Department of Mathematical Sciences
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PhD Thesis:

Classification of non-simple C^*-algebras

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Abstract

The main theme of this thesis is classification of non-simple C^*-algebras. It is based on the work obtained in the author's Master's thesis (*cf.* [Res03] and [Res06]). The thesis is divided into two parts: a text-part and an appendices-part consisting of four papers (Articles A–D).

The text-part contains two main results. The first one is a generalization of Bonkat's Universal Coefficient Theorem (UCT) (*cf.* [Bon02]) for Kirchberg's ideal-related KK-theory with two specified ideals. Invoking results of Kirchberg, this gives classification of certain purely infinite C^*-algebras with exactly two non-trivial ideals. The second result consists of a development of a notion of ideal-related K-theory with coefficients with one specified ideal, which has shown to be of relevance to classification of automorphisms of non-simple C^*-algebras (with Eilers and Ruiz).

In Article A and B (with Eilers and with Ruiz) we classify essential extensions of Kirchberg algebras and characterize the range of the invariants. In Article C (with Eilers and Ruiz) we classify certain extensions of algebras belonging to certain classes, where we can lift positive, invertible KK-elements to $*$-isomorphisms. Also we have some applications of this to Matsumoto algebras and graph algebras. In Article D (with Eilers and Ruiz) we give a series of examples showing that the naïve guess of how to define ideal-related K-theory with coefficients does not work. Also we give an example showing that Bonkat's UCT does not split, in general.

Sammenfatning

Hovedtemaet for denne afhandling er klassifikation af ikke-simple C^*-algebraer. Den tager udgangspunkt i forfatterens speciale (jævnfør [Res03] samt [Res06]). Afhandlingen består af to dele: en del tekst samt tillæg bestående af fire artikler (artikel A–D).

Tekstdelen har to hovedresultater. Det første er en generalisering af Bonkats universal koefficient sætning (UCT) (*cf.* [Bon02]) for ideal relateret KK-teori med to specificerede idealer. Ved anvendelse af Kirchbergs resultater giver dette en klassifikation af visse rent uendelige C^*-algebraer med præcis to ikke-trivielle idealer. Det andet består af udvikling af en form for ideal relateret K-teori med koefficienter med eet specificeret ideal, hvilket har vist sig at være relevant for klassifikation af automorfier af ikke-simple C^*-algebraer (med Eilers og Ruiz).

I artikel A og B (med Eilers og med Ruiz) klassificerer vi essentielle udvidelser af Kirchberg algebraer og beskriver billedet af invarianterne. I artikel C (med Eilers og Ruiz) klassificerer vi visse udvidelser af algebraer, der tilhører visse klasser, hvori vi kan løfte positive, invertible KK-elementer til $*$-isomorfier. Endvidere anvender vi resultaterne på Matsumoto algebraer samt graf algebraer. I artikel D (med Eilers og Ruiz) gives en række eksempler, der viser, at den naive måde at definere ideal relateret K-teori med koefficienter på ikke er brugbar. Desuden vises, at Bonkats UCT ikke splitter generelt.

Samandráttur

Høvuðstemað í hesari ritgerð er klassifikatión av ikki-simplum C^*-algebraum, og tekur hon útgangsstøði í serritgerð høvundsins (sí [Res03] og [Res06]). Ritgerðin er í tveimum lutum: fyrri partur er tekstur og seinri partur er eitt uppískoyti við fýra greinum (grein A–D).

Í tekstpartinum eru tvinni høvuðsúrslit. Fyrra er ein generalisering av universal koeffisient setningi (UCT) Bonkats (sí [Bon02]) fyri ideal relateraða KK-teori við tveimum spesifiseraðum idealum. Við at nýta úrslit Kirchbergs fáa vit soleiðis klassifiserað ávísar reint óendaligar C^*-algebrair við júst tveimum idealum. Seinra umfatar menning av einum slagi av ideal relateraðari K-teori við koeffisientum við eittans spesifiseraðum ideali — hetta hevur víst seg at vera nær tengt at klassifikatión av automorfium av ikki-simplum C^*-algebraum (við Eilers og Ruiz).

Í grein A og B (við Eilers og við Ruiz) klassifisera vit vesentligar víðkanir av Kirchberg algebraum og karakterisera myndina av invariantunum. Í grein C (við Eilers og Ruiz) klassificera vit ávísar víðkanir av algebraum úr klassum, har til ber at lyfta positivar, invertiblar KK-lutir til $*$-isomorfiir. Harumframt nýta vit úrslitini í sambandi við Matsumoto algebrair og graf algebrair. Í grein D (við Eilers og Ruiz) hyggja vit at eini røð av dømum, har vit lýsa, hví naivi mátin at skapa eina ideal relateraða K-teori við koeffisientum ikki kann nýtast. Eisini verður víst, at Bonkatsar UCT ikki splittar generelt.

Preface

Here I will describe the process which lead to this thesis. In the first chapter, I will give a description and overview of the contents of the thesis.

In the autumn of 2003, I wrote my Master's thesis in mathematics at the University of Copenhagen. The subject of the thesis was classification of Cuntz-Krieger algebras — with an emphasis on the non-simple case. Cuntz-Krieger algebras were introduced by Cuntz and Krieger, and have since shown to give important examples of C^*-algebras as well as they have established interplay between C^*-algebra-theory and shift spaces. In the nineties Rørdam and Huang classified the Cuntz-Krieger algebras of type (II) in three cases: (i) the simple case, (ii) the case with exactly one non-trivial ideal, and (iii) the case where the K_1-group is trivial. Type (II) corresponds to pure infiniteness.

Using recent results of Boyle and Huang, I was able to generalize the results to include all Cuntz-Krieger algebras of type (II). The main ingredients were proven in my Master's thesis (*cf.* [Res03]), and the results later published in Crelle's Journal (*cf.* [Res06]). Since this article has mainly been worked out prior to the start of my PhD-studies, this is not a part of the thesis.

During the summer of 2004, I and my family moved back to the Faroe Islands. At the same time, I won, by lottery, a grant from Valdemar Andersen's *Rejselegat for Matematikere* (*Travel scholarship for Mathematicians*), which would pay all expenses for me and my family for a year of studies in mathematics abroad.

I applied for more money to supplement this to a full PhD-project, and got the remaining part from Faculty of Science, University of Copenhagen, and the Faroese Research Council — they are paying one year each.

In June 2005, I started my PhD-studies, and I and my family went one year to Canada — mainly Toronto — where I visited Professor George A. Elliott, University of Toronto, and the Fields Institute, which provided me with excellent working conditions.

The second and third year I have been at the Department of Mathematical Sciences, University of Copenhagen — travelling a lot between the Faroe Islands and Copenhagen. Also during this period I was on a ten weeks leave due to my youngest daughter's birth.

The thesis is based on the work of my Master's thesis, and all the way the main objective is to classify non-simple C^*-algebras. The starting point was to try to generalize the UCT for ideal-related KK-theory obtained by Bonkat in the case where we specify one ideal (*cf.* [Bon02]). I used a lot of my time in Toronto on understanding Bonkat's work and trying to generalize it. One of the main problems were to make a suitable framework for doing homological algebra with the invariants. During the year in Toronto, I obtained a UCT for ideal-related KK-theory with two specified ideals (linearly ordered). Partly because I wanted to generalize it, and partly because of teaching duties and other research projects, I did not really use so much time on writing this down until the beginning of January 2007. In the end of January 2007, I got an e-mail from Ryszard Nest informing me that he in collaboration with Ralf Meyer had obtained a UCT for all finite ideal lattices.

My proof was quite long, and, moreover, quite specialized to the case with only two ideals (though it probably would be easy to generalize it to the case of a linear ideal lattice). Since Meyer and Nest claimed to have a proof, which worked in the general settings, I stopped working on this project. In November 2007, I learned from Ralf Meyer that they did not have a proof for the general case, and, moreover, it was somewhat unclear whether they still believed it to hold in general. Thus I decided to finish writing my proof down, and include it as a part of my thesis. Partly because it is not written in a format suitable for publishing, and partly because it already has been generalized by Meyer and

Nest. I have not tried to make any further improvements of this part (the theorems are in Chapter 5).

The latest I know about the status of the UCT in general, is that they can prove it for (finite) linearly ordered ideal lattices, and can disprove it for most other more complicated cases. The reader is referred to the work of Meyer and Nest (*cf.* [MNa, MNb]).

During my time in Toronto, I started collaborating with my advisor, Søren Eilers, and with Efren Ruiz, who was a post-doc. at University of Toronto at that time. This evolved to an ongoing project, where we have looked at aspects of classification theory for non-simple C^*-algebras. The rest of the thesis, Chapter 6 and the articles in the appendices, are joint work with them (in different constellations of the three of us). We have one work in progress which is based on the results of Chapter 6 (*cf.* [ERR]). As the existing preprint is very preliminary and hard to read, it is not included.

Acknowledgement

Thanks are due to many people and institutions. First I would like to express my gratitude to my late teacher, Gert Kjærgård Pedersen, for being the person he was, and for his always inspiring and aesthetical way of teaching us the theory of operator algebras.

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A special thank is due to my advisor, Søren Eilers, for his guidance during the past years. I am especially grateful for his great commitment and his interest in my work, the many interesting discussions and our fruitful collaboration. Last, I am greatly indebted to my family for their love, encourage and patience, especially my beloved wife, Joana, and my three lovely children, Brestir, Anna Birna and Poulina.

København, the 15th Day of July, 2008

Gunnar Restorff

Contents

Chapter 1

Introduction

Correspondences, which satisfy that objects that are thought of being the same (equivalent or isomorphic) are mapped to the same object (or at least equivalent or isomorphic objects) are called **invariants** (modulo the specific equivalence relation).

With respect to classification of objects, invariants can in general only be used to tell objects apart: if two objects have different invariants (non-equivalent or non-isomorphic), then the objects have to be different (non-equivalent or non-isomorphic). We cannot, in general, deduce that objects with the same invariant have to be the same. For instance, the cardinality of a group is an (isomorphism) invariant, but it is certainly not true that the equipotent groups are isomorphic.

For vector spaces over $\mathbb{C}$, the dimension is an invariant (the cardinality of a basis — we will always assume the Axiom of Choice) of vector spaces up to isomorphism. But here we can make the opposite conclusion: any two vector spaces with the same dimension are isomorphic. Such invariants are called **complete invariants**.

It is very common that invariants of categories are given as functors — although this is not always the case.[1] A functor is automatically an invariant (up to isomorphism). We call a functor a **classification functor** if it gives rise to a complete invariant.

With respect to classification theory, it is very natural to consider the following properties for a functor $\mathsf{F}\colon \mathbf{C} \to \mathbf{D}$.

- For every morphism $\alpha\colon \mathsf{F}(X) \to \mathsf{F}(Y)$, there is a morphism $\phi\colon X \to Y$ such that $\mathsf{F}(\phi) = \alpha$ (for all objects X and Y). Such a functor is called a **full functor** in category theory.

- For every isomorphism $\alpha\colon \mathsf{F}(X) \to \mathsf{F}(Y)$, there is an isomorphism $\phi\colon X \to Y$ such that $\mathsf{F}(\phi) = \alpha$ (for all objects X and Y). In this case, we call F a **strong classification functor**.

- For every object Y of $\mathbf{D}$, there is an object X of $\mathbf{C}$ such that $\mathsf{F}(X)$ is (isomorphic to) Y. Such a functor is called **essentially surjective** in category theory.

Of course, we may also ask for necessary or sufficient conditions for two homomorphisms (or isomorphisms) to induce the same morphism, or specifically, we might ask for a characterization of the morphisms inducing the identity morphism.[2]

We call a functor **faithful**, if it is injective on morphisms. The reason we omitted these above, is that these are rarely interesting from a classification point of view: usually the objective in classification theory is that it should be easier to compare the invariants than the original objects.

The program of classifying C^*-algebras was essentially initiated by George A. Elliott, and is now known as the *Elliott program.* A large number of classes of C^*-algebras have successfully been classified using some flavour of K-theoretical invariant, and for these many of the above questions have been

[1] Actually, we can think of any invariant of a category as induced by a functor on the largest subcategory, which is groupoid, *i.e.*, every morphism is an isomorphism.

[2] It is evident that the automorphisms of an object that induce the identity morphism form a normal subgroup of the automorphism group of that object, so if the functor is a strong classification functor, the automorphism group can be described as a group extension.

answered. Usually we restrict ourselves to classify only separable, nuclear C^*-algebras. For a good overview of parts of the topics of the Elliott program and the classification theory for simple, nuclear C^*-algebras the reader is referred to [Rør02].

Overview of the thesis

The thesis is organized as follows. The thesis is in two parts — one text-part and one part with four appendices. The text-part contains six numbered chapters, of which the last two chapters contain the (somewhat independent) main results.

Chapter 1. Introduction. This is the current chapter, and it serves to give an overview of the contents of the thesis.

Chapter 2. Quivers. Here the notion of quivers with relations is introduced. Along the way, we introduce representations of quivers with relations over the ring $\mathbb{Z}$, and the path algebra of a quiver with relations. Basic facts are shown. This is quite similar to the theory considered in [ARS97, Section III.1] — where they instead work over special fields (instead of the ring $\mathbb{Z}$). This chapter is needed for reading Chapters 4 and 5.

Chapter 3. Mapping cones. The chapter starts with basic definitions and results about suspensions, cones, and mapping cones. Then it goes on with more technical lemmata needed later in the thesis. It includes a survey of the definitions and results on homology and cohomology theories on C^*-algebras (based on Blackadar's book, [Bla98]). We explore the interplay between such theories and mapping cone sequences of $*$-homomorphisms. This chapter is needed for the last three chapters (though the technical Lemmata 3.1.13 and 3.1.14 are only need for the last chapter).

Chapter 4. Invariants for C^*-algebras with a distinguished system of ideals. Here we review the different — equivalent — pictures of C^*-algebras with a distinguished system of ideals. In terms of a functor, we define an invariant of such systems (for a fixed index set), and a category which serves as the codomain for this functor — this category is defined using the framework of representations of quivers developed earlier. This chapter is needed for Chapter 5.

Chapter 5. A UCT for ideal-related KK-theory. Here we proceed as in [Bon02] to prove a UCT for ideal-related KK-theory, in the case that we have two specified ideals (linearly ordered). One of the main difficulties in proving this was to establish that all objects in the range of the invariant have projective and injective dimension 1 (and to characterize the projective and injective objects of the category). The last section contains classification results for certain purely infinite, nuclear, separable C^*-algebras with exactly two non-trivial ideals. This is obtained by combining work of Kirchberg with the UCT we have obtained here. This is done analogous to the papers in Appendices A and B (and therefore it might be preferable for the reader to take a look at them before Section 5.6). This chapter is independent of Chapter 6.

Chapter 6. Ideal-related K-theory with coefficients. In this chapter we develop a notion of ideal-related K-theory with coefficients. The goal is to use this to prove a Universal Multi-Coefficient Theorem (UMCT) along the lines of Dadarlat and Loring for ideal-related KK-theory with one specified ideal. The series of examples in the paper in Appendix D gives the motivation for these definitions. The largest part of this chapter is devoted to obtain some new groups and commutative diagrams involving the new groups, and the cyclic six term exact sequences in K-theory with coefficients. These diagrams will be used in a forthcoming paper where we prove a 'limited UMCT' for a class of C^*-algebras with one specified ideal — this class includes all Cuntz-Krieger algebras of type (II) with exactly one non-trivial ideal (*cf.* [ERR]). It might be a good motivation for the reader to read the paper in Appendix D first (although it is not a prerequisite). This chapter is independent of Chapter 5.

Appendices. The four appendices, Appendices A–D, consist of four papers. These four papers can be read independently of the text-part of the thesis. Articles A, C, and D can, indeed, be read independently — while Article B builds on Article A.

Appendix A. *On Rørdam's classification of certain C^*-algebras with one non-trivial ideal* (with Eilers). This is a published paper (*cf.* [ER06]). Using Bonkat's UCT, we prove that the classification functor obtained by Rørdam for stable, essential extensions of Kirchberg algebras in the bootstrap category $\mathcal{N}$ is in fact a strong classification functor (*i.e.*, we allow for lifting of isomorphisms). We generalize a trick used by Rørdam for the classification of unital, simple Cuntz-Krieger algebras of type

(II), which allows us — in certain cases — to deduce a unital classification from a strong classification of the stabilization. Using these results, we also get a unital classification of unital, essential extensions of Kirchberg algebras in the bootstrap category $\mathcal{N}$.

Appendix B. *On Rørdam's classification of certain C^*-algebras with one non-trivial ideal, II* (with Ruiz). This is a published paper (*cf.* [RR07]). In this paper we extend the results from the paper in Appendix A. We prove that the obtained classification functor for the unital, essential extensions of Kirchberg algebras in the bootstrap category $\mathcal{N}$ is in fact a strong classification functor. We also prove a classification result for the non-stable, non-unital essential extensions of Kirchberg algebras in the bootstrap category $\mathcal{N}$, and we characterize the range in both cases. The invariants are cyclic six term exact sequences together with the class of some unit.

Appendix C. *Classification of extensions of classifiable C^*-algebras* (with Eilers and Ruiz). This is an unpublished preprint. Most likely, it will be reorganized before submission (to make it shorter and more concise). For a certain class of extensions $e\colon \mathfrak{B} \hookrightarrow \mathfrak{E} \twoheadrightarrow \mathfrak{A}$ of C^*-algebras in which $\mathfrak{B}$ and $\mathfrak{A}$ belong to a classifiable class of C^*-algebras, we show that the functor which sends e to its associated six term exact sequence in K-theory and the positive cones of $K_0(\mathfrak{B})$ and $K_0(\mathfrak{A})$ is a classification functor. We give two independent applications addressing the classification of a class of C^*-algebras arising from substitutional shift spaces on one hand and of graph algebras on the other.

Appendix D. *Non-splitting in Kirchberg's ideal-related KK-theory* (with Eilers and Ruiz). This paper has been accepted for publication in the Canadian Mathematical Bulletin. Bonkat proved that his UCT for Kirchberg's ideal-related KK-theory splits, unnaturally, under certain conditions. Employing certain K-theoretical information derivable from the given operator algebras in a way introduced here, we shall demonstrate that Bonkat's UCT does not split in general. Related methods lead to information on the complexity of the K-theory which must be used to classify $*$-isomorphisms for purely infinite C^*-algebras with exactly one non-trivial ideal.

Notation

A few words on the used notation. We will use $\hookrightarrow$ and $\twoheadrightarrow$ for injective and surjective homomorphisms, resp. Unless the contrary is explicitly stated, we will use the following conventions: ideals are two-sided; ideals of C^*-algebras are furthermore closed; modules are left modules. We let $\mathbb{N}$ denote the positive integers, $\{1, 2, 3, \ldots\}$. Otherwise, most of the notation should be self explanatory to operator algebraists.

We have tried to make an effort to unify notation throughout (the text-part of) the thesis. Also results and definitions from articles and books have been included, when it felt natural and made the text easier to read and more self-contained. However, the thesis relies heavily upon other works, and uses, of course, notation and results herefrom — especially the thesis of Bonkat (*cf.* [Bon02]).

Chapter 2

Quivers

In this chapter we develop some framework for the homological considerations leading to the UCT. We will consider quivers over the ring $\mathbb{Z}$, their representations, and the corresponding path algebras. Much of this chapter is a generalization of [ARS97, Section III.1].

2.1 Quivers

Remark 2.1.1. Throughout this chapter, we will work only with quivers over the ring $\mathbb{Z}$. Many of the definitions and results can, however, be formulated for modules over any non-trivial, commutative ring with an identity. In fact, one can generalize much of Section III.1 in [ARS97] about the connection between representations of quivers and modules over the path algebra to include commutative rings — but we will not need this.

This is sometimes in the literature also called the algebra of the enveloping category (see Fei Xu's thesis, [Xu06]).

Definition 2.1.2. A **quiver** $\Gamma = (\Gamma_0, \Gamma_1)$ is an oriented graph, where Γ_0 is the set of vertices and Γ_1 is the set of arrows between vertices. We say that the quiver Γ is **finite** if both Γ_0 and Γ_1 are finite sets. We denote by $s\colon \Gamma_1 \to \Gamma_0$ and $t\colon \Gamma_1 \to \Gamma_0$ the source and target maps, resp.

A **path** in the quiver Γ is either an ordered non-empty finite sequence of arrows $p = \alpha_n \cdots \alpha_1$ with $t(\alpha_i) = s(\alpha_{i+1})$, for $i = 1, \ldots, n-1$, or the symbol e_i, for $i \in \Gamma_0$. We call the paths e_i the **trivial paths** and we define $s(e_i) = t(e_i) = i$. For a non-trivial path $p = \alpha_n \cdots \alpha_1$ we define $s(p) = s(\alpha_1)$ and $t(p) = t(\alpha_n)$. A non-trivial path p is said to be an **oriented cycle** if $s(p) = t(p)$.

Definition 2.1.3. A **representation** $\mathbf{M}_\bullet = (\mathsf{M}, \mathsf{m})$ of a quiver Γ (over the ring $\mathbb{Z}$) is a system of abelian groups $\{\mathsf{M}_i \mid i \in \Gamma_0\}$ together with homomorphisms $\mathsf{m}_\alpha\colon \mathsf{M}_i \to \mathsf{M}_j$, for every arrow $\alpha\colon i \to j$ in Γ.

A **morphism** $\boldsymbol{\phi}_\bullet\colon (\mathsf{M}, \mathsf{m}) \to (\mathsf{N}, \mathsf{n})$ between two representations of Γ is a family $(\phi_i\colon \mathsf{M}_i \to \mathsf{N}_i)_{i\in\Gamma_0}$ of homomorphisms such that for each arrow $\alpha\colon i \to j$ in Γ the diagram

$$\begin{tikzcd} \mathsf{M}_i \arrow[r, "\phi_i"] \arrow[d, "\mathsf{m}_\alpha"] & \mathsf{N}_i \arrow[d, "\mathsf{n}_\alpha"] \\ \mathsf{M}_j \arrow[r, "\phi_j"] & \mathsf{N}_j \end{tikzcd}$$

commutes. If $\boldsymbol{\phi}_\bullet\colon (\mathsf{L}, \mathsf{l}) \to (\mathsf{M}, \mathsf{m})$ and $\boldsymbol{\psi}_\bullet\colon (\mathsf{M}, \mathsf{m}) \to (\mathsf{N}, \mathsf{n})$ are morphisms between representations then the composite morphism $\boldsymbol{\psi}_\bullet\boldsymbol{\phi}_\bullet$ is defined to be the family $(\psi_i\phi_i\colon \mathsf{L}_i \to \mathsf{N}_i)_{i\in\Gamma_0}$. This gives us the **category of representations** of Γ (over $\mathbb{Z}$), which we denote by $\operatorname{Rep}_{\mathbb{Z}}\Gamma$, or just $\operatorname{Rep}\Gamma$.

Definition 2.1.4 (Structure on $\operatorname{Rep}\Gamma$)**.** Let $\boldsymbol{\phi}_\bullet\colon (\mathsf{M}, \mathsf{m}) \to (\mathsf{N}, \mathsf{n})$ and $\boldsymbol{\psi}_\bullet\colon (\mathsf{M}, \mathsf{m}) \to (\mathsf{N}, \mathsf{n})$ be morphisms between representations. Then we define $\boldsymbol{\phi}_\bullet + \boldsymbol{\psi}_\bullet$ as the family $(\phi_i + \psi_i)_{i\in\Gamma_0}$. Clearly, $\operatorname{Hom}((\mathsf{M}, \mathsf{m}), (\mathsf{N}, \mathsf{n}))$ is an abelian group under this addition, and the composition is bilinear.

We call the representation (M, m) where $\mathsf{M}_i = 0$, for all $i \in \Gamma_0$, and $\mathsf{m}_\alpha = 0$, for all $\alpha \in \Gamma_1$, the **null** (or zero) **object** of $\operatorname{Rep}\Gamma$.

We say that an object (N, n) is a **subobject** of a representation (M, m) if N_i is a subgroup of M_i, for all $i \in \Gamma_0$, and $\mathsf{n}_\alpha = \mathsf{m}_\alpha|_{\mathsf{N}_i}$, for each arrow $\alpha \colon i \to j$.

For a morphism $\boldsymbol{\phi}_\bullet \colon (\mathsf{M}, \mathsf{m}) \to (\mathsf{N}, \mathsf{n})$ we define the **kernel**, $\ker \boldsymbol{\phi}_\bullet$, to be the subobject (L, l) of (M, m), where $\mathsf{L}_i = \ker \phi_i$, for all $i \in \Gamma_0$, and $\mathsf{l}_\alpha = \mathsf{m}_\alpha|_{\mathsf{L}_i}$, for each arrow $\alpha \colon i \to j$ (for this to be well-defined, one needs to check that $\mathsf{m}_\alpha(\ker \phi_i) \subseteq \ker \phi_j$ for each arrow $\alpha \colon i \to j$); let $\iota_{\ker \boldsymbol{\phi}_\bullet}$ denote the canonical morphism from $\ker \boldsymbol{\phi}_\bullet$ to (M, m).

For a morphism $\boldsymbol{\phi}_\bullet \colon (\mathsf{M}, \mathsf{m}) \to (\mathsf{N}, \mathsf{n})$ we define the **image**, $\operatorname{im} \boldsymbol{\phi}_\bullet$, to be the subobject (L, l) of (N, n), where $\mathsf{L}_i = \operatorname{im} \phi_i$, for all $i \in \Gamma_0$, and $\mathsf{l}_\alpha = \mathsf{n}_\alpha|_{\mathsf{L}_i}$, for each arrow $\alpha \colon i \to j$ (one needs to check that $\mathsf{n}_\alpha(\operatorname{im} \phi_i) \subseteq \operatorname{im} \phi_j$ for each arrow $\alpha \colon i \to j$); let $\iota_{\operatorname{im} \boldsymbol{\phi}_\bullet}$ denote the canonical morphism from $\operatorname{im} \boldsymbol{\phi}_\bullet$ to (N, n).

For a morphism $\boldsymbol{\phi}_\bullet \colon (\mathsf{M}, \mathsf{m}) \to (\mathsf{N}, \mathsf{n})$ we define the **cokernel**, $\operatorname{cok} \boldsymbol{\phi}_\bullet$, to be the object (L, l), where $\mathsf{L}_i = \operatorname{cok} \phi_i = \mathsf{N}_i / \operatorname{im} \phi_i$, for all $i \in \Gamma_0$, and $\mathsf{l}_\alpha \colon \mathsf{L}_i \to \mathsf{L}_j$ is the lifting of n_α, for each arrow $\alpha \colon i \to j$ (we use of course that $\mathsf{n}_\alpha(\operatorname{im} \phi_i) \subseteq \operatorname{im} \phi_j$); let $\pi_{\operatorname{cok} \boldsymbol{\phi}_\bullet}$ denote the canonical morphism (obtained from the quotient maps) from (N, n) to $\operatorname{cok} \boldsymbol{\phi}_\bullet$.

Let $\mathbf{M}_\bullet = (\mathsf{M}, \mathsf{m})$ and $\mathbf{N}_\bullet = (\mathsf{N}, \mathsf{n})$ be two representations. We define the **product** and **sum** of (M, m) and (N, n) as $\mathbf{M}_\bullet \times \mathbf{N}_\bullet = (\mathsf{M}_i \times \mathsf{N}_i)_{i \in \Gamma_0}$ together with the maps $(\mathsf{m}_\alpha \times \mathsf{n}_\alpha)_{\alpha \in \Gamma_1}$ and $\mathbf{M}_\bullet \oplus \mathbf{N}_\bullet = (\mathsf{M}_i \oplus \mathsf{N}_i)_{i \in \Gamma_0}$ with $(\mathsf{m}_\alpha \oplus \mathsf{n}_\alpha)_{\alpha \in \Gamma_1}$, resp. The product and the sum are canonically isomorphic. It is evident that we have canonical morphisms (corresponding to injections and projections) $\iota_{\mathbf{M}_\bullet, \mathbf{M}_\bullet \times \mathbf{N}_\bullet} \colon \mathbf{M}_\bullet \to \mathbf{M}_\bullet \times \mathbf{N}_\bullet$, $\iota_{\mathbf{N}_\bullet, \mathbf{M}_\bullet \times \mathbf{N}_\bullet} \colon \mathbf{N}_\bullet \to \mathbf{M}_\bullet \times \mathbf{N}_\bullet$, $\pi_{\mathbf{M}_\bullet \times \mathbf{N}_\bullet, \mathbf{M}_\bullet} \colon \mathbf{M}_\bullet \times \mathbf{N}_\bullet \to \mathbf{M}_\bullet$, and $\pi_{\mathbf{M}_\bullet \times \mathbf{N}_\bullet, \mathbf{N}_\bullet} \colon \mathbf{M}_\bullet \times \mathbf{N}_\bullet \to \mathbf{N}_\bullet$ (and correspondingly for the sum).

An object is said to be **indecomposable** if it is not isomorphic to the sum of two non-zero subobjects.

Definition 2.1.5 (Quivers with relations)**.** A **relation** σ on a quiver Γ is a $\mathbb{Z}$-linear combination of non-trivial paths $\sigma = a_1 p_1 + \cdots + a_n p_n$, where $n \in \mathbb{N}$, $a_1, \ldots, a_n \in \mathbb{Z}$, $s(p_1) = \cdots = s(p_n)$ and $t(p_1) = \cdots = t(p_n)$.[1] If $\rho = (\sigma_t)_{t \in T}$ is a family of relations, then the pair (Γ, ρ) is called a **quiver with relations**.

Recall that a subcategory $\mathbf{D}$ of $\mathbf{C}$ is called full, if every morphisms in $\mathbf{C}$ between objects of $\mathbf{D}$ is a morphism in $\mathbf{D}$.

Definition 2.1.6 ($\operatorname{Rep}(\Gamma, \rho)$)**.** Let (Γ, ρ) be a quiver with relations. For each representation $\mathbf{M}_\bullet$ of the quiver Γ and for each non-trivial path $p = \alpha_n \cdots \alpha_1$, we define m_p to be the homomorphism $\mathsf{m}_{\alpha_n} \cdots \mathsf{m}_{\alpha_1}$, and let $\mathsf{m}_{e_i} = \mathrm{id}_{\mathsf{M}_i}$, for all $i \in \Gamma_0$.

By $\operatorname{Rep}(\Gamma, \rho)$ we denote the full subcategory of $\operatorname{Rep}\Gamma$, whose objects $\mathbf{M}_\bullet$ satisfy

$$a_1 \mathsf{m}_{p_1} + \cdots + a_n \mathsf{m}_{p_n} = 0,$$

for each relation $\sigma = a_1 p_1 + \cdots + a_n p_n$ in ρ.

Example 2.1.7 (A special representation)**.** Let (Γ, ρ) be a quiver with relations (where we allow ρ to be the empty family in which case this just denotes the quiver).

Let G be an abelian group and let $i_0 \in \Gamma_0$ be a vertex. Then we define the representation $\mathbf{M}_\bullet = \mathbf{Compl}_\bullet^{G, i_0}$ by $\mathsf{M}_{i_0} = G$, $\mathsf{M}_i = \{0\}$, for all $i \neq i_0$, and $\mathsf{m}_\alpha = 0$, for every arrow $\alpha \in \Gamma_1$. Then $\mathbf{M}_\bullet$ is in $\operatorname{Rep}(\Gamma, \rho)$.

2.2 Some preliminaries

Definition 2.2.1. An **algebra** over the ring $\mathbb{Z}$ is a $\mathbb{Z}$-module A with an associative multiplication such that the map $A \times A \ni (a, a') \mapsto aa'$ is bilinear. This is equivalent to say that A is a ring (where we allow for non-unital rings).

[1] In [ARS97] they assume that the number of arrows in each of these paths is at least 2 — but for our purposes, we do not need that assumption.

We say that an element e of a ring is **idempotent** if $e^2 = e$. We say that a system $(e_1, \dots, e_n)$ of idempotents is **orthogonal** if $e_i e_j = 0$ whenever $i \neq j$. We say that a non-zero idempotent e is **primitive** if e cannot be written as the sum of two non-zero orthogonal idempotents. When we say module, we mean left module (except when we explicitly say right module).

The following lemma is inspired by [ARS97, Proposition I.4.8].

Lemma 2.2.2. *Let R be a non-trivial ring with identity, and let e be a non-zero idempotent in R.*

(a) Let $(e_1, \dots, e_n)$ be an orthogonal system of non-zero idempotents such that $e = e_1 + \cdots + e_n$. Then Re_i is a submodule of Re, for all $i = 1, \dots, n$, and $Re = Re_1 \oplus \cdots \oplus Re_n$.

(b) Re is a projective R-module.

(c) e is primitive if and only if Re is an indecomposable R-module.

Proof. (a): We have that $e_i e = e_i e_1 + \cdots + e_i e_n = e_i$. Hence each e_i is in Re and so each $Re_i \subseteq Re$. We now want to show, that each element x of Re can be written uniquely as a sum $x_1 + \cdots + x_n$ with $x_i \in Re_i$, for all $i = 1, \dots, n$. Let $x \in Re$ be given. Clearly, x can be written as such a sum ($x = xe = xe_1 + \cdots + xe_n$). Let now $x = r_1 e_1 + \cdots + r_n e_n = r'_1 e_1 + \cdots + r'_n e_n$ with $r_i, r'_i \in R$, for all $i = 1, \dots, n$. From this we get $r_1 e_1 e_i + \cdots + r_n e_n e_i = r'_1 e_1 e_i + \cdots + r'_n e_n e_i$, which is the same as $r_i e_i = r'_i e_i$. This proves the uniqueness.

(b): If $e = 1$ this is clear, so assume that $e \neq 1$. Clearly, $1 - e$ and e are non-zero orthogonal idempotents. Thus we have from (a) that $R = R1 = R(1 - e) \oplus Re$. So Re is a direct summand in the free R-module R, consequently, it is projective.

(c): If e is not primitive, then we have two non-zero orthogonal idempotents e_1, e_2 such that $e = e_1 + e_2$. From (a) we have that $Re = Re_1 \oplus Re_2$, and clearly neither Re_1 nor Re_2 are zero.

Contrary, if Re is decomposable, then we have two non-zero submodules M_1, M_2 of Re such that $Re = M_1 \oplus M_2$. So there exist unique elements $e_1 \in M_1$ and $e_2 \in M_2$ such that $e = e_1 + e_2$. We also have that $e_1 e = e_1 e_1 + e_1 e_2 = e_1$ and $e_2 e = e_2 e_1 + e_2 e_2 = e_2$, since M_1 and M_2 are submodules of Re. By uniqueness we have $e_1^2 = e_1$, $e_1 e_2 = 0 = e_2 e_1$ and $e_2^2 = e_2$ (since $e_j e_i \in M_i$, for all $i, j \in \{1, 2\}$). It is easy to see that $e_1 \neq 0 \neq e_2$, hence e is not primitive. ●●●

The following definition is taken from [HS97, §I.8].

Definition 2.2.3. Let R be a non-trivial ring with identity. Let M be a right R-module and let G be an abelian group. Then we can equip the abelian group $\mathrm{Hom}_{\mathbb{Z}}(M, G)$ with a (left) R-module structure as follows:

$$(r\varphi)(x) = \varphi(xr), \quad x \in M, r \in R, \varphi \in \mathrm{Hom}_{\mathbb{Z}}(M, G).$$

It is an easy exercise to verify that this makes $\mathrm{Hom}_{\mathbb{Z}}(M, G)$ into a (left) R-module.

Definition 2.2.4. Let R be a non-trivial ring with identity. Let M be an R-module and let G be an abelian group. Then $M \otimes_{\mathbb{Z}} G$ is clearly a $\mathbb{Z}$-module. But for each $r \in R$ we can uniquely define a $\mathbb{Z}$-module homomorphism by

$$f_r \colon M \otimes_{\mathbb{Z}} G \ni x \otimes g \mapsto rx \otimes g \in M \otimes_{\mathbb{Z}} G.$$

By uniqueness we see, that for all $r_1, r_2 \in R$ we have $f_{r_1 + r_2} = f_{r_1} + f_{r_2}$, $f_{r_1 r_2} = f_{r_1} \circ f_{r_2}$, $f_{\mathbb{1}} = \mathrm{id}_{M \otimes_{\mathbb{Z}} G}$. Consequently, the left action of R on $M \otimes_{\mathbb{Z}} G$ given by $R \times (M \otimes_{\mathbb{Z}} G) \ni (r, x) \mapsto f_r(x) \in M \otimes_{\mathbb{Z}} G$ makes $M \otimes_{\mathbb{Z}} G$ into an R-module.

Inspired by [HS97, Proposition I.8.1], we prove the following two propositions.

Proposition 2.2.5. *Let R be a non-trivial ring with identity, and let e be a non-zero idempotent in R. Regard eR as a right R-module. Let M be an R-module, and let G be an abelian group. Regard $\mathrm{Hom}_{\mathbb{Z}}(eR, G)$ as an R-module as above. Then we have a functorial isomorphism*

$$\eta_M \colon \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbb{Z}}(eR, G)) \to \mathrm{Hom}_{\mathbb{Z}}(eM, G)$$

of abelian groups.

Proof. The proof of this proposition is quite similar to the proof of [HS97, Proposition I.8.1] (where $e = 1$). For each $\varphi \in \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbb{Z}}(eR, G))$, we define $\eta_M(\varphi) = \varphi' \in \mathrm{Hom}_{\mathbb{Z}}(eM, G)$ by

$$\varphi'(x) = (\varphi(x))(e), \quad x \in eM.$$

For each $\psi \in \mathrm{Hom}_{\mathbb{Z}}(eM, G)$, we define a map $\psi' \colon M \to \mathrm{Hom}_{\mathbb{Z}}(eR, G)$ by

$$(\psi'(x))(r) = \psi(rx), \quad r \in eR, x \in M$$

(clearly, $\psi'(x)$ is a $\mathbb{Z}$-module homomorphism). Clearly, ψ' is a $\mathbb{Z}$-module homomorphism, and for $r \in R$, $r' \in eR$ and $x \in M$ we have

$$\psi'(rx)(r') = \psi(r'rx) = \psi'(x)(r'r) = (r(\psi'(x)))(r').$$

Hence, ψ' is an R-module homomorphism.

It is evident, that $\varphi \mapsto \varphi'$ and $\psi \mapsto \psi'$ are $\mathbb{Z}$-module homomorphisms. Moreover, $(\varphi')' = \varphi$ and $(\psi')' = \psi$. First, we see immediately that $(\psi')'(x) = (\psi'(x))(e) = \psi(ex) = \psi(x)$ for all $x \in eM$. Moreover, for $x \in M$ and $r \in eR$ we have

$$((\varphi')'(x))(r) = \varphi'(rx) = (\varphi(rx))(e) = (r(\varphi(x)))(e) = (\varphi(x))(er) = (\varphi(x))(r).$$

Thus $\eta_M = [\varphi \mapsto \varphi']$ is a $\mathbb{Z}$-module isomorphism, with inverse $\psi \mapsto \psi'$. Functoriality is straightforward to check. ●●●

Proposition 2.2.6. *Let R be a non-trivial ring with identity, and let e be a non-zero idempotent in R. Let M be an R-module, and let G be an abelian group. Regard $Re \otimes_{\mathbb{Z}} G$ as an R-module as above. Then we have a functorial isomorphism*

$$\eta_M \colon \mathrm{Hom}_R(Re \otimes_{\mathbb{Z}} G, M) \to \mathrm{Hom}_{\mathbb{Z}}(G, eM)$$

of abelian groups.

Proof. For each $\varphi \in \mathrm{Hom}_R(Re \otimes_{\mathbb{Z}} G, M)$, we define $\varphi' \in \mathrm{Hom}_{\mathbb{Z}}(G, eM)$ by

$$\varphi'(g) = \varphi(e \otimes g) = e\varphi(1 \otimes g), \quad g \in G.$$

Using the universal property for tensor products, for each $\psi \in \mathrm{Hom}_{\mathbb{Z}}(G, eM)$, we define a unique $\mathbb{Z}$-module homomorphism $\psi' \colon Re \otimes_{\mathbb{Z}} G \to M$ by

$$\psi'(r \otimes g) = r\psi(g), \quad \text{for all } r \in Re, g \in G.$$

It is straightforward to check that ψ' is an R-module homomorphism.

Clearly, $\varphi \mapsto \varphi'$ and $\psi \mapsto \psi'$ are $\mathbb{Z}$-module homomorphisms. Moreover, $(\varphi')' = \varphi$ and $(\psi')' = \psi$:

$$(\varphi')'(r \otimes g) = r\varphi'(g) = r\varphi(e \otimes g) = \varphi(r(e \otimes g)) = \varphi(r \otimes g), \quad r \in Re, g \in G,$$

$$(\psi')'(g) = \psi'(e \otimes g) = e\psi(g) = \psi(g), \quad g \in G.$$

Thus $\eta_M = [\varphi \mapsto \varphi']$ is a $\mathbb{Z}$-module isomorphism, with inverse $\psi \mapsto \psi'$. Functoriality is straightforward to check. ●●●

Proposition 2.2.7. *Let R be a non-trivial ring with identity, let $e \in R$ be a non-zero idempotent. For each projective $\mathbb{Z}$-module P (i.e., free abelian group), the R-module $Re \otimes_{\mathbb{Z}} P$ is projective. For each injective $\mathbb{Z}$-module I (i.e., divisible abelian group), the R-module $\mathrm{Hom}_{\mathbb{Z}}(eR, I)$ is injective.*

Proof. If $0 \to L \to M \to N \to 0$ is a short exact sequence of R-modules, the induced sequence $0 \to eL \to eM \to eN \to 0$ abelian groups is also short exact. Since $\mathrm{Hom}_{\mathbb{Z}}(P, -)$ is exact when P is projective, and $\mathrm{Hom}_{\mathbb{Z}}(-, I)$ is exact when I is injective, the results follow from the functorial isomorphisms of Propositions 2.2.5 and 2.2.6. ●●●

We will be using some notation from category theory — a good reference for category theory is the monography [ML98] (all what we use will be found there).

Proposition 2.2.8. *Let $\mathsf{F}\colon \mathbf{C} \to \mathbf{D}$ be an equivalence of categories. Then the following three properties hold.*

(a) For all objects X and Y of $\mathbf{C}$ the induced function $\mathrm{Hom}_{\mathbf{C}}(X, Y) \to \mathrm{Hom}_{\mathbf{D}}(\mathsf{F}(X), \mathsf{F}(Y))$ is injective (i.e., the functor F is faithful).

(b) For all objects X and Y of $\mathbf{C}$ the induced function $\mathrm{Hom}_{\mathbf{C}}(X, Y) \to \mathrm{Hom}_{\mathbf{D}}(\mathsf{F}(X), \mathsf{F}(Y))$ is surjective (i.e., the functor F is full).

(c) For all objects Z of $\mathbf{D}$ there is an object X of $\mathbf{C}$ such that $\mathsf{F}(X)$ and Z are isomorphic in $\mathbf{D}$ (i.e., the functor F is essentially surjective).

Proof. Let $\mathsf{G}\colon \mathbf{D} \to \mathbf{C}$ denote the inverse of F. Let $\theta\colon \mathsf{GF} \to \mathrm{id}_{\mathbf{C}}$ and $\eta\colon \mathsf{FG} \to \mathrm{id}_{\mathbf{D}}$ be the natural isomorphisms corresponding to this equivalence.

Assume $f, g\colon X \to Y$ with $\mathsf{F}(f) = \mathsf{F}(g)$. Then

$$f = \theta_Y \circ \mathsf{GF}(f) \circ \theta_X^{-1} = \theta_Y \circ \mathsf{GF}(g) \circ \theta_X^{-1} = g.$$

Let $g\colon \mathsf{F}(X) \to \mathsf{F}(Y)$ be an arbitrary morphism. Set

$$h = \eta_{\mathsf{F}(Y)} \circ \mathsf{F}(\theta_Y^{-1}) \circ g \circ \mathsf{F}(\theta_X) \circ \eta_{\mathsf{F}(X)}^{-1},$$

and set

$$f = \theta_Y \circ \mathsf{G}(h) \circ \theta_X^{-1}.$$

Then

$$\mathsf{F}(f) = \mathsf{F}(\theta_Y) \circ \mathsf{FG}(h) \circ \mathsf{F}(\theta_X^{-1}) = \mathsf{F}(\theta_Y) \circ \eta_{\mathsf{F}(Y)}^{-1} \circ h \circ \eta_{\mathsf{F}(X)} \circ \mathsf{F}(\theta_X^{-1}) = g.$$

Let Z be an arbitrary object of $\mathbf{D}$. Then η_Z is an isomorphism from $\mathsf{F}(\mathsf{G}(Z))$ to Z. ●●●

Proposition 2.2.9. *Let $\mathsf{F}\colon \mathbf{C} \to \mathbf{D}$ be an equivalence of categories with $\mathsf{G}\colon \mathbf{D} \to \mathbf{C}$ as inverse. Let $\theta\colon \mathsf{GF} \to \mathrm{id}_{\mathbf{C}}$ and $\eta\colon \mathsf{FG} \to \mathrm{id}_{\mathbf{D}}$ be the natural isomorphisms corresponding to this equivalence.*

Let X be an object of $\mathbf{C}$ and let f be an arrow in $\mathbf{C}$. Then

(a) X is an initial (terminal, or zero) object of $\mathbf{C}$ if and only if $\mathsf{F}(X)$ is an initial (terminal, or zero) object of $\mathbf{D}$.

(b) f is a monic (epic, or iso-) morphism in $\mathbf{C}$ if and only if $\mathsf{F}(f)$ is a monic (epic, or iso-) morphism in $\mathbf{D}$.

(c) X is projective (or injective) if and only if $\mathsf{F}(X)$ is projective (or injective).

(d) The image of a product (or coproduct) is — in the canonical way — again a product (or coproduct).

(e) The image of a kernel (or cokernel) of f is — in the canonical way — a kernel (or cokernel) of $\mathsf{F}(f)$.

(f) F is an exact functor (i.e., maps exact sequences to exact sequences).

(g) If $\mathbf{C}$ is an additive category (or abelian category), then $\mathbf{D}$ may be turned into an additive category in such a way that F becomes an additive functor. On the other hand, if $\mathbf{C}$ and $\mathbf{D}$ both are additive categories (or abelian categories), then F and G are automatically additive.

Proof. We found this on the internet without any reference nor any proof.[2] Using the previous proposition, the proof is quite straightforward (at least until the last part), and we leave the long and tedious proof to the reader. ●●●

[2] In order to not being as bad, we should refer to the homepage `http://en.wikipedia.org/wiki/Equivalence_of_categories` where we found it.

2.3 The path algebra of a quiver

Definition 2.3.1 (Path algebra of a quiver)**.** For the ring $\mathbb{Z}$, let $\mathbb{Z}\Gamma$ be the (free) $\mathbb{Z}$-module with the paths of Γ as $\mathbb{Z}$-basis. Thus as $\mathbb{Z}$-module $\mathbb{Z}\Gamma$ is just the set $\mathbb{Z}^{(\mathrm{path}(\Gamma))}$ of functions with finite support from the set of paths of Γ to $\mathbb{Z}$ equipped with the pointwise addition.

For paths $p = \alpha_m \cdots \alpha_1$ and $q = \beta_n \cdots \beta_1$ with $t(q) = s(p)$ we let pq denote the concatenated path $\alpha_m \cdots \alpha_1 \beta_n \cdots \beta_1$, and $pq = p$ (resp. $pq = q$) if q (resp. p) is trivial. Then we can define a multiplication $\star$ on $\mathbb{Z}\Gamma$ as follows:

$$(f \star g)(r) = \sum_{pq=r} f(p)g(q), \quad \text{for every path } r.$$

It is easy to verify, that the map $\mathbb{Z}\Gamma \times \mathbb{Z}\Gamma \ni (f, g) \mapsto f \star g \in \mathbb{Z}\Gamma$ is bilinear and that the multiplication is associative. Consequently, $\mathbb{Z}\Gamma$ is a $\mathbb{Z}$-algebra (or, equivalently, $(\mathbb{Z}\Gamma, +, \star)$ is a ring). This algebra is called the **path algebra** of Γ (over $\mathbb{Z}$). Clearly, $\mathbb{1} = \sum_{i \in \Gamma_0} e_i$ is an identity for the ring $\mathbb{Z}\Gamma$ if Γ is a finite quiver. It is evident from the definition that a finite quiver Γ has no oriented cycles if and only if $\mathbb{Z}\Gamma$ is finitely generated as a $\mathbb{Z}$-module.

For a path p we will also let p denote the characteristic function of $\{p\}$. Note that with this notation $pq = p \star q$ for any two paths, with the convention, that $pq = 0$ if $t(q) \neq s(p)$.

Definition 2.3.2. Let there be given a finite quiver Γ, and assume for notational convenience that $\Gamma_0 = \{1, \ldots, n\}$. We are looking at the category of representations of Γ (over $\mathbb{Z}$), $\operatorname{Rep}\Gamma$, and at the category of $\mathbb{Z}\Gamma$-modules, $\operatorname{Mod}(\mathbb{Z}\Gamma)$ (we do not only consider the finitely generated ones as the authors of [ARS97] do). We now want to construct functors $\mathsf{F}\colon \operatorname{Rep}\Gamma \to \operatorname{Mod}(\mathbb{Z}\Gamma)$ and $\mathsf{G}\colon \operatorname{Mod}(\mathbb{Z}\Gamma) \to \operatorname{Rep}\Gamma$.

Let $\mathbf{M}_\bullet$ be an object of $\operatorname{Rep}\Gamma$. Define $\mathsf{F}(\mathbf{M}_\bullet)$ to be $V = \oplus_{i \in \Gamma_0} \mathsf{M}_i$ as an abelian group. Let $f \in \mathbb{Z}\Gamma$ and $v = (v_i)_{i \in \Gamma_0} \in V$. Then we define a left action by

$$fv = \left(\sum_{p \in \mathrm{path}(\Gamma), t(p)=i} f(p) \mathsf{m}_p(v_{s(p)}) \right)_{i \in \Gamma_0}.$$

It is straightforward to show, that V is a $\mathbb{Z}\Gamma$-module under this action.

Now let $\boldsymbol{\phi}_\bullet\colon \mathbf{M}_\bullet \to \mathbf{N}_\bullet$ be a morphism in $\operatorname{Rep}\Gamma$. For each $i \in \Gamma_0$ we have a group homomorphism $\phi_i\colon \mathsf{M}_i \to \mathsf{N}_i$. Clearly this induces a group homomorphism from $\mathsf{F}(\mathbf{M}_\bullet)$ to $\mathsf{F}(\mathbf{N}_\bullet)$. Let $\mathsf{F}(\boldsymbol{\phi}_\bullet)$ be this homomorphism. It is easy to check that this in fact is a $\mathbb{Z}\Gamma$-module homomorphism. Clearly F is a functor.

Now let V be a given $\mathbb{Z}\Gamma$-module. Since $\mathbb{1} = e_1 + \cdots + e_n$ is a sum of orthogonal idempotents in $\mathbb{Z}\Gamma$, we get the abelian group V as a direct sum $V = \bigoplus_{i=1}^n e_i V$. For any arrow $\alpha\colon i \to j$ we have $\alpha e_i = \alpha = e_j \alpha$ — therefore we have a group homomorphism $\mathsf{m}_\alpha\colon e_i V \ni v \mapsto \alpha v \in e_j V$. Now let $\mathsf{G}(V)$ be the representation $\mathbf{M}_\bullet$ consisting of the abelian groups $(e_i V)_{i \in \Gamma_0}$ together with the homomorphisms $(\mathsf{m}_\alpha)_{\alpha \in \Gamma_1}$.

Now let $\phi\colon V \to V'$ be a $\mathbb{Z}\Gamma$-module homomorphism. Let $\mathbf{M}_\bullet = \mathsf{G}(V)$ and $\mathbf{N}_\bullet = \mathsf{G}(V')$. Then $\phi(e_i V) = e_i \phi(V) \subseteq e_i V'$. Thus we can define $\phi_i\colon e_i V \to e_i V'$ to be the restriction of ϕ to $e_i V$, for every $i = 1, \ldots, n$. Clearly these are group homomorphisms. Let $\alpha\colon i \to j$ be an arrow. Then $\phi_j(\mathsf{m}_\alpha(v)) = \phi(\alpha v) = \alpha\phi(v) = \mathsf{n}_\alpha(\phi_i(v))$, for every $v \in e_i V$. Consequently, $(\phi_i)_{i \in \Gamma_0}$ is a morphism in $\operatorname{Rep}\Gamma$, we denote it $\mathsf{G}(\phi)$. Clearly G is a functor.

Similarly as in [ARS97], we prove that F and G are equivalences of categories:

Lemma 2.3.3. *Let Γ be a finite quiver. The functors F and G are inverse equivalences of categories.*

Proof. Let $\mathbf{M}_\bullet$ be a representation. Let $\iota_i\colon \mathsf{M}_i \to \oplus_{j \in \Gamma_0} \mathsf{M}_j$ be the canonical embedding, and let $\theta_i^{\mathbf{M}_\bullet}$ be the corestriction of ι_i to the image of ι_i. Then clearly $\theta_i^{\mathbf{M}_\bullet}$ is an isomorphism, for every $i \in \Gamma_0$. In fact, we have an isomorphism $\boldsymbol{\theta}_\bullet^{\mathbf{M}_\bullet}\colon \mathbf{M}_\bullet \to \mathsf{GF}(\mathbf{M}_\bullet) = \mathbf{N}_\bullet$ — for this we only need to show that $\theta_j^{\mathbf{M}_\bullet} \mathsf{m}_\alpha = \mathsf{n}_\alpha \theta_i^{\mathbf{M}_\bullet}$, for every arrow $\alpha\colon i \to j$. So let $\alpha\colon i \to j$ be a given arrow. Then for $m_i \in \mathsf{M}_i$ we have

$$\begin{aligned} \mathsf{n}_\alpha \theta_i^{\mathbf{M}_\bullet}(m_i) &= \mathsf{n}_\alpha((\delta_{i,k} m_i)_{k \in \Gamma_0}) = \alpha(\delta_{i,k} m_i)_{k \in \Gamma_0} = (\delta_{j,k} \mathsf{m}_\alpha(m_i))_{k \in \Gamma_0} \\ &= \theta_j^{\mathbf{M}_\bullet}(\mathsf{m}_\alpha(m_i)). \end{aligned}$$

We want to prove that $\boldsymbol{\theta}_\bullet^-$ is a natural transformation from $\mathrm{id}_{\mathrm{Rep}\,\Gamma}$ to GF. To show this, let $\mathbf{M}_\bullet$ and $\mathbf{N}_\bullet$ be representations, and let $\boldsymbol{\phi}_\bullet\colon \mathbf{M}_\bullet \to \mathbf{N}_\bullet$ be a morphism. Then we have to show that $\mathsf{GF}(\boldsymbol{\phi}_\bullet)\boldsymbol{\theta}_\bullet^{\mathbf{M}_\bullet} = \boldsymbol{\theta}_\bullet^{\mathbf{N}_\bullet}\boldsymbol{\phi}_\bullet$, which is clear since we for $m_i \in \mathsf{M}_i$ have

$$\begin{aligned}\mathsf{GF}(\boldsymbol{\phi}_\bullet)_i\theta_i^{\mathbf{M}_\bullet}(m_i) &= \mathsf{GF}(\boldsymbol{\phi}_\bullet)_i((\delta_{i,k}m_i)_{k\in\Gamma_0}) = \mathsf{F}(\boldsymbol{\phi}_\bullet)((\delta_{i,k}m_i)_{k\in\Gamma_0})\\ &= (\delta_{i,k}\phi_i(m_i))_{k\in\Gamma_0} = \theta_i^{\mathbf{N}_\bullet}(\phi_i(m_i)).\end{aligned}$$

Now let V and W be $\mathbb{Z}\Gamma$-modules, and let $\phi\colon V \to W$ be a $\mathbb{Z}\Gamma$-module homomorphism. Let $\phi_i\colon e_iV \to e_iW$ denote the restriction. Then we have the following commutative diagram

$$\begin{tikzcd} V \arrow[r, "\phi"] \arrow[d, "\cong"] & W \arrow[d, "\cong"] \\ \mathsf{FG}(V) = \bigoplus_{i\in\Gamma_0} e_iV \arrow[r, "\mathsf{FG}(\phi)=\phi_1\oplus\phi_2\oplus\cdots\oplus\phi_n"] & \mathsf{FG}(W) = \bigoplus_{i\in\Gamma_0} e_iW \end{tikzcd},$$

where the vertical $\mathbb{Z}\Gamma$-module isomorphisms are the canonical ones. From this it follows that we have an isomorphism of functors from $\mathrm{id}_{\mathrm{Mod}\,\mathbb{Z}\Gamma}$ to FG. ●●●

Definition 2.3.4 (Path algebra of a quiver with relations)**.** Let (Γ, ρ) be a quiver with relations. Associated with (Γ, ρ) is the **path algebra** $\mathbb{Z}(\Gamma, \rho) = \mathbb{Z}\Gamma/ < \rho >$, where $< \rho >$ denotes the ideal of $\mathbb{Z}\Gamma$ generated by the set ρ of relations.

Corollary 2.3.5. *Let (Γ, ρ) be a finite quiver with relations. Then the functor F induces an equivalence between the categories $\mathrm{Rep}(\Gamma, \rho)$ and $\mathrm{Mod}(\mathbb{Z}(\Gamma, \rho))$.*

Proof. Let $\mathbf{M}_\bullet$ be in $\mathrm{Rep}(\Gamma, \rho)$, and let $\sigma = a_1p_1 + \cdots + a_kp_k \in \rho$. Let $i = s(p_1)$ and $j = t(p_1)$. Then $\mathsf{m}_\sigma = 0$. So for each $v = (v_k)_{k\in\Gamma_0} \in \mathsf{F}(\mathbf{M}_\bullet)$ is

$$\sigma v = a_1p_1v + \cdots + a_kp_kv = (\delta_{j,k}(a_1\mathsf{m}_{p_1}(v_i) + \cdots + a_k\mathsf{m}_{p_k}(v_i)))_{k\in\Gamma_0} = 0.$$

From this we see that $\mathsf{F}(\mathbf{M}_\bullet)$ is a $\mathbb{Z}(\Gamma, \rho)$-module ($\mathrm{Mod}\,\mathbb{Z}(\Gamma, \rho)$ is a subcategory of $\mathrm{Mod}\,\mathbb{Z}\Gamma$).

If conversely $\mathsf{F}(\mathbf{M}_\bullet)$ is a $\mathbb{Z}(\Gamma, \rho)$-module, then $\sigma\mathsf{F}(\mathbf{M}_\bullet) = 0$, for all $\sigma \in \rho$. So — by a similar calculation — $\mathsf{m}_\sigma = 0$, for all $\sigma \in \rho$. Hence $\mathbf{M}_\bullet$ is in $\mathrm{Rep}(\Gamma, \rho)$, and therefore it is easy to see, that also $\mathsf{GF}(\mathbf{M}_\bullet)$ is in $\mathrm{Rep}(\Gamma, \rho)$. So the above lemma finishes our proof. ●●●

Definition 2.3.6. We say that a sequence $\mathbf{L}_\bullet \xrightarrow{\boldsymbol{\phi}_\bullet} \mathbf{M}_\bullet \xrightarrow{\boldsymbol{\psi}_\bullet} \mathbf{N}_\bullet$ of morphisms is exact if $\mathrm{im}\,\boldsymbol{\phi}_\bullet = \ker\boldsymbol{\psi}_\bullet$. Clearly this is the case if and only if $\mathsf{L}_i \xrightarrow{\phi_i} \mathsf{M}_i \xrightarrow{\psi_i} \mathsf{N}_i$ is exact, for all $i \in \Gamma_0$. We extend this to define exactness of any sequence, and to define short exact sequences in the obvious way.

