , -27.6)	(29.3 , 30.8)	(44.5 , 75.1)	(60.4 , 86.3)

Table 2.5: Relative pricing errors for barrier options in %.

2.4.2 Barrier option valuation

We now turn to the main question: How well do the different models perform when it comes to valuing barrier options? Implied volatilities for barrier options are not well-defined and their raw prices differ several orders of magnitude. Therefore we will report relative price errors for the barrier options. To fix notation, the relative error for the i'th observation for the j'th model at horizon h is

$$\xi^{j}_{i,h} = \frac{B^{\text{model } j}(i; \hat{\vartheta}_j(t(i) - h)) - B^{\text{obs}}(i)}{B^{\text{obs}}(i)},$$

where the notational philosophy is as before, and the $B^{\cdot}(i; \cdot)$'s denote values of barrier options with the appropriate characteristics. For comparison we report relative errors of plain vanilla options too. These are defined analogously and denoted by $e^{j}_{i,h}$.

For plain vanilla options, the estimation procedure ensures that each model's average errors are small.[9] Thus plain vanilla comparisons should be made based on some measure of dispersion such as standard or mean absolute deviation. For barrier options both average errors and their dispersion are relevant measures of a model's quality. Therefore our tables (Table 2.5 for barrier options, Table 2.4 for plain vanilla) report sample averages of both errors and absolute errors. For the errors, † and ‡ indicate that there is no significant difference from 0 at, respectively, the 1% and 5% levels. For the absolute errors, $*$ and $**$ indicate that errors are not significantly different than those of the Heston model; this is the result of a paired test based on absolute error differences.

In table 2.4 we have sub-divided the relative plain vanilla option pricing errors into out-of-the-money (OTM) and at-the-money (ATM) errors. General for all five models (except Heston's errors at horizon 0), is that they produce larger relative pricing errors for options out-of-the-money. This is in line with the previous observation that a given implied volatility error corresponds to a larger relative price error for an out-of-the-money option than for an option at-the-money.

From table 2.5 we see that the three continuous-path models (Black-Scholes, CEV and Heston) have quite similar behaviour when it comes

[9] We minimize averages of absolute differences of implied volatilities. Therefore average errors (raw and particularly relative) are not *exactly* zero.

to barrier option valuation. The CEV model is most accurate with regards to average price errors (0.1% vs. -3.5% for Heston and 2.7% for Black-Scholes), while the Heston model has the lowest dispersion (average absolute error of 7.8% vs. 8.2% for Black-Scholes and 8.5% for CEV). One could explain the good behaviour of the Black-Scholes model as a self-fulfilling prophesy; market participants use Black-Scholes('ish) formulas because that is what is on their computers. But if that were the only reason, we would expect to see rapid deterioration in the Black-Scholes model's predictive quality. We do not; conclusions are invariant to the prediction horizon. Looking at the plain vanilla benchmark in table 2.4 we see that the continuous-path models' error dispersions for barrier options are two to five times larger than for plain vanilla options. Or differently put, barrier options are half an order of magnitude harder to price.

For the jump-models (Merton and Variance Gamma), the story is quite different. The Merton model is bad (dispersion of 29%; about four times that of the continuous-path models), and the Variance Gamma model is worse (dispersion of 79%; a ten-fold increase). Again, this holds at all horizons. One could argue that "that is because continuous-path models systematically underestimate knock-out probabilities. You should still use models with jumps." But that fails to explain why the Merton model undervalues the barrier options (by 23% on average) and the Variance Gamma model overvalues them (by 67% on average). For plain vanilla options, the Merton model was arguably one of the best performing models, and Variance Gamma was on par with Black-Scholes and CEV. Thus their poor ability to explain barrier option values again emphasizes the model risk aspect pointed out by Schoutens et al. (2004), Hirsa et al. (2002), Detlefsen & Härdle (2007) and numerous other papers: models may produce very similar prices of plain vanilla options yet differ markedly for exotic options.

To understand the models' pricing performance for barrier options we have analysed inter-model differences; we simply changed "observed values" to "Black-Scholes values" in the definition of relative errors. This reveals that the Black-Scholes and CEV model values typically are closer to each other, than they each are to data (the average absolute CEV-to-Black-Scholes error is 4.8% compared to about 8% for each model with observed values as reference point), while the average absolute Heston-to-

Black-Scholes error is 8.4%.

To further detect patterns, we have sub-divided errors according to different criteria: reverse vs. straight and up vs. down. Results are also reported in table 2.5; the numbers in parentheses. First, we see that error dispersions are markedly larger for reverse barrier options than for the straight ones; sample averages of absolute errors are 2-3 times higher. Given the difficulties in hedging the reverse options due to their exploding Greeks, this increased dispersion between market and model values may be understandable, but it should be noted that there is no clear pattern for the average hedge errors from the reverse-straight stratification. There is little effect from the up-down split which we interpret as more evidence that exchange rate markets are reasonably symmetric.

With respect to barrier option pricing, the Bates model performs better than the Merton and VG models but significantly worse than the Heston model with average absolute errors of 24.7% ($h = 0$) and 25.6% ($h = 5$). So adding jumps to the Heston model merely worsens the model's barrier option valuation abilities. The same is true when introducing stochastic volatility into the VG model as in the VG-CIR model.

2.5 Conclusion

We investigated empirical barrier option values, and found that in general the continuous-path models did equally well in explaining the market data, while models with jumps turned out to be quite inaccurate, this despite the jump-diffusion model being – arguably – the best performing model for plain vanilla options.

A logical next step is to investigate how well the barrier options can be hedged, dynamically, statically or by some hybrid hereof. A particularly interesting question, that the barrier option data-set allows us to shed (some) light on, is the benefit of applying a portfolio, rather that "each option on its own", approach to hedging.

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3. Constant Proportion Portfolio Insurance: Discrete-time Trading and Gap Risk Coverage[1]

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Abstract: This paper studies constant proportion portfolio insurance (CPPI) in a setup that accounts for market frictions. Trading costs, fees and borrowing restrictions are incorporated, and the assumption of continuous portfolio rebalancing is relaxed. The main goals are to cover issuer's gap risk and to maximize CPPI performance over possible multipliers; the proportionality factor that determines the risky exposure of a CPPI. Investment objectives are described by the Sortino ratio and alternatively by a kinked constant relative risk aversion utility function. Investors with either objective will choose a lower multiplier than if CPPI performance is measured by the expected return. Discrete-time trading requires a portfolio rebalancing rule, which affects both performance and gap risk. Two commonly applied strategies, rebalancing at equidistant time steps and rebalancing based on market movements, are compared to a new rule, which takes trading costs into account. While the new and the market-based rules deliver similar CPPI performance, the new rebalancing rule achieves this by fewer trading interventions. Issuer's gap risk can be covered by a fee charge, hedging or by an artificial floor. A new approach for determining the artificial floor is introduced. Even though all three methods reduce losses from gap events effectively, the artificial floor and hedging are less costly to the investor.

Keywords: Gap risk, discrete rebalancing, Sortino ratio, kinked CRRA utility

[1] I thank Rolf Poulsen for helpful comments.

3.1 Introduction

Portfolio insurance techniques will appeal to investors with fear of large losses as recently experienced on the financial markets. A popular portfolio insurance strategy is constant proportion portfolio insurance (CPPI), see Black & Perold (1992) and the references therein. A CPPI provides a capital guarantee by dynamically allocating wealth between two assets: a risky asset, which gives the investor an upside potential, and a risk free asset. The relative portfolio weight of risky assets is determined by a *multiplier* $m > 1$. If the portfolio value falls below the *floor*, defined as the present value of the capital guarantee, all funds are invested in the risk free asset in order not to jeopardize the guarantee any further. A CPPI hereby allows investors with a lower risk appetite to invest in alternative asset classes and still benefit from favourable market scenarios.

Existing CPPI-literature often make the assumption of continuous time trading to preserve analytical tractability. In a general Lévy framework Cont & Tankov (2009) provide analytical expressions for quantities describing *gap risk*: the risk of the portfolio value breaking the floor such that the guarantee cannot be fully honoured. When market frictions such as trading costs are introduced, the assumption of continuous trading must be relaxed, thereby possibly compromising analytical tractability.

The CPPI strategy has been studied in a setup including trading costs and discrete-time trading in a number of papers; among these are Black & Perold (1992), Boulier & Kanniganti (1995), Hamidi, Jurczenko & Maillet (2009), Balder, Brandl & Mahayni (2009), Constantinou & Khuman (2009) and Paulot & Lacroze (2009). However, none of these consider the choice of portfolio rebalancing rule explicitly. Yet, the rebalancing rule is important since too rare rebalancing increases issuer's gap risk and too frequent trading imposes high trading costs. In this paper I provide a first attempt to address this problem. I compare weekly portfolio rebalancing to two customized trading strategies which take movements in underlying variables into account. One strategy is simply to rebalance when the change in the risky asset value exceeds a carefully chosen tolerance level. Secondly, I introduce a new trading rule known from option replication (Whalley & Wilmott (1997)), which can be applied in a CPPI context after a useful link between the CPPI portfolio and a position in down-and-out barrier options is established. While both customized trading strategies are found to outperform weekly rebalancing, the new trading rule has the advantages that it avoids the delicate choice of tolerance levels and requires fewer trading interventions.

The issuer's risk[2] when managing a CPPI deal is the gap risk; a gap event implies a loss to the issuer, since the capital guarantee issued in the CPPI contract must still be honoured. Therefore the issuer will take actions to cover or reduce this risk. One approach suggested by Cont & Tankov (2009) is to hedge gap risk using short maturity put options or gap options. A second approach typically used in actual CPPI issuances, and to my knowledge not previously studied in academic literature, is to cover losses due to gap events by charging a fee. The size of the fee is chosen based on a risk analysis of the issuer's exposure. A third way to handle gap risk is to introduce an artificial floor above the true floor, which gives the CPPI manager a buffer to absorb potential losses. To determine the buffer size, I apply a result for pricing discretely monitored barrier options by Broadie et al. (1997). The artificial floor approach is compared to coverage by hedging and charging a fee, and while all three methods reduce losses due to gap events effectively, covering gap risk by the two former approaches are less costly with respect to expected CPPI return.

Once the issuing entity has covered its risks and administration costs the CPPI multiplier can be chosen to accommodate the CPPI investor's risk/return profile. Assuming risk averse investors, I consider two measures for the CPPI performance. The Sortino ratio advocated by Pedersen & Satchell (2002) measures the upside potential of an investment relative to its downside risk. An alternative is to maximize investor's expected utility, here described by a kinked constant relative risk aversion utility function as suggested by Sharpe (2006) for protected investment products. I find that the two performance measures lead to similar choices of multiplier, and lower than the multiplier maximizing expected CPPI return.

Capital restrictions are important to include in a CPPI analysis, since banks are often reluctant to provide additional capital for gearing a risky position and if doing so typically charge a spread above the risk free rate. I find that borrowing restrictions improve CPPI performance, because

[2]Counterparty risk, i.e. the risk that the issuing entity cannot meet its obligations, will not be considered.

trading costs and costs of capital exceed the increase in return obtained from higher risky exposure.

Finally, I investigate the effects of adding a *profit lock in* feature to the CPPI strategy. Profit lock in (also known as ratcheting) ensures part of the investor's return in a favourable market situation by raising the floor. Profit lock in has previously been included in e.g. the studies of Paulot & Lacroze (2009), Boulier & Kanniganti (1995) and MKaouar & Prigent (2007*a*). I find that in a setup with market frictions, the investor does not benefit from locking in profits, neither with respect to expected return, utility or Sortino ratio.

The rest of the paper is organized as follows. The next section describes the CPPI investment strategy and incorporates market frictions. Special attention is paid to gap risk coverage and discrete portfolio rebalancing. Section 3.3 describes the models for the underlying asset and for investor's preferences. In section 3.4, numerical experiments are conducted to study the effects of market frictions, especially what consequences the choices of discrete portfolio rebalancing rule and gap risk coverage approach have. Section 3.5 sums up.

3.2 CPPI strategy incorporating market frictions

Trading costs, fees and capital restrictions are unavoidable in an actual implementation of a CPPI. This section considers adjustments in the stylized strategy, which are necessary when incorporating market frictions.

3.2.1 Stylized CPPI

A CPPI is an investment strategy that guarantees a fixed amount of capital G at expiration T. Let I denote the initial investment. The guarantee must satisfy $G \leq Ip(0,T)$, where $p(s,t)$ denotes the time s price of a zero-coupon bond maturing at time t. The value $Gp(t,T)$ is referred to as the floor and is the smallest amount that will guarantee a portfolio value of G at expiry. For simplicity, assume $G = I$ and normalize the initial investment to $I = 1$. Let $(V_t)_{t\in[0,T]}$ denote the CPPI portfolio value process and $(S_t)_{t\in[0,T]}$ the value of the underlying risky asset. At initiation $t = 0$, $V_0 = I > p(0,T)$ if interest rates are positive. As long as

$V_t > p(t,T)$, the stylized CPPI trading strategy will maintain the *exposure*, e, to the risky asset equal to

$$e_t = mC_t := m\big(V_t - p(t,T)\big). \tag{3.1}$$

C is referred to as the *cushion* and $m > 1$ the pre-specified, constant multiplier. If the floor is broken at time $\tau \in [0,T]$, $V_\tau \leq p(\tau,T)$, the entire portfolio value must be invested in zero-coupon bonds in order not to jeopardize the capital guarantee any further. This position is held until expiry: $e_t = 0$ for $t \in (\tau, T]$. The stylized CPPI is self financing: any dividends or coupons are assumed reinvested in the CPPI portfolio.

3.2.2 Capital restrictions

In case $mC_t > V_t$ the portfolio value does not cover the funds required to invest in the risky asset. The issuer may be reluctant to provide additional liquidity and set a borrowing limit b, which restricts exposure to $e_t = \min\{mC_t, bV_t\}$. Furthermore, the cost of additional capital is typically a spread δ above the risk free rate.

3.2.3 Trading costs

Trading costs are also introduced; assume a proportional cost of ε per dollar risky asset traded. The magnitude of ε reflects the liquidity of the underlying asset. When adjusting the CPPI portfolio, the new exposure, depending on the portfolio value according to (3.1), is computed ex-post trading costs. In case the position in risky assets is increased, $e_{t-} < mC_t$, the portfolio value is

$$\begin{aligned} V_t &= V_{t-} - \varepsilon(mC_t - e_{t-}) = V_{t-} - \varepsilon\big(m(V_t - p(t,T)) - e_{t-}\big) \qquad \Rightarrow \\ V_t &= \frac{1}{1+m\varepsilon}\big(V_{t-} + m\varepsilon p(t,T) + \varepsilon e_{t-}\big). \end{aligned}$$

If $e_{t-} > mC_t$, the expression holds with opposite sign on ε.

3.2.4 Discrete-time rebalancing strategies

The assumption of continuous portfolio rebalancing must be relaxed when introducing trading costs, and a rebalancing rule is therefore required. A simple strategy is to rebalance daily, weekly or monthly as done in e.g. Balder et al. (2009) and Cesari & Cremonini (2003). However, CPPI performance can be improved significantly, as will be verified in section 3.4.2, by employing a trading strategy that takes movements in underlying variables into account. Two such strategies are introduced here.

Rebalancing depending on movements in the underlying asset

Boulier & Kanniganti (1995) and Black & Perold (1992) suggest rebalancing the CPPI portfolio whenever the underlying asset has moved a fixed percentage. Others, e.g. Paulot & Lacroze (2009), Hamidi et al. (2009) and MKaouar & Prigent (2007*b*), apply a rebalancing rule that depends on divergence of actual exposure e_t from target exposure mC_t. If ignoring interest accruals, the two approaches are in fact equivalent as stated by Black & Perold (1992): a relative move of α in S is equivalent to a change of $-\frac{(m-1)\alpha}{1+m\alpha}$ in $\frac{e_t}{C_t}$. This statement is proved here in order to clarify where approximations are made.

Suppose $S_t = (1+\alpha)S_s$, where $s < t$ is the latest rebalancing date prior to time t, i.e. $e_s = mC_s$. This implies $e_t = (1+\alpha)e_s$. The relative change in the cushion is approximately $m\alpha$, since

$$\begin{aligned} C_t &= V_t - p(t,T) = e_t + (V_t - e_t) - p(t,T) \\ &= (1+\alpha)e_s + (V_s - e_s)e^{\int_s^t r_u du} - p(s,T)e^{\int_s^t r_u du} \\ &\approx C_s + \alpha e_s = (1+m\alpha)C_s, \end{aligned} \tag{3.2}$$

where $(r_t)_{t\in[0,T]}$ is the risk free interest rate process. Precision of the approximation relies on a small interest rate such that changes in the floor, $p(t,T)$, and in the risk free asset, $V_t - e_t$, between portfolio adjustments are small. If the position in risk free assets is positive, interest earned by the risk free investment will to some extent cancel the increase in the floor. The relative movement in $\frac{e_t}{C_t}$ is then, as claimed

$$\frac{e_t}{C_t} = \frac{(1+\alpha)e_s}{(1+m\alpha)C_s} = \left(1 - \frac{(m-1)\alpha}{1+m\alpha}\right)\frac{e_s}{C_s}. \tag{3.3}$$

A CPPI rebalancing rule $\{\alpha_u, \alpha_d\}$ gives upper and lower bounds on relative changes in the underlying asset that do not require an adjustment of the CPPI portfolio. In other words, adjust the risky exposure e_t to equal target exposure mC_t if

$$S_t \notin \big[(1-\alpha_d)S_s, (1+\alpha_u)S_s\big].$$

Bertrand & Prigent (2002) show that the maximum drop in the underlying asset value the CPPI portfolio value can sustain without hitting the floor is $\frac{1}{m}$. Thus, for a given multiplier m the restrictions $\alpha_d \in \big(0, \frac{1}{m}\big)$, $\alpha_u \in (0, \infty)$ are imposed.

This rebalancing rule is equivalent to a rule $\{\tau_u, \tau_d\}$ that sets upper and lower bounds on actual exposure's e_t divergence from target exposure mC_t

$$e_t \notin \big[(1-\tau_d)mC_t, (1+\tau_u)mC_t\big], \tag{3.4}$$

where $\tau_d = \frac{(m-1)\alpha_u}{1+m\alpha_u}$ and $\tau_u = \frac{(m-1)\alpha_d}{1-m\alpha_d}$ are found by applying (3.3). Restrictions on α_d and α_u translate to $\tau_d \in \big(0, 1-\frac{1}{m}\big)$, $\tau_u \in (0, \infty)$.

The strategy (3.4) will be referred to as the *market-based rebalancing rule.*

Rebalancing strategy depending on cushion size

A new rebalancing rule that takes the cushion size into account is introduced here. The market-based strategy has the potential weakness that the tolerance levels τ_d, τ_u do not depend on the cushion size. Thus, this rule does not allow for more frequent downward adjustments of exposure when the portfolio value is close to the floor. This issue is recognized from barrier option replication, where frequent rebalancing of the hedge portfolio is needed when the underlying is close to the barrier.

Observe that locally, i.e. until next rebalancing, the risky part of the CPPI portfolio is equivalent to a position in barrier options: let η_t be the number of underlying shares held in the CPPI portfolio at time t, then the risky CPPI exposure is equivalent to η_t down-and-out call options on the underlying asset S with expiry T, strike equal to zero and time dependent rebate R_u and barrier B_u given by $R_u = B_u = \frac{1}{\eta_t}\big(p(u,T) - b_u\big)$, $u > t$. Here, $b_u := \big(V_t - \eta_t S_t\big)e^{\int_t^u r_s ds}$ is the value of the position in risk free assets.

Whalley & Wilmott (1997) consider replication of plain vanilla options in a Black Scholes model extended to include transaction costs. Poulsen

& Siven (2008) apply these results to barrier options. If trading costs are present, the Δ-hedge portfolio derived in a frictionless market setting is shown to be suboptimal to adjust, while the number of shares Δ lies within the following no-trading band:

$$(\Delta_t - \xi_t, \Delta_t + \xi_t) \quad \text{for} \quad \xi_t := \varepsilon^{\frac{1}{3}} \left(\frac{3\Gamma_t^2 S_t e^{-r(t-s)}}{2\omega} \right)^{\frac{1}{3}}. \tag{3.5}$$

$\Gamma_t = \frac{\partial^2 V_t}{\partial S_t^2} = \frac{\partial \Delta_t}{\partial S_t}$ is the gamma of the derivative with value V_t to be hedged, and s is the latest portfolio rebalancing time prior to time t. The parameter ω controls the trade-off between aversion to costs and to the risk of deviating from the replicating strategy in a frictionless market. If $\omega < 1$ the aversion to costs dominates. Whalley & Wilmott (1997) argue that if the boundary of the no-trading band is crossed, the minimum number of risky shares necessary to bring the position back to the edge of the no-trading band should be bought or sold.

The derivation of this portfolio rebalancing rule rests on Black-Scholes assumptions. These affect both the expression for the bandwidth ξ and the calculation of the barrier option greeks. The rule (3.5) will be implemented as a rebalancing rule for the CPPI portfolio, even though the setting in the analysis to come not necessarily is that of Black-Scholes. Thereby, it cannot be claimed that (3.5) is an optimal rebalancing strategy for the CPPI portfolio, yet it is a reasonable simplification.

For a continuously adjusted CPPI portfolio, Δ and Γ can be calculated. Δ is merely the number of shares held in the stylized version: $\Delta_t = \frac{mC_t}{S_t}$. In the present setup $\Delta_t \neq \eta_t$ since the process (η_t) is piecewise constant with jumps only at rebalancing dates. Furthermore

$$\Gamma_t = \frac{\partial \Delta_t}{\partial S_t} = \frac{S_t \frac{\partial (mC_t)}{\partial S_t} - mC_t \frac{\partial S_t}{\partial S_t}}{S_t^2} = \frac{m}{S_t}\frac{mC_t}{S_t} - \frac{mC_t}{S_t^2} = \frac{\Delta_t(m-1)}{S_t}.$$

The trading strategy (3.5) is referred to as the *bandwidth rebalancing rule*, and states to adjust the number of shares to equal $\Delta_t \pm \xi_t$, if $\eta_t \gtrless \Delta_t \pm \xi_t$.

The bandwidth rebalancing rule implies more frequent portfolio adjustments when the cushion is small, because the Γ of a CPPI is an increasing function of the cushion. If the floor is stochastic (as would be the case with stochastic interest rates) having a cushion dependent rebalancing strategy appears crucial.

3.2.5 Gap risk coverage

The CPPI issuer will charge certain fees for managing the CPPI deal. An upfront fee f_u can be deducted from the initial investment thereby lowering the highest possible capital guarantee to $G < I(1 - f_u)p(0,T)$. In the following, a potential upfront fee is assumed to cover administration costs. The issuer may also impose modifications of the CPPI strategy to cover or reduce gap risk. Three approaches for covering losses from gap events are studied in this section.

Covering gap risk by fee charge

As compensation for losses arising from potential gap events, the issuer may simply charge a running fee f_r. Let the fee be given as a percentage of initial investment and suppose the size depends on the multiplier m. The value of the issuer's engagement F_T at expiry T is the sum of fee payments at dates $t_1 < t_2 < \cdots < t_n < T$ as long as no gap event has occurred, plus the value of the cushion if a gap event occurs:

$$F_T = \sum_{i=1}^{n} \mathbf{1}_{\{t_i<\tau\}} f_r(m) e^{\int_{t_i}^{T} r_u du} + \mathbf{1}_{\{\tau<T\}} C_\tau e^{\int_{\tau}^{T} r_u du}, \quad \tau = \inf\big\{t < T | C_t < 0\big\}.$$

The upfront fee payment is excluded in this calculation since it is supposed to cover only administration costs. The determination of the fee f_r at initiation of the CPPI deal is based on some risk measure of the issuer's engagement. Popular risk measures are expected loss given a loss occurs $E[F_T|F_T < 0]$, probability of loss $P[F_T < 0]$, Value at Risk (VaR) and expected shortfall (ES):

$$\begin{aligned} VaR_\alpha[F_T] &= \sup\big\{l \in \mathbb{R} \,|\, P[F_T < l] < 1 - \alpha\big\} := q_{(1-\alpha)} \\ ES_\alpha[F_T] &= \frac{1}{1-\alpha}\Big(E[F_T \mathbf{1}_{\{F_T < q_{(1-\alpha)}\}}] + q_{(1-\alpha)}(P[F_T \geq q_{(1-\alpha)}] - \alpha)\Big). \end{aligned}$$

$q_{(1-\alpha)}$ is the lower $(1-\alpha)$'th quantile in the distribution of F_T. The running fee $f_r(m)$ is chosen as the smallest fee charge that fulfils the issuer's risk management requirement. A possible condition is that the expected shortfall is above some lower bound.

Hedging gap risk

Gap risk can be hedged using gap options (studied in Tankov (2008)) or short maturity put options as suggested by Cont & Tankov (2009). In practice this hedging approach requires that such derivatives on the underlying asset are liquidly traded. Here, put options are chosen as hedge instruments, since these are more commonly available than gap options.

Consider the hedge position entered at time t, and let Δt denote the length of the hedge intervals. Assuming the floor has not been broken, a new put position expiring at time $t + \Delta t$ is entered as the existing hedge portfolio expires. The strike K must be chosen such that the put options are in the money, $K > S_{t+\Delta t}$, if the floor is broken, $C_{t+\Delta t} < 0$. When Δt is small, it is reasonable to assume that no trading takes place between time t and $t + \Delta t$. If the CPPI portfolio is rebalanced when new hedge-positions are entered (i.e. at equidistant time steps $t + h\Delta t$, $h = 1, 2, ..$), K can be found by applying equation (3.2) as done in Cont & Tankov (2009). However, if the portfolio is rebalanced more rarely, one must allow for $e_t \neq mC_t$ when determining the strike price:

$$\begin{aligned} & 0 > C_{t+\Delta t} = V_{\Delta t} - p(t + \Delta t, T) \approx V_t + (S_{t+\Delta t} - S_t)\eta_t - p(t, T) \\ \Leftrightarrow \quad & S_{t+\Delta t} < \frac{1}{\eta_t}\big(p(t, T) - V_t\big) + S_t. \end{aligned}$$

Investing in η_t put options with strike $K = \frac{1}{\eta_t}\big(p(t, T) - V_t\big) + S_t$ (almost) eliminates gap risk in the period $[t, t + \Delta t]$. Hedging costs are deducted from the CPPI portfolio value. A small cushion or a large position in risky assets, η_t, increases the cost of hedging gap risk. Tankov (2008) finds that (next to continuous re-hedging) this hedge strategy performs best if the hedge position is unwound immediately after a gap event. Hedging cannot eliminate gap risk completely due to approximations in the calculations above and due to market frictions not considered in the construction of the hedge.

Artificial floor

Gap risk can also be reduced by introducing an artificial floor above the true floor, such that the risky exposure is unwound if the artificial floor is broken. The true floor is otherwise applied for determining the risky exposure. This gives the issuer a buffer to absorb potential losses. The artificial floor approach has an indirect cost; it eliminates the chance of the CPPI recovering in scenarios where the artificial floor is broken but the true floor is not.

Paulot & Lacroze (2009) compare the reduction in gap risk from four arbitrarily chosen buffer sizes. Inspiration for applying an appropriate buffer can be found in the barrier option literature. In a Black-Scholes setup, Broadie et al. (1997) provide a price approximation for discretely monitored barrier options, where the regular barrier option pricing formula is used with an adjusted barrier. In particular, they find that for down-and-out call options the first monitoring instant, where the underlying asset value is observed below the barrier B, the expected difference is $B(1 - e^{-\beta\sigma\sqrt{\Delta t}})$; this difference is referred to as the asset's undershoot of the barrier. σ is the Black-Scholes volatility, Δt is the time between monitoring instants and β is a constant $\beta = \frac{-\zeta(\frac{1}{2})}{\sqrt{2\pi}} \approx 0.5826$, where ζ denotes the Riemann zeta function.

The barrier level of the down-and-out call position equivalent to the CPPI portfolio is $B_t = \frac{1}{\eta_t}\big(p(t,T) - b_t\big)$. By applying the results of Broadie et al. (1997), the CPPI issuer can avoid the average loss in a gap event if employing the artificial barrier $\tilde{B} = B\big(2 - e^{-\beta\sigma\sqrt{\Delta t}}\big)$, which corresponds to an artificial floor at

$$b_t + \eta_t \tilde{B}_t = b_t + \big(p(t,T) - b_t\big)\left(2 - e^{-\beta\sigma\sqrt{\Delta t}}\right) > p(t,T). \tag{3.6}$$

The average undershoot found by Broadie et al. (1997) is relevant for pricing purposes, however, for the risk management purpose considered here, the issuer should preferably cover losses arising in case of the maximum (or some higher quantile) undershoot of the barrier. Furthermore, the results of Broadie et al. (1997) rely on Black-Scholes assumptions, which are not necessary the assumptions in the forthcoming analysis. An application of (3.6) for an underlying asset with heavier tails in the return distribution will therefore possibly underestimate the average undershoot of the barrier. These issues are considered further in section 3.4.3, but for now it is merely noted that gap risk is not completely eliminated by employing the artificial floor given in (3.6).

3.2.6 Extending the strategy by profit lock in

Profit lock in is an additional feature which can be added to the CPPI strategy to secure part of the return against future depreciations in the underlying asset. If at time $\tau_1 \in [0, T]$ the portfolio value reaches some level $(1 + \theta_V)V_0$, a part of the profit is locked in by raising the floor to $(1 + \theta_F)p(\tau_1, T)$. Profit may be locked in several times: if at time τ_i the portfolio value crosses $(1 + \theta_V)V_{\tau_{i-1}} = (1 + \theta_V)^i V_0$, the floor is raised to $(1 + \theta_F)^i p(\tau_i, T)$.

Whereas the original floor is equal to the present value of the capital guarantee covered by the issuer, the higher floor following a profit lock in event is only a temporary floor intended to secure returns and is not considered a new guarantee in this analysis.

3.3 Model setup

3.3.1 The underlying risky asset

Consider an arbitrage-free model represented by a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where P denotes the real-world probability measure of market scenarios. By the assumption of no arbitrage, there exists some pricing measure $Q \sim P$. The risk free asset has the dynamics $dR_t = r_t R_t dt$. In the following, the interest rate process r is assumed to be constant, although this is by no means necessary.

The underlying asset is modelled as a stochastic volatility process with jumps in asset value. This process allows for heavier tails in the return distribution than the geometric Brownian motion, and has this as a special case. Under the real-world probability measure P the dynamics of S is given by

$$\begin{aligned} \frac{dS_t}{S_t} &= \mu dt + \sqrt{v_t} dW_t^1 + (Z_t - 1) dN_t \\ dv_t &= \kappa(\theta - v_t)dt + \xi \sqrt{v_t} dW_t^2, \end{aligned} \tag{3.7}$$

where W^1, W^2 are Brownian motions with correlation ρ and N is a Poisson process with intensity λ and independent of W^1, W^2. The relative jump size $\log Z_t \sim \mathcal{N}(\mu_J, \sigma_J^2)$ is assumed normally distributed with mean μ_J and volatility σ_J^2. Cont & Tankov (2009) argue that it is important for an assessment of gap risk to include jumps in the model because gap risk due to jumps cannot be eliminated even by continuous rebalancing.

3.3.2 CPPI investor

The investor's optimal portfolio choice is here restricted to a search among CPPI portfolios; i.e. consider the CPPI investor's problem of choosing an optimal multiplier m. Two approaches for describing the investment objectives are considered: by means of a utility function and by a risk/return profile.

Utility function

A commonly employed utility function U describing the preferences of an agent as a function of wealth w is the constant relative risk aversion (CRRA) utility function[3]

$$U^{CRRA}(w) = \frac{w^{1-\gamma} - 1}{1 - \gamma}. \tag{3.8}$$

Black & Perold (1992) and El Karoui, Jeanblanc & Lacoste (2005) study optimal portfolio choice problems for CPPIs and for general portfolios with a capital guarantee, respectively, using a CRRA utility function under a minimum consumption constraint.

The notion of risk aversion concerns any deviation, positive and negative, from the expected wealth. Work in behavioural finance suggests that simple utility functions such as (3.8) do not fully capture investors' behaviour. In their seminal studies, Kahneman & Tversky (1992) find that decisions are more driven by loss aversion and the prospect of terminal wealth being lower than the current.

Maringer (2008) suggests that loss aversion can be introduced as a kink in the utility function by placing more emphasis on losses in form of a loss aversion parameter. This way, the utility curve becomes steeper to the left of the initial level of wealth. Sharpe (2006) equips agents, who invest in protected investment products, with a CRRA utility function with two kinks, which arise from applying different risk aversion parameters in

[3]The nominator is modified by subtracting a constant compared to the standard form to give a well-defined limit as $\gamma \to 1$.

different wealth regions. Here, a CRRA utility function with a kink at the initial wealth level (equal to the capital guarantee G) is applied

$$U^{kCRRA}(w) = \begin{cases} \frac{w^{1-\gamma_1}-1}{1-\gamma_1} + C & w \leq G \\ \frac{w^{1-\gamma_2}-1}{1-\gamma_2} & G < w \end{cases}, \tag{3.9}$$

where $0 \leq \gamma_1 \leq \gamma_2$ are the risk aversion coefficients and C is a constant, which ensures a continuous utility function. The intuition is that investors have a higher risk aversion once the desired level of wealth is obtained. This type of kinked CRRA utility is also applied by Jessen & Jørgensen (2009) for investments in structured bonds. The utility functions (3.8) and (3.9) are shown in figure 3.1.

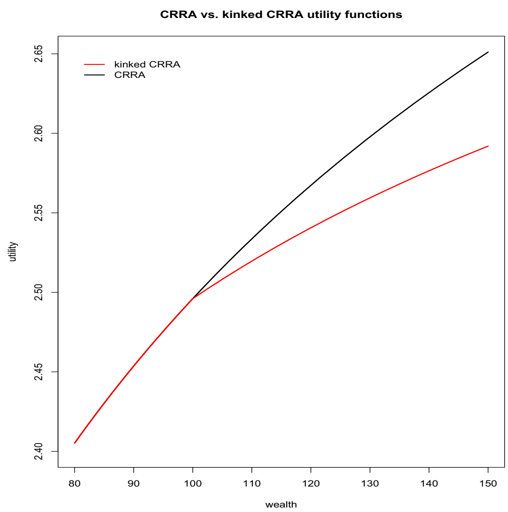

Figure 3.1: Kinked CRRA utility with $\gamma_1 = 1.2$, $\gamma_2 = 1.3$ compared to CRRA utility with $\gamma = 1.2$.

The objective of the CPPI investor is to maximize expected utility of terminal wealth over possible multipliers $m > 1$ given the set of conditions $\Phi(m)$ in the CPPI contract. The issuer sets the fees $f_u, f_r(m)$, borrowing limit b, spread δ charged for providing additional capital, rebalancing rule

$RR \in \{E, M, B\}$ (rebalancing at *e*quidistant time steps, *m*arket-based or *b*andwidth rebalancing) and the approach for gap risk coverage $GRC \in \{FC, H, AF\}$ (*f*ee *c*harge, *h*edging or *a*rtificial *f*loor). Trading costs ε reflects the liquidity of the risky asset. The utility maximization problem of the investor is

$$\max_{m>1} E_0[U(W_T(m))] \qquad \text{given} \qquad \Phi(m) = \{RR, GRC, f_u, f_r(m), \delta, b, \varepsilon\},$$

where $W_T(m)$ is the wealth at time T from investing in a CPPI with multiplier m, assuming this investment is held until expiry T.

Risk/return profile

An alternative to the utility maximization approach is to evaluate an investment by its risk-adjusted return, i.e. by measuring expected return relative to some risk measure. A well-known example is the Sharpe ratio, which measures expected excess return relative to its standard deviation. However, standard deviation as risk measure can be misleading, since it punishes large losses and gains equally. This is avoided by the Sortino ratio, which measures expected excess return relative to the square root of the second lower partial moment (also known as semi-standard deviation). Denote by r^{CPPI} the T-year logarithmic return of a CPPI, and let the excess return be given over a T-year risk free investment. Then the Sortino ratio SR is given by

$$SR_x = \frac{E[r^{\mathrm{CPPI}}] - rT}{\sqrt{E\left[(r^{\mathrm{CPPI}})^2 | r^{\mathrm{CPPI}} < x\right]}}. \tag{3.10}$$

Pedersen & Satchell (2002) gives the Sortino ratio a theoretical foundation as a performance measure by relating it to the so-called maximum principle. The capital guarantee of a CPPI ensures $r^{CPPI} \geq 0$, so the choice $x = 0$ implies $SR_0 = \infty$. Instead, setting $x = rT$ (referred to as modified Sortino ratio by Pedersen & Satchell (2002)) reflects the investor's expectation that the CPPI will outperform the risk free investment. By using the semi-standard deviation as risk measure, the Sortino ratio is sensitive to asymmetric return distributions as e.g. the CPPI's. The Sortino ratio has been applied as CPPI performance measure by Constantinou & Khuman (2009) and Cesari & Cremonini (2003).

Like in the utility maximization approach, an investor with this risk/return profile seeks to maximize the Sortino ratio of the CPPI investment over possible multipliers $m > 1$.

3.4 Numerical experiments

The performance of the CPPI strategy is analysed by Monte Carlo simulations. The aim is to investigate implications of introducing market frictions into the setup, and in particular to compare the rebalancing rules and the three approaches for gap risk coverage. Expected values, risk measures, etc. reported here are based on 100 000 simulations.

3.4.1 Base case

The risky asset dynamics is simulated using the following set of parameters reported by Eraker (2004), who estimates S&P 500 index return data over a 3-year period:

$$\begin{array}{lllll} \mu = 0.066 & \rho = -0.586 & \lambda = 0.504 & \mu_J = -0.004 & \sigma_J = 0.066 \\ v_0 = 0.042 & \theta = 0.042 & \kappa = 4.788 & \xi = 0.512 & r = 0.02 \end{array}$$

The base case is a CPPI contract with $T = 5$ years to expiry. An upfront fee $f_u = 1\%$ of initial investment is assumed, cost of capital is $\delta = 1\%$ and a borrowing limit at $b = 2$ is imposed. Proportional transaction cost for trading in the underlying asset is set to $\varepsilon = 0.5\%$. The portfolio is evaluated once a day, but only rebalanced so that $e_t = mC_t$ according to the market-based strategy (3.4), for which the choices of τ_u and τ_d are discussed in section 3.4.2. No means to cover gap risk are taken yet.

The top panel in figure 3.2 shows that a pure index investment outperforms the CPPI with respect to expected return. The CPPI investor indirectly pays a cost of insurance: the cost of forfeiting higher returns from a direct investment in the underlying index. The cost of insurance over the 5-year period ranges from 18–24 percentage points. As anticipated the CPPI gives a higher return than the risk free investment. The highest CPPI return is accomplished for multipliers $m \in (4, 6)$. In contrast, in a frictionless Black-Scholes model the expected return of a CPPI portfolio

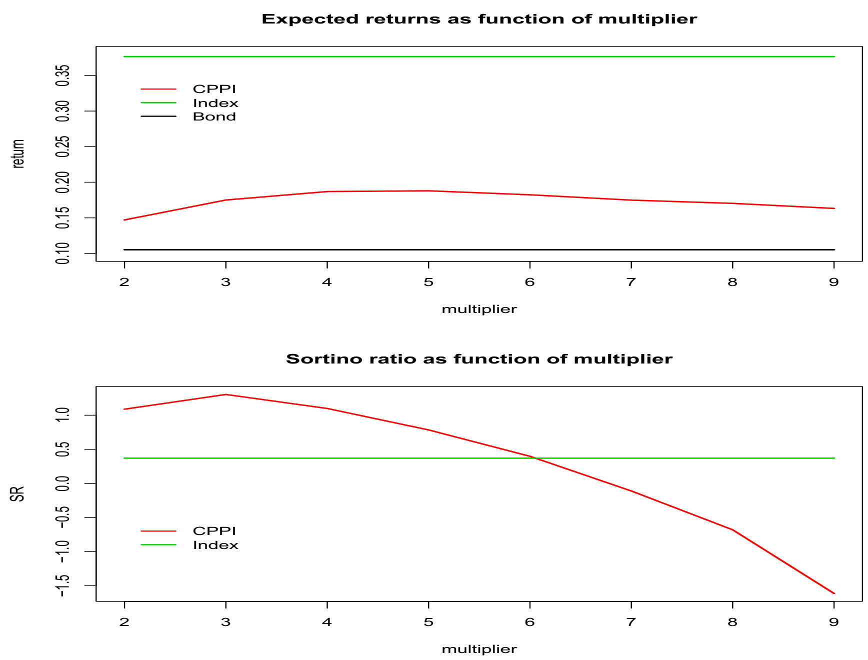

Figure 3.2: Top: Expected returns of underlying stock index, risk free investment and CPPI with market-based rebalancing as function of the multiplier. Bottom: Sortino ratio of index and CPPI investments.

can be increased indefinitely by choosing a high enough multiplier (shown in e.g. Cont & Tankov (2009)).