Corollary 2.3.7. *Let (Γ, ρ) be a finite quiver with relations. Then $\mathbf{M}_\bullet$ is projective (resp. injective, indecomposable) in $\mathrm{Rep}(\Gamma, \rho)$, if and only if $\mathsf{F}(\mathbf{M}_\bullet)$ is projective (resp. injective, indecomposable) in $\mathrm{Mod}\,\mathbb{Z}(\Gamma, \rho)$, if and only if $\mathsf{GF}(\mathbf{M}_\bullet)$ is projective (resp. injective, indecomposable) in $\mathrm{Rep}(\Gamma, \rho)$.*

Moreover, a sequence $\mathbf{L}_\bullet \to \mathbf{M}_\bullet \to \mathbf{N}_\bullet$ in $\mathrm{Rep}(\Gamma, \rho)$ is exact, if and only if the induced sequence $\mathsf{F}(\mathbf{L}_\bullet) \to \mathsf{F}(\mathbf{M}_\bullet) \to \mathsf{F}(\mathbf{N}_\bullet)$ is exact in $\mathrm{Mod}\,\mathbb{Z}(\Gamma, \rho)$, if and only if $\mathsf{GF}(\mathbf{L}_\bullet) \to \mathsf{GF}(\mathbf{M}_\bullet) \to \mathsf{GF}(\mathbf{N}_\bullet)$ is exact in $\mathrm{Rep}(\Gamma, \rho)$.

Proof. This is a direct consequence of Proposition 2.2.9 and Corollary 2.3.5. ●●●

Assume that (Γ, ρ) is a finite quiver with relations. From Proposition 2.2.9 — and Corollary 2.3.5 — it also follows, that $\mathrm{Rep}(\Gamma, \rho)$ is an abelian category. Moreover, it follows that $\mathrm{Rep}(\Gamma, \rho)$ has enough projectives and injectives, so it makes sense to define derived functors such as Ext^1 (actually, it is not so hard to prove these results directly, but the proof is long and boring). It is easy to check that the socalled null object, products, resp. sums, are in fact null object, products, resp. coproducts in the category theoretical sense. Also, it is easy to see that the kernel, image and cokernel (in $\mathrm{Rep}(\Gamma, \rho)$) correspond to the usual definitions in $\mathrm{Mod}(\mathbb{Z}(\Gamma, \rho))$ via this equivalence.

Using this equivalence and Proposition 2.2.9, we also see that a morphism $\boldsymbol{\phi}_\bullet\colon \mathbf{M}_\bullet \to \mathbf{N}_\bullet$ is a monic morphism, an epic morphism, or an isomorphism if and only if for all $i \in \Gamma_0$ the $\mathbb{Z}$-module homomorphism ϕ_i is injective, surjective, or bijective, resp. (*cf.* [HS97, Propositions I.6.1 and I.6.2])

Definition 2.3.8. Let (Γ, ρ) be a finite quiver with relations, and denote $\Lambda = \mathbb{Z}(\Gamma, \rho)$. Note that if every cycle belongs to $< \rho >$, then $\mathsf{F}(\mathbf{Compl}_{\bullet}^{G,i})$ is canonically isomorphic to $\overline{e}_i \Lambda \overline{e}_i \otimes_{\mathbb{Z}} G$.

Form the Λ-module $\Lambda \overline{e}_i \otimes_{\mathbb{Z}} G$ (*cf.* Definition 2.2.4). We let $\mathbf{Free}_{\bullet}^{G,i}$ denote the representation $\mathsf{G}(\Lambda \overline{e}_i \otimes_{\mathbb{Z}} G)$. Note that $\mathbf{Free}_{\bullet}^{G,i}$ is projective if G is a projective $\mathbb{Z}$-module (*cf.* Proposition 2.2.7).

Form the Λ-module $\mathrm{Hom}_{\mathbb{Z}}(\overline{e}_i \Lambda, G)$ (*cf.* Definition 2.2.3). We let $\mathbf{Cofree}_{\bullet}^{G,i}$ denote the representation $\mathsf{G}(\mathrm{Hom}_{\mathbb{Z}}(\overline{e}_i \Lambda, G))$. Note that $\mathbf{Cofree}_{\bullet}^{G,i}$ is injective if G is an injective $\mathbb{Z}$-module (*cf.* Proposition 2.2.7).

2.4 Examples

Remark 2.4.1. Let (Γ, ρ) be a finite quiver with relations. Let Λ denote the ring $\mathbb{Z}(\Gamma, \rho)$. Then we can write the identity of the ring Λ as a sum of orthogonal, non-zero idempotents $\mathbb{1} = \sum_{i \in \Gamma_0} \overline{e}_i$ (for a path p we let $\overline{p}$ denote the class in $\mathbb{Z}(\Gamma, \rho)$ containing p). It is clear that $(\overline{e}_i)_{i \in \Gamma_0}$ is an orthogonal system of idempotents. Since every path in a relation in ρ is assumed to be non-trivial, it is easy to see that $e_i \notin < \rho >$, for $i \in \Gamma_0$.

Therefore, $\Lambda \overline{e}_i$ is a (non-zero) projective module over Λ, for every $i \in \Gamma$ (*cf.* Lemma 2.2.2). So $\Lambda \overline{e}_i$ is generated by all paths starting at vertex i. Also, $\mathrm{Hom}_{\mathbb{Z}}(\overline{e}_i \Lambda, \mathbb{Q}/\mathbb{Z})$ is an injective Λ-module.

Let us look at some examples.

Example 2.4.2. Let Γ be the quiver

$$v_1 \xrightarrow{\alpha} v_2 \xrightarrow{\beta} v_3 \xrightarrow{\gamma} v_4.$$

Then $\{e_1, e_2, e_3, e_4, \alpha, \beta, \gamma, \beta\alpha, \gamma\beta, \gamma\beta\alpha\}$ is a $\mathbb{Z}$-basis for the path algebra $\mathbb{Z}\Gamma$. Let there also be given the relations $\rho = \{\beta\alpha, \gamma\beta\}$. Then $\{\overline{e}_1, \overline{e}_2, \overline{e}_3, \overline{e}_4, \overline{\alpha}, \overline{\beta}, \overline{\gamma}\}$ is a $\mathbb{Z}$-basis for $\mathbb{Z}(\Gamma, \rho)$, while $\{\beta\alpha, \gamma\beta, \gamma\beta\alpha\}$ is a $\mathbb{Z}$-basis for the ideal $< \rho >$.

Let $\Lambda = \mathbb{Z}(\Gamma, \rho)$. The four projective Λ-modules $\Lambda \overline{e}_i$, for $i = 1, \ldots, 4$, have as $\mathbb{Z}$-basis $\{\overline{e}_1, \overline{\alpha}\}$, $\{\overline{e}_2, \overline{\beta}\}$, $\{\overline{e}_3, \overline{\gamma}\}$, and $\{\overline{e}_4\}$, resp. The corresponding four representations are:

$$\mathbb{Z} = \mathbb{Z} \longrightarrow 0 \longrightarrow 0, \qquad 0 \longrightarrow \mathbb{Z} = \mathbb{Z} \longrightarrow 0,$$

$$0 \longrightarrow 0 \longrightarrow \mathbb{Z} = \mathbb{Z}, \qquad 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z}, \qquad \text{resp.}$$

The four injective Λ-modules $\mathrm{Hom}_{\mathbb{Z}}(\overline{e}_i \Lambda, \mathbb{Q}/\mathbb{Z})$, for $i = 1, \ldots, 4$, correspond to the four representations:

$$\mathbb{Q}/\mathbb{Z} \longrightarrow 0 \longrightarrow 0 \longrightarrow 0, \qquad \mathbb{Q}/\mathbb{Z} = \mathbb{Q}/\mathbb{Z} \longrightarrow 0 \longrightarrow 0,$$

$$0 \longrightarrow \mathbb{Q}/\mathbb{Z} = \mathbb{Q}/\mathbb{Z} \longrightarrow 0, \qquad 0 \longrightarrow 0 \longrightarrow \mathbb{Q}/\mathbb{Z} = \mathbb{Q}/\mathbb{Z}, \qquad \text{resp.}$$

Consider the subring R of the ring of all 4×4 matrices over $\mathbb{Z}$ consisting of lower triangular matrices. This ring is isomorphic to $\mathbb{Z}\Gamma$ (the matrix

$$\begin{pmatrix} e_1 & 0 & 0 & 0 \\ \alpha & e_2 & 0 & 0 \\ \beta\alpha & \beta & e_3 & 0 \\ \gamma\beta\alpha & \gamma\beta & \gamma & e_4 \end{pmatrix}$$

indicates how). The ring Λ is of course isomorphic to the quotient of this ring by the ideal consisting of matrices of the form $\left(\begin{smallmatrix} 0&0&0&0 \\ 0&0&0&0 \\ *&0&0&0 \\ *&*&0&0 \end{smallmatrix}\right)$.

Example 2.4.3. Let Γ be the quiver

$$v_1 \circlearrowright \alpha \ .$$

Then $\{e_1, \alpha, \alpha^2, \ldots\}$ is a $\mathbb{Z}$-basis for the path algebra $\mathbb{Z}\Gamma$. Clearly $\mathbb{Z}\Gamma$ is isomorphic to the polynomial ring $\mathbb{Z}[X]$ in one variable over $\mathbb{Z}$. Let there be given the relation $\rho = \{\alpha\alpha\}$, and let $\Lambda = \mathbb{Z}(\Gamma, \rho)$.

Then $\{\overline{e}_1, \overline{\alpha}\}$ is a $\mathbb{Z}$-basis for Λ, while $\{\alpha^2, \alpha^3, \alpha^4, \ldots\}$ is a $\mathbb{Z}$-basis for $< \rho >$. So Λ is isomorphic to $\mathbb{Z}^2$ equipped with the multiplication $(x_1, y_1)(x_2, y_2) = (x_1x_2, x_1y_2 + x_2y_1)$. Also $\Lambda\overline{e}_1 = \Lambda$, and the corresponding representation is $\begin{tikzcd} \mathbb{Z}^2 \arrow[loop right, "f"] \end{tikzcd}$, where $f(x, y) = (0, x)$. We see that the injective Λ-module $\mathrm{Hom}_{\mathbb{Z}}(\overline{e}_1\Lambda, \mathbb{Q}/\mathbb{Z})$ corresponds to the representation $\begin{tikzcd} (\mathbb{Q}/\mathbb{Z})^2 \arrow[loop right, "f"] \end{tikzcd}$, where $f(x, y) = (y, 0)$.

Example 2.4.4. Let Γ be the quiver

$$\begin{tikzcd} & v_2 \arrow[dr, "\beta"] & \\ v_1 \arrow[ur, "\alpha"] & & v_3 . \arrow[ll, "\gamma"] \end{tikzcd}$$

Let $p_1 = \gamma\beta\alpha$, $p_2 = \alpha\gamma\beta$ and $p_3 = \beta\alpha\gamma$. Then

$$\{p_1^i, p_2^i, p_3^i, \alpha p_1^i, \beta p_2^i, \gamma p_3^i, \beta\alpha p_1^i, \gamma\beta p_2^i, \alpha\gamma p_3^i \mid i \geq 0\}$$

is a $\mathbb{Z}$-basis for the path algebra $\mathbb{Z}\Gamma$, where $p_i^0 = e_i$. Let there be given the relations $\rho = \{\beta\alpha, \gamma\beta, \alpha\gamma\}$, and let $\Lambda = \mathbb{Z}(\Gamma, \rho)$. Then $\{\overline{e}_1, \overline{e}_2, \overline{e}_3, \overline{\alpha}, \overline{\beta}, \overline{\gamma}\}$ is a $\mathbb{Z}$-basis for Λ.

Also $\{\overline{e}_1, \overline{\alpha}\}$, $\{\overline{e}_2, \overline{\beta}\}$, and $\{\overline{e}_3, \overline{\gamma}\}$ are $\mathbb{Z}$-bases for $\Lambda\overline{e}_1$, $\Lambda\overline{e}_2$, and $\Lambda\overline{e}_3$, resp. — and they correspond to the representations

$$\begin{tikzcd} & \mathbb{Z} \arrow[dr] & \\ \mathbb{Z} \arrow[ur, equal] & & 0, \arrow[ll] \end{tikzcd} \quad \begin{tikzcd} & \mathbb{Z} \arrow[dr, equal] & \\ 0 \arrow[ur] & & \mathbb{Z}, \arrow[ll] \end{tikzcd} \quad \begin{tikzcd} & 0 \arrow[dr] & \\ \mathbb{Z} \arrow[ur] & & \mathbb{Z} \arrow[ll, equal] \end{tikzcd} \quad \text{resp.}$$

The injective Λ-modules $\mathrm{Hom}_{\mathbb{Z}}(\overline{e}_1\Lambda, \mathbb{Q}/\mathbb{Z})$, $\mathrm{Hom}_{\mathbb{Z}}(\overline{e}_2\Lambda, \mathbb{Q}/\mathbb{Z})$, and $\mathrm{Hom}_{\mathbb{Z}}(\overline{e}_3\Lambda, \mathbb{Q}/\mathbb{Z})$ correspond to the representations

$$\begin{tikzcd} & 0 \arrow[dr] & \\ \mathbb{Q}/\mathbb{Z} \arrow[ur] & & \mathbb{Q}/\mathbb{Z}, \arrow[ll, equal] \end{tikzcd} \quad \begin{tikzcd} & \mathbb{Q}/\mathbb{Z} \arrow[dr] & \\ \mathbb{Q}/\mathbb{Z} \arrow[ur, equal] & & 0, \arrow[ll] \end{tikzcd} \quad \begin{tikzcd} & \mathbb{Q}/\mathbb{Z} \arrow[dr, equal] & \\ 0 \arrow[ur] & & \mathbb{Q}/\mathbb{Z} \arrow[ll] \end{tikzcd} \quad \text{resp.}$$

Example 2.4.5. Let Γ be the quiver

$$\begin{tikzcd} v_1 \arrow[r, "\alpha"] \arrow[d, "\beta"'] & v_2 \arrow[d, "\delta"] \\ v_3 \arrow[r, "\gamma"'] & v_4 . \end{tikzcd}$$

Then $\{e_1, e_2, e_3, e_4, \alpha, \beta, \delta, \gamma, \delta\alpha, \gamma\beta\}$ is a $\mathbb{Z}$-basis for the path algebra $\mathbb{Z}\Gamma$. Let there be given the relation $\rho = \{\delta\alpha - \gamma\beta\}$, and let $\Lambda = \mathbb{Z}(\Gamma, \rho)$. Then $\{\overline{e}_1, \overline{e}_2, \overline{e}_3, \overline{e}_4, \overline{\alpha}, \overline{\beta}, \overline{\delta}, \overline{\gamma}, \overline{\delta\alpha}\}$ is a $\mathbb{Z}$-basis for Λ.

Moreover $\{\overline{e}_1, \overline{\alpha}, \overline{\beta}, \overline{\delta\alpha}\}$, $\{\overline{e}_2, \overline{\delta}\}$, $\{\overline{e}_3, \overline{\gamma}\}$, and $\{\overline{e}_4\}$ are $\mathbb{Z}$-bases for $\Lambda\overline{e}_1$, $\Lambda\overline{e}_2$, $\Lambda\overline{e}_3$, and $\Lambda\overline{e}_4$, resp. — and they correspond to the representations

$$\begin{tikzcd} \mathbb{Z} \arrow[r, equal] \arrow[d, equal] & \mathbb{Z} \arrow[d, equal] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd} \quad \begin{tikzcd} 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d, equal] \\ 0 \arrow[r] & \mathbb{Z} \end{tikzcd} \quad \begin{tikzcd} 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd} \quad \begin{tikzcd} 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \mathbb{Z} \end{tikzcd} \quad \text{resp.}$$

The injective Λ-modules $\mathrm{Hom}_{\mathbb{Z}}(\overline{e}_1\Lambda, \mathbb{Q}/\mathbb{Z})$, for $i = 1, 2, 3, 4$, correspond to the representations

$$\begin{tikzcd} \mathbb{Q}/\mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & 0 \end{tikzcd} \quad \begin{tikzcd} \mathbb{Q}/\mathbb{Z} \arrow[r, equal] \arrow[d] & \mathbb{Q}/\mathbb{Z} \arrow[d] \\ 0 \arrow[r] & 0 \end{tikzcd} \quad \begin{tikzcd} \mathbb{Q}/\mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \arrow[d] \\ \mathbb{Q}/\mathbb{Z} \arrow[r] & 0 \end{tikzcd} \quad \begin{tikzcd} \mathbb{Q}/\mathbb{Z} \arrow[r, equal] \arrow[d, equal] & \mathbb{Q}/\mathbb{Z} \arrow[d, equal] \\ \mathbb{Q}/\mathbb{Z} \arrow[r, equal] & \mathbb{Q}/\mathbb{Z} \end{tikzcd} \quad \text{resp.}$$

Example 2.4.6. Let Γ be the quiver

$$v_1 \xrightarrow{\alpha} v_2 .$$

Then $\{e_1, e_2, \alpha\}$ is a $\mathbb{Z}$-basis for the path algebra $\mathbb{Z}\Gamma$. This ring is isomorphic to the 2×2 lower triangular matrices over $\mathbb{Z}$. Let there be given the relation $\rho = \{2\alpha\}$. Then $\{\overline{e}_1, \overline{e}_2, \overline{\alpha}\}$ generates $\mathbb{Z}(\Gamma, \rho)$ as a $\mathbb{Z}$-module, while $\{2\alpha\}$ is a $\mathbb{Z}$-basis for the ideal $< \rho >$.

Let $\Lambda = \mathbb{Z}(\Gamma, \rho)$. The two projective Λ-modules $\Lambda_1 = \Lambda\overline{e}_1$ and $\Lambda_2 = \Lambda\overline{e}_2$, have as generating sets $\{\overline{e}_1, \overline{\alpha}\}$ and $\{\overline{e}_2\}$, resp. The corresponding two representations are:

$$\mathbb{Z} \xrightarrow{x \mapsto [x]} \mathbb{Z}/2, \qquad 0 \longrightarrow \mathbb{Z}, \qquad \text{resp.}$$

The two injective Λ-modules, $\mathrm{Hom}_{\mathbb{Z}}(\overline{e}_1\Lambda, \mathbb{Q}/\mathbb{Z})$ and $\mathrm{Hom}_{\mathbb{Z}}(\overline{e}_2\Lambda, \mathbb{Q}/\mathbb{Z})$, correspond to the two representations:

$$\mathbb{Q}/\mathbb{Z} \longrightarrow 0, \qquad \mathbb{Z}/2\mathbb{Z} \xrightarrow{[x] \mapsto \left[\frac{1}{2}x\right]} \mathbb{Q}/\mathbb{Z}, \qquad \text{resp.}$$

This example shows that it is certainly possible to get torsion into the path algebra. But in the cases we will be considering there will be no torsion.

It is natural to ask whether every projective (or injective) object is a direct sum (or direct product) of such basic projective (or injective) objects as above (in the case that every oriented cycle belongs to $< \rho >$, say). We do believe that this is true, but we do not know how to prove this in general. We have asked some specialists in this area, but they could not answer this question. One of the problems is that the Krull-Schmidt Theorem does not apply for these rings. In later sections we will study the projective and the injective objects in more detail. In order to do so, we will need the constructions from Definitions 2.2.3 and 2.2.4, and some more facts about them.

Remark 2.4.7. Let (Γ, ρ) be a finite quiver with relations, and denote $\Lambda = \mathbb{Z}(\Gamma, \rho)$. Let G be an abelian group, and let $i \in \Gamma_0$. Let, moreover, V be a Λ-module, and let $\phi\colon G \to \overline{e}_i V$ be a $\mathbb{Z}$-module homomorphism. Then it follows from Proposition 2.2.6 (and its proof) that there exists exactly one Λ-module homomorphism $\phi_{G,i}\colon \Lambda\overline{e}_i \otimes_{\mathbb{Z}} G \to V$ such that

$$\phi_{G,i}(\overline{e}_i \otimes g) = \phi(g), \quad \text{for all } g \in G.$$

It is immediate from the proof of Proposition 2.2.6, that $\mathsf{G}(\phi_{G,i})_i$ is surjective whenever ϕ is surjective, and that ϕ is injective whenever $\mathsf{G}(\phi_{G,i})_i$ is injective. But $\mathsf{G}(\phi_{G,i})_i$ need not be injective even if ϕ is injective, nor need ϕ be surjective even if $\mathsf{G}(\phi_{G,i})$ is surjective.

Remark 2.4.8. Let (Γ, ρ) be a finite quiver with relations. Let G be an abelian group, let $\mathbf{M}_\bullet$ be a representation, let $i_0 \in \Gamma_0$, and let $\phi\colon G \to \mathsf{M}_{i_0}$ be a $\mathbb{Z}$-module homomorphism.

Using the preceding remark, we will consider the morphism $\boldsymbol{\phi}_\bullet = \mathsf{G}(\phi_{G,i_0})\colon \mathbf{Free}_\bullet^{G,i_0} \to \mathbf{M}_\bullet$ which is induced by ϕ. We also see that if $\boldsymbol{\phi}_\bullet, \boldsymbol{\psi}_\bullet\colon \mathbf{Free}_\bullet^{G,i_0} \to \mathbf{M}_\bullet$ are induced by $\phi, \psi\colon G \to \mathsf{M}_{i_0}$, resp., then $\boldsymbol{\phi}_\bullet = \boldsymbol{\psi}_\bullet$ if and only if $\phi = \psi$.

Now assume that every cycle belongs to $< \rho >$, so in particular, $\overline{e}_{i_0}\mathbb{Z}(\Gamma, \rho)\overline{e}_{i_0}$ is canonically isomorphic to $\mathbb{Z}$. According to the previous remark, there exists a unique morphism $\boldsymbol{\phi}_\bullet = \mathsf{G}(\phi_{G,i_0})$ from $\mathbf{Free}_\bullet^{G,i_0}$ to $\mathbf{M}_\bullet$ such that $\phi_{i_0} = \phi$ (where we, in the canonical way, identify $\overline{e}_{i_0}(\mathbb{Z}(\Gamma, \rho)\overline{e}_{i_0} \otimes_{\mathbb{Z}} G)$ with G). So in this case, injectivity (resp. surjectivity) of ϕ is equivalent to injectivity (resp. surjectivity) of $\phi_{i_0} = \mathsf{G}(\phi_{G,i_0})_{i_0}$.

Remark 2.4.9. Let (Γ, ρ) be a finite quiver with relations, and denote $\Lambda = \mathbb{Z}(\Gamma, \rho)$. Let G be an abelian group, and let $i \in \Gamma_0$. Let, moreover, V be a Λ-module, and let $\phi\colon \overline{e}_i V \to G$ be a $\mathbb{Z}$-module homomorphism. Then it follows from Proposition 2.2.5 (and its proof) that there exists exactly one Λ-module homomorphism $\phi_{G,i}\colon V \to \mathrm{Hom}_{\mathbb{Z}}(\overline{e}_i\Lambda, G)$ such that

$$(\phi_{G,i}(x))(\overline{e}_i) = \phi(\overline{e}_i x), \quad \text{for all } x \in V.$$

Surely, the existence is clear from this proposition (and its proof), but also the uniqueness is clear, since for $\phi_{G,i}$ satisfying this we have for all $x \in V$ and $\lambda \in \Lambda$ that

$$(\phi_{G,i}(x))(\overline{e}_i\lambda) = (\lambda(\phi_{G,i}(x)))(\overline{e}_i) = (\phi_{G,i}(\lambda x))(\overline{e}_i) = \phi(\overline{e}_i\lambda x).$$

It is immediate from the proof of Proposition 2.2.5, that $\mathsf{G}(\phi_{G,i})_i$ is injective whenever ϕ is injective, and that ϕ is surjective whenever $\mathsf{G}(\phi_{G,i})_i$ is surjective. But $\mathsf{G}(\phi_{G,i})_i$ need not be surjective even if ϕ is surjective, nor need ϕ be injective even if $\mathsf{G}(\phi_{G,i})_i$ is injective.

Remark 2.4.10. Let (Γ, ρ) be a finite quiver with relations. Let G be an abelian group, let $\mathbf{M}_\bullet$ be a representation, let $i_0 \in \Gamma_0$, and let $\phi\colon \mathsf{M}_{i_0} \to G$ be a $\mathbb{Z}$-module homomorphism.

Using the preceding remark, we will consider the morphism $\boldsymbol{\phi}_\bullet = \mathsf{G}(\phi_{G,i_0})\colon \mathbf{M}_\bullet \to \mathbf{Cofree}_\bullet^{G,i_0}$ which is induced by ϕ. We also see that if $\boldsymbol{\phi}_\bullet, \boldsymbol{\psi}_\bullet\colon \mathbf{M}_\bullet \to \mathbf{Cofree}_\bullet^{G,i_0}$ are induced by $\phi, \psi\colon \mathsf{M}_{i_0} \to G$, resp., then $\boldsymbol{\phi}_\bullet = \boldsymbol{\psi}_\bullet$ if and only if $\phi = \psi$.

Now assume that every cycle belongs to $< \rho >$, so in particular $\overline{e}_{i_0}\mathbb{Z}(\Gamma, \rho)\overline{e}_{i_0}$ is canonically isomorphic to $\mathbb{Z}$. According to the previous remark, there exists a unique morphism $\boldsymbol{\phi}_\bullet = \mathsf{G}(\phi_{G,i_0})$ from $\mathbf{M}_\bullet$ to $\mathbf{Cofree}_\bullet^{G,i_0}$ such that $\phi_{i_0} = \phi$ (where we, in the canonical way, identify $\overline{e}_{i_0}\operatorname{Hom}_{\mathbb{Z}}(\overline{e}_{i_0}\mathbb{Z}(\Gamma, \rho), G)$ with G). So in this case, injectivity (resp. surjectivity) of ϕ is equivalent to injectivity (resp. surjectivity) of $\phi_{i_0} = \mathsf{G}(\phi_{G,i_0})_{i_0}$.

Chapter 3

Mapping cones

In this chapter we examine the interplay between mapping cone sequences and homology and cohomology theories on C^*-algebras (like K-functors and KK-functors).

3.1 Preliminaries: Suspensions, cones, and mapping cones

In this section, we define basic concepts like cones, suspensions, and mapping cones. We prove some fundamental results, which will be needed later. Also we prove two technical lemmata (Lemmata 3.1.13 and 3.1.14), which are needed for Chapter 6.

Definition 3.1.1. Let $\mathfrak{A}$ be a C^*-algebra. Then we define the **suspension** and the **cone**[1] of $\mathfrak{A}$ as

$$\begin{aligned} \mathsf{S}\mathfrak{A} &= \{\, f \in C([0,1],\mathfrak{A}) \mid f(0)=0, f(1)=0 \,\}\,, \\ \mathsf{C}\mathfrak{A} &= \{\, f \in C([0,1],\mathfrak{A}) \mid f(0)=0 \,\}\,, \quad \text{resp.} \end{aligned}$$

Remark 3.1.2. For each C^*-algebras $\mathfrak{A}$, we have a canonical short exact sequence:

$$\mathsf{S}\mathfrak{A} \hookrightarrow \mathsf{C}\mathfrak{A} \twoheadrightarrow \mathfrak{A}.$$

It is well-known, that S and C are exact functors.

Notation 3.1.3. Whenever convenient we will identify $\mathsf{CC}\mathfrak{A}$, $\mathsf{SC}\mathfrak{A}$, $\mathsf{CS}\mathfrak{A}$, and $\mathsf{SS}\mathfrak{A}$ with subalgebras of $C([0,1]^2,\mathfrak{A})$ by writing $f(x,y)$ for $(f(x))(y)$. In this way $\mathrm{ev}_1(f)$ will become $f(1,-)$ while $(\mathsf{S}\,\mathrm{ev}_1)(f)$ or $(\mathsf{C}\,\mathrm{ev}_1)(f)$ will be $f(-,1)$.

We let flip denote the operation on $C([0,1]^2,\mathfrak{A})$ that *flips* a function on $[0,1]^2$ along the diagonal, *i.e.*, $\mathrm{flip}(f)(x,y) = f(y,x)$.

Definition 3.1.4. Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras, and let $\phi\colon \mathfrak{A} \to \mathfrak{B}$ be a $*$-homomorphism. The **mapping cone of** ϕ, C_ϕ, is the pullback of the maps $\mathfrak{A} \xrightarrow{\phi} \mathfrak{B}$ and $\mathsf{C}\mathfrak{B} \xrightarrow{\mathrm{ev}_1} \mathfrak{B}$.

As usual, we may realize the pullback as the restricted direct sum:

$$\mathsf{C}_\phi = \mathfrak{A} \oplus_{\phi,\mathrm{ev}_1} \mathsf{C}\mathfrak{B} = \{\, (x,y) \in \mathfrak{A} \oplus \mathsf{C}\mathfrak{B} \mid \phi(x) = \mathrm{ev}_1(y) = y(1) \,\}\,.$$

Remark 3.1.5. Let $\phi\colon \mathfrak{A} \to \mathfrak{B}$ be a $*$-homomorphism between C^*-algebras. Then there is a canonical short exact sequence

$$\mathsf{S}\mathfrak{B} \hookrightarrow \mathsf{C}_\phi \twoheadrightarrow \mathfrak{A}$$

called the **mapping cone sequence**. This sequence is natural in $\mathfrak{A}$ and $\mathfrak{B}$, *i.e.*, if we have a commuting diagram

$$\begin{tikzcd} \mathfrak{A}_1 \arrow[r, "\phi_1"] \arrow[d, "f"'] & \mathfrak{B}_1 \arrow[d, "g"] \\ \mathfrak{A}_2 \arrow[r, "\phi_2"] & \mathfrak{B}_2 \end{tikzcd}$$

[1]Note that some authors place the algebra at 0 rather than 1 — *e.g.* Blackadar in [Bla98]

then there is a (canonical) $*$-homomorphism $\omega\colon \mathsf{C}_{\phi_1} \to \mathsf{C}_{\phi_2}$ making the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathsf{S}\mathfrak{B}_1 \arrow[r] \arrow[d, "\mathsf{S}g"'] & \mathsf{C}_{\phi_1} \arrow[r] \arrow[d, "\omega"] & \mathfrak{A}_1 \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & \mathsf{S}\mathfrak{B}_2 \arrow[r] & \mathsf{C}_{\phi_2} \arrow[r] & \mathfrak{A}_2 \arrow[r] & 0
\end{tikzcd}$$

commutative (*cf.* [Bla98, Section 19.4]). Actually, we have a concrete description of ω as follows: $\omega(a,h) = (f(a), g \circ h)$ for all $(a,h) \in \mathfrak{A}_1 \oplus_{\phi_1,\mathrm{ev}_1} \mathsf{C}\mathfrak{B}_1 = \mathsf{C}_{\phi_1}$.

Remark 3.1.6. The mapping cone sequence of the identity homomorphism $\mathrm{id}_{\mathfrak{A}}$ is the canonical sequence $\mathsf{S}\mathfrak{A} \hookrightarrow \mathsf{C}\mathfrak{A} \twoheadrightarrow \mathfrak{A}$. For each $*$-homomorphism $\phi\colon \mathfrak{A} \to \mathfrak{B}$ between C^*-algebras, we have canonical $*$-isomorphisms S-flip from $\mathsf{S}\mathsf{C}_\phi$ to $\mathsf{C}_{\mathsf{S}\phi}$, and C-flip from $\mathsf{C}\mathsf{C}_\phi$ to $\mathsf{C}_{\mathsf{C}\phi}$, given by

$$\mathsf{S}\mathsf{C}_\phi = \mathsf{S}(\mathfrak{A} \oplus_{\phi,\mathrm{ev}_1} \mathsf{C}\mathfrak{B}) \ni (x,y) \mapsto (x, \mathrm{flip}(y)) \in \mathsf{S}\mathfrak{A} \oplus_{\mathsf{S}\phi,\mathrm{ev}_1} \mathsf{C}\mathsf{S}\mathfrak{B} = \mathsf{C}_{\mathsf{S}\phi},$$
$$\mathsf{C}\mathsf{C}_\phi = \mathsf{C}(\mathfrak{A} \oplus_{\phi,\mathrm{ev}_1} \mathsf{C}\mathfrak{B}) \ni (x,y) \mapsto (x, \mathrm{flip}(y)) \in \mathsf{C}\mathfrak{A} \oplus_{\mathsf{C}\phi,\mathrm{ev}_1} \mathsf{C}\mathsf{C}\mathfrak{B} = \mathsf{C}_{\mathsf{C}\phi},$$

resp. See Definition 3.1.10 and Lemma 3.1.11 for more on these isomorphisms.

Definition 3.1.7. We define functors $\mathfrak{mc}$, S and C on the category of all extensions of C^*-algebras (with the morphisms being triples of $*$-homomorphisms making the obvious diagram commutative) as follows. For an extension $e\colon \mathfrak{A}_0 \overset{\iota}{\hookrightarrow} \mathfrak{A}_1 \overset{\pi}{\twoheadrightarrow} \mathfrak{A}_2$ we set

$$\mathfrak{mc}(e)\colon \mathsf{S}\mathfrak{A}_2 \overset{\iota_{\mathfrak{mc}}}{\hookrightarrow} \mathsf{C}_\pi \overset{\pi_{\mathfrak{mc}}}{\twoheadrightarrow} \mathfrak{A}_1,$$
$$\mathsf{S}(e) = \mathsf{S}e\colon \mathsf{S}\mathfrak{A}_0 \overset{\mathsf{S}\iota}{\hookrightarrow} \mathsf{S}\mathfrak{A}_1 \overset{\mathsf{S}\pi}{\twoheadrightarrow} \mathsf{S}\mathfrak{A}_2,$$
$$\mathsf{C}(e) = \mathsf{C}e\colon \mathsf{C}\mathfrak{A}_0 \overset{\mathsf{C}\iota}{\hookrightarrow} \mathsf{C}\mathfrak{A}_1 \overset{\mathsf{C}\pi}{\twoheadrightarrow} \mathsf{C}\mathfrak{A}_2.$$

For a morphism $\phi = (\phi_0, \phi_1, \phi_2)$ from e to e', we let $\mathfrak{mc}(\phi)$ be the morphism $(\mathsf{S}\phi_2, \omega, \phi_1)$ defined using the naturality of the mapping cone construction (see above), we let $\mathsf{S}(\phi) = \mathsf{S}\phi$ be the morphism $(\mathsf{S}\phi_0, \mathsf{S}\phi_1, \mathsf{S}\phi_2)$, and we let $\mathsf{C}(\phi) = \mathsf{C}\phi$ be the morphism $(\mathsf{C}\phi_0, \mathsf{C}\phi_1, \mathsf{C}\phi_2)$.

It is easy to verify that these are functors. Moreover, one easily verifies, that they are exact (*i.e.*, they send short exact sequences of extensions to short exact sequences of extensions).

Definition 3.1.8. Let there be given an extension $e\colon \mathfrak{A}_0 \overset{\iota}{\hookrightarrow} \mathfrak{A}_1 \overset{\pi}{\twoheadrightarrow} \mathfrak{A}_2$ of C^*-algebras. Then we construct two new extensions, $\mathfrak{i}(e)$ and $\mathfrak{q}(e)$, from e as follows. Let $\mathfrak{i}(e)$ denote the extension $\mathfrak{A}_0 = \mathfrak{A}_0 \twoheadrightarrow 0$, and let $\mathfrak{q}(e)$ denote the extension $0 \hookrightarrow \mathfrak{A}_2 = \mathfrak{A}_2$. Then we have a canonical short exact sequence of extensions: $\mathfrak{i}(e) \overset{\mathfrak{i}_e}{\hookrightarrow} e \overset{\mathfrak{q}_e}{\twoheadrightarrow} \mathfrak{q}(e)$.

Remark 3.1.9. If we have an extension

$$e\colon \mathfrak{A}_0 \overset{\iota}{\hookrightarrow} \mathfrak{A}_1 \overset{\pi}{\twoheadrightarrow} \mathfrak{A}_2$$

of C^*-algebras, then we get a commuting diagram

$$\begin{tikzcd}
0 \arrow[r, hook] \arrow[d, hook] & \mathsf{S}\mathfrak{A}_2 \arrow[r, equal] \arrow[d, hook, "\iota_{\mathfrak{mc}}"] & \mathsf{S}\mathfrak{A}_2 \arrow[d, hook] \\
\mathfrak{A}_0 \arrow[r, hook, "f_e"] \arrow[d, equal] & \mathsf{C}_\pi \arrow[r, two heads] \arrow[d, two heads, "\pi_{\mathfrak{mc}}"] & \mathsf{C}\mathfrak{A}_2 \arrow[d, two heads, "\mathrm{ev}_1"] \\
\mathfrak{A}_0 \arrow[r, hook, "\iota"] & \mathfrak{A}_1 \arrow[r, two heads, "\pi"] & \mathfrak{A}_2
\end{tikzcd}$$

with short exact rows and columns. The map $f_e\colon \mathfrak{A}_0 \to \mathsf{C}_\pi$ induces isomorphism on the level of K-theory (actually, this hold more generally for additive, homotopy-invariant, half-exact functors, *cf.* [Bla98, Proposition 21.4.1]).

Actually, this diagram is nothing but the short exact sequence $\mathfrak{mc}(\mathfrak{i}(e)) \overset{\mathfrak{mc}(\mathfrak{i}_e)}{\hookrightarrow} \mathfrak{mc}(e) \overset{\mathfrak{mc}(\mathfrak{q}_e)}{\twoheadrightarrow} \mathfrak{mc}(\mathfrak{q}(e))$ induced by applying the functor $\mathfrak{mc}$ to the short exact sequence $\mathfrak{i}(e) \overset{\mathfrak{i}_e}{\hookrightarrow} e \overset{\mathfrak{q}_e}{\twoheadrightarrow} \mathfrak{q}(e)$.

Definition 3.1.10. Let there be given an extension $e\colon \mathfrak{A}_0 \xhookrightarrow{\iota} \mathfrak{A}_1 \xrightarrow{\pi} \mathfrak{A}_2$ of C^*-algebras. Form the extensions $\mathsf{S}(\mathfrak{mc}(e))$, $\mathfrak{mc}(\mathsf{S}(e))$, $\mathsf{C}(\mathfrak{mc}(e))$, and $\mathfrak{mc}(\mathsf{C}(e))$ as above. Then we define morphisms θ_e from $\mathsf{S}(\mathfrak{mc}(e))$ to $\mathfrak{mc}(\mathsf{S}e)$ and η_e from $\mathsf{C}(\mathfrak{mc}(e))$ to $\mathfrak{mc}(\mathsf{C}e)$ as follows:

$$\begin{tikzcd}
\mathsf{S}(\mathfrak{mc}(e))\colon \arrow[d, "\cong"', "\theta_e"] & 0 \arrow[r] & \mathsf{S}\mathsf{S}\mathfrak{A}_2 \arrow[r, "\mathsf{S}(\iota_{\mathfrak{mc}})"] \arrow[d, "\cong"', "\text{flip}"] & \mathsf{S}\mathsf{C}_\pi \arrow[r, "\mathsf{S}(\iota_{\mathfrak{mc}})"] \arrow[d, "\cong"', "\mathsf{S}\text{-flip}"] & \mathsf{S}\mathfrak{A}_1 \arrow[r] \arrow[d, equal] & 0 \\
\mathfrak{mc}(\mathsf{S}e)\colon & 0 \arrow[r] & \mathsf{S}\mathsf{S}\mathfrak{A}_2 \arrow[r, "(\mathsf{S}\iota)_{\mathfrak{mc}}"] & \mathsf{C}_{\mathsf{S}\pi} \arrow[r, "(\mathsf{S}\pi)_{\mathfrak{mc}}"] & \mathsf{S}\mathfrak{A}_1 \arrow[r] & 0,
\end{tikzcd}$$

$$\begin{tikzcd}
\mathsf{C}(\mathfrak{mc}(e))\colon \arrow[d, "\cong"', "\eta_e"] & 0 \arrow[r] & \mathsf{C}\mathsf{S}\mathfrak{A}_2 \arrow[r, "\mathsf{C}(\iota_{\mathfrak{mc}})"] \arrow[d, "\cong"', "\text{flip}"] & \mathsf{C}\mathsf{C}_\pi \arrow[r, "\mathsf{C}(\iota_{\mathfrak{mc}})"] \arrow[d, "\cong"', "\mathsf{C}\text{-flip}"] & \mathsf{C}\mathfrak{A}_1 \arrow[r] \arrow[d, equal] & 0 \\
\mathfrak{mc}(\mathsf{C}e)\colon & 0 \arrow[r] & \mathsf{S}\mathsf{C}\mathfrak{A}_2 \arrow[r, "(\mathsf{C}\iota)_{\mathfrak{mc}}"] & \mathsf{C}_{\mathsf{C}\pi} \arrow[r, "(\mathsf{C}\pi)_{\mathfrak{mc}}"] & \mathsf{C}\mathfrak{A}_1 \arrow[r] & 0,
\end{tikzcd}$$

where the $*$-homomorphisms $\mathsf{S}\mathsf{C}_\pi \to \mathsf{C}_{\mathsf{S}\pi}$ and $\mathsf{C}\mathsf{C}_\pi \to \mathsf{C}_{\mathsf{C}\pi}$ are the canonical isomorphisms from Remark 3.1.6.

Lemma 3.1.11. *The above morphisms, θ_e and η_e, are functorial, i.e., they implement isomorphisms from the functor $\mathsf{S} \circ \mathfrak{mc}$ to the functor $\mathfrak{mc} \circ \mathsf{S}$ and from the functor $\mathsf{C} \circ \mathfrak{mc}$ to the functor $\mathfrak{mc} \circ \mathsf{C}$, respectively.*

Proof. This is a long, straightforward verification. ●●●

Lemma 3.1.12. *Let e be an extension of C^*-algebras. Then we have an isomorphism of short exact sequences of extensions as follows:*

$$\begin{tikzcd}
0 \arrow[r] & \mathsf{S}\mathfrak{mc}(e) \arrow[r] \arrow[d, "\theta_e"] & \mathsf{C}\mathfrak{mc}(e) \arrow[r] \arrow[d, "\eta_e"] & \mathfrak{mc}(e) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathfrak{mc}(\mathsf{S}e) \arrow[r] & \mathfrak{mc}(\mathsf{C}e) \arrow[r] & \mathfrak{mc}(e) \arrow[r] & 0.
\end{tikzcd}$$

Proof. This is a straightforward verification. ●●●

Lemma 3.1.13. *Let there be given a commutative diagram*

$$\begin{tikzcd}
\mathfrak{X} \arrow[r, "\phi_1"] \arrow[d, "\phi_2"'] & \mathfrak{Y}_1 \arrow[d, "\psi_1"] \\
\mathfrak{Y}_2 \arrow[r, "\psi_2"'] & \mathfrak{Z}
\end{tikzcd}$$

of C^*-algebras and $*$-homomorphisms. We get canonical induced $*$-homomorphisms $\mathsf{C}_{\phi_1} \to \mathsf{C}_{\psi_2}$ and $\mathsf{C}_{\phi_2} \to \mathsf{C}_{\psi_1}$. The mapping cones $\mathsf{C}_{\mathsf{C}_{\phi_1}\to\mathsf{C}_{\psi_2}}$ and $\mathsf{C}_{\mathsf{C}_{\phi_2}\to\mathsf{C}_{\psi_1}}$ are canonically isomorphic to

$$\left\{ (x, f_1, f_2, h) \in \mathfrak{X} \oplus \mathsf{C}\mathfrak{Y}_1 \oplus \mathsf{C}\mathfrak{Y}_2 \oplus \mathsf{C}\mathsf{C}\mathfrak{Z} \;\middle|\; \begin{matrix} \phi_1(x) = f_1(1), & \psi_1 \circ f_1(-) = h(1,-), \\ \phi_2(x) = f_2(1), & \psi_2 \circ f_2(-) = h(-,1) \end{matrix} \right\}$$

$$\left\{ (x, f_2, f_1, h) \in \mathfrak{X} \oplus \mathsf{C}\mathfrak{Y}_2 \oplus \mathsf{C}\mathfrak{Y}_1 \oplus \mathsf{C}\mathsf{C}\mathfrak{Z} \;\middle|\; \begin{matrix} \phi_1(x) = f_1(1), & \psi_1 \circ f_1(-) = h(-,1), \\ \phi_2(x) = f_2(1), & \psi_2 \circ f_2(-) = h(1,-) \end{matrix} \right\}$$

resp. So $(x, f_1, f_2, h) \mapsto (x, f_2, f_1, \text{flip}(h))$ is an isomorphism from $\mathsf{C}_{\mathsf{C}_{\phi_1}\to\mathsf{C}_{\psi_2}}$ to $\mathsf{C}_{\mathsf{C}_{\phi_2}\to\mathsf{C}_{\psi_1}}$.

Proof. This is straightforward to check by writing out the mapping cones as restricted direct sums. Note that we only need to check the first statement, since the second follows by symmetry (by interchanging 1 and 2). ●●●

For a morphism $\phi = (\phi_0, \phi_1, \phi_2)$ between extensions of C^*-algebras, we let C_ϕ denote the object $\mathsf{C}_{\phi_0} \hookrightarrow \mathsf{C}_{\phi_1} \twoheadrightarrow \mathsf{C}_{\phi_2}$ (*cf.* also [Bon02, Definition 3.4.1] and Section 5.2.1).

Lemma 3.1.14. *Let there be given a commuting diagram*

$$\begin{tikzcd}
\mathfrak{A}_0 \arrow[r, hook, "x_0"] \arrow[d, hook, "\iota_{\mathfrak{A}}"] & \mathfrak{B}_0 \arrow[r, two heads, "y_0"] \arrow[d, hook, "\iota_{\mathfrak{B}}"] & \mathfrak{C}_0 \arrow[d, hook, "\iota_{\mathfrak{C}}"] \\
\mathfrak{A}_1 \arrow[r, hook, "x_1"] \arrow[d, two heads, "\pi_{\mathfrak{A}}"] & \mathfrak{B}_1 \arrow[r, two heads, "y_1"] \arrow[d, two heads, "\pi_{\mathfrak{B}}"] & \mathfrak{C}_1 \arrow[d, two heads, "\pi_{\mathfrak{C}}"] \\
\mathfrak{A}_2 \arrow[r, hook, "x_2"] & \mathfrak{B}_2 \arrow[r, two heads, "y_2"] & \mathfrak{C}_2
\end{tikzcd}$$

with the rows and columns being short exact sequences of C^*-algebras — we will write this short as $e_{\mathfrak{A}} \xhookrightarrow{\mathbf{x}} e_{\mathfrak{B}} \xrightarrow{\mathbf{y}} e_{\mathfrak{C}}$. Then we have an isomorphism $\xi_{\mathbf{y}}$ from $\mathsf{C}_{\mathfrak{mc}(\mathbf{y})}$ to $\mathfrak{mc}(\mathsf{C}_{\mathbf{y}})$ given as follows:

$$\begin{tikzcd}
\mathsf{C}_{\mathfrak{mc}(\mathbf{y})}: \arrow[d, "\xi_{\mathbf{y}}", "\cong"'] & 0 \arrow[r] & \mathsf{C}_{\mathsf{S}y_2} \arrow[r] \arrow[d, "\text{S-flip}", "\cong"'] & \mathsf{C}_{\mathsf{C}_{\pi_{\mathfrak{B}}} \to \mathsf{C}_{\pi_{\mathfrak{C}}}} \arrow[r] \arrow[d, "\cong"'] & \mathsf{C}_{y_1} \arrow[r] \arrow[d, equal] & 0 \\
\mathfrak{mc}(\mathsf{C}_{\mathbf{y}}): & 0 \arrow[r] & \mathsf{S}\mathsf{C}_{y_2} \arrow[r] & \mathsf{C}_{\mathsf{C}_{y_1} \to \mathsf{C}_{y_2}} \arrow[r] & \mathsf{C}_{y_1} \arrow[r] & 0
\end{tikzcd}$$

where the isomorphism from $\mathsf{C}_{\mathsf{C}_{\pi_{\mathfrak{B}}} \to \mathsf{C}_{\pi_{\mathfrak{C}}}}$ to $\mathsf{C}_{\mathsf{C}_{y_1} \to \mathsf{C}_{y_2}}$ is given as in the above lemma. Moreover, the map given by the matrix

$$\begin{pmatrix} 0 & \theta_{e_{\mathfrak{C}}} & \theta_{e_{\mathfrak{C}}} \\ \mathrm{id} & \xi_{\mathbf{y}} & \eta_{e_{\mathfrak{C}}} \\ \mathrm{id} & \mathrm{id} & \mathrm{id} \end{pmatrix}$$

between the standard diagrams

$$\begin{tikzcd}
0 \arrow[r, hook] \arrow[d, hook] & \mathsf{S}\mathfrak{mc}(e_{\mathfrak{C}}) \arrow[r, equal] \arrow[d, hook] & \mathsf{S}\mathfrak{mc}(e_{\mathfrak{C}}) \arrow[d, hook] \\
\mathfrak{mc}(e_{\mathfrak{A}}) \arrow[r, hook] \arrow[d, equal] & \mathsf{C}_{\mathfrak{mc}(\mathbf{y})} \arrow[r, two heads] \arrow[d, two heads] & \mathsf{C}\mathfrak{mc}(e_{\mathfrak{C}}) \arrow[d, two heads] \\
\mathfrak{mc}(e_{\mathfrak{A}}) \arrow[r, hook] & \mathfrak{mc}(e_{\mathfrak{B}}) \arrow[r, two heads] & \mathfrak{mc}(e_{\mathfrak{C}})
\end{tikzcd}
\quad \textit{and} \quad
\begin{tikzcd}
0 \arrow[r, hook] \arrow[d, hook] & \mathfrak{mc}(\mathsf{S}e_{\mathfrak{C}}) \arrow[r, equal] \arrow[d, hook] & \mathfrak{mc}(\mathsf{S}e_{\mathfrak{C}}) \arrow[d, hook] \\
\mathfrak{mc}(e_{\mathfrak{A}}) \arrow[r, hook] \arrow[d, equal] & \mathfrak{mc}(\mathsf{C}_{\mathbf{y}}) \arrow[r, two heads] \arrow[d, two heads] & \mathfrak{mc}(\mathsf{C}e_{\mathfrak{C}}) \arrow[d, two heads] \\
\mathfrak{mc}(e_{\mathfrak{A}}) \arrow[r, hook] & \mathfrak{mc}(e_{\mathfrak{B}}) \arrow[r, two heads] & \mathfrak{mc}(e_{\mathfrak{C}})
\end{tikzcd}$$

makes everything commutative.

Proof. Using the above, we have that $\mathsf{C}_{\mathsf{C}_{\pi_{\mathfrak{B}}} \to \mathsf{C}_{\pi_{\mathfrak{C}}}}$ is isomorphic to

$$\left\{ (x, f_2, f_1, h) \in \mathfrak{B}_1 \oplus \mathsf{C}\mathfrak{B}_2 \oplus \mathsf{C}\mathfrak{C}_1 \oplus \mathsf{C}\mathsf{C}\mathfrak{C}_2 \,\middle|\, \begin{matrix} y_1(x) = f_1(1), & \pi_{\mathfrak{C}} \circ f_1(-) = h(-,1), \\ \pi_{\mathfrak{B}}(x) = f_2(1), & y_2 \circ f_2(-) = h(1,-) \end{matrix} \right\}$$

and $\mathsf{C}_{\mathsf{C}_{y_1} \to \mathsf{C}_{y_2}}$ is isomorphic to

$$\left\{ (x, f_1, f_2, h) \in \mathfrak{B}_1 \oplus \mathsf{C}\mathfrak{C}_1 \oplus \mathsf{C}\mathfrak{B}_2 \oplus \mathsf{C}\mathsf{C}\mathfrak{C}_2 \,\middle|\, \begin{matrix} y_1(x) = f_1(1), & \pi_{\mathfrak{C}} \circ f_1(-) = h(1,-), \\ \pi_{\mathfrak{B}}(x) = f_2(1), & y_2 \circ f_2(-) = h(-,1) \end{matrix} \right\}$$

and, moreover,

$$\begin{aligned}
\mathsf{C}_{\mathsf{S}y_2} &= \{ (f_2, h) \in \mathsf{S}\mathfrak{B}_2 \oplus \mathsf{C}\mathsf{S}\mathfrak{C}_2 \mid y_2 \circ f_2(-) = h(1,-) \}, \\
\mathsf{S}\mathsf{C}_{y_2} &= \{ (f_2, h) \in \mathsf{S}\mathfrak{B}_2 \oplus \mathsf{S}\mathsf{C}\mathfrak{C}_2 \mid y_2 \circ f_2(-) = h(-,1) \}, \\
\mathsf{C}_{y_1} &= \{ (x, f_1) \in \mathfrak{B}_1 \oplus \mathsf{C}\mathfrak{C}_1 \mid y_1(x) = f_1(1) \}.
\end{aligned}$$

Using these identifications, we compute the extensions:

$$\mathsf{C}_{\mathfrak{mc}(\mathbf{y})}\colon \qquad 0 \longrightarrow \mathsf{C}_{\mathsf{S}y_2} \xrightarrow{(f_2,h)\mapsto(0,f_2,0,h)} \mathsf{C}_{\mathsf{C}_{\pi_{\mathfrak{B}}}\to\mathsf{C}_{\pi_{\mathfrak{C}}}} \xrightarrow{(x,f_2,f_1,h)\mapsto(x,f_1)} \mathsf{C}_{y_1} \longrightarrow 0,$$

$$\mathfrak{mc}(\mathsf{C}_{\mathbf{y}})\colon \qquad 0 \longrightarrow \mathsf{S}\mathsf{C}_{y_2} \xrightarrow{(f_2,h)\mapsto(0,0,f_2,h)} \mathsf{C}_{\mathsf{C}_{y_1}\to\mathsf{C}_{y_2}} \xrightarrow{(x,f_1,f_2,h)\mapsto(x,f_1)} \mathsf{C}_{y_1} \longrightarrow 0.$$

Now it is routine to check that the given diagram commutes.

Second part: The above results show that every square which not involves $\mathsf{C}_{\mathfrak{mc}(\mathbf{y})}$ and $\mathfrak{mc}(\mathsf{C}_{\mathbf{y}})$ commutes. The long and straightforward proof of the commutativity of the remaining four squares of morphisms of extension is left to the reader. ●●●

3.2 Homology and cohomology theories for C^*-algebras

This section is essentially contained in [Bla98, Chapters 21 and 22] (these two chapters are primarily due to Cuntz, Higson, Rosenberg, and Schochet — see the monography for further references). These definitions and results will be very important to us in the sequel. Because of this and because it is not completely standard how to define the connecting homomorphisms, we have chosen to include this.

Definition 3.2.1. Let $\mathbf{S}$ be a subcategory of the category of all C^*-algebras, which is closed under quotients, extensions, and closed under suspension in the sense that if $\mathfrak{A}$ is an object of $\mathbf{S}$ then the suspension $\mathsf{S}\mathfrak{A}$ of $\mathfrak{A}$ is also an object of $\mathbf{S}$, $\mathsf{S}\phi$ is a morphism in $\mathbf{S}$ whenever ϕ is, $\mathsf{S}\mathbb{C}$ is an object of $\mathbf{S}$ and every $*$-homomorphism from $\mathsf{S}\mathbb{C}$ to every object of $\mathbf{S}$ is a morphism in $\mathbf{S}$.

Let $\mathbf{Ab}$ denote the category of abelian groups. We will consider functors F from $\mathbf{S}$ to $\mathbf{Ab}$. Such functors may or may not satisfy each of the following axioms:

(H) **Homotopy-invariance**. If $\phi,\psi\colon\mathfrak{A}\to\mathfrak{B}$ are homotopic, then $\mathsf{F}(\phi)=\mathsf{F}(\psi)$.

(S) **Stability**. The canonical embedding $\kappa\colon\mathfrak{A}\to\mathfrak{A}\otimes\mathbb{K}$ induces an isomorphism $\mathsf{F}(\kappa)$, whenever $\mathfrak{A}$ is in $\mathbf{S}$ (here we assume, moreover, that $\mathbf{S}$ is closed under tensoring by $\mathbb{K}$).

(A) **σ-additivity**. The subcategory $\mathbf{S}$ is closed under finite direct sums and (countable) inductive limits (and so it is closed also under (countable) direct sums), and the canonical maps

- $\mathsf{F}(\mathfrak{A}_i)\to\mathsf{F}(\bigoplus_i\mathfrak{A}_i)$ induce an isomorphism $\bigoplus_i\mathsf{F}(\mathfrak{A}_i)\to\mathsf{F}(\bigoplus_i\mathfrak{A}_i)$ for every countable family $(\mathfrak{A}_i)$ of C^*-algebras, if F is covariant, and
- $\mathsf{F}(\bigoplus_i\mathfrak{A}_i)\to\mathsf{F}(\mathfrak{A}_i)$ induce an isomorphism $\mathsf{F}(\bigoplus_i\mathfrak{A}_i)\to\bigoplus_i\mathsf{F}(\mathfrak{A}_i)$ for every countable family $(\mathfrak{A}_i)$ of C^*-algebras, if F is contravariant.

If we replace countable by arbitrary, we say the functor is **completely additive**. If we replace countable by finite, we say the functor is **additive**.

(HX) **Half-exactness**. If

$$\mathfrak{A}_0\hookrightarrow\mathfrak{A}_1\twoheadrightarrow\mathfrak{A}_2$$

is a short exact sequence of C^*-algebras in $\mathbf{S}$, then the induced sequence $\mathsf{F}(\mathfrak{A}_0)\to\mathsf{F}(\mathfrak{A}_1)\to\mathsf{F}(\mathfrak{A}_2)$ is exact, if F is covariant (resp. $\mathsf{F}(\mathfrak{A}_2)\to\mathsf{F}(\mathfrak{A}_1)\to\mathsf{F}(\mathfrak{A}_0)$ is exact, if F is contravariant).

Definition 3.2.2. A **homology theory** on $\mathbf{S}$ is a sequence (h_n) of covariant functors from $\mathbf{S}$ to $\mathbf{Ab}$ satisfying (H) and if

$$\mathfrak{A}_0\overset{\iota}{\hookrightarrow}\mathfrak{A}_1\overset{\pi}{\twoheadrightarrow}\mathfrak{A}_2$$

is a short exact sequence in $\mathbf{S}$, then for each n there is a connecting map $\partial_n\colon h_n(\mathfrak{A}_2)\to h_{n-1}(\mathfrak{A}_0)$ making exact the long sequence

$$\cdots\xrightarrow{\partial_{n+1}}h_n(\mathfrak{A}_0)\xrightarrow{h_n(\iota)}h_n(\mathfrak{A}_1)\xrightarrow{h_n(\pi)}h_n(\mathfrak{A}_2)\xrightarrow{\partial_n}h_{n-1}(\mathfrak{A}_0)\xrightarrow{h_{n-1}(\iota)}\cdots$$

where ∂_n are natural with respect to morphisms of short exact sequences.

Definition 3.2.3. A **cohomology theory** on $\mathbf{S}$ is a sequence (h^n) of contravariant functors from $\mathbf{S}$ to $\mathbf{Ab}$ satisfying (H) and if

$$\mathfrak{A}_0 \overset{\iota}{\hookrightarrow} \mathfrak{A}_1 \overset{\pi}{\twoheadrightarrow} \mathfrak{A}_2$$

is a short exact sequence in $\mathbf{S}$, then for each n there is a connecting map $\partial^n \colon h^n(\mathfrak{A}_0) \to h^{n+1}(\mathfrak{A}_2)$ making exact the long sequence

$$\cdots \xrightarrow{\partial^{n-1}} h^n(\mathfrak{A}_2) \xrightarrow{h^n(\pi)} h^n(\mathfrak{A}_1) \xrightarrow{h^n(\iota)} h^n(\mathfrak{A}_0) \xrightarrow{\partial^n} h^{n+1}(\mathfrak{A}_2) \xrightarrow{h^{n+1}(\pi)} \cdots$$

where ∂^n are natural with respect to morphisms of short exact sequences.

Definition 3.2.4. Let F be an additive functor from $\mathbf{S}$ to $\mathbf{Ab}$ satisfying (H) and (HX).