If the investor is risk averse, (s)he will care not only about expected returns of an investment but also about higher moments of the return distribution. A higher CPPI multiplier implies higher risky exposure, and thereby an increased variance in CPPI returns. With the fairly low borrowing restriction imposed here, variance of the returns of a CPPI with $m = 9$ is still slightly lower than the variance in pure index returns. However, without the borrowing restriction the variance of CPPI returns (and the expected CPPI return) is much higher. Furthermore, since CPPI returns are bounded below the return distribution will have a positive skew.

By applying the semi-standard deviation as risk measure, the Sortino ratio (3.10) is sensitive to the asymmetry in the CPPI return distribution. The bottom panel in figure 3.2 shows that investors with a risk/return profile described by the Sortino ratio will prefer a CPPI with multiplier $m \in (3, 4)$. Even though such investors favour the downside protection provided by a CPPI, they would choose the uninsured index investment over a CPPI with $m > 6$.

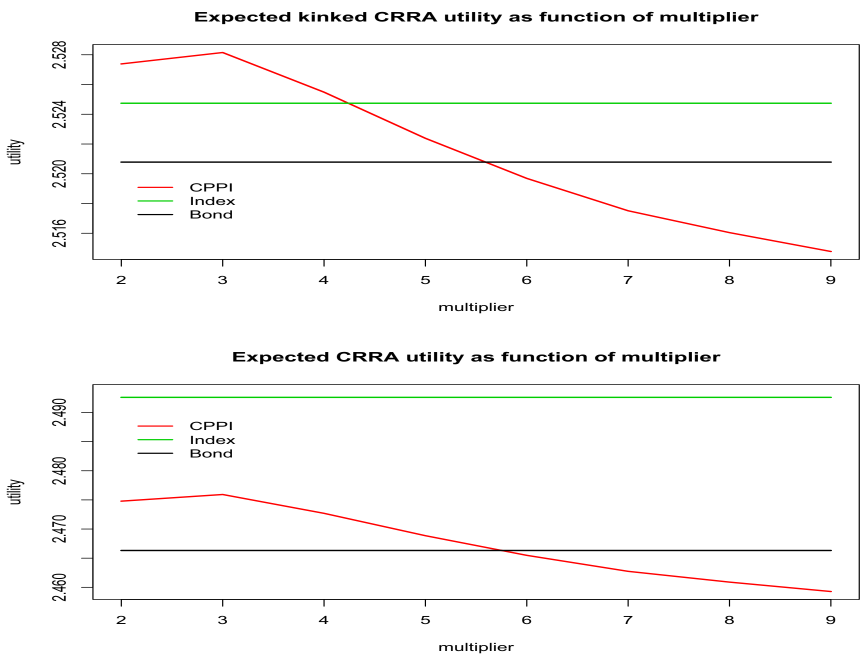

Figure 3.3: Top: Investor's expected kinked CRRA utility with $\gamma_1 = 1.2$ and $\gamma_2 = 1.3$ from investing in a CPPI, the underlying index and the risk free investment. Bottom: Expected CRRA utility with risk aversion parameter $\gamma = 1.25$ from investing in a CPPI, the underlying index and the risk free investment.

Figure 3.3 illustrates the ranking of the index, bond and CPPIs with different multipliers when a utility maximization approach is taken. The top panel shows expected kinked CRRA utility (3.9) with risk aversion parameters $\gamma_1 = 1.2$ and $\gamma_2 = 1.3$. An investor with such preferences would choose a CPPI with multiplier $m = 3$ and prefer the uninsured index investment to a CPPI with multiplier $m > 4$. Other risk aversion parameters will alter the investor's ranking of the three investments, however the shape of the CPPI expected utility curve as a function of multipliers will remain more or less unchanged.

A quantification of the difference in expected utility obtained from two investments can be given in terms of the certainty equivalent: the risk free amount one investment should be compensated by on average to obtain the same expected utility as the other investment. With kinked CRRA preferences the pure index investor should be compensated by 1.2% of initial investment and the risk free investor by 3.4% to be as well off as an investor in a CPPI with $m = 3$.

The bottom panel in figure 3.3 shows the expected CRRA utility (3.8) with risk aversion parameter $\gamma = \frac{1}{2}(\gamma_1 + \gamma_2) = 1.25$ for the CPPI, index and risk free investments. For this risk aversion parameter a CPPI investment is not preferred to the uninsured index for any multiplier. A more risk averse investor would invest in a CPPI; equipped with a risk aversion parameter $\gamma > 1.9$, the investor would choose a CPPI with the smallest possible multiplier $m = 2$.

Since the CPPI investment chosen based on the Sortino ratio and kinked CRRA preferences are similar, the former will be applied henceforth. Applying the utility maximization approach with kinked CRRA utility would not alter the overall conclusions.

3.4.2 Comparing rebalancing strategies

Now consider the two rebalancing strategies, market-based (base case) and bandwidth rebalancing, suggested in section 3.2.4. These are compared to weekly rebalancing, where the CPPI portfolio is adjusted once a week irrespectively of changes in underlying variables. For all three strategies the risky portfolio is unwound as soon as the floor is observed broken.

An implementation of the market-based strategy (3.4) requires a choice of tolerance levels for the divergence of exposure, e, from its target, mC. Based on numerical experiments seeking to maximize expected CPPI return the tolerance levels $\tau_d = 0.04$ and $\tau_u = 0.5$ are applied. For $m = 5$

this corresponds to the following tolerance levels for movements in the index: $\alpha_u = 1.1\%$ and $\alpha_d = 7.2\%$.[4] Note that the condition $\tau_d < 1 - \frac{1}{m}$ given in section 3.2.4 is satisfied for all m. τ_d controlls the buy region where risky exposure is increased, and the fact that $\tau_d < \tau_u$ implies more frequent portfolio adjustments in rising markets than in falling markets. This is a natural consequence of choosing tolerance levels based on expected returns, without taking gap risk into consideration.

Bandwidth rebalancing (3.5) is employed with $\omega = 0.25$. This risk versus cost aversion ω was found experimentally to give the highest expected return among the range of values tried. Recall that in this respect the notion of risk refers to deviation from the stylized CPPI strategy. A choice of $\omega < 1$, which reflects a higher aversion to costs, is therefore not surprising. At initiation of a CPPI with $m = 5$, $\omega = 0.25$ corresponds to a no-trading band of $\pm 9.4\%$ around Δ_t. This translates to a $\pm 9.4\%$ move in the underlying index, although this number will change as the underlying variables change.

The expected return and Sortino ratio of a CPPI with weekly, market-based and bandwidth rebalancing are shown in figure 3.4. Weekly rebalancing cannot compete with the two customized strategies.[5] With respect to expected return the market-based strategy perform slightly better than bandwidth rebalancing for the low range of multipliers, while the opposite is true for multipliers $m \geq 5$. When performance is measured by the Sortino ratio bandwidth rebalancing is preferred.

CPPI investors with kinked CRRA preferences would arrive at a ranking of the three rebalancing strategies for different multipliers almost identical to that of the Sortino ratio. In terms of certainty equivalents, an investor in a CPPI employing market-based rebalancing should be compensated by -0.36% of initial investment for $m = 2$ and by 1.6% for $m = 9$ to be as well off as if employing bandwidth rebalancing. Correspondingly, a CPPI investment with weekly rebalancing needs 0.78–1.9% compensation to provide the same utility as with bandwidth rebalancing.

With respect to trading interventions, the cushion dependence of band-

[4] For given τ_u, τ_d, both $\alpha_u(\cdot)$ and $\alpha_d(\cdot)$ are decreasing in m.

[5] Daily rebalancing results in even lower expected returns. Monthly rebalancing is a competitor to the two customized strategies with respect to expected CPPI returns, although at the cost of large losses to the issuer. These alternatives are therefore disregarded.

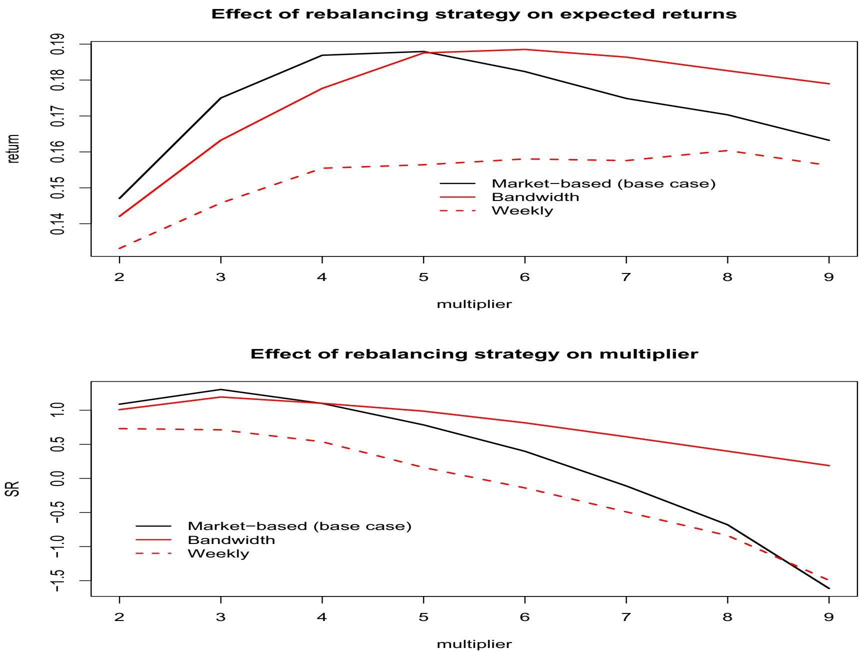

Figure 3.4: CPPI investor's expected return and Sortino ratio for CPPIs with portfolios adjusted according to market-based, bandwidth and weekly rebalancing rules.

width rebalancing makes this strategy more efficient than market-based rebalancing. The market-based strategy requires 16–250 trading interventions (increasing in m) over the 5-year period, whereas bandwidth rebalancing achieves comparable results by only 16–33 trades. If trading costs were introduced as a fixed cost per trade or a combination of fixed and proportional trading costs, bandwidth rebalancing would be advantageous.

3.4.3 Covering gap risk

Figure 3.5 shows risk measures of the issuer's engagement when managing a CPPI portfolio adjusted according to the market-based, bandwidth and weekly rebalancing strategies. Weekly rebalancing is not only more expensive to the investor but also to the issuer in terms of higher losses. This

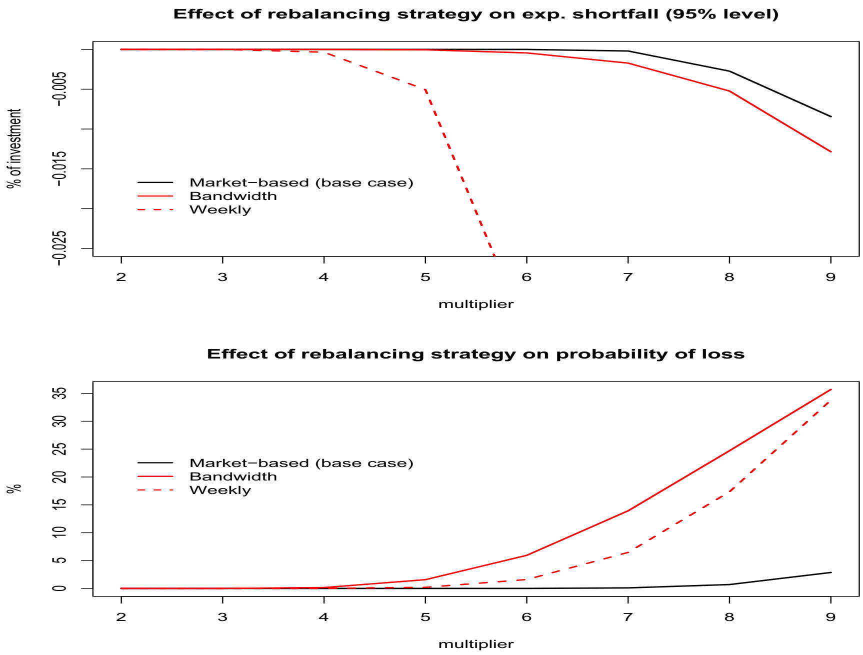

Figure 3.5: CPPI issuer's expected shortfall and probability of loss when the CPPI portfolio is adjusted according to market-based, bandwidth and weekly rebalancing.

result is not surprising since weekly rebalancing does not allow for rapid reduction of exposure during downside market moves. Although the probability of loss (bottom panel) is higher for bandwidth rebalancing than for market-based, the expected shortfall produced by the two strategies are comparable. Due to its previously observed efficiency with respect to number of trading interventions, bandwidth rebalancing will be employed henceforth.

The three approaches for reducing issuer's gap risk suggested in section 3.2.5 are now implemented. First, the issuer can charge a semi-annual running fee as compensation for potential losses. The fee size is here set such that the expected shortfall at the 95% level of the issuer's engagement is positive: $\min f_r(m)$ subject to $ES_{95\%}[F_T] > 0$. The fee charge is determined based on numerical experiments and the results are reported in table 3.1.[6]

m	2	3	4	5	6	7	8	9
f_r (%)	0	0	0.002	0.003	0.005	0.01	0.7	1.0

Table 3.1: Semi-annual running fee.

Alternatively, the issuer can reduce gap risk by introducing the artificial floor given in (3.6) with volatility parameter $\sigma = \sqrt{v_t}$. Broadie et al. (1997) notice that the approximation works poorly close to the barrier; i.e. when the CPPI exposure is close to zero. Therefore a multiplier dependent lower bound on exposure $\bar{e}_t = \max\{m, e_t\}$ is applied in the expression for the risk free holdings, $b_t = V_t - \bar{e}_t$, when calculating the artificial barrier of the down-and-out call position.

The third possibility introduced is to hedge gap risk using put options – here options with two weeks to expiry are used. Trading costs on derivatives ε_d are typically higher than on the underlying, and therefore $\varepsilon_d = 2\varepsilon$ is assumed.[7] Implementation of the hedging strategy requires hedge instruments to be priced. Eraker (2004) also estimates the dynamics of the underlying asset under the pricing measure Q used by the market, and reports the following Q-parameters:

$$\kappa^Q = 2.772 \qquad \theta^Q = 0.269^2 \qquad \mu_J^Q = -0.020 \qquad \mu^Q = r - \lambda\mu_J^Q = 0.050.$$

Figure 3.6 illustrates expected return and issuer's expected shortfall of CPPI investments incorporating the three approaches for gap risk reduction. The top panel shows that investor's expected return will be lowered only marginally, if the issuer chooses to reduce gap risk by hedging and even less if choosing an artificial floor. In a market where derivatives are less liquidly traded and trading costs are higher, the hedging approach would be more expensive. Covering potential losses by a running fee charge is seen to be both costly and ineffective for high multipliers.

[6] For such an analysis of tail events Monte Carlo simulation is not the best tool, and the results in table 3.1 should only be considered a crude approximation of the optimal solution to the minimization problem.

[7] This corresponds to a bid/ask spread of ± 0.002 on implied volatility for at-the-money-options.

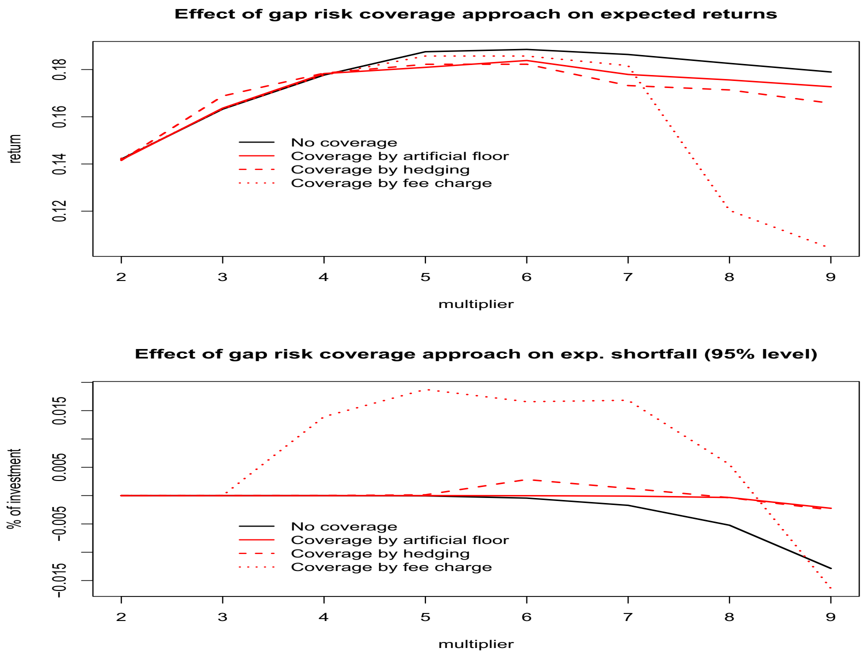

Figure 3.6: CPPI investor's expected return and issuer's expected shortfall when gap risk is covered by fee charge, hedging and artificial floor.

Another problem in covering gap risk by a fee charge is the cumbersome determination of the appropriate fee size: the procedure is not automatic and the fee size must be re-calculated for a different CPPI issuance.

The effectiveness of the three approaches for gap risk coverage is studied in the bottom panel in figure 3.6.[8] For multipliers $m > 7$ none of the three approaches are able to eliminate issuer's risk completely. Both hedging and the artificial floor approach reduce the losses significantly. Furthermore, a numerical investigation shows that the artificial floor is capable of practically eliminating issuer's risk at a small additional cost

[8]The expected shortfall, when covering gap risk by a fee charge, is positive for the middle section of multipliers due to inaccuracy in the numerical approximation.

to the investor, if placing an extra margin[9] in the floor. This supports the conjecture in section 3.2.5 that this approach can be improved by applying a buffer, that covers more than the average undershoot, as given in equation (3.6).

In the following the bandwidth rebalancing strategy with gap risk covered by an artificial floor is adapted as a reference case. By choosing this approach, concerns about whether the required short maturity put options for hedging gap risk are available can be ignored.

3.4.4 Effects of market frictions

This section explores the direct effects of introducing market frictions such as trading costs and capital restrictions in form of borrowing constraint and cost of capital.

Capital restrictions

The effects of capital restrictions imposed by the CPPI issuer on expected return and Sortino ratio are illustrated in figure 3.7. More precisely, the situations where an issuer provides no borrowing facility ($b = 1$), unlimited borrowing ($b = \infty$) and imposes no capital restrictions at all ($b = \infty$, $\delta = 0$) are compared.

Figure 3.7 shows that the borrowing restrictions have a considerable influence on the CPPI performance. While removing the borrowing facility completely has no visible effect on expected CPPI return, investors with a risk/return profile described by the Sortino ratio would actually prefer not to have any additional capital available. The reason is that without additional capital, exposure will be kept at a moderate level even for higher multipliers. There is even a minor improvement in the Sortino ratio from investing in a CPPI with a high multiplier relative to a small, whereas the opposite is true in the reference case.

If the issuer facilitates unlimited additional capital, the exposure is allowed to become extremely high, which implies possibly high returns. Since a consequence of higher exposure is higher variance in returns, the CPPI portfolio value hits the floor in the majority of scenarios (80% for $m = 9$). Therefore the expected CPPI return is lower than in the reference

[9] For $m = 9$ an artificial floor at $1.015p(t, T)$ will practically eliminate gap risk.