Let $e \colon \mathfrak{A}_0 \overset{\iota}{\hookrightarrow} \mathfrak{A}_1 \overset{\pi}{\twoheadrightarrow} \mathfrak{A}_2$ be a given extension. Then for each $n \in \mathbb{N}_0$, we set

$$\mathsf{F}_n = \mathsf{F} \circ \mathsf{S}^n, \quad \text{and} \quad \partial_{n+1} = \mathsf{F}_n(f_e)^{-1} \circ \mathsf{F}_n(\iota_{\mathfrak{mc}}) \colon \mathsf{F}_{n+1}(\mathfrak{A}_2) \to \mathsf{F}_n(\mathfrak{A}_0), \text{ if } \mathsf{F} \text{ is covariant,}$$

$$\mathsf{F}^n = \mathsf{F} \circ \mathsf{S}^n, \quad \text{and} \quad \partial^n = \mathsf{F}^n(\iota_{\mathfrak{mc}}) \circ \mathsf{F}^n(f_e)^{-1} \colon \mathsf{F}^n(\mathfrak{A}_0) \to \mathsf{F}^{n+1}(\mathfrak{A}_2), \text{ if } \mathsf{F} \text{ is contravariant,}$$

where $\iota_{\mathfrak{mc}} \colon \mathsf{S}\mathfrak{A}_2 \to \mathsf{C}_\pi$ and $f_e \colon \mathfrak{A}_0 \to \mathsf{C}_\pi$ are the canonical $*$-homomorphisms.[2]

From [Bla98, Theorem 21.4.3] we have the following theorem.

Theorem 3.2.5. *Let F be an additive functor from $\mathbf{S}$ to $\mathbf{Ab}$ satisfying (H) and (HX). If F is covariant, then $(\mathsf{F}_n)_{n=0}^\infty$ is a homology theory. If F is contravariant, then $(\mathsf{F}^n)_{n=0}^\infty$ is a cohomology theory.*

Corollary 3.2.6. *If F is an additive functor from $\mathbf{S}$ to $\mathbf{Ab}$ satisfying (H) and (HX), then F is split-exact, i.e., F sends split-exact sequences from $\mathbf{S}$ to split-exact sequences of abelian groups.*

Proof. Let $\mathfrak{A}_0 \overset{\iota}{\hookrightarrow} \mathfrak{A}_1 \overset{\pi}{\twoheadrightarrow} \mathfrak{A}_2$ be a split-exact sequence of C^*-algebras, and assume that F is covariant. It is clear that $\mathsf{F}(\pi)$ and $\mathsf{F}(\mathsf{S}\pi)$ are surjective (since F and $\mathsf{F} \circ \mathsf{S}$ are functors). From preceding theorem it follows that $\partial_1 = 0$, so $\mathsf{F}\iota$ is injective. The proof in the contravariant case is dual. ●●●

The following theorem is taken from [Bla98, Corollary 22.3.2].

Theorem 3.2.7. *Let F be an additive functor from $\mathbf{S}$ to $\mathbf{Ab}$ satisfying (H), (S), and (HX). Then F is naturally isomorphic to $\mathsf{F} \circ \mathsf{S}^2$.*

Definition 3.2.8. Let F be an additive functor from $\mathbf{S}$ to $\mathbf{Ab}$ satisfying (H), (S), and (HX), and let $\beta_{\mathfrak{A}} \colon \mathsf{F}(\mathfrak{A}) \to \mathsf{F}(\mathsf{S}^2\mathfrak{A})$ denote the natural isomorphism. Then for each short exact sequence

$$e \colon \mathfrak{A}_0 \overset{\iota}{\hookrightarrow} \mathfrak{A}_1 \overset{\pi}{\twoheadrightarrow} \mathfrak{A}_2$$

of C^*-algebras we make the following definition. If F is covariant, then we define $\partial_0 \colon \mathsf{F}(\mathfrak{A}_2) \to \mathsf{F}(\mathsf{S}\mathfrak{A}_0)$ as the composition of the homomorphisms

$$\mathsf{F}(\mathfrak{A}_2) \xrightarrow{\beta_{\mathfrak{A}_2}} \mathsf{F}(\mathsf{S}^2\mathfrak{A}_2) \xrightarrow{\partial_2} \mathsf{F}(\mathsf{S}\mathfrak{A}_0).$$

If F is contravariant, then we define $\widetilde{\partial^1} \colon \mathsf{F}(\mathsf{S}\mathfrak{A}_0) \to \mathsf{F}(\mathfrak{A}_2)$ as the composition of the homomorphisms

$$\mathsf{F}(\mathsf{S}\mathfrak{A}_0) \xrightarrow{\partial^1} \mathsf{F}(\mathsf{S}^2\mathfrak{A}_2) \xrightarrow{\beta_{\mathfrak{A}_2}^{-1}} \mathsf{F}(\mathfrak{A}_2).$$

So with each such short exact sequence we have associated a cyclic six term exact sequence

$$\begin{tikzcd} \mathsf{F}(\mathfrak{A}_0) \arrow[r, "\mathsf{F}(\iota)"] & \mathsf{F}(\mathfrak{A}_1) \arrow[r, "\mathsf{F}(\pi)"] & \mathsf{F}(\mathfrak{A}_2) \arrow[d, "\partial_0"] \\ \mathsf{F}(\mathsf{S}\mathfrak{A}_2) \arrow[u, "\partial_1"] & \mathsf{F}(\mathsf{S}\mathfrak{A}_1) \arrow[l, "\mathsf{F}(\mathsf{S}\pi)"] & \mathsf{F}(\mathsf{S}\mathfrak{A}_0) \arrow[l, "\mathsf{F}(\mathsf{S}\iota)"] \end{tikzcd} \quad \text{resp.} \quad \begin{tikzcd} \mathsf{F}(\mathfrak{A}_2) \arrow[r, "\mathsf{F}(\pi)"] & \mathsf{F}(\mathfrak{A}_1) \arrow[r, "\mathsf{F}(\iota)"] & \mathsf{F}(\mathfrak{A}_0) \arrow[d, "\partial^0"] \\ \mathsf{F}(\mathsf{S}\mathfrak{A}_0) \arrow[u, "\widetilde{\partial^1}"] & \mathsf{F}(\mathsf{S}\mathfrak{A}_1) \arrow[l, "\mathsf{F}(\mathsf{S}\iota)"] & \mathsf{F}(\mathsf{S}\mathfrak{A}_2) \arrow[l, "\mathsf{F}(\mathsf{S}\pi)"] \end{tikzcd}$$

which is natural with respect to morphisms of short exact sequences of C^*-algebras. We will occasionally misuse the notation and write ∂^1 instead of $\widetilde{\partial^1}$ (which should not cause any confusions).

[2] Note that S^n denotes the composition of S with itself n times, while the superscript in F^n indicates that this is some kind of n'th cohomology.

Remark 3.2.9. While it is obvious how to generalize homotopy-invariance, stability, additivity, and split-exactness for a functor from $\mathbf{S}$ to an additive category $\mathbf{A}$, it is not obvious how to generalize half-exactness.

In Section 21.4 in [Bla98], Blackadar defines half-exactness for such functors (*i.e.*, $\mathrm{Hom}_{\mathbf{A}}(X, \mathsf{F}(-))$ and $\mathrm{Hom}_{\mathbf{A}}(\mathsf{F}(-), X)$ should be half-exact for all objects X). It is natural to ask whether this *extends* the original definition, and the answer is no. This is seen by applying $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_3, K_1(-))$ to the short exact sequence $\mathsf{SM}_3 \hookrightarrow \mathbb{I}_{3,0} \twoheadrightarrow \mathbb{C}$ (*cf.* Definition 6.1.1). On the other hand, for the category of modules over a unital ring, $\mathrm{Hom}_R(R, M)$ is naturally isomorphic to M — so this property is stronger than the ordinary half-exactness. To avoid confusions, we will not use this terminology.

3.3 (Co-)Homology theories and mapping cone sequences

In this section we show exactly how the cyclic six term exact sequence of the mapping cone sequence for an extension of C^*-algebras is related to the cyclic six term exact sequence of the original extension (when they are defined as in the previous section).

First we will need the following lemma, which Bonkat uses a version of in the proof of [Bon02, Lemma 7.3.1]. The proof given here is much more elementary.

Lemma 3.3.1. *Let F_0 and F_1 be covariant additive functors from the category $\mathbf{S}$ to the category $\mathbf{Ab}$, which have the properties (H), (S), and (HX). Assume that ∂_0^- and ∂_1^- are boundary maps making $(\mathsf{F}_i, \partial_i)_{i=0}^1$ into a cyclic homology theory on $\mathbf{S}$. Let there also be given a commuting diagram*

$$\begin{tikzcd}
\mathfrak{A}_0 \arrow[r, hook] \arrow[d, hook] & \mathfrak{A}_1 \arrow[r, two heads] \arrow[d, hook] & \mathfrak{A}_2 \arrow[d, hook] \\
\mathfrak{B}_0 \arrow[r, hook] \arrow[d, two heads] & \mathfrak{B}_1 \arrow[r, two heads] \arrow[d, two heads] & \mathfrak{B}_2 \arrow[d, two heads] \\
\mathfrak{C}_0 \arrow[r, hook] & \mathfrak{C}_1 \arrow[r, two heads] & \mathfrak{C}_2
\end{tikzcd}$$

with the rows and columns being short exact sequences of C^*-algebras. Let $e_{\mathfrak{A}}, e_{\mathfrak{B}}$ and $e_{\mathfrak{C}}$ denote the three horizontal extensions, while e_0, e_1 and e_2 denote the three vertical extensions. Then we get a diagram

$$\begin{tikzcd}
 & {} \arrow[d, "\partial_1^{e_0}"] & {} \arrow[d, "\partial_1^{e_1}"] & {} \arrow[d, "\partial_1^{e_2}"] & {} \arrow[d, "\partial_0^{e_0}"] & {} \arrow[d, "\partial_0^{e_1}"] & {} \arrow[d, "\partial_0^{e_2}"] & \\
{} \arrow[r, "\partial_1^{e_{\mathfrak{A}}}"] & \mathsf{F}_0(\mathfrak{A}_0) \arrow[r] \arrow[d] & \mathsf{F}_0(\mathfrak{A}_1) \arrow[r] \arrow[d] & \mathsf{F}_0(\mathfrak{A}_2) \arrow[r, "\partial_0^{e_{\mathfrak{A}}}"] \arrow[d] & \mathsf{F}_1(\mathfrak{A}_0) \arrow[r] \arrow[d] & \mathsf{F}_1(\mathfrak{A}_1) \arrow[r] \arrow[d] & \mathsf{F}_1(\mathfrak{A}_2) \arrow[r, "\partial_1^{e_{\mathfrak{A}}}"] \arrow[d] & {} \\
{} \arrow[r, "\partial_1^{e_{\mathfrak{B}}}"] & \mathsf{F}_0(\mathfrak{B}_0) \arrow[r] \arrow[d] & \mathsf{F}_0(\mathfrak{B}_1) \arrow[r] \arrow[d] & \mathsf{F}_0(\mathfrak{B}_2) \arrow[r, "\partial_0^{e_{\mathfrak{B}}}"] \arrow[d] & \mathsf{F}_1(\mathfrak{B}_0) \arrow[r] \arrow[d] & \mathsf{F}_1(\mathfrak{B}_1) \arrow[r] \arrow[d] & \mathsf{F}_1(\mathfrak{B}_2) \arrow[r, "\partial_1^{e_{\mathfrak{B}}}"] \arrow[d] & {} \\
{} \arrow[r, "\partial_1^{e_{\mathfrak{C}}}"] & \mathsf{F}_0(\mathfrak{C}_0) \arrow[r] \arrow[d, "\partial_0^{e_0}"] & \mathsf{F}_0(\mathfrak{C}_1) \arrow[r] \arrow[d, "\partial_0^{e_1}"] & \mathsf{F}_0(\mathfrak{C}_2) \arrow[r, "\partial_0^{e_{\mathfrak{C}}}"] \arrow[d, "\partial_0^{e_2}"] & \mathsf{F}_1(\mathfrak{C}_0) \arrow[r] \arrow[d, "\partial_1^{e_0}"] & \mathsf{F}_1(\mathfrak{C}_1) \arrow[r] \arrow[d, "\partial_1^{e_1}"] & \mathsf{F}_1(\mathfrak{C}_2) \arrow[r, "\partial_1^{e_{\mathfrak{C}}}"] \arrow[d, "\partial_1^{e_2}"] & {} \\
{} \arrow[r, "\partial_0^{e_{\mathfrak{A}}}"] & \mathsf{F}_1(\mathfrak{A}_0) \arrow[r] \arrow[d] & \mathsf{F}_1(\mathfrak{A}_1) \arrow[r] \arrow[d] & \mathsf{F}_1(\mathfrak{A}_2) \arrow[r, "\partial_1^{e_{\mathfrak{A}}}"] \arrow[d] & \mathsf{F}_0(\mathfrak{A}_0) \arrow[r] \arrow[d] & \mathsf{F}_0(\mathfrak{A}_1) \arrow[r] \arrow[d] & \mathsf{F}_0(\mathfrak{A}_2) \arrow[r, "\partial_0^{e_{\mathfrak{A}}}"] \arrow[d] & {} \\
{} \arrow[r, "\partial_0^{e_{\mathfrak{B}}}"] & \mathsf{F}_1(\mathfrak{B}_0) \arrow[r] \arrow[d] & \mathsf{F}_1(\mathfrak{B}_1) \arrow[r] \arrow[d] & \mathsf{F}_1(\mathfrak{B}_2) \arrow[r, "\partial_1^{e_{\mathfrak{B}}}"] \arrow[d] & \mathsf{F}_0(\mathfrak{B}_0) \arrow[r] \arrow[d] & \mathsf{F}_0(\mathfrak{B}_1) \arrow[r] \arrow[d] & \mathsf{F}_0(\mathfrak{B}_2) \arrow[r, "\partial_0^{e_{\mathfrak{B}}}"] \arrow[d] & {} \\
{} \arrow[r, "\partial_0^{e_{\mathfrak{C}}}"] & \mathsf{F}_1(\mathfrak{C}_0) \arrow[r] \arrow[d, "\partial_1^{e_0}"] & \mathsf{F}_1(\mathfrak{C}_1) \arrow[r] \arrow[d, "\partial_1^{e_1}"] & \mathsf{F}_1(\mathfrak{C}_2) \arrow[r, "\partial_1^{e_{\mathfrak{C}}}"] \arrow[d, "\partial_1^{e_2}"] & \mathsf{F}_0(\mathfrak{C}_0) \arrow[r] \arrow[d, "\partial_0^{e_0}"] & \mathsf{F}_0(\mathfrak{C}_1) \arrow[r] \arrow[d, "\partial_0^{e_1}"] & \mathsf{F}_0(\mathfrak{C}_2) \arrow[r, "\partial_0^{e_{\mathfrak{C}}}"] \arrow[d, "\partial_0^{e_2}"] & {} \\
 & {} & {} & {} & {} & {} & {} &
\end{tikzcd}$$

with the cyclic six term exact sequence both horizontally and vertically. The two squares

$$\begin{tikzcd} \mathsf{F}_0(\mathfrak{C}_2) \arrow[r, "\partial_0^{e_\mathfrak{C}}"] \arrow[d, "\partial_0^{e_2}"] & \mathsf{F}_1(\mathfrak{C}_0) \arrow[d, "\partial_1^{e_0}"] \\ \mathsf{F}_1(\mathfrak{A}_2) \arrow[r, "\partial_1^{e_\mathfrak{A}}"] & \mathsf{F}_0(\mathfrak{A}_0) \end{tikzcd} \qquad \begin{tikzcd} \mathsf{F}_1(\mathfrak{C}_2) \arrow[r, "\partial_1^{e_\mathfrak{C}}"] \arrow[d, "\partial_1^{e_2}"] & \mathsf{F}_0(\mathfrak{C}_0) \arrow[d, "\partial_0^{e_0}"] \\ \mathsf{F}_0(\mathfrak{A}_2) \arrow[r, "\partial_0^{e_\mathfrak{A}}"] & \mathsf{F}_1(\mathfrak{A}_0) \end{tikzcd}$$

anticommute, while all the other squares (in the big diagram) commute.

If F *is contravariant instead, the dual statement holds.*

Proof. That all the other squares commute, is evident (using that F_0 and F_1 are functors and that the maps ∂_0 and ∂_1 are natural).

Let $\mathfrak{D}$ denote the pullback of $\mathfrak{C}_2$ along $\mathfrak{B}_2 \to \mathfrak{C}_2$ and $\mathfrak{C}_1 \to \mathfrak{C}_2$. Then we have short exact sequences

$$e_{\text{sum}}: \qquad \mathfrak{A}_0 \hookrightarrow \mathfrak{A}_1 + \mathfrak{B}_0 \twoheadrightarrow \mathfrak{A}_2 \oplus \mathfrak{C}_0$$

$$e_{\text{pullback}}: \qquad \mathfrak{A}_2 \oplus \mathfrak{C}_0 \hookrightarrow \mathfrak{D} \twoheadrightarrow \mathfrak{C}_2,$$

where we identify $\mathfrak{A}_1$ and $\mathfrak{B}_0$ with their images inside $\mathfrak{B}_1$. Split-exactness of F_0 and F_1, *cf.* Corollary 3.2.6, and naturality of ∂_0 and ∂_1 together with the morphisms of extensions

$$\begin{tikzcd} \mathfrak{A}_0 \arrow[r, hook] \arrow[d, equal] & \mathfrak{B}_0 \arrow[r, two heads] \arrow[d] & \mathfrak{C}_0 \arrow[d] \\ \mathfrak{A}_0 \arrow[r, hook] & \mathfrak{A}_1+\mathfrak{B}_0 \arrow[r, two heads] & \mathfrak{A}_2\oplus\mathfrak{C}_0 \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{A}_0 \arrow[r, hook] \arrow[d, equal] & \mathfrak{A}_1 \arrow[r, two heads] \arrow[d] & \mathfrak{A}_2 \arrow[d] \\ \mathfrak{A}_0 \arrow[r, hook] & \mathfrak{A}_1+\mathfrak{B}_0 \arrow[r, two heads] & \mathfrak{A}_2\oplus\mathfrak{C}_0 \end{tikzcd}$$

$$\begin{tikzcd} \mathfrak{A}_2\oplus\mathfrak{C}_0 \arrow[r, hook] \arrow[d] & \mathfrak{D} \arrow[r, two heads] \arrow[d] & \mathfrak{C}_2 \arrow[d, equal] \\ \mathfrak{A}_2 \arrow[r, hook] & \mathfrak{B}_2 \arrow[r, two heads] & \mathfrak{C}_2 \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{A}_2\oplus\mathfrak{C}_0 \arrow[r, hook] \arrow[d] & \mathfrak{D} \arrow[r, two heads] \arrow[d] & \mathfrak{C}_2 \arrow[d, equal] \\ \mathfrak{C}_0 \arrow[r, hook] & \mathfrak{C}_1 \arrow[r, two heads] & \mathfrak{C}_2 \end{tikzcd}$$

give that the map $\partial_{1-j}^{e_{\text{sum}}} \partial_j^{e_{\text{pullback}}} : \mathsf{F}_j(\mathfrak{C}_2) \to \mathsf{F}_j(\mathfrak{A}_0)$ is exactly $\partial_{1-j}^{e_\mathfrak{A}} \partial_j^{e_2} + \partial_{1-j}^{e_0} \partial_j^{e_\mathfrak{C}}$, for $j = 0, 1$. But it turns out that $\partial_{1-j}^{e_{\text{sum}}} \partial_j^{e_{\text{pullback}}} = 0$ proving anticommutativity. For we have the following commuting diagram with short exact rows and columns

$$\begin{tikzcd} \mathfrak{A}_0 \arrow[r, hook] \arrow[d, equal] & \mathfrak{A}_1 + \mathfrak{B}_0 \arrow[r, two heads] \arrow[d, hook] & \mathfrak{A}_2 \oplus \mathfrak{C}_0 \arrow[d, hook] \\ \mathfrak{A}_0 \arrow[r, hook] \arrow[d, two heads] & \mathfrak{B}_1 \arrow[r, two heads] \arrow[d, two heads] & \mathfrak{D} \arrow[d, two heads] \\ 0 \arrow[r, hook] & \mathfrak{C}_2 \arrow[r, equal] & \mathfrak{C}_2, \end{tikzcd}$$

so the map $\partial_j^{e_{\text{pullback}}}$ factors through $\mathsf{F}_{1-j}(\mathfrak{A}_1 + \mathfrak{B}_0) \to \mathsf{F}_{1-j}(\mathfrak{A}_2 \oplus \mathfrak{C}_0)$.

The proof in the case that F is contravariant is dual. ●●●

Lemma 3.3.2. *Let* F *be an additive functor from the category* $\mathbf{S}$ *to the category* $\mathbf{Ab}$*, which has the properties (H), (S), and (HX). Let* $\mathfrak{A}$ *be an arbitrary* C^**-algebra. The standard cyclic six term exact sequence*[3] *associated with* $\mathsf{S}\mathfrak{A} \hookrightarrow \mathsf{C}\mathfrak{A} \twoheadrightarrow \mathfrak{A}$ *is the sequence*

$$\begin{tikzcd} \mathsf{F}(\mathsf{S}\mathfrak{A}) \arrow[r] & 0 \arrow[r] & \mathsf{F}(\mathfrak{A}) \arrow[d, "\cong"', "-\beta_\mathfrak{A}"] \\ \mathsf{F}(\mathsf{S}\mathfrak{A}) \arrow[u, "-\mathrm{id}"] & 0 \arrow[l] & \mathsf{F}(\mathsf{S}^2\mathfrak{A}), \arrow[l] \end{tikzcd}$$

[3] as defined in Definitions 3.2.4 and 3.2.8

in the covariant case, and the sequence

$$\begin{tikzcd}
\mathsf{F}(\mathfrak{A}) \arrow[r] & 0 \arrow[r] & \mathsf{F}(\mathsf{S}\mathfrak{A}) \arrow[d, "-\mathrm{id}"] \\
\mathsf{F}(\mathsf{S}^2\mathfrak{A}) \arrow[u, "\cong"', "-\beta_{\mathfrak{A}}^{-1}"] & 0 \arrow[l] & \mathsf{F}(\mathsf{S}\mathfrak{A}), \arrow[l]
\end{tikzcd}$$

in the contravariant case.

Proof. Assume that F is covariant. Since the cone, $\mathsf{C}\mathfrak{A}$, of $\mathfrak{A}$ is homotopy equivalent to the zero C^*-algebra, $\mathsf{F}(\mathsf{C}\mathfrak{A}) \cong \mathsf{F}(\mathsf{S}\mathsf{C}\mathfrak{A}) \cong 0$ (*cf.* [RLL00, Example 4.1.5]).

We have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r, hook] \arrow[d, hook] & \mathsf{S}\mathfrak{A} \arrow[r, equal] \arrow[d, hook, "\iota_{\mathfrak{mc}}"] & \mathsf{S}\mathfrak{A} \arrow[d, hook] \\
\mathsf{S}\mathfrak{A} \arrow[r, hook, "f"] \arrow[d, equal] & \mathsf{C}_\pi \arrow[r, two heads] \arrow[d, two heads, "\pi_{\mathfrak{mc}}"] & \mathsf{C}\mathfrak{A} \arrow[d, two heads] \\
\mathsf{S}\mathfrak{A} \arrow[r, hook, "\iota"] & \mathsf{C}\mathfrak{A} \arrow[r, two heads, "\pi"] & \mathfrak{A}
\end{tikzcd}$$

with short exact rows and columns. Note that C_π is realized as $\{(x, y) \in \mathsf{C}\mathfrak{A} \oplus \mathsf{C}\mathfrak{A} \mid x(1) = y(1)\}$. Using this picture we get a $*$-homomorphism $\varphi\colon \mathsf{C}\mathfrak{A} \ni x \mapsto (x, x) \in \mathsf{C}_\pi$. Note that the composed $*$-homomorphism $\varphi \circ \iota$ is just $f + \iota_{\mathfrak{mc}}$. Since $\mathsf{F}(\mathsf{C}\mathfrak{A}) = 0$, we must have $\mathsf{F}(\varphi \circ \iota) = 0$. Using the split-exactness of F (*cf.* Corollary 3.2.6), we get a canonical identification of $\mathsf{F}(\mathsf{S}\mathfrak{A} \oplus \mathsf{S}\mathfrak{A})$ with $\mathsf{F}(\mathsf{S}\mathfrak{A}) \oplus \mathsf{F}(\mathsf{S}\mathfrak{A})$. Under this identification, we get

$$\begin{tikzcd}
\mathsf{F}(\mathsf{S}\mathfrak{A}) \arrow[r] \arrow[dr, "x \mapsto (x,x)"'] & \mathsf{F}(\mathsf{S}\mathfrak{A} \oplus \mathsf{S}\mathfrak{A}) \arrow[r] \arrow[d, "\cong"'] & \mathsf{F}(\mathsf{C}_\pi) \\
 & \mathsf{F}(\mathsf{S}\mathfrak{A}) \oplus \mathsf{F}(\mathsf{S}\mathfrak{A}) \arrow[ur, "(x,y) \mapsto \mathsf{F}(f)(x) + \mathsf{F}(\iota_{\mathfrak{mc}})(y)"'] &
\end{tikzcd}$$

Consequently,

$$\mathsf{F}(f) + \mathsf{F}(\iota_{\mathfrak{mc}}) = \mathsf{F}(f + \iota_{\mathfrak{mc}}) = \mathsf{F}(\varphi \circ \iota) = 0,$$

and hence $\mathsf{F}(f) = -\mathsf{F}(\iota_{\mathfrak{mc}})$. Therefore, we have $\partial_1 = \mathsf{F}(f)^{-1} \circ \mathsf{F}(\iota_{\mathfrak{mc}}) = -\mathrm{id}$.

The map $\partial_0\colon \mathsf{F}(\mathfrak{A}) \to \mathsf{F}(\mathsf{S}^2\mathfrak{A})$ is the composition of the maps

$$\mathsf{F}(\mathfrak{A}) \xrightarrow{\beta_{\mathfrak{A}}} \mathsf{F}(\mathsf{S}^2\mathfrak{A}) \xrightarrow{\partial_2} \mathsf{F}(\mathsf{S}^2\mathfrak{A}),$$

where $\partial_2 = \mathsf{F}(\mathsf{S}f)^{-1} \circ \mathsf{F}(\mathsf{S}\iota_{\mathfrak{mc}})$. It is easy to see that the matrix

$$\begin{pmatrix} 0 & \mathrm{flip} & \mathrm{flip} \\ \mathrm{flip} & (\mathrm{flip}, \mathrm{flip}) & \mathrm{flip} \\ \mathrm{flip} & \mathrm{flip} & \mathrm{id} \end{pmatrix}$$

implements a map between the diagrams

$$\begin{tikzcd}
0 \arrow[r, hook] \arrow[d, hook] & \mathsf{S}\mathsf{S}\mathfrak{A} \arrow[r, equal] \arrow[d, hook, "\mathsf{S}\iota_{\mathfrak{mc}}"] & \mathsf{S}\mathsf{S}\mathfrak{A} \arrow[d, hook] \\
\mathsf{S}\mathsf{S}\mathfrak{A} \arrow[r, hook, "\mathsf{S}f"] \arrow[d, equal] & \mathsf{S}\mathsf{C}_\pi \arrow[r, two heads] \arrow[d, two heads, "\mathsf{S}\pi_{\mathfrak{mc}}"] & \mathsf{S}\mathsf{C}\mathfrak{A} \arrow[d, two heads] \\
\mathsf{S}\mathsf{S}\mathfrak{A} \arrow[r, hook, "\mathsf{S}\iota"] & \mathsf{S}\mathsf{C}\mathfrak{A} \arrow[r, two heads, "\mathsf{S}\pi"] & \mathsf{S}\mathfrak{A}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
0 \arrow[r, hook] \arrow[d, hook] & \mathsf{S}(\mathsf{S}\mathfrak{A}) \arrow[r, equal] \arrow[d, hook] & \mathsf{S}(\mathsf{S}\mathfrak{A}) \arrow[d, hook] \\
\mathsf{S}(\mathsf{S}\mathfrak{A}) \arrow[r, hook] \arrow[d, equal] & \mathsf{C}_\rho \arrow[r, two heads] \arrow[d, two heads] & \mathsf{C}(\mathsf{S}\mathfrak{A}) \arrow[d, two heads] \\
\mathsf{S}(\mathsf{S}\mathfrak{A}) \arrow[r, hook] & \mathsf{C}(\mathsf{S}\mathfrak{A}) \arrow[r, two heads, "\rho"] & \mathsf{S}\mathfrak{A}
\end{tikzcd}$$

such that everything commutes. So by the above, we have $\partial_0 = -\beta_{\mathfrak{A}}$.

The proof when F is contravariant is dual. ●●●

Proposition 3.3.3. *Let* F *be an additive functor from* $\mathbf{S}$ *to the category* $\mathbf{Ab}$*, which has the properties (H), (S), and (HX). Let there be given an extension*

$$e : \mathfrak{A}_0 \overset{\iota}{\hookrightarrow} \mathfrak{A}_1 \overset{\pi}{\twoheadrightarrow} \mathfrak{A}_2.$$

Then we have isomorphism of cyclic six term exact sequences as follows:

$$\begin{tikzcd}
{} \arrow[r, "-\mathsf{F}(\mathsf{S}\pi)"] & \mathsf{F}(\mathsf{S}\mathfrak{A}_2) \arrow[r, "\partial_1^e"] \arrow[d, equal] & \mathsf{F}(\mathfrak{A}_0) \arrow[r, "\mathsf{F}(\iota)"] \arrow[d, "\mathsf{F}(f_e)", "\cong"'] & \mathsf{F}(\mathfrak{A}_1) \arrow[r, "-\mathsf{F}(\pi)"] \arrow[d, equal] & \mathsf{F}(\mathfrak{A}_2) \arrow[r, "\partial_0^e"] \arrow[d, "\beta_{\mathfrak{A}_2}", "\cong"'] & \mathsf{F}(\mathsf{S}\mathfrak{A}_0) \arrow[r, "\mathsf{F}(\mathsf{S}\iota)"] \arrow[d, "\mathsf{F}(\mathsf{S}f_e)", "\cong"'] & \mathsf{F}(\mathsf{S}\mathfrak{A}_1) \arrow[r, "-\mathsf{F}(\mathsf{S}\pi)"] \arrow[d, equal] & {} \\
{} \arrow[r, "\partial_1^{\mathfrak{mc}(e)}"] & \mathsf{F}(\mathsf{S}\mathfrak{A}_2) \arrow[r, "\mathsf{F}(\iota_{\mathfrak{mc}})"] & \mathsf{F}(\mathsf{C}_\pi) \arrow[r, "\mathsf{F}(\pi_{\mathfrak{mc}})"] & \mathsf{F}(\mathfrak{A}_1) \arrow[r, "\partial_0^{\mathfrak{mc}(e)}"] & \mathsf{F}(\mathsf{S}\mathsf{S}\mathfrak{A}_2) \arrow[r, "\mathsf{F}(\mathsf{S}\iota_{\mathfrak{mc}})"] & \mathsf{F}(\mathsf{S}\mathsf{C}_\pi) \arrow[r, "\mathsf{F}(\mathsf{S}\pi_{\mathfrak{mc}})"] & \mathsf{F}(\mathsf{S}\mathfrak{A}_1) \arrow[r, "\partial_1^{\mathfrak{mc}(e)}"] & {}
\end{tikzcd}$$

in the covariant case, and

$$\begin{tikzcd}
{} \arrow[r, "-\mathsf{F}(\pi)"] & \mathsf{F}(\mathfrak{A}_1) \arrow[r, "\mathsf{F}(\iota)"] & \mathsf{F}(\mathfrak{A}_0) \arrow[r, "\partial_e^0"] & \mathsf{F}(\mathsf{S}\mathfrak{A}_2) \arrow[r, "-\mathsf{F}(\mathsf{S}\pi)"] & \mathsf{F}(\mathsf{S}\mathfrak{A}_1) \arrow[r, "\mathsf{F}(\mathsf{S}\iota)"] & \mathsf{F}(\mathsf{S}\mathfrak{A}_0) \arrow[r, "\partial_e^1"] & \mathsf{F}(\mathfrak{A}_2) \arrow[r, "-\mathsf{F}(\pi)"] & {} \\
{} \arrow[r, "\partial^1_{\mathfrak{mc}(e)}"] & \mathsf{F}(\mathfrak{A}_1) \arrow[r, "\mathsf{F}(\pi_{\mathfrak{mc}})"] \arrow[u, equal] & \mathsf{F}(\mathsf{C}_\pi) \arrow[r, "\mathsf{F}(\iota_{\mathfrak{mc}})"] \arrow[u, "\mathsf{F}(f_e)"', "\cong"] & \mathsf{F}(\mathsf{S}\mathfrak{A}_2) \arrow[r, "\partial^0_{\mathfrak{mc}(e)}"] \arrow[u, equal] & \mathsf{F}(\mathsf{S}\mathfrak{A}_1) \arrow[r, "\mathsf{F}(\mathsf{S}\pi_{\mathfrak{mc}})"] \arrow[u, equal] & \mathsf{F}(\mathsf{S}\mathsf{C}_\pi) \arrow[r, "\mathsf{F}(\mathsf{S}\iota_{\mathfrak{mc}})"] \arrow[u, "\mathsf{F}(\mathsf{S}f_e)"', "\cong"] & \mathsf{F}(\mathsf{S}\mathsf{S}\mathfrak{A}_2) \arrow[r, "\partial^1_{\mathfrak{mc}(e)}"] \arrow[u, "\beta^{-1}_{\mathfrak{A}_2}"', "\cong"] & {}
\end{tikzcd}$$

in the contravariant case.

Proof. Assume that F is covariant. The diagram

$$\begin{tikzcd}
\mathfrak{A}_1 \arrow[r, "\pi"] \arrow[d, "\pi"] & \mathfrak{A}_2 \arrow[d, equal] \\
\mathfrak{A}_2 \arrow[r, equal] & \mathfrak{A}_2
\end{tikzcd}$$

induces the morphism of extensions

$$\begin{tikzcd}
0 \arrow[r] & \mathsf{S}\mathfrak{A}_2 \arrow[r] \arrow[d, equal] & \mathsf{C}_\pi \arrow[r] \arrow[d, "\omega"] & \mathfrak{A}_1 \arrow[r] \arrow[d, "\pi"] & 0 \\
0 \arrow[r] & \mathsf{S}\mathfrak{A}_2 \arrow[r] & \mathsf{C}_{\mathrm{id}_{\mathfrak{A}_2}} \arrow[r] & \mathfrak{A}_2 \arrow[r] & 0.
\end{tikzcd}$$

Note that $\mathsf{C}_{\mathrm{id}_{\mathfrak{A}_2}}$ is canonical isomorphic to $\mathsf{C}\mathfrak{A}_2$. According to Lemma 3.3.2, this induces a morphism between cyclic six term exact sequences:

$$\begin{tikzcd}
{} \arrow[r, "\partial_1^{e\,\mathfrak{mc}}"] & \mathsf{F}(\mathsf{S}\mathfrak{A}_2) \arrow[r, "\mathsf{F}(\iota_{\mathfrak{mc}})"] \arrow[d, equal] & \mathsf{F}(\mathsf{C}_\pi) \arrow[r, "\mathsf{F}(\pi_{\mathfrak{mc}})"] \arrow[d] & \mathsf{F}(\mathfrak{A}_1) \arrow[r, "\partial_0^{e\,\mathfrak{mc}}"] \arrow[d, "\mathsf{F}(\pi)"] & \mathsf{F}(\mathsf{S}\mathsf{S}\mathfrak{A}_2) \arrow[r, "\mathsf{F}(\mathsf{S}\iota_{\mathfrak{mc}})"] \arrow[d, equal] & \mathsf{F}(\mathsf{S}\mathsf{C}_\pi) \arrow[r, "\mathsf{F}(\mathsf{S}\iota_{\mathfrak{mc}})"] \arrow[d] & \mathsf{F}(\mathsf{S}\mathfrak{A}_1) \arrow[r, "\partial_1^{e\,\mathfrak{mc}}"] \arrow[d, "\mathsf{F}(\mathsf{S}\pi)"] & {} \\
{} \arrow[r, "\cong", "-\mathrm{id}"'] & \mathsf{F}(\mathsf{S}\mathfrak{A}_2) \arrow[r] & 0 \arrow[r] & \mathsf{F}(\mathfrak{A}_2) \arrow[r, "\cong", "-\beta_{\mathfrak{A}_2}"'] & \mathsf{F}(\mathsf{S}\mathsf{S}\mathfrak{A}_2) \arrow[r] & 0 \arrow[r] & \mathsf{F}(\mathsf{S}\mathfrak{A}_2) \arrow[r, "\cong", "-\mathrm{id}"'] & {}
\end{tikzcd}$$

This takes care of the commutativity of two of the six squares.

Commutativity of

$$\begin{tikzcd}
\mathsf{F}(\mathfrak{A}_0) \arrow[r, "\mathsf{F}(\iota)"] \arrow[d, "\mathsf{F}(f_e)", "\cong"'] & \mathsf{F}(\mathfrak{A}_1) \arrow[d, equal] \\
\mathsf{F}(\mathsf{C}_\pi) \arrow[r, "\mathsf{F}(\pi_{\mathfrak{mc}})"] & \mathsf{F}(\mathfrak{A}_1)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{F}(\mathsf{S}\mathfrak{A}_0) \arrow[r, "\mathsf{F}(\mathsf{S}(\iota))"] \arrow[d, "\mathsf{F}(\mathsf{S}f_e)", "\cong"'] & \mathsf{F}(\mathsf{S}\mathfrak{A}_1) \arrow[d, equal] \\
\mathsf{F}(\mathsf{S}\mathsf{C}_\pi) \arrow[r, "\mathsf{F}(\mathsf{S}\pi_{\mathfrak{mc}})"] & \mathsf{F}(\mathsf{S}\mathfrak{A}_1)
\end{tikzcd}$$

follows directly from the 3×3-diagram above.

Now we only need to check commutativity of

$$
\begin{tikzcd}
\mathsf{F}(\mathsf{S}\mathfrak{A}_2) \arrow[r, "\partial_1^e"] \arrow[d, equal] & \mathsf{F}(\mathfrak{A}_0) \arrow[d, "\mathsf{F}(f_e)", "\cong"'] \\
\mathsf{F}(\mathsf{S}\mathfrak{A}_2) \arrow[r, "\mathsf{F}(\iota_{\mathfrak{mc}})"] & \mathsf{F}(\mathsf{C}_\pi)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{F}(\mathfrak{A}_2) \arrow[r, "-\partial_0^e"] \arrow[d, "\beta_{\mathfrak{A}_2}", "\cong"'] & \mathsf{F}(\mathsf{S}\mathfrak{A}_0) \arrow[d, "\mathsf{F}(\mathsf{S}f_e)", "\cong"'] \\
\mathsf{F}(\mathsf{S}\mathsf{S}\mathfrak{A}_2) \arrow[r, "\mathsf{F}(\mathsf{S}\iota_{\mathfrak{mc}})"] & \mathsf{F}(\mathsf{S}\mathsf{C}_\pi)
\end{tikzcd}
$$

Since C_π is the pullback, we get a canonical map $\mathsf{C}\mathfrak{A}_1 \to \mathsf{C}_\pi$ and commuting diagrams

$$
\begin{tikzcd}
\mathsf{S}\mathfrak{A}_0 \arrow[r, hook] \arrow[d, hook] & \mathsf{C}\mathfrak{A}_0 \arrow[r, two heads] \arrow[d, hook] & \mathfrak{A}_0 \arrow[d, hook] \\
\mathsf{S}\mathfrak{A}_1 \arrow[r, hook] \arrow[d, two heads] & \mathsf{C}\mathfrak{A}_1 \arrow[r, two heads] \arrow[d, two heads] & \mathfrak{A}_1 \arrow[d, two heads] \\
\mathsf{S}\mathfrak{A}_2 \arrow[r, hook] & \mathsf{C}\mathfrak{A}_2 \arrow[r, two heads] & \mathfrak{A}_2
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{S}\mathfrak{A}_0 \arrow[r, equal] \arrow[d, hook] & \mathsf{S}\mathfrak{A}_0 \arrow[r, two heads] \arrow[d, hook] & 0 \arrow[d, hook] \\
\mathsf{C}\mathfrak{A}_0 \arrow[r, hook] \arrow[d, two heads] & \mathsf{C}\mathfrak{A}_1 \arrow[r, two heads] \arrow[d, two heads] & \mathsf{C}\mathfrak{A}_2 \arrow[d, equal] \\
\mathfrak{A}_0 \arrow[r, hook, "f_e"] & \mathsf{C}_\pi \arrow[r, two heads] & \mathsf{C}\mathfrak{A}_2
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{S}\mathfrak{A}_0 \arrow[r, equal] \arrow[d, hook] & \mathsf{S}\mathfrak{A}_0 \arrow[r, two heads] \arrow[d, hook] & 0 \arrow[d, hook] \\
\mathsf{S}\mathfrak{A}_1 \arrow[r, hook] \arrow[d, two heads] & \mathsf{C}\mathfrak{A}_1 \arrow[r, two heads] \arrow[d, two heads] & \mathfrak{A}_1 \arrow[d, equal] \\
\mathsf{S}\mathfrak{A}_2 \arrow[r, hook] & \mathsf{C}_\pi \arrow[r, two heads] & \mathfrak{A}_1
\end{tikzcd}
$$

with exact rows and columns. Using Lemma 3.3.1 and Lemma 3.3.2, these diagrams give rise to the following commutative diagrams

$$
\begin{tikzcd}
\mathsf{F}(\mathfrak{A}_2) \arrow[r, "-\beta_{\mathfrak{A}_2}"] \arrow[d, "\partial_0^e"'] & \mathsf{F}(\mathsf{S}\mathsf{S}\mathfrak{A}_2) \arrow[d, "\partial_1^{\mathsf{S}e}"] \\
\mathsf{F}(\mathsf{S}\mathfrak{A}_0) \arrow[r, equal] & \mathsf{F}(\mathsf{S}\mathfrak{A}_0)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{F}(\mathfrak{A}_0) \arrow[r, "\mathsf{F}(f_e)"] \arrow[d, "-\beta_{\mathfrak{A}_0}"'] & \mathsf{F}(\mathsf{C}_\pi) \arrow[d, "\partial_0^{e'}"] \\
\mathsf{F}(\mathsf{S}\mathsf{S}\mathfrak{A}_0) \arrow[r, equal] & \mathsf{F}(\mathsf{S}\mathsf{S}\mathfrak{A}_0)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{F}(\mathsf{S}\mathfrak{A}_2) \arrow[r, "\mathsf{F}(\iota_{\mathfrak{mc}})"] \arrow[d, "\partial_0^{\mathsf{S}e}"'] & \mathsf{F}(\mathsf{C}_\pi) \arrow[d, "\partial_0^{e'}"] \\
\mathsf{F}(\mathsf{S}\mathsf{S}\mathfrak{A}_0) \arrow[r, equal] & \mathsf{F}(\mathsf{S}\mathsf{S}\mathfrak{A}_0)
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathsf{F}(\mathsf{S}\mathfrak{A}_2) \arrow[r, equal] \arrow[d, "\partial_1^e"'] & \mathsf{F}(\mathsf{S}\mathfrak{A}_2) \arrow[d, "\partial_0^{\mathsf{S}e}"] \\
\mathsf{F}(\mathfrak{A}_0) \arrow[r, "-\beta_{\mathfrak{A}_0}"'] & \mathsf{F}(\mathsf{S}\mathsf{S}\mathfrak{A}_0)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{F}(\mathsf{S}\mathfrak{A}_0) \arrow[r, "-\mathsf{F}(\mathsf{S}f_e)"] \arrow[d, equal] & \mathsf{F}(\mathsf{S}\mathsf{C}_\pi) \arrow[d, "\partial_1^{e'}"] \\
\mathsf{F}(\mathsf{S}\mathfrak{A}_0) \arrow[r, equal] & \mathsf{F}(\mathsf{S}\mathfrak{A}_0)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{F}(\mathsf{S}\mathsf{S}\mathfrak{A}_2) \arrow[r, "\mathsf{F}(\mathsf{S}\iota_{\mathfrak{mc}})"] \arrow[d, "\partial_1^{\mathsf{S}e}"'] & \mathsf{F}(\mathsf{S}\mathsf{C}_\pi) \arrow[d, "\partial_1^{e'}"] \\
\mathsf{F}(\mathsf{S}\mathfrak{A}_0) \arrow[r, equal] & \mathsf{F}(\mathsf{S}\mathfrak{A}_0)
\end{tikzcd}
$$

where e' denotes the extension $\mathsf{S}\mathfrak{A}_0 \hookrightarrow \mathsf{C}\mathfrak{A}_1 \twoheadrightarrow \mathsf{C}_\pi$. Consequently,

$$
\begin{aligned}
\mathsf{F}(\iota_{\mathfrak{mc}}) &= (\partial_0^{e'})^{-1} \circ \partial_0^{\mathsf{S}e} = -\mathsf{F}(f_e) \circ (\beta_{\mathfrak{A}_0}^{-1}) \circ \partial_0^{\mathsf{S}e} \\
&= \mathsf{F}(f_e) \circ \beta_{\mathfrak{A}_0}^{-1} \circ \beta_{\mathfrak{A}_0} \circ \partial_1^e = \mathsf{F}(f_e) \circ \partial_1^e, \\
\mathsf{F}(\mathsf{S}\iota_{\mathfrak{mc}}) &= (\partial_1^{e'})^{-1} \circ \partial_1^{\mathsf{S}e} = -\mathsf{F}(\mathsf{S}f_e) \circ \partial_1^{\mathsf{S}e} \\
&= \mathsf{F}(\mathsf{S}f_e) \circ \partial_0^e \circ \beta_{\mathfrak{A}_2}^{-1}.
\end{aligned}
$$

The proof in the contravariant case is dual. ●●●

Corollary 3.3.4. *Let* F *be an additive functor from* $\mathbf{S}$ *to the category* $\mathbf{Ab}$*, which has the properties (H), (S), and (HX). Let there be given a* $*$*-homomorphism*

$$\phi : \mathfrak{A} \to \mathfrak{B}$$

from a C^**-algebra* $\mathfrak{A}$ *to a* C^**-algebra* $\mathfrak{B}$*, and let*

$$e \colon \mathsf{S}\mathfrak{B} \xhookrightarrow{\iota_{\mathfrak{mc}}} \mathsf{C}_\phi \xrightarrow{\pi_{\mathfrak{mc}}} \mathfrak{A}.$$

denote the corresponding mapping cone sequence.

Then we have isomorphism of cyclic six term exact sequences as follows:

$$
\begin{tikzcd}
{} \arrow[r, "-\mathsf{F}(\mathsf{S}\phi)"] & \mathsf{F}(\mathsf{S}\mathfrak{B}) \arrow[r, "\mathsf{F}(\iota_{\mathfrak{mc}})"] \arrow[d, equal] & \mathsf{F}(\mathsf{C}_\phi) \arrow[r, "\mathsf{F}(\pi_{\mathfrak{mc}})"] \arrow[d, equal] & \mathsf{F}(\mathfrak{A}) \arrow[r, "-\mathsf{F}(\phi)"] \arrow[d, equal] & \mathsf{F}(\mathfrak{B}) \arrow[r, "\mathsf{F}(\mathsf{S}\iota_{\mathfrak{mc}}) \circ \beta_{\mathfrak{B}}"] \arrow[d, "\beta_{\mathfrak{B}}", "\cong"'] & \mathsf{F}(\mathsf{S}\mathsf{C}_\phi) \arrow[r, "\mathsf{F}(\mathsf{S}\pi_{\mathfrak{mc}})"] \arrow[d, equal] & \mathsf{F}(\mathsf{S}\mathfrak{A}) \arrow[r, "-\mathsf{F}(\mathsf{S}\phi)"] \arrow[d, equal] & {} \\
{} \arrow[r, "\partial_1^e"] & \mathsf{F}(\mathsf{S}\mathfrak{B}) \arrow[r, "\mathsf{F}(\iota_{\mathfrak{mc}})"] & \mathsf{F}(\mathsf{C}_\phi) \arrow[r, "\mathsf{F}(\pi_{\mathfrak{mc}})"] & \mathsf{F}(\mathfrak{A}) \arrow[r, "\partial_0^e"] & \mathsf{F}(\mathsf{S}\mathsf{S}\mathfrak{B}) \arrow[r, "\mathsf{F}(\mathsf{S}\iota_{\mathfrak{mc}})"] & \mathsf{F}(\mathsf{S}\mathsf{C}_\phi) \arrow[r, "\mathsf{F}(\mathsf{S}\pi_{\mathfrak{mc}})"] & \mathsf{F}(\mathsf{S}\mathfrak{A}) \arrow[r, "\partial_1^e"] & {}
\end{tikzcd}
$$

in the covariant case, and

$$\begin{tikzcd} {} \arrow[r, "-\mathsf{F}(\phi)"] & \mathsf{F}(\mathfrak{A}) \arrow[r, "\mathsf{F}(\pi_{\mathfrak{mc}})"] \arrow[d, equal] & \mathsf{F}(\mathsf{C}_\phi) \arrow[r, "\mathsf{F}(\iota_{\mathfrak{mc}})"] \arrow[d, equal] & \mathsf{F}(\mathsf{S}\mathfrak{B}) \arrow[r, "-\mathsf{F}(\mathsf{S}\phi)"] \arrow[d, equal] & \mathsf{F}(\mathsf{S}\mathfrak{A}) \arrow[r, "\mathsf{F}(\mathsf{S}\pi_{\mathfrak{mc}})"] \arrow[d, equal] & \mathsf{F}(\mathsf{SC}_\phi) \arrow[r, "\mathsf{F}(\mathsf{S}\iota_{\mathfrak{mc}})"] \arrow[d, equal] & \mathsf{F}(\mathfrak{B}) \arrow[r, "-\mathsf{F}(\phi)"] \arrow[d, "\beta_{\mathfrak{B}}", "\cong"'] & {} \\ {} \arrow[r, "\partial_e^1"] & \mathsf{F}(\mathfrak{A}) \arrow[r, "\mathsf{F}(\pi_{\mathfrak{mc}})"] & \mathsf{F}(\mathsf{C}_\phi) \arrow[r, "\mathsf{F}(\iota_{\mathfrak{mc}})"] & \mathsf{F}(\mathsf{S}\mathfrak{B}) \arrow[r, "\partial_e^0"] & \mathsf{F}(\mathsf{S}\mathfrak{A}) \arrow[r, "\mathsf{F}(\mathsf{S}\pi_{\mathfrak{mc}})"] & \mathsf{F}(\mathsf{SC}_\phi) \arrow[r, "\beta_{\mathfrak{B}} \circ \mathsf{F}(\mathsf{S}\iota_{\mathfrak{mc}})"] & \mathsf{F}(\mathsf{SS}\mathfrak{B}) \arrow[r, "\partial_e^1"] & {} \end{tikzcd}$$

in the contravariant case.

Proof. This follows from the first part of the proof of the previous proposition. ●●●

3.4 Examples of concrete homology and cohomology theories

Example 3.4.1. Let $\mathbf{S}$ be the full subcategory of the category of all C^*-algebras, consisting of separable, nuclear algebras. For each separable C^*-algebra $\mathfrak{A}$, both $KK(-,\mathfrak{A})$ and $KK(\mathfrak{A},-)$ are additive functors from $\mathbf{S}$ to $\mathbf{Ab}$, which have the properties (H), (S), and (HX). The first one is contravariant while the second is covariant. So the above theory applies to these, and identifies the cyclic six term exact sequences associated with extensions in these two cases (as defined in [Bla98]).

Example 3.4.2. The functors K_0 and K_1 are additive, covariant functors from the category of all separable C^*-algebras to the category $\mathbf{Ab}$, which have the properties (H), (S), and (HX). So the above theory applies to these two functors.

We have also a standard cyclic six term exact sequence in K-theory (as defined in [RLL00]). To avoid confusions, we write δ_0 and δ_1 for the exponential map and the index maps, resp. We will recall the definition here. We have an isomorphism θ_- of functors from $K_1(-)$ to $K_0(\mathsf{S}(-))$, *i.e.*, for each C^*-algebra $\mathfrak{A}$ we have an isomorphism $\theta_{\mathfrak{A}}\colon K_1(\mathfrak{A}) \to K_0(\mathsf{S}\mathfrak{A})$ and, moreover, for all C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ and all $*$-homomorphisms $\varphi\colon \mathfrak{A} \to \mathfrak{B}$, the diagram

$$\begin{tikzcd} K_1(\mathfrak{A}) \arrow[r, "K_1(\varphi)"] \arrow[d, "\theta_{\mathfrak{A}}"'] & K_1(\mathfrak{B}) \arrow[d, "\theta_{\mathfrak{B}}"] \\ K_0(\mathsf{S}\mathfrak{A}) \arrow[r, "K_0(\mathsf{S}\varphi)"'] & K_0(\mathsf{S}\mathfrak{B}) \end{tikzcd}$$

commutes (*cf.* [RLL00, Theorem 10.1.3]).

The exponential map $\delta_0\colon K_0(\mathfrak{A}_2) \to K_1(\mathfrak{A}_0)$ associated with a short exact sequence $\mathfrak{A}_0 \hookrightarrow \mathfrak{A}_1 \twoheadrightarrow \mathfrak{A}_2$ is defined as the composition of the maps

$$K_0(\mathfrak{A}_2) \xrightarrow{\beta_{\mathfrak{A}_2}} K_1(\mathsf{S}\mathfrak{A}_2) \xrightarrow{\overline{\delta_1}} K_0(\mathsf{S}\mathfrak{A}_0) \xrightarrow{\theta_{\mathfrak{A}_0}^{-1}} K_1(\mathfrak{A}_0),$$

where $\overline{\delta_1}$ is the index map associated with the short exact sequence

$$\mathsf{S}\mathfrak{A}_0 \hookrightarrow \mathsf{S}\mathfrak{A}_1 \twoheadrightarrow \mathsf{S}\mathfrak{A}_2.$$

Lemma 3.4.3. *Let $\mathfrak{A}$ be a C^*-algebra. The standard cyclic six term exact sequence in K-theory associated with $\mathsf{S}\mathfrak{A} \hookrightarrow \mathsf{C}\mathfrak{A} \twoheadrightarrow \mathfrak{A}$ (as in [RLL00]) is the sequence*

$$\begin{tikzcd} K_0(\mathsf{S}\mathfrak{A}) \arrow[r] & 0 \arrow[r] & K_0(\mathfrak{A}) \arrow[d, "-\beta_{\mathfrak{A}}", "\cong"'] \\ K_1(\mathfrak{A}) \arrow[u, "\theta_{\mathfrak{A}}", "\cong"'] & 0 \arrow[l] & K_1(\mathsf{S}\mathfrak{A}). \arrow[l] \end{tikzcd}$$

Proof. Since the cone, $\mathsf{C}\mathfrak{A}$, of $\mathfrak{A}$ is homotopy equivalent to the zero C^*-algebra, $K_0(\mathsf{C}\mathfrak{A}) \cong K_1(\mathsf{C}\mathfrak{A}) \cong 0$ (*cf.* [RLL00, Example 4.1.5]).

That the index map is $\theta_{\mathfrak{A}}$ follows directly from the definition of $\theta_{\mathfrak{A}}$ (*cf.* [RLL00, Proof of Theorem 10.1.3]).

The exponential map $\delta_0\colon K_0(\mathfrak{A}) \to K_1(\mathsf{S}\mathfrak{A})$ is defined as the composition of the maps

$$K_0(\mathfrak{A}) \xrightarrow{\beta_{\mathfrak{A}}} K_1(\mathsf{S}\mathfrak{A}) \xrightarrow{\overline{\delta_1}} K_0(\mathsf{S}(\mathsf{S}\mathfrak{A})) \xrightarrow{\theta_{\mathsf{S}\mathfrak{A}}^{-1}} K_1(\mathsf{S}\mathfrak{A}),$$

where $\overline{\delta_1}$ is the index map associated with the short exact sequence

$$\mathsf{S}(\mathsf{S}\mathfrak{A}) \hookrightarrow \mathsf{S}(\mathsf{C}\mathfrak{A}) \twoheadrightarrow \mathsf{S}\mathfrak{A}.$$

We have a commuting diagram

$$\begin{tikzcd}
\mathsf{S}(\mathsf{S}\mathfrak{A}) \arrow[r, hook] \arrow[d, hook] & \mathsf{S}(\mathsf{C}\mathfrak{A}) \arrow[r, two heads] \arrow[d, hook] & \mathsf{S}(\mathfrak{A}) \arrow[d, hook] \\
\mathsf{C}(\mathsf{S}\mathfrak{A}) \arrow[r, hook] \arrow[d, two heads] & \mathsf{C}(\mathsf{C}\mathfrak{A}) \arrow[r, two heads] \arrow[d, two heads] & \mathsf{C}(\mathfrak{A}) \arrow[d, two heads] \\
\mathsf{S}\mathfrak{A} \arrow[r, hook] & \mathsf{C}\mathfrak{A} \arrow[r, two heads] & \mathfrak{A}
\end{tikzcd}$$

with exact rows and columns. This gives — by Lemma 3.3.1 and the above (applied to $\mathsf{S}\mathfrak{A}$ instead of $\mathfrak{A}$) — rise to an anticommuting square

$$\begin{tikzcd}
K_0(\mathfrak{A}) \arrow[r, "\delta_0", "\cong"'] \arrow[d, "\delta_0", "\cong"'] & K_1(\mathsf{S}\mathfrak{A}) \arrow[d, "\theta_{\mathsf{S}\mathfrak{A}}"] \\
K_1(\mathsf{S}(\mathfrak{A})) \arrow[r, "\overline{\delta_1}"] & K_0(\mathsf{S}(\mathsf{S}\mathfrak{A}))
\end{tikzcd}$$

Consequently, $\overline{\delta_1} = -\theta_{\mathsf{S}\mathfrak{A}}$. Now it follows that $\delta_0 = -\beta_{\mathfrak{A}}$. ●●●

Since the index and exponential maps are unique up to signs (*cf.* [WO93, Exercise 9.F]), we have that the standard cyclic six term exact sequence in K-theory as defined here differs from the cyclic six term exact sequence defined as above by change of sign of the index map (under the identification θ_- of K_1 with $K_0 \circ \mathsf{S}$).

Thus we get the following corollaries:

Corollary 3.4.4. *Let there be given an extension*

$$e : \mathfrak{A}_0 \overset{\iota}{\hookrightarrow} \mathfrak{A}_1 \overset{\pi}{\twoheadrightarrow} \mathfrak{A}_2.$$

Then we have isomorphism of cyclic six term exact sequences as follows:

$$\begin{tikzcd}
{} \arrow[r] & K_1(\mathfrak{A}_2) \arrow[r, "-\delta_1^e"] \arrow[d, "\theta_{\mathfrak{A}_2}", "\cong"'] & K_0(\mathfrak{A}_0) \arrow[r, "K_0(\iota)"] \arrow[d, "K_0(f_e)", "\cong"'] & K_0(\mathfrak{A}_1) \arrow[r, "-K_0(\pi)"] \arrow[d, equal] & K_0(\mathfrak{A}_2) \arrow[r, "\delta_0^e"] \arrow[d, "\beta_{\mathfrak{A}_2}", "\cong"'] & K_1(\mathfrak{A}_0) \arrow[r, "K_1(\iota)"] \arrow[d, "K_1(f_e)", "\cong"'] & K_1(\mathfrak{A}_1) \arrow[r, "K_1(\pi)"] \arrow[d, equal] & {} \\
{} \arrow[r] & K_0(\mathsf{S}\mathfrak{A}_2) \arrow[r] & K_0(\mathsf{C}_\pi) \arrow[r] & K_0(\mathfrak{A}_1) \arrow[r] & K_1(\mathsf{S}\mathfrak{A}_2) \arrow[r] & K_1(\mathsf{C}_\pi) \arrow[r] & K_1(\mathfrak{A}_1) \arrow[r] & {}
\end{tikzcd}$$

the second sequence is the standard cyclic six term exact sequence in K-theory associated with $\mathfrak{mc}(e)$.

Corollary 3.4.5. *Let there be given a $*$-homomorphism*

$$\phi : \mathfrak{A} \to \mathfrak{B}$$

from a C^*-algebra $\mathfrak{A}$ to a C^*-algebra $\mathfrak{B}$, and let

$$e\colon \mathsf{S}\mathfrak{B} \xhookrightarrow{\iota_{\mathrm{mc}}} \mathsf{C}_\phi \xrightarrow{\pi_{\mathrm{mc}}} \mathfrak{A}$$

denote the mapping cone sequence.

Then we have isomorphism of exact sequences as follows:

$$\begin{tikzcd}
{} \arrow[r] & K_1(\mathfrak{B}) \arrow[r] \arrow[d, "\cong"', "\theta_{\mathfrak{B}}"] & K_0(\mathsf{C}_\phi) \arrow[r] \arrow[d, equal] & K_0(\mathfrak{A}) \arrow[r, "-K_0(\phi)"] \arrow[d, equal] & K_0(\mathfrak{B}) \arrow[r] \arrow[d, "\cong"', "\beta_{\mathfrak{B}}"] & K_1(\mathsf{C}_\phi) \arrow[r] \arrow[d, equal] & K_1(\mathfrak{A}) \arrow[r, "K_1(\phi)"] \arrow[d, equal] & {} \\
{} \arrow[r] & K_0(\mathsf{S}\mathfrak{B}) \arrow[r] & K_0(\mathsf{C}_\phi) \arrow[r] & K_0(\mathfrak{A}) \arrow[r] & K_1(\mathsf{S}\mathfrak{B}) \arrow[r] & K_1(\mathsf{C}_\phi) \arrow[r] & K_1(\mathfrak{A}) \arrow[r] & {}
\end{tikzcd}$$

the second sequence is the standard cyclic six term exact sequence in K-theory associated with e.

Remark 3.4.6. Note that the way Bonkat associates cyclic six term exact sequences in ideal-related KK-theory with short exact sequences with completely positive contractive coherent splittings is completely analogous to the definitions of Section 3.2 (*cf.* [Bon02, Section 3.4]).

Example 3.4.7. An instructive example to get a better understanding of Lemma 3.3.1 is to look at

$$\begin{tikzcd}
\mathsf{S} \otimes \mathsf{S} \arrow[r, hook] \arrow[d, hook] & \mathsf{S} \otimes \mathsf{C} \arrow[r, two heads] \arrow[d, hook] & \mathsf{S} \otimes \mathbb{C} \arrow[d, hook] \\
\mathsf{C} \otimes \mathsf{S} \arrow[r, hook] \arrow[d, two heads] & \mathsf{C} \otimes \mathsf{C} \arrow[r, two heads] \arrow[d, two heads] & \mathsf{C} \otimes \mathbb{C} \arrow[d, two heads] \\
\mathbb{C} \otimes \mathsf{S} \arrow[r, hook] & \mathbb{C} \otimes \mathsf{C} \arrow[r, two heads] & \mathbb{C} \otimes \mathbb{C}
\end{tikzcd}$$

where $\mathsf{S} = \mathsf{S}\mathbb{C}$ and $\mathsf{C} = \mathsf{C}\mathbb{C}$. It is tempting to guess that the maps

$$K_0(\mathbb{C} \otimes \mathbb{C}) \to K_1(\mathsf{S} \otimes \mathbb{C}) \to K_0(\mathsf{S} \otimes \mathsf{S})$$

$$K_0(\mathbb{C} \otimes \mathbb{C}) \to K_1(\mathbb{C} \otimes \mathsf{S}) \to K_0(\mathsf{S} \otimes \mathsf{S})$$

are equal (after all, $\mathsf{S} \otimes \mathbb{C}$ is canonically isomorphic to $\mathbb{C} \otimes \mathsf{S}$) — but this is not the case. One map gives the Bott map while the other gives the anti-Bott map. After some thought this seems reasonable after all, since the map $\mathsf{S} \otimes \mathsf{S} \ni x \otimes y \mapsto y \otimes x \in \mathsf{S} \otimes \mathsf{S}$ corresponds to the flip along the diagonal in $C_0((0,1) \times (0,1))$, which induces the automorphism $-\mathrm{id}$ on K_0.

Chapter 4

Invariants for C^*-algebras with a distinguished system of ideals

In this chapter, we first review the different — equivalent — pictures of C^*-algebras with a distinguished system of ideals. For each fixed index set, we define a quiver with relations, and we give some examples of representations of this quiver for different index sets. The category of representations over this quiver with relations serves as the codomain of the invariant we define for C^*-algebras with a distinguished system of ideals (for a fixed index set). This invariant is used in the next chapter to obtain a UCT for the case with two specified (linearly ordered) ideals.

4.1 Categories of systems of C^*-algebras

We will consider C^*-algebras with a distinguished system of ideals. Bonkat also considers such systems, but he prefers to view them as a special example of projective systems. Each of the different pictures in use in [Bon02] has its own advantages and disadvantages. Since this thesis is very closely related to Bonkat's thesis, we will define the different pictures that we will use, and explain their interrelation.

The following definition is an amalgamation of [Bon02, Definitions 1.1.1–1.1.5]

Definition 4.1.1. Let I be an ordered set. Assume that I has a countable cofinal subset (a subset $I_0 \subseteq I$ is cofinal in I if for every $i \in I$ there is an $i_0 \in I_0$ such that $i_0 \leq i$). A **projective system over** I (of C^*-algebras) is a family $(\mathfrak{A}_i)_{i\in I}$ of C^*-algebras together with surjective $*$-homomorphisms $\alpha_{ij}\colon \mathfrak{A}_i \to \mathfrak{A}_j$, for all $i \leq j$, such that $\alpha_{ii} = \mathrm{id}_{\mathfrak{A}_i}$, for all $i \in I$, and $\alpha_{jk} \circ \alpha_{ij} = \alpha_{ik}$, for all $i \leq j \leq k$.

A **morphism** from a projective system $(\mathfrak{A}_i)_{i\in I}$ to a projective system $(\mathfrak{B}_i)_{i\in I}$ is a family of completely positive linear maps $f_i\colon \mathfrak{A}_i \to \mathfrak{B}_i$, for $i \in I$, such that $f_j \circ \alpha_{ij} = \beta_{ij} \circ f_i$ whenever $i \leq j$ (where α_{ij} and β_{ij} denote the connecting morphisms of $(\mathfrak{A}_i)_{i\in I}$ and $(\mathfrak{B}_i)_{i\in I}$, resp.). We also call such a morphism a completely positive linear map (between projective systems). If each f_i is a $*$-homomorphism, then we call $(f_i)_{i\in I}$ a $\mathcal{C}_I$**-homomorphism** (or just a homomorphism).

In the obvious way, this gives us the **category of projective systems over** I, $\mathcal{C}_I$[1]. The objects of $\mathcal{C}_I$ are the projective systems over I, and the morphisms are the completely positive linear maps.

We will also consider the subcategory, $\mathbf{S}\mathcal{C}_I$, of projective systems of separable C^*-algebras with $\mathcal{C}_I$-homomorphisms as morphisms.

These categories satisfy the axioms ($\mathcal{C}1$) to ($\mathcal{C}4$) on [Bon02, p. 25]. From these, it follows, that $\mathcal{C}_I$ has a null-object, $\mathcal{C}_I$ is closed under ideals, quotients, (finite) direct sums and products, pullbacks and it is also closed under tensoring by a nuclear C^*-algebra. Moreover, a convex combination of completely positive maps is again completely positive (*cf.* [Bon02, Section 2.2]).

For these systems (and more general systems) Bonkat develops a KK-theory. But first we want to compare the definition of projective systems for certain index sets with the notion of distinguished systems of ideals of a C^*-algebra. The following definition is from [Bon02, Definition 6.1.1].

[1]Bonkat considers two different categories of projective systems over a fixed index set, *cf.* [Bon02, pp. 30–31] — we will only use this one

Definition 4.1.2. Let I be an ordered set containing a least element, $i_{\min}$, and let $\mathfrak{A}$ be a C^*-algebra. A **distinguished system of ideals over** I in $\mathfrak{A}$ is an order preserving map

$$\Psi_{\mathfrak{A}} \colon I \to \mathcal{I}(\mathfrak{A}), \qquad \text{satisfying} \qquad \Psi_{\mathfrak{A}}(i_{\min}) = \{0\},$$

where $\mathcal{I}(\mathfrak{A})$ denotes the lattice of ideals of $\mathfrak{A}$.

A completely positive linear map $f \colon \mathfrak{A} \to \mathfrak{B}$ between two C^*-algebras with distinguished systems of ideals over I is called **Ψ-equivariant** if $f(\Psi_{\mathfrak{A}}(i)) \subseteq \Psi_{\mathfrak{B}}(i)$, for all $i \in I$.

Let $\mathbf{C^*Id(I)}$ denote the category of C^*-algebras with a distinguished system of ideals over I with the Ψ-equivariant completely positive linear maps as morphisms. Let $\mathbf{SC^*Id(I)}$ denote the subcategory of separable C^*-algebras with a distinguished system of ideals over I with the Ψ-equivariant $*$-homomorphisms as morphisms.

Remark 4.1.3. Let I be an ordered set with a least element, $i_{\min}$. We let $\tilde{I} = I \cup \{i_{\max}\}$, where $i_{\max}$ is a distinguished element (not in I), such that $i \lneq i_{\max}$ for all $i \in I$. There is a one-one correspondence between the order preserving maps $\Psi_{\mathfrak{A}} \colon I \to \mathcal{I}(\mathfrak{A})$ satisfying $\Psi_{\mathfrak{A}}(i_{\min}) = \{0\}$ and the order preserving maps $\Psi_{\mathfrak{A}} \colon \tilde{I} \to \mathcal{I}(\mathfrak{A})$ satisfying $\Psi_{\mathfrak{A}}(i_{\min}) = \{0\}$ and $\Psi_{\mathfrak{A}}(i_{\max}) = \mathfrak{A}$.

Remark 4.1.4. This is a natural generalization of Kirchberg's action of a (locally complete T_0) topological space (here $\tilde{I}$ is the lattice of the open sets of the space). Every ordered set I can be enlarged by a least element $i_{\min}$. So I could as well be an ordered set without a least element. Also, I could be an ordered set with a least element i_0 without having $\Psi_{\mathfrak{A}}(i_0) = \{0\}$. We could even have I to be just a set, then add an element $i_{\min}$ to the set and impose the order $i_{\min} \leq i$, for all $i \in I$.

In [Bon02, Satz 6.1.2], Bonkat proves the following proposition.

Proposition 4.1.5. *Let I be an ordered set with a least element $i_{\min}$. Then we define functors $\mathsf{G} \colon \mathbf{C^*Id(I)} \to \mathcal{C}_I$ and $\mathsf{H} \colon \mathcal{C}_I \to \mathbf{C^*Id(I)}$ as follows. Set $\mathsf{G}(\mathfrak{A}) = (\mathfrak{A}/\Psi_{\mathfrak{A}}(i))_{i \in I}$, for all $\mathfrak{A}$ in $\mathbf{C^*Id(I)}$, and let $\alpha_{ij} \colon \mathfrak{A}/\Psi_{\mathfrak{A}}(i) \twoheadrightarrow \mathfrak{A}/\Psi_{\mathfrak{A}}(j)$ be the surjective homomorphism induced by the quotient map $\mathfrak{A} \to \mathfrak{A}/\Psi_{\mathfrak{A}}(j)$ whenever $i \leq j$; and let $\mathsf{H}((\mathfrak{A}_i)_{i \in I})$ be the C^*-algebra $\mathfrak{A}_{i_{\min}}$ together with the action $\Psi_{\mathfrak{A}_{i_{\min}}}(i) = \ker(\mathfrak{A}_{i_{\min}} \twoheadrightarrow \mathfrak{A}_i)$, for $i \in I$. The functors act on morphisms in the obvious way.*

Then the pair (G, H) is an equivalence between the categories $\mathbf{C^*Id(I)}$ and $\mathcal{C}_I$. The restrictions G_0 and H_0 to the subcategories $\mathbf{SC^*Id(I)}$ and $\mathbf{S}\mathcal{C}_I$ give an equivalence between these two categories.

§ 4.1.6. Let I be an ordered set with a least element $i_{\min}$. Let $(\mathfrak{A}_i)_{i \in I}$ be a projective system over I. For each pair of elements $i, j \in \tilde{I}$ with $i \lneq j$, we have a C^*-algebra $\ker \alpha_{ij}$. For each triple $i, j, k \in \tilde{I}$ with $i \lneq j \lneq k$, we have a short exact sequence

$$\ker \alpha_{ij} \hookrightarrow \ker \alpha_{ik} \twoheadrightarrow \ker \alpha_{jk}.$$

Moreover, for each quadruple $i, j, k, l \in \tilde{I}$ with $i \lneq j \lneq k \lneq l$, we have a commutative diagram with short exact rows and columns:

$$\begin{tikzcd}
\ker \alpha_{ij} \arrow[r, hook] \arrow[d, equal] & \ker \alpha_{ik} \arrow[r, two heads] \arrow[d, hook] & \ker \alpha_{jk} \arrow[d, hook] \\
\ker \alpha_{ij} \arrow[r, hook] & \ker \alpha_{il} \arrow[r, two heads] \arrow[d, two heads] & \ker \alpha_{jl} \arrow[d, two heads] \\
& \ker \alpha_{kl} \arrow[r, equal] & \ker \alpha_{kl}
\end{tikzcd}$$

Note that here we have used $\tilde{I}$ instead of I to unify and shorten the notation. Of course, all the algebras, $(\mathfrak{A}_i)_{i \in I}$ are included in these diagrams; for each $i \in I$, we have $\ker \alpha_{i i_{\max}} = \mathfrak{A}_i$.

Remark 4.1.7. Let I be an ordered set with a least element $i_{\min}$, and let $(\mathfrak{A}_i)_{i \in I}$ and $(\mathfrak{B}_i)_{i \in I}$ be projective systems over I. For every completely positive map $f \colon (\mathfrak{A}_i)_{i \in I} \to (\mathfrak{B}_i)_{i \in I}$, there is a unique linear map from $\ker\alpha_{ij}$ to $\ker\beta_{ij}$ such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & \ker\alpha_{ij} \arrow[r] \arrow[d, dotted] & \mathfrak{A}_i \arrow[r] \arrow[d] & \mathfrak{A}_j \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \ker\beta_{ij} \arrow[r] & \mathfrak{B}_i \arrow[r] & \mathfrak{B}_j \arrow[r] & 0
\end{tikzcd}$$

commutes, whenever $i \lneq j$. This map will automatically be completely positive, and if f is a $\mathcal{C}_I$-homomorphism, then this map is a $*$-homomorphism. It is an easy exercise to check commutativity of the following diagram for $i \lneq j \lneq k$:

$$\begin{tikzcd}[column sep=small]
\ker\alpha_{ij} \arrow[rr, hook] \arrow[dr] \arrow[dd, equal] & & \ker\alpha_{ik} \arrow[rr, two heads] \arrow[dr] \arrow[dd, hook] & & \ker\alpha_{jk} \arrow[dr] \arrow[dd, hook] & \\
& \ker\beta_{ij} \arrow[rr, hook] \arrow[dd, equal] & & \ker\beta_{ik} \arrow[rr, two heads] \arrow[dd, hook] & & \ker\beta_{jk} \arrow[dd, hook] \\
\ker\alpha_{ij} \arrow[rr, hook] \arrow[dr] & & \mathfrak{A}_i \arrow[rr, two heads] \arrow[dr] \arrow[dd, two heads] & & \mathfrak{A}_j \arrow[dr] \arrow[dd, two heads] & \\
& \ker\beta_{ij} \arrow[rr, hook] & & \mathfrak{B}_i \arrow[rr, two heads] \arrow[dd, two heads] & & \mathfrak{B}_j \arrow[dd, two heads] \\
& & \mathfrak{A}_k \arrow[rr, equal] \arrow[dr] & & \mathfrak{A}_k \arrow[dr] & \\
& & & \mathfrak{B}_k \arrow[rr, equal] & & \mathfrak{B}_k
\end{tikzcd}$$

From this, it is clear that the maps between the corresponding diagrams as given in previous paragraph, will give a commutative diagram. So looking at the category of projective systems over I, at the category of C^*-algebras with a distinguished system of ideals over I, or at the category of families of extensions (as outlined above) is equivalent. So we choose the picture which is most convenient to us in the particular case considered. If we have only one distinguished ideal (except for the zero ideal), then it seems more natural to work with extensions. The picture with projective systems has at least one nice feature: since the connecting homomorphisms are surjective, it makes perfectly sense to define the multiplier system of a projective system in the obvious way — while for extensions, it is not the multiplier algebra of the ideal you want to consider. While it seems more natural (from C^*-algebraic point of view) to use the projective systems, there is a specific reason for viewing them as systems of extensions — this reason will be clear later, when we define the invariant we will use.

Definition 4.1.8. As in [Bon02, Sections 6.2 and 7.1], we let $\mathcal{E}$ denote the category of extensions of C^*-algebras, with the morphisms being triples of completely positive maps making the obvious diagram commutative. We let $S\mathcal{E}$ denote the subcategory consisting of extensions of separable C^*-algebras, with the morphisms being triples of $*$-homomorphisms. It is clear that $S\mathcal{E}$ is canonically equivalent to $\mathbf{SC^*Id(I)}$, where $I = \{0,1\}$ (*cf.* Remark 4.1.7).

Definition 4.1.9. We define the category $\mathcal{E}_2$ as follows. An object $\mathfrak{A}_\bullet$ of $\mathcal{E}_2$ is a commuting diagram

$$\begin{tikzcd}
\mathfrak{A}_1 \arrow[r, hook, "\alpha_{1,2}"] \arrow[d, equal] & \mathfrak{A}_2 \arrow[r, two heads, "\alpha_{2,4}"] \arrow[d, hook, "\alpha_{2,3}"] & \mathfrak{A}_4 \arrow[d, hook, "\alpha_{4,5}"] \\
\mathfrak{A}_1 \arrow[r, hook, "\alpha_{1,3}"] & \mathfrak{A}_3 \arrow[r, two heads, "\alpha_{3,5}"] \arrow[d, two heads, "\alpha_{3,6}"] & \mathfrak{A}_5 \arrow[d, two heads, "\alpha_{5,6}"] \\
& \mathfrak{A}_6 \arrow[r, equal] & \mathfrak{A}_6
\end{tikzcd}$$

with the rows and columns being extensions of C^*-algebras. The morphisms in $\mathcal{E}_2$ are six completely positive maps making the obvious diagram commutative. Let $S\mathcal{E}_2$ denote the subcategory of $\mathcal{E}_2$ consisting of diagrams involving only separable C^*-algebras, where the morphisms are six $*$-homomorphisms. It is clear that $S\mathcal{E}_2$ is canonically equivalent to $\mathbf{SC^*Id(I)}$, where $I = \{0,1,2\}$ (*cf.* Remark 4.1.7).

Similarly to $S\mathcal{E}$ we define the following:

Definition 4.1.10 (Constructions with systems of extensions)**.** Let $\mathfrak{D}$ be a nuclear C^*-algebra. Then we define the functor $-\otimes\mathfrak{D}$ from $S\mathcal{E}_2$ to $S\mathcal{E}_2$ as follows. Let $\mathfrak{A}_\bullet$ be an object of $S\mathcal{E}_2$. We let $\mathfrak{A}_\bullet\otimes\mathfrak{D}$ denote the object $(\mathfrak{A}_i\otimes\mathfrak{D})_{i=1}^6$ with the canonical maps, $(\alpha_{i,j}\otimes\mathrm{id}_{\mathfrak{D}})$. If $\mathbf{\Phi}_\bullet\colon\mathfrak{A}_\bullet\to\mathfrak{B}_\bullet$ is a morphism in $S\mathcal{E}_2$, then $\mathbf{\Phi}_\bullet\otimes\mathfrak{D}$ denotes the morphism $(\phi_i\otimes\mathrm{id}_{\mathfrak{D}})_{i=1}^6$.

In this way we define the **suspension** of $\mathfrak{A}_\bullet$, $\mathsf{S}\mathfrak{A}_\bullet=\mathfrak{A}_\bullet\otimes\mathsf{S}=\mathfrak{A}_\bullet\otimes C_0((0,1))$, the **cone** of $\mathfrak{A}_\bullet$, $\mathsf{C}\mathfrak{A}_\bullet=\mathfrak{A}_\bullet\otimes\mathsf{C}=\mathfrak{A}_\bullet\otimes C_0((0,1])$, and the **stabilization** of $\mathfrak{A}_\bullet$, $\mathfrak{A}_\bullet\otimes\mathbb{K}$, for each object $\mathfrak{A}_\bullet$ of $S\mathcal{E}_2$.

Remark 4.1.11. We will only use the category where the morphism correspond to $*$-homomorphisms. But for consistency, we use the same definitions as Bonkat does.

4.2 Definition of a quiver with relations from an ordered set

One of our goals is to define an invariant of projective systems of C^*-algebras over a fixed ordered set (with a least element), which fits in a UCT with Kirchberg's ideal-related KK-theory. This invariant is going to be defined as a functor, so first we want to define a suitable category which is going to be the codomain of this functor. Thus for each ordered set (with a least element), we want to define a quiver with relations. The category we are interested in, will then be the category of representations over this quiver with relations. First let us define the quiver.

Assumption 4.2.1. In this section, let $(I,\leq)$ be an ordered set with a least element, $i_{\min}$. Moreover, let $(\tilde{I},\leq)$ be the ordered set, obtained from $(I,\leq)$ by adjoining a greatest element, $i_{\max}$ (even if $(I,\leq)$ already has a greatest element).

Definition 4.2.2. Now we associate a quiver, $\Gamma=(\Gamma_0,\Gamma_1)$, with the given ordered set I as follows. For each pair $i,i'\in\tilde{I}$ with $i\lneq i'$, we let $(i'/i)_0$ and $(i'/i)_1$ denote vertices of Γ. For each triple $i,i',i''\in\tilde{I}$ with $i\lneq i'\lneq i''$ we have arrows as indicated

$$\begin{tikzcd}
(i'/i)_0 \arrow[r, "\iota_0^{(i,i''),(i,i')}"] & (i''/i)_0 \arrow[r, "\pi_0^{(i',i''),(i,i'')}"] & (i''/i')_0 \arrow[d, "\delta_0^{(i,i'),(i',i'')}"] \\
(i''/i')_1 \arrow[u, "\delta_1^{(i,i'),(i',i'')}"] & (i''/i)_1 \arrow[l, "\pi_1^{(i',i''),(i,i'')}"] & (i'/i)_1. \arrow[l, "\iota_1^{(i,i''),(i,i')}"]
\end{tikzcd}$$

Remark 4.2.3. Note that from the indices included in the labels of the arrows, we easily read of a lot of information.

- ▶ The source (resp. target) of an arrow is immediately read of from the grading and second pair (resp. first pair). Thus two arrows can be 'composed' if and only if the grading and the meeting pair match up.
- ▶ Moreover, we can immediately read of whether a given arrow exists. More specific for $*=0,1$ we have
  - $\iota_*^{(i,i'),(i'',i''')}$ exists if and only if $i=i''\lneq i'''\lneq i'$,
  - $\pi_*^{(i,i'),(i'',i''')}$ exists if and only if $i''\lneq i\lneq i'=i'''$,
  - $\delta_*^{(i,i'),(i'',i''')}$ exists if and only if $i\lneq i'=i''\lneq i'''$.

We will use the convention, that $\iota_*^{(i,i'),(i'',i''')}$ and $\pi_*^{(i,i'),(i'',i''')}$, for $i=i''\lneq i'=i'''$, will denote the trivial path from $(i'/i)_*$ to $(i'''/i'')_*=(i'/i)_*$.

Definition 4.2.4. Now we want to define a family, ρ, of relations on the quiver Γ associated with I. For every triple $i,i',i''\in\tilde{I}$ with $i\lneq i'\lneq i''$ we include for $*=0,1$ the relations

$$\iota_*^{(i,i''),(i,i')}\delta_{1-*}^{(i,i'),(i',i'')},\quad \pi_*^{(i',i'')(i,i'')}\iota_*^{(i,i''),(i,i')},\quad \delta_*^{(i,i'),(i',i'')}\pi_*^{(i',i''),(i,i'')}.$$

Moreover, for every quadruple $i, i', i'', i''' \in \tilde{I}$ with $i \lneq i' \lneq i'' \lneq i'''$ we include for $* = 0, 1$ the relations

$$\iota_*^{(i,i'''),(i,i'')} \iota_*^{(i,i''),(i,i')} - \iota_*^{(i,i'''),(i,i')},$$

$$\pi_*^{(i'',i'''),(i',i''')} \pi_*^{(i',i'''),(i,i''')} - \pi_*^{(i'',i'''),(i,i''')},$$

$$\iota_*^{(i',i'''),(i',i'')} \pi_*^{(i',i''),(i,i'')} - \pi_*^{(i',i'''),(i,i''')} \iota_*^{(i,i'''),(i,i'')}$$

$$\delta_*^{(i,i'),(i',i''')} \iota_*^{(i',i'''),(i',i'')} - \delta_*^{(i,i'),(i',i'')},$$

$$\pi_{1-*}^{(i',i''),(i,i'')} \delta_*^{(i,i''),(i'',i''')} - \delta_*^{(i',i''),(i'',i''')},$$

$$\delta_*^{(i,i''),(i'',i''')} \pi_*^{(i'',i'''),(i',i''')} - \iota_{1-*}^{(i,i''),(i,i')} \delta_*^{(i,i'),(i',i''')}.$$

Definition 4.2.5. Let $M, N \in \mathbb{N}$, let $m_1, \ldots, m_M, n_1, \ldots, n_N \in \mathbb{Z}$, and let $p_1, \ldots, p_M, q_1, \ldots, q_N$ be paths in the quiver Γ associated with I. We write

$$m_1 p_1 + m_2 p_2 + \cdots + m_M p_M \sim n_1 q_1 + n_2 q_2 + \cdots + n_N q_N$$

whenever

$$s(p_1) = s(p_2) = \cdots = s(p_M) = s(q_1) = s(q_2) = \cdots = s(q_N),$$
$$t(p_1) = t(p_2) = \cdots = t(p_M) = t(q_1) = t(q_2) = \cdots = t(q_N), \quad \text{and}$$
$$m_1 p_1 + m_2 p_2 + \cdots + m_M p_M - (n_1 q_1 + n_2 q_2 + \cdots + n_N q_N) \in < \rho >.$$

Remark 4.2.6. So if p and q are paths, then $p \sim q$ exactly when $s(p) = s(q)$, $t(p) = t(q)$, and $p - q \in < \rho >$. Clearly, $\sim$ is an equivalence relation. Also we easily show that

$$p \sim q \text{ and } p' \sim q' \Longrightarrow \begin{cases} np + n'p' \sim nq + n'q' & \text{if } s(p) = s(q), t(p) = t(q), \\ p'p \sim q'q & \text{if } t(p) = s(p'). \end{cases}$$

So, in particular, $p \sim q \Longrightarrow p_1 p p_0 \sim p_1 q p_0$ whenever $t(p_0) = s(p) = s(q)$ and $t(p) = t(q) = s(p_1)$.

Proposition 4.2.7. *Let $(I, \leq)$ and $(\tilde{I}, \leq)$ be as in Assumption 4.2.1, and let (Γ, ρ) be the quiver with relations associated with I (as in Definitions 4.2.2 and 4.2.4). Then the following holds:*

(a) For every pair of vertices $v, v' \in \Gamma_0$ there is at most one arrow $\alpha\colon v \to v'$.

(b) There is no arrow $\alpha \in \Gamma_1$ with $s(\alpha) = t(\alpha)$.

(c) For each triple $(i, i', i'') \in \tilde{I}^3$ with $i \lneq i' \lneq i''$ there is a canonical functor $\mathsf{F}_{(i,i',i'')}$ from $\operatorname{Rep} \mathbb{Z}(\Gamma, \rho)$ to the category of complexes of $\mathbb{Z}$-modules.

(d) Let p be a non-trivial path in the 0-layer from $(y/x)_0$ to $(y'/x')_0$. Then $x \leq x'$ and $y \leq y'$, and

$$p \sim \begin{cases} \iota_0^{(x',y'),(x,y)} & \text{if } x = x', \\ \pi_0^{(x',y'),(x,y)} & \text{if } y = y', \\ \pi_0^{(x',y'),(x,y')} \iota_0^{(x,y'),(x,y)} & \text{if } x \neq x' \text{ and } y \neq y'. \end{cases}$$

With the convention mentioned above, we can use the last expression for all the cases. The corresponding statement about the 1-layer also holds.

(e) Let $i, i', i'' \in \tilde{I}$ with $i \lneq i' \lneq i''$.

$$\delta_0^{(i,i'),(i',i'')} \sim \pi_1^{(i,i'),(i_{\min},i')} \delta_0^{(i_{\min},i')(i',i_{\max})} \iota_0^{(i',i_{\max}),(i',i'')},$$

$$\delta_1^{(i,i'),(i',i'')} \sim \pi_0^{(i,i'),(i_{\min},i')} \delta_1^{(i_{\min},i')(i',i_{\max})} \iota_1^{(i',i_{\max}),(i',i'')}.$$

(f) Let p_0 be a path in the 1-layer, let p be a path, and assume that

$$p \sim \delta_1^{(a,b),(b,c)} p_0 \delta_0^{(d,e),(e,f)}.$$

Then $p \sim 0$. We also have the corresponding statement for the 0-layer.

(g) If p is a (non-trivial) oriented cycle, then $p \in < \rho >$.

Proof. Part (a), (b), and (c): This follows easily from the construction of (Γ, ρ).

Part (d): The path p clearly consists of compositions of ι_0's and/or π_0's. Therefore it is clear that $x \leq x'$ and $y \leq y'$. We may w.l.o.g. assume that there never are two consecutive ι_0's nor two consecutive π_0's (using the relations ρ). If

$$p = p_2 \iota_0^{(b,d)(b,c)} \pi_0^{(b,c),(a,c)} p_1 \tag{4.1}$$

for some paths p_1, p_2, then

$$p \sim p_2 \pi_0^{(b,d),(a,d)} \iota_0^{(a,d)(a,c)} p_1.$$

Again doing concatenation of ι_0's and π_0's, we get the same type of representation of p, and if p_1 or p_2 is non-trivial, we have reduced the total number of ι_0's and π_0's in p. If both p_1 and p_2 are trivial, then p is in the desired form. If p cannot be written in the form (4.1), then

$$p \sim \iota_0^{(x',y'),(x,y)}, \quad p \sim \pi_0^{(x',y'),(x,y)}, \quad \text{or} \quad p \sim \pi_0^{(x',y'),(x,y')} \iota_0^{(x,y'),(x,y)}.$$

By induction we see that p can be written in the desired form.

Part (e): We show this for the 0-layer only. If $i \neq i_{\min}$, then

$$\delta_0^{(i,i'),(i',i'')} \sim \pi_1^{(i,i'),(i_{\min},i')} \delta_0^{(i_{\min},i'),(i',i'')}.$$

If $i'' \neq i_{\max}$, then

$$\delta_0^{(i_{\min},i'),(i',i'')} \sim \delta_0^{(i_{\min},i'),(i',i_{\max})} \iota_0^{(i',i_{\max}),(i',i'')}.$$

Part (f): From part (e) it follows that

$$p \sim p_3 \delta_1^{(i_{\min},b),(b,i_{\max})} p_2 \delta_0^{(i_{\min},e)(e,i_{\max})} p_1,$$

where p_1 and p_3 are paths in the 0-layer and p_2 is a path in the 1-layer. Clearly p_2 is non-trivial (because otherwise would $i_{\max} = e \lneq i_{\max}$). We note that $b \neq i_{\min}$ and $e \neq i_{\max}$, because $a \lneq b$ and $e \lneq f$. If we assume that $p_2 \notin < \rho >$, then it follows from (d) that

$$p_2 \sim \pi_1^{(b,i_{\max}),(i_{\min},i_{\max})} \iota_1^{(i_{\min},i_{\max}),(i_{\min},e)}$$

– but $\iota_1^{(i_{\min},i_{\max}),(i_{\min},e)} \delta_0^{(i_{\min},e)(e,i_{\max})} \sim 0$.

Part (g): Let p be a non-trivial oriented cycle. From part (d) it follows that p has to visit both levels. Now it is evident from part (f) that $p \in < \rho >$. ●●●

Definition 4.2.8 (Exact representation)**.** We call a representation $\mathbf{M}_\bullet$ **exact** if $\mathsf{F}_{(i,i',i'')}(\mathbf{M}_\bullet)$ is exact for every triple $(i,i',i'') \in \tilde{I}^3$ satisfying $i \lneq i' \lneq i''$.

4.3 Examples

Example 4.3.1. Let $I = \{0\}$, and let 1 denote $i_{\max}$, so that we write $\tilde{I} = \{0, 1\}$. Then the representations of the quiver associated with $(I, \leq)$ are exactly the $\mathbb{Z}_2$-graded groups.

Example 4.3.2. Let $I = \{0, 1\}$ with the usual order, and let 2 denote $i_{\max}$, so that we write $\tilde{I} = \{0, 1, 2\}$. Then the representations of the quiver associated with $(I, \leq)$ are exactly the cyclic six term complexes (considered by Bonkat).

Among other things, Bonkat shows, that a cyclic six term complex is projective (resp. injective) if and only if it is exact and every entry is a projective (resp. injective) $\mathbb{Z}$-module.

Example 4.3.3. Let $I = \{0, 1, 2\}$ with the usual order, and let 3 denote $i_{\max}$, so that we write $\tilde{I} = \{0, 1, 2, 3\}$. Then the representations of the quiver associated with $(I, \leq)$ correspond exactly to the commuting diagrams

$$\begin{tikzcd}
{} & {} & {} \arrow[d, "\partial_{12,2}"] & {} \arrow[d, "\partial_{12,4}"] & {} & {} \arrow[d, "\partial_{6,8}"] & {} \arrow[d, "\partial_{6,10}"] & {} \\
{} \arrow[r, "\partial_{10,1}"] & L_1 \arrow[r] \arrow[d, equal] & L_2 \arrow[r] \arrow[d] & L_4 \arrow[r] \arrow[d] & L_7 \arrow[r] \arrow[d, equal] & L_8 \arrow[r] \arrow[d] & L_{10} \arrow[r, "\partial_{10,1}"] \arrow[d] & {} \\
{} \arrow[r, "\partial_{11,1}"] & L_1 \arrow[r] & L_3 \arrow[r] \arrow[d] & L_5 \arrow[r] \arrow[d] & L_7 \arrow[r] & L_9 \arrow[r] \arrow[d] & L_{11} \arrow[r, "\partial_{11,1}"] \arrow[d] & {} \\
& & L_6 \arrow[r, equal] \arrow[d, "\partial_{6,8}"] & L_6 \arrow[d, "\partial_{6,10}"] & & L_{12} \arrow[r, equal] \arrow[d, "\partial_{12,2}"] & L_{12} \arrow[d, "\partial_{12,4}"] & \\
& & {} & {} & & {} & {} &
\end{tikzcd}$$

with the rows and columns being cyclic six term complexes with the additional conditions:

$$\partial_{6,8}\partial_{5,6} = \partial_{7,8}\partial_{5,7} \quad \text{and} \quad \partial_{12,2}\partial_{11,12} = \partial_{1,2}\partial_{11,1}.$$

We will later show (*cf.* Theorem 5.1.8) that an object is projective (resp. injective), if and only if the four mentioned complexes are exact and all the $\mathbb{Z}$-modules $L_1, \ldots, L_{12}$ are projective (resp. injective).

Example 4.3.4. Let $I = \{0, a, b\}$, with $0 \leq a, 0 \leq b, a \not\leq b, b \not\leq a$, and denote $\tilde{I} = \{0, a, b, 1\}$. Then a representation corresponds to two overlapping cyclic six term complexes:

$$\begin{tikzcd}
{} \arrow[r, "\partial_6'"] & L_1' \arrow[dr, "\partial_1'"] & & L_3 \arrow[r, "\partial_3"] & L_4 \arrow[dr, "\partial_4"] & & L_6' \arrow[r, "\partial_6'"] & {} \\
& & L_2 \arrow[ur, "\partial_2"] \arrow[dr, "\partial_2'"] & & & L_5 \arrow[ur, "\partial_5'"] \arrow[dr, "\partial_5"] & & \\
{} \arrow[r, "\partial_6"] & L_1 \arrow[ur, "\partial_1"] & & L_3' \arrow[r, "\partial_3'"] & L_4' \arrow[ur, "\partial_4'"] & & L_6 \arrow[r, "\partial_6"] & {}
\end{tikzcd}$$

So we have the following examples of representations:

$$\begin{tikzcd}
{} \arrow[r] & 0 \arrow[dr] & & 0 \arrow[r] & \mathbb{Z} \arrow[dr, equal] & & 0 \arrow[r] & {} \\
& & 0 \arrow[ur] \arrow[dr] & & & \mathbb{Z} \arrow[ur] \arrow[dr] & & \\
{} \arrow[r] & 0 \arrow[ur] & & 0 \arrow[r] & \mathbb{Z} \arrow[ur, equal] & & 0 \arrow[r] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r] & 0 \arrow[dr] & & 0 \arrow[r] & \mathbb{Z} \arrow[dr, "x \mapsto (x,0)"] & & 0 \arrow[r] & {} \\
& & 0 \arrow[ur] \arrow[dr] & & & \mathbb{Z} \oplus \mathbb{Z} \arrow[ur] \arrow[dr] & & \\
{} \arrow[r] & 0 \arrow[ur] & & 0 \arrow[r] & \mathbb{Z} \arrow[ur, "y \mapsto (0,y)"'] & & 0 \arrow[r] & {}
\end{tikzcd}$$

We have an epic morphism from the second one onto the first one, given by identity on "L_4 and L_4'" and $(x, y) \mapsto x + y$ on "L_5". Clearly this cannot split, so the first representation is not projective, even though both complexes are exact and every entry is a projective $\mathbb{Z}$-module.

Similarly, one can show that the representation

$$\begin{tikzcd}
{} \arrow[r] & 0 \arrow[dr] & & \mathbb{Q}/\mathbb{Z} \arrow[r] & 0 \arrow[dr] & & 0 \arrow[r] & {} \\
& & \mathbb{Q}/\mathbb{Z} \arrow[ur, equal] \arrow[dr, equal] & & & 0 \arrow[ur] \arrow[dr] & & \\
{} \arrow[r] & 0 \arrow[ur] & & \mathbb{Q}/\mathbb{Z} \arrow[r] & 0 \arrow[ur] & & 0 \arrow[r] & {}
\end{tikzcd}$$

is not injective (even though both complexes are exact and every entry is an injective $\mathbb{Z}$-module).

Remark 4.3.5. One should note that the two representations in the above example cannot be obtained as the invariant (see the next section) of any distinguished system of ideals over $I = \{0, a, b\}$, since this corresponds to a direct sum (if the ideal lattice of $\mathfrak{A}$ is isomorphic to $\tilde{I}$ via $\Psi_{\mathfrak{A}}$).

Moreover, we do believe that the objects coming from distinguished systems of ideals over I have homological dimension 1 in general. For the above example it is easy to show, but we have not been able to prove this in general.

4.4 An invariant

Here we define an extension of the full filtered K-theory introduced in [Res06].

Definition 4.4.1. Let $(I, \leq)$ be an ordered set with a least element, $i_{\min}$. Let, moreover, $(\tilde{I}, \leq)$ be the ordered set, obtained from $(I, \leq)$ by adjoining a greatest element, $i_{\max}$ (even if $(I, \leq)$ already has a greatest element).

We will consider distinguished systems of ideals over I in a C^*-algebra (*cf.* Definition 4.1.2). These systems $\Psi_{\mathfrak{A}} : I \to \mathcal{I}(\mathfrak{A})$ are in one-to-one correspondence with the distinguished systems of ideals over $\tilde{I}$ in $\mathfrak{A}$ satisfying $\Psi_{\mathfrak{A}}(i_{\max}) = \mathfrak{A}$ — and we will freely shift between these points of view.

Now we want to construct a functor $K_{\circledast}$ from the category $\mathbf{SC^*Id(I)}$ to the category $\operatorname{Rep}(\Gamma, \rho)$, where (Γ, ρ) is the quiver associated with I. This is done as follows.

Let $\Psi_{\mathfrak{A}} : I \to \mathcal{I}(\mathfrak{A})$ be an object of $\mathbf{SC^*Id(I)}$.[2] Now we associate an object $\mathbf{M}_\bullet = K_{\circledast}(\Psi_{\mathfrak{A}})$ with $\Psi_{\mathfrak{A}}$ as follows. For each pair $i, i' \in \tilde{I}$ with $i \lneq i'$ we let $\mathsf{M}_{(i'/i)_0} = K_0(\Psi_{\mathfrak{A}}(i')/\Psi_{\mathfrak{A}}(i))$ and $\mathsf{M}_{(i'/i)_1} = K_1(\Psi_{\mathfrak{A}}(i')/\Psi_{\mathfrak{A}}(i))$. For each triple $i, i', i'' \in \tilde{I}$ with $i \lneq i' \lneq i''$ we let

$$\begin{tikzcd}
\mathsf{M}_{(i'/i)_0} \arrow[r, "\mathsf{m}_{\iota_0^{(i,i''),(i,i')}}"] & \mathsf{M}_{(i''/i)_0} \arrow[r, "\mathsf{m}_{\pi_0^{(i',i''),(i,i'')}}"] & \mathsf{M}_{(i''/i')_0} \arrow[d, "\mathsf{m}_{\delta_0^{(i,i'),(i',i'')}}"] \\
\mathsf{M}_{(i''/i')_1} \arrow[u, "\mathsf{m}_{\delta_1^{(i,i'),(i',i'')}}"] & \mathsf{M}_{(i''/i)_1} \arrow[l, "\mathsf{m}_{\pi_1^{(i',i''),(i,i'')}}"] & \mathsf{M}_{(i'/i)_1} \arrow[l, "\mathsf{m}_{\iota_1^{(i,i''),(i,i')}}"]
\end{tikzcd}$$

be the standard cyclic six term exact sequence[3]

$$\begin{tikzcd}
K_0(\Psi_{\mathfrak{A}}(i')/\Psi_{\mathfrak{A}}(i)) \arrow[r, "K_0(\iota)"] & K_0(\Psi_{\mathfrak{A}}(i'')/\Psi_{\mathfrak{A}}(i)) \arrow[r, "K_0(\pi)"] & K_0(\Psi_{\mathfrak{A}}(i'')/\Psi_{\mathfrak{A}}(i')) \arrow[d, "\delta_0"] \\
K_1(\Psi_{\mathfrak{A}}(i'')/\Psi_{\mathfrak{A}}(i')) \arrow[u, "\delta_1"] & K_1(\Psi_{\mathfrak{A}}(i'')/\Psi_{\mathfrak{A}}(i)) \arrow[l, "K_1(\pi)"] & K_1(\Psi_{\mathfrak{A}}(i')/\Psi_{\mathfrak{A}}(i)) \arrow[l, "K_1(\iota)"]
\end{tikzcd}$$

in K-theory induced by the extension

$$\Psi_{\mathfrak{A}}(i')/\Psi_{\mathfrak{A}}(i) \xhookrightarrow{\iota} \Psi_{\mathfrak{A}}(i'')/\Psi_{\mathfrak{A}}(i) \xrightarrow{\pi} \Psi_{\mathfrak{A}}(i'')/\Psi_{\mathfrak{A}}(i').$$

Using naturality of the index and exponential map, it is easy to verify that $\mathbf{M}_\bullet$ satisfies the relations ρ, *i.e.*, $\mathbf{M}_\bullet$ is really in $\operatorname{Rep}(\Gamma, \rho)$ and not only in $\operatorname{Rep}(\Gamma)$. Note that $K_{\circledast}(\Psi_{\mathfrak{A}})$ is always exact.

Let $\Psi_{\mathfrak{A}} : I \to \mathcal{I}(\mathfrak{A})$ and $\Psi_{\mathfrak{B}} : I \to \mathcal{I}(\mathfrak{B})$ be objects of $\mathbf{SC^*Id(I)}$, and let $\varphi \colon \mathfrak{A} \to \mathfrak{B}$ be a morphism in $\mathbf{SC^*Id(I)}$, *i.e.*, a Ψ-equivariant $*$-homomorphism. For each pair $i, i' \in \tilde{I}$ with $i \lneq i'$ this induces a $*$-homomorphism $\varphi_{(i,i')} \colon \Psi_{\mathfrak{A}}(i')/\Psi_{\mathfrak{A}}(i) \to \Psi_{\mathfrak{B}}(i')/\Psi_{\mathfrak{B}}(i)$. We define the morphism $\boldsymbol{\varphi}_\bullet = K_{\circledast}(\varphi)$ by setting $\varphi_{(i'/i)_0} = K_0(\varphi_{(i,i')})$ and $\varphi_{(i'/i)_1} = K_1(\varphi_{(i,i')})$. It is easy to verify that $\boldsymbol{\varphi}_\bullet$ is a morphism.

We define the functor $K_{\circledast+1}$ from the category $\mathbf{SC^*Id(I)}$ to the category $\operatorname{Rep}(\Gamma, \rho)$ in exactly the same way, just interchanging K_0 and K_1 everywhere and interchanging the index map, δ_1, with the exponential map, δ_0, everywhere. It is easy to check (using the results from previous chapter), that $K_{\circledast+1}$ and $K_{\circledast} \circ \mathsf{S}$ are naturally isomorphic (using the canonical isomorphisms).

Remark 4.4.2. Every object in the range of the invariant is exact. The reason we use this larger category, is that if we restrict to only exact representations, this is not an abelian category (the kernel of a morphisms from an exact representation to another exact representation is not exact, in general).

[2] Recall that we let $\Psi_{\mathfrak{A}}(i_{\max}) = \mathfrak{A}$ where $i_{\max}$ is the greatest element of $\tilde{I}$.

[3] We define the index and exponential maps according to [RLL00]

Chapter 5

A UCT for ideal-related KK-theory

In this chapter, we prove the first main result of this thesis: a Universal Coefficient Theorem (UCT) for ideal-related KK-theory with two specified ideals, of which one is included in the other. The main ideas of the proof are along the lines of the proofs of the UCT of Rosenberg and Schochet (*cf.* [RS87] and [Bla98]) and the UCT of Bonkat (*cf.* [Bon02]). The main difficulty has been to establish a suitable framework for the homological algebra needed in the proof — in particular to characterize the projective and injective objects in the category where the invariant lives and to prove that all objects in the image of the invariant have projective and injective dimension at most one. The UCT is used together with results of Kirchberg to prove classification theorems for certain purely infinite C^*-algebras with exactly two non-trivial ideals. The main results are in the three last sections.

It seems that Meyer and Nest recently have generalized this UCT to include all finite, linearly ordered ideal lattices (*cf.* [MNa] and [MNb]). Also they claim that there are obstructions for having a UCT for many other finite ideal lattices (earlier Dadarlat and Eilers have pointed out to the author that there are such obstructions in the case with infinitely many specified ideals). It seems that the invariant used by Meyer and Nest is more abstract, and it is not clear to the author whether it coincides with (or is equivalent to) the invariant introduced here. For these reasons, no attempts have been made to generalize the proofs of this chapter to cover other specified ideal structures (*e.g.*, finite, linearly ordered lattices).

5.1 Projective and injective objects

In this section, we characterize the projective and injective objects (in the case of two specified ideals, linearly ordered). It turns out, that these are, indeed, the objects which are exact and have every entry to be a projective resp. injective abelian group. Using this, we can prove that an object is exact if and only if it has projective dimension at most one if and only if it has injective dimension at most one. This result plays a crucial rôle in the sequel.

Assumption 5.1.1. We assume that $I = \{0, 1, 2\}$ and $\tilde{I} = \{0, 1, 2, 3\}$ with the usual order. Much of what we will do from now on can be done in general, but it is much easier to consider this special case. And, moreover, we cannot prove all the results generally (as stated, they are not even true in general, *cf.* Example 4.3.4).

Let (Γ, ρ) denote the finite quiver with relations associated with $(I, \leq)$. From Proposition 4.2.7(g) we also know that every oriented cycle is in the ideal $< \rho >$ generated by the relations ρ. Therefore the ring $\Lambda = \mathbb{Z}(\Gamma, \rho)$ is finitely generated free as a $\mathbb{Z}$-module.

Corollary 5.1.2. *If F is a projective $\mathbb{Z}$-module (i.e., a free abelian group), then $\mathbf{Free}_\bullet^{F,v}$ is a projective object for all $v \in \Gamma_0$. If D is an injective $\mathbb{Z}$-module (i.e., a divisible abelian group), then $\mathbf{Cofree}_\bullet^{D,v}$ is an injective object for all $v \in \Gamma_0$.*

Proof. We already know this (*cf.* Proposition 2.2.7 and Definition 2.3.8). ●●●

Lemma 5.1.3. *Let G be a $\mathbb{Z}$-module, and let $x, y \in \tilde{I}$ with $x \lneq y$. Then* $\mathbf{Free}_\bullet^{G,(y/x)_0}$, $\mathbf{Free}_\bullet^{G,(y/x)_1}$, $\mathbf{Cofree}_\bullet^{G,(y/x)_0}$, *and* $\mathbf{Cofree}_\bullet^{G,(y/x)_1}$ *are exact objects.*

Proof. It is easy to check all the cases by bare hand. It seems very lengthy to prove this in general, but we expect this to be true in general. ●●●

As in [Bon02, Lemma 7.2.6] we get the following.

Lemma 5.1.4. *Let $\mathbf{M}_\bullet$ be an object of* $\operatorname{Rep}(\Gamma, \rho)$. *Then there exists an exact, projective object $\mathbf{P}_\bullet$ in* $\operatorname{Rep}(\Gamma, \rho)$ *and an epic morphism $\boldsymbol{\phi}_\bullet \colon \mathbf{P}_\bullet \to \mathbf{M}_\bullet$, and there exists an exact, injective object $\mathbf{I}_\bullet$ in* $\operatorname{Rep}(\Gamma, \rho)$ *and a monic morphism $\boldsymbol{\phi}_\bullet \colon \mathbf{M}_\bullet \to \mathbf{I}_\bullet$.*

Proof. For each $v \in \Gamma_0$ there exists a projective $\mathbb{Z}$-module P^v and a surjective group homomorphism $\varphi_v^v \colon P^v \to \mathsf{M}_v$. This induces a morphisms $\boldsymbol{\varphi}_\bullet^v \colon \mathbf{Proj}_\bullet^{P^v, v} \to \mathbf{M}_\bullet$ for every $v \in \Gamma_0$.

Let now $\mathbf{P}_\bullet$ be the direct sum of the family $(\mathbf{Proj}_\bullet^{P^v, v})_{v \in \Gamma_0}$ of representations. Using the universal property of the direct sum, define a morphism $\boldsymbol{\phi}_\bullet \colon \mathbf{P}_\bullet \to \mathbf{M}_\bullet$ from the family $(\boldsymbol{\varphi}_\bullet^v)_{v \in \Gamma_0}$ of morphisms.

Note that $\boldsymbol{\phi}_\bullet$ is epic (we just need to check that ϕ_v is surjective for each $v \in \Gamma_0$, which is clear from the universal construction because $\phi_v^v \colon \mathsf{P}_v \to \mathsf{M}_v$ is surjective). Note also that F_r is additive, so $\mathsf{F}_r(\mathbf{P}_\bullet)$ is a (finite) direct sum of exact complexes, hence it is exact.

Dualize the proof to get a proof for the part involving injectivity. ●●●

Following [Bon02, Korollar 7.2.7], we prove the corollary:

Corollary 5.1.5. *Every projective object and every injective object of* $\operatorname{Rep}(\Gamma, \rho)$ *is exact.*

Proof. Let $\mathbf{P}_\bullet$ be a projective object of $\operatorname{Rep}(\Gamma, \rho)$. From preceding lemma we know that there exists an exact representation $\mathbf{M}_\bullet$ and an epic morphism $\boldsymbol{\phi}_\bullet \colon \mathbf{M}_\bullet \to \mathbf{P}_\bullet$. Then there exists a morphism $\boldsymbol{\psi}_\bullet \colon \mathbf{P}_\bullet \to \mathbf{M}_\bullet$ such that the diagram

$$\begin{tikzcd} & \mathbf{M}_\bullet \arrow[d, "\boldsymbol{\phi}_\bullet"] \\ \mathbf{P}_\bullet \arrow[ur, "\boldsymbol{\psi}_\bullet"] \arrow[r, equal] & \mathbf{P}_\bullet \end{tikzcd}$$

commutes. Let $r = (i, i', i'') \in \mathfrak{I}^3$ with $i \lneq i' \lneq i''$ be given. Then it is enough to show that $\mathsf{F}_r(\mathbf{P}_\bullet)$ is exact. By standard homological algebra we need only to show that the homology groups are zero, *i.e.*, $H_n(\mathsf{F}_r(\mathbf{P}_\bullet)) = 0$, for all $n \in \mathbb{Z}$. But this is clear from the induced diagram

$$\begin{tikzcd} & 0 \arrow[d, "H_n(\mathsf{F}_r(\boldsymbol{\phi}_\bullet))"] \\ H_n(\mathsf{F}_r(\mathbf{P}_\bullet)) \arrow[ur, "H_n(\mathsf{F}_r(\boldsymbol{\psi}_\bullet))"] \arrow[r, equal] & H_n(\mathsf{F}_r(\mathbf{P}_\bullet)) \end{tikzcd}$$

– which commutes since $H_n(\mathsf{F}_r(\boldsymbol{\phi}_\bullet)) \circ H_n(\mathsf{F}_r(\boldsymbol{\psi}_\bullet)) = H_n(\mathsf{F}_r(\mathrm{id})) = \mathrm{id}$.

The statement involving injectivity is proved by dualizing the proof. ●●●

Part of [Bon02, Proposition 7.2.8] corresponds to the following:

Proposition 5.1.6. *If $\mathbf{P}_\bullet$ is a projective object of* $\operatorname{Rep}(\Gamma, \rho)$, *then $\mathbf{P}_\bullet$ is exact and P_v is a projective $\mathbb{Z}$-module (i.e., free abelian group) for every $v \in \Gamma_0$. If $\mathbf{I}_\bullet$ is an injective object of* $\operatorname{Rep}(\Gamma, \rho)$, *then $\mathbf{I}_\bullet$ is exact and I_v is an injective $\mathbb{Z}$-module (i.e., divisible abelian group) for every $v \in \Gamma_0$.*

Proof. By preceding corollary, $\mathbf{P}_\bullet$ is exact.

Let $\mathbf{P}'_\bullet$ be the exact projective object and let $\boldsymbol{\phi}_\bullet \colon \mathbf{P}'_\bullet \to \mathbf{P}_\bullet$ be the epic morphism constructed in the proof of Lemma 5.1.4. Then there exists a morphism $\boldsymbol{\psi}_\bullet \colon \mathbf{P}_\bullet \to \mathbf{P}'_\bullet$ such that $\boldsymbol{\phi}_\bullet \circ \boldsymbol{\psi}_\bullet = \mathrm{id}$. So $\mathbf{P}_\bullet$ is a direct summand of $\mathbf{P}'_\bullet$. Consequently, P_v is a subgroup of P'_v, and hence projective (for all $v \in \Gamma_0$).

This proof is also dualizable (since the quotient of an injective $\mathbb{Z}$-module is injective). ●●●

Proposition 5.1.7 (A Mayer-Vietoris sequence)**.** *Let $(M_n, f_n)_{n\in\mathbb{Z}}$ and $(N_n, g_n)_{n\in\mathbb{Z}}$ be ordinary chain complexes, let $(\varphi_n)_{n\in\mathbb{Z}}\colon (M_n)_{n\in\mathbb{Z}} \to (N_n)_{n\in\mathbb{Z}}$ be a chain homomorphism, and assume that*

$$M_{3n} = N_{3n}, \quad \varphi_{3n} = \mathrm{id},$$

for all $n \in \mathbb{Z}$, i.e., we have a commuting diagram:

$$\begin{tikzcd}
\cdots \arrow[r, "f_{-3}"] & M_{-2} \arrow[r, "f_{-2}"] \arrow[d, "\varphi_{-2}"] & M_{-1} \arrow[r, "f_{-1}"] \arrow[d, "\varphi_{-1}"] & M_0 \arrow[r, "f_0"] \arrow[d, equal] & M_1 \arrow[r, "f_1"] \arrow[d, "\varphi_1"] & M_2 \arrow[r, "f_2"] \arrow[d, "\varphi_2"] & M_3 \arrow[r, "f_3"] \arrow[d, equal] & \cdots \\
\cdots \arrow[r, "g_{-3}"] & N_{-2} \arrow[r, "g_{-2}"] & N_{-1} \arrow[r, "g_{-1}"] & N_0 \arrow[r, "g_0"] & N_1 \arrow[r, "g_1"] & N_2 \arrow[r, "g_2"] & N_3 \arrow[r, "g_3"] & \cdots
\end{tikzcd}$$

Then the two sequences

$$\cdots \longrightarrow N_{-1} \xrightarrow{x\mapsto f_0 g_{-1} x} M_1 \xrightarrow{x\mapsto (f_1 x, \varphi_1 x)} M_2 \oplus N_1 \xrightarrow{(x,y)\mapsto g_1 y - \varphi_2 x}$$
$$\xrightarrow{(x,y)\mapsto g_1 y - \varphi_2 x} N_2 \xrightarrow{x\mapsto f_3 g_2 x} M_4 \longrightarrow \cdots$$

and

$$\cdots \longrightarrow N_{-1} \xrightarrow{x\mapsto f_0 g_{-1} x} M_1 \xrightarrow{x\mapsto (f_1 x, -\varphi_1 x)} M_2 \oplus N_1 \xrightarrow{(x,y)\mapsto g_1 y + \varphi_2 x}$$
$$\xrightarrow{(x,y)\mapsto g_1 y + \varphi_2 x} N_2 \xrightarrow{x\mapsto f_3 g_2 x} M_4 \longrightarrow \cdots$$

are complexes. If, moreover, $(M_n)_{n\in\mathbb{Z}}$ and $(N_n)_{n\in\mathbb{Z}}$ are exact, then these two sequences are also exact.

Proof. First of all, the second sequence follows from the first by considering the chain homomorphism between complexes given by:

$$\begin{tikzcd}
\cdots \arrow[r, "f_{-3}"] & M_{-2} \arrow[r, "f_{-2}"] \arrow[d, "-\varphi_{-2}"] & M_{-1} \arrow[r, "-f_{-1}"] \arrow[d, "-\varphi_{-1}"] & M_0 \arrow[r, "f_0"] \arrow[d, equal] & M_1 \arrow[r, "f_1"] \arrow[d, "-\varphi_1"] & M_2 \arrow[r, "-f_2"] \arrow[d, "-\varphi_2"] & M_3 \arrow[r, "f_3"] \arrow[d, equal] & \cdots \\
\cdots \arrow[r, "-g_{-3}"] & N_{-2} \arrow[r, "g_{-2}"] & N_{-1} \arrow[r, "g_{-1}"] & N_0 \arrow[r, "-g_0"] & N_1 \arrow[r, "g_1"] & N_2 \arrow[r, "g_2"] & N_3 \arrow[r, "-g_3"] & \cdots
\end{tikzcd}$$

It is straightforward to verify that the sequence is a complex. So assume that $(M_n)_{n\in\mathbb{Z}}$ and $(N_n)_{n\in\mathbb{Z}}$ are exact sequences.

Diagram chases:

(1) Assume $M_1 \ni x \mapsto (f_1 x, \varphi_1 x) = 0 \in M_2 \oplus N_1$. Then we have

$$\begin{tikzcd}
(M_n)_{n\in\mathbb{Z}} \arrow[d, "(\varphi_n)_{n\in\mathbb{Z}}"] & & x_0 \arrow[r, mapsto, "1"] \arrow[d, mapsto, "2"] & x \arrow[r, mapsto, "0"] \arrow[d, mapsto, "0"] & 0 \\
(N_n)_{n\in\mathbb{Z}} & z \arrow[r, mapsto, "4"] & \varphi_0 x_0 = x_0 \arrow[r, mapsto, "3"] & 0,
\end{tikzcd}$$

and, consequently, $x = f_0 g_{-1} z$.