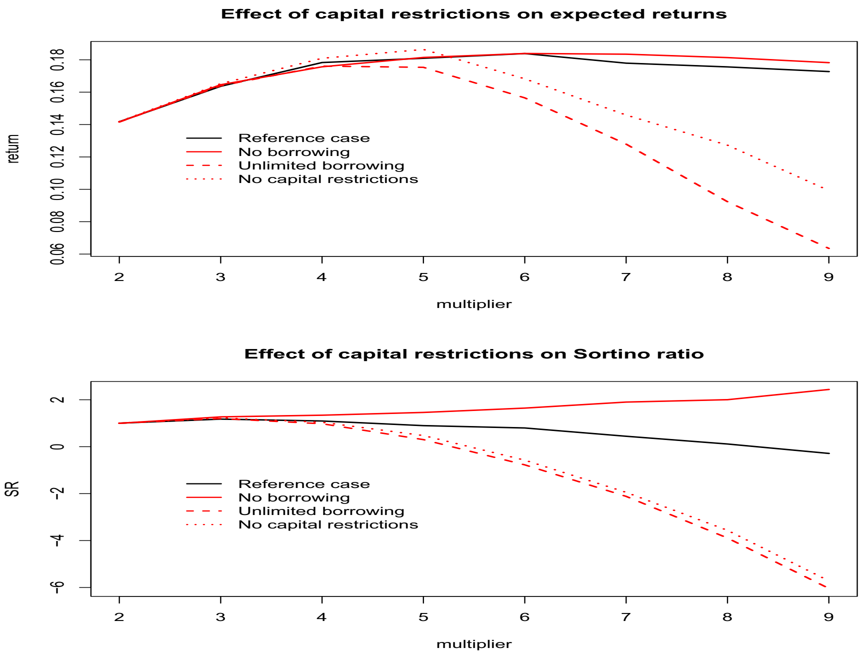

Figure 3.7: Effects of capital restrictions on expected return and Sortino ratio.

case for $m > 4$. If the issuer also removes the cost of capital, $\delta = 0$, there is a clear positive effect on expected CPPI return, although a poorer performance than in the reference case is still reported.

Trading costs also play an important role for the observation that investors are better off without the borrowing facility. This role is investigated next.

Trading costs and continuous rebalancing

The most common simplification in analyses of the CPPI strategy is to ignore trading costs[10] and allow for continuous rebalancing of the portfo-

[10] The importance of including trading costs in a CPPI analysis is also considered by MKaouar & Prigent (2007*b*).

lio. With $\varepsilon = 0$ the bandwidth trading strategy automatically rebalances the portfolio daily, which is a natural consequence of frictionless trading. The effects of this simplification on expected CPPI return and issuer's expected shortfall are investigated in figure 3.8. Two cases are considered: frictionless trading with the borrowing constraint at $b = 2$ reintroduced and frictionless trading combined with frictionless capital markets ($b = \infty$, $\delta = 0$).

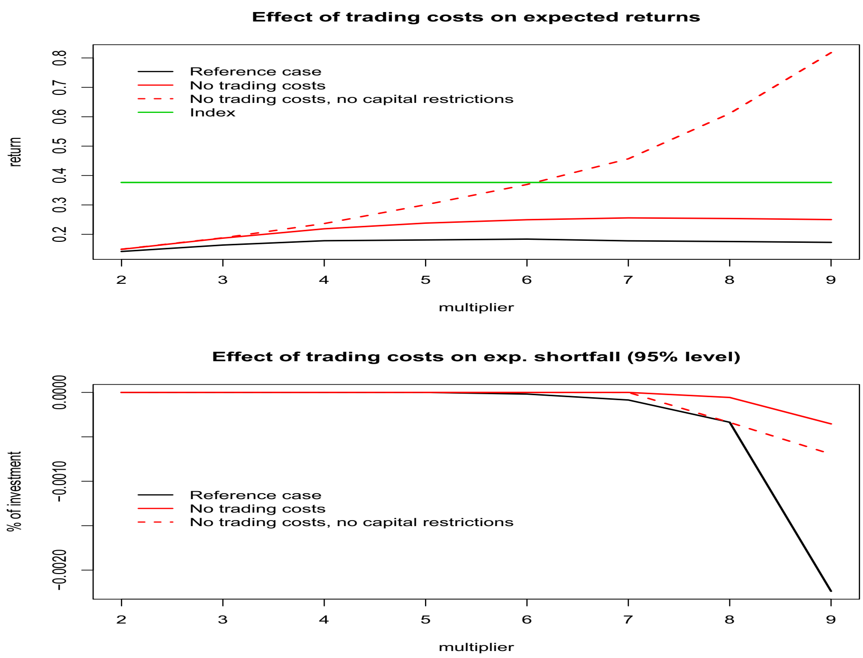

Figure 3.8: Effects of trading costs on investor's expected return and issuer's expected shortfall.

If trading in the underlying index was costless, the solid red curve in the top panel shows that expected CPPI return would improve significantly (notice the scale on the ordinate axis). If further assuming frictionless capital markets the CPPI would even outperform the pure index investment, when measured by expected return.

The bottom panel shows that daily portfolio rebalancing reduces gap risk significantly, since exposure is kept closer to its target level. If capital restrictions are removed, the high of exposure is seen to cause higher losses, although still significantly lower than the expected shortfall in the reference case.

3.4.5 Model risk

A popular choice of model setup preserving analytical tractability in the CPPI analysis is that of Black-Scholes. The effects of choosing this simpler model specification for the underlying asset is studied in figure 3.9. The volatility parameter of the geometric Brownian motion reported by Eraker (2004), $\sigma_{\mathrm{GBM}} = 0.202$, is employed.

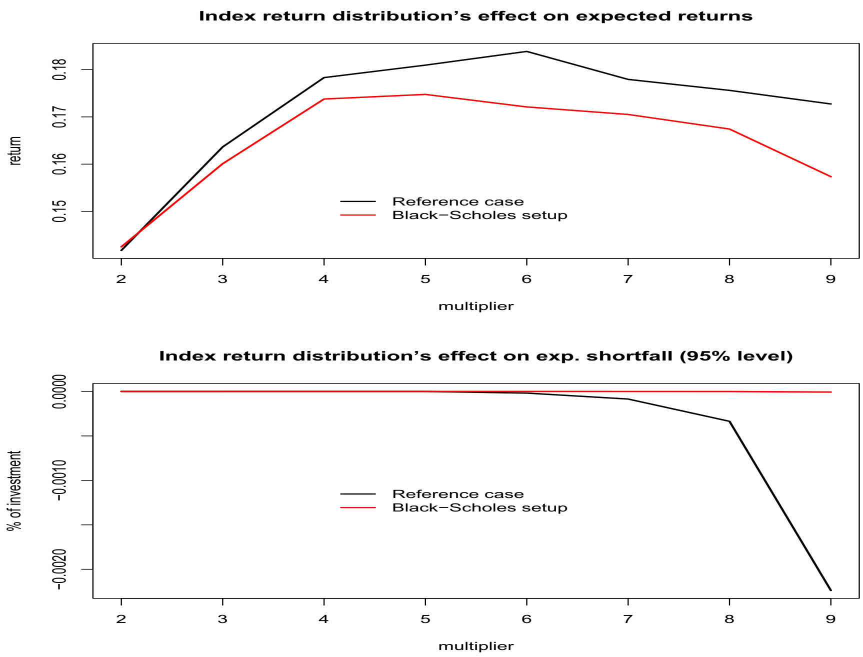

Figure 3.9: Effects on expected return and issuer's expected shortfall when modelling the underlying asset as a geometric Brownian motion.

The top panel shows that the lighter right tail of the return distribution in a Black-Scholes setup implies a lower CPPI return compared to the reference case. Furthermore, the underlying asset dynamics has a crucial effect on gap risk: with log-normally distributed returns the gap risk is essentially zero (even if removing gap risk coverage, not shown in figure 3.9), and consequently, a CPPI analysis in a Black-Scholes setup will possibly underestimate the true risk facing the issuer.

3.4.6 Profit lock in

The profit lock in feature is now added to the standard CPPI strategy. The effects of two choices of lock-in levels $\theta_V^1 = 10\%, \theta_F^1 = 2\%$ and $\theta_V^2 = 50\%, \theta_F^2 = 10\%$ are investigated, where the first gives frequent profit lock in events and the second more rare.

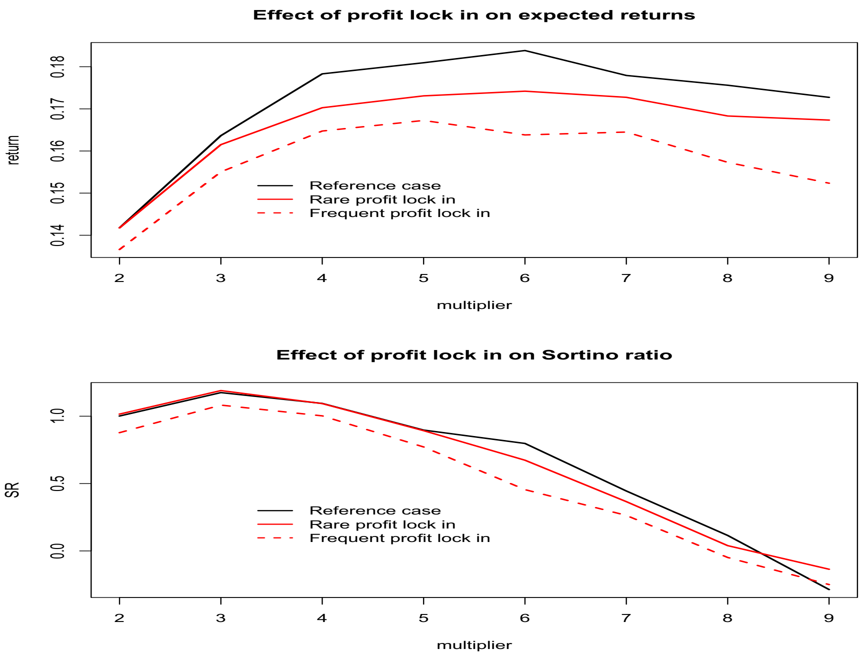

Figure 3.10: Effects of profit lock in on expected CPPI return and Sortino ratio.

Figure 3.10 depicts expected return and the Sortino ratio of CPPIs with profit lock in compared to a regular CPPI. For $\theta_V^1 = 10\%$, $\theta_F^1 = 2\%$, profits are locked in at least once in 70–47% (decreasing in m) of the scenarios. For the rare profit lock in, where the barrier is higher, the portfolio value of CPPIs with small multipliers will have a lower probability of reaching this level: profit is locked in for 4–17% (increasing in m) of scenarios. Figure 3.10 shows that neither expected return nor Sortino ratio are improved from this extension of the regular CPPI strategy; both measures indicate poorer performance. The rare profit lock in performs slightly better than frequent lock in. In a scenario where the borrowing constraint is not binding,[11] a profit lock in event will reduce exposure to the underlying. Based on the poorer performance observed in figure 3.10, this effect does not seem desirable to the CPPI investor.

Yet, profit lock in is commonly encountered in the CPPI literature and in actual implementations. In a more volatile market (e.g. for $\theta = v_0 = 0.09$), the rare profit lock in would indeed give better performance than the regular CPPI when measured by the Sortino ratio – although not with respect to expected return or kinked CRRA utility.

3.5 Summary

This paper studied the CPPI strategy in a setting taking market frictions into account. Trading costs, fees and borrowing restrictions were introduced, and most importantly the assumption of continuous portfolio rebalancing was relaxed.

The choice of discrete-time rebalancing rule was shown to play an important role for the expected return delivered by the CPPI strategy. I found that market-based and bandwidth rebalancing resulted in similar CPPI performance, although bandwidth rebalancing achieved this with fewer trading interventions. Moreover, the cushion-dependence of the bandwidth rebalancing rule may give this a further advantage over the marked-based rule if the floor is stochastic, e.g. in a setup with stochastic interest rates. An investigation of this is left for future research.

[11] If the borrowing constraint is binding, actual exposure will be lower than target exposure. In this situation, a profit lock in event will typically only reduce the difference, but have no direct effect on actual exposure.

From the perspective of a CPPI issuer the main objective is to limit gap risk. I found that hedging using short maturity put options and introduction of an artificial floor reduce gap risk effectively at a small cost to the investor. The artificial floor approach has the advantage that it does not depend on specific hedge instruments being available. Gap risk could possibly be further reduced by the artificial floor approach if applying a higher quantile instead of the mean of the risky asset's undershoot of the barrier. Future research also includes an investigation of the implications of applying the barrier option results for a Black-Scholes model in the stochastic volatility model with jumps.

Given the conditions in a CPPI contract imposed by the issuer, the investor will choose the CPPI multiplier according to his/her investment objectives. With expected return as performance measure, I found that the multipliers $m \in (4, 6)$ provided the best performance. Furthermore, risk averse investors with kinked CRRA preferences and investors with a risk/return profile given by the Sortino ratio had similar CPPI investment objectives. Both investor types would choose a multiplier $m \in (3, 4)$. This is well below the upper bound on multipliers $m \in (10, 17)$ found by Bertrand & Prigent (2002) and similar to the time and risk dependent multiplier found by Hamidi et al. (2009).[12] Since CPPI investors are found to prefer fairly low multipliers, the problems involved with covering gap risk for higher multipliers become less relevant.

In a setup that includes trading costs and cost of capital, I found that investors would prefer a CPPI with no additional capital available, both when measuring performance by expected returns and by the Sortino ratio. Such a restriction is also desirable for the issuer, since it reduces gap risk.

Finally, I studied the effects of extending the CPPI strategy by a profit lock in feature. In the setup considered here, the investor did not benefit from this extension neither with respect to expected return or Sortino ratio. However, profit lock in could be preferable to some investors in a more volatile market. Therefore, if profit lock in is provided, it should be carefully designed to meet specific investor demands in the given market scenario.

[12] To my knowledge these are the only papers considering the choice of CPPI multiplier. For illustration purposes Balder et al. (2009), Cont & Tankov (2009), Paulot & Lacroze (2009) and Boulier & Kanniganti (1995) apply multipliers $m \in (2, 10)$.

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4. Constant Proportion Debt Obligations: Modelling and Risk Analysis[1]

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Abstract: Constant proportion debt obligations (CPDOs) are structured credit derivatives indexed on a portfolio of investment grade debt, which generate high coupon payments by dynamically leveraging a position in an underlying portfolio of index default swaps. CPDO coupons and principal notes received high initial ratings from the major rating agencies, based on complex models for the joint transition of ratings and spreads for underlying names. We propose a parsimonious model for analysing the performance of CPDOs using a top-down approach which captures essential risk factors of the CPDO. Our analysis allows to compute default probabilities, loss distributions and other tail risk measures for the CPDO strategy and to analyse the dependence of these risk measures on parameters describing the risk factors. Though the probability of the CPDO defaulting on its coupon payments is found to be small, the ratings obtained strongly depend on the credit environment. CPDO loss distributions are found to be bimodal and our results point to a heterogeneous range of tail risk measures inside a given rating category, suggesting that credit ratings for such complex leveraged strategies should be complemented by other risk measures for the purpose of performance analysis. A worst-case scenario analysis indicates that CPDOs have a high exposure to persistent spread-widening scenarios. By calculating rating transition probabilities we find that ratings can be quite unstable during the lifetime of the CPDO.

Keywords: CPDO, credit risk, credit derivatives, top-down models

[1] We thank William Morokoff, William Dellal and Eric Raiten for helpful comments.

4.1 Introduction

Constant Proportion Debt Obligations (CPDOs) are leveraged credit investment strategies which appeared in the low credit spread environment of 2006 with the aim of generating high coupons while investing in investment grade credit. The asset side of the CPDO contains two positions: a money market account and leveraged credit exposure via index default swaps on indices of corporate names, typically the ITRAXX and DJ CDX. The dynamically adjusted risky exposure is chosen such as to ensure that the CPDO generates enough income to meet its promised liabilities and also to cover for fees, expenses and credit losses due to defaults in the reference portfolio and mark-to-market losses linked to the fair value of the index default swap contract.

The CPDO strategy involves high initial leverage but reduces its risky exposure as the gap between portfolio value and present value of liabilities narrows. In case losses are incurred leverage is increased in order to regain some of the lost capital. With this leverage rule a CPDO has no upside potential but it has an added ability to recover from negative positions at the cost of not having principal protection as for example the CPPI strategy. The "constant proportion" in the name CPDO refers to the fact that it operates with a piecewise constant leverage rule.

The first CPDO launched by ABN Amro paid coupons at 100bp above Euribor and later versions of the CPDO have paid up to a 200bp spread. Yet CPDO coupons and principal notes initially received top ratings from the major rating agencies. This top rating gave rise to an intense discussion among market participants, because standard top-rated products such as treasury bonds pay significantly lower coupons and also because the pool of corporate names on which the CPDO sells protection has significantly lower average rating.

When first issued, there were several studies on the risk and performance of CPDOs conducted by rating agencies (Wong et al. (2007) and Jobst et al. (2007)) and by issuers (Varloot et al. (2007)). The sensitivity analysis conducted in these studies suggested that the CPDO strategy is fairly robust and could overcome most historical credit stresses prior to the 2007-2008 financial crisis with low default rates (Lucas, Goodman & Fabozzi (2007)). However, one concern of agencies which chose not to rate this product was the potentially high level of model risk involved in the analysis of the CPDO strategy, given the large number of factors and parameters in these models. Another major concern was the limited extent of historical data for backtesting the strategy: spread data for the ITRAXX and CDX indices are only a few years in length (only a fraction of the risk horizon of CPDOs) and in this period credit markets had not been under serious stress. In hindsight this was a serious drawback since the 2007 credit crunch hit the markets quite suddenly and the following steep increase in ITRAXX and CDX spreads caused heavy CPDO losses. The continued market distress has forced many structures to unwind.

Rating agencies (Linden et al. (2007), Wong et al. (2007) and Morokoff (2007)) have analysed CPDOs using high-dimensional models for co-movements of ratings and spreads for all names in the reference portfolio. Defaults in the underlying index are generated through a detailed modelling of rating migrations of the underlying names and the index spread is modelled as a stochastic process depending on the average rating of the names in the index. This modelling approach leads to hundreds of state variables and is not accessible to entities other than rating agencies due to lack of historical data on ratings.

We argue that such a complex framework may not be necessary. We show that the main risk and performance drivers can be parsimoniously modelled using a *top-down* approach where the underlying credit portfolio is modelled in terms of its aggregate default loss. We model the rate of occurrence of defaults in the underlying index using a default intensity process and thereby arrive at a description of the cumulative loss process of the portfolio. This setting allows to study the key risk factors associated with CPDOs, while keeping estimation and simulation of the model at a simple level and enabling a meaningful sensitivity analysis. Our analysis allows an independent assessment of the credit ratings assigned by agencies, allows to compute default probabilities, loss distributions and other tail risk measures for the CPDO strategy.

The paper is organized as follows. Section 4.2 describes the CPDO strategy and the cash flows involved. Risk factors influencing these cash flows are analysed in section 4.3 and based on this we setup a one factor top-down model for the default intensity. We analyse the performance of CPDOs in a Monte Carlo framework in section 4.4 by studying ratings and risk measures in different credit market environments, by conducting a sensitivity analysis and evaluating transition probabilities for ratings. Section 4.5 summarizes our results and discusses some implications of our analysis.

4.2 The CPDO strategy

A CPDO is a dynamically leveraged credit trading strategy which aims at generating high coupon payments (typically 100-200bp above LIBOR rate) by selling default protection on a portfolio of investment-grade obligors with low default probabilities. The idea is to achieve this objective by dynamically adjusting a leveraged exposure to a credit index.

4.2.1 Description

An investor in a CPDO provides initial capital (normalized to 1 in the sequel) and receives periodic coupon payments of a contractual spread above the LIBOR rate until expiry T of the deal. The CPDO manager sells protection on some credit index via index default swaps on the notional which is leveraged up with respect to initial placement. The CPDO portfolio is composed of two positions: a short term investment, such as a money market account, denoted $(A_t)_{0\leq t\leq T}$ and a position in a T^I-year index default swap (typically the 5-year index default swap). The sum of the value of the swap contracts and the money market account is denoted by $(V_t)_{t\in[0,T]}$.

Initially, the notional paid by the investor, minus an eventual arrangement fee ($\simeq 1\%$) is invested in the money market account: $A_0 = 0.99$. The money market account earns interest at the LIBOR rate: we denote $L(t,s)$ the spot LIBOR rate quoted at t for maturity $s > t$.

The investor receives coupons at dates $\mathbf{CD} = \{t_l \leq T \,|\, l = 1, 2, ...\}$. CPDO coupons are paid out as a spread δ over LIBOR

$$c_{t_l} = \Delta(t_l)\big[L(t_{l-1}, t_l) + \delta\big],$$

where $\Delta(t) = t - \max\{t_l \in \mathbf{CD} |\, t_l < t\}$ is the time elapsed since last coupon payment date. The present value of these liabilities is called the *target value*:

$$TV_t = B(t,T) + \sum_{t_l \in \mathbf{CD} \cap [t,T]} E^Q \left[c_{t_l} e^{-\int_t^{t_l} r_s ds} | \mathcal{F}_t \right],$$

where $B(t,u) = E^Q \left[e^{-\int_t^u r_s ds} | \mathcal{F}_t \right]$ is the discount factor associated with some short rate process r and $\mathcal{F}_t$ is the market information at time t. If $V_t \geq TV_t$ then the CPDO manager can meet her obligations by simply investing (part of) the fund in the money market.

To be able to meet the coupon payments, the CPDO manager sells protection on a reference credit index (ITRAXX, CDX,..) by maintaining a position in index default swaps on the investor's notional that is leveraged by a factor m (the leverage ratio). This position generates income for the CPDO by earning a periodic spread, denoted $S(t, T^I)$ for the spread observed at time t of a swap expiring at time T^I. We denote by P_t the present value of these spread payments; i.e. P_t is equal to the present value of the premium leg of the index default swap at time t.

If a name in the underlying index defaults, the CPDO manager incurs a loss, which is magnified through leverage. We denote by $\mathbf{DT} = \{\tau_1 \leq \tau_2 \leq ... \leq \tau_{N_I}\}$ the set of default times in the index: τ_i represents the date of the i-th default event, N^I denotes the number of names in the underlying index ($N^I = 250$ for a CPDO referencing the ITRAXX and CDX), and $N_t = \sum_{i=1}^{N^I} \mathbf{1}_{\{\tau_i \leq t\}}$ is the number of defaults in the index up to time t.

The CPDO is said to *cash in* if the portfolio value reaches a value sufficient to meet future liabilities, i.e. $V_t \geq TV_t$. In this event all swap contracts are liquidated and the CPDO portfolio consists only of the money market account.

If, on the other hand, the value falls below a threshold k, $V_t \leq k$ (e.g. $k = 10\%$ of the investor's initial placement) the CPDO is said to *cash out*. In this case the CPDO unwinds all its risky exposures, ends coupon payments and returns the remaining funds to the investor.

A CPDO can default on its payments either by cashing out and thereby defaulting on both remaining coupon payments and principal note, or by simply failing to repay par to investor at maturity, in which case it defaults on its principal note. Default clustering in the reference portfolio or sudden spread-widening may result in a cash out event where the money market account is not sufficient to settle the swap contracts. This loss is covered by the CPDO issuer and the risk of such a scenario (known as "gap risk") is reduced by setting the cash out threshold strictly above zero.

Until expiry, a cash-in or a cash-out event occurs, the manager readjusts the leveraged position in index default swaps in order to ensure that she can meet future coupon payments. We will now describe the rule used for adjusting the leverage.

4.2.2 Leverage rule

At initiation there is a shortfall between the net value of assets, V_t, and the target value TV_t: $TV_0 > V_0$. To close this shortfall, target leverage m_t is chosen such that the income generated by the swap, P_t, equals the shortfall:

$$m_t = \beta \frac{TV_t - V_t}{P_t}. \tag{4.1}$$

β denotes a gearing factor that controls the aggressiveness of strategy. A more aggressive strategy can alternatively be achieved by including a cushion μ, such that the shortfall applied is $TV_t + \mu - V_t$.

The actual leverage factor is not adjusted continuously as this would involve significant trading costs in practice. The underlying index rolls into new indices every six months and it is therefore natural to update actual leverage $\left(\bar{m}^i\right)_{i=1,2,..}$ to equal target leverage on index roll dates **RD**:

$$\bar{m}^{i(t)} = m_t, \quad \text{for} \quad t \in \mathbf{RD} = \left\{ T_j \,\middle|\, T_j = \frac{j}{2},\, j = 1, ..., 2T \right\},$$

where $i(t) \in \mathbb{N}$ denotes the leverage factor index employed at time t. The leverage factor is also adjusted if it differs more than ε (usually $\varepsilon = 25\%$) from target leverage:

$$\bar{m}^{i(t)} = m_t, \quad \text{if} \quad \bar{m}^{i(t)-1} \notin \left[(1-\varepsilon)m_t, (1+\varepsilon)m_t\right].$$

The set of these dates rebalancing dates (excluding roll dates) will be denoted **RBD**. The actual leverage factor is automatically adjusted on default dates as the number of names in the underlying index is reduced by one until next roll date:

$$\bar{m}^{i(t)} = \frac{N^I - N_t}{N^I - N_{t-}} \bar{m}^{i(t)-1}, \quad \text{for} \quad t \in \mathbf{DT}.$$

The leverage factor is capped at a maximum level M in order to reduce the overall possible loss (usually $M = 15$).

By this strategy, the leverage factor employed by a CPDO is piecewise constant, hence the name "constant proportion" debt obligations. The leverage adjustment rule leads to an increase in leverage if losses occur in the index, and a decrease in leverage if the shortfall is reduced. It is therefore a "buy low, sell high" strategy as opposed, for instance, to more popular CPPI strategies (Cont & Tankov (2009)), which lead to a "buy high, sell low" strategy.

4.2.3 Cash flow structure

Spread income generated by the CPDO is determined by the average spread on the swap contracts held. Contracted spread changes every time the CPDO enters new swap contracts and is thereby a piecewise constant process denoted $(\bar{S}^i)_{i=1,2,\ldots}$. Initially, contracted spread is equal to observed spread: $\bar{S}^0 = S(0, T^I)$. On index roll dates existing swap contracts on the off-the-run index are liquidated and new on-the-run contracts are entered, i.e.

$$\bar{S}^{i(t)} = S(t, t + T^I) \quad \text{for} \quad t \in \mathbf{RD}.$$

At rebalancing dates on which the leverage factor is increased, the new contracts entered contribute to the contracted spread. For $t \in \mathbf{RBD}$

$$\bar{S}^{i(t)} = \begin{cases} \bar{S}^{i(t)-1}, & \bar{m}^{i(t)} < \bar{m}^{i(t)-1} \\ w\bar{S}^{i(t)-1} + (1-w)S\big(t, T_{j(t)} + T^I\big), & \bar{m}^{i(t)} > \bar{m}^{i(t)-1} \end{cases},$$

where $w = \frac{\bar{m}^{i(t)-1}}{\bar{m}^{i(t)}}$ is the relative weight of old contracts in the swap portfolio after releveraging, and $T_{j(t)}$ denotes the latest roll date prior to time t: $j(t) := \max\{j \,|\, T_j < t,\, T_j \in \mathbf{RD}\}$.

A change in the observed index default swap spread implies a change in the mark-to-market value, denoted MtM_t, of the swap contracts. Mark-to-market is the value of entering an offsetting swap with the same expiry and coupon dates:

$$MtM_t = \big(\bar{S}^{i(t)} - S(t, T_{j(t)} + T^I)\big) D_t^{\text{swap}},$$

where

$$D_t^{\text{swap}} := E^Q \left[\sum_{t_l \in \mathbf{CD} \cap [t, T^I]} e^{-\int_t^{t_l} r_s ds} \Delta(t_l) \left(1 - \frac{E^Q \left[N_{t_l} | \mathcal{F}_t \right]}{N^I} \right) \right]$$

is the duration of the swap contract. The value of the CPDO portfolio is given as the sum of the money market account and the value of swap contracts: $V_t = A_t + MtM_t$.

Liquidating (part of) the position in swap contracts leads to a profit or loss which is balanced by the money market account. On roll dates the entire position of swap contracts is liquidated and the profit/loss is

$$\bar{m}^{i(t)} \Big(\bar{S}^{i(t)} - S(t, T_{j(t)} + T^I) \Big) D_t^{\text{swap}}, \quad t \in \mathbf{RD}.$$

Note that on roll date $t \in \mathbf{RD}$ the spread at which protection on the off-the-run is bought back is $S(t, t + T^I - \frac{1}{2})$, whereas the spread of new on-the-run contracts is $S(t, t + T^I)$; new contracts have six months longer to expiry.

At rebalancing dates on which the leverage factor is decreased ($t \in \mathbf{RBD} \cap \{m_t < \bar{m}^{i(t)-1}\}$) a part of the swap contracts are liquidated giving the following profit/loss to the money market account:

$$\big(\bar{m}^{i(t)-1} - m_t\big) \Big(\bar{S}^{i(t)} - S(t, T_{j(t)} + T^I) \Big) D_t^{\text{swap}}.$$

In summary the cash flows of a CPDO can be decomposed into:

1. **Interest payments** $t \in [0, T]$: $A_{t-\Delta} L(t - \Delta, t) \Delta$, where Δ is time between interest payment dates.

2. **Coupon payments** $t_l \in \mathbf{CD}$: $-c_{t_l}$.

3. **Spread income** $t_l \in \mathbf{CD}$: $\bar{m}^{i(t_l)} \bar{S}^{i(t_l)} \Delta(t_l)$ (assuming spread premiums are paid on the same dates as CPDO coupons).

4. **Default loss $\tau \in \mathbf{DT}$**: $-\bar{m}^{i(\tau)}\frac{(1-R)}{N^I}$, where R is the recovery rate on a single default event.

5. **Liquidation of swap contracts**:

$$\bar{m}^{i(t)}\Big(\bar{S}^{i(t)} - S(t, T_{j(t)} + T^I)\Big)D_t^{\text{swap}}\mathbf{1}_{\mathbf{RD}}(t)$$
$$+\big(\bar{m}^{i(t)-1} - m_t\big)\Big(\bar{S}^{i(t)} - S(t, T_{j(t)} + T^I)\Big)D_t^{\text{swap}}\mathbf{1}_{(\mathbf{RBD}\cap\{m_t<\bar{m}^{i(t)-1}\})}(t).$$

Given that the value of the money market account and the CPDO portfolio is known up to but not at time t, A_t and V_t can be calculated in the following way:

$$\begin{aligned}
A_t &= A_{t-\Delta}\big(1 + L(t-\Delta, t)\Delta\big) + \Big(\bar{m}^{i(t)}\bar{S}^{i(t)}\Delta(t) - c_t\Big)\mathbf{1}_{\mathbf{CD}}(t) \qquad (4.2)\\
&\quad -m^{i(t)}\frac{(1-R)}{N^I}\mathbf{1}_{\mathbf{DT}}(t) + \bar{m}^{i(t)}\Big(\bar{S}^{i(t)} - S(t, T_{j(t)} + T^I)\Big)D_t^{\text{swap}}\mathbf{1}_{\mathbf{RD}}(t)\\
&\quad +\big(\bar{m}^{i(t)-1} - m_t\big)\Big(\bar{S}^{i(t)} - S(t, T_{j(t)} + T^I)\Big)D_t^{\text{swap}}\mathbf{1}_{(\mathbf{RBD}\cap\{m_t<\bar{m}^{i(t)-1}\})}(t)\\
V_t &= A_t + MtM_t. \qquad (4.3)
\end{aligned}$$

4.2.4 Risk factors

Based on the description above we can identify the following risk factors influencing the cash flows of the CPDO strategy:

- Spread risk
  The main determinant of the CPDO cash flows is the index default swap spread. With a deterministic spread, the target leverage rule is designed such that the CPDO is certain to cash in prior to expiry, given that there are no further defaults in the underlying portfolio. Therefore a stochastic model for the swap spread is essential for capturing the spread risk of the strategy.

  The swap spread evolution not only affects the swap premium income but also the profits/losses on roll and rebalancing dates. A sudden spread change will give rise to a single cash flow on roll dates, but it will have long term effects on the spread income; which of these effects that will dominate is not immediate clear.

The index roll will typically result in a downward jump in the swap spread since the downgraded names that are removed contribute with higher spreads than the investment grade names they are replaced by. On average this negative jump implies a mark-to-market loss on roll dates.

- Default risk
  The default rate in the underlying portfolio determines the average number of defaults during the lifetime of the CPDO. A higher default rate is negative for the CPDO performance due to higher expected credit losses.

  The recovery level affects the size of credit losses incurred at default dates although this is to some extent offset by its effect on the spread income, since lower recovery level implies higher swap spread and thereby higher spread premium income to the CPDO. Since recovery data is sparse a constant recovery level $R = 0.4$ is chosen.

- Interest rates
  The term structure of interest rates has two main effects on the cash flows. First, higher LIBOR rates imply higher coupon payments to the investor but this effect will more or less be offset by the higher interest accruing to the money market account. The interest rate also influences present value calculations via the discount factor, for example when determining the target value. The stochastic evolution of the interest rate can easily be incorporated in our framework but in the remainder of the paper we will focus on a constant term structure since the effect is of second order with respect to the credit spreads and their volatility.

- Liquidity risk
  The liquidity of the index default swaps also affects the cash flows via the bid/ask spread of the index. Note however that most CPDOs reference the most liquid indices, ITRAXX and DJ CDX. In the following we do not explicitly model liquidity risk though this can be done by introducing a bid/ask spread of the index at roll dates.

Rating of a CPDO structure

A CPDO is a structured product with leverage effects, and it is not straightforward to assign a credit rating to it. Ratings have been assigned to CPDOs by major rating agencies by comparing the default probability or the expected loss of the structure to thresholds which are typically adjusted versions of bond default probabilities (Linden et al. (2007) and Wong et al. (2007)). These ratings follow similar procedures adopted for CDO tranches (Cont & Moussa (2008)) and share many of their drawbacks. As will become clear in the sequel, we do not necessarily condone the use of such 'ratings' as an appropriate metric for a complex product such as a CPDO. However, given their widespread use, we will compute sample ratings in various examples and examine their properties in the case of CPDOs.

Separate ratings are assigned to the coupons and the principal note of a CPDO. In the sequel we will focus on the approach based on default probabilities.

The rating on the coupon note is based on the probability of the CPDO cashing out. This probability can be found by Monte Carlo simulations and is translated into a rating according to the ratingthresholds, an example of which is given in table 4.1. Both cash out scenarios and scenarios in which the CPDO survives until expiry but is unable to repay par in full will result in default on principal note and the probability of this is likewise found by Monte Carlo simulations. Thresholds in table 4.1 are used to translate the probability of default on principal into a rating.

%	**AAA**	**AA+**	**AA**	**AA-**	**A+**	**A**	**A-**
10 year PD	0.73	1.01	1.49	1.88	2.29	2.72	3.56
	BBB+	**BBB**	**BBB-**	**BB+**	**BB**	**BB-**	**B+**
10 year PD	4.78	7.10	12.31	14.63	19.94	26.18	32.76

Table 4.1: Standard & Poor's CDO rating thresholds in terms of default probabilities. Source: Gilkes et al. (2005).

Major rating agencies (Linden et al. (2007), Wong et al. (2007) and Morokoff (2007)) have analysed CPDOs using high-dimensional models for co-movements of ratings and spreads for all names in the reference portfolio. In such models the defaults in the underlying index are generated through a detailed modelling of rating migrations of the underlying names, and the index spread is modelled as a stochastic process depending on the average rating of names in the index. Such detailed joint modelling of rating and spread movements is not accessible to entities other than rating agencies due to lack of historical data on ratings. We will argue below that in fact such a complex framework may not be necessary: the main features of CPDOs can be captured with a low-dimensional model, which can be more readily estimated, simulated and analysed.

4.3 Top-down modelling of CPDOs

The above considerations show that the risk and performance of a CPDO strategy mainly depend on

- the behaviour of the index default swap spread
- the number of defaults/the total loss in the reference portfolio
- index roll effects

CPDO cashflows do not depend directly on features such as individual name ratings, the identity of the defaulting entities, the spreads of individual names, etc. This suggests that the risk of CPDOs can be parsimoniously modelled by describing defaults at the portfolio level using a top-down model.

We consider an arbitrage-free market model represented by a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ where P denotes the real-world probability of market scenarios (statistical measure). We consider as numeraire the zero-coupon bond $B(t,T)$ and denote by $Q \sim P$ the forward measure associated with this numeraire (Geman, El-Karoui & Rochet (1995)). The spot yield curve $s \mapsto R(t,s)$ at date t is defined by

$$B(t,s) = \exp[-(s-t)R(t,s)]$$

and the LIBOR rates at date t are given by $L(t,s) = \frac{\frac{1}{B(t,s)}-1}{s-t}$. In the examples we shall use a flat term structure $B(t,s) = e^{-r(s-t)}$ but this is by no means necessary.

Denote by N_t the number of defaults in the underlying portfolio up to time $t \leq T$: $(N_t)_{t\in[0,T]}$ is a point process. As we shall see below, we need to model the dynamics of N_t under P *and* Q. These dynamics will be described by specifying an *intensity* for N_t under each probability measure.

The T^I-year index default swap spread is determined such that the risk-neutral expected value of the default leg of the swap is equal to the expected value of the premium leg. Denote by $(L_t)_{t\in[0,T]}$ the loss process. Assuming a constant recovery level R across all names in the underlying index, we have $L_t = \frac{(1-R)}{N^I} N_t$. The default leg of the index default swap is a stream of payments that cover the portfolio losses as they occur. At time $t \leq T^I$ the cumulative discounted losses are given by

$$\begin{aligned}
D_t &= E^Q\left[\int_t^{T^I} B(t,s)dL_s \Big| \mathcal{F}_t\right] \\
&= B(t,T^I)E^Q\left[L_{T^I}|\mathcal{F}_t\right] - L_t - \int_t^{T^I} \overbrace{\frac{\partial}{\partial s}B(t,s)}^{-R(t,s)B(t,s)}\ E^Q\left[L_s|\mathcal{F}_t\right]ds \\
&= \frac{(1-R)}{N^I}\left(B(t,T^I)E^Q\left[N_{T^I}|\mathcal{F}_t\right] - N_t + \int_t^{T^I} R(t,s)B(t,s)E^Q\left[N_s|\mathcal{F}_t\right]ds\right).
\end{aligned}$$

The value of the premium leg at time t as a function of the index default swap spread S is

$$\begin{aligned}
P_t(S) &= S \sum_{t_l\in\mathbf{CD}\cap[t,T^I]} B(t,t_l)\Delta(t_l)\left(1 - \frac{E^Q\left[N_{t_l}|\mathcal{F}_t\right]}{N^I}\right) \\
&:= S\, D_t^{\text{swap}}.
\end{aligned}$$

Finally, the swap spread contracted at time t for a swap expiring at T^I is

$$S(t,T^I) = \frac{\frac{(1-R)}{N^I}\left(B(t,T^I)E^Q\left[N_{T^I}|\mathcal{F}_t\right] - N_t + \int_t^{T^I} R(t,s)B(t,s)E^Q\left[N_s|\mathcal{F}_t\right]ds\right)}{D_t^{\text{swap}}}.$$

4.3.1 Modelling default risk

The main ingredient to the model is the dynamics of the number of defaults N_t. We propose here two reduced-form approaches for modelling N_t. In the first approach, N_t is modelled as a Cox process: conditionally on some underlying market factor $(X_t)_{t\in[0,T]}$, N_t follows an inhomogeneous Poisson process with intensity $(\lambda(X_t))_{t\in[0,T]}$.

In the second approach, the occurrence of defaults is specified via the aggregate default intensity, defined as the conditional probability per unit time of a default in the portfolio. This intensity-based approach has been used in the recent literature to model portfolio credit risk by Cont & Minca (2008) and Giesecke & Kim (2007).

The choice of dynamics for the risk-neutral default intensity and/or hazard rate determines the slope of the term structure of credit spreads. This influences the CPDO performance via the profit/loss from liquidation of swap contracts on roll dates, since at these dates the CPDO manager buys back protection of a $(T^I - \frac{1}{2})$-tenor swap, protection that was initially sold with a T^I-year tenor. An upward (downward) sloping term structure will on average imply a profit (loss) on roll dates. Empirically, we typically observe an upward sloping term structure.

In both approaches, it is crucial to be able to compute the default swap spread in an efficient manner in the simulations and cash flow computations. As noted above, the expression for the swap spread requires computation of the expected number of defaults and/or the survival probabilities efficiently. These computations, especially the computation of the T^I-year swap spread, will be made tractable by choosing affine processes for the hazard rate (in the first approach) or the intensity (in the second approach) under Q.