(2) Assume $M_2 \oplus N_1 \ni (x, y) \mapsto g_1 y - \varphi_2 x = 0 \in N_2$. Then we have

$$\begin{tikzcd}
(M_n)_{n\in\mathbb{Z}} \arrow[d, "(\varphi_n)_{n\in\mathbb{Z}}"] & x_1 \arrow[r, mapsto, "6"] & x \arrow[r, mapsto, "2"] \arrow[d, mapsto, "0"] & f_2 x \underset{5}{=} 0 \arrow[d, mapsto, "3"] \\
(N_n)_{n\in\mathbb{Z}} & y \arrow[r, mapsto, "0"] & g_1 y \underset{0}{=} \varphi_2 x \arrow[r, mapsto, "1"] & 0 \underset{4}{=} f_2 x,
\end{tikzcd}$$

$$\begin{tikzcd}
(M_n)_{n\in\mathbb{Z}} \arrow[d, "(\varphi_n)_{n\in\mathbb{Z}}"] & y_0 \arrow[r, mapsto, "10"] \arrow[d, mapsto, "9"] & f_0 y_0 \arrow[d, mapsto, "11"] & \\
(N_n)_{n\in\mathbb{Z}} & y_0 \arrow[r, mapsto, "8"] & y - \varphi_1 x_1 \arrow[r, mapsto, "7"] & 0.
\end{tikzcd}$$

and, consequently, $x = f_1x_1 = f_1(f_0y_0 + x_1)$ and $y = y - \varphi_1x_1 + \varphi_1x_1 = \varphi_1(f_0y_0 + x_1)$.

(3) Assume $N_2 \ni x \mapsto f_3g_2x = 0 \in M_4$. Then we have

$$\begin{tikzcd}
(M_n)_{n\in\mathbb{Z}} \arrow[d, "(\varphi_n)_{n\in\mathbb{Z}}"] & x_2 \arrow[r, mapsto, "1"] & g_2x \arrow[r, mapsto, "0"] \arrow[d, mapsto, "0"] & 0 \\
(N_n)_{n\in\mathbb{Z}} & x \arrow[r, mapsto, "0"] & g_2x, &
\end{tikzcd}$$

$$\begin{tikzcd}
(M_n)_{n\in\mathbb{Z}} \arrow[d, "(\varphi_n)_{n\in\mathbb{Z}}"] & & & \\
(N_n)_{n\in\mathbb{Z}} & y_1 \arrow[r, mapsto, "3"] & x - \varphi_2x_2 \arrow[r, mapsto, "2"] & 0,
\end{tikzcd}$$

and, consequently, $x = x - \varphi_2x_2 + \varphi_2x_2 = g_1y_1 - \varphi_2(-x_2)$. ●●●

Analogous to [Bon02, Propositionen 7.2.8 und 7.2.9] we have the following main result — but the proof is not analogous.

Theorem 5.1.8. *An object* $\mathbf{M}_\bullet$ *of* $\mathrm{Rep}(\Gamma, \rho)$ *is projective (resp. injective) if and only if* $\mathbf{M}_\bullet$ *is exact and* M_v *is a projective (resp. injective)* $\mathbb{Z}$*-module for all* $v \in \Gamma_0$.

Proof. The "only if" part has already been proved above (Proposition 5.1.6).

So assume that $\mathbf{M}_\bullet$ is exact and that M_v is a projective (resp. injective) $\mathbb{Z}$-module for all $v \in \Gamma_0$. As we saw in Example 4.3.3, we may visualize objects $\mathbf{L}_\bullet$ of $\mathrm{Rep}(\Gamma, \rho)$ as a commuting diagram

$$\begin{tikzcd}
 & & \arrow[d, "\partial_{12,2}"] & \arrow[d, "\partial_{12,4}"] & & \arrow[d, "\partial_{6,8}"] & \arrow[d, "\partial_{6,10}"] & \\
 \arrow[r, "\partial_{10,1}"] & L_1 \arrow[r] \arrow[d, equal] & L_2 \arrow[r] \arrow[d] & L_4 \arrow[r] \arrow[d] & L_7 \arrow[r] \arrow[d, equal] & L_8 \arrow[r] \arrow[d] & L_{10} \arrow[r, "\partial_{10,1}"] \arrow[d] & {} \\
 \arrow[r, "\partial_{11,1}"] & L_1 \arrow[r] & L_3 \arrow[r] \arrow[d] & L_5 \arrow[r] \arrow[d] & L_7 \arrow[r] & L_9 \arrow[r] \arrow[d] & L_{11} \arrow[r, "\partial_{11,1}"] \arrow[d] & {} \\
 & & L_6 \arrow[r, equal] \arrow[d, "\partial_{6,8}"] & L_6 \arrow[d, "\partial_{6,10}"] & & L_{12} \arrow[r, equal] \arrow[d, "\partial_{12,2}"] & L_{12} \arrow[d, "\partial_{12,4}"] & \\
 & & {} & {} & & {} & {} &
\end{tikzcd} \tag{5.1}$$

with the rows and columns being cyclic six term complexes, and with the additional conditions:

$$\partial_{6,8}\partial_{5,6} = \partial_{7,8}\partial_{5,7} \quad \text{and} \quad \partial_{12,2}\partial_{11,12} = \partial_{1,2}\partial_{11,1}$$

— and with the obvious notion of morphisms. We may equivalently write the diagram as:

$$\begin{tikzcd}
 & & \arrow[d, "\partial_{4,5}"] & \arrow[d, "\partial_{12,4}"] & & \arrow[d, "\partial_{10,11}"] & \arrow[d, "\partial_{10,1}"] & \\
 \arrow[r, "\partial_{1,3}"] & L_3 \arrow[r] \arrow[d, equal] & L_5 \arrow[r] \arrow[d] & L_7 \arrow[r] \arrow[d] & L_9 \arrow[r] \arrow[d, equal] & L_{11} \arrow[r] \arrow[d] & L_1 \arrow[r, "\partial_{1,3}"] \arrow[d] & {} \\
 \arrow[r, "\partial_{2,3}"] & L_3 \arrow[r] & L_6 \arrow[r] \arrow[d] & L_8 \arrow[r] \arrow[d] & L_9 \arrow[r] & L_{12} \arrow[r] \arrow[d] & L_2 \arrow[r, "\partial_{2,3}"] \arrow[d] & {} \\
 & & L_{10} \arrow[r, equal] \arrow[d, "\partial_{10,11}"] & L_{10} \arrow[d, "\partial_{10,1}"] & & L_4 \arrow[r, equal] \arrow[d, "\partial_{4,5}"] & L_4 \arrow[d, "\partial_{12,4}"] & \\
 & & {} & {} & & {} & {} &
\end{tikzcd} \tag{5.2}$$

with the additional conditions:

$$\partial_{3,5}\partial_{2,3} = \partial_{4,5}\partial_{2,4} \quad \text{and} \quad \partial_{10,11}\partial_{8,10} = \partial_{9,11}\partial_{8,9}.$$

Also the object $\mathbf{L}_\bullet$ is exact if and only if all of the four cyclic six term complexes are exact (in either of the diagrams). Although we in the sequel will switch between these two pictures, when convenient, we are always referring to the former picture, when we explicitly write out an object.

For every exact sequence $A \xrightarrow{f} B \xrightarrow{g} C$ of projective (resp. injective) $\mathbb{Z}$-modules,

$$0 \longrightarrow \operatorname{im} f \hookrightarrow B \xrightarrow{g} \operatorname{im} g \longrightarrow 0 \tag{5.3}$$

is exact. As a submodule of a projective $\mathbb{Z}$-module, $\operatorname{im} g$ is projective (resp. as a quotient of an injective $\mathbb{Z}$-module, $\operatorname{im} f$ is injective). Hence the sequence (5.3) splits, so $B = \operatorname{im} f \oplus \tilde{B}$ for some submodule $\tilde{B}$ of B. Note that $\tilde{B}$ is also projective (resp. injective), and that $g(b) = 0$ for a $b \in \tilde{B}$ only if $b = 0$.

The strategy of the proof is to write $\mathbf{M}_\bullet$ as a direct sum of 12 objects which are already known to be projective (resp. injective). The first main problem is to get started.

Step 1: Write

$$\begin{aligned} M_1 &= \operatorname{im} \partial_{11,1} \oplus \tilde{M}_1, \\ M_{12} &= \operatorname{im} \partial_{11,12} \oplus \tilde{M}_{12}. \end{aligned}$$

Clearly, $\operatorname{im} \partial_{10,1} \subseteq \operatorname{im} \partial_{11,1}$ and $\operatorname{im} \partial_{9,12} \subseteq \operatorname{im} \partial_{11,12}$. From the Mayer-Vietoris sequence, Proposition 5.1.7, we have a cyclic six term exact sequence

$$\longrightarrow M_{11} \longrightarrow M_{12} \oplus M_1 \longrightarrow M_2 \longrightarrow M_5 \longrightarrow M_6 \oplus M_7 \longrightarrow M_8 \longrightarrow$$

From this we get a decomposition

$$M_2 = (\operatorname{im} \partial_{1,2} + \operatorname{im} \partial_{12,2}) \oplus \tilde{M}_2.$$

Write

$$\begin{aligned} M_3 &= \operatorname{im} \partial_{2,3} \oplus \tilde{M}_3 \\ M_4 &= \operatorname{im} \partial_{2,4} \oplus \tilde{M}_4. \end{aligned}$$

Claims:

$$\operatorname{im} \partial_{1,2} = \operatorname{im}(\partial_{1,2}\partial_{11,1}) \oplus \partial_{1,2}\tilde{M}_1 \tag{5.4}$$
$$\operatorname{im} \partial_{12,2} = \operatorname{im}(\partial_{1,2}\partial_{11,1}) \oplus \partial_{12,2}\tilde{M}_{12} \tag{5.5}$$
$$M_2 = \operatorname{im}(\partial_{1,2}\partial_{11,1}) \oplus \partial_{1,2}\tilde{M}_1 \oplus \partial_{12,2}\tilde{M}_{12} \oplus \tilde{M}_2 \tag{5.6}$$
$$M_3 = \partial_{2,3}\tilde{M}_2 \oplus \partial_{1,3}\tilde{M}_1 \oplus \tilde{M}_3 \tag{5.7}$$
$$M_4 = \partial_{2,4}\tilde{M}_2 \oplus \partial_{12,4}\tilde{M}_{12} \oplus \tilde{M}_4. \tag{5.8}$$

"Equation (5.4)": Clearly, $\operatorname{im} \partial_{1,2} \supseteq \operatorname{im}(\partial_{1,2}\partial_{11,1}) + \partial_{1,2}\tilde{M}_1$. Let $z \in M_1$, and write $z = \partial_{11,1}x + y$, where $x \in M_{11}$ and $y \in \tilde{M}_1$. Then $\partial_{1,2}z = \partial_{1,2}\partial_{11,1}x + \partial_{1,2}y$, so $\operatorname{im} \partial_{1,2} \subseteq \operatorname{im}(\partial_{1,2}\partial_{11,1}) + \partial_{1,2}\tilde{M}_1$ also holds. Now let there be given $x \in M_{11}$ and $y \in \tilde{M}_1$ with $\partial_{1,2}\partial_{11,1}x = \partial_{1,2}y$. Then

$$\partial_{1,3}y = \partial_{2,3}\partial_{1,2}y = \partial_{2,3}\partial_{1,2}\partial_{11,1}x = \partial_{1,3}\partial_{11,1}x = 0,$$

and, consequently, this is a direct sum.

"Equation (5.5)": This is proven analogously.

"Equation (5.6)": It is enough to show that

$$\operatorname{im} \partial_{1,2} + \operatorname{im} \partial_{12,2} = \operatorname{im}(\partial_{1,2}\partial_{11,1}) \oplus \partial_{1,2}\tilde{M}_1 \oplus \partial_{12,2}\tilde{M}_{12}.$$

Clearly,

$$\operatorname{im}\partial_{1,2}+\operatorname{im}\partial_{12,2}\supseteq\operatorname{im}(\partial_{1,2}\partial_{11,1})+\partial_{1,2}\tilde{M}_1+\partial_{12,2}\tilde{M}_{12}. \tag{5.9}$$

On the other hand, let $z_1\in M_1$ and $z_{12}\in M_{12}$. Write $z_1=\partial_{11,1}x_1+y_1$ and $z_{12}=\partial_{11,12}x_{12}+y_{12}$, where $x_1,x_{12}\in M_{11}$, $y_1\in\tilde{M}_1$, and $y_{12}\in\tilde{M}_{12}$. Then

$$\partial_{1,2}z_1+\partial_{12,2}z_{12}=\partial_{1,2}\partial_{11,1}(x_1+x_{12})+\partial_{1,2}y_1+\partial_{12,2}y_{12}.$$

So "$\subseteq$" in Equation (5.9) holds as well. To prove that this is in fact a direct sum we need to show that the sum is unique. For this, let $x\in M_{11}$, $y\in\tilde{M}_1$, and $z\in\tilde{M}_{12}$ and assume that $w=\partial_{1,2}\partial_{11,1}x+\partial_{1,2}y+\partial_{12,2}z=0$. Then

$$0=\partial_{2,4}w=\partial_{2,4}\partial_{1,2}\partial_{11,1}x+\partial_{2,4}\partial_{1,2}y+\partial_{2,4}\partial_{12,2}z=\partial_{12,4}\partial_{11,12}x+0+\partial_{12,4}z=\partial_{12,4}z.$$

Hence $z=0$. Analogously, $y=0$, and hence $\partial_{1,2}\partial_{11,1}x=0$.

"Equation (5.7)": It is enough to show that

$$\operatorname{im}\partial_{2,3}=\partial_{2,3}\tilde{M}_2\oplus\partial_{1,3}\tilde{M}_1.$$

Clearly,

$$\operatorname{im}\partial_{2,3}\supseteq\partial_{2,3}\tilde{M}_2+\partial_{1,3}\tilde{M}_1. \tag{5.10}$$

On the other hand, let $z\in M_2$, and write

$$z=\partial_{1,2}\partial_{11,1}x_{11}+\partial_{1,2}x_1+\partial_{12,2}x_{12}+x_2,$$

where $x_{11}\in M_{11}$, $x_1\in\tilde{M}_1$, $x_{12}\in\tilde{M}_{12}$, and $x_2\in\tilde{M}_2$. Then

$$\partial_{2,3}z=\partial_{1,3}\partial_{11,1}x_{11}+\partial_{1,3}x_1+\partial_{2,3}\partial_{12,2}x_{12}+\partial_{2,3}x_2=0+\partial_{1,3}x_1+0+\partial_{2,3}x_2.$$

So "$\subseteq$" in Equation (5.10) also holds. Now assume that we have $x\in\tilde{M}_2$ and $y\in\tilde{M}_1$ such that $z=\partial_{2,3}x=\partial_{1,3}y$. Then $\partial_{3,5}\partial_{2,3}x=\partial_{3,5}\partial_{1,3}y=0$, so $x=0$ (according to the construction of $\tilde{M}_2$). Hence $z=0$, so this is a direct sum.

"Equation (5.8)": This claim is proven similarly.

Let $\mathbf{M}^1_\bullet$ be the object corresponding to

$$\begin{tikzcd}
{} & & {} \arrow[d] & {} \arrow[d] & & {} \arrow[d] & {} \arrow[d] & \\
{} \arrow[r] & \tilde{M}_1 \arrow[r] \arrow[d, equal] & \partial_{1,2}\tilde{M}_1 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \tilde{M}_1 \arrow[r] & \partial_{1,3}\tilde{M}_1 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & {} \\
& & 0 \arrow[r, equal] \arrow[d] & 0 \arrow[d] & & 0 \arrow[r, equal] \arrow[d] & 0 \arrow[d] & \\
& & {} & {} & & {} & {} &
\end{tikzcd}$$

and let $\mathbf{M}^{12}_\bullet$ be the object corresponding to

$$\begin{tikzcd}
{} & & {} \arrow[d] & {} \arrow[d] & & {} \arrow[d] & {} \arrow[d] & \\
{} \arrow[r] & 0 \arrow[r] \arrow[d, equal] & \partial_{12,2}\tilde{M}_{12} \arrow[r] \arrow[d] & \partial_{12,4}\tilde{M}_{12} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & {} \\
& & 0 \arrow[r, equal] \arrow[d] & 0 \arrow[d] & & \tilde{M}_{12} \arrow[r, equal] \arrow[d] & \tilde{M}_{12} \arrow[d] & \\
& & {} & {} & & {} & {} &
\end{tikzcd}$$

and let $\mathbf{M}'_\bullet$ be the object corresponding to

$$\begin{tikzcd}
& & {} \arrow[d] & {} \arrow[d, "0"] & & {} \arrow[d] & {} \arrow[d] & \\
{} \arrow[r] & \operatorname{im}\partial_{11,1} \arrow[r] \arrow[d, equal] & \operatorname{im}(\partial_{1,2}\partial_{11,1})\oplus\tilde M_2 \arrow[r] \arrow[d] & \partial_{2,4}\tilde M_2\oplus\tilde M_4 \arrow[r] \arrow[d] & M_7 \arrow[r] \arrow[d, equal] & M_8 \arrow[r] \arrow[d] & M_{10} \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \operatorname{im}\partial_{11,1} \arrow[r, "0"] & \partial_{2,3}\tilde M_2\oplus\tilde M_3 \arrow[r] \arrow[d] & M_5 \arrow[r] \arrow[d] & M_7 \arrow[r] & M_9 \arrow[r] \arrow[d] & M_{11} \arrow[r] \arrow[d] & {} \\
& & M_6 \arrow[r, equal] \arrow[d] & M_6 \arrow[d] & & \operatorname{im}\partial_{11,12} \arrow[r, equal] \arrow[d] & \operatorname{im}\partial_{11,12} \arrow[d, "0"] & \\
& & {} & {} & & {} & {} &
\end{tikzcd}$$

Claim: These are exact, and each entry is a projective (resp. injective) $\mathbb{Z}$-module. That the entries are projective (resp. injective) $\mathbb{Z}$-modules is clear. Moreover, these are clearly objects if we can show that the maps are well-defined. This is clear for $\mathbf{M}^1_\bullet$ and $\mathbf{M}^{12}_\bullet$. By checking that the image of each map is inside the stated codomain, this is also seen to be true for $\mathbf{M}'_\bullet$. To check that $\mathbf{M}^1_\bullet$ and $\mathbf{M}^{12}_\bullet$ are exact, we check that the three non-trivial maps in each object are isomorphisms (which is easy). A straightforward lengthy computation shows that $\mathbf{M}'_\bullet$ also is exact.

Moreover,

$$\mathbf{M}_\bullet = \mathbf{M}^1_\bullet \oplus \mathbf{M}^{12}_\bullet \oplus \mathbf{M}'_\bullet.$$

Step 2: Analogously, we show that (with M_3, M_4, and M_5 in M_1's, M_{12}'s, and M_2's rôle, resp.)

$$\mathbf{M}'_\bullet = \mathbf{M}^3_\bullet \oplus \mathbf{M}^4_\bullet \oplus \mathbf{M}''_\bullet,$$

where $\mathbf{M}^3_\bullet$ is the object corresponding to

$$\begin{tikzcd}
& & {} \arrow[d] & {} \arrow[d] & & {} \arrow[d] & {} \arrow[d] & \\
{} \arrow[r] & 0 \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & 0 \arrow[r] & \tilde M_3 \arrow[r] \arrow[d] & \partial_{3,5}\tilde M_3 \arrow[r] \arrow[d] & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & {} \\
& & \partial_{3,6}\tilde M_3 \arrow[r, equal] \arrow[d] & \partial_{3,6}\tilde M_3 \arrow[d] & & 0 \arrow[r, equal] \arrow[d] & 0, \arrow[d] & \\
& & {} & {} & & {} & {} &
\end{tikzcd}$$

$\mathbf{M}^4_\bullet$ is the object corresponding to

$$\begin{tikzcd}
& & {} \arrow[d] & {} \arrow[d] & & {} \arrow[d] & {} \arrow[d] & \\
{} \arrow[r] & 0 \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d] & \tilde M_4 \arrow[r] \arrow[d] & \partial_{4,7}\tilde M_4 \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & \partial_{4,5}\tilde M_4 \arrow[r] \arrow[d] & \partial_{4,7}\tilde M_4 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & {} \\
& & 0 \arrow[r, equal] \arrow[d] & 0 \arrow[d] & & 0 \arrow[r, equal] \arrow[d] & 0, \arrow[d] & \\
& & {} & {} & & {} & {} &
\end{tikzcd}$$

and $\mathbf{M}''_\bullet$ is the object corresponding to

$$\begin{tikzcd}
& & {} \arrow[d] & {} \arrow[d, "0"] & & {} \arrow[d] & {} \arrow[d] & \\
{} \arrow[r] & \operatorname{im}\partial_{11,1} \arrow[r] \arrow[d, equal] & \operatorname{im}(\partial_{1,2}\partial_{11,1})\oplus\tilde{M}_2 \arrow[r] \arrow[d] & \partial_{2,4}\tilde{M}_2 \arrow[r, "0"] \arrow[d] & \partial_{5,7}\tilde{M}_5\oplus\tilde{M}_7 \arrow[r] \arrow[d, equal] & M_8 \arrow[r] \arrow[d] & M_{10} \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \operatorname{im}\partial_{11,1} \arrow[r, "0"] & \partial_{2,3}\tilde{M}_2 \arrow[r] \arrow[d, "0"] & \operatorname{im}(\partial_{4,5}\partial_{2,4})\oplus\tilde{M}_5 \arrow[r] \arrow[d] & \partial_{5,7}\tilde{M}_5\oplus\tilde{M}_7 \arrow[r] & M_9 \arrow[r] \arrow[d] & M_{11} \arrow[r] \arrow[d] & {} \\
& & \partial_{5,6}\tilde{M}_5\oplus\tilde{M}_6 \arrow[r, equal] \arrow[d] & \partial_{5,6}\tilde{M}_5\oplus\tilde{M}_6 \arrow[d] & & \operatorname{im}\partial_{11,12} \arrow[r, equal] \arrow[d] & \operatorname{im}\partial_{11,12}. \arrow[d, "0"] & \\
& & {} & {} & & {} & {} &
\end{tikzcd}$$

These are all exact and all entries of these objects are projective (resp. injective) $\mathbb{Z}$-modules.

Step 3: Now we apply the same argument with M_7, M_6, and M_8 in M_1's, M_{12}'s, and M_2's rôle, resp. We see that

$$\mathbf{M}''_\bullet = \mathbf{M}^7_\bullet \oplus \mathbf{M}^6_\bullet \oplus \mathbf{M}'''_\bullet,$$

where $\mathbf{M}^6_\bullet$ is the object corresponding to

$$\begin{tikzcd}
& & {} \arrow[d] & {} \arrow[d] & & {} \arrow[d] & {} \arrow[d] & \\
{} \arrow[r] & 0 \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d, equal] & \partial_{6,8}\tilde{M}_6 \arrow[r] \arrow[d] & \partial_{6,10}\tilde{M}_6 \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & {} \\
& & \tilde{M}_6 \arrow[r, equal] \arrow[d] & \tilde{M}_6 \arrow[d] & & 0 \arrow[r, equal] \arrow[d] & 0, \arrow[d] & \\
& & {} & {} & & {} & {} &
\end{tikzcd}$$

$\mathbf{M}^7_\bullet$ is the object corresponding to

$$\begin{tikzcd}
& & {} \arrow[d] & {} \arrow[d] & & {} \arrow[d] & {} \arrow[d] & \\
{} \arrow[r] & 0 \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \tilde{M}_7 \arrow[r] \arrow[d, equal] & \partial_{7,8}\tilde{M}_7 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \tilde{M}_7 \arrow[r] & \partial_{7,9}\tilde{M}_7 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & {} \\
& & 0 \arrow[r, equal] \arrow[d] & 0 \arrow[d] & & 0 \arrow[r, equal] \arrow[d] & 0, \arrow[d] & \\
& & {} & {} & & {} & {} &
\end{tikzcd}$$

and $\mathbf{M}'''_\bullet$ is the object corresponding to

$$\begin{tikzcd}
& & {} \arrow[d] & {} \arrow[d, "0"] & & {} \arrow[d] & {} \arrow[d, "0"] & \\
{} \arrow[r] & \operatorname{im}\partial_{11,1} \arrow[r] \arrow[d, equal] & \operatorname{im}(\partial_{1,2}\partial_{11,1})\oplus\tilde{M}_2 \arrow[r] \arrow[d] & \partial_{2,4}\tilde{M}_2 \arrow[r, "0"] \arrow[d] & \partial_{5,7}\tilde{M}_5 \arrow[r] \arrow[d, equal] & \operatorname{im}(\partial_{7,8}\partial_{5,7})\oplus\tilde{M}_8 \arrow[r] \arrow[d] & \partial_{8,10}\tilde{M}_8\oplus\tilde{M}_{10} \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \operatorname{im}\partial_{11,1} \arrow[r, "0"] & \partial_{2,3}\tilde{M}_2 \arrow[r] \arrow[d, "0"] & \operatorname{im}(\partial_{4,5}\partial_{2,4})\oplus\tilde{M}_5 \arrow[r] \arrow[d] & \partial_{5,7}\tilde{M}_5 \arrow[r, "0"] & \partial_{8,9}\tilde{M}_8\oplus\tilde{M}_9 \arrow[r] \arrow[d] & M_{11} \arrow[r] \arrow[d] & {} \\
& & \partial_{5,6}\tilde{M}_5 \arrow[r, equal] \arrow[d] & \partial_{5,6}\tilde{M}_5 \arrow[d, "0"] & & \operatorname{im}\partial_{11,12} \arrow[r, equal] \arrow[d] & \operatorname{im}\partial_{11,12}. \arrow[d, "0"] & \\
& & {} & {} & & {} & {} &
\end{tikzcd}$$

These are all exact and all entries of these objects are projective (resp. injective) $\mathbb{Z}$-modules.

Step 4: Now we apply a similar argument (with M_9, M_{10}, and M_{11} in M_1's, M_{12}'s, and M_2's rôle, resp.). We see that

$$\mathbf{M}_\bullet''' = \mathbf{M}_\bullet^9 \oplus \mathbf{M}_\bullet^{10} \oplus \mathbf{M}_\bullet'''',$$

where $\mathbf{M}_\bullet^9$ is the object corresponding to

$$\begin{tikzcd}
 & & {} \arrow[d] & {} \arrow[d] & & {} \arrow[d] & {} \arrow[d] & \\
{} \arrow[r] & 0 \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] & \tilde{M}_9 \arrow[r] \arrow[d] & \partial_{9,11}\tilde{M}_9 \arrow[r] \arrow[d] & {} \\
 & & 0 \arrow[r, equal] \arrow[d] & 0 \arrow[d] & & \partial_{9,12}\tilde{M}_9 \arrow[r, equal] \arrow[d] & \partial_{9,12}\tilde{M}_9, \arrow[d] & \\
 & & {} & {} & & {} & {} &
\end{tikzcd}$$

$\mathbf{M}_\bullet^{10}$ is the object corresponding to

$$\begin{tikzcd}
 & & {} \arrow[d] & {} \arrow[d] & & {} \arrow[d] & {} \arrow[d] & \\
{} \arrow[r] & \partial_{10,1}\tilde{M}_{10} \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d] & \tilde{M}_{10} \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \partial_{10,1}\tilde{M}_{10} \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & \partial_{10,11}\tilde{M}_{10} \arrow[r] \arrow[d] & {} \\
 & & 0 \arrow[r, equal] \arrow[d] & 0 \arrow[d] & & 0 \arrow[r, equal] \arrow[d] & 0, \arrow[d] & \\
 & & {} & {} & & {} & {} &
\end{tikzcd}$$

and $\mathbf{M}_\bullet''''$ is the object corresponding to

$$\begin{tikzcd}
 & & {} \arrow[d] & {} \arrow[d, "0"] & & {} \arrow[d] & {} \arrow[d, "0"] & \\
{} \arrow[r, "0"] & \partial_{11,1}\tilde{M}_{11} \arrow[r] \arrow[d, equal] & \mathrm{im}(\partial_{1,2}\partial_{11,1}) \oplus \tilde{M}_2 \arrow[r] \arrow[d] & \partial_{2,4}\tilde{M}_2 \arrow[r, "0"] \arrow[d] & \partial_{5,7}\tilde{M}_5 \arrow[r] \arrow[d, equal] & \mathrm{im}(\partial_{7,8}\partial_{5,7}) \oplus \tilde{M}_8 \arrow[r] \arrow[d] & \partial_{8,10}\tilde{M}_8 \arrow[r, "0"] \arrow[d] & {} \\
{} \arrow[r] & \partial_{11,1}\tilde{M}_{11} \arrow[r, "0"] & \partial_{2,3}\tilde{M}_2 \arrow[r] \arrow[d, "0"] & \mathrm{im}(\partial_{4,5}\partial_{2,4}) \oplus \tilde{M}_5 \arrow[r] \arrow[d] & \partial_{5,7}\tilde{M}_5 \arrow[r, "0"] & \partial_{8,9}\tilde{M}_8 \arrow[r] \arrow[d, "0"] & \mathrm{im}(\partial_{10,11}\partial_{8,10}) \oplus \tilde{M}_{11} \arrow[r] \arrow[d] & {} \\
 & & \partial_{5,6}\tilde{M}_5 \arrow[r, equal] \arrow[d] & \partial_{5,6}\tilde{M}_5 \arrow[d, "0"] & & \partial_{11,12}\tilde{M}_{11} \arrow[r, equal] \arrow[d] & \partial_{11,12}\tilde{M}_{11}. \arrow[d, "0"] & \\
 & & {} & {} & & {} & {} &
\end{tikzcd}$$

These are all exact and all entries of these objects are projective (resp. injective) $\mathbb{Z}$-modules.

Step 5: Now, it is elementary to see that

$$\mathbf{M}_\bullet'''' = \mathbf{M}_\bullet^2 \oplus \mathbf{M}_\bullet^5 \oplus \mathbf{M}_\bullet^8 \oplus \mathbf{M}_\bullet^{11},$$

where $\mathbf{M}_\bullet^2$ is the object corresponding to

$$\begin{tikzcd}
& & {} \arrow[d] & {} \arrow[d] & & {} \arrow[d] & {} \arrow[d] & \\
{} \arrow[r] & 0 \arrow[r] \arrow[d, equal] & \tilde{M}_2 \arrow[r] \arrow[d] & \partial_{2,4}\tilde{M}_2 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & 0 \arrow[r] & \partial_{2,3}\tilde{M}_2 \arrow[r] \arrow[d] & \mathrm{im}(\partial_{4,5}\partial_{2,4}) \arrow[r] \arrow[d] & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & {} \\
& & 0 \arrow[r, equal] \arrow[d] & 0 \arrow[d] & & 0 \arrow[r, equal] \arrow[d] & 0, \arrow[d] & \\
& & {} & {} & & {} & {} &
\end{tikzcd}$$

$\mathbf{M}_\bullet^5$ is the object corresponding to

$$\begin{tikzcd}
& & {} \arrow[d] & {} \arrow[d] & & {} \arrow[d] & {} \arrow[d] & \\
{} \arrow[r] & 0 \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \partial_{5,7}\tilde{M}_5 \arrow[r] \arrow[d, equal] & \mathrm{im}(\partial_{7,8}\partial_{5,7}) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & \tilde{M}_5 \arrow[r] \arrow[d] & \partial_{5,7}\tilde{M}_5 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & {} \\
& & \partial_{5,6}\tilde{M}_5 \arrow[r, equal] \arrow[d] & \partial_{5,6}\tilde{M}_5 \arrow[d] & & 0 \arrow[r, equal] \arrow[d] & 0, \arrow[d] & \\
& & {} & {} & & {} & {} &
\end{tikzcd}$$

$\mathbf{M}_\bullet^8$ is the object corresponding to

$$\begin{tikzcd}
& & {} \arrow[d] & {} \arrow[d] & & {} \arrow[d] & {} \arrow[d] & \\
{} \arrow[r] & 0 \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d, equal] & \tilde{M}_8 \arrow[r] \arrow[d] & \partial_{8,10}\tilde{M}_8 \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] & \partial_{8,9}\tilde{M}_8 \arrow[r] \arrow[d] & \mathrm{im}(\partial_{10,11}\partial_{8,10}) \arrow[r] \arrow[d] & {} \\
& & 0 \arrow[r, equal] \arrow[d] & 0 \arrow[d] & & 0 \arrow[r, equal] \arrow[d] & 0, \arrow[d] & \\
& & {} & {} & & {} & {} &
\end{tikzcd}$$

and $\mathbf{M}_\bullet^{11}$ is the object corresponding to

$$\begin{tikzcd}
& & {} \arrow[d] & {} \arrow[d] & & {} \arrow[d] & {} \arrow[d] & \\
{} \arrow[r] & \partial_{11,1}\tilde{M}_{11} \arrow[r] \arrow[d, equal] & \mathrm{im}(\partial_{1,2}\partial_{11,1}) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \partial_{11,1}\tilde{M}_{11} \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & \tilde{M}_{11} \arrow[r] \arrow[d] & {} \\
& & 0 \arrow[r, equal] \arrow[d] & 0 \arrow[d] & & \partial_{11,12}\tilde{M}_{11} \arrow[r, equal] \arrow[d] & \partial_{11,12}\tilde{M}_{11}. \arrow[d] & \\
& & {} & {} & & {} & {} &
\end{tikzcd}$$

Step 6: Now we have written $\mathbf{M}_\bullet$ as a direct sum of standard projective (resp. injective) objects:

$$\mathbf{M}_\bullet = \bigoplus_{i=1}^{12} \mathbf{M}_\bullet^i.$$

Hence $\mathbf{M}_\bullet$ is projective (resp. injective). ●●●

From [Bon02, Lemma 7.2.13] we have the following lemma:

Lemma 5.1.9. *Let* $\mathbf{L}_\bullet \hookrightarrow \mathbf{M}_\bullet \twoheadrightarrow \mathbf{N}_\bullet$ *be a short exact sequence of objects of* $\operatorname{Rep}(\Gamma, \rho)$*. If two of the objects are exact, then all three are exact.*

Proof. This follows directly from the corresponding statement for chain complexes $\mathbf{L}_\bullet$, $\mathbf{M}_\bullet$, and $\mathbf{N}_\bullet$ — which is proved by diagram chase (*cf.* also [Bon02, Lemma 7.2.13 and Lemma A.1.2]). ●●●

Analogous to [Bon02, Korollar 7.2.14], we characterize the exact objects.

Proposition 5.1.10. *Let* $\mathbf{M}_\bullet$ *be an element of* $\operatorname{Rep}(\Gamma, \rho)$*. The following are equivalent:*

(1) The object $\mathbf{M}_\bullet$ *is exact.*

(2) The projective dimension of $\mathbf{M}_\bullet$ *is at most* 1*.*

(3) The projective dimension of $\mathbf{M}_\bullet$ *is finite.*

(4) The injective dimension of $\mathbf{M}_\bullet$ *is at most* 1*.*

(5) The injective dimension of $\mathbf{M}_\bullet$ *is finite.*

Proof. The proof of this is analogous to the proof of [Bon02, Korollar 7.2.14].

"(1)⇒(2)": Assume that $\mathbf{M}_\bullet$ is exact. Let $\mathbf{P}_\bullet$ be a projective object and $\boldsymbol{\phi}_\bullet \colon \mathbf{P}_\bullet \to \mathbf{M}_\bullet$ an epic morphism. Then $\ker \boldsymbol{\phi}_\bullet \hookrightarrow \mathbf{P}_\bullet \twoheadrightarrow \mathbf{M}_\bullet$ is a short exact sequence. The object $\mathbf{P}'_\bullet = \ker \boldsymbol{\phi}_\bullet$ is exact (*cf.* Corollary 5.1.5 and Lemma 5.1.9), and P'_v is projective for all $v \in \Gamma_0$ (because P'_v is a subgroup of the projective abelian group P_v, *cf.* Proposition 5.1.6). Consequently, $\mathbf{P}'_\bullet = \ker \boldsymbol{\phi}_\bullet$ is projective (*cf.* Theorem 5.1.8).

"(2)⇒(3)": Trivial.

"(3)⇒(1)": Let there be given an exact sequence:

$$0 \longrightarrow \mathbf{P}^n_\bullet \xrightarrow{\boldsymbol{\varphi}^n_\bullet} \mathbf{P}^{n-1}_\bullet \xrightarrow{\boldsymbol{\varphi}^{n-1}_\bullet} \cdots \xrightarrow{\boldsymbol{\varphi}^2_\bullet} \mathbf{P}^1_\bullet \xrightarrow{\boldsymbol{\varphi}^1_\bullet} \mathbf{P}^0_\bullet \xrightarrow{\boldsymbol{\varphi}^0_\bullet} \mathbf{M}_\bullet \longrightarrow 0,$$

where $n \in \mathbb{N}_0$ and $\mathbf{P}^0_\bullet, \ldots, \mathbf{P}^n_\bullet$ are projective objects. Then we have a short exact sequence

$$\ker \boldsymbol{\varphi}^k_\bullet \hookrightarrow \mathbf{P}^k_\bullet \overset{\boldsymbol{\varphi}^k_\bullet}{\twoheadrightarrow} \operatorname{im} \boldsymbol{\varphi}^k_\bullet$$

for each $k = 0, 1, \ldots, n$. We have that $\mathbf{P}^0_\bullet, \mathbf{P}^1_\bullet, \ldots, \mathbf{P}^{n-1}_\bullet, \mathbf{P}^n_\bullet \cong \operatorname{im} \boldsymbol{\varphi}^n_\bullet$ are exact (*cf.* Corollary 5.1.5). So, using $\operatorname{im} \boldsymbol{\varphi}^k_\bullet = \ker \boldsymbol{\varphi}^{k-1}_\bullet$, for $k = 1, \ldots, n$, and Lemma 5.1.9, we see by induction that $\ker \boldsymbol{\varphi}^k_\bullet$ is exact for all $k = 0, 1, \ldots, n$. So now it follows from the short exact sequence

$$\ker \boldsymbol{\varphi}^0_\bullet \hookrightarrow \mathbf{P}^0_\bullet \overset{\boldsymbol{\varphi}^0_\bullet}{\twoheadrightarrow} \operatorname{im} \boldsymbol{\varphi}^0_\bullet = \mathbf{M}_\bullet$$

and Lemma 5.1.9 that $\mathbf{M}_\bullet$ is exact.

The proofs of "(1)⇒(4)", "(4)⇒(5)", and "(5)⇒(1)" are dual. ●●●

5.2 Geometric resolution

As in the proof of the usual UCT, we need to be able to construct geometric resolutions for each object $\mathfrak{A}_\bullet$ (both projective and injective), *i.e.*, we need to construct a projective (resp. injective) resolution of $K_\circledast(\mathfrak{A}_\bullet)$ coming from a short exact sequence of C^*-algebras. In the construction of the geometric resolutions, we need some definitions and some basic C^*-algebra results, which will be given in the first subsection — some of these results may already be known by the reader. In the second subsection, we construct the geometric resolutions.

5.2.1 Some preliminaries

Definition 5.2.1. Let $\mathfrak{A}_\bullet \hookrightarrow \mathfrak{B}_\bullet \twoheadrightarrow \mathfrak{C}_\bullet$ be a short exact sequence of objects of $\mathcal{SE}_2$. Define the morphisms $\boldsymbol{\Delta}_\bullet^\circledast \colon K_\circledast(\mathfrak{C}_\bullet) \to K_{\circledast+1}(\mathfrak{A}_\bullet)$ and $\boldsymbol{\Delta}_\bullet^{\circledast+1} \colon K_{\circledast+1}(\mathfrak{C}_\bullet) \to K_\circledast(\mathfrak{A}_\bullet)$ as follows. We let $\Delta_i^\circledast = \partial_0^{e_i}$ and $\Delta_{i+6}^\circledast = \partial_1^{e_i}$, for $i = 1, 2, 3, 4, 5, 6$, and similarly we let $\Delta_i^{\circledast+1} = \partial_1^{e_i}$ and $\Delta_{i+6}^{\circledast+1} = \partial_0^{e_i}$, where e_i is the extension $\mathfrak{A}_i \hookrightarrow \mathfrak{B}_i \twoheadrightarrow \mathfrak{C}_i$, for $i = 1, \ldots, 6$.[1]

As in [Bon02, Lemma 7.3.1], every short exact sequence of objects of $\mathcal{SE}_2$ induces a cyclic six term exact sequence in the invariant:

Lemma 5.2.2. *Let $\mathfrak{A}_\bullet \overset{\boldsymbol{\Phi}_\bullet}{\hookrightarrow} \mathfrak{B}_\bullet \overset{\boldsymbol{\Pi}_\bullet}{\twoheadrightarrow} \mathfrak{C}_\bullet$ be a short exact sequence of objects of $\mathcal{SE}_2$. Then we have the following cyclic six term exact sequence*

$$\begin{tikzcd}
K_\circledast(\mathfrak{A}_\bullet) \arrow[r, "K_\circledast(\boldsymbol{\Phi}_\bullet)"] & K_\circledast(\mathfrak{B}_\bullet) \arrow[r, "K_\circledast(\boldsymbol{\Pi}_\bullet)"] & K_\circledast(\mathfrak{C}_\bullet) \arrow[d, "\boldsymbol{\Delta}_\bullet^\circledast"] \\
K_{\circledast+1}(\mathfrak{C}_\bullet) \arrow[u, "\boldsymbol{\Delta}_\bullet^{\circledast+1}"] & K_{\circledast+1}(\mathfrak{B}_\bullet) \arrow[l, "K_{\circledast+1}(\boldsymbol{\Pi}_\bullet)"] & K_{\circledast+1}(\mathfrak{A}_\bullet) \arrow[l, "K_{\circledast+1}(\boldsymbol{\Phi}_\bullet)"]
\end{tikzcd}$$

Proof. The only part which is not obvious is that $\boldsymbol{\Delta}_\bullet^\circledast$ and $\boldsymbol{\Delta}_\bullet^{\circledast+1}$ are morphisms. This follows from Lemma 3.3.1 and Section 3.4. Here the difference of signs for ∂_i and δ_i (*i.e.* $\partial_i = (-1)^i\delta_i$) is, of course, crucial. ●●●

Definition 5.2.3 (Mapping cones)**.** Let $\boldsymbol{\Phi}_\bullet \colon \mathfrak{A}_\bullet \to \mathfrak{B}_\bullet$ a morphism between objects of $\mathcal{SE}_2$. Then the **mapping cone**, $\mathsf{C}_{\boldsymbol{\Phi}_\bullet}$, of $\boldsymbol{\Phi}_\bullet$ is the object

$$\begin{tikzcd}
\mathsf{C}_{\Phi_1} \arrow[r, hook] \arrow[d, equal] & \mathsf{C}_{\Phi_2} \arrow[r, two heads] \arrow[d, hook] & \mathsf{C}_{\Phi_4} \arrow[d, hook] \\
\mathsf{C}_{\Phi_1} \arrow[r, hook] & \mathsf{C}_{\Phi_3} \arrow[r, two heads] \arrow[d, two heads] & \mathsf{C}_{\Phi_5} \arrow[d, two heads] \\
& \mathsf{C}_{\Phi_6} \arrow[r, equal] & \mathsf{C}_{\Phi_6}
\end{tikzcd}$$

where C_{Φ_i} is the mapping cone of Φ_i, for $i = 1, \ldots, 6$. It follows from [Bon02, Korollar A.1.5] that this diagram consists of short exact sequences (both horizontally and vertically). It is easy to show that this diagram is commutative (using the concrete description of the maps $\mathsf{C}_{\Phi_i} \to \mathsf{C}_{\Phi_j}$ mentioned in Section 3.1).

Remark 5.2.4. Let $\boldsymbol{\Omega}_\bullet \colon \mathsf{C}_{\boldsymbol{\Phi}_\bullet} \to \mathfrak{A}_\bullet$ be the standard morphism (the naturality of the mapping cone implies that this is a morphism). The kernel of $\boldsymbol{\Omega}_\bullet$ is canonically isomorphic to $\mathsf{S}\mathfrak{B}_\bullet$, so we have a short exact sequence

$$\mathsf{S}\mathfrak{B}_\bullet \hookrightarrow \mathsf{C}_{\boldsymbol{\Phi}_\bullet} \overset{\boldsymbol{\Omega}_\bullet}{\twoheadrightarrow} \mathfrak{A}_\bullet.$$

This sequence is natural in $\mathfrak{A}_\bullet$ and $\mathfrak{B}_\bullet$, *i.e.*, if we have a commuting diagram

$$\begin{tikzcd}
\mathfrak{A}_\bullet \arrow[r, "\boldsymbol{\Phi}_\bullet"] \arrow[d, "\boldsymbol{f}_\bullet"'] & \mathfrak{B}_\bullet \arrow[d, "\boldsymbol{g}_\bullet"] \\
\mathfrak{A}'_\bullet \arrow[r, "\boldsymbol{\Phi}'_\bullet"] & \mathfrak{B}'_\bullet
\end{tikzcd}$$

[1] Here ∂_0 and ∂_1 are the connecting homomorphisms defined as in Section 3.2, and, as seen in Section 3.4, we have $\partial_0 = \delta_0$ and $\partial_1 = -\delta_1$ (up to the standard identification θ_- of K_1 with $K_0 \circ \mathsf{S}$).

then there is a (canonical) homomorphism $\boldsymbol{\omega}_\bullet \colon \mathsf{C}_{\boldsymbol{\Phi}_\bullet} \to \mathsf{C}_{\boldsymbol{\Phi}'_\bullet}$ making the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathsf{S}\mathfrak{B}_\bullet \arrow[r] \arrow[d, "\mathsf{S}g_\bullet"'] & \mathsf{C}_{\boldsymbol{\Phi}_\bullet} \arrow[r] \arrow[d, "\boldsymbol{\omega}_\bullet"] & \mathfrak{A}_\bullet \arrow[r] \arrow[d, "\boldsymbol{f}_\bullet"] & 0 \\ 0 \arrow[r] & \mathsf{S}\mathfrak{B}'_\bullet \arrow[r] & \mathsf{C}_{\boldsymbol{\Phi}'_\bullet} \arrow[r] & \mathfrak{A}'_\bullet \arrow[r] & 0 \end{tikzcd}$$

commutative. If we have an extension

$$\mathfrak{A}_\bullet \hookrightarrow \mathfrak{B}_\bullet \xrightarrow{\boldsymbol{\pi}_\bullet} \mathfrak{C}_\bullet$$

of objects of $S\mathcal{E}_2$, then we get a commuting diagram

$$\begin{tikzcd} 0 \arrow[r, hook] \arrow[d, hook] & \mathsf{S}\mathfrak{C}_\bullet \arrow[r, equal] \arrow[d, hook] & \mathsf{S}\mathfrak{C}_\bullet \arrow[d, hook] \\ \mathfrak{A}_\bullet \arrow[r, hook, "\boldsymbol{f}_\bullet"] \arrow[d, equal] & \mathsf{C}_{\boldsymbol{\pi}_\bullet} \arrow[r, two heads] \arrow[d, two heads] & \mathsf{C}\mathfrak{C}_\bullet \arrow[d, two heads] \\ \mathfrak{A}_\bullet \arrow[r, hook] & \mathfrak{B}_\bullet \arrow[r, two heads, "\boldsymbol{\pi}_\bullet"] & \mathfrak{C}_\bullet \end{tikzcd}$$

with short exact rows and columns. The map $\boldsymbol{f}_\bullet \colon \mathfrak{A}_\bullet \to \mathsf{C}_{\boldsymbol{\pi}_\bullet}$ induces isomorphism on the level of K-theory.

In [Cun98, Section 2], Cuntz constructs a universal extension for each separable C^*-algebra. This construction will be useful for constructing geometric resolutions — for convenience we will restate the result here.

Proposition 5.2.5. *Let $\mathfrak{A}_2$ be a separable C^*-algebra. Then there exists an extension*

$$J\mathfrak{A}_2 \hookrightarrow T\mathfrak{A}_2 \twoheadrightarrow \mathfrak{A}_2$$

in $S\mathcal{E}_2$ with the property that, given any extension

$$\mathfrak{A}_0 \hookrightarrow \mathfrak{A}_1 \twoheadrightarrow \mathfrak{A}_2$$

in $S\mathcal{E}_2$ there is a morphism of extensions

$$\begin{tikzcd} 0 \arrow[r] & J\mathfrak{A}_2 \arrow[r] \arrow[d] & T\mathfrak{A}_2 \arrow[r] \arrow[d] & \mathfrak{A}_2 \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathfrak{A}_0 \arrow[r] & \mathfrak{A}_1 \arrow[r] & \mathfrak{A}_2 \arrow[r] & 0. \end{tikzcd}$$

Moreover, $T\mathfrak{A}_2$ is contractible, and hence has trivial K-theory.

Definition 5.2.6. When $\mathfrak{A}$ is a C^*-algebra, we let $\tilde{\mathfrak{A}}$ denote the C^*-algebra obtained from $\mathfrak{A}$ by adjoining a unit (even if $\mathfrak{A}$ is unital). Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras and let $\varphi\colon \mathfrak{A} \to \mathfrak{B}$ be a $*$-homomorphism. Then there exists a unique $*$-homomorphism $\tilde{\varphi}\colon \tilde{\mathfrak{A}} \to \tilde{\mathfrak{B}}$ such that

$$\begin{tikzcd} \mathfrak{A} \arrow[r] \arrow[d, "\varphi"] & \tilde{\mathfrak{A}} \arrow[r] \arrow[d, "\tilde{\varphi}"] & \mathbb{C} \arrow[d, equal] \\ \mathfrak{B} \arrow[r] & \tilde{\mathfrak{B}} \arrow[r] & \mathbb{C} \end{tikzcd}$$

commutes, where the left hand horizontal maps are the canonical embeddings, and the right hand horizontal maps are the corresponding quotient maps.

In [RLL00, Section 8.1], K_1 of a C^*-algebra $\mathfrak{A}$ is defined as $\mathcal{U}_\infty(\tilde{\mathfrak{A}})/\sim_1$, where $\mathcal{U}_\infty(\tilde{\mathfrak{A}}) = \bigcup_{n=1}^\infty \mathcal{U}_n(\tilde{\mathfrak{A}})$, $\mathcal{U}_n(\tilde{\mathfrak{A}})$ is the unitary group of $\mathrm{Mat}_n(\tilde{\mathfrak{A}})$, and $u \sim_1 v$ if and only if $u \oplus \mathbb{1}_{r-m} \sim_h v \oplus \mathbb{1}_{r-n}$ in $\mathcal{U}_r(\tilde{\mathfrak{A}})$ for an $r \geq m, n$ whenever $u \in \mathcal{U}_m(\tilde{\mathfrak{A}})$ and $v \in \mathcal{U}_n(\tilde{\mathfrak{A}})$. We will need the following description.

Lemma 5.2.7. *Let $\mathfrak{A}$ be a C^*-algebra. Then*

$$K_1(\mathfrak{A}) = \{[u]_1 \in K_1(\mathfrak{A}) \mid \exists n \in \mathbb{N} \exists a \in \mathrm{Mat}_n(\mathfrak{A})\colon u = \mathbb{1} + a \in \mathcal{U}_n(\tilde{\mathfrak{A}})\}.$$

Proof. Let $u \in \mathcal{U}_n(\tilde{\mathfrak{A}})$ be given. Then it is enough to find some $a \in \mathrm{Mat}_n(\mathfrak{A})$ such that $\mathbb{1} + a \in \mathcal{U}_n(\tilde{\mathfrak{A}})$ and $u \sim_1 \mathbb{1} + a$. We proceed in two steps.

Write $u = \lambda + a$ where $\lambda \in \mathrm{Mat}_n(\mathbb{C}\mathbb{1}_{\tilde{\mathfrak{A}}}) \cong \mathrm{Mat}_n(\mathbb{C})$ and $a \in \mathrm{Mat}_n(\mathfrak{A})$. Then it is easy to verify that λ is unitary. So there exists a unitary w in $\mathrm{Mat}_n(\mathbb{C}\mathbb{1}_{\tilde{\mathfrak{A}}})$ such that $w\lambda w^*$ is diagonal. So

$$u = w^* w u \sim_1 w u w^* = w\lambda w^* + w a w^*.$$

So we may w.l.o.g. assume that λ is diagonal: $\lambda = \mathrm{diag}(e^{2\pi i\theta_1}\mathbb{1}_{\tilde{\mathfrak{A}}}, \ldots, e^{2\pi i\theta_n}\mathbb{1}_{\tilde{\mathfrak{A}}})$, for some $\theta_1, \ldots, \theta_n$ in $[0, 1)$. Let

$$\lambda_t = \mathrm{diag}(e^{-2\pi i\theta_1 t}\mathbb{1}_{\tilde{\mathfrak{A}}}, \ldots, e^{-2\pi i\theta_n t}\mathbb{1}_{\tilde{\mathfrak{A}}}),$$

for $t \in [0, 1]$, and let $u_t = \lambda_t u$, for $t \in [0, 1]$. Then $[0, 1] \ni t \mapsto u_t \in \mathcal{U}_n(\tilde{\mathfrak{A}})$ is continuous, $u_1 = \mathbb{1}_{\tilde{\mathfrak{A}}} + \lambda_1 a$, and $u_0 = u$. So u_t is a homotopy (in $\mathcal{U}_n(\tilde{\mathfrak{A}})$) from u to u_1. ●●●

Proposition 5.2.8. *We have a $*$-isomorphism*

$$\{g \in C(\mathbb{T}) \mid g(1) = 0\} \ni f \mapsto [\theta \mapsto f(e^{2\pi i\theta})] \in C_0((0, 1)).$$

Identifying $C_0((0, 1))$ with its image, $C_0(\mathbb{T} \setminus \{1\})$, in $C(\mathbb{T})$ under this $*$-homomorphism, we consider $C(\mathbb{T})$ as $\widetilde{C_0((0, 1))}$.

Let there be given a C^*-algebra $\mathfrak{A}$. Then the map

$$\begin{aligned}\{\varphi\colon C_0(\mathbb{T} \setminus \{1\}) \to \mathfrak{A} \mid \varphi \textit{ is a } *\textit{-homomorphism}\} \ni \\ \varphi \mapsto \tilde{\varphi}([z \mapsto z]) \\ \in \{u \in \mathcal{U}(\tilde{\mathfrak{A}}) \mid \mathbb{1}_{\tilde{\mathfrak{A}}} - u \in \mathfrak{A}\}\end{aligned}$$

is bijective. Moreover, if $\varphi\colon C_0(\mathbb{T} \setminus \{1\}) \to \mathfrak{A}$ is a $*$-homomorphism, then

$$K_1(\varphi)\colon K_1(C_0(\mathbb{T} \setminus \{1\})) \to K_1(\mathfrak{A})$$

is given on the generator $[z \mapsto z]_1$ of $K_1(C_0((0, 1)))(\cong \mathbb{Z})$ as

$$K_1(\varphi)([z \mapsto z]_1) = [\tilde{\varphi}(z \mapsto z)]_1.$$

Proof. Clearly, $f_0 = [z \mapsto z]$ is a unitary in $C(\mathbb{T})$. Let $\varphi\colon C_0(\mathbb{T}\setminus\{1\}) \to \mathfrak{A}$ be a given $*$-homomorphism. Since $\tilde{\varphi}$ is unital, $\tilde{\varphi}(f_0)$ is a unitary in $\tilde{\mathfrak{A}}$. Moreover, $(f_0 - \mathbb{1}_{C(\mathbb{T})})(1) = 0$ and

$$\mathfrak{A} \ni \varphi(f_0 - \mathbb{1}_{C(\mathbb{T})}) = \tilde{\varphi}(f_0 - \mathbb{1}_{C(\mathbb{T})}) = \tilde{\varphi}(f_0) - \mathbb{1}_{\tilde{\mathfrak{A}}}.$$

If $\psi\colon C_0(\mathbb{T}\setminus\{1\}) \to \mathfrak{A}$ satisfies $\tilde{\varphi}(f_0) = \tilde{\psi}(f_0)$, then $\varphi = \psi$ (because f_0 generates $C(\mathbb{T})$ as a C^*-algebra).

Now suppose that $u = \mathbb{1}_{\mathfrak{A}} + a$, for an $a \in \mathfrak{A}$, is unitary. Using Gelfand's Theorem one easily shows that $C(\mathbb{T}) = C^*(f_0)$ is the universal C^*-algebra generated by a unitary. Thus there is a unique $*$-homomorphism $\varphi\colon C(\mathbb{T}) \to \tilde{\mathfrak{A}}$ such that $\varphi(f_0) = u$. Clearly, $\varphi|_{C_0(\mathbb{T}\setminus\{1\})}$ is a $*$-homomorphism. Let $f \in C_0(\mathbb{T} \setminus \{1\})$ be given. To show that $\varphi|_{C_0(\mathbb{T}\setminus\{1\})}$ is a $*$-homomorphism from $C_0(\mathbb{T} \setminus \{1\})$ to $\mathfrak{A}$, we only need to show that $\varphi(f) \in \mathfrak{A}$. For each $\varepsilon > 0$, we can find $N \in \mathbb{N}$ and $a_{-N}, \ldots, a_N \in \mathbb{C}$ such that $\|f - g\|_u < \varepsilon$, where $g = \sum_{n=-N}^N a_n f_0^n$, $g(1) = 0$ (by Stone-Weierstraß' Theorem for locally compact spaces, see *e.g.* [Ped89]). We have

$$\varphi(g) = \sum_{n=-N}^{N} a_n \varphi(f_0)^n = \sum_{n=-N}^{N} a_n u^n = a' + \sum_{n=-N}^{N} a_n \mathbb{1}_{\tilde{\mathfrak{A}}} = a',$$

for some $a' \in \mathfrak{A}$, so $\varphi(g) = a' \in \mathfrak{A}$. By continuity, $\varphi(f) \in \mathfrak{A}$.

The rest of the proposition is well-known. ●●●

Lemma 5.2.9. *Let $\mathfrak{B}$ be a separable C^*-algebra, and let S be a countable set of generators for $K_1(\mathfrak{B})$. Then there exists a $*$-homomorphism $\varphi\colon \bigoplus_{s\in S} C_0((0,1)) \to \mathfrak{B}\otimes\mathbb{K}$ such that*

$$K_1(\varphi)\colon K_1(\bigoplus_{s\in S} C_0((0,1))) \to K_1(\mathfrak{B}\otimes\mathbb{K}) \cong K_1(\mathfrak{B})$$

is surjective. Moreover, $K_1(\bigoplus_{s\in S} C_0((0,1))) \cong \bigoplus_{s\in S}\mathbb{Z}$.

Proof. Let $f_0 = [z\mapsto z] \in C(\mathbb{T})$. Since $\mathfrak{B}$ is separable, $K_1(\mathfrak{B})$ is countably generated. For each $s\in S$ there is a unitary $u_s \in \widetilde{\operatorname{Mat}_{n_s}}(\mathfrak{B})$ such that $\mathbb{1}_{\widetilde{\operatorname{Mat}_{n_s}(\mathfrak{B})}} - u_s \in \operatorname{Mat}_{n_s}(\mathfrak{B})$ and $s = [u_s]_1$ (use Lemma 5.2.7).

According to Proposition 5.2.8 there is, for each $s\in S$, a $*$-homomorphism φ_s from $C_0((0,1))$ to $\operatorname{Mat}_{n_s}(\mathfrak{B})$ such that $K_1(\varphi_s)([f_0]_1) = [u_s]_1$, where we view $C(\mathbb{T})$ as the unitization of the C^*-algebra $C_0((0,1)) \cong C_0(\mathbb{T}\setminus\{1\})$ (*cf.* also Proposition 5.2.8).

Define $\psi\colon \bigoplus_{s\in S} C_0((0,1)) \to \bigoplus_{s\in S}\operatorname{Mat}_{n_s}(\mathfrak{B})$ as the direct sum $\bigoplus_{s\in S}\varphi_s$. If S is finite, then we clearly have an embedding

$$\iota\colon \bigoplus_{s\in S}\operatorname{Mat}_{n_s}(\mathfrak{B}) \hookrightarrow \operatorname{Mat}_{\sum_{s\in S} n_s}(\mathfrak{B}) \hookrightarrow \mathfrak{B}\otimes\mathbb{K}.$$

In this case, let $\varphi = \iota\circ\psi$.

Now assume the S is infinite. Choose a bijection $\alpha\colon \mathbb{N}\to S$. Let $N_i = \sum_{j=1}^{i} n_{\alpha(j)}$. It is well-known, that the inductive limit of $\operatorname{Mat}_{N_1}(\mathfrak{B}) \to \operatorname{Mat}_{N_2}(\mathfrak{B}) \to \cdots$ is (isomorphic to) $\mathfrak{B}\otimes\mathbb{K}$. For each $i\in\mathbb{N}$ we have a canonical $*$-homomorphism ψ_i of $C_0((0,1))$ into "the lower right corner of $\operatorname{Mat}_{N_i}(\mathfrak{B})$" induced by $\varphi_{\alpha(i)}$. Then $C_0((0,1)) \xrightarrow{\psi_i} \operatorname{Mat}_{N_i}(\mathfrak{B}) \longrightarrow \mathfrak{B}\otimes\mathbb{K}$ is a commuting family if $*$-homomorphisms. Thus it induces a $*$-homomorphism $\varphi\colon \bigoplus_{s\in S} C_0((0,1)) \to \mathfrak{B}\otimes\mathbb{K}$.

Since K_1 is completely additive, $K_1(\varphi)([f_0\delta_{s,s_0}]_1) = [u_{s_0}]_1 = s_0$ and $K_1(\bigoplus_{s\in S} C_0((0,1)))$ is isomorphic to $\bigoplus_{s\in S}\mathbb{Z}$ (*cf.* [Bla98, §21.1]). ●●●

5.2.2 Geometric resolution

We now prove the following proposition, which is similar to [Bon02, Proposition 7.4.1].

Proposition 5.2.10. *Let $\mathfrak{A}_\bullet$ be an object of $\mathcal{SE}_2$. Then there exists an object $\mathfrak{M}_\bullet$ of $\mathcal{SE}_2$ with $K_0(\mathfrak{M}_i)$ and $K_1(\mathfrak{M}_i)$ countable free abelian groups, for $i=1,\ldots,6$, such that there exists a morphism $\mathbf{\Phi}_\bullet\colon \mathfrak{M}_\bullet \to \mathsf{S}\mathfrak{A}_\bullet\otimes\mathbb{K}$ with $K_0(\Phi_i)$ and $K_1(\Phi_i)$ surjective, for $i=1,\ldots,6$. In other words, $K_{\circledast}(\mathfrak{M}_\bullet)$ is a projective object, and $K_{\circledast}(\mathbf{\Phi}_\bullet)$ is an epic morphism. If the K-theory of $\mathfrak{A}_\bullet$ is finitely generated, then we can choose $\mathfrak{M}_\bullet$ to have finitely generated K-theory.*

Proof. Choose a countable set S_1 of generators for $K_1(\mathfrak{A}_1)$, and construct a homomorphism φ_1 from $\mathfrak{B} = \bigoplus_{s\in S_1} C_0((0,1))$ to $\mathfrak{A}_1\otimes\mathbb{K}$ as in Lemma 5.2.9. Let $\mathfrak{B}_\bullet$ denote the object

$$\begin{tikzcd}
\mathfrak{B} \arrow[r, equal] \arrow[d, equal] & \mathfrak{B} \arrow[r, two heads] \arrow[d, equal] & 0 \arrow[d, hook] \\
\mathfrak{B} \arrow[r, equal] & \mathfrak{B} \arrow[r, two heads] \arrow[d, two heads] & 0 \arrow[d, two heads] \\
 & 0 \arrow[r, equal] & 0
\end{tikzcd}$$

and let $\mathbf{\Phi}_\bullet^{\mathfrak{B}_\bullet} = (\Phi_1^{\mathfrak{B}_\bullet},\ldots,\Phi_6^{\mathfrak{B}_\bullet})$ denote the morphism $(\varphi_1, \alpha_{1,2}\varphi_1, \alpha_{1,3}\varphi_1, 0,0,0)$.

Choose a countable set S_2 of generators for $K_1(\mathfrak{A}_2)$, and construct a homomorphism φ_2 from

$\mathfrak{C} = \bigoplus_{s\in S_2} C_0((0,1))$ to $\mathfrak{A}_2 \otimes \mathbb{K}$ as in Lemma 5.2.9. Let $\mathfrak{C}_\bullet$ denote the object

$$\begin{tikzcd}
0 \arrow[r, hook] \arrow[d, equal] & \mathfrak{C} \arrow[r, equal] \arrow[d, equal] & \mathfrak{C} \arrow[d, equal] \\
0 \arrow[r, hook] & \mathfrak{C} \arrow[r, equal] \arrow[d] & \mathfrak{C} \arrow[d] \\
 & 0 \arrow[r, equal] & 0
\end{tikzcd}$$

and let $\mathbf{\Phi}_\bullet^{\mathfrak{C}_\bullet} = (\Phi_1^{\mathfrak{C}_\bullet}, \ldots, \Phi_6^{\mathfrak{C}_\bullet})$ denote the morphism $(0, \varphi_2, \alpha_{2,3}\varphi_2, \alpha_{2,4}\varphi_2, \alpha_{2,5}\varphi_2, 0)$.

Choose a countable set S_3 of generators for $K_1(\mathfrak{A}_3)$, and construct a homomorphism φ_3 from $\mathfrak{D} = \bigoplus_{s\in S_3} C_0((0,1))$ to $\mathfrak{A}_3 \otimes \mathbb{K}$ as in Lemma 5.2.9. Let $\mathfrak{D}_\bullet$ denote the object

$$\begin{tikzcd}
0 \arrow[r, hook] \arrow[d, equal] & 0 \arrow[r, two heads] \arrow[d, hook] & 0 \arrow[d, hook] \\
0 \arrow[r, hook] & \mathfrak{D} \arrow[r, equal] \arrow[d, equal] & \mathfrak{D} \arrow[d, equal] \\
 & \mathfrak{D} \arrow[r, equal] & \mathfrak{D}
\end{tikzcd}$$

and let $\mathbf{\Phi}_\bullet^{\mathfrak{D}_\bullet} = (\Phi_1^{\mathfrak{D}_\bullet}, \ldots, \Phi_6^{\mathfrak{D}_\bullet})$ denote the morphism $(0, 0, \varphi_3, 0, \alpha_{3,5}\varphi_3, \alpha_{3,6}\varphi_3)$.

Choose a countable set S_4 of generators for $K_1(\mathfrak{A}_4)$, and construct a homomorphism φ_4 from $\mathfrak{E} = \bigoplus_{s\in S_4} C_0((0,1))$ to $\mathfrak{A}_4 \otimes \mathbb{K}$ as in Lemma 5.2.9. Let $\mathfrak{E}_\bullet$ denote the object

$$\begin{tikzcd}
J(\mathfrak{E}) \arrow[r, hook] \arrow[d, equal] & T(\mathfrak{E}) \arrow[r, two heads] \arrow[d, equal] & \mathfrak{E} \arrow[d, equal] \\
J(\mathfrak{E}) \arrow[r, hook] & T(\mathfrak{E}) \arrow[r, two heads] \arrow[d] & \mathfrak{E} \arrow[d] \\
 & 0 \arrow[r, equal] & 0
\end{tikzcd}$$

and let $\mathbf{\Phi}_\bullet^{\mathfrak{E}_\bullet} = (\Phi_1^{\mathfrak{E}_\bullet}, \ldots, \Phi_6^{\mathfrak{E}_\bullet})$, where $\Phi_1^{\mathfrak{E}_\bullet}$ and $\Phi_2^{\mathfrak{E}_\bullet}$ are the $*$-homomorphisms induced by $\Phi_4^{\mathfrak{E}_\bullet} = \varphi_4$, and $\Phi_3^{\mathfrak{E}_\bullet} = \alpha_{2,3}\Phi_2^{\mathfrak{E}_\bullet}$, $\Phi_5^{\mathfrak{E}_\bullet} = \alpha_{4,5}\varphi_4$, and $\Phi_6^{\mathfrak{E}_\bullet} = 0$ (where $J(\mathfrak{E})$ and $T(\mathfrak{E})$ are as in Proposition 5.2.5 and the induced maps also are according to this proposition). Then $\mathbf{\Phi}_\bullet^{\mathfrak{E}_\bullet}$ is a morphism.

Choose a countable set S_5 of generators for $K_1(\mathfrak{A}_5)$, and construct a homomorphism φ_5 from $\mathfrak{F} = \bigoplus_{s\in S_5} C_0((0,1))$ to $\mathfrak{A}_5 \otimes \mathbb{K}$ as in Lemma 5.2.9. Let $\mathfrak{F}_\bullet$ denote the object

$$\begin{tikzcd}
J(\mathfrak{F}) \arrow[r, equal] \arrow[d, equal] & J(\mathfrak{F}) \arrow[r, two heads] \arrow[d, hook] & 0 \arrow[d, hook] \\
J(\mathfrak{F}) \arrow[r, hook] & T(\mathfrak{F}) \arrow[r, two heads] \arrow[d] & \mathfrak{F} \arrow[d, equal] \\
 & \mathfrak{F} \arrow[r, equal] & \mathfrak{F}
\end{tikzcd}$$

and let $\mathbf{\Phi}_\bullet^{\mathfrak{F}_\bullet} = (\Phi_1^{\mathfrak{F}_\bullet}, \ldots, \Phi_6^{\mathfrak{F}_\bullet})$, where $\Phi_1^{\mathfrak{F}_\bullet}$ and $\Phi_3^{\mathfrak{F}_\bullet}$ are the $*$-homomorphisms induced by $\Phi_5^{\mathfrak{F}_\bullet} = \varphi_5$, and $\Phi_2^{\mathfrak{F}_\bullet} = \alpha_{1,2}\Phi_1^{\mathfrak{F}_\bullet}$, $\Phi_4^{\mathfrak{F}_\bullet} = 0$, and $\Phi_6^{\mathfrak{F}_\bullet} = \alpha_{5,6}\varphi_5$ (where $J(\mathfrak{F})$ and $T(\mathfrak{F})$ are as in Proposition 5.2.5 and the induced maps also are according to this proposition). Then $\mathbf{\Phi}_\bullet^{\mathfrak{F}_\bullet}$ is a morphism.

Choose a countable set S_6 of generators for $K_1(\mathfrak{A}_6)$, and construct a homomorphism φ_6 from $\mathfrak{G} = \bigoplus_{s \in S_6} C_0((0,1))$ to $\mathfrak{A}_6 \otimes \mathbb{K}$ as in Lemma 5.2.9. Let $\mathfrak{G}_\bullet$ denote the object

$$\begin{tikzcd}
0 \arrow[r, hook] \arrow[d, equal] & J(\mathfrak{G}) \arrow[r, equal] \arrow[d, hook] & J(\mathfrak{G}) \arrow[d, hook] \\
0 \arrow[r, hook] & T(\mathfrak{G}) \arrow[r, equal] \arrow[d, two heads] & T(\mathfrak{G}) \arrow[d, two heads] \\
 & \mathfrak{G} \arrow[r, equal] & \mathfrak{G}
\end{tikzcd}$$

and let $\mathbf{\Phi}_\bullet^{\mathfrak{G}_\bullet} = (\Phi_1^{\mathfrak{G}_\bullet}, \ldots, \Phi_6^{\mathfrak{G}_\bullet})$, where $\Phi_2^{\mathfrak{G}_\bullet}$ and $\Phi_3^{\mathfrak{G}_\bullet}$ are the $*$-homomorphisms induced by $\Phi_6^{\mathfrak{G}_\bullet} = \varphi_6$, and $\Phi_1^{\mathfrak{G}_\bullet} = 0$, $\Phi_4^{\mathfrak{G}_\bullet} = \alpha_{2,4}\Phi_2^{\mathfrak{G}_\bullet}$, and $\Phi_5^{\mathfrak{G}_\bullet} = \alpha_{3,5}\Phi_3^{\mathfrak{G}_\bullet}$ (where $J(\mathfrak{G})$ and $T(\mathfrak{G})$ are as in Proposition 5.2.5 and the induced maps also are according to this proposition). Then $\mathbf{\Phi}_\bullet^{\mathfrak{G}_\bullet}$ is a morphism.

We let $\mathfrak{H}_\bullet$ denote the object $\mathfrak{B}_\bullet \oplus \mathfrak{C}_\bullet \oplus \mathfrak{D}_\bullet \oplus \mathfrak{E}_\bullet \oplus \mathfrak{F}_\bullet \oplus \mathfrak{G}_\bullet$, and we let $\mathbf{\Phi}_\bullet^{\mathfrak{H}_\bullet}$ denote the morphism $\mathbf{\Phi}_\bullet^{\mathfrak{B}_\bullet} \oplus \mathbf{\Phi}_\bullet^{\mathfrak{C}_\bullet} \oplus \mathbf{\Phi}_\bullet^{\mathfrak{D}_\bullet} \oplus \mathbf{\Phi}_\bullet^{\mathfrak{E}_\bullet} \oplus \mathbf{\Phi}_\bullet^{\mathfrak{F}_\bullet} \oplus \mathbf{\Phi}_\bullet^{\mathfrak{G}_\bullet}$ — so we have

$$\mathbf{\Phi}_\bullet^{\mathfrak{H}_\bullet} \colon \mathfrak{H}_\bullet \to \bigoplus_{i=1}^{6} \mathfrak{A}_\bullet \otimes \mathbb{K} \subseteq \mathrm{Mat}_6(\mathfrak{A}_\bullet \otimes \mathbb{K}) = \mathfrak{A}_\bullet \otimes \mathrm{Mat}_6(\mathbb{K}).$$

Now it is evident that $K_1(\Phi_i^{\mathfrak{H}_\bullet})$ is surjective for all $i = 1, \ldots, 6$, and that $K_{\circledast}(\mathfrak{H}_\bullet)$ is projective (using Theorem 5.1.8 and Proposition 5.2.5).

Use the same construction for $\mathsf{S}\mathfrak{A}_\bullet$ and get an object $\mathfrak{I}_\bullet$ and a morphism $\mathbf{\Phi}_\bullet^{\mathfrak{I}_\bullet} \colon \mathfrak{I}_\bullet \to \mathsf{S}\mathfrak{A}_\bullet \otimes \mathrm{Mat}_6(\mathbb{K})$ such that $K_1(\Phi_i^{\mathfrak{I}_\bullet}) \colon K_1(\mathfrak{I}_i) \to K_1(\mathsf{S}\mathfrak{A}_i \otimes \mathrm{Mat}_6(\mathbb{K}))$ is surjective, for $i = 1, \ldots, 6$. Let $\mathfrak{M}_\bullet = \mathsf{S}\mathfrak{H}_\bullet \oplus \mathfrak{I}_\bullet$, and let

$$\mathbf{\Phi}_\bullet^{\mathfrak{M}_\bullet} = \mathsf{S}\mathbf{\Phi}_\bullet^{\mathfrak{H}_\bullet} \oplus \mathbf{\Phi}_\bullet^{\mathfrak{I}_\bullet} \colon \mathfrak{M}_\bullet \to \mathsf{S}\mathfrak{A}_\bullet \otimes \mathrm{Mat}_6(\mathbb{K}) \oplus \mathsf{S}\mathfrak{A}_\bullet \otimes \mathrm{Mat}_6(\mathbb{K}) \subseteq \mathsf{S}\mathfrak{A}_\bullet \otimes \mathrm{Mat}_{12}(\mathbb{K}) \cong \mathsf{S}\mathfrak{A}_\bullet \otimes \mathbb{K}.$$

Clearly, $K_1(\Phi_i^{\mathfrak{M}_\bullet}) \colon K_1(\mathfrak{M}_i) \to K_1(\mathsf{S}\mathfrak{A}_i \otimes \mathbb{K})$ is surjective, for all $i = 1, \ldots, 6$. Moreover, we have that $K_0(\Phi_i^{\mathfrak{M}_\bullet}) = K_0(\mathsf{S}\Phi_i^{\mathfrak{H}_\bullet}) \oplus K_0(\Phi_i^{\mathfrak{I}_\bullet})$, for $i = 1, \ldots, 6$. Because we know that $K_1(\Phi_i^{\mathfrak{H}_\bullet})$ is surjective, for $i = 1, \ldots, 6$, it follows from the above and that $K_1(-) \cong K_0(\mathsf{S}(-))$ that $K_0(\Phi_i^{\mathfrak{M}_\bullet})$ is surjective, for $i = 1, \ldots, 6$.

Moreover, $K_0(\mathfrak{M}_i)$ and $K_1(\mathfrak{M}_i)$ are countable free abelian groups, for $i = 1, \ldots, 6$. They are finitely generated, if $K_0(\mathfrak{A}_i)$ and $K_1(\mathfrak{A}_i)$ are finitely generated, for all $i = 1, \ldots, 6$. ●●●

Dually, we obtain the following (which is analogous to [Bon02, Proposition 7.4.2]).

Proposition 5.2.11. *Let $\mathfrak{A}_\bullet$ be an object of $\mathcal{SE}_2$. Then there exists an object $\mathfrak{N}_\bullet$ of $\mathcal{SE}_2$ with $K_0(\mathfrak{N}_i)$ and $K_1(\mathfrak{N}_i)$ countable divisible abelian groups, for $i = 1, \ldots, 6$, such that there exists a morphism $\mathbf{\Psi}_\bullet \colon \mathsf{S}\mathsf{S}\mathfrak{A}_\bullet \to \mathfrak{N}_\bullet$ with $K_0(\Psi_i)$ and $K_1(\Psi_i)$ injective, for $i = 1, \ldots, 6$. In other words, $K_{\circledast}(\mathfrak{N}_\bullet)$ is an injective object, and $K_{\circledast}(\mathbf{\Psi}_\bullet)$ is a monic morphism.*

Proof. Let $\mathbf{\Phi}_\bullet \colon \mathfrak{M}_\bullet \to \mathsf{S}\mathfrak{A}_\bullet \otimes \mathbb{K}$ be as in Proposition 5.2.10. The mapping cone, $\mathsf{C}_{\mathbf{\Phi}_\bullet}$, of $\mathbf{\Phi}_\bullet$ is

$$\begin{tikzcd}
\mathsf{C}_{\Phi_1} \arrow[r, hook] \arrow[d, equal] & \mathsf{C}_{\Phi_2} \arrow[r, two heads] \arrow[d, hook] & \mathsf{C}_{\Phi_4} \arrow[d, hook] \\
\mathsf{C}_{\Phi_1} \arrow[r, hook] & \mathsf{C}_{\Phi_3} \arrow[r, two heads] \arrow[d, two heads] & \mathsf{C}_{\Phi_5} \arrow[d, two heads] \\
 & \mathsf{C}_{\Phi_6} \arrow[r, equal] & \mathsf{C}_{\Phi_6}
\end{tikzcd}$$

where C_{Φ_i} is the mapping cone of Φ_i, for $i = 1, \ldots, 6$.

Let $\mathbf{\Omega}_\bullet^s \colon \mathsf{C}_{\mathbf{\Phi}_\bullet} \to \mathfrak{M}_\bullet$ be the standard morphism. Let R be the UHF-algebra with dimension group $\mathbb{Q}$, and let $R\mathfrak{M}_\bullet = \mathfrak{M}_\bullet \otimes R$. Define $\mathbf{\Omega}_\bullet^t \colon \mathfrak{M}_\bullet \to R\mathfrak{M}_\bullet$ by $\Omega_i^t(x) = x \otimes \mathbb{1}$, for all $x \in \mathfrak{M}_i$ and $i = 1, \ldots, 6$.

By the Künneth Theorem, $K_j(R\mathfrak{M}_i) = K_j(\mathfrak{M}_i) \otimes \mathbb{Q}$ and $K_j(\Omega_i^t)$ is injective, for $i = 1, \ldots, 6$ and $j = 0, 1$ (for each C^*-algebra $\mathfrak{B}$ identify $\mathfrak{B}$ with $\mathfrak{B} \otimes \mathbb{C}$ in the canonical way). The mapping cone sequence for $\boldsymbol{\Omega}_\bullet^t \boldsymbol{\Omega}_\bullet^s \colon \mathsf{C}_{\boldsymbol{\Phi}_\bullet} \to R\mathfrak{M}_\bullet$,

$$\mathsf{S}R\mathfrak{M}_\bullet \xhookrightarrow{\Delta_p} \mathsf{C}_{\boldsymbol{\Omega}_\bullet^t \boldsymbol{\Omega}_\bullet^s} \xrightarrow{\Delta_i} \mathsf{C}_{\boldsymbol{\Phi}_\bullet},$$

induces (up to a sign)

$$\longrightarrow K_\circledast(\mathsf{C}_{\boldsymbol{\Omega}_\bullet^t \boldsymbol{\Omega}_\bullet^s}) \xrightarrow{K_\circledast(\Delta_p)} K_\circledast(\mathsf{C}_{\boldsymbol{\Phi}_\bullet}) \xrightarrow{K_\circledast(\boldsymbol{\Omega}_\bullet^t \boldsymbol{\Omega}_\bullet^s)} K_\circledast(R\mathfrak{M}_\bullet) \longrightarrow K_{\circledast+1}(\mathsf{C}_{\boldsymbol{\Omega}_\bullet^t \boldsymbol{\Omega}_\bullet^s}) \longrightarrow \tag{5.11}$$

(*cf.* Section 3.3). Since $K_\circledast(\boldsymbol{\Omega}_\bullet^t \boldsymbol{\Omega}_\bullet^s)$ is injective, this sequence degenerates to two short exact sequences:

$$K_\circledast(\mathsf{C}_{\boldsymbol{\Phi}_\bullet}) \xhookrightarrow{K_\circledast(\boldsymbol{\Omega}_\bullet^t \boldsymbol{\Omega}_\bullet^s)} K_\circledast(R\mathfrak{M}_\bullet) \twoheadrightarrow K_{\circledast+1}(\mathsf{C}_{\boldsymbol{\Omega}_\bullet^t \boldsymbol{\Omega}_\bullet^s}).$$

This implies that the K-groups of $\mathsf{C}_{\Omega_i^t \Omega_i^s}$, for $i = 1, \ldots, 6$, are divisible (since quotients of divisible groups are divisible, and $G \otimes \mathbb{Q}$ is divisible for every group G). Consequently, $K_{\circledast+1}(\mathsf{C}_{\boldsymbol{\Omega}_\bullet^t \boldsymbol{\Omega}_\bullet^s})$ is an injective object.

We have a commuting diagram

$$\begin{tikzcd}
\mathsf{C}_{\boldsymbol{\Phi}_\bullet} \arrow[r, "\boldsymbol{\Omega}_\bullet^s"] \arrow[d, equal] & \mathfrak{M}_\bullet \arrow[d, "\boldsymbol{\Omega}_\bullet^t"] \\
\mathsf{C}_{\boldsymbol{\Phi}_\bullet} \arrow[r, "\boldsymbol{\Omega}_\bullet^t \boldsymbol{\Omega}_\bullet^s"] & R\mathfrak{M}_\bullet .
\end{tikzcd}$$

Naturality now gives the commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathsf{S}\mathfrak{M}_\bullet \arrow[r] \arrow[d, "\mathsf{S}\boldsymbol{\Omega}_\bullet^t"] & \mathsf{C}_{\boldsymbol{\Omega}_\bullet^s} \arrow[r] \arrow[d, "\boldsymbol{\omega}_\bullet"] & \mathsf{C}_{\boldsymbol{\Phi}_\bullet} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathsf{S}R\mathfrak{M}_\bullet \arrow[r] & \mathsf{C}_{\boldsymbol{\Omega}_\bullet^t \boldsymbol{\Omega}_\bullet^s} \arrow[r] & \mathsf{C}_{\boldsymbol{\Phi}_\bullet} \arrow[r] & 0.
\end{tikzcd}$$

This induces a commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & K_\circledast(\mathsf{C}_{\boldsymbol{\Phi}_\bullet}) \arrow[r, "K_\circledast(\boldsymbol{\Omega}_\bullet^s)"] \arrow[d, equal] & K_\circledast(\mathfrak{M}_\bullet) \arrow[r] \arrow[d, "K_\circledast(\boldsymbol{\Omega}_\bullet^t)"] & K_{\circledast+1}(\mathsf{C}_{\boldsymbol{\Omega}_\bullet^s}) \arrow[r] \arrow[d, "K_{\circledast+1}(\boldsymbol{\omega}_\bullet)"] & 0 \\
0 \arrow[r] & K_\circledast(\mathsf{C}_{\boldsymbol{\Phi}_\bullet}) \arrow[r, "K_\circledast(\boldsymbol{\Omega}_\bullet^t \boldsymbol{\Omega}_\bullet^s)"] & K_\circledast(R\mathfrak{M}_\bullet) \arrow[r] & K_{\circledast+1}(\mathsf{C}_{\boldsymbol{\Omega}_\bullet^t \boldsymbol{\Omega}_\bullet^s}) \arrow[r] & 0
\end{tikzcd}$$

Using that $K_j(\Omega_i^t)$ is injective, a simple diagram chase shows that $K_j(\omega_i)$ is injective, for $i = 1, \ldots, 6$ and $j = 0, 1$. Consequently, $K_\circledast(\boldsymbol{\omega}_\bullet)$ is monic.

Since $\mathsf{S}(\mathsf{S}\mathfrak{A}_\bullet \otimes \mathbb{K})$ is the kernel of $\boldsymbol{\Omega}_\bullet^s$ we get a commuting diagram of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & \mathsf{S}(\mathsf{S}\mathfrak{A}_\bullet \otimes \mathbb{K}) \arrow[r, "\boldsymbol{\Omega}_\bullet^v"] \arrow[d, equal] & \mathsf{C}_{\boldsymbol{\Omega}_\bullet^s} \arrow[r] \arrow[d] & \mathsf{C}\mathfrak{M}_\bullet \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathsf{S}(\mathsf{S}\mathfrak{A}_\bullet \otimes \mathbb{K}) \arrow[r] & \mathsf{C}_{\boldsymbol{\Phi}_\bullet} \arrow[r, "\boldsymbol{\Omega}_\bullet^s"] & \mathfrak{M}_\bullet \arrow[r] & 0.
\end{tikzcd}$$

Since $K_\circledast(\mathsf{C}\mathfrak{M}_\bullet)$ is the zero object, $K_\circledast(\boldsymbol{\Omega}_\bullet^v)$ is an isomorphism. We let $\mathfrak{N}_\bullet = \mathsf{C}_{\boldsymbol{\Omega}_\bullet^t \boldsymbol{\Omega}_\bullet^s}$, and we let $\boldsymbol{\Psi}_\bullet \colon \mathsf{S}\mathsf{S}\mathfrak{A}_\bullet \to \mathfrak{N}_\bullet$ denote the composition of the morphisms

$$\mathsf{S}\mathsf{S}\mathfrak{A}_\bullet \xrightarrow{\mathsf{S}\mathsf{S}\boldsymbol{\kappa}_\bullet} \mathsf{S}(\mathsf{S}\mathfrak{A}_\bullet \otimes \mathbb{K}) \xrightarrow{\boldsymbol{\Omega}_\bullet^v} \mathsf{C}_{\boldsymbol{\Omega}_\bullet^s} \xrightarrow{\boldsymbol{\omega}_\bullet} \mathsf{C}_{\boldsymbol{\Omega}_\bullet^t \boldsymbol{\Omega}_\bullet^s}.$$

Note that $K_\circledast(\boldsymbol{\Psi}_\bullet)$ is monic and $K_\circledast(\mathfrak{N}_\bullet)$ is injective. ●●●

Now we are prepared to construct a geometric injective resolution (*cf.* also [Bon02, Proposition 7.4.3]). A geometric projective resolution (which is needed to prove that the UCT is natural, *cf.* the proof of Proposition 5.5.2) is constructed in the dual way.

Proposition 5.2.12. *Let $\mathfrak{A}_\bullet$ be an object of $S\mathcal{E}_2$. Then there exist objects $\mathfrak{C}_\bullet$ and $\mathfrak{D}_\bullet$ of $S\mathcal{E}_2$ and morphisms $\mathbf{\Phi}_\bullet\colon \mathfrak{D}_\bullet \to \mathfrak{C}_\bullet$ and $\mathbf{\Psi}_\bullet\colon \mathfrak{C}_\bullet \to \mathsf{SS}\mathfrak{A}$ and a completely positive contraction $\mathbf{\Theta}_\bullet\colon \mathsf{SS}\mathfrak{A}_\bullet \to \mathfrak{C}_\bullet$ such that*

$$\mathfrak{D}_\bullet \overset{\mathbf{\Phi}_\bullet}{\hookrightarrow} \mathfrak{C}_\bullet \overset{\mathbf{\Psi}_\bullet}{\twoheadrightarrow} \mathsf{SS}\mathfrak{A}_\bullet$$

is exact and $\mathbf{\Psi}_\bullet\mathbf{\Theta}_\bullet = \mathbf{id}_\bullet$ and such that this short exact sequence induces a short exact sequence

$$K_\circledast(\mathfrak{A}_\bullet) \hookrightarrow K_{\circledast+1}(\mathfrak{D}_\bullet) \twoheadrightarrow K_{\circledast+1}(\mathfrak{C}_\bullet)$$

with $K_{\circledast+1}(\mathfrak{D}_\bullet)$ and $K_{\circledast+1}(\mathfrak{C}_\bullet)$ being injective.

Proof. Let $\mathbf{\Psi}_\bullet\colon \mathsf{SS}\mathfrak{A}_\bullet \to \mathfrak{N}_\bullet$ be as in Proposition 5.2.11. Let $\mathfrak{C}_\bullet = \mathsf{C}_{\mathbf{\Psi}_\bullet}$ and $\mathfrak{D}_\bullet = \mathsf{S}\mathfrak{N}_\bullet$.

Then we have a short exact sequence

$$\mathsf{S}\mathfrak{N}_\bullet \overset{\mathbf{\Theta}_\bullet}{\hookrightarrow} \mathsf{C}_{\mathbf{\Psi}_\bullet} \overset{\mathbf{\Omega}_\bullet}{\twoheadrightarrow} \mathsf{SS}\mathfrak{A}_\bullet.$$

This induces a cyclic six term exact sequence

$$\begin{tikzcd}
{} \arrow[r, "0"] & K_{\circledast+1}(\mathsf{SS}\mathfrak{A}_\bullet) \arrow[r] \arrow[d, equal] & K_\circledast(\mathsf{S}\mathfrak{N}_\bullet) \arrow[r] \arrow[d, "\cong"] & K_\circledast(\mathsf{C}_{\mathbf{\Psi}_\bullet}) \arrow[r, "0"] \arrow[d, equal] & K_\circledast(\mathsf{SS}\mathfrak{A}_\bullet) \arrow[r] \arrow[d, equal] & K_{\circledast+1}(\mathsf{S}\mathfrak{N}_\bullet) \arrow[r] \arrow[d, "\cong"] & {} \\
{} \arrow[r, "0"] & K_{\circledast+1}(\mathsf{SS}\mathfrak{A}_\bullet) \arrow[r, "-K_{\circledast+1}(\mathbf{\Psi}_\bullet)"] & K_{\circledast+1}(\mathfrak{N}_\bullet) \arrow[r] & K_\circledast(\mathsf{C}_{\mathbf{\Psi}_\bullet}) \arrow[r, "0"] & K_\circledast(\mathsf{SS}\mathfrak{A}_\bullet) \arrow[r, "-K_\circledast(\mathbf{\Psi}_\bullet)"] & K_\circledast(\mathfrak{N}_\bullet) \arrow[r] & {}
\end{tikzcd}$$

So we have such a short exact sequence

$$\mathfrak{D}_\bullet \overset{\mathbf{\Phi}_\bullet}{\hookrightarrow} \mathfrak{C}_\bullet \overset{\mathbf{\Psi}_\bullet}{\twoheadrightarrow} \mathsf{SS}\mathfrak{A}_\bullet$$

which induces a short exact sequence

$$K_\circledast(\mathfrak{A}_\bullet) \hookrightarrow K_{\circledast+1}(\mathfrak{D}_\bullet) \twoheadrightarrow K_{\circledast+1}(\mathfrak{C}_\bullet)$$

with $K_{\circledast+1}(\mathfrak{D}_\bullet)$ and $K_{\circledast+1}(\mathfrak{C}_\bullet)$ being injective (here we of course use the main theorem from Section 5.1, Theorem 5.1.8, and Proposition 5.1.10).

Now define $\Phi_i\colon \mathsf{SS}\mathfrak{A}_i \to \mathfrak{C}_i = \mathsf{C}_{\Psi_i}$ by

$$\Phi_i(x) = (x, [t \mapsto t\Psi_i(x)]) \in \mathfrak{C}_i, \quad x \in \mathsf{SS}\mathfrak{A}_i, i = 1, \ldots, 6.$$

Clearly, Φ_i is linear (but not necessarily a $*$-homomorphism). Moreover, $\Omega_i\Phi_i = \mathrm{id}_{\mathsf{SS}\mathfrak{A}_i}$ and Φ_i is contractive, for $i = 1, \ldots, 6$. Using that $t\Psi_i(x)\Psi(y)^* = (\sqrt{t}\Psi_i(x))(\sqrt{t}\Psi_i(y))^*$ it is straightforward to show that Φ_i is completely positive. Of course, one also needs to check that $\Phi_j \circ \mathsf{SS}\alpha_{i,j} = \gamma_{i,j} \circ \Phi_i$, where $\gamma_{i,j}$ are the maps of $\mathfrak{C}_\bullet$. ●●●

5.3 The $KK_{\mathcal{E}_2}$-groups in certain cases

In this section we will, in certain cases, relate the $KK_{\mathcal{E}_2}$-groups to $KK_{\mathcal{E}}$-groups (in a way similar to [Bon02, Section 7.1]).

We have four canonical functors $F_i\colon S\mathcal{E}_2 \to S\mathcal{E}$, $i = 1, 2, 3, 4$, defined by $F_1(\mathfrak{A}_\bullet)$, $F_2(\mathfrak{A}_\bullet)$, $F_3(\mathfrak{A}_\bullet)$, and $F_4(\mathfrak{A}_\bullet)$ is the extension

$$\mathfrak{A}_1 \hookrightarrow \mathfrak{A}_2 \twoheadrightarrow \mathfrak{A}_4,$$

$$\mathfrak{A}_1 \hookrightarrow \mathfrak{A}_3 \twoheadrightarrow \mathfrak{A}_5,$$

$$\mathfrak{A}_2 \hookrightarrow \mathfrak{A}_3 \twoheadrightarrow \mathfrak{A}_6,$$

$$\mathfrak{A}_4 \hookrightarrow \mathfrak{A}_5 \twoheadrightarrow \mathfrak{A}_6,$$

resp. By universality, these will induce functors G_1, G_2, G_3, and G_4 from $\mathbf{KK}_{\mathcal{E}_2}$ to $\mathbf{KK}_{\mathcal{E}}$ such that

$$\begin{tikzcd} S\mathcal{E}_2 \arrow[r, "F_i"] \arrow[d, "KK_{\mathcal{E}_2}"'] & S\mathcal{E} \arrow[d, "KK_{\mathcal{E}}"] \\ \mathbf{KK}_{\mathcal{E}_2} \arrow[r, "G_i"'] & \mathbf{KK}_{\mathcal{E}}. \end{tikzcd}$$

The proofs of the following two lemmata are straightforward (the lemmata correspond to the easy parts of [Bon02, Proposition 7.1.2 und Proposition 7.1.3]). See [Bon02, Definition 3.1.1] for the definition of Kasparov-$\mathfrak{A}_\bullet$-$\mathfrak{B}_\bullet$-modules.

Lemma 5.3.1. *Let $\mathfrak{A}_\bullet$ and $\mathfrak{B}_\bullet$ be objects of $S\mathcal{E}_2$. Let $x = ((E_3 \twoheadrightarrow E_5 \twoheadrightarrow E_6), (\phi_3, \phi_5, \phi_6), (F_3, F_5, F_6))$ be a Kasparov-$(\mathfrak{A}_3, \mathfrak{A}_5, \mathfrak{A}_6)$-$(\mathfrak{B}_3, \mathfrak{B}_5, \mathfrak{B}_6)$-module. Then clearly*

$$\begin{aligned} x_2 &:= ((E_3 \twoheadrightarrow E_5), (\phi_3, \phi_5), (F_3, F_5)) && \text{is a Kasparov-}(\mathfrak{A}_3, \mathfrak{A}_5)\text{-}(\mathfrak{B}_3, \mathfrak{B}_5)\text{-module,} \\ x_3 &:= ((E_3 \twoheadrightarrow E_6), (\phi_3, \phi_6), (F_3, F_6)) && \text{is a Kasparov-}(\mathfrak{A}_3, \mathfrak{A}_6)\text{-}(\mathfrak{B}_3, \mathfrak{B}_6)\text{-module,} \\ x_4 &:= ((E_5 \twoheadrightarrow E_6), (\phi_5, \phi_6), (F_5, F_6)) && \text{is a Kasparov-}(\mathfrak{A}_5, \mathfrak{A}_6)\text{-}(\mathfrak{B}_5, \mathfrak{B}_6)\text{-module.} \end{aligned}$$

Lemma 5.3.2. *We have that $G_2(x) = x_2$, $G_3(x) = x_3$, and $G_4(x) = x_4$.*

Analogous to [Bon02, Lemma 7.1.5], we prove the following proposition.

Proposition 5.3.3. *Let $\mathfrak{A}_\bullet$ and $\mathfrak{B}_\bullet$ be objects of $S\mathcal{E}_2$. Then the following holds:*

(a) $G_2\colon KK_{\mathcal{E}_2}(\mathfrak{A}_\bullet, \mathfrak{B}_\bullet) \to KK_{\mathcal{E}}(F_2(\mathfrak{A}_\bullet), F_2(\mathfrak{B}_\bullet))$ is an isomorphism whenever $\mathfrak{A}_4 = 0$ or $\mathfrak{B}_6 = 0$.

(b) $G_3\colon KK_{\mathcal{E}_2}(\mathfrak{A}_\bullet, \mathfrak{B}_\bullet) \to KK_{\mathcal{E}}(F_3(\mathfrak{A}_\bullet), F_3(\mathfrak{B}_\bullet))$ is an isomorphism whenever $\mathfrak{A}_1 = 0$ or $\mathfrak{B}_4 = 0$.

(c) $G_4\colon KK_{\mathcal{E}_2}(\mathfrak{A}_\bullet, \mathfrak{B}_\bullet) \to KK_{\mathcal{E}}(F_4(\mathfrak{A}_\bullet), F_4(\mathfrak{B}_\bullet))$ is an isomorphism whenever $\mathfrak{B}_1 = 0$.

Proof. We first prove surjectivity.

Assume that $\mathfrak{B}_6 = 0$. Let

$$y = ((E_3 \twoheadrightarrow E_5), (\phi_3, \phi_5), (F_3, F_5))$$

be an arbitrary Kasparov-$(\mathfrak{A}_3, \mathfrak{A}_5)$-$(\mathfrak{B}_3, \mathfrak{B}_5)$-module. Now define

$$x := ((E_3 \twoheadrightarrow E_5 \twoheadrightarrow 0), (\phi_3, \phi_5, 0), (F_3, F_5, 0)),$$

which clearly is a Kasparov-$(\mathfrak{A}_3, \mathfrak{A}_5, \mathfrak{A}_6)$-$(\mathfrak{B}_3, \mathfrak{B}_5, \mathfrak{B}_6)$-module. Then it is clear that $x_2 = y$.

Assume that $\mathfrak{A}_4 = 0$. So α_{56} is an isomorphism. Let

$$y = ((E_3 \twoheadrightarrow E_5), (\phi_3, \phi_5), (F_3, F_5))$$

be an arbitrary Kasparov-$(\mathfrak{A}_3, \mathfrak{A}_5)$-$(\mathfrak{B}_3, \mathfrak{B}_5)$-module. Now define

$$x := ((E_3 \twoheadrightarrow E_5 \twoheadrightarrow E_5 \otimes_{\beta_{56}} \mathfrak{B}_6), (\phi_3, \phi_5, \phi_5 \circ \alpha_{56}^{-1} \otimes_{\beta_{56}} 1), (F_3, F_5, F_5 \otimes_{\beta_{56}} 1)).$$

Using the results from [Bon02, Section 1.2], one easily shows that x is a Kasparov-$(\mathfrak{A}_3, \mathfrak{A}_5, \mathfrak{A}_6)$-$(\mathfrak{B}_3, \mathfrak{B}_5, \mathfrak{B}_6)$-module, such that $x_2 = y$.

Assume that $\mathfrak{A}_1 = 0$. So α_{35} is an isomorphism. Let

$$y = ((E_3 \twoheadrightarrow E_6), (\phi_3, \phi_6), (F_3, F_6))$$

be an arbitrary Kasparov-$(\mathfrak{A}_3, \mathfrak{A}_6)$-$(\mathfrak{B}_3, \mathfrak{B}_6)$-module. Now — using results from [Bon02, Section 1.2] — we can define a Kasparov-$(\mathfrak{A}_3, \mathfrak{A}_5, \mathfrak{A}_6)$-$(\mathfrak{B}_3, \mathfrak{B}_5, \mathfrak{B}_6)$-module by

$$x := ((E_3 \twoheadrightarrow E_3 \otimes_{\beta_{35}} \mathfrak{B}_5 \twoheadrightarrow E_6), (\phi_3, \phi_3 \circ \alpha_{35}^{-1} \otimes_{\beta_{35}} 1, \phi_6), (F_3, F_3 \otimes_{\beta_{35}} 1, F_6)).$$

Clearly, we have that $x_3 = y$.

Assume that $\mathfrak{B}_1 = 0$. So β_{35} is an isomorphism. Let

$$y = ((E_5 \twoheadrightarrow E_6), (\phi_5, \phi_6), (F_5, F_6))$$

be an arbitrary Kasparov-$(\mathfrak{A}_5, \mathfrak{A}_6)$-$(\mathfrak{B}_5, \mathfrak{B}_6)$-module. Now define

$$x := ((E_5 \twoheadrightarrow E_5 \twoheadrightarrow E_6), (\phi_5 \circ \alpha_{35}, \phi_5, \phi_6), (F_5, F_5, F_6)).$$

It is straightforward to show that x is a Kasparov-$(\mathfrak{A}_3, \mathfrak{A}_5, \mathfrak{A}_6)$-$(\mathfrak{B}_3, \mathfrak{B}_5, \mathfrak{B}_6)$-module with $x_4 = y$.

Assume that $\mathfrak{B}_4 = 0$. So β_{56} is an isomorphism. Let

$$y = ((E_3 \twoheadrightarrow E_6), (\phi_3, \phi_6), (F_3, F_6))$$

be an arbitrary Kasparov-$(\mathfrak{A}_3, \mathfrak{A}_6)$-$(\mathfrak{B}_3, \mathfrak{B}_6)$-module. Now define

$$x := ((E_3 \twoheadrightarrow E_6 \twoheadrightarrow E_6), (\phi_3, \phi_6 \circ \alpha_{56}, \phi_6), (F_3, F_6, F_6)).$$

It is straightforward to show that x is a Kasparov-$(\mathfrak{A}_3, \mathfrak{A}_5, \mathfrak{A}_6)$-$(\mathfrak{B}_3, \mathfrak{B}_5, \mathfrak{B}_6)$-module with $x_3 = y$.

We now prove injectivity. We use the picture of $KK_{\mathcal{E}_2}(\mathfrak{A}_\bullet, \mathfrak{B}_\bullet)$ given as unitary equivalence classes of Kasparov-$(\mathfrak{A}_3, \mathfrak{A}_5, \mathfrak{A}_6)$-$(\mathfrak{B}_3, \mathfrak{B}_5, \mathfrak{B}_6)$-modules modulo homotopy (*cf.* [Bon02, Definition 3.1.3]).

Assume that $\mathfrak{B}_6 = 0$. So assume that there are given two Kasparov-$(\mathfrak{A}_3, \mathfrak{A}_5, \mathfrak{A}_6)$-$(\mathfrak{B}_3, \mathfrak{B}_5, 0)$-modules

$$x^i = ((E_3^i \twoheadrightarrow E_5^i \twoheadrightarrow E_6^i), (\phi_3^i, \phi_5^i, \phi_6^i), (F_3^i, F_5^i, F_6^i)), \quad \text{for } i = 0, 1,$$

such that

$$(x^i)_2 = ((E_3^i \twoheadrightarrow E_5^i), (\phi_3^i, \phi_5^i), (F_3^i, F_5^i)), \quad \text{for } i = 0, 1,$$

are homotopic, *i.e.*, there is a Kasparov-$(\mathfrak{A}_3, \mathfrak{A}_5)$-$(C([0,1], \mathfrak{B}_3), C([0,1], \mathfrak{B}_5))$-module

$$((E_3 \twoheadrightarrow E_5, (\phi_3, \phi_5), (F_3, F_5))$$

such that

$$((E_3 \otimes_{\mathrm{ev}_i} \mathfrak{B}_3 \twoheadrightarrow E_5 \otimes_{\mathrm{ev}_i} \mathfrak{B}_5, (\phi_3 \otimes_{\mathrm{ev}_i} 1, \phi_5 \otimes_{\mathrm{ev}_i} 1), (F_3 \otimes_{\mathrm{ev}_i} 1, F_5 \otimes_{\mathrm{ev}_i} 1)) = (x^i)_2, \quad \text{for } i = 0, 1.$$

Since $\mathfrak{B}_6 = 0$, we have $E_6^0 = 0 = E_6^1$, and, consequently, $\phi_6^0 = 0 = \phi_6^1$ and $F_6^0 = 0 = F_6^1$.

As above, we can lift this to a Kasparov-$(\mathfrak{A}_3, \mathfrak{A}_5, \mathfrak{A}_6)$-$(C([0,1], \mathfrak{B}_3), C([0,1], \mathfrak{B}_5), 0)$-module

$$((E_3 \twoheadrightarrow E_5 \twoheadrightarrow 0), (\phi_3, \phi_5, 0), (F_3, F_5, 0)).$$

This means that we have homotopic Kasparov-$(\mathfrak{A}_3, \mathfrak{A}_5, \mathfrak{A}_6)$-$(\mathfrak{B}_3, \mathfrak{B}_5, 0)$-modules

$$\begin{aligned} x^i &= ((E_3^i \twoheadrightarrow E_5^i \twoheadrightarrow 0), (\phi_3^i, \phi_5^i, 0), (F_3^i, F_5^i, 0)) \\ &= ((E_3 \otimes_{\mathrm{ev}_i} \mathfrak{B}_3 \twoheadrightarrow E_5 \otimes_{\mathrm{ev}_i} \mathfrak{B}_5 \twoheadrightarrow 0), (\phi_3 \otimes_{\mathrm{ev}_i} 1, \phi_5 \otimes_{\mathrm{ev}_i} 1, 0), (F_3 \otimes_{\mathrm{ev}_i} 1, F_5 \otimes_{\mathrm{ev}_i} 1, 0)), \quad \text{for } i = 0, 1. \end{aligned}$$

Assume that $\mathfrak{B}_4 = 0$. So β_{56} is an isomorphism. For convenience, we assume that $\mathfrak{B}_5 = \mathfrak{B}_6$ and $\beta_{56} = \mathrm{id}_{\mathfrak{B}_6}$. Assume first that there is given a Kasparov-$(\mathfrak{A}_3, \mathfrak{A}_5, \mathfrak{A}_6)$-$(\mathfrak{B}_3, \mathfrak{B}_6, \mathfrak{B}_6)$-module

$$x = ((E_3 \twoheadrightarrow E_5 \twoheadrightarrow E_6), (\phi_3, \phi_5, \phi_6), (F_3, F_5, F_6)).$$

Then ϵ_{56} is a module isomorphism, and E_5 and E_6 are identical as Hilbert-$\mathfrak{B}_6$-modules under this isomorphism, and, moreover, $\phi_5 = \phi_6 \circ \alpha_{56}$ and $T_5 = T_6$ under this identification (this follows easily from [Bon02, Lemmata 1.2.3 und 1.2.4]). So x is actually unitarily equivalent to

$$((E_3 \twoheadrightarrow E_6 \twoheadrightarrow E_6), (\phi_3, \phi_6 \circ \alpha_{56}, \phi_6), (F_3, F_6, F_6)).$$

So assume that there are given two Kasparov-$(\mathfrak{A}_3, \mathfrak{A}_5, \mathfrak{A}_6)$-$(\mathfrak{B}_3, \mathfrak{B}_6, \mathfrak{B}_6)$-modules

$$x^i = ((E_3^i \twoheadrightarrow E_6^i \twoheadrightarrow E_6^i), (\phi_3^i, \phi_6^i \circ \alpha_{56}, \phi_6^i), (F_3^i, F_6^i, F_6^i)), \quad \text{for } i = 0, 1,$$

such that

$$(x^i)_3 = ((E_3^i \twoheadrightarrow E_6^i), (\phi_3^i, \phi_6^i), (F_3^i, F_6^i)), \quad \text{for } i = 0, 1,$$

are homotopic, *i.e.*, there is a Kasparov-$(\mathfrak{A}_3, \mathfrak{A}_6)$-$(C([0,1], \mathfrak{B}_3), C([0,1], \mathfrak{B}_6))$-module

$$((E_3 \twoheadrightarrow E_6, (\phi_3, \phi_6), (F_3, F_6))$$

such that

$$((E_3 \otimes_{\mathrm{ev}_i} \mathfrak{B}_3 \twoheadrightarrow E_6 \otimes_{\mathrm{ev}_i} \mathfrak{B}_6, (\phi_3 \otimes_{\mathrm{ev}_i} 1, \phi_6 \otimes_{\mathrm{ev}_i} 1), (F_3 \otimes_{\mathrm{ev}_i} 1, F_6 \otimes_{\mathrm{ev}_i} 1)) = (x^i)_3, \quad \text{for } i = 0, 1.$$

As above, we can lift this to a Kasparov-$(\mathfrak{A}_3, \mathfrak{A}_5, \mathfrak{A}_6)$-$(C([0,1], \mathfrak{B}_3), C([0,1], \mathfrak{B}_6), C([0,1], \mathfrak{B}_6))$-module

$$((E_3 \twoheadrightarrow E_6 \twoheadrightarrow E_6), (\phi_3, \phi_6 \circ \alpha_{56}, \phi_6), (F_3, F_6, F_6)).$$

This means that we have homotopic Kasparov-$(\mathfrak{A}_3, \mathfrak{A}_5, \mathfrak{A}_6)$-$(\mathfrak{B}_3, \mathfrak{B}_5, \mathfrak{B}_6)$-modules

$$\begin{aligned} x^i &= ((E_3^i \twoheadrightarrow E_6^i \twoheadrightarrow E_6^i), (\phi_3^i, \phi_6^i \circ \alpha_{56}, \phi_6^i), (F_3^i, F_6^i, F_6^i)) \\ &= ((E_3 \otimes_{\mathrm{ev}_i} \mathfrak{B}_3 \twoheadrightarrow E_6 \otimes_{\mathrm{ev}_i} \mathfrak{B}_6 \twoheadrightarrow E_6 \otimes_{\mathrm{ev}_i} \mathfrak{B}_6), \\ &\qquad (\phi_3 \otimes_{\mathrm{ev}_i} 1, \phi_6 \circ \alpha_{56} \otimes_{\mathrm{ev}_i} 1, \phi_6 \otimes_{\mathrm{ev}_i} 1), (F_3 \otimes_{\mathrm{ev}_i} 1, F_6 \otimes_{\mathrm{ev}_i} 1, F_6 \otimes_{\mathrm{ev}_i} 1)), \quad \text{for } i = 0, 1. \end{aligned}$$

Assume that $\mathfrak{B}_1 = 0$. So β_{35} is an isomorphism. For convenience, we assume that $\mathfrak{B}_3 = \mathfrak{B}_5$ and $\beta_{35} = \mathrm{id}_{\mathfrak{B}_5}$. Assume first that there is given a Kasparov-$(\mathfrak{A}_3, \mathfrak{A}_5, \mathfrak{A}_6)$-$(\mathfrak{B}_5, \mathfrak{B}_5, \mathfrak{B}_6)$-module

$$x = ((E_3 \twoheadrightarrow E_5 \twoheadrightarrow E_6), (\phi_3, \phi_5, \phi_6), (F_3, F_5, F_6)).$$

Then ϵ_{35} is a module isomorphism, and E_3 and E_5 are identical as Hilbert-$\mathfrak{B}_6$-modules under the isomorphism, and, moreover, $\phi_3 = \phi_5 \circ \alpha_{35}$ and $T_3 = T_5$ under this identification (this follows easily from [Bon02, Lemmata 1.2.3 und 1.2.4]). So x is actually unitarily equivalent to

$$((E_5 \twoheadrightarrow E_5 \twoheadrightarrow E_6), (\phi_5 \circ \alpha_{35}, \phi_5, \phi_6), (F_5, F_5, F_6)).$$

So assume that there are given two Kasparov-$(\mathfrak{A}_3, \mathfrak{A}_5, \mathfrak{A}_6)$-$(\mathfrak{B}_5, \mathfrak{B}_5, \mathfrak{B}_6)$-modules

$$x^i = ((E_5^i \twoheadrightarrow E_5^i \twoheadrightarrow E_6^i), (\phi_5^i \circ \alpha_{35}, \phi_5^i, \phi_6^i), (F_5^i, F_5^i, F_6^i)), \quad \text{for } i = 0, 1,$$

such that

$$(x^i)_4 = ((E_5^i \twoheadrightarrow E_6^i), (\phi_5^i, \phi_6^i), (F_5^i, F_6^i)), \quad \text{for } i = 0, 1,$$

are homotopic, *i.e.*, there is a Kasparov-$(\mathfrak{A}_5, \mathfrak{A}_6)$-$(C([0,1], \mathfrak{B}_5), C([0,1], \mathfrak{B}_6))$-module

$$((E_5 \twoheadrightarrow E_6, (\phi_5, \phi_6), (F_5, F_6))$$

such that

$$((E_5 \otimes_{\mathrm{ev}_i} \mathfrak{B}_5 \twoheadrightarrow E_6 \otimes_{\mathrm{ev}_i} \mathfrak{B}_6, (\phi_5 \otimes_{\mathrm{ev}_i} 1, \phi_6 \otimes_{\mathrm{ev}_i} 1), (F_5 \otimes_{\mathrm{ev}_i} 1, F_6 \otimes_{\mathrm{ev}_i} 1)) = (x^i)_4, \quad \text{for } i = 0, 1.$$

As above, we can lift this to a Kasparov-$(\mathfrak{A}_3, \mathfrak{A}_5, \mathfrak{A}_6)$-$(C([0,1], \mathfrak{B}_5), C([0,1], \mathfrak{B}_5), C([0,1], \mathfrak{B}_6))$-module

$$((E_5 \twoheadrightarrow E_5 \twoheadrightarrow E_6), (\phi_5 \circ \alpha_{35}, \phi_5, \phi_6), (F_5, F_5, F_6)).$$

This means that we have homotopic Kasparov-$(\mathfrak{A}_3, \mathfrak{A}_5, \mathfrak{A}_6)$-$(\mathfrak{B}_5, \mathfrak{B}_5, \mathfrak{B}_6)$-modules

$$\begin{aligned} x^i &= ((E_5^i \twoheadrightarrow E_5^i \twoheadrightarrow E_6^i), (\phi_3^i \circ \alpha_{35}, \phi_5^i, \phi_6^i), (F_5^i, F_5^i, F_6^i)) \\ &= ((E_5 \otimes_{\mathrm{ev}_i} \mathfrak{B}_5 \twoheadrightarrow E_5 \otimes_{\mathrm{ev}_i} \mathfrak{B}_5 \twoheadrightarrow E_6 \otimes_{\mathrm{ev}_i} \mathfrak{B}_6), \\ &\qquad (\phi_5 \circ \alpha_{35} \otimes_{\mathrm{ev}_i} 1, \phi_5 \otimes_{\mathrm{ev}_i} 1, \phi_6 \otimes_{\mathrm{ev}_i} 1), (F_5 \otimes_{\mathrm{ev}_i} 1, F_5 \otimes_{\mathrm{ev}_i} 1, F_6 \otimes_{\mathrm{ev}_i} 1)), \quad \text{for } i = 0, 1. \end{aligned}$$

Assume that $\mathfrak{A}_1 = 0$. So α_{35} is an isomorphism. For convenience, we assume that $\mathfrak{A}_3 = \mathfrak{A}_5$ and $\alpha_{35} = \mathrm{id}_{\mathfrak{A}_5}$. Assume first that there is given a Kasparov-$(\mathfrak{A}_5, \mathfrak{A}_5, \mathfrak{A}_6)$-$(\mathfrak{B}_3, \mathfrak{B}_5, \mathfrak{B}_6)$-module

$$x = ((E_3 \twoheadrightarrow E_5 \twoheadrightarrow E_6), (\phi_3, \phi_5, \phi_6), (F_3, F_5, F_6)).$$

Then E_5 can be identified with $E_3 \otimes_{\beta_{35}} \mathfrak{B}_5$, and under this identification, ϕ_5 becomes $\phi_3 \otimes_{\beta_{35}} 1$, and F_5 becomes $F_3 \otimes_{\beta_{35}} 1$ (this follows easily from [Bon02, Lemmata 1.2.3 und 1.2.4]). So x is actually unitarily equivalent to

$$((E_3 \twoheadrightarrow E_3 \otimes_{\beta_{35}} \mathfrak{B}_5 \twoheadrightarrow E_6), (\phi_3, \phi_3 \otimes_{\beta_{35}} 1, \phi_6), (F_3, F_3 \otimes_{\beta_{35}} 1, F_6)).$$

So assume that there are given two Kasparov-$(\mathfrak{A}_3, \mathfrak{A}_5, \mathfrak{A}_6)$-$(\mathfrak{B}_3, \mathfrak{B}_5, \mathfrak{B}_6)$-modules

$$x^i = ((E_3^i \twoheadrightarrow E_3^i \otimes_{\beta_{35}} \mathfrak{B}_5 \twoheadrightarrow E_6^i), (\phi_3^i, \phi_3^i \otimes_{\beta_{35}} 1, \phi_6^i), (F_3^i, F_3^i \otimes_{\beta_{35}} 1, F_6^i)), \quad \text{for } i = 0, 1,$$

such that

$$(x^i)_3 = ((E_3^i \twoheadrightarrow E_6^i), (\phi_3^i, \phi_6^i), (F_3^i, F_6^i)), \quad \text{for } i = 0, 1,$$

are homotopic, *i.e.*, there is a Kasparov-$(\mathfrak{A}_3, \mathfrak{A}_6)$-$(C([0,1], \mathfrak{B}_3), C([0,1], \mathfrak{B}_6))$-module

$$((E_3 \twoheadrightarrow E_6, (\phi_3, \phi_6), (F_3, F_6))$$

such that

$$((E_3 \otimes_{\mathrm{ev}_i} \mathfrak{B}_3 \twoheadrightarrow E_6 \otimes_{\mathrm{ev}_i} \mathfrak{B}_6, (\phi_3 \otimes_{\mathrm{ev}_i} 1, \phi_6 \otimes_{\mathrm{ev}_i} 1), (F_3 \otimes_{\mathrm{ev}_i} 1, F_6 \otimes_{\mathrm{ev}_i} 1)) = (x^i)_3, \quad \text{for } i = 0, 1.$$

As above, we can lift this to a Kasparov-$(\mathfrak{A}_5, \mathfrak{A}_5, \mathfrak{A}_6)$-$(C([0,1], \mathfrak{B}_3), C([0,1], \mathfrak{B}_5), C([0,1], \mathfrak{B}_6))$-module

$$((E_3 \twoheadrightarrow E_3 \otimes_{\beta_{35}} \mathfrak{B}_5 \twoheadrightarrow E_6), (\phi_3, \phi_3 \otimes_{\beta_{35}} 1, \phi_6), (F_3, F_3 \otimes_{\beta_{35}} 1, F_6)).$$

This means that we have homotopic Kasparov-$(\mathfrak{A}_5, \mathfrak{A}_5, \mathfrak{A}_6)$-$(\mathfrak{B}_3, \mathfrak{B}_5, \mathfrak{B}_6)$-modules

$$\begin{aligned} x^i &= ((E_3^i \twoheadrightarrow E_3^i \otimes_{\beta_{35}} \mathfrak{B}_5 \twoheadrightarrow E_6^i), (\phi_3^i, \phi_3^i \otimes_{\beta_{35}} 1, \phi_6^i), (F_3^i, F_3^i \otimes_{\beta_{35}} 1, F_6^i)) \\ &= ((E_3 \otimes_{\mathrm{ev}_i} \mathfrak{B}_3 \twoheadrightarrow (E_3 \otimes_{\beta_{35}} \mathfrak{B}_5) \otimes_{\mathrm{ev}_i} \mathfrak{B}_5 \twoheadrightarrow E_6 \otimes_{\mathrm{ev}_i} \mathfrak{B}_6), \\ &\quad (\phi_3 \otimes_{\mathrm{ev}_i} 1, (\phi_3 \otimes_{\beta_{35}} 1) \otimes_{\mathrm{ev}_i} 1, \phi_6 \otimes_{\mathrm{ev}_i} 1), (F_3 \otimes_{\mathrm{ev}_i} 1, (F_3 \otimes_{\beta_{35}} 1) \otimes_{\mathrm{ev}_i} 1, F_6 \otimes_{\mathrm{ev}_i} 1)), \quad \text{for } i = 0, 1. \end{aligned}$$

Assume that $\mathfrak{A}_4 = 0$. So α_{56} is an isomorphism. For convenience, we assume that $\mathfrak{A}_5 = \mathfrak{A}_6$ and $\alpha_{56} = \mathrm{id}_{\mathfrak{A}_5}$. Assume first that there is given a Kasparov-$(\mathfrak{A}_3, \mathfrak{A}_5, \mathfrak{A}_5)$-$(\mathfrak{B}_3, \mathfrak{B}_5, \mathfrak{B}_6)$-module

$$x = ((E_3 \twoheadrightarrow E_5 \twoheadrightarrow E_6), (\phi_3, \phi_5, \phi_6), (F_3, F_5, F_6)).$$

Then E_6 can be identified with $E_5 \otimes_{\beta_{56}} \mathfrak{B}_6$, and under this identification, ϕ_6 becomes $\phi_5 \otimes_{\beta_{56}} 1$, and F_6 becomes $F_5 \otimes_{\beta_{56}} 1$ (this follows easily from [Bon02, Lemmata 1.2.3 und 1.2.4]). So x is actually unitarily equivalent to

$$((E_3 \twoheadrightarrow E_5 \twoheadrightarrow E_5 \otimes_{\beta_{56}} \mathfrak{B}_6), (\phi_3, \phi_5, \phi_5 \otimes_{\beta_{56}} 1), (F_3, F_5, F_6 \otimes_{\beta_{56}} 1)).$$

So assume that there are given two Kasparov-$(\mathfrak{A}_3, \mathfrak{A}_5, \mathfrak{A}_5)$-$(\mathfrak{B}_3, \mathfrak{B}_5, \mathfrak{B}_6)$-modules

$$x^i = ((E_3^i \twoheadrightarrow E_5^i \twoheadrightarrow E_5^i \otimes_{\beta_{56}} \mathfrak{B}_6), (\phi_3^i, \phi_5^i, \phi_5^i \otimes_{\beta_{56}} 1), (F_3^i, F_5^i, F_5^i \otimes_{\beta_{56}} 1)), \quad \text{for } i = 0, 1,$$

such that

$$(x^i)_2 = ((E_3^i \twoheadrightarrow E_5^i), (\phi_3^i, \phi_5^i), (F_3^i, F_5^i)), \quad \text{for } i = 0, 1,$$

are homotopic, *i.e.*, there is a Kasparov-$(\mathfrak{A}_3, \mathfrak{A}_5)$-$(C([0,1], \mathfrak{B}_3), C([0,1], \mathfrak{B}_5))$-module

$$((E_3 \twoheadrightarrow E_5, (\phi_3, \phi_5), (F_3, F_5))$$

such that

$$((E_3 \otimes_{\mathrm{ev}_i} \mathfrak{B}_3 \twoheadrightarrow E_5 \otimes_{\mathrm{ev}_i} \mathfrak{B}_5, (\phi_3 \otimes_{\mathrm{ev}_i} 1, \phi_5 \otimes_{\mathrm{ev}_i} 1), (F_3 \otimes_{\mathrm{ev}_i} 1, F_5 \otimes_{\mathrm{ev}_i} 1)) = (x^i)_2, \quad \text{for } i = 0, 1.$$

As above, we can lift this to a Kasparov-$(\mathfrak{A}_3, \mathfrak{A}_5, \mathfrak{A}_5)$-$(C([0,1], \mathfrak{B}_3), C([0,1], \mathfrak{B}_5), C([0,1], \mathfrak{B}_6))$-module

$$((E_3 \twoheadrightarrow E_5 \twoheadrightarrow E_5 \otimes_{\beta_{56}} \mathfrak{B}_6), (\phi_3, \phi_5, \phi_5 \otimes_{\beta_{56}} 1), (F_3, F_5, F_6 \otimes_{\beta_{35}} 1)).$$

This means that we have homotopic Kasparov-$(\mathfrak{A}_3, \mathfrak{A}_5, \mathfrak{A}_5)$-$(\mathfrak{B}_3, \mathfrak{B}_5, \mathfrak{B}_6)$-modules

$$\begin{aligned} x^i &= ((E_3^i \twoheadrightarrow E_5^i \twoheadrightarrow E_5^i \otimes_{\beta_{56}} \mathfrak{B}_6), (\phi_3^i, \phi_5^i, \phi_5^i \otimes_{\beta_{56}} 1), (F_3^i, F_5^i, F_5^i \otimes_{\beta_{56}} 1)) \\ &= ((E_3 \otimes_{\mathrm{ev}_i} \mathfrak{B}_3 \twoheadrightarrow E_5 \otimes_{\mathrm{ev}_i} \mathfrak{B}_5 \twoheadrightarrow (E_5 \otimes_{\beta_{56}} \mathfrak{B}_6) \otimes_{\mathrm{ev}_i} \mathfrak{B}_6), \\ &\qquad (\phi_3 \otimes_{\mathrm{ev}_i} 1, \phi_5 \otimes_{\mathrm{ev}_i} 1, (\phi_5 \otimes_{\beta_{56}} 1) \otimes_{\mathrm{ev}_i} 1), (F_3 \otimes_{\mathrm{ev}_i} 1, F_5 \otimes_{\mathrm{ev}_i} 1, (F_5 \otimes_{\beta_{56}} 1) \otimes_{\mathrm{ev}_i} 1)), \quad \text{for } i = 0, 1. \end{aligned}$$

●●●

In the above, we did not prove anything for G_1. We expect similar results to hold for G_1, and we expect them to be somewhat harder to prove than for G_2, G_3, and G_4. The examination of this case is left out, since we will not need these (potential) results.