Under an equivalent probability measure $P \sim Q$, the point process (N_t) will in general have a different intensity process (Brémaud (1981, Theorem VI.2.)) of the form $\lambda_t^Q = \vartheta_t \lambda_t^P$, where ϑ is a strictly positive predictable process which characterizes the risk premium for the uncertainty associated with the timing of defaults. In accordance with empirical observations that suggest a roughly constant ratio between historical and risk neutral default rates (Berndt, Douglas, Duffie, Ferguson & Schranz (2005)), we assume that the statistical default intensity is proportional to the risk neutral intensity: $\lambda_t^P = \frac{1}{\vartheta}\lambda_t^Q$ where ϑ is a risk premium parameter. We typically expect $\vartheta > 1$.

4.3.2 Hazard rates and Cox processes

A common approach in reduced-form modelling is to model the default counting process N as a Cox process (Lando (1998)). Let X be a Markov process on a probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ designating a risk factor and define $\mathcal{G}_t = \sigma\{X_s | s \leq t\}$. We model the hazard rate by $\lambda_t := \lambda(X_t)$ where $\lambda : \mathbb{R}^n \to \mathbb{R}_+$ is a non-negative function. We assume

$$\Lambda_t := \int_0^t \lambda_s ds < \infty \quad \text{almost surely} \qquad t \in [0, T].$$

Let $\tilde{N}$ be a standard unit rate Poisson process independent of $\mathcal{G}_t$. A Cox process N with intensity (λ_t) can be constructed as

$$N_t := \tilde{N}_{\Lambda_t}.$$

We may interpret the hazard rate λ_t as the conditional probability per unit time of the next default event given the evolution of the risk factor:

$$\lambda_t = \lim_{\Delta t \to 0} \frac{1}{\Delta t} P\big(N_{t+\Delta t} = N_{t-} + 1 | \mathcal{G}_t\big).$$

It is straightforward to check that $N_t - \int_0^t \lambda_s ds$ is a $\mathcal{G}_t$-martingale and thereby (λ_t) is a $\mathcal{G}_t$-intensity for N_t. Default times can then be simulated/generated successively as

$$\tau_i = \inf \left\{ t > \tau_{i-1} \big| \int_{\tau_{i-1}}^t \lambda_s ds \geq E_i \right\}$$

where $(E_i)_{i=1,..,N^I}$ is a sequence of independent, identically distributed standard exponential variables.

Conditions for preserving the Cox setting under an equivalent change of measure and computing the hazard rate under the new measure are provided by Duffie (2005). We assume here that the risk-adjusted dynamics of N_t under Q is also described by a Cox process with hazard rate $\lambda_t^Q = \vartheta \lambda_t$ under Q. In our case, the risk neutral portfolio hazard rate can be interpreted as the short-term spread for protection against the next default in the portfolio.

Closed-form expressions for the swap spread may be readily obtained by assuming that the risk neutral default intensity (λ_t^Q) is an affine process:

$$d\lambda_t^Q = \mu(\lambda_t^Q)dt + \sigma(\lambda_t^Q)dW_t + \eta dZ_t, \tag{4.4}$$

where the coefficients $\mu(\cdot)$, $\sigma^2(\cdot)$ and the intensity of the jump process Z are affine functions of λ_t^Q. Transform methods can be applied to give an explicit expression for the swap spread as done in Errais, Giesecke & Goldberg (2009). To compute $E^Q[N_s|\mathcal{G}_t]$ for $s \in]t, T^I]$ consider the 2-dimensional process $Y_t = (\lambda_t^Q, N_t)'$. Y is of the general affine form (4.4) and the drift function can be written $K_0 + K_1 Y_t$, the volatility $H_0 + H_1 Y_t$ and the jump intensity of the 2-dimensional jump process $\Lambda_0 + \Lambda_1 Y_t$. Define the Laplace transform $\theta : \mathbb{C}^2 \to \mathbb{C}$ of ν by

$$\theta(c) = \int_{\mathbb{R}^2} e^{c \cdot z} d\nu(z).$$

In affine models (Duffie, Pan & Singleton (2000)) the conditional expectation of the number of defaults can be expressed as an affine function of the state variable

$$E^Q[N_s|\mathcal{G}_t] = E^Q[v'Y_s|\mathcal{G}_t] = A(t) + B(t)Y_t$$

for $v = (0 \quad 1)'$, where $A : [0, s] \to \mathbb{R}$ and $B : [0, s] \to \mathbb{R}^2$ are determined by the following differential equations

$$\partial_t B(t) \;=\; -K_1' B(t) - \Lambda_1 \nabla\theta(0) \cdot \eta B(t) \tag{4.5}$$

$$\partial_t A(t) \;=\; -K_0 \cdot B(t) - \Lambda_0 \nabla\theta(0) \cdot \eta B(t) \tag{4.6}$$

with terminal conditions $A(s) = 0$ and $B(s) = v$. These expressions can in turn be used to compute

$$E^Q[N_s|\mathcal{F}_t] = E^Q[E^Q[N_s|\mathcal{G}_t]|\mathcal{F}_t] = A(t) + B(t)Y_t.$$

In special cases (4.5)-(4.6) can be solved analytically, providing an analytic expression for $E^Q[N_s|\mathcal{G}_t]$ and thereby for the swap spread.

Example 4.3.1 (A Cox process with CIR hazard rate) *Let the risk-neutral hazard rate (i.e. under Q) be defined by the CIR dynamics:*

$$d\lambda_t^Q = \kappa\big(\theta - \lambda_t^Q\big)dt + \sigma\sqrt{\lambda_t^Q}dW_t, \qquad \lambda_0^Q > 0. \tag{4.7}$$

This model leads to a mean-reverting and non-negative short term spread if $2\kappa\theta \geq \sigma^2$. For calculating the swap spread, we make use of the affine property of the CIR process which implies a closed form expression for the expected number of defaults: For $B(t) = (B_1(t), B_2(t))'$, $B_2(t) = 1$ and

$$
\begin{aligned}
B_1(t) &= -\frac{1}{\kappa}\big(e^{-\kappa(T^I-t)} - 1\big) \\
A(t) &= \int_t^{T^I} \kappa\theta B_1(s)ds = \frac{\theta}{\kappa}\left(e^{-\kappa(T^I-t)} - 1\right) + \theta(T^I - t),
\end{aligned}
$$

we have $E^Q[N_{T^I}|\mathcal{G}_t] = A(t) + B_1(t)\lambda_t^Q + N_t$ and thereby an analytic expression for the T^I-year swap spread.

◇

Another choice of hazard rate is the exponential OU process:

Example 4.3.2 (Exponential Ornstein-Uhlenbeck process) *In this model the hazard rate is assumed to follow*

$$\frac{d\lambda_t^Q}{\lambda_t^Q} = \alpha\big(\beta - \ln\lambda_t^Q\big)dt + \xi dW_t. \tag{4.8}$$

This process is mean-reverting and non-negative, which are desirable qualities for a hazard rate process, it has a log-normal distribution and is stationary for long time horizons. The exponential OU process produces heavier tails in the distribution of the default intensity than the CIR model, for which increments follow a χ^2-distribution. The process (4.8) is not affine and the index default swap spread needs to be computed via quadrature.

◇

4.3.3 Intensity-based approach

One restriction implied by specifying the intensity as a Cox process is that the default intensity process is not affected by the occurrence of default events. This leads to an underestimation of default clustering effects (Das, Duffie, Kapadia & Saita (2007)) due to the fact that in the Cox framework the hazard rate λ_t depends only on the history of the factor process X but not on the default itself.

An alternative approach is to model the default events via the *default intensity* γ_t, defined as the $\mathcal{F}_t$-intensity of the default process N_t, where $\mathcal{F}_t$ designates the market history up to t, *including* observations of past defaults. Intuitively, the default intensity γ_t is the conditional probability per unit time of the next default event, given past market history:

$$\gamma_t = \lim_{\Delta t \to 0} \frac{1}{\Delta t} P\big(N_{t+\Delta t} = N_{t-} + 1 | \mathcal{F}_t\big).$$

This naturally leads to a default intensity which jumps at default dates. Thereby the default process becomes self-affecting in that one default may have spill-over effects on other names and trigger a cluster of defaults.

Example 4.3.3 (Markovian defaults) *A simple way to model the impact of past defaults on the default rate is to model the default intensity as a function of the total number of defaults:*

$$\gamma_t = f(t, N_t).$$

This leads to a Markov process for N_t which is easy to simulate and in which loss distributions and other quantities may be computed by solving a system of linear ordinary differential equations. Cont & Minca (2008) show that the intensity function f implied from market prices of CDO tranches exhibit a strong, non-monotone, dependence of the default intensity on the number of defaults. However this model is too simple for our purpose since it leads to piecewise-deterministic spread dynamics between default dates, whereas the CPDO is sensitive to spread volatility.

Example 4.3.4 (Self-exciting defaults) *An example of a self-exciting default process is given by the model of Giesecke & Kim (2007) where the default intensity jumps up by a magnitude proportional to the loss at defaults and follows a diffusive process between default times. The intensity process is given by*

$$d\gamma_t^Q = \kappa\big(\theta - \gamma_t^Q\big)dt + \sigma\sqrt{\gamma_t^Q}dW_t + \eta dL_t, \qquad \gamma_0^Q > 0 \qquad (4.9)$$

where L denotes the loss process. The intensity of the default counting process N is thus updated at each default and undergoes a jump. Between *default events the intensity reverts back to its long term level θ exponentially in mean at a rate $\kappa \geq 0$ with diffusive fluctuations driven by a Brownian motion. The default counting process N is self-exciting because the intensity of N_t increases at each default event. This property captures the feedback effects (contagion) of defaults observed in the credit market. As in the CIR case, $2\kappa\theta \geq \sigma^2$ is required to ensure $\gamma_t^Q > 0$ almost surely. The process (4.9) also belongs to the class of affine processes, where, with the same notations as in Example 4.3.1,*

$$
\begin{aligned}
B_1(t) &= -\frac{1}{\kappa + \eta\left(\frac{1-R}{N^I}\right)}\left(e^{-\left(\kappa+\eta\left(\frac{1-R}{N^I}\right)\right)(T^I-t)} - 1\right) \\
A(t) &= \frac{\kappa\theta}{\left(\kappa + \eta\left(\frac{1-R}{N^I}\right)\right)^2}\left(e^{-\left(\kappa+\eta\left(\frac{1-R}{N^I}\right)\right)(T^I-t)} - 1\right) + \frac{\kappa\theta}{\kappa + \eta\left(\frac{1-R}{N^I}\right)}(T^I - t).
\end{aligned}
$$

◇

4.3.4 Modelling the index roll

When modelling the default intensity of the underlying index, it is crucial to take the semi–annual rolling of the index into account: the replacement of downgraded names results in a negative jump in the default intensity and thereby also in the swap spread on average implying a loss when liquidating swap contracts. In the long run, rolling the index also has a positive effect as the portfolio default risk and thereby the portfolio loss is lowered.
We consider two possible approaches for modelling the roll over effect.

The simplest model is to include a constant proportional jump size in the default intensity on each roll date. However, empirical observations show variation in the jump sizes, so extending this setup to allow for two possible jump sizes $h_1, h_2 \in [0, 1]$, not necessarily taken with equal probability, is more realistic.

To model in more detail the index roll, we assume that the index is homogeneous, such that all individual name default intensities $(\Lambda^1, ..., \Lambda^{N^I})$ are independent with identical distribution denoted F. For a given roll date $T_j \in \mathbf{RD}$ let $(\bar{\Lambda}^1_{T_j}, ..., \bar{\Lambda}^{N^I}_{T_j})$ be a realization of N^I independent F distributed variables. Rolling the index corresponds to removing a number of the highest realizations and replacing these by new independent draws from the F distribution. The roll over effect is then given as the difference between the average intensity before and after the roll. This setup requires an assessment of the average number of names removed on each roll date and of the individual default intensity distribution.

Assuming that the term structure is upward sloping, the effect from rolling down the credit curve will counteract the effect from rolling over the index. Empirically these two effects are more or less observed to offset each other. In the following, the first mentioned approach for modeling the roll over effect will be taken and the jump sizes $\{h_1, h_2\}$ are chosen such as on average to cancel the roll down effect implied by the dynamics of the default intensity. Note that, the proportional jump in the default intensity does not affect the calculation of the index spread, since this references the current index, not the rolling index.

4.4 Performance and risk analysis

We analyse the performance of the CPDO strategy by Monte Carlo simulations. The aim is not only to assess the rating based on the default and cash out probabilities but also to study other risk measures such as the loss distribution and the expected shortfall. Further, we wish to identify key parameters and study the dependence of the CPDO performance on these parameters.

4.4.1 Simulation results

We model the risk neutral hazard rate λ_t^Q as a CIR process (4.7) following example 4.3.1 and assume the P-hazard rate is given by $\lambda^Q = \vartheta\lambda^P$. We conduct the analysis in three credit market configurations. The first is a benign credit environment with a relatively low number of obligors defaulting (on average 5 defaults during the lifetime of the CPDO) and a tight index spread around 40bp. This setup would correspond to empirical observations in 2005–2006. Parameters for the risk neutral hazard rate corresponding to such a setup could be

$$\theta = \lambda_0^Q = 1.6 \qquad \kappa = 0.2 \qquad \sigma = 0.8 \quad \text{and} \quad \vartheta = 2.5.$$

According to the empirical findings of Berndt et al. (2005) we expect the risk neutral default intensity to be 2-3 times higher than the statistical intensity but ϑ may become much higher in stressed credit environments.

Let $r = 0.05$ and $R = 0.4$. For the roll over effect, we choose a fairly low jump size $h_1 = 0.05$ in most scenarios (95%) and occasional (5%) large downward jumps of $h_2 = 0.2$ on roll dates. We examine the case of a CPDO contract paying coupons of 100bp above LIBOR employing a maximum leverage of $M = 15$, $\varepsilon = 0.25$ and cash out threshold $k = 0.1$. The aggressiveness of the target leverage rule (4.1) is determined by the gearing factor $\beta = 1.5$ and cushion $\mu = 0.05$.

Based on 10 000 simulation runs we find the probability of the CPDO defaulting to be 3.7%. According to Standard & Poor's default probability thresholds given in table 4.1, this will earn the CPDO principal note a BBB+ rating. The probability of the CPDO cashing out and thereby defaulting on both coupons and principal note is 0.24%, thereby giving the coupon payments a AAA rating. The expected loss conditional on default occurring, LGD, is 22% of notional.

Another useful risk measure is the expected shortfall defined at a given level α by

$$ES_\alpha = E\big[L|L > VaR_\alpha\big] \qquad \text{for} \qquad VaR_\alpha = \inf\big\{l \,|\, P(L > l) < 1 - \alpha\big\},$$

where L denotes the loss of the CPDO. In this credit environment we find $ES_{0.99} = 55\%$ of note notional. That is, in the worst 1% of scenarios, the investor expects to recover about half of the initial investment. If not defaulting the CPDO cashes in quite fast, on average after 2.6 years, even though the chosen strategy is not very aggressive. Results are given in table 4.2.

Market	PD (%)	Cash Out (%)	Rating –	LGD (%)	ES_{99} (%)	Cash In (years)	N_T –
Benign	3.7	0.24	BBB+	22.3	54.8	2.6	4.8
Stressed	1.2	0.49	AA	40.2	47.3	3.0	8.6
Historical	2.5	0.70	A	33.8	75.0	3.1	7.0

Table 4.2: Summary of results.

Figure 4.1 illustrates a typical scenario in the benign credit market. The first graph shows the portfolio default intensity λ^P, at the top right is the on-the-run index default swap spread $S(t, T^I)$ versus the contracted spread ($\bar{S}_t$), at the bottom left is the target (m_t) and actual leverage factor ($\bar{m}_t$) and the last graph is the evolution in the CPDO portfolio value (V_t).

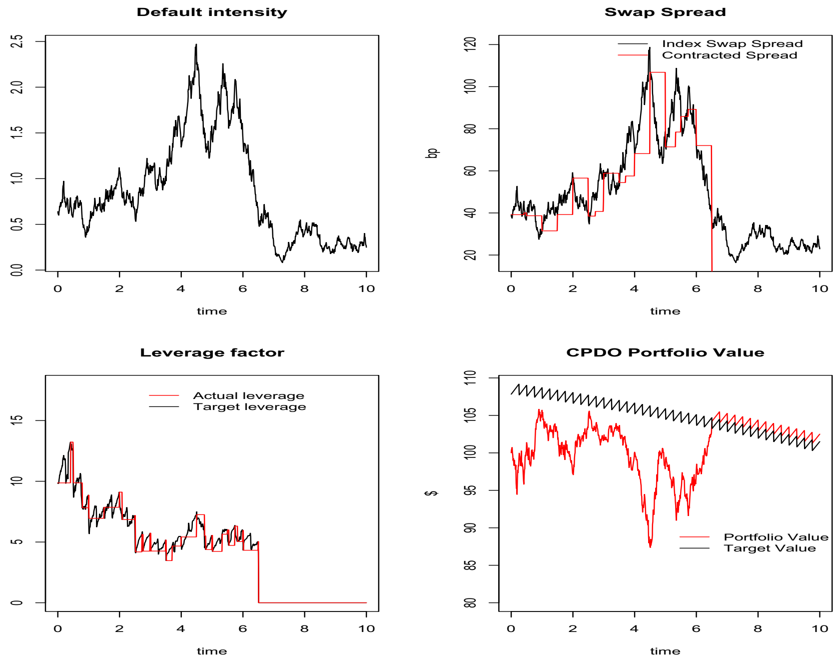

Figure 4.1: Benign credit market scenario.

Figure 4.1 clearly shows how the spread widening during the years 3-5 and the implied mark-to-market losses result in decreasing CPDO portfolio value. However, the consecutive spread tightening allows the CPDO to cash in after approximately 6.5 years.

The second model configuration studied is a stressed credit environment with an average of 9 defaults among the 250 names in the underlying portfolio over a 10 year period. The average index default swap spread is around 100bp. This could illustrate a market during a crisis, as experienced during the second half of 2007 and in 2008. CIR parameters corresponding to such a setup could be $\theta = 4.0$, $\kappa = 0.2$ and $\sigma = 1.0$. Consistent with empirical findings in Berndt et al. (2005) let $\vartheta = 3.5$.

With a higher spread level it is reasonable to expect the CPDO to pay higher coupons, therefore in this credit environment consider a CPDO paying a 200bp spread above LIBOR. The gearing factor is $\beta = 2$ and the remaining parameters left unchanged. The probability of the CPDO defaulting is 1.2% corresponding to AA and the cash out probability is 0.5% giving the coupons a AAA rating in this credit environment. The expected shortfall is 47% of note notional and close to the tail loss in the benign credit market. Here, the CPDO takes a little longer to cash in, on average 3 years. Figure 4.2 shows a typical scenario in the stressed credit environment.

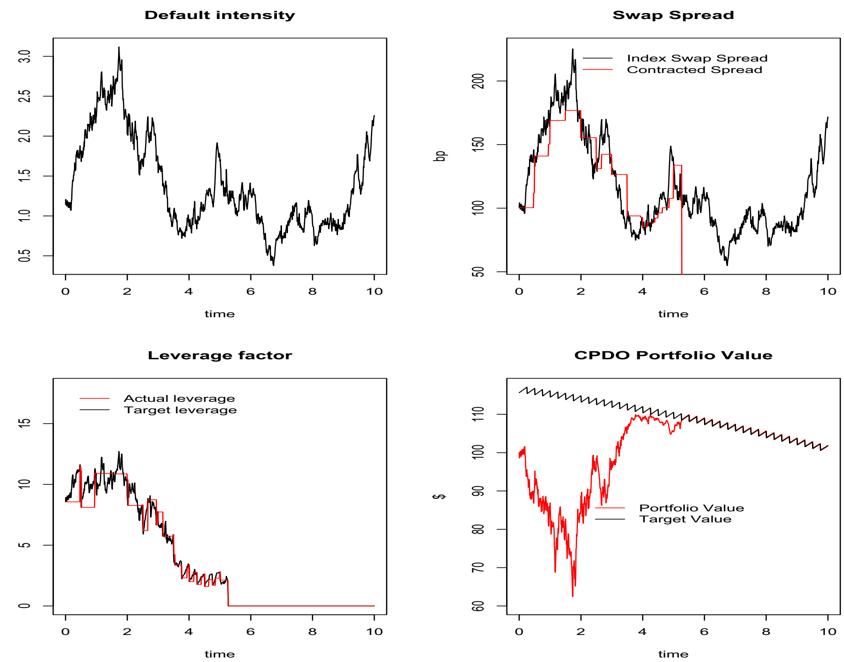

Figure 4.2: Stressed credit market scenario.