5.4 The UCT for $KK_{\mathcal{E}_2}$

In this section we will state and prove the Universal Coefficient Theorem (UCT) for $KK_{\mathcal{E}_2}$. First we define the bootstrap class (*cf.* also [Bon02, Definition 7.5.2]).

Definition 5.4.1. Let $\mathcal{N}_{\mathcal{E}_2}$ be the smallest class of objects $\mathfrak{A}_\bullet$ of $S\mathcal{E}_2$ with $\mathfrak{A}_1, \mathfrak{A}_2, \dots, \mathfrak{A}_6$ nuclear satisfying the following:

(1) If $\mathfrak{A}_\bullet$ is an object of $S\mathcal{E}_2$ with $\mathfrak{A}_1, \mathfrak{A}_2, \dots, \mathfrak{A}_6$ nuclear and in the bootstrap class $\mathcal{N}$, then $\mathfrak{A}_\bullet$ is in $\mathcal{N}_{\mathcal{E}_2}$.

(2) If $(\mathfrak{A}_\bullet^n)_{n=1}^\infty$ is an inductive system in $\mathcal{N}_{\mathcal{E}_2}$, then the inductive limit is in $\mathcal{N}_{\mathcal{E}_2}$.

(3) Let $\mathfrak{A}_\bullet \hookrightarrow \mathfrak{C}_\bullet \twoheadrightarrow \mathfrak{D}_\bullet$ be a short exact sequence in $S\mathcal{E}_2$. If two of the objects in the short exact sequence belong to $\mathcal{N}_{\mathcal{E}_2}$, then also the third one belongs to $\mathcal{N}_{\mathcal{E}_2}$.

(4) If $\mathfrak{A}_\bullet$ and $\mathfrak{A}'_\bullet$ are $KK_{\mathcal{E}_2}$-equivalent, then $\mathfrak{A}_\bullet$ belongs to $\mathcal{N}_{\mathcal{E}_2}$ if and only if $\mathfrak{A}'_\bullet$ belongs to $\mathcal{N}_{\mathcal{E}_2}$.

Now we state one of our main theorem, the generalization of Bonkat's UCT ([Bon02, Satz 7.5.3]). The proof will be carried out in two steps (in the next two propositions, which correspond to [Bon02, Proposition 7.5.5] and [Bon02, pp. 170–172], resp.).

Theorem 5.4.2 (The Universal Coefficient Theorem (UCT) for $KK_{\mathcal{E}_2}$). *Let $\mathfrak{A}_\bullet$ and $\mathfrak{B}_\bullet$ be objects of $S\mathcal{E}_2$, and assume that $\mathfrak{A}_\bullet$ belongs to the class $\mathcal{N}_{\mathcal{E}_2}$.*

Then there is a short exact sequence

$$\operatorname{Ext}(K_\circledast(\mathfrak{A}_\bullet), K_{\circledast+1}(\mathfrak{B}_\bullet)) \xhookrightarrow{\Delta_{\mathfrak{A}_\bullet,\mathfrak{B}_\bullet}} KK_{\mathcal{E}_2}(\mathfrak{A}_\bullet, \mathfrak{B}_\bullet) \xrightarrow{\Gamma_{\mathfrak{A}_\bullet,\mathfrak{B}_\bullet}} \operatorname{Hom}(K_\circledast(\mathfrak{A}_\bullet), K_\circledast(\mathfrak{B}_\bullet)).$$

Proposition 5.4.3. *Let $\mathfrak{A}_\bullet$ be a fixed given object of $S\mathcal{E}_2$. Assume that for all objects $\mathfrak{B}_\bullet$ of $S\mathcal{E}_2$ with $K_\circledast(\mathfrak{B}_\bullet)$ injective, the natural map*

$$\Gamma_{\mathfrak{A}_\bullet,\mathfrak{B}_\bullet} \colon KK_{\mathcal{E}_2}(\mathfrak{A}_\bullet, \mathfrak{B}_\bullet) \to \operatorname{Hom}(K_\circledast(\mathfrak{A}_\bullet), K_\circledast(\mathfrak{B}_\bullet))$$

is a (group) isomorphism. Then the UCT holds with $\mathfrak{A}_\bullet$ and $\mathfrak{B}_\bullet$ for all objects $\mathfrak{B}_\bullet$ of $S\mathcal{E}_2$.

Proof. Let $\mathfrak{B}_\bullet$ be an arbitrary object of $S\mathcal{E}_2$. Form a geometric injective resolution

$$0 \longrightarrow \mathfrak{D}_\bullet \xrightarrow{\Phi_\bullet} \mathfrak{C}_\bullet \xrightarrow{\Psi_\bullet} \mathsf{SS}\mathfrak{B}_\bullet \longrightarrow 0$$

as in Proposition 5.2.12.

The cyclic six term exact sequence for $KK_{\mathcal{E}_2}$ (*cf.* [Bon02, Korollar 3.4.6]) is

$$\begin{tikzcd} KK_{\mathcal{E}_2}(\mathfrak{A}_\bullet, \mathfrak{D}_\bullet) \arrow[r, "KK_{\mathcal{E}_2}(\mathfrak{A}_\bullet,\Phi_\bullet)"] & KK_{\mathcal{E}_2}(\mathfrak{A}_\bullet, \mathfrak{C}_\bullet) \arrow[r] & KK_{\mathcal{E}_2}(\mathfrak{A}_\bullet, \mathsf{SS}\mathfrak{B}_\bullet) \arrow[d] \\ KK_{\mathcal{E}_2}(\mathfrak{A}_\bullet, \mathsf{SSS}\mathfrak{B}_\bullet) \arrow[u] & KK_{\mathcal{E}_2}(\mathfrak{A}_\bullet, \mathsf{S}\mathfrak{C}_\bullet) \arrow[l] & KK_{\mathcal{E}_2}(\mathfrak{A}_\bullet, \mathsf{S}\mathfrak{D}_\bullet) \arrow[l, "KK_{\mathcal{E}_2}(\mathfrak{A}_\bullet,\mathsf{S}\Phi_\bullet)"'] \end{tikzcd}$$

We let $\boldsymbol{\omega}_\bullet = KK_{\mathcal{E}_2}(\mathfrak{A}_\bullet, \boldsymbol{\Phi}_\bullet)$ and $\boldsymbol{\omega}_\bullet^1 = KK_{\mathcal{E}_2}(\mathfrak{A}_\bullet, \mathsf{S}\boldsymbol{\Phi}_\bullet)$. If we use the the standard identification of $KK_{\mathcal{E}_2}(\mathfrak{A}_\bullet, \mathsf{SS}\mathfrak{B}_\bullet)$ with $KK_{\mathcal{E}_2}(\mathfrak{A}_\bullet, \mathfrak{B}_\bullet)$, this will give rise to a short exact sequence

$$\operatorname{cok}\boldsymbol{\omega}_\bullet \hookrightarrow KK_{\mathcal{E}_2}(\mathfrak{A}_\bullet, \mathfrak{B}_\bullet) \twoheadrightarrow \ker\boldsymbol{\omega}_\bullet^1.$$

By assumption, we have

$$\begin{tikzcd}
KK_{\mathcal{E}_2}(\mathfrak{A}_\bullet, \mathsf{S}\mathfrak{D}_\bullet) \arrow[r, "\boldsymbol{\omega}_\bullet^1"] \arrow[d, "\cong"', "\Gamma_{\mathfrak{A}_\bullet,\mathsf{S}\mathfrak{D}_\bullet}"] & KK_{\mathcal{E}_2}(\mathfrak{A}_\bullet, \mathsf{S}\mathfrak{C}_\bullet) \arrow[d, "\cong"', "\Gamma_{\mathfrak{A}_\bullet,\mathsf{S}\mathfrak{C}_\bullet}"] \\
\operatorname{Hom}(K_\circledast(\mathfrak{A}_\bullet), K_\circledast(\mathsf{S}\mathfrak{D}_\bullet)) \arrow[r, "K_\circledast(\mathsf{S}\boldsymbol{\Phi}_\bullet)\circ-"] & \operatorname{Hom}(K_\circledast(\mathfrak{A}_\bullet), K_\circledast(\mathsf{S}\mathfrak{C}_\bullet)),
\end{tikzcd}$$

which is commutative (by universality). Since $\operatorname{Hom}(K_\circledast(\mathfrak{A}_\bullet), -)$ is left-exact, it follows from the short exact sequence

$$K_\circledast(\mathfrak{B}_\bullet) \hookrightarrow K_{\circledast+1}(\mathfrak{D}_\bullet) \overset{K_{\circledast+1}(\boldsymbol{\Phi}_\bullet)}{\twoheadrightarrow} K_{\circledast+1}(\mathfrak{C}_\bullet),$$

that we have

$$\ker(\boldsymbol{\omega}_\bullet^1) \cong \ker(K_\circledast(\mathsf{S}\boldsymbol{\Phi}_\bullet)\circ -) \cong \ker(K_{\circledast+1}(\boldsymbol{\Phi}_\bullet)\circ -) \cong \operatorname{Hom}(K_\circledast(\mathfrak{A}_\bullet), K_\circledast(\mathfrak{B}_\bullet)).$$

Similarly, we have

$$\begin{tikzcd}
KK_{\mathcal{E}_2}(\mathfrak{A}_\bullet, \mathfrak{D}_\bullet) \arrow[r, "\boldsymbol{\omega}_\bullet"] \arrow[d, "\cong"', "\Gamma_{\mathfrak{A}_\bullet,\mathfrak{D}_\bullet}"] & KK_{\mathcal{E}_2}(\mathfrak{A}_\bullet, \mathfrak{C}_\bullet) \arrow[d, "\cong"', "\Gamma_{\mathfrak{A}_\bullet,\mathfrak{C}_\bullet}"] \\
\operatorname{Hom}(K_\circledast(\mathfrak{A}_\bullet), K_\circledast(\mathfrak{D}_\bullet)) \arrow[r, "K_\circledast(\boldsymbol{\Phi}_\bullet)\circ-"] & \operatorname{Hom}(K_\circledast(\mathfrak{A}_\bullet), K_\circledast(\mathfrak{C}_\bullet)),
\end{tikzcd}$$

which is commutative (by universality). Since

$$K_{\circledast+1}(\mathfrak{B}_\bullet) \hookrightarrow K_\circledast(\mathfrak{D}_\bullet) \overset{K_\circledast(\boldsymbol{\Phi}_\bullet)}{\twoheadrightarrow} K_\circledast(\mathfrak{C}_\bullet)$$

is an injective resolution of $K_{\circledast+1}(\mathfrak{B}_\bullet)$, it follows from the definition of Ext^1, that

$$\operatorname{cok}\boldsymbol{\omega}_\bullet \cong \operatorname{cok}(K_\circledast(\boldsymbol{\Phi}_\bullet)\circ -) \cong \operatorname{Ext}^1(K_\circledast(\mathfrak{A}_\bullet), K_{\circledast+1}(\mathfrak{B}_\bullet)).$$ ●●●

Proposition 5.4.4. *Let $\mathfrak{B}_\bullet$ be a fixed given object of $S\mathcal{E}_2$ with $K_\circledast(\mathfrak{B}_\bullet)$ injective. Then for all objects $\mathfrak{A}_\bullet$ of $\mathcal{N}_{\mathcal{E}_2}$ the natural map*

$$\Gamma_{\mathfrak{A}_\bullet,\mathfrak{B}_\bullet}\colon KK_{\mathcal{E}_2}(\mathfrak{A}_\bullet, \mathfrak{B}_\bullet) \longrightarrow \operatorname{Hom}(K_\circledast(\mathfrak{A}_\bullet), K_\circledast(\mathfrak{B}_\bullet))$$

is a (group) isomorphism.

Proof. It is enough to show that the class of objects $\mathfrak{A}_\bullet$ for which $\Gamma_{\mathfrak{A}_\bullet,\mathfrak{B}_\bullet}$ is an isomorphism satisfies the corresponding conditions (1)–(4) in Definition 5.4.1.

(1) Let $\mathfrak{A}_\bullet$ be an object of $S\mathcal{E}_2$ with $\mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_6$ nuclear and in the bootstrap class $\mathcal{N}$. Let $\mathfrak{A}_\bullet^1$, $\mathfrak{A}_\bullet^2$, $\mathfrak{A}_\bullet^3$, and $\mathfrak{A}_\bullet^4$ denote the objects

$$\begin{tikzcd}
\mathfrak{A}_1 \arrow[r, hook, "\mathrm{id}"] \arrow[d, equal] & \mathfrak{A}_1 \arrow[r, two heads] \arrow[d, hook, "\mathrm{id}"] & 0 \arrow[d, hook] \\
\mathfrak{A}_1 \arrow[r, hook, "\mathrm{id}"] & \mathfrak{A}_1 \arrow[r, two heads] \arrow[d, two heads] & 0 \arrow[d, two heads] \\
& 0 \arrow[r, equal] & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r, hook] \arrow[d, equal] & \mathfrak{A}_4 \arrow[r, two heads, "\mathrm{id}"] \arrow[d, hook, "\mathrm{id}"] & \mathfrak{A}_4 \arrow[d, hook, "\mathrm{id}"] \\
0 \arrow[r, hook] & \mathfrak{A}_4 \arrow[r, two heads, "\mathrm{id}"] \arrow[d, two heads] & \mathfrak{A}_4 \arrow[d, two heads] \\
& 0 \arrow[r, equal] & 0
\end{tikzcd}$$

$$
\begin{tikzcd}
0 \arrow[r, hook] \arrow[d, equal] & 0 \arrow[r, two heads] \arrow[d, hook] & 0 \arrow[d, hook] \\
0 \arrow[r, hook] & \mathfrak{A}_6 \arrow[r, two heads, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & \mathfrak{A}_6 \arrow[d, "\mathrm{id}"] \\
& \mathfrak{A}_6 \arrow[r, equal] & \mathfrak{A}_6
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r, hook] \arrow[d, equal] & \mathfrak{A}_4 \arrow[r, two heads, "\mathrm{id}"] \arrow[d, hook] & \mathfrak{A}_4 \arrow[d, hook] \\
0 \arrow[r, hook] & \mathfrak{A}_5 \arrow[r, two heads, "\mathrm{id}"] \arrow[d] & \mathfrak{A}_5 \arrow[d] \\
& \mathfrak{A}_6 \arrow[r, equal] & \mathfrak{A}_6
\end{tikzcd}
$$

resp. An argument similar to that on page 170 in [Bon02]— using Proposition 5.3.3 and [Bon02, Lemma 7.1.5] — it follows that $\Gamma_{\mathfrak{A}^1_\bullet,\mathfrak{B}_\bullet}$, $\Gamma_{\mathfrak{A}^2_\bullet,\mathfrak{B}_\bullet}$, and $\Gamma_{\mathfrak{A}^3_\bullet,\mathfrak{B}_\bullet}$ are isomorphisms. Since we have short exact sequences $\mathfrak{A}^1_\bullet \hookrightarrow \mathfrak{A}_\bullet \twoheadrightarrow \mathfrak{A}^4_\bullet$ and $\mathfrak{A}^2_\bullet \hookrightarrow \mathfrak{A}^4_\bullet \twoheadrightarrow \mathfrak{A}^3_\bullet$, it follows from part (3) below that $\Gamma_{\mathfrak{A}_\bullet,\mathfrak{B}_\bullet}$ is an isomorphism.

(2) Let $(\mathfrak{A}^n_\bullet)_{n=1}^\infty$ be an inductive system. Assume that for each $n \in \mathbb{N}$, $\mathfrak{A}^n_\bullet$ is nuclear and that $\Gamma_{\mathfrak{A}^n_\bullet,\mathfrak{B}_\bullet}$ is an isomorphism. Let $\mathfrak{A}_\bullet$ denote the inductive limit.

By use of the Milnor-$\lim^1$-sequence for $KK_{\mathcal{E}_2}(-,\mathfrak{B}_\bullet)$ and $\mathrm{Hom}(K_\circledast(-), K_\circledast(\mathfrak{B}_\bullet))$ (*cf.* [Bon02, Satz 4.3.2 und Bemerkung 4.3.3]) and the Five Lemma, it follows that $\Gamma_{\mathfrak{A}_\bullet,\mathfrak{B}_\bullet}$ is an isomorphism.

(3) Let $\mathfrak{A}_\bullet \hookrightarrow \mathfrak{C}_\bullet \twoheadrightarrow \mathfrak{D}_\bullet$ be a short exact sequence in $S\mathcal{E}_2$ with $\mathfrak{A}_i$, $\mathfrak{C}_i$, and $\mathfrak{D}_i$ nuclear, for all $i = 1, 2, \ldots, 6$.

By Lemma 5.2.2, the above short exact sequence induces a cyclic six term exact sequence of $K_\circledast$-objects. Since $K_\circledast(\mathfrak{B}_\bullet)$ is injective, $\mathrm{Hom}(-, K_\circledast(\mathfrak{B}_\bullet))$ is an exact functor. Consequently, it maps the cyclic six term exact sequence of $K_\circledast$-objects to a cyclic six term exact sequence of abelian groups.

The natural homomorphisms $\Gamma_{-,\mathfrak{B}_\bullet}$ give a morphism from the cyclic six term exact sequence in $KK_{\mathcal{E}_2}$ to the above mentioned cyclic six term exact sequence (the reason the boundary maps commute is that they both are defined in the same way — here it is important that we make exactly the choices we do in Definition 4.4.1 and Definition 5.2.1).

Now it follows from the Five Lemma, that if two of the maps $\Gamma_{\mathfrak{A}_\bullet,\mathfrak{B}_\bullet}$, $\Gamma_{\mathfrak{C}_\bullet,\mathfrak{B}_\bullet}$, and $\Gamma_{\mathfrak{D}_\bullet,\mathfrak{B}_\bullet}$ are isomorphisms then also the third is an isomorphism.

(4) Assume that $\Gamma_{\mathfrak{A}_\bullet,\mathfrak{B}_\bullet}$ is an isomorphism, and that $y \in KK_{\mathcal{E}_2}(\mathfrak{A}_\bullet, \mathfrak{A}'_\bullet)$ is a $KK_{\mathcal{E}_2}$-equivalence. Since Γ actually is a functor, $\Gamma_{\mathfrak{A}_\bullet,\mathfrak{A}'_\bullet}(y)$ is an isomorphism and the diagram

$$
\begin{tikzcd}
KK_{\mathcal{E}_2}(\mathfrak{A}'_\bullet, \mathfrak{B}_\bullet) \arrow[r, "\Gamma_{\mathfrak{A}'_\bullet,\mathfrak{B}_\bullet}"] \arrow[d, "y \otimes_{KK_{\mathcal{E}_2}} -"', "\cong"] & \mathrm{Hom}(K_\circledast(\mathfrak{A}'_\bullet), K_\circledast(\mathfrak{B}_\bullet)) \arrow[d, "\cong"', "- \circ \Gamma_{\mathfrak{A}'_\bullet,\mathfrak{B}_\bullet}(y)"] \\
KK_{\mathcal{E}_2}(\mathfrak{A}_\bullet, \mathfrak{B}_\bullet) \arrow[r, "\Gamma_{\mathfrak{A}_\bullet,\mathfrak{B}_\bullet}"] & \mathrm{Hom}(K_\circledast(\mathfrak{A}_\bullet), K_\circledast(\mathfrak{B}_\bullet))
\end{tikzcd}
$$

commutes. Since $\Gamma_{\mathfrak{A}_\bullet,\mathfrak{B}_\bullet}$ is an isomorphism (by assumption), we have that $\Gamma_{\mathfrak{A}'_\bullet,\mathfrak{B}_\bullet}$ is also an isomorphism. ●●●

5.5 Naturality of the UCT

Definition 5.5.1. Let $\mathcal{N}'_{\mathcal{E}_2}$ denote the class of objects $\mathfrak{A}_\bullet$ of $S\mathcal{E}_2$ with $\mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_6$ are nuclear such that the UCT holds for $(\mathfrak{A}_\bullet, \mathfrak{B}_\bullet)$ for every object $\mathfrak{B}_\bullet$ of $S\mathcal{E}_2$. Clearly (by the UCT), the class $\mathcal{N}_{\mathcal{E}_2}$ is contained in $\mathcal{N}'_{\mathcal{E}_2}$.

The UCT for $KK_{\mathcal{E}_2}$ is also natural in both variables (*cf.* also [Bon02, Satz 7.7.1]).

Proposition 5.5.2. *The UCT for $KK_{\mathcal{E}_2}$ is natural in both variables with respect to the $KK_{\mathcal{E}_2}$-product, i.e., if $\mathfrak{A}_\bullet$ and $\mathfrak{A}'_\bullet$ are objects of $\mathcal{N}'_{\mathcal{E}_2}$, x is an element of $KK_{\mathcal{E}_2}(\mathfrak{A}_\bullet, \mathfrak{A}'_\bullet)$, and y is an element of*

$KK_{\mathcal{E}_2}(\mathfrak{B}_\bullet, \mathfrak{B}'_\bullet)$, *then the we have the following commutative diagrams:*

$$\begin{tikzcd}
\mathrm{Ext}(K_\circledast(\mathfrak{A}'_\bullet), K_{\circledast+1}(\mathfrak{B}_\bullet)) \arrow[r, hook, "\Delta_{\mathfrak{A}'_\bullet,\mathfrak{B}_\bullet}"] \arrow[d, "\Gamma_{\mathfrak{A}_\bullet,\mathfrak{A}'_\bullet}(x)^*"] & KK_{\mathcal{E}_2}(\mathfrak{A}'_\bullet, \mathfrak{B}_\bullet) \arrow[r, two heads, "\Gamma_{\mathfrak{A}'_\bullet,\mathfrak{B}_\bullet}"] \arrow[d, "x\otimes_{\mathfrak{A}'_\bullet}-"] & \mathrm{Hom}(K_\circledast(\mathfrak{A}'_\bullet), K_\circledast(\mathfrak{B}_\bullet)) \arrow[d, "-\circ\Gamma_{\mathfrak{A}_\bullet,\mathfrak{A}'_\bullet}(x)"] \\
\mathrm{Ext}(K_\circledast(\mathfrak{A}_\bullet), K_{\circledast+1}(\mathfrak{B}_\bullet)) \arrow[r, hook, "\Delta_{\mathfrak{A}_\bullet,\mathfrak{B}_\bullet}"] & KK_{\mathcal{E}_2}(\mathfrak{A}_\bullet, \mathfrak{B}_\bullet) \arrow[r, two heads, "\Gamma_{\mathfrak{A}_\bullet,\mathfrak{B}_\bullet}"] & \mathrm{Hom}(K_\circledast(\mathfrak{A}_\bullet), K_\circledast(\mathfrak{B}_\bullet))
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Ext}(K_\circledast(\mathfrak{A}_\bullet), K_{\circledast+1}(\mathfrak{B}_\bullet)) \arrow[r, hook, "\Delta_{\mathfrak{A}_\bullet,\mathfrak{B}_\bullet}"] \arrow[d, "\Gamma_{\mathfrak{B}_\bullet,\mathfrak{B}'_\bullet}(y)_*"] & KK_{\mathcal{E}_2}(\mathfrak{A}_\bullet, \mathfrak{B}_\bullet) \arrow[r, two heads, "\Gamma_{\mathfrak{A}_\bullet,\mathfrak{B}_\bullet}"] \arrow[d, "-\otimes_{\mathfrak{B}_\bullet}y"] & \mathrm{Hom}(K_\circledast(\mathfrak{A}_\bullet), K_\circledast(\mathfrak{B}_\bullet)) \arrow[d, "\Gamma_{\mathfrak{B}_\bullet,\mathfrak{B}'_\bullet}(y)\circ-"] \\
\mathrm{Ext}(K_\circledast(\mathfrak{A}_\bullet), K_{\circledast+1}(\mathfrak{B}'_\bullet)) \arrow[r, hook, "\Delta_{\mathfrak{A}_\bullet,\mathfrak{B}'_\bullet}"] & KK_{\mathcal{E}_2}(\mathfrak{A}_\bullet, \mathfrak{B}'_\bullet) \arrow[r, two heads, "\Gamma_{\mathfrak{A}_\bullet,\mathfrak{B}'_\bullet}"] & \mathrm{Hom}(K_\circledast(\mathfrak{A}_\bullet), K_\circledast(\mathfrak{B}'_\bullet))
\end{tikzcd}$$

Proof. Naturality of the quotient map is clear, since Γ is a functor. Naturality of Δ is proved analogous to [RS87, Theorem 4.4] (*cf.* also [Bon02, Satz 7.7.1]) — here we will need the geometric projective resolutions (as mentioned above). ●●●

As usually, we use this to prove that isomorphisms on the invariant level can be lifted to a $KK_{\mathcal{E}_2}$-equivalence (*cf.* [Bon02, Proposition 7.7.2]).

Proposition 5.5.3. *Let $\mathfrak{A}_\bullet$ and $\mathfrak{B}_\bullet$ be objects of $\mathcal{N}'_{\mathcal{E}_2}$ and let $x \in KK_{\mathcal{E}_2}(\mathfrak{A}_\bullet, \mathfrak{B}_\bullet)$. Then x is a $KK_{\mathcal{E}_2}$-equivalence if and only if $\Gamma_{\mathfrak{A}_\bullet,\mathfrak{B}_\bullet}(x)$ is an isomorphism.*

Proof. Since Γ is a functor, the “only if”-part is clear. The “if”-part follows from the naturality of the UCT and the Five Lemma (for abelian groups). ●●●

5.6 Classification of purely infinite algebras

In this section we extend the classification of purely infinite C^*-algebras to include (stable and unital) purely infinite C^*-algebras with exactly two non-trivial ideals (assuming the bootstrap class, separability, and nuclearity, of course). See also [Rør97, Bon02, Kir00, ER06, RR07, Phi00, Kir].

Theorem 5.6.1. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras with exactly two non-trivial ideals each, $\mathfrak{I}_1$ and $\mathfrak{I}_2$ resp. $\mathfrak{J}_1$ and $\mathfrak{J}_2$. Assume, moreover, that the ideal lattices of $\mathfrak{A}$ and $\mathfrak{B}$ are totally ordered. Assume that $\mathfrak{I}_1$, $\mathfrak{I}_2/\mathfrak{I}_1$, $\mathfrak{A}/\mathfrak{I}_2$, $\mathfrak{J}_1$, $\mathfrak{J}_2/\mathfrak{J}_1$, and $\mathfrak{B}/\mathfrak{J}_2$ are Kirchberg algebras in the bootstrap class $\mathcal{N}$. Let $\mathfrak{A}_\bullet$ and $\mathfrak{B}_\bullet$ be the corresponding objects of $S\mathcal{E}_2$.*

If $\mathfrak{A}$ and $\mathfrak{B}$ are stable, then every isomorphism from $K_\circledast(\mathfrak{A}_\bullet)$ to $K_\circledast(\mathfrak{B}_\bullet)$ can be lifted to an isomorphism from $\mathfrak{A}_\bullet$ to $\mathfrak{B}_\bullet$.

If $\mathfrak{A}$ and $\mathfrak{B}$ are unital, then every isomorphism from $K_\circledast(\mathfrak{A}_\bullet)$ to $K_\circledast(\mathfrak{B}_\bullet)$ sending $[\mathbb{1}_\mathfrak{A}]_0 \in K_0(\mathfrak{A})$ to $[\mathbb{1}_\mathfrak{B}]_0 \in K_0(\mathfrak{B})$ can be lifted to an isomorphism from $\mathfrak{A}_\bullet$ to $\mathfrak{B}_\bullet$.

Proof. The stable case follows from the UCT and [Kir00, Folgerung 4.3] as in the one ideal case (*cf.* [ER06, Theorem 5]). The only thing which might not be clear, is that $\mathfrak{A}$ and $\mathfrak{B}$ are strongly purely infinite. Strong pure infiniteness is considered in [KR02], and it is shown that a separable, stable and nuclear C^*-algebra $\mathfrak{C}$ is strongly purely infinite, if and only if $\mathfrak{C}$ absorbs $\mathcal{O}_\infty$, i.e. if and only if $\mathfrak{C} \cong \mathfrak{C} \otimes \mathcal{O}_\infty$. Now it follows from [TW07, Theorem 4.3] that $\mathfrak{A}$ and $\mathfrak{B}$ are strongly purely infinite.

The unital case now follows from the stable case and [ER06, Theorem 11] and [RR07, Theorem 2.1] by noting that $\mathfrak{A}$ and $\mathfrak{B}$ are properly infinite (by [KR00, Theorem 4.16 and Theorem 4.19]). ●●●

A classification result has been obtained in the case of one non-trivial ideal when the algebra is neither unital nor stable by including the class of the unit in the quotient (*cf.* the paper in Appendix B).

Question 1. *Do we have a (strong) classification in the cases that the algebras are neither unital nor stable (by including the class of the relevant unit)?*

In the one ideal case, Rørdam has characterized the range of the six term exact sequences of extensions of stable Kirchberg algebras as the cyclic six term exact sequences of countable abelian groups, *cf.* [Rør97] (the cases with non-stable quotients have also been characterized, *cf.* Appendix B).

Question 2. *What is the range of the invariant?*

Of course, it is also natural to ask to what extent the invariant can be used for classification of purely infinite C^*-algebras with other (finite) ideal lattices.

Chapter 6

Ideal-related K-theory with coefficients

with Søren Eilers and Efren Ruiz

To characterize the automorphism groups of purely infinite C^*-algebras up to, say, approximate unitary equivalence, one naturally looks at the work of Dadarlat and Loring, which gave such a characterization of the automorphism groups of certain stably finite C^*-algebras of real rank zero as a corollary to their Universal Multi-Coefficient Theorem (UMCT), *cf.* [DL96]. But even for nuclear, separable, purely infinite C^*-algebras with real rank zero, finitely generated K-theory and only one non-trivial ideal, there are substantial problems in doing so. The work of Rørdam (*cf.* [Rør97]) clearly indicates that the right invariant contains the associated six term exact sequence in K-theory, and the work of Dadarlat and Loring indicates that one should consider K-theory with coefficients in a similar way.

In the paper in Appendix D, we have given a series of examples showing that the naïve approach — of combining the six term exact sequence with total K-theory — does not work. There are several obstructions given in the paper, and they can even be obtained using Cuntz-Krieger algebras of type (II) with exactly one non-trivial ideal.

With this as motivation, we have defined a new invariant, which we believe should be thought of as the substitute for total K-theory, when working with C^*-algebras with exactly one non-trivial ideal. We call it *ideal-related K-theory with coefficients*, and introduce it in this chapter. It is easy to show that all the obstructions from the paper in Appendix D vanish when using this invariant. We furthermore exhibit a lot of diagrams which are part of the new invariant (though not all of it). These diagrams can — in many cases — be of big help when computing the new groups which go into the invariant. Also these diagrams are used in a work in progress by the three authors, where we show a UMCT for $KK_{\mathcal{E}}$ for a class of C^*-algebras including all Cuntz-Krieger algebras of type (II) with one specified ideal (*cf.* [ERR]) — in this case, the invariant can actually be relaxed quite much.

6.1 An invariant

In this section, we introduce the new invariant.

Definition 6.1.1. Let $n \in \mathbb{N}_{\geq 2}$. We let $\mathbb{I}_{n,0}$ denote the (non-unital) **dimension-drop interval**, *i.e.*, $\mathbb{I}_{n,0}$ is the mapping cone of the unital $*$-homomorphism from $\mathbb{C}$ to M_n.

Definition 6.1.2. Let $n \in \mathbb{N}_{\geq 2}$. We let $\mathfrak{e}_{n,0}$ denote the mapping cone sequence

$$\mathfrak{e}_{n,0} \colon \mathsf{SM}_n \hookrightarrow \mathbb{I}_{n,0} \twoheadrightarrow \mathbb{C}$$

corresponding to the unital $*$-homomorphism from $\mathbb{C}$ to M_n. We let, moreover, $\mathfrak{e}_{n,i} = \mathfrak{mc}^i(\mathfrak{e}_{n,0})$, for all $i \in \mathbb{N}$. We write

$$\mathfrak{e}_{n,1} \colon \mathsf{S}\mathbb{C} \hookrightarrow \mathbb{I}_{n,1} \twoheadrightarrow \mathbb{I}_{n,0},$$
$$\mathfrak{e}_{n,i} \colon \mathsf{S}\mathbb{I}_{n,i-2} \hookrightarrow \mathbb{I}_{n,i} \twoheadrightarrow \mathbb{I}_{n,i-1}, \text{ for } i \geq 2.$$

Similarly, we set $\mathfrak{f}_{1,0}\colon \mathbb{C} \xhookrightarrow{\mathrm{id}} \mathbb{C} \twoheadrightarrow 0$ and $\mathfrak{f}_{n,0}\colon \mathbb{I}_{n,0} \xhookrightarrow{\mathrm{id}} \mathbb{I}_{n,0} \twoheadrightarrow 0$, for all $n \in \mathbb{N}_{\geq 2}$. Moreover, we set $\mathfrak{f}_{n,i} = \mathfrak{mc}^i(\mathfrak{f}_{n,0})$ for all $n \in \mathbb{N}$ and all $i \in \mathbb{N}$.

Definition 6.1.3. Let K_{six} denote the functor, which to each extension of C^*-algebras associates the corresponding standard cyclic six term exact sequence (as defined in [RLL00] — *cf.* Example 3.4.2). We let $\mathrm{Hom}_{\mathrm{six}}(K_{\mathrm{six}}(e_1), K_{\mathrm{six}}(e_2))$ denote the group of cyclic chain homomorphisms.

As in [DL96], we let $K_i(-;\mathbb{Z}_n) = KK^i(\mathbb{I}_{n,0},-)$. Moreover, we let $\underline{K}$ denote total K-theory as defined in [DL96].

Remark 6.1.4. As is easily seen, the above cyclic six term exact sequence in K-theory differs from that defined by Bonkat in [Bon02, §7.3] by the index and exponential maps having the opposite signs. This makes no difference for the arguments and results in [Bon02] (the important thing here is that we change the sign of either the index map or the exponential map compared with the definition of the connecting homomorphisms in KK-theory).

By applying Lemma 3.3.1 and Lemma 3.4.3 to

$$\begin{tikzcd}
\mathsf{S}\mathfrak{A}_0 \arrow[r, hook, "\mathsf{S}\iota"] \arrow[d, hook] & \mathsf{S}\mathfrak{A}_1 \arrow[r, two heads, "\mathsf{S}\pi"] \arrow[d, hook] & \mathsf{S}\mathfrak{A}_2 \arrow[d, hook] \\
\mathsf{C}\mathfrak{A}_0 \arrow[r, hook, "\mathsf{C}\iota"] \arrow[d, two heads] & \mathsf{C}\mathfrak{A}_1 \arrow[r, two heads, "\mathsf{C}\pi"] \arrow[d, two heads] & \mathsf{C}\mathfrak{A}_2 \arrow[d, two heads] \\
\mathfrak{A}_0 \arrow[r, hook, "\iota"] & \mathfrak{A}_1 \arrow[r, two heads, "\pi"] & \mathfrak{A}_2
\end{tikzcd}$$

we get a commuting diagram

$$\begin{tikzcd}
{} \arrow[r, "\delta_1^e"] & K_0(\mathfrak{A}_0) \arrow[r, "K_0(\iota)"] \arrow[d, "\beta_{\mathfrak{A}_0}", "\cong"'] & K_0(\mathfrak{A}_1) \arrow[r, "K_0(\pi)"] \arrow[d, "\beta_{\mathfrak{A}_1}", "\cong"'] & K_0(\mathfrak{A}_2) \arrow[r, "\delta_0^e"] \arrow[d, "\beta_{\mathfrak{A}_2}", "\cong"'] & K_1(\mathfrak{A}_0) \arrow[r, "K_1(\iota)"] \arrow[d, "\beta_{\mathfrak{A}_0}", "\cong"'] & K_1(\mathfrak{A}_1) \arrow[r, "K_1(\pi)"] \arrow[d, "\beta_{\mathfrak{A}_1}", "\cong"'] & K_1(\mathfrak{A}_2) \arrow[r, "\delta_1^e"] \arrow[d, "\beta_{\mathfrak{A}_2}", "\cong"'] & {} \\
{} \arrow[r, "\delta_0^{\mathsf{S}e}"] & K_1(\mathsf{S}\mathfrak{A}_0) \arrow[r, "K_1(\mathsf{S}\iota)"] & K_1(\mathsf{S}\mathfrak{A}_1) \arrow[r, "K_1(\mathsf{S}\pi)"] & K_1(\mathsf{S}\mathfrak{A}_2) \arrow[r, "\delta_1^{\mathsf{S}e}"] & K_0(\mathsf{S}\mathfrak{A}_0) \arrow[r, "K_0(\mathsf{S}\iota)"] & K_0(\mathsf{S}\mathfrak{A}_1) \arrow[r, "K_0(\mathsf{S}\pi)"] & K_0(\mathsf{S}\mathfrak{A}_2) \arrow[r, "\delta_0^{\mathsf{S}e}"] & {}
\end{tikzcd}$$

Consequently, the definition of "$(K_{*+1}A_i)$" in [Bon02] is just $K_{\mathrm{six}}(\mathsf{S}e)$ (up to canonical identification with our terminology). The same argument works if we choose to work with the slightly different cyclic six term exact sequence defined in [Bon02]. Note also that this is not true if we define the cyclic six term sequence using the abstract machinery of Section 3.2.

Definition 6.1.5. For each extension e of separable C^*-algebras, we define the **ideal-related K-theory with coefficients**, $\underline{K}_{\mathcal{E}}(e)$, of e to be the (graded) group

$$\underline{K}_{\mathcal{E}}(e) = \bigoplus_{i=0}^{2}\bigoplus_{j=0}^{1}\left(KK_{\mathcal{E}}^{j}(\mathfrak{f}_{1,i},e) \oplus \bigoplus_{n=2}^{\infty} KK_{\mathcal{E}}^{j}(\mathfrak{e}_{n,i},e) \oplus KK_{\mathcal{E}}^{j}(\mathfrak{f}_{n,i},e)\right).$$

A homomorphism from $\underline{K}_{\mathcal{E}}(e_1)$ to $\underline{K}_{\mathcal{E}}(e_2)$ is a group homomorphism α from $\underline{K}_{\mathcal{E}}(e_1)$ to $\underline{K}_{\mathcal{E}}(e_2)$ respecting the direct sum decomposition and the natural homomorphisms induced by the elements of $KK_{\mathcal{E}}^{j}(e,e')$, for $j=0,1$, where e and e' are in $\{\,\mathfrak{e}_{n,i},\mathfrak{f}_{n,i},\mathfrak{f}_{1,i} \mid n \in \mathbb{N}_{\geq 2}, i=0,1,2\,\}$. Every morphism from an extension e_1 to another extension e_2 (of separable C^*-algebras) induces a homomorphism from $\underline{K}_{\mathcal{E}}(e_1)$ to $\underline{K}_{\mathcal{E}}(e_2)$. In this way $\underline{K}_{\mathcal{E}}$ becomes a functor.

Remark 6.1.6. For extensions $e_1 : \mathfrak{A}_0 \hookrightarrow \mathfrak{A}_1 \twoheadrightarrow \mathfrak{A}_2$ and $e_2 : \mathfrak{B}_0 \hookrightarrow \mathfrak{B}_1 \twoheadrightarrow \mathfrak{B}_2$ of separable C^*-algebras, we have natural homomorphisms $G_i\colon KK_{\mathcal{E}}(e_1,e_2) \longrightarrow KK(\mathfrak{A}_i,\mathfrak{B}_i)$, for $i=0,1,2$.

As in the proof of [Bon02, Satz 7.5.6], the obvious diagram

$$\begin{tikzcd}
\mathrm{Ext}_{\mathrm{six}}(K_{\mathrm{six}}(e_1),K_{\mathrm{six}}(\mathsf{S}e_2)) \arrow[r, hook] \arrow[d] & KK_{\mathcal{E}}(e_1,e_2) \arrow[r, two heads] \arrow[d, "G_i"] & \mathrm{Hom}_{\mathrm{six}}(K_{\mathrm{six}}(e_1),K_{\mathrm{six}}(e_2)) \arrow[d] \\
\mathrm{Ext}(K_0(\mathfrak{A}_i),K_1(\mathfrak{B}_i)) \oplus \mathrm{Ext}(K_1(\mathfrak{A}_i),K_0(\mathfrak{B}_i)) \arrow[r, hook] & KK(\mathfrak{A}_i,\mathfrak{B}_i) \arrow[r, two heads] & \mathrm{Hom}(K_0(\mathfrak{A}_i),K_0(\mathfrak{B}_i)) \oplus \mathrm{Hom}(K_1(\mathfrak{A}_i),K_1(\mathfrak{B}_i))
\end{tikzcd}$$

commutes and is natural in e_2, for $i = 0, 1, 2$ — provided that e_1 belongs to the UCT class considered by Bonkat.

Let $e\colon \mathfrak{A}_0 \overset{\iota}{\hookrightarrow} \mathfrak{A}_1 \overset{\pi}{\twoheadrightarrow} \mathfrak{A}_2$ and $e'\colon \mathfrak{B}_0 \overset{\iota'}{\hookrightarrow} \mathfrak{B}_1 \overset{\pi'}{\twoheadrightarrow} \mathfrak{B}_2$ be two given extensions. Then we define

$$\Lambda_{e,e'}\colon \operatorname{Hom}_{\mathrm{six}}(K_{\mathrm{six}}(e), K_{\mathrm{six}}(e')) \longrightarrow \operatorname{Hom}_{\mathrm{six}}(K_{\mathrm{six}}(\mathfrak{mc}(e)), K_{\mathrm{six}}(\mathfrak{mc}(e')))$$

as follows: Let $(\alpha_i)_{i=0}^5$ in $\operatorname{Hom}_{\mathrm{six}}(K_{\mathrm{six}}(e), K_{\mathrm{six}}(e'))$ be given. Then by Corollary 3.4.4 the diagram

$$\begin{tikzcd}
{} \arrow[r, "\delta_1^{\mathfrak{mc}(e)}"] & K_0(\mathsf{S}\mathfrak{A}_2) \arrow[r, "K_0(\iota_{\mathfrak{mc}})"] \arrow[d, "\theta_{\mathfrak{A}_2}^{-1}"] & K_0(\mathsf{C}_\pi) \arrow[r, "K_0(\pi_{\mathfrak{mc}})"] \arrow[d, "K_0(f_e)^{-1}"] & K_0(\mathfrak{A}_1) \arrow[r, "\delta_0^{\mathfrak{mc}(e)}"] \arrow[d, equal] & K_1(\mathsf{S}\mathfrak{A}_2) \arrow[r, "K_1(\iota_{\mathfrak{mc}})"] \arrow[d, "\beta_{\mathfrak{A}_2}^{-1}"] & K_1(\mathsf{C}_\pi) \arrow[r, "K_1(\pi_{\mathfrak{mc}})"] \arrow[d, "K_1(f_e)^{-1}"] & K_1(\mathfrak{A}_1) \arrow[r, "\delta_1^{\mathfrak{mc}(e)}"] \arrow[d, equal] & {} \\
{} \arrow[r, "K_1(\pi)"] & K_1(\mathfrak{A}_2) \arrow[r, "-\delta_1^e"] \arrow[d, "\alpha_5"] & K_0(\mathfrak{A}_0) \arrow[r, "K_0(\iota)"] \arrow[d, "\alpha_0"] & K_0(\mathfrak{A}_1) \arrow[r, "-K_0(\pi)"] \arrow[d, "\alpha_1"] & K_0(\mathfrak{A}_2) \arrow[r, "\delta_0^e"] \arrow[d, "\alpha_2"] & K_1(\mathfrak{A}_0) \arrow[r, "K_1(\iota)"] \arrow[d, "\alpha_3"] & K_1(\mathfrak{A}_1) \arrow[r, "K_1(\pi)"] \arrow[d, "\alpha_4"] & {} \\
{} \arrow[r, "K_1(\pi')"] & K_1(\mathfrak{B}_2) \arrow[r, "-\delta_1^{e'}"] \arrow[d, "\theta_{\mathfrak{B}_2}"] & K_0(\mathfrak{B}_0) \arrow[r, "K_0(\iota')"] \arrow[d, "K_0(f_{e'})"] & K_0(\mathfrak{B}_1) \arrow[r, "-K_0(\pi')"] \arrow[d, equal] & K_0(\mathfrak{B}_2) \arrow[r, "\delta_0^{e'}"] \arrow[d, "\beta_{\mathfrak{B}_2}"] & K_1(\mathfrak{B}_0) \arrow[r, "K_1(\iota')"] \arrow[d, "K_1(f_{e'})"] & K_1(\mathfrak{B}_1) \arrow[r, "K_1(\pi')"] \arrow[d, equal] & {} \\
{} \arrow[r, "\delta_1^{\mathfrak{mc}(e')}"] & K_0(\mathsf{S}\mathfrak{B}_2) \arrow[r, "K_0(\iota'_{\mathfrak{mc}})"] & K_0(\mathsf{C}_{\pi'}) \arrow[r, "K_0(\pi'_{\mathfrak{mc}})"] & K_0(\mathfrak{B}_1) \arrow[r, "\delta_0^{\mathfrak{mc}(e')}"] & K_1(\mathsf{S}\mathfrak{B}_2) \arrow[r, "K_1(\iota'_{\mathfrak{mc}})"] & K_1(\mathsf{C}_{\pi'}) \arrow[r, "K_1(\pi'_{\mathfrak{mc}})"] & K_1(\mathfrak{B}_1) \arrow[r, "\delta_1^{\mathfrak{mc}(e')}"] & {}
\end{tikzcd}$$

commutes. Let $\Lambda_{e,e'}((\alpha_i)_{i=0}^5)$ denote the composition of these maps. Clearly, $\Lambda_{e,e'}$ is an isomorphism. A computation shows that Λ from $\operatorname{Hom}_{\mathrm{six}}(K_{\mathrm{six}}(e), K_{\mathrm{six}}(-))$ to $\operatorname{Hom}_{\mathrm{six}}(K_{\mathrm{six}}(\mathfrak{mc}(e)), K_{\mathrm{six}}(\mathfrak{mc}(-)))$ defined by $\Lambda(e') = \Lambda_{e,e'}$ is a natural transformation such that $\Lambda_{e,e}(K_{\mathrm{six}}(\mathrm{id}_e)) = K_{\mathrm{six}}(\mathrm{id}_{\mathfrak{mc}(e)})$.

Let $S\mathcal{E}$ be the subcategory of $\mathcal{E}$ consisting only of extensions of separable C^*-algebras and morphism being triples of $*$-homomorphisms such that the obvious diagram commutes (*cf.* Definition 4.1.8). Consider the category $\mathbf{KK}_{\mathcal{E}}$ whose objects are the objects of $S\mathcal{E}$ and the group of morphisms is $KK_{\mathcal{E}}(e_1, e_2)$. Consider the the composed functor $KK_{\mathcal{E}} \circ \mathfrak{mc}$ from $S\mathcal{E}$ to $\mathbf{KK}_{\mathcal{E}}$, which sends an object e of $S\mathcal{E}$ to $\mathfrak{mc}(e)$, and sends a morphism (ϕ_0, ϕ_1, ϕ_2) of $S\mathcal{E}$ to $KK_{\mathcal{E}}(\mathfrak{mc}((\phi_0, \phi_1, \phi_2)))$. This is a stable, homotopy invariant, split exact functor, so by [Bon02, Satz 3.5.10 und Satz 6.2.4], there exists a unique functor $\widehat{\mathfrak{mc}}$ from $\mathbf{KK}_{\mathcal{E}}$ to $\mathbf{KK}_{\mathcal{E}}$ such that the diagram

$$\begin{tikzcd}
S\mathcal{E} \arrow[r, "\mathfrak{mc}"] \arrow[d, "KK_{\mathcal{E}}"'] & S\mathcal{E} \arrow[d, "KK_{\mathcal{E}}"] \\
\mathbf{KK}_{\mathcal{E}} \arrow[r, "\widehat{\mathfrak{mc}}"'] & \mathbf{KK}_{\mathcal{E}}
\end{tikzcd}$$

commutes. By the universal property, the diagram

$$\begin{tikzcd}
KK_{\mathcal{E}}(e, e') \arrow[r] \arrow[d, "\widehat{\mathfrak{mc}}"'] & \operatorname{Hom}_{\mathrm{six}}(K_{\mathrm{six}}(e), K_{\mathrm{six}}(e')) \arrow[d, "\Lambda(e')"] \\
KK_{\mathcal{E}}(\mathfrak{mc}(e), \mathfrak{mc}(e')) \arrow[r] & \operatorname{Hom}_{\mathrm{six}}(K_{\mathrm{six}}(\mathfrak{mc}(e)), K_{\mathrm{six}}(\mathfrak{mc}(e')))
\end{tikzcd}$$

commutes, where the horizontal arrows are the natural maps in the UCT.

Lemma 6.1.7. *Let e and e' be extensions of separable, nuclear C^*-algebras in the bootstrap category $\mathcal{N}$. Then $\widehat{\mathfrak{mc}}$ induces an isomorphism from $KK_{\mathcal{E}}(e, e')$ to $KK_{\mathcal{E}}(\mathfrak{mc}(e), \mathfrak{mc}(e'))$, which is natural in both variables.*

Proof. Let $\alpha_{e,e'}$ denote the map from $KK_{\mathcal{E}}(e, e')$ to $KK_{\mathcal{E}}(\mathfrak{mc}(e), \mathfrak{mc}(e'))$ induced by the functor $\widehat{\mathfrak{mc}}$. Since $\widehat{\mathfrak{mc}}$ is a functor, clearly the map is going to be natural (in both variables).

From Proposition 3.5.6 in [Bon02] (*cf.* also [Hig87, Lemma 3.2]), it follows that $\widehat{\mathfrak{mc}}$ is a group homomorphism.

Since $\Lambda_{e,e'}$ is an isomorphism, from the above diagram and the UCT of Bonkat [Bon02], we have that $\alpha_{e,e'}$ is an isomorphism whenever $K_{\mathrm{six}}(e')$ is injective.

When e' is an arbitrary extension, then by [Bon02, Proposition 7.4.3], there exist an injective geometric resolution $e_1 \hookrightarrow e_2 \twoheadrightarrow \mathsf{S}e'$ of e', *i.e.*, there exists a short exact sequence $e_1 \hookrightarrow e_2 \twoheadrightarrow \mathsf{S}e'$ of extensions from $S\mathcal{E}$, with a completely positive contractive coherent splitting, such that the induced six term exact K_{six}-sequence degenerates to a short exact sequence $K_{\mathrm{six}}(\mathsf{SS}e') \hookrightarrow K_{\mathrm{six}}(e_1) \twoheadrightarrow K_{\mathrm{six}}(e_2)$, which is an injective resolution of $K_{\mathrm{six}}(\mathsf{SS}e')$.

The cyclic six term exact sequences in $KK_{\mathcal{E}}$-theory give a commuting diagram

$$\begin{tikzcd}
KK_{\mathcal{E}}(e,\mathsf{S}e_1) \arrow[r, "\alpha_{e,\mathsf{S}e_1}"] \arrow[d] & KK_{\mathcal{E}}(\mathfrak{mc}(e),\mathfrak{mc}(\mathsf{S}e_1)) \arrow[r] \arrow[d] & KK_{\mathcal{E}}(\mathfrak{mc}(e),\mathsf{S}\mathfrak{mc}(e_1)) \arrow[d] \\
KK_{\mathcal{E}}(e,\mathsf{S}e_2) \arrow[r, "\alpha_{e,\mathsf{S}e_2}"] \arrow[d] & KK_{\mathcal{E}}(\mathfrak{mc}(e),\mathfrak{mc}(\mathsf{S}e_2)) \arrow[r] \arrow[d] & KK_{\mathcal{E}}(\mathfrak{mc}(e),\mathsf{S}\mathfrak{mc}(e_2)) \arrow[d] \\
KK_{\mathcal{E}}(e,\mathsf{SS}e') \arrow[r, "\alpha_{e,\mathsf{SS}e'}"] \arrow[d] & KK_{\mathcal{E}}(\mathfrak{mc}(e),\mathfrak{mc}(\mathsf{SS}e')) \arrow[r] \arrow[d] & KK_{\mathcal{E}}(\mathfrak{mc}(e),\mathsf{S}\mathfrak{mc}(\mathsf{S}e')) \arrow[d] \\
KK_{\mathcal{E}}(e,e_1) \arrow[r, "\alpha_{e,e_1}"] \arrow[d] & KK_{\mathcal{E}}(\mathfrak{mc}(e),\mathfrak{mc}(e_1)) \arrow[r, equal] \arrow[d] & KK_{\mathcal{E}}(\mathfrak{mc}(e),\mathfrak{mc}(e_1)) \arrow[d] \\
KK_{\mathcal{E}}(e,e_2) \arrow[r, "\alpha_{e,e_2}"] & KK_{\mathcal{E}}(\mathfrak{mc}(e),\mathfrak{mc}(e_2)) \arrow[r, equal] & KK_{\mathcal{E}}(\mathfrak{mc}(e),\mathfrak{mc}(e_2))
\end{tikzcd}$$

with exact columns. Naturality of $\alpha_{e,-}$ gives us commutativity of the squares on the left hand side, while naturality of the isomorphism from the functor $\mathfrak{mc}\circ\mathsf{S}$ to the functor $\mathsf{S}\circ\mathfrak{mc}$ gives us commutativity of the squares on the right hand side (*cf.* Lemma 3.1.11). The remaining rectangle is seen to commute by using the definition of the connecting homomorphisms and Lemma 3.1.14. By the Five Lemma, we have that $\alpha_{e,\mathsf{SS}e'}$ is an isomorphism. Therefore also $\alpha_{e,e'}$. ●●●

Remark 6.1.8. Similarly, there exists a unique functor $\widehat{\mathsf{S}}$ from $\mathbf{KK}_{\mathcal{E}}$ to $\mathbf{KK}_{\mathcal{E}}$ such that the diagram

$$\begin{tikzcd}
S\mathcal{E} \arrow[r, "\mathsf{S}"] \arrow[d, "KK_{\mathcal{E}}"'] & S\mathcal{E} \arrow[d, "KK_{\mathcal{E}}"] \\
\mathbf{KK}_{\mathcal{E}} \arrow[r, "\widehat{\mathsf{S}}"'] & \mathbf{KK}_{\mathcal{E}}
\end{tikzcd}$$

commutes.