Calibrating the model to market data is not straightforward, since data for the index default swap spread is only available a few years back in time. During this period credit markets have been fairly benign (up to the summer of 2007) and may therefore result in an incorrect risk profile for a CPDO strategy. To overcome this issue, previous studies (Gilkes et al. (2005) and Jobst et al. (2007)) have used different proxies for the index default swap data in past periods. In these studies, the 5-year index spread is directly modeled as an OU-process as given in example 4.3.2:

$$\frac{dS(t,T^I)}{S(t,T^I)} = \alpha(\beta - \ln S(t,T^I)\ )dt + \xi dW_t. \tag{4.10}$$

To compare our model to estimates found by calibrating (4.10) to index proxy data, we simulated 1 000 scenarios for the 5-year index spread implied by our top-down model and calibrated each path to a spread process of the form (4.10). Thereby we arrive at a historical model configuration generated by the following parameters for the risk neutral hazard rate: $\kappa = 0.2$, $\theta = 2.8$ and $\sigma = 0.9$.

We examine the case of a CPDO paying out a 200bp spread. The risk premium parameter is set to $\vartheta = 3.0$ corresponding to 7 defaults on average among the 250 names over a 10 year period. Again we set the gearing factor $\beta = 2$. The probability of default is 2.5% which corresponds to a A rating. The cash out probability is 0.7% still earning the coupon payments a AAA rating. The expected shortfall is now somewhat higher than in the previous two model configurations, namely 75% of notional.

A typical market evolution in this setup is shown in figure 4.3. We see that even though the CPDO portfolio value falls below half of its principal value and the leverage factor is at its maximum for 3-4 years, a significant spread tightening makes it possible for the CPDO to cash in after 6 years.

Self-exciting defaults

The effects of allowing for a self-exciting default intensity as described in example 4.3.4 can also be studied in a Monte Carlo framework. To do this, we use the historical parameters for κ, θ and σ and set $\eta = 3$ in accordance with the findings in Giesecke & Kim (2007). This means that a default event causes a jump in the default intensity of three times the loss. Including this feed-back effect from defaults on the default intensity results in a poorer CPDO performance. The default probability increases to 3.0% corresponding to A- and the cash out probability to 0.9% lowering the rating of the coupons to AA+. We also find a negative effect on the

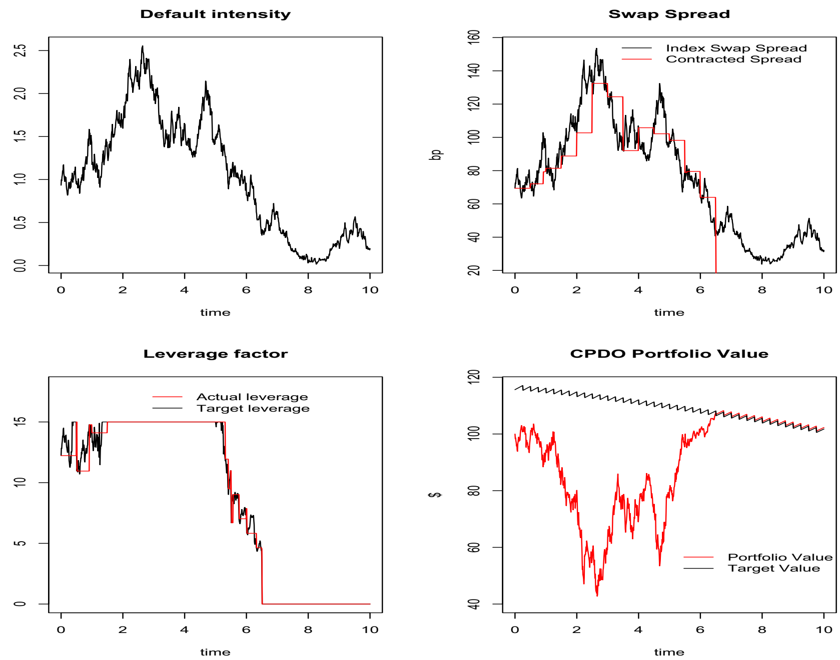

Figure 4.3: Scenario in the credit market based on historical parameters.

expected shortfall which increases to almost 90%. CPDO performance for different choices of η are given in appendix A. Although allowing for feedback effects from default events does result in a poorer performance it does not change the overall qualitative conclusions of our analysis.

4.4.2 Sensitivity analysis

To assess the impact of the various parameters on the performance of the CPDO, we have carried out a sensitivity analysis of the dependence of risk measures on model parameters. Simulation results from the three market configurations are found in appendix A. The main findings are summarized below.

Default intensity
The default risk premium parameter ϑ clearly has a significant influence on the CPDO performance, since ϑ determines the average level of spread income relative to the credit losses incurred. Even small changes in ϑ may imply rating changes of several categories. The dependence of the CPDO performance on the risk premium is illustrated in figure 4.4 showing the probability of default and expected loss given default in the historical credit market as a function of the risk premium ϑ based on 10 000 simulations. Not surprisingly, we see downward sloping curves in both cases – especially for $\vartheta \in [0, 2]$, the slopes are very steep.

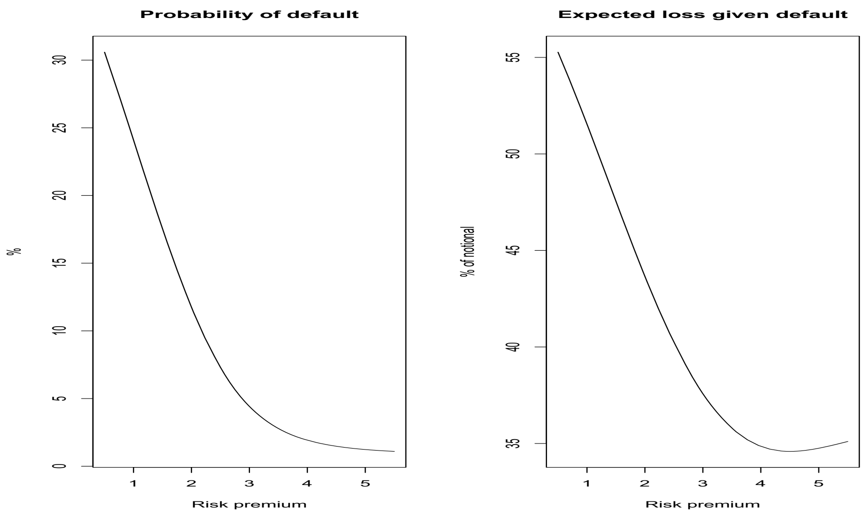

Figure 4.4: Left: Dependence of CPDO default rate on risk premium ϑ. Right: Dependence of CPDO loss given default on risk premium ϑ.

The parameter underlying the intensity process affecting the CPDO performance the most is the long term mean θ: the higher level of spread income implied by a higher value of θ more than compensates for the increase in credit losses which is another implication of increasing θ. This is most clearly seen when comparing the benign, stressed and historical credit markets. The portfolio swap spread level, mainly determined by

θ, will typically set the coupon size a CPDO can offer its investors. The analysis also shows that an initial spread widening (tightening) and the resulting mark-to-market losses (gains) have a large effect on the probability of default, although less effect on the expected shortfall.

The volatility of the hazard rate affecting the spread volatility is less important, but we do see that that a higher volatility is harmful for the CPDO performance both when it comes to the probabilities of default and cash out as well as the expected shortfall. The reason is that higher volatility leads to more extreme scenarios, which for a CPDO that has no upside potential means more extreme loss scenarios and therefore a higher probability of cashing out.

Another important parameter is the mean reversion speed κ. A higher κ reduces the probability of default and leads to a lower expected shortfall. With a higher mean reversion speed the index spread fluctuates more tightly around its long term mean level thereby reducing the mark-to-market losses incurred by the CPDO. This dependence is investigated in figure 4.5, where the default rate and expected loss as a function of the mean reversion speed κ are shown. We see an improved performance, both with respect to the CPDO default rate and the expected loss given default, for increasing values of κ. The dependence is close to linear.

Not surprisingly, we also find that the magnitude of the (negative) jump in the spread at index roll dates affects the performance significantly.

Recovery rate

The level of recovery R upon default in the underlying portfolio affects the outcomes in two ways. First, a higher recovery will increase the CPDO default probability due to the fact that a higher recovery implies a lower index default swap spread. On the contrary, a high recovery rate will lead to fewer cash out events and lower losses given default.

Interest rates

The level of interest rates affects the probabilities of default and cash out in opposite directions: A higher interest rate implies lower probability of default at the cost of more cash out events and higher expected shortfalls. The average cash in time is also reduced as a consequence of higher interest rates.

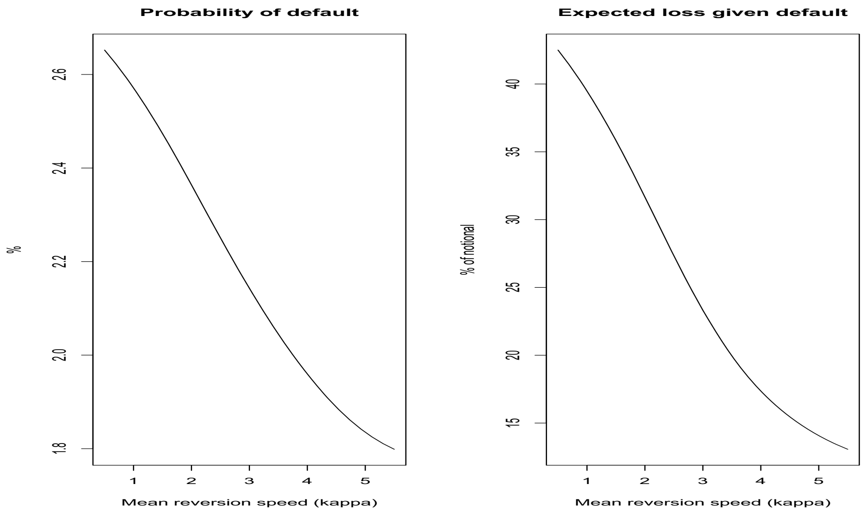

Figure 4.5: Left: Dependence of CPDO default rate on mean reversion speed κ. Right: Dependence of CPDO loss given default on mean reversion speed κ.

Leverage strategy

The leverage strategy employed has significant effects on the CPDO performance. Starting off with a high initial leverage is important, and increasing the maximum leverage M will result in lower default rates at the cost of higher losses. This is exactly the reason for capping the leverage factor, namely to reduce the overall possible loss.

The aggressiveness of the strategy described by the gearing factor β (and to some extent the cushion) can be used to balance the CPDO default rate versus the expected loss. Some versions of the CPDO strategy even employs a time dependent gearing factor. Alternatively, the risky duration of the liabilities of the CPDO could be used when calculating target leverage instead of the duration of assets. This would result in a less aggressive strategy the first couple of years, but in case the CPDO has not cashed in closer to maturity, this leverage rule becomes more aggressive.

Employing a simplified version of the leverage strategy according to which leverage is only adjusted on roll dates does not change the performance significantly. In the benign and stressed markets we observe slightly lower probabilities of the CPDO defaulting on its principal note at the cost of higher expected shortfall. In the historical environment, both measures are reduced when employing this strategy. The reason for these observations is that as long as a scenario evolves favourable, i.e. as long as the shortfall and thereby the target leverage is being reduced, this alternative strategy operates more aggressively than the regular leverage rule due to fewer deleveraging events. However, if the CPDO closer to expiry is falling short of meeting the repayment of principal, the simplified strategy is not as flexible and the result is heavier losses and more defaults.

Including a 1% administration fee at initiation of the CPDO does not have a significant effect in neither of the three credit market configurations.

In this simple model we are also able to study the loss distribution for the CPDO, as shown in figure 4.6 in a credit environment generated by historical parameters. We see that in a large part of the default scenarios, only small losses of 0-20% of note notional are incurred. In these cases the CPDO is typically not under distress toward expiry but the employed leverage strategy is not aggressive enough to allow a cash in prior to expiry. The heavy right tail of the loss distribution corresponds to cases where the CPDO cashes out or is very close to cashing out.

4.4.3 Scenario analysis

The model allows to determine market scenarios that are most harmful for the CPDO performance by studying scenarios in the historical credit market in which the CPDO cashes out. An example of such a scenario is given in figure 4.7. In this case a 100bp spread widening between year 2 and 4 causes the portfolio value to drop to 20% of notional. Two defaults in the underlying portfolio at the end of this period cause even more distress. Another 50bp spread widening within a few months and finally a default results in a cash out event around year 5.

In general, the main reason causing a CPDO to cash out is continued spread widening. The average cash out time, given that there is a cash out event, in this credit environment is 4.1 years. Since the default intensity is mean reverting, a significant spread widening will at some point be

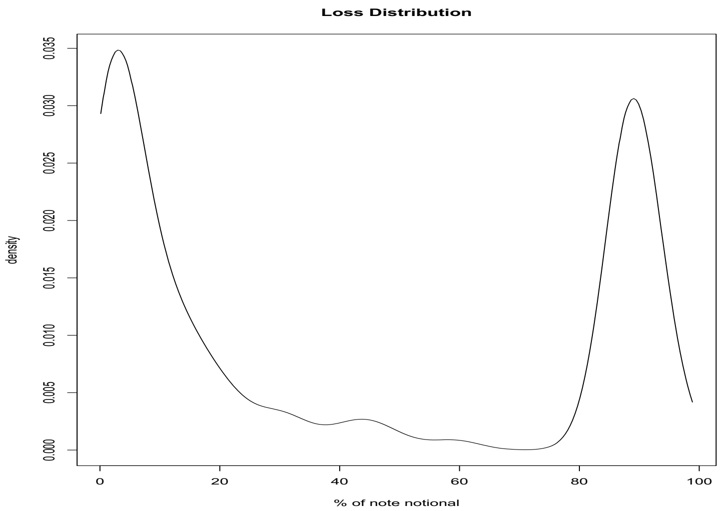

Figure 4.6: Kernel estimator of loss density based on 10 000 simulated scenarios.

followed by a similar spread tightening (as is also the case in figure 4.7). Thereby, if the CPDO survives the period of spread widening it will benefit from the consecutive spread tightening and may thereby regain a large part of the lost capital.

Defaults in the underlying portfolio also have some effect, although not as large as spread widening. One default causes a reduction of portfolio value (if leveraged up to 15x) of 3.6%, whereas a 15bp spread widening causes almost 10% reduction of the portfolio value. In our model, the main determinant of possible spread widenings is the mean reversion parameter κ. A cluster of defaults (3 or 4 defaults over a short period of time) in the reference portfolio may cause the CPDO to cash out if it is already under distress. Figure 4.8 illustrates a scenario with many obligor defaults, in this case 22 defaults over the 10 year period. Periods with particularly many defaults are year 2-2.5 with four defaults, year 5.5-6 with five defaults and finally four defaults during the last six months. In this case the reason for the CPDO defaulting on its principal value is due to heavy credit losses during its entire 10 year lifetime. This example also illustrates the fact

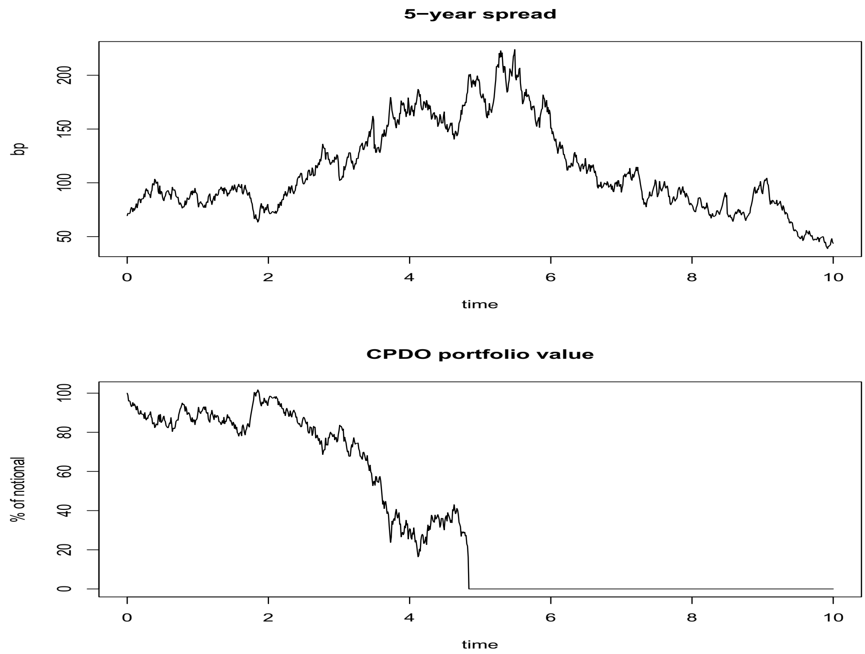

Figure 4.7: A cash-out scenario.

that CPDOs are able to recover (or in this case, partly recover) from bad positions; after 1.5 years only 20% of notional is left and three years later it has recovered 90%.

Similarly, one can determine the best scenarios for the CPDO. This reveals that initial spread tightening combined with no defaults in the reference portfolio, results in the fastest cash in and the lowest maximum shortfall, i.e. the difference between minimum portfolio value and target value. A cash in scenario is shown in figure 4.9. In this example, cash in is not a result of significant spread tightening and it shows that in relatively stable spread scenarios the CPDO will cash in within a few years.

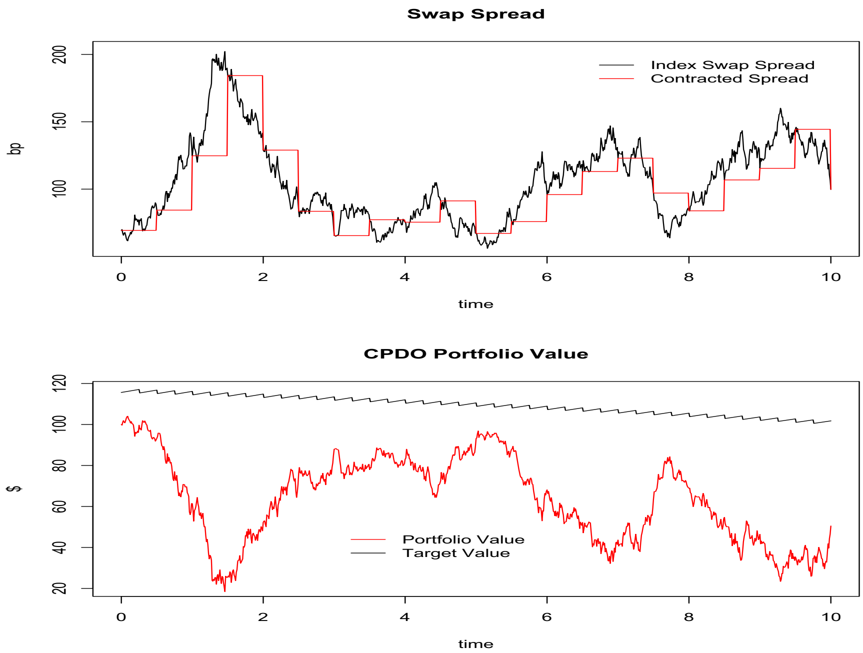

Figure 4.8: A scenario with many defaults in the underlying portfolio.

4.4.4 Variability of ratings and downgrade probabilities

Given that CPDOs are leveraged and path-dependent instruments, the initial rating of CPDO notes gives only a partial idea of the risk of the instrument. As in the case of CDO tranches studied by Cont & Moussa (2008), a CPDO with initial AAA rating may have a probability of being downgraded which is much higher than a AAA bond. It is therefore interesting to examine the probability of rating downgrades during the lifetime of the CPDO. This can be done by a nested Monte Carlo simulation as suggested by Morokoff (2007).

Suppose that at initiation the strategy is given rating R_0 corresponding to a T-year probability of default $PD \in [L_0^T, U_0^T[$. We want to re-assess the rating after $T_1 < T$ years. In the outer loop of the simulation N_O paths of λ, V, A, $\bar{m}$, etc. up to time T_1 is generated. For each path that has not

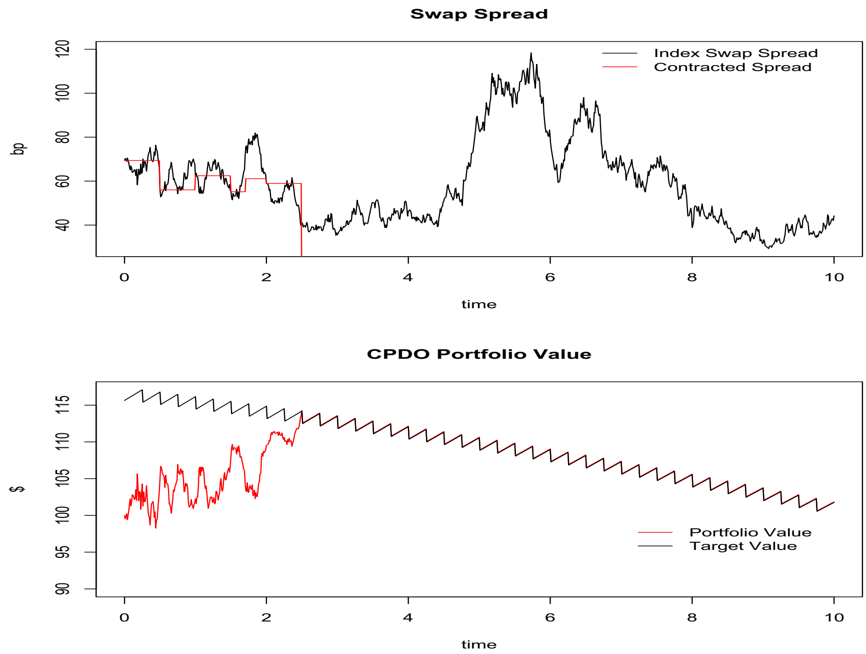

Figure 4.9: A cash-in scenario.

cashed out at time T_1, a second Monte Carlo simulation is performed in order to assess the default probability at T_1 and thereby a possibly new rating R_1. This is done by simulating N_I paths from time T_1 to expiry T, using the starting values λ_{T_1}, V_{T_1}, A_{T_1}, etc. found in the outer loop. The rating transition probability is estimated by:

$$P(R_1 \neq R_0) \approx \frac{1}{N_O} \sum_{i=1}^{N_O} \mathbf{1}_{\left\{\widehat{PD}_i \notin [L_0^{T_2}, U_0^{T_2}[\right\}}, \tag{4.11}$$

where $T_2 = T - T_1$ and $\widehat{PD}_i$ denotes the estimated probability of default in the i'th outer loop. Now, $P(\widehat{PD}_i = \frac{j}{N_I}) = P_{\text{bin}}(j; N_I, PD_i)$ where P_{bin} denotes the binomial point probability in j for N_I trials and success rate equal to the true default probability PD_i. Then $\widehat{PD}_i \to PD_i$ for $N_I \to \infty$

implying

$$
\begin{aligned}
E\big[\mathbf{1}_{\{\widehat{PD}_i \notin [L_0^{T_2}, U_0^{T_2}[\}}\big] &= \sum_{j=0}^{N_I} \mathbf{1}_{\{\frac{j}{N_I} \notin [L_0^{T_2}, U_0^{T_2}[\}} P_{\text{bin}}(j; N_I, PD_i) \\
&\to \mathbf{1}_{\{PD_i \notin [L_0^{T_2}, U_0^{T_2}[\}} \qquad \text{for } N_I \to \infty,
\end{aligned}
$$

i.e. the expected estimated transition indicator converges to the true transition indicator. Hereby it follows, that the estimation in (4.11) can be performed summing $E\big[\mathbf{1}_{\{\widehat{PD}_i \notin [L_0^{T_2}, U_0^{T_2}[\}}\big] = P\big(\widehat{PD}_i \notin [L_0^{T_2}, U_0^{T_2}[\big) := p_T^i$ over $i = 1, ..., N_O$.

Define the simulated rating transition indicator by

$$
y_i = \begin{cases} 1 & \text{if } \widehat{PD}_i \notin [L_0^{T_2}, U_0^{T_2}[\\ 0 & \text{otherwise} \end{cases}
$$

and let

$$
w_i = \begin{cases} y_i & \text{with probability } p_T^i \\ 1 - y_i & \text{with probability } 1 - p_T^i. \end{cases}
$$

Now $w_i y_i + (1 - w_i)(1 - y_i)$ is equal to 1 with probability p_T^i and the rating transition probability can then be calculated as

$$
P(R_1 \neq R_0) \approx \frac{1}{N_O} \sum_{i=1}^{N_O} \left(w_i y_i + (1 - w_i)(1 - y_i) \right).
$$

With $N_O = 1\,000$ and $N_I = 10\,000$ the rating transition probabilities are given in table 4.3. Our results indicate high probabilities for rating

	$P(R_1 \neq R_0)$	$P(R_1 < R_0)$
$T_1 = 1$	98.2%	8.2%
$T_1 = 5$	98.5%	9.4%

Table 4.3: Rating transition probabilities.

downgrades. Since the average cash in time is less than 5 years, for a large part of the scenarios the CPDO has already cashed in before year 5, earning the CPDO a AAA rating, which differs from the initial A rating obtained in this market configuration. More interestingly, almost 8% of the contracts have been downgraded after one year and this number increases up to almost 10% at year 5. Since less than 3% of the contracts end up in default, this is an indication of the CPDO being able to recover even after severe losses. The high rating volatility documented here clearly distinguishes CPDOs from similarly rated investment grade bonds, which typically are expected to maintain their original rating during the lifetime.

4.5 Discussion

We have presented a parsimonious model for analysing the performance and risks of CPDO strategies. We consider a variety of specifications for a one factor top-down model for the index default intensity and show that they allow to study credit ratings, default probabilities, loss distributions and different tail risk measures for the CPDO and capture its risk features in a meaningful yet simple way.

Our results indicate that while coupon notes have a low probability of default (compatible with AAA or AA given pre-crisis market parameters), principal notes have typically a much higher probability of default, leading to A ratings under the same market conditions. Also, our scenario analysis identifies a high exposure to credit-spread widening scenarios, similar to those observed recently in the market.

Perhaps the most important insight from our study is that CPDOs are less sensitive to default risk than to movements of spreads and behave in this respect more like path-dependent derivatives on the index spread. Our scenario analysis clearly indicates that the worst case scenario for a CPDO manager is that of a sustained period of spread widening. This scenario has precisely happened in the second half of 2007 and in 2008, and has resulted in the forced unwinding of many CPDOs (Wood (2007)) as predicted by our analysis.

In line with the findings of rating agencies, we have found the CPDO structure to be very parameter sensitive. Relatively small changes in certain parameters may result in a jump of several notches in the rating. Accordingly, we conclude that over the lifetime of the CPDO this leads to very high variability of the rating compared to standard top-rated products and is one of the main criticisms of CPDOs receiving a top rating. The parameters with respect to which we observed the highest sensitivity are the (long-term) spread level θ setting the spread income generated by the CPDO, the mean reversion speed κ determining the possible spread widening and the risk premium parameter ϑ which governs the discrepancy between market-implied and historical default rates.

Another insight from our analysis is the influence of the aggressiveness of leverage strategy employed. Following a more aggressive leverage rule results in fewer defaults at the cost of higher tail losses. An actively managed CPDO or a time/state dependent gearing factor would therefore possibly result in a better performing CPDO in some scenarios and could be designed to accommodate the risk aversion of the investor. Analyzing risks and performance of such a product would require a subtle specification of actions taken by the manager.

There are straightforward extensions and refinements, e.g. a two factor model for the joint evolution of the default intensity and interest rate or including a time depending risk premium ϑ. Yet we believe that the top-down model in the basic form introduced here captures the essential risk factors of CPDOs.

It is difficult to compare directly our results with the ratings/default probabilities given by rating agencies since many parameters and contract details enter into these computations. But our model makes it clear that the main factors are the term structure of credit spreads, which determines the roll-down effect, the behaviour of the index spread at each roll date (explicitly modelled here using parameters compatible with historical data) and the dynamics of the spread (mean-reversion, widening/tightening).

More importantly, our analysis shows that within a given rating category a wide range of expected shortfalls may be observed, leading us to conclude that basing the risk analysis of such complex products as CPDOs on ratings or default probabilities alone is not sensible: credit ratings should be complemented by other risk measures such as tail conditional expectation or other measures of downside risk, in agreement with similar conclusions drawn from studies of CDO tranche ratings (Cont & Moussa (2008)).

The question of "credit ratings" for such leveraged, path-dependent products such as CPDOs raises various methodological issues. Credit ratings are usually presented to investors as a metric for credit/default risk (as opposed to indicators of "market risk"). However, in the context of structured credit products such as CPDOs, it is clear that the rating will be based on scenario simulations incorporating various market risk factors such as volatility of spreads, the level and volatility of interest rates, etc., blurring the (non-existent?) borderline between credit and market risk and raising questions about the interpretation of such ratings by investors. Our results suggest that for such complex products ratings tend to be misleading and cannot replace a detailed risk analysis. Indeed, some rating agencies have refused to rate CPDO deals.

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Appendix A

Results of the sensitivity analysis in the three credit environments are given in this appendix. Ratings in the tables refer to the principal note and are given according to the CDO default matrix of Standard & Poor's (Gilkes et al. (2005)).

The standard parameters in the benign credit environment (setup 1) are:

$$\theta = \lambda_0^Q = 1.6 \qquad \kappa = 0.2 \qquad \sigma = 0.8 \quad \text{and} \quad \vartheta = 2.5.$$

We study a CPDO paying a 100bp spread and the strategy chosen has gearing factor $\beta = 1.5$. Note that in the sensitivity analysis in table 4.4 it is not possible to choose the parameters freely, since the condition $2\kappa\theta \geq \sigma^2$ should be fulfilled in a CIR model. For the above parameters the condition holds with equality and therefore σ cannot be chosen higher and κ and θ not lower without changing other parameters as well.

	PD (%)	Cash Out (%)	Rating –	LGD (%)	ES_{99} (%)	sd(LGD) –	Cash In (years)	S_0 (bp)	N_T –
setup 1	3.7	0.24	BBB+	22.3	54.8	23.9	2.6	39	4.8
simple	3.5	0.42	A-	30.5	67.2	27.1	2.2	39	4.8
1% adm. fee	4.8	0.31	BBB+	23.2	62.9	24.1	2.7	39	4.9
$\vartheta = 2.0$	5.2	0.46	BBB	27.2	83.5	26.3	2.7	39	5.9
$\vartheta = 3.0$	2.6	0.10	A	15.6	32.3	19.5	2.5	39	4.0
$\sigma = 0.6$	4.3	0.02	BBB+	12.0	33.4	14.3	3.2	39	4.8
$\theta = \lambda_0^Q = 2.0$	3.3	0.19	A-	16.6	43.0	23.2	2.8	49	6.0
$\lambda_0^Q = 1.0$	5.9	0.50	BBB	23.7	70.8	24.6	2.7	30	4.1
$\lambda_0^Q = 2.0$	2.8	0.26	A-	22.5	50.8	26.7	2.6	46	5.3
$\kappa = 0.4$	3.2	0	A-	9.8	23.2	11.5	3.4	39	5.2
$R = 0.2$	3.0	0.40	A-	23.4	56.6	29.6	2.6	52	4.9
$R = 0.6$	8.0	0.05	BBB-	17.0	47.1	15.0	2.7	26	4.8
$r = 0.02$	4.3	0.34	BBB+	21.3	59.4	25.1	2.8	39	4.9
$r = 0.1$	3.2	0.31	A-	30.2	63.7	26.4	2.2	39	4.8
$(h_1, h_2) = (0, 0)$	2.5	0.22	A	15.0	32.9	24.5	2.7	39	6.5
$(h_1, h_2) = (0.3, 0.02)$	3.2	0.31	A-	21.7	54.4	26.6	2.6	39	5.3
$\delta = 0.015$	7.8	0.58	BBB-	25.0	78.5	24.8	3.2	39	4.9
$M = 10$	7.3	0.04	BBB-	16.4	44.8	14.7	2.7	39	4.9
$M = 20$	2.9	0.36	A-	23.0	54.7	28.9	2.6	39	4.8
$\beta = 1$	7.1	0.10	BBB	13.3	43.9	15.5	3.4	39	4.8
$\beta = 2$	3.4	0.38	A-	29.9	63.4	26.1	2.1	39	4.8

Table 4.4: Sensitivity analysis: Benign credit market.

The standard parameters in the stressed credit market (setup 2, table 4.5) are:

$$\theta = \lambda_0^Q = 4.0 \qquad \kappa = 0.2 \qquad \sigma = 1.0 \quad \text{and} \quad \vartheta = 3.5.$$

The CPDO studied pays a 200bp spread and the strategy chosen has gearing factor $\beta = 2$.

	PD (%)	Cash Out (%)	Rating –	LGD (%)	ES_{99} (%)	sd(LGD) –	Cash In (years)	S_0 (bp)	N_T –
setup 2	1.2	0.49	AA	40.2	47.3	41.1	3.0	100	8.6
simple	1.1	0.87	AA	73.6	79.2	32.3	2.1	100	8.6
1% adm. fee	1.4	0.70	AA	47.8	65.8	41.9	3.1	100	8.6
$\vartheta = 3.0$	1.6	0.68	AA-	42.0	64.5	41.6	3.1	100	10.1
$\vartheta = 4.0$	1.0	0.45	AA+	42.2	42.6	42.5	3.0	100	7.5
$\sigma = 0.8$	0.89	0.09	AA+	13.6	13.6	25.9	3.5	100	8.6
$\sigma = 1.2$	1.7	1.4	AA-	73.7	90.8	33.2	2.6	100	8.6
$\theta = \lambda_0^Q = 3.5$	1.4	0.75	AA	50.9	70.2	41.2	2.9	87	7.6
$\theta = \lambda_0^Q = 4.5$	0.96	0.31	AA+	32.8	32.8	39.3	3.1	114	9.7
$\lambda_0^Q = 3.5$	1.4	0.57	AA	40.6	55.8	40.7	3.1	92	8.1
$\lambda_0^Q = 4.5$	0.97	0.39	AA+	40.5	40.5	41.5	3.0	110	9.0
$\kappa = 0.15$	1.3	0.83	AA	58.3	76.9	39.9	2.8	100	8.2
$\kappa = 0.4$	0.37	0	AAA	3.5	3.5	3.8	3.8	100	9.4
$R = 0.2$	1.2	0.65	AA	50.9	60.8	42.1	3.0	134	8.7
$R = 0.6$	1.4	0.25	AA	24.5	34.4	32.4	3.1	67	8.6
$r = 0.02$	2.0	0.71	A+	36.6	69.5	41.1	3.4	100	8.6
$r = 0.1$	0.51	0.24	AAA	42.8	42.8	39.5	2.5	101	8.6
$(h_1, h_2) = (0, 0)$	0.61	0.35	AAA	52.3	52.3	42.2	2.9	100	11.4
$(h_1, h_2) = (0.3, 0.02)$	0.93	0.43	AA+	43.4	43.4	42.4	3.0	100	9.5
$\delta = 0.015$	0.64	0.21	AAA	32.4	32.4	40.3	2.7	100	8.6
$\delta = 0.025$	1.9	0.80	AA-	42.9	76.3	40.9	3.3	100	8.6
$M = 10$	1.4	0.26	AA	25.1	35.0	32.8	3.1	100	8.6
$M = 20$	1.2	0.53	AA	41.9	50.6	43.0	3.0	100	8.6
$\beta = 1$	12.9	0.02	BB+	6.8	27.5	8.0	5.1	100	8.6
$\beta = 3$	1.3	1.2	AA	85.3	89.4	16.3	2.0	100	8.6

Table 4.5: Sensitivity analysis: Stressed credit market.

The standard parameters in the historical credit environment (setup 3, table 4.6) are:

$$\theta = \lambda_0^Q = 2.8 \qquad \kappa = 0.2 \qquad \sigma = 0.9 \quad \text{and} \quad \vartheta = 3.0.$$

The CPDO premium is 200bp above LIBOR and the strategy chosen has gearing factor $\beta = 2$.

	PD (%)	Cash Out (%)	Rating –	LGD (%)	ES_{99} (%)	sd(LGD) –	Cash In (years)	S_0 (bp)	N_T –
setup 3	2.5	0.70	A	33.8	75.0	36.4	3.1	69	7.0
simple	2.1	0.96	A+	52.5	59.4	36.1	2.2	69	7.0
1% adm. fee	2.8	0.90	A-	38.6	89.6	37.6	3.1	69	7.0
$\vartheta = 2.5$	3.9	1.3	BBB+	40.5	89.5	37.0	3.1	69	8.3
$\vartheta = 3.5$	2.0	0.70	A+	40.9	75.0	39.4	2.9	69	6.0
$\sigma = 0.7$	2.5	0.09	A	12.2	26.2	18.6	3.6	69	7.0
$\sigma = 1.0$	2.9	1.5	A-	53.3	89.9	38.7	2.8	69	7.0
$\theta = \lambda_0^Q = 2.1$	4.8	1.0	BBB+	35.2	89.6	32.2	2.9	52	5.3
$\theta = \lambda_0^Q = 3.5$	1.8	0.48	AA-	29.9	51.0	37.1	3.2	87	8.7
$\lambda_0^Q = 2.1$	4.1	1.1	BBB+	34.2	89.2	35.3	3.1	58	6.3
$\lambda_0^Q = 3.5$	1.7	0.60	AA-	35.3	59.1	38.7	2.9	81	7.8
$\kappa = 0.15$	3.2	1.4	A-	47.4	90.4	39.3	2.8	69	6.8
$\kappa = 0.4$	1.6	0.01	AA-	7.3	11.1	11.6	3.9	69	7.6
$R = 0.2$	2.3	1.1	A+	46.8	89.6	41.0	3.0	93	7.0
$R = 0.6$	5.3	0.26	BBB	20.9	59.1	22.4	3.4	46	7.0
$r = 0.02$	3.4	1.0	A-	36.1	91.1	38.8	3.3	69	7.0
$r = 0.1$	1.9	0.55	AA-	37.8	65.8	34.2	2.5	70	7.0
$(h_1, h_2) = (0, 0)$	1.1	0.42	AA	37.7	41.4	41.2	3.0	69	9.3
$(h_1, h_2) = (0.3, 0.02)$	2.0	0.79	A+	40.1	75.5	40.2	3.0	69	7.8
$\delta = 0.015$	1.1	0.34	AA	37.4	40.3	37.9	2.7	69	7.0
$\delta = 0.025$	4.2	1.3	BBB+	37.8	90.6	38.0	3.3	69	7.0
$M = 10$	5.7	0.27	BBB	20.8	60.1	21.9	3.4	69	7.0
$M = 20$	2.4	1.2	A	48.1	90.0	41.6	3.0	69	7.0
$\beta = 1$	15.4	0.09	BB	9.5	41.6	11.2	5.0	69	7.0
$\beta = 3$	2.1	1.0	A+	55.2	62.4	89.1	2.3	69	7.0

Table 4.6: Sensitivity analysis: Historical parameters.

For the self exciting process (4.9) in example 4.3.4 we use the historical parameters and vary the index roll parameter η according to table 4.7.

	PD (%)	Cash Out (%)	Rating –	LGD (%)	ES_{99} (%)	sd(LGD) –	Cash In (years)	S_0 (bp)	N_T –
$\eta = 0$	2.5	0.81	A	38.4	81.2	37.2	3.04	69.4	7.0
$\eta = 1$	2.8	1.0	A-	39.6	89.1	38.2	3.0	69.1	7.0
$\eta = 3$	3.0	0.92	A-	37.2	87.5	37.4	3.1	68.5	7.0
$\eta = 5$	3.3	0.95	A-	35.8	88.8	36.7	3.1	67.9	7.1
$\eta = 10$	4.2	1.2	BBB+	36.4	89.5	36.3	3.1	66.5	7.2

Table 4.7: Sensitivity analysis: Self-exciting default.