6.2 Some diagrams

In this section we construct 19 diagrams involving the groups of the new invariant. These diagrams can in many cases be used to determine the new groups introduced in the invariant. They will also be used in a forthcoming paper, where the three authors prove a UMCT for a certain class of C^*-algebras with one specified ideal, which includes all the Cuntz-Krieger algebras of type (II) with one specified ideal. The *long* proof of these diagrams is outlined in the next section.

Assumption 6.2.1. Throughout this section, $e\colon \mathfrak{A}_0 \overset{\iota}{\hookrightarrow} \mathfrak{A}_1 \overset{\pi}{\twoheadrightarrow} \mathfrak{A}_2$ is a fixed extension of separable C^*-algebras.

Definition 6.2.2. Set $F_{1,i} = KK_{\mathcal{E}}(\mathfrak{f}_{1,i},e)$, $F_{n,i} = KK_{\mathcal{E}}(\mathfrak{f}_{n,i},e)$, and $H_{n,i} = KK_{\mathcal{E}}(\mathfrak{e}_{n,i},e)$, for all $n \in \mathbb{N}_{\geq 2}$ and all $i = 0,1,2,3,4,5$. For convenience, we will identify indices modulo 6, *i.e.*, we write $F_{n,6} = F_{n,0}, F_{n,7} = F_{n,1}$ etc.

Remark 6.2.3. Let $e\colon \mathfrak{A}_0 \xhookrightarrow{\iota} \mathfrak{A}_1 \xrightarrow{\pi} \mathfrak{A}_2$ be a given extension of C^*-algebras. Then we consider the two extensions

$$\mathfrak{mc}^3(e)\colon \mathsf{SC}_\pi \xhookrightarrow{((\iota_{\mathfrak{mc}})_{\mathfrak{mc}})_{\mathfrak{mc}}} \mathsf{C}_{(\pi_{\mathfrak{mc}})_{\mathfrak{mc}}} \xrightarrow{((\pi_{\mathfrak{mc}})_{\mathfrak{mc}})_{\mathfrak{mc}}} \mathsf{C}_{\pi_{\mathfrak{mc}}}$$

and

$$\mathsf{S}(e)\colon \mathsf{S}\mathfrak{A}_0 \xhookrightarrow{\mathsf{S}\iota} \mathsf{S}\mathfrak{A}_1 \xrightarrow{\mathsf{S}\pi} \mathsf{S}\mathfrak{A}_2.$$

We have canonical $*$-homomorphisms $\mathsf{S}\mathfrak{A}_0 \to \mathsf{SC}_\pi$, $\mathsf{S}\mathfrak{A}_1 \to \mathsf{C}_{(\pi_{\mathfrak{mc}})_{\mathfrak{mc}}}$, and $\mathsf{S}\mathfrak{A}_2 \to \mathsf{C}_{\pi_{\mathfrak{mc}}}$, which all induce isomorphisms on the level of K-theory. But these do *not*, in general, induce a morphism of extensions — in fact not even of the corresponding cyclic six term exact sequences. Using Corollary 3.4.4, we easily see, that the diagram

$$\begin{tikzcd}[column sep=small]
{} \arrow[r, "\delta_1^{\mathsf{S}e}"] & K_0(\mathsf{S}\mathfrak{A}_0) \arrow[r, "K_0(\mathsf{S}\iota)"] \arrow[d, "\alpha_0"', "\cong"] & K_0(\mathsf{S}\mathfrak{A}_1) \arrow[r, "K_0(\mathsf{S}\pi)"] \arrow[d, "-\alpha_1"', "\cong"] & K_0(\mathsf{S}\mathfrak{A}_2) \arrow[r, "\delta_0^{\mathsf{S}e}"] \arrow[d, "\alpha_2"', "\cong"] & K_1(\mathsf{S}\mathfrak{A}_0) \arrow[r, "K_1(\mathsf{S}\iota)"] \arrow[d, "\alpha_3"', "\cong"] & K_1(\mathsf{S}\mathfrak{A}_1) \arrow[r, "K_1(\mathsf{S}\pi)"] \arrow[d, "-\alpha_4"', "\cong"] & K_1(\mathsf{S}\mathfrak{A}_2) \arrow[r, "\delta_1^{\mathsf{S}e}"] \arrow[d, "\alpha_5"', "\cong"] & {} \\
{} \arrow[r, "\delta_1^{\mathfrak{mc}^3(e)}"] & K_0(\mathsf{SC}_\pi) \arrow[r, "K_0(\iota')"] & K_0(\mathsf{C}_{(\pi_{\mathfrak{mc}})_{\mathfrak{mc}}}) \arrow[r, "K_0(\pi')"] & K_0(\mathsf{C}_{\pi_{\mathfrak{mc}}}) \arrow[r, "\delta_0^{\mathfrak{mc}^3(e)}"] & K_1(\mathsf{SC}_\pi) \arrow[r, "K_1(\iota')"] & K_1(\mathsf{C}_{(\pi_{\mathfrak{mc}})_{\mathfrak{mc}}}) \arrow[r, "K_1(\pi')"] & K_1(\mathsf{C}_{\pi_{\mathfrak{mc}}}) \arrow[r, "\delta_1^{\mathfrak{mc}^3(e)}"] & {}
\end{tikzcd}$$

commutes, where α_i are the induced maps as mentioned above, and ι' and π' denote the maps $((\iota_{\mathfrak{mc}})_{\mathfrak{mc}})_{\mathfrak{mc}}$ and $((\pi_{\mathfrak{mc}})_{\mathfrak{mc}})_{\mathfrak{mc}}$, resp.[1] We expect that it is possible to find a functorial way to implement the $KK_{\mathcal{E}}$-equivalences between $\mathfrak{mc}^3(e)$ and $\mathsf{S}e$, but can not see how to do this — not even how to make a canonical choice of $KK_{\mathcal{E}}$-equivalences.

Definition 6.2.4. The previous remark showed that $\mathfrak{mc}^3(e)$ and $\mathsf{S}e$ are $KK_{\mathcal{E}}$-equivalent (assuming the UCT). Though, the remark did not give us a canonical way to choose a specific $KK_{\mathcal{E}}$-equivalence (so we get a functorial identification of the two functors).

For our purposes, it is enough to have the following lemma. Let $e\colon \mathfrak{A}_0 \xhookrightarrow{\iota} \mathfrak{A}_1 \xrightarrow{\pi} \mathfrak{A}_2$ be a given extension of separable, nuclear C^*-algebras in the bootstrap category $\mathcal{N}$. Assume, moreover, that $\operatorname{Ext}^1_{\mathrm{six}}(K_{\mathrm{six}}(e), K_{\mathrm{six}}(\mathsf{S}e))$ is the trivial group. For each such extension e, we can define

$$\mathbf{x}_e \in KK_{\mathcal{E}}(\mathsf{S}e, \mathfrak{mc}^3(e))$$

to be the unique element inducing $(\alpha_0, -\alpha_1, \alpha_2, \alpha_3, -\alpha_4, \alpha_5)$ in $\operatorname{Hom}_{\mathrm{six}}(K_{\mathrm{six}}(\mathsf{S}e), K_{\mathrm{six}}(\mathfrak{mc}^3(e)))$ (as defined in the preceding remark).

Lemma 6.2.5. *Let e and e' be two given extensions of separable, nuclear C^*-algebras in the bootstrap category $\mathcal{N}$. Assume, moreover, that $\operatorname{Ext}^1_{\mathrm{six}}(K_{\mathrm{six}}(e), K_{\mathrm{six}}(\mathsf{S}e))$, $\operatorname{Ext}^1_{\mathrm{six}}(K_{\mathrm{six}}(e'), K_{\mathrm{six}}(\mathsf{S}e'))$, and $\operatorname{Ext}^1_{\mathrm{six}}(K_{\mathrm{six}}(e), K_{\mathrm{six}}(\mathsf{S}e'))$ are trivial groups. Let ϕ be a morphism from e to e', and set $\mathbf{x} = KK_{\mathcal{E}}(\phi)$ in $KK_{\mathcal{E}}(e, e')$. Then*

$$KK_{\mathcal{E}}(\mathsf{S}\phi) \times \mathbf{x}_{e'} = \widehat{\mathsf{S}}\mathbf{x} \times \mathbf{x}_{e'} = \mathbf{x}_e \times \widehat{\mathfrak{mc}}^3(\mathbf{x}) = \mathbf{x}_e \times KK_{\mathcal{E}}(\mathfrak{mc}^3(\phi)).$$

Proof. From the assumptions and the UCT of Bonkat, we see that the canonical homomorphisms

$$KK_{\mathcal{E}}(e, e') \longrightarrow \operatorname{Hom}_{\mathrm{six}}(K_{\mathrm{six}}(e), K_{\mathrm{six}}(e')),$$

$$KK_{\mathcal{E}}(\mathsf{S}e, \mathsf{S}e') \longrightarrow \operatorname{Hom}_{\mathrm{six}}(K_{\mathrm{six}}(\mathsf{S}e), K_{\mathrm{six}}(\mathsf{S}e')),$$

$$KK_{\mathcal{E}}(\mathfrak{mc}^3(e), \mathfrak{mc}^3(e')) \longrightarrow \operatorname{Hom}_{\mathrm{six}}(K_{\mathrm{six}}(\mathfrak{mc}^3(e)), K_{\mathrm{six}}(\mathfrak{mc}^3(e'))),$$

$$KK_{\mathcal{E}}(\mathsf{S}e, \mathfrak{mc}^3(e')) \longrightarrow \operatorname{Hom}_{\mathrm{six}}(K_{\mathrm{six}}(\mathsf{S}e), K_{\mathrm{six}}(\mathfrak{mc}^3(e')))$$

are functorial isomorphisms. Consequently, it is enough to prove that the result holds for the induced maps in K-theory, *i.e.*,

$$K_{\mathrm{six}}(\mathbf{x}_{e'}) \circ K_{\mathrm{six}}(\widehat{\mathsf{S}}\mathbf{x}) = K_{\mathrm{six}}(\widehat{\mathfrak{mc}}^3(\mathbf{x})) \circ K_{\mathrm{six}}(\mathbf{x}_e).$$

[1] Here we also use that the canonical identifications $K_i(\mathfrak{A}_j) \to K_{1-i}(\mathsf{S}\mathfrak{A}_j)$ give an isomorphism of the corresponding cyclic six term exact sequences.

Again to prove this, it is enough to show that

$$\psi_i' \circ \mathsf{S}\phi_i = (\mathfrak{mc}^3(\phi))_i \circ \psi_i,$$

for $i = 0, 1, 2$, where ψ_0 (ψ_1, and ψ_2, resp.) is the canonical $*$-homomorphisms from the ideal (the extension, and the quotient, resp.) of $\mathsf{S}e$ to the ideal (the extension, and the quotient, resp.) of $\mathfrak{mc}^3(e)$ — and correspondingly for ψ_i'. This equation is straightforward to check. ●●●

Remark 6.2.6. Let e be an extensions of separable, nuclear C^*-algebras in the bootstrap category $\mathcal{N}$, and assume that $\operatorname{Ext}^1_{\mathrm{six}}(K_{\mathrm{six}}(e), K_{\mathrm{six}}(\mathsf{S}e))$ is the trivial group. Then we get a $KK_{\mathcal{E}}$-equivalence $\widehat{\mathsf{S}}\mathbf{x}_e \times \mathbf{x}_{\mathfrak{mc}^3(e)}$ from $\mathsf{SS}e$ to $\mathfrak{mc}^6(e)$. Composed with the standard $KK_{\mathcal{E}}$-equivalence from e to $\mathsf{SS}e$, this gives a canonical $KK_{\mathcal{E}}$-equivalence from e to $\mathfrak{mc}^6(e)$.

It is also easy to show, that we have that

$$\mathbf{x}_{\mathfrak{mc}(e)} = -KK_{\mathcal{E}}(\theta_e) \times \widehat{\mathfrak{mc}}(\mathbf{x}_e).$$

Definition 6.2.7. For an extension e, we let $\mathbf{b}_e$ denote the element of $KK_{\mathcal{E}}(e, \mathsf{SS}e)$ induced by the Bott element — this is a $KK_{\mathcal{E}}$-equivalence. Moreover, we let $\mathbf{z}_n$ denote the $KK_{\mathcal{E}}$-equivalence in $KK_{\mathcal{E}}(\mathsf{S}\mathfrak{f}_{1,0}, \mathfrak{i}(\mathfrak{e}_{n,0}))$ induced by the canonical embedding $\mathbb{C} \to \mathsf{M}_n$. We let $\mathbf{w}_n$ denote the $KK_{\mathcal{E}}$-equivalence from $0 \xhookrightarrow{0} \mathsf{SM}_n \xrightarrow{\mathrm{id}} \mathsf{SM}_n$ to $\mathfrak{q}(\mathfrak{e}_{n,2})$ induced by the canonical embedding $\mathsf{SM}_n \to \mathbb{I}_{n,1}$.

For each $n \in \mathbb{N}_{\geq 2}$, we will, during the following three definitions, define 36 homomorphisms,

$$F_{1,i+1} \xrightarrow{h^{1,1,in}_{n,i}} H_{n,i} \xrightarrow{h^{1,1,out}_{n,i}} F_{1,i+3}$$

$$F_{n,i} \xrightarrow{h^{n,1,in}_{n,i}} H_{n,i} \xrightarrow{h^{n,1,out}_{n,i}} F_{1,i+2}$$

$$F_{1,i+2} \xrightarrow{h^{1,n,in}_{n,i}} H_{n,i} \xrightarrow{h^{1,n,out}_{n,i}} F_{n,i+1},$$

where we identify indices modulo 6 (so we write *e.g.* $h^{*,*,*}_{n,6} = h^{*,*,*}_{n,0}$).

Definition 6.2.8. For each $n \in \mathbb{N}_{\geq 2}$, we have a short exact sequence $\mathfrak{i}(\mathfrak{e}_{n,0}) \xhookrightarrow{\mathfrak{i}_{\mathfrak{e}_{n,0}}} \mathfrak{e}_{n,0} \xrightarrow{\mathfrak{q}_{\mathfrak{e}_{n,0}}} \mathfrak{q}(\mathfrak{e}_{n,0})$ of extensions. We define $h^{1,1,in}_{n,0}$ and $h^{1,1,out}_{n,0}$ by

$$\begin{tikzcd}
F_{1,1} \arrow[r, "h^{1,1,in}_{n,0}"] \arrow[d, equal] & H_{n,0} \arrow[r, "h^{1,1,out}_{n,0}"] \arrow[d, equal] & F_{1,3} \arrow[d, equal] \\
KK_{\mathcal{E}}(\mathfrak{q}(\mathfrak{e}_{n,0}), e) \arrow[r, "KK_{\mathcal{E}}(\mathfrak{q}_{\mathfrak{e}_{n,0}})\times -"'] & KK_{\mathcal{E}}(\mathfrak{e}_{n,0}, e) \arrow[r, "\mathbf{x}^{-1}_{\mathfrak{f}_{1,0}}\times \mathbf{z}_n \times KK_{\mathcal{E}}(\mathfrak{i}_{\mathfrak{e}_{n,0}})\times -"'] & KK_{\mathcal{E}}(\mathfrak{f}_{1,3}, e).
\end{tikzcd}$$

By applying the functor $\widehat{\mathfrak{mc}}$, we define $h^{1,1,in}_{n,i}$ and $h^{1,1,out}_{n,i}$, for $i = 1, 2, 3, 4, 5$, *i.e.*,

$$\begin{aligned}
h^{1,1,in}_{n,i} &= KK_{\mathcal{E}}(\widehat{\mathfrak{mc}}^i(KK_{\mathcal{E}}(\mathfrak{q}_{\mathfrak{e}_{n,0}})), e), \\
h^{1,1,out}_{n,i} &= KK_{\mathcal{E}}(\widehat{\mathfrak{mc}}^i(\mathbf{x}^{-1}_{\mathfrak{f}_{1,0}} \times \mathbf{z}_n \times KK_{\mathcal{E}}(\mathfrak{i}_{\mathfrak{e}_{n,0}})), e),
\end{aligned}$$

for all $i = 0, 1, 2, 3, 4, 5$ (of course we use the canonical $KK_{\mathcal{E}}$-equivalences from Remark 6.2.6 to identify $KK_{\mathcal{E}}(\mathfrak{f}_{1,j+6}, e)$ with $KK_{\mathcal{E}}(\mathfrak{f}_{1,j}, e)$).

Definition 6.2.9. For each $n \in \mathbb{N}_{\geq 2}$, we have a short exact sequence $\mathfrak{i}(\mathfrak{e}_{n,1}) \xhookrightarrow{\mathfrak{i}_{\mathfrak{e}_{n,1}}} \mathfrak{e}_{n,1} \xrightarrow{\mathfrak{q}_{\mathfrak{e}_{n,1}}} \mathfrak{q}(\mathfrak{e}_{n,1})$ of extensions. We define $h^{n,1,in}_{n,1}$ and $h^{n,1,out}_{n,1}$ by

$$\begin{tikzcd}
F_{n,1} \arrow[r, "h^{n,1,in}_{n,1}"] \arrow[d, equal] & H_{n,1} \arrow[r, "h^{n,1,out}_{n,1}"] \arrow[d, equal] & F_{1,3} \arrow[d, equal] \\
KK_{\mathcal{E}}(\mathfrak{q}(\mathfrak{e}_{n,1}), e) \arrow[r, "KK_{\mathcal{E}}(\mathfrak{q}_{\mathfrak{e}_{n,1}})\times -"'] & KK_{\mathcal{E}}(\mathfrak{e}_{n,1}, e) \arrow[r, "\mathbf{x}^{-1}_{\mathfrak{f}_{1,0}}\times KK_{\mathcal{E}}(\mathfrak{i}_{\mathfrak{e}_{n,1}})\times -"'] & KK_{\mathcal{E}}(\mathfrak{f}_{1,3}, e).
\end{tikzcd}$$

By applying the functor $\widehat{\mathfrak{mc}}$, we define $h^{n,1,in}_{n,i}$ and $h^{n,1,out}_{n,i}$, for $i = 0,2,3,4,5$, *i.e.*,

$$\begin{aligned} h^{n,1,in}_{n,i} &= KK_{\mathcal{E}}(\widehat{\mathfrak{mc}}^{i-1}(KK_{\mathcal{E}}(\mathfrak{q}_{\mathfrak{e}_{n,1}})), e), \\ h^{n,1,out}_{n,i} &= KK_{\mathcal{E}}(\widehat{\mathfrak{mc}}^{i-1}(\mathbf{x}^{-1}_{\mathfrak{f}_{1,0}} \times KK_{\mathcal{E}}(\mathfrak{i}_{\mathfrak{e}_{n,1}})), e), \end{aligned}$$

for all $i = 1,2,3,4,5,6$.

Definition 6.2.10. For each $n \in \mathbb{N}_{\geq 2}$, we have a short exact sequence $\mathfrak{i}(\mathfrak{e}_{n,2}) \xhookrightarrow{\mathfrak{i}_{\mathfrak{e}_{n,2}}} \mathfrak{e}_{n,2} \xrightarrow{\mathfrak{q}_{\mathfrak{e}_{n,2}}} \mathfrak{q}(\mathfrak{e}_{n,2})$ of extensions. We define $h^{1,n,in}_{n,2}$ and $h^{1,n,out}_{n,2}$ by

$$\begin{tikzcd} F_{1,4} \arrow[r, "h^{1,n,in}_{n,2}"] \arrow[d, equal] & H_{n,2} \arrow[r, "h^{1,n,out}_{n,2}"] \arrow[d, equal] & F_{n,3} \arrow[d, equal] \\ KK_{\mathcal{E}}(\mathfrak{f}_{1,4}, e) \arrow[r, "KK_{\mathcal{E}}(\mathfrak{q}_{\mathfrak{e}_{n,2}}) \times \mathbf{w}_n^{-1} \times \widehat{\mathfrak{mc}}(\mathbf{z}_n^{-1}) \times \mathbf{x}_{\mathfrak{f}_{1,1}} \times -"'] & KK_{\mathcal{E}}(\mathfrak{e}_{n,2}, e) \arrow[r, "\mathbf{x}^{-1}_{\mathfrak{f}_{n,0}} \times KK_{\mathcal{E}}(\mathfrak{i}_{\mathfrak{e}_{n,2}}) \times -"'] & KK_{\mathcal{E}}(\mathfrak{f}_{n,3}, e). \end{tikzcd}$$

By applying the functor $\widehat{\mathfrak{mc}}$, we define $h^{1,n,in}_{n,i}$ and $h^{1,n,out}_{n,i}$, for $i = 0,1,3,4,5$, *i.e.*,

$$\begin{aligned} h^{1,n,in}_{n,i} &= KK_{\mathcal{E}}(\widehat{\mathfrak{mc}}^{i-2}(KK_{\mathcal{E}}(\mathfrak{q}_{\mathfrak{e}_{n,2}}) \times \mathbf{w}_n^{-1} \times \widehat{\mathfrak{mc}}(\mathbf{z}_n^{-1}) \times \mathbf{x}_{\mathfrak{f}_{1,1}}), e), \\ h^{1,n,out}_{n,i} &= KK_{\mathcal{E}}(\widehat{\mathfrak{mc}}^{i-2}(\mathbf{x}^{-1}_{\mathfrak{f}_{n,0}} \times KK_{\mathcal{E}}(\mathfrak{i}_{\mathfrak{e}_{n,2}})), e), \end{aligned}$$

for all $i = 2,3,4,5,6,7$.

Definition 6.2.11. Now, we define homomorphisms $f_{n,i}$ from $F_{n,i}$ to $F_{n,i+1}$, for all $n \in \mathbb{N}$ and $i = 0,1,2,3,4,5$. We set

$$\begin{tikzcd} KK_{\mathcal{E}}(\mathsf{SS}\mathfrak{f}_{n,0}, e) \arrow[rr] & & KK_{\mathcal{E}}(\mathfrak{f}_{n,1}, e) \arrow[rr] & & KK_{\mathcal{E}}(\mathfrak{f}_{n,2}, e) \arrow[ddd] \\ & F_{n,0} \arrow[r, "f_{n,0}"] \arrow[ul, "\mathbf{b}^{-1}_{\mathfrak{f}_{n,0}} \times -"', "\cong"] & F_{n,1} \arrow[r, "f_{n,1}"] \arrow[u, equal] & F_{n,2} \arrow[d, "f_{n,2}"] \arrow[ur, equal] & \\ & F_{n,5} \arrow[u, "f_{n,5}"] \arrow[dl, "\mathbf{x}_{\mathfrak{f}_{n,2}} \times -", "\cong"'] & F_{n,4} \arrow[l, "f_{n,4}"] \arrow[d, "\mathbf{x}_{\mathfrak{f}_{n,1}} \times -", "\cong"'] & F_{n,3} \arrow[l, "f_{n,3}"] \arrow[dr, "\mathbf{x}_{\mathfrak{f}_{n,0}} \times -", "\cong"'] & \\ KK_{\mathcal{E}}(\mathsf{S}\mathfrak{f}_{n,2}, e) \arrow[uuu] & & KK_{\mathcal{E}}(\mathsf{S}\mathfrak{f}_{n,1}, e) \arrow[ll] & & KK_{\mathcal{E}}(\mathsf{S}\mathfrak{f}_{n,0}, e) \arrow[ll] \end{tikzcd}$$

where the outer sequence is the cyclic six term exact sequence in $KK_{\mathcal{E}}$-theory induced by the short exact sequence $\mathfrak{i}(\mathfrak{f}_{n,2}) \hookrightarrow \mathfrak{f}_{n,2} \twoheadrightarrow \mathfrak{q}(\mathfrak{f}_{n,2})$ (which is exactly $\mathsf{S}\mathfrak{f}_{n,0} \hookrightarrow \mathfrak{f}_{n,2} \twoheadrightarrow \mathfrak{f}_{n,1}$).

Definition 6.2.12. Now, we will define the Bockstein operations,

$$F_{1,i} \xrightarrow{\rho_{n,i}} F_{n,i} \xrightarrow{\beta_{n,i}} F_{1,i+3}\ ,$$

for all $n \in \mathbb{N}_{\geq 2}$ and $i = 0,1,2,3,4,5$.

The extension $\mathfrak{e}_{n,0} \colon \mathsf{SM}_n \hookrightarrow \mathbb{I}_{n,0} \twoheadrightarrow \mathbb{C}$ induces a short exact sequence $\mathfrak{i}(\mathfrak{e}_{n,0}) \xhookrightarrow{x} \mathfrak{f}_{n,0} \xrightarrow{y} \mathfrak{f}_{1,0}$. We set

$$\begin{tikzcd} F_{1,0} \arrow[r, "\rho_{n,0}"] \arrow[d, equal] & F_{n,0} \arrow[r, "\beta_{n,0}"] \arrow[d, equal] & F_{1,3} \arrow[d, equal] \\ KK_{\mathcal{E}}(\mathfrak{f}_{1,0}, e) \arrow[r, "KK_{\mathcal{E}}(y) \times -"'] & KK_{\mathcal{E}}(\mathfrak{f}_{n,0}, e) \arrow[r, "\mathbf{x}^{-1}_{\mathfrak{f}_{1,0}} \times \mathbf{z}_n \times KK_{\mathcal{E}}(x) \times -"'] & KK_{\mathcal{E}}(\mathfrak{f}_{1,3}, e). \end{tikzcd}$$

By applying the functor $\widehat{\mathfrak{mc}}$, we define $\rho_{n,i}$ and $\beta_{n,i}$, for $i = 1,2,3,4,5$, *i.e.*,

$$\begin{aligned}
\rho_{n,i} &= KK_{\mathcal{E}}(\widehat{\mathfrak{mc}}^{i}(KK_{\mathcal{E}}(y)), e), \\
\beta_{n,i} &= KK_{\mathcal{E}}(\widehat{\mathfrak{mc}}^{i}(\mathbf{x}_{\mathfrak{f}_{1,0}}^{-1} \times \mathbf{z}_n \times KK_{\mathcal{E}}(x)), e),
\end{aligned}$$

for all $i = 0,1,2,3,4,5$ (of course we use the canonical $KK_{\mathcal{E}}$-equivalences from Remark 6.2.6 to make identifications modulo 6).

Definition 6.2.13. For each $n \in \mathbb{N}$, we set $\tilde{f}_{n,i} = f_{n,i}$ for $i = 1,2,4,5$ and $\tilde{f}_{n,i} = -f_{n,i}$ for $i = 0,3$.

Theorem 6.2.14. *For all $n \in \mathbb{N}$ and all $i = 0,1,2,3,4,5$,*

$$F_{n,i-1} \xrightarrow{f_{n,i-1}} F_{n,i} \xrightarrow{f_{n,i}} F_{n,i+1}$$

is exact. For all $n \in \mathbb{N}_{\geq 2}$ and all $i = 0,1,2,3,4,5$,

$$\begin{tikzcd}
F_{1,i+1} \arrow[r, "h_{n,i}^{1,1,in}"] & H_{n,i} \arrow[r, "h_{n,i}^{1,1,out}"] & F_{1,i+3} \arrow[d, "nf_{1,i+3}"'] \\
F_{1,i} \arrow[u, "nf_{1,i}"'] & H_{n,i+3} \arrow[l, "h_{n,i+3}^{1,1,out}"] & F_{1,i+4}, \arrow[l, "h_{n,i+3}^{1,1,in}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
F_{n,i} \arrow[r, "h_{n,i}^{n,1,in}"] & H_{n,i} \arrow[r, "h_{n,i}^{n,1,out}"] & F_{1,i+2} \arrow[d, "f_{n,i+2}\circ\rho_{n,i+2}"'] \\
F_{1,i+5} \arrow[u, "f_{n,i+5}\circ\rho_{n,i+5}"'] & H_{n,i+3} \arrow[l, "h_{n,i+3}^{n,1,out}"] & F_{n,i+3}, \arrow[l, "h_{n,i+3}^{n,1,in}"]
\end{tikzcd}$$

$$\begin{tikzcd}
F_{1,i+2} \arrow[r, "h_{n,i}^{1,n,in}"] & H_{n,i} \arrow[r, "h_{n,i}^{1,n,out}"] & F_{n,i+1} \arrow[d, "\beta_{n,i+2}\circ f_{n,i+1}"'] \\
F_{n,i+4} \arrow[u, "\beta_{n,i+5}\circ f_{n,i+4}"'] & H_{n,i+3} \arrow[l, "h_{n,i+3}^{1,n,out}"] & F_{1,i+5}, \arrow[l, "h_{n,i+3}^{1,n,in}"]
\end{tikzcd}
\quad \textit{and} \quad
\begin{tikzcd}
F_{1,i} \arrow[r, "\rho_{n,i}"] & F_{n,i} \arrow[r, "\beta_{n,i}"] & F_{1,i+3} \arrow[d, "\times n"'] \\
F_{1,i} \arrow[u, "\times n"'] & F_{n,i+3} \arrow[l, "\beta_{n,i+3}"] & F_{1,i+3} \arrow[l, "\rho_{n,i+3}"]
\end{tikzcd}$$

are exact. Moreover, all the three diagrams

$$\begin{tikzcd}
F_{1,i} \arrow[r, "\tilde{f}_{1,i}"] \arrow[d, "\rho_{n,i}"'] & F_{1,i+1} \arrow[d, "h_{n,i}^{1,1,in}"'] \arrow[dr, "\tilde{f}_{1,i+1}"] & \\
F_{n,i} \arrow[r, "h_{n,i}^{n,1,in}"] \arrow[dr, "\beta_{n,i}"'] & H_{n,i} \arrow[r, "h_{n,i}^{n,1,out}"'] \arrow[d, "h_{n,i}^{1,1,out}"] & F_{1,i+2} \arrow[d, "\tilde{f}_{1,i+2}"] \\
& F_{1,i+3} \arrow[r, "\times n"'] & F_{1,i+3}
\end{tikzcd} \tag{6.1}$$

$$\begin{tikzcd}
F_{1,i+1} \arrow[r, "\times n"] \arrow[d, "\tilde{f}_{1,i+1}"'] & F_{1,i+1} \arrow[d, "h_{n,i}^{1,1,in}"'] \arrow[dr, "\rho_{n,i+1}"] & \\
F_{1,i+2} \arrow[r, "h_{n,i}^{1,n,in}"] \arrow[dr, "\tilde{f}_{1,i+2}"'] & H_{n,i} \arrow[r, "h_{n,i}^{1,n,out}"'] \arrow[d, "h_{n,i}^{1,1,out}"] & F_{n,i+1} \arrow[d, "-\beta_{n,i+1}"] \\
& F_{1,i+3} \arrow[r, "\tilde{f}_{1,i+3}"'] & F_{1,i+4}
\end{tikzcd} \tag{6.2}$$

$$\begin{tikzcd}
F_{n,i+5} \arrow[r, "\tilde f_{n,i+5}"] \arrow[d, "-\beta_{n,i+5}"'] & F_{n,i} \arrow[d, "h_{n,i}^{n,1,in}"'] \arrow[dr, "\tilde f_{n,i}"] & \\
F_{1,i+2} \arrow[r, "h_{n,i}^{1,n,in}"] \arrow[dr, "\times n"'] & H_{n,i} \arrow[r, "h_{n,i}^{1,n,out}"'] \arrow[d, "h_{n,i}^{n,1,out}"] & F_{n,i+1} \arrow[d, "\tilde f_{n,i+1}"] \\
& F_{1,i+2} \arrow[r, "\rho_{n,i+2}"'] & F_{n,i+2}
\end{tikzcd} \tag{6.3}$$

commute.

Proof. See next section. ●●●

Corollary 6.2.15. *For each $i = 0, 1, 2, 3, 4, 5$, the two squares*

$$\begin{tikzcd}
F_{1,i} \arrow[r, "\tilde f_{1,i}"] \arrow[d, "\rho_{n,i}"'] & F_{1,i+1} \arrow[d, "\rho_{n,i+1}"] \\
F_{n,i} \arrow[r, "\tilde f_{n,i}"'] & F_{n,i+1}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
F_{n,i} \arrow[r, "\tilde f_{n,i}"] \arrow[d, "-\beta_{n,i}"'] & F_{n,i+1} \arrow[d, "\beta_{n,i+1}"] \\
F_{1,i+3} \arrow[r, "\tilde f_{1,i+3}"'] & F_{1,i+4}
\end{tikzcd}$$

commute.

Proof. This follows directly from the previous theorem:

$$\begin{aligned}
\tilde f_{n,i} \circ \rho_{n,i} &= \tilde f_{n,i} \circ h_{n,i-1}^{1,n,out} \circ h_{n,i-1}^{1,1,in} && \text{by } (6.2) \\
&= \rho_{n,i+1} \circ h_{n,i-1}^{n,1,out} \circ h_{n,i-1}^{1,1,in} && \text{by } (6.3) \\
&= \rho_{n,i+1} \circ \tilde f_{1,i} && \text{by } (6.1) \\
\beta_{n,i+1} \circ \tilde f_{n,i} &= \beta_{n,i+1} \circ h_{n,i}^{1,n,out} \circ h_{n,i}^{n,1,in} && \text{by } (6.3) \\
&= -\tilde f_{1,i+3} \circ h_{n,i}^{1,1,out} \circ h_{n,i}^{n,1,in} && \text{by } (6.2) \\
&= -\tilde f_{1,i+3} \circ \beta_{n,i} && \text{by } (6.1)
\end{aligned}$$

●●●

Remark 6.2.16. From the preceding theorem and corollary, it follows that, for each $i = 0, 1, 2$, we have the following — both horizontally and vertically six term cyclic — commuting diagrams with exact rows and columns:

$$\begin{tikzcd}
& {} \arrow[d, "\times n"] & {} \arrow[d, "\times n"] & {} \arrow[d, "\times n"] & {} \arrow[d, "\times n"] & {} \arrow[d, "\times n"] & {} \arrow[d, "\times n"] & \\
{} \arrow[r, "\tilde f_{1,5}"] & F_{1,0} \arrow[r, "\tilde f_{1,0}"] \arrow[d, "\rho_{n,0}"] & F_{1,1} \arrow[r, "\tilde f_{1,1}"] \arrow[d, "\rho_{n,1}"] & F_{1,2} \arrow[r, "\tilde f_{1,2}"] \arrow[d, "\rho_{n,2}"] & F_{1,3} \arrow[r, "\tilde f_{1,3}"] \arrow[d, "\rho_{n,3}"] & F_{1,4} \arrow[r, "\tilde f_{1,4}"] \arrow[d, "\rho_{n,4}"] & F_{1,5} \arrow[r, "\tilde f_{1,5}"] \arrow[d, "\rho_{n,5}"] & {} \\
{} \arrow[r, "\tilde f_{n,5}"] & F_{n,0} \arrow[r, "\tilde f_{n,0}"] \arrow[d, "\beta_{n,0}"] & F_{n,1} \arrow[r, "\tilde f_{n,1}"] \arrow[d, "-\beta_{n,1}"] & F_{n,2} \arrow[r, "\tilde f_{n,2}"] \arrow[d, "\beta_{n,2}"] & F_{n,3} \arrow[r, "\tilde f_{n,3}"] \arrow[d, "-\beta_{n,3}"] & F_{n,4} \arrow[r, "\tilde f_{n,4}"] \arrow[d, "\beta_{n,4}"] & F_{n,5} \arrow[r, "\tilde f_{n,5}"] \arrow[d, "-\beta_{n,5}"] & {} \\
{} \arrow[r, "\tilde f_{1,2}"] & F_{1,3} \arrow[r, "\tilde f_{1,3}"] \arrow[d, "\times n"] & F_{1,4} \arrow[r, "\tilde f_{1,4}"] \arrow[d, "\times n"] & F_{1,5} \arrow[r, "\tilde f_{1,5}"] \arrow[d, "\times n"] & F_{1,0} \arrow[r, "\tilde f_{1,0}"] \arrow[d, "\times n"] & F_{1,1} \arrow[r, "\tilde f_{1,1}"] \arrow[d, "\times n"] & F_{1,2} \arrow[r, "\tilde f_{1,2}"] \arrow[d, "\times n"] & {} \\
& {} & {} & {} & {} & {} & {} &
\end{tikzcd} \tag{D_0}$$

$$\begin{tikzcd}
{} & {} \arrow[d, "\times n"] & {} \arrow[d, "n\tilde f_{1,i}"] & {} \arrow[d] & {} \arrow[d, "\times n"] & {} \arrow[d, "n\tilde f_{1,i+3}"] & {} \arrow[d] & {} \\
{} \arrow[r, "\tilde f_{1,i+5}"] & F_{1,i} \arrow[r, "\tilde f_{1,i}"] \arrow[d, "\rho_{n,i}"] & F_{1,i+1} \arrow[r, "\tilde f_{1,i+1}"] \arrow[d, "h^{1,1,in}_{n,i}"] & F_{1,i+2} \arrow[r, "\tilde f_{1,i+2}"] \arrow[d, equal] & F_{1,i+3} \arrow[r, "\tilde f_{1,i+3}"] \arrow[d, "\rho_{n,i+3}"] & F_{1,i+4} \arrow[r, "\tilde f_{1,i+4}"] \arrow[d, "h^{1,1,in}_{n,i+3}"] & F_{1,i+5} \arrow[r, "\tilde f_{1,i+5}"] \arrow[d, equal] & {} \\
{} \arrow[r, "\rho_{n,i}\circ\tilde f_{1,i+5}"] & F_{n,i} \arrow[r, "h^{n,1,in}_{n,i}"] \arrow[d, "\beta_{n,i}"] & H_{n,i} \arrow[r, "h^{n,1,out}_{n,i}"] \arrow[d, "h^{1,1,out}_{n,i}"] & F_{1,i+2} \arrow[r, "\rho_{n,i+3}\circ\tilde f_{1,i+2}"] \arrow[d] & F_{n,i+3} \arrow[r, "h^{n,1,in}_{n,i+3}"] \arrow[d, "\beta_{n,i+3}"] & H_{n,i+3} \arrow[r, "h^{n,1,out}_{n,i+3}"] \arrow[d, "h^{1,1,out}_{n,i+3}"] & F_{1,i+5} \arrow[r, "\rho_{n,i}\circ\tilde f_{1,i+5}"] \arrow[d] & {} \\
{} \arrow[r] & F_{1,i+3} \arrow[r, equal] \arrow[d, "\times n"] & F_{1,i+3} \arrow[r] \arrow[d, "n\tilde f_{1,i+3}"] & 0 \arrow[r] \arrow[d] & F_{1,i} \arrow[r, equal] \arrow[d, "\times n"] & F_{1,i} \arrow[r] \arrow[d, "n\tilde f_{1,i}"] & 0 \arrow[r] \arrow[d] & {} \\
{} & {} & {} & {} & {} & {} & {} & {}
\end{tikzcd} \tag{D_1}$$

$$\begin{tikzcd}
{} & {} \arrow[d] & {} \arrow[d, "n\tilde f_{1,i}"] & {} \arrow[d, "\tilde f_{1,i}"] & {} \arrow[d] & {} \arrow[d, "n\tilde f_{1,i+3}"] & {} \arrow[d, "\tilde f_{1,i+3}"] & {} \\
{} \arrow[r] & 0 \arrow[r] \arrow[d] & F_{1,i+1} \arrow[r, equal] \arrow[d, "h^{1,1,in}_{n,i}"] & F_{1,i+1} \arrow[r] \arrow[d, "\tilde f_{1,i+1}"] & 0 \arrow[r] \arrow[d] & F_{1,i+4} \arrow[r, equal] \arrow[d, "h^{1,1,in}_{n,i+3}"] & F_{1,i+4} \arrow[r] \arrow[d, "\tilde f_{1,i+4}"] & {} \\
{} \arrow[r, "\rho_{n,i}\circ\tilde f_{1,i+5}"] & F_{n,i} \arrow[r, "h^{n,1,in}_{n,i}"] \arrow[d, equal] & H_{n,i} \arrow[r, "h^{n,1,out}_{n,i}"] \arrow[d, "h^{1,1,out}_{n,i}"] & F_{1,i+2} \arrow[r, "\rho_{n,i+3}\circ\tilde f_{1,i+2}"] \arrow[d, "\tilde f_{1,i+2}"] & F_{n,i+3} \arrow[r, "h^{n,1,in}_{n,i+3}"] \arrow[d, equal] & H_{n,i+3} \arrow[r, "h^{n,1,out}_{n,i+3}"] \arrow[d, "h^{1,1,out}_{n,i+3}"] & F_{1,i+5} \arrow[r, "\rho_{n,i}\circ\tilde f_{1,i+5}"] \arrow[d, "\tilde f_{1,i+5}"] & {} \\
{} \arrow[r, "\rho_{n,i}"] & F_{n,i} \arrow[r, "\beta_{n,i}"] \arrow[d] & F_{1,i+3} \arrow[r, "\times n"] \arrow[d, "n\tilde f_{1,i+3}"] & F_{1,i+3} \arrow[r, "\rho_{n,i+3}"] \arrow[d, "\tilde f_{1,i+3}"] & F_{n,i+3} \arrow[r, "\beta_{n,i+3}"] \arrow[d] & F_{1,i} \arrow[r, "\times n"] \arrow[d, "n\tilde f_{1,i}"] & F_{1,i} \arrow[r, "\rho_{n,i}"] \arrow[d, "\tilde f_{1,i}"] & {} \\
{} & {} & {} & {} & {} & {} & {} & {}
\end{tikzcd} \tag{$D_1^\star$}$$

$$\begin{tikzcd}
{} & {} \arrow[d, "\tilde f_{1,i}"] & {} \arrow[d, "n\tilde f_{1,i}"] & {} \arrow[d] & {} \arrow[d, "\tilde f_{1,i+3}"] & {} \arrow[d, "n\tilde f_{1,i+3}"] & {} \arrow[d] & {} \\
{} \arrow[r, "\beta_{n,i+4}"] & F_{1,i+1} \arrow[r, "\times n"] \arrow[d, "\tilde f_{1,i+1}"] & F_{1,i+1} \arrow[r, "\rho_{n,i+1}"] \arrow[d, "h^{1,1,in}_{n,i}"] & F_{n,i+1} \arrow[r, "\beta_{n,i+1}"] \arrow[d, equal] & F_{1,i+4} \arrow[r, "\times n"] \arrow[d, "\tilde f_{1,i+4}"] & F_{1,i+4} \arrow[r, "\rho_{n,i+4}"] \arrow[d, "h^{1,1,in}_{n,i+3}"] & F_{n,i+4} \arrow[r, "\beta_{n,i+4}"] \arrow[d, equal] & {} \\
{} \arrow[r, "\tilde f_{1,i+1}\circ\beta_{n,i+4}"] & F_{1,i+2} \arrow[r, "h^{1,n,in}_{n,i}"] \arrow[d, "\tilde f_{1,i+2}"] & H_{n,i} \arrow[r, "h^{1,n,out}_{n,i}"] \arrow[d, "h^{1,1,out}_{n,i}"] & F_{n,i+1} \arrow[r, "\tilde f_{1,i+4}\circ\beta_{n,i+1}"] \arrow[d] & F_{1,i+5} \arrow[r, "h^{1,n,in}_{n,i+3}"] \arrow[d, "\tilde f_{1,i+5}"] & H_{n,i+3} \arrow[r, "h^{1,n,out}_{n,i+3}"] \arrow[d, "h^{1,1,out}_{n,i+3}"] & F_{n,i+4} \arrow[r, "\tilde f_{1,i+1}\circ\beta_{n,i+4}"] \arrow[d] & {} \\
{} \arrow[r] & F_{1,i+3} \arrow[r, equal] \arrow[d, "\tilde f_{1,i+3}"] & F_{1,i+3} \arrow[r] \arrow[d, "n\tilde f_{1,i+3}"] & 0 \arrow[r] \arrow[d] & F_{1,i} \arrow[r, equal] \arrow[d, "\tilde f_{1,i}"] & F_{1,i} \arrow[r] \arrow[d, "n\tilde f_{1,i}"] & 0 \arrow[r] \arrow[d] & {} \\
{} & {} & {} & {} & {} & {} & {} & {}
\end{tikzcd} \tag{D_2}$$

$$\begin{tikzcd}
{} & {} \arrow[d] & {} \arrow[d, "n\tilde f_{1,i}"] & {} \arrow[d, "\times n"] & {} \arrow[d] & {} \arrow[d, "n\tilde f_{1,i+3}"] & {} \arrow[d, "\times n"] & {} \\
{} \arrow[r] & 0 \arrow[r] \arrow[d] & F_{1,i+1} \arrow[r, equal] \arrow[d, "h^{1,1,in}_{n,i}"] & F_{1,i+1} \arrow[r] \arrow[d, "\rho_{n,i+1}"] & 0 \arrow[r] \arrow[d] & F_{1,i+4} \arrow[r, equal] \arrow[d, "h^{1,1,in}_{n,i+3}"] & F_{1,i+4} \arrow[r] \arrow[d, "\rho_{n,i+4}"] & {} \\
{} \arrow[r, "-\tilde f_{1,i+1}\circ\beta_{n,i+4}"] & F_{1,i+2} \arrow[r, "h^{1,n,in}_{n,i}"] \arrow[d, equal] & H_{n,i} \arrow[r, "h^{1,n,out}_{n,i}"] \arrow[d, "h^{1,1,out}_{n,i}"] & F_{n,i+1} \arrow[r, "-\tilde f_{1,i+4}\circ\beta_{n,i+1}"] \arrow[d, "-\beta_{n,i+1}"] & F_{1,i+5} \arrow[r, "h^{1,n,in}_{n,i+3}"] \arrow[d, equal] & H_{n,i+3} \arrow[r, "h^{1,n,out}_{n,i+3}"] \arrow[d, "h^{1,1,out}_{n,i+3}"] & F_{n,i+4} \arrow[r, "-\tilde f_{1,i+1}\circ\beta_{n,i+4}"] \arrow[d, "-\beta_{n,i+4}"] & {} \\
{} \arrow[r, "\tilde f_{1,i+1}"] & F_{1,i+2} \arrow[r, "\tilde f_{1,i+2}"] \arrow[d] & F_{1,i+3} \arrow[r, "\tilde f_{1,i+3}"] \arrow[d, "n\tilde f_{1,i+3}"] & F_{1,i+4} \arrow[r, "\tilde f_{1,i+4}"] \arrow[d, "\times n"] & F_{1,i+5} \arrow[r, "\tilde f_{1,i+5}"] \arrow[d] & F_{1,i} \arrow[r, "\tilde f_{1,i}"] \arrow[d, "n\tilde f_{1,i}"] & F_{1,i+1} \arrow[r, "\tilde f_{1,i+1}"] \arrow[d, "\times n"] & {} \\
{} & {} & {} & {} & {} & {} & {} & {}
\end{tikzcd} \tag{$D_2^\star$}$$

$$
\begin{tikzcd}
{} & {} \arrow[d, "\rho_{n,i+5}"] & {} \arrow[d, "\tilde{f}_{n,i+5}\circ\rho_{n,i+5}"] & {} \arrow[d] & {} \arrow[d, "\rho_{n,i+2}"] & {} \arrow[d, "\tilde{f}_{n,i+2}\circ\rho_{n,i+2}"] & {} \arrow[d] & {} \\
{} \arrow[r, "\tilde{f}_{n,i+4}"] & F_{n,i+5} \arrow[r, "\tilde{f}_{n,i+5}"] \arrow[d, "-\beta_{n,i+5}"] & F_{n,i} \arrow[r, "\tilde{f}_{n,i}"] \arrow[d, "h^{n,1,in}_{n,i}"] & F_{n,i+1} \arrow[r, "\tilde{f}_{n,i+1}"] \arrow[d, equal] & F_{n,i+2} \arrow[r, "\tilde{f}_{n,i+2}"] \arrow[d, "-\beta_{n,i+2}"] & F_{n,i+3} \arrow[r, "\tilde{f}_{n,i+3}"] \arrow[d, "h^{n,1,in}_{n,i+3}"] & F_{n,i+4} \arrow[r, "\tilde{f}_{n,i+4}"] \arrow[d, equal] & {} \\
{} \arrow[r] & F_{1,i+2} \arrow[r, "h^{1,n,in}_{n,i}"] \arrow[d, "\times n"] & H_{n,i} \arrow[r, "h^{1,n,out}_{n,i}"] \arrow[d, "h^{n,1,out}_{n,i}"] & F_{n,i+1} \arrow[r] \arrow[d] & F_{1,i+5} \arrow[r, "h^{1,n,in}_{n,i+3}"] \arrow[d, "\times n"] & H_{n,i+3} \arrow[r, "h^{1,n,out}_{n,i+3}"] \arrow[d, "h^{n,1,out}_{n,i+3}"] & F_{n,i+4} \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & F_{1,i+2} \arrow[r, equal] \arrow[d, "\rho_{n,i+2}"] & F_{1,i+2} \arrow[r] \arrow[d, "\tilde{f}_{n,i+2}\circ\rho_{n,i+2}"] & 0 \arrow[r] \arrow[d] & F_{1,i+5} \arrow[r, equal] \arrow[d, "\rho_{n,i+5}"] & F_{1,i+5} \arrow[r] \arrow[d, "\tilde{f}_{n,i+5}\circ\rho_{n,i+5}"] & 0 \arrow[r] \arrow[d] & {} \\
{} & {} & {} & {} & {} & {} & {} & {}
\end{tikzcd}
\tag{D_3}
$$

$$
\begin{tikzcd}
{} & {} \arrow[d] & {} \arrow[d, "\tilde{f}_{n,i+5}\circ\rho_{n,i+5}"] & {} \arrow[d, "\tilde{f}_{n,i+5}"] & {} \arrow[d] & {} \arrow[d, "\tilde{f}_{n,i+2}\circ\rho_{n,i+2}"] & {} \arrow[d, "\tilde{f}_{n,i+2}"] & {} \\
{} \arrow[r] & 0 \arrow[r] \arrow[d] & F_{n,i} \arrow[r, equal] \arrow[d, "h^{n,1,in}_{n,i}"] & F_{n,i} \arrow[r] \arrow[d, "\tilde{f}_{n,i}"] & 0 \arrow[r] \arrow[d] & F_{n,i+3} \arrow[r, equal] \arrow[d, "h^{n,1,in}_{n,i+3}"] & F_{n,i+3} \arrow[r] \arrow[d, "\tilde{f}_{n,i+3}"] & {} \\
{} \arrow[r] & F_{1,i+2} \arrow[r, "h^{1,n,in}_{n,i}"] \arrow[d, equal] & H_{n,i} \arrow[r, "h^{1,n,out}_{n,i}"] \arrow[d, "h^{n,1,out}_{n,i}"] & F_{n,i+1} \arrow[r] \arrow[d, "\tilde{f}_{n,i+1}"] & F_{1,i+5} \arrow[r, "h^{1,n,in}_{n,i+3}"] \arrow[d, equal] & H_{n,i+3} \arrow[r, "h^{1,n,out}_{n,i+3}"] \arrow[d, "h^{n,1,out}_{n,i+3}"] & F_{n,i+4} \arrow[r] \arrow[d, "\tilde{f}_{n,i+4}"] & {} \\
{} \arrow[r, "\beta_{n,i+5}"] & F_{1,i+2} \arrow[r, "\times n"] \arrow[d] & F_{1,i+2} \arrow[r, "\rho_{n,i+2}"] \arrow[d, "\tilde{f}_{n,i+2}\circ\rho_{n,i+2}"] & F_{n,i+2} \arrow[r, "\beta_{n,i+2}"] \arrow[d, "\tilde{f}_{n,i+2}"] & F_{1,i+5} \arrow[r, "\times n"] \arrow[d] & F_{1,i+5} \arrow[r, "\rho_{n,i+5}"] \arrow[d, "\tilde{f}_{n,i+5}\circ\rho_{n,i+5}"] & F_{n,i+5} \arrow[r, "\beta_{n,i+5}"] \arrow[d, "\tilde{f}_{n,i+5}"] & {} \\
{} & {} & {} & {} & {} & {} & {} & {}
\end{tikzcd}
\tag{$D_3^\star$}
$$

Remark 6.2.17. Just like with Diagrams (5.1) and (5.2), we see that Diagrams (D_i) and $(D_i^\star)$ with two extra conditions each are equivalent, for $i = 1, 2, 3$.

6.3 Proof of Theorem 6.2.14

In this section, we prove Theorem 6.2.14. First we need some results, which will be useful in the proof.

Remark 6.3.1. Let $\mathfrak{A}$ be a separable, nuclear C^*-algebra in the bootstrap category $\mathcal{N}$. Set $e_0 \colon \mathfrak{A} \xhookrightarrow{\mathrm{id}} \mathfrak{A} \twoheadrightarrow 0$, and set $e_i = \mathfrak{mc}^i(e_0)$. As earlier we know that

$$e_0 \colon \mathfrak{A} \xhookrightarrow{\mathrm{id}} \mathfrak{A} \twoheadrightarrow 0,$$

$$e_1 \colon 0 \hookrightarrow \mathfrak{A} \xrightarrow{\mathrm{id}}\!\!\!\!\rightarrow \mathfrak{A},$$

$$e_2 \colon \mathsf{S}\mathfrak{A} \xhookrightarrow{\iota} \mathsf{C}\mathfrak{A} \xrightarrow{\mathrm{ev}_1}\!\!\!\!\rightarrow \mathfrak{A},$$

$$e_3 \colon \mathsf{S}\mathfrak{A} \xhookrightarrow{(0,\iota)} \mathsf{C}\mathfrak{A} \oplus_{\mathrm{ev}_1,\mathrm{ev}_1} \mathsf{C}\mathfrak{A} \xrightarrow{\pi_1}\!\!\!\!\rightarrow \mathsf{C}\mathfrak{A},$$

where π_1 is the projection onto the first coordinate.

We have a canonical morphism, $\phi = (\mathrm{id}, (0, \iota), 0)$, from $\mathsf{S}e_0$ to e_3, which induces a $KK_{\mathcal{E}}$-equivalence. It is evident that $KK_{\mathcal{E}}(\phi)$ is exactly $\mathbf{x}_{e_0}$ in the case that $\mathrm{Ext}^1_{\mathbb{Z}}(K_i(\mathfrak{A}), K_{1-i}(\mathfrak{A})) = 0$, for $i = 0, 1$. Also we see that in this case, $KK_{\mathcal{E}}(\mathfrak{mc}(\phi)) = -\mathbf{x}_{e_1}$ (according to Remark 6.2.6).

Note that $\mathfrak{i}(e_2) = \mathsf{S}e_0$, $\mathfrak{q}(e_2) = e_1$, and $\mathfrak{mc}(\mathfrak{i}(e_2)) = \mathsf{S}e_1$. So if we apply $\mathfrak{mc}^0$, $\mathfrak{mc}^1$, and $\mathfrak{mc}^2$ to the short exact sequence $\mathfrak{i}(e_2) \xhookrightarrow{\mathfrak{i}_{e_2}} e_2 \xrightarrow{\mathfrak{q}_{e_2}}\!\!\!\!\rightarrow \mathfrak{q}(e_2)$, we get just $\mathsf{S}e_0 \xhookrightarrow{\mathfrak{i}_{e_2}} e_2 \xrightarrow{\mathfrak{q}_{e_2}}\!\!\!\!\rightarrow e_1$, $\mathsf{S}e_1 \xhookrightarrow{\mathfrak{mc}(\mathfrak{i}_{e_2})} e_3 \xrightarrow{\mathfrak{mc}(\mathfrak{q}_{e_2})}\!\!\!\!\rightarrow e_2$, and $\mathfrak{mc}\mathsf{S}e_1 \xhookrightarrow{\mathfrak{mc}^2(\mathfrak{i}_{e_2})} e_4 \xrightarrow{\mathfrak{mc}^2(\mathfrak{q}_{e_2})}\!\!\!\!\rightarrow e_3$, resp.

Proposition 6.3.2. *Let $\mathfrak{A}$ be a separable, nuclear C^*-algebra in the bootstrap category $\mathcal{N}$ satisfying $\operatorname{Ext}^1_{\mathbb{Z}}(K_i(\mathfrak{A}), K_{1-i}(\mathfrak{A})) = 0$, for $i = 0, 1$, and let e be an extension of separable C^*-algebras. Set $e_0 \colon \mathfrak{A} \xhookrightarrow{\mathrm{id}} \mathfrak{A} \twoheadrightarrow 0$, and set $e_i = \mathfrak{mc}^i(e_0)$. Then we have*

$$\begin{tikzcd}[column sep=small]
KK_{\mathcal{E}}(\mathsf{S}\mathsf{S}e_1, e) \arrow[rr] \arrow[dr, "\mathbf{b}_{e_1}\times -", "\cong"'] & & KK_{\mathcal{E}}(e_2, e) \arrow[rr] \arrow[d, equal] \arrow[dl, phantom, "\textit{Anti-commutes}"] \arrow[dr, phantom, "\textit{Commutes}"] & & KK_{\mathcal{E}}(e_3, e) \arrow[ddd] \arrow[dl, "\mathbf{x}_{e_0}\times -", "\cong"'] \\
\arrow[d, phantom, "\textit{Commutes}"] & KK_{\mathcal{E}}(e_1, e) \arrow[r] & KK_{\mathcal{E}}(e_2, e) \arrow[r] & KK_{\mathcal{E}}(\mathsf{S}e_0, e) \arrow[d] & \arrow[d, phantom, "\textit{Commutes}"] \\
{} & KK_{\mathcal{E}}(\mathsf{S}\mathsf{S}e_0, e) \arrow[u] \arrow[dr, phantom, "\textit{Commutes}"] & KK_{\mathcal{E}}(\mathsf{S}e_2, e) \arrow[l] \arrow[d, equal] & KK_{\mathcal{E}}(\mathsf{S}e_1, e) \arrow[l] \arrow[dr, equal] \arrow[dl, phantom, "\textit{Anti-commutes}"] & {} \\
KK_{\mathcal{E}}(\mathsf{S}e_3, e) \arrow[uuu] \arrow[ur, "\widehat{\mathsf{S}}\mathbf{x}_{e_0}\times -", "\cong"'] & & KK_{\mathcal{E}}(\mathsf{S}e_2, e) \arrow[ll] & & KK_{\mathcal{E}}(\mathsf{S}e_1, e) \arrow[ll]
\end{tikzcd}$$

where the inner and outer sequences are the cyclic six term exact sequences in $KK_{\mathcal{E}}$-theory induced by $\mathfrak{i}(e_2) \xhookrightarrow{\mathfrak{i}_{e_2}} e_2 \xtwoheadrightarrow{\mathfrak{q}_{e_2}} \mathfrak{q}(e_2)$ and $\mathfrak{mc}(\mathfrak{i}(e_2)) \xhookrightarrow{\mathfrak{mc}(\mathfrak{i}_{e_2})} \mathfrak{mc}(e_2) \xtwoheadrightarrow{\mathfrak{mc}(\mathfrak{q}_{e_2})} \mathfrak{mc}(\mathfrak{q}(e_2))$, resp. Moreover, we have that

$$\begin{tikzcd}[column sep=small]
KK_{\mathcal{E}}(\mathsf{S}\mathfrak{mc}\mathsf{S}e_1, e) \arrow[rr] \arrow[dr, "\mathbf{b}_{e_2}\times KK_{\mathcal{E}}(\mathsf{S}\theta_{e_1})\times -", "\cong"'] & & KK_{\mathcal{E}}(e_3, e) \arrow[rr] \arrow[d, equal] \arrow[dl, phantom, "\textit{Anti-commutes}"] \arrow[dr, phantom, "\textit{Anti-commutes}"] & & KK_{\mathcal{E}}(e_4, e) \arrow[ddd] \arrow[dl, "\mathbf{x}_{e_1}\times -", "\cong"'] \\
\arrow[d, phantom, "\textit{Anti-commutes}"] & KK_{\mathcal{E}}(e_2, e) \arrow[r] & KK_{\mathcal{E}}(e_3, e) \arrow[r] & KK_{\mathcal{E}}(\mathsf{S}e_1, e) \arrow[d] & \arrow[d, phantom, "\textit{Anti-commutes}"] \\
{} & KK_{\mathcal{E}}(\mathsf{S}\mathsf{S}e_1, e) \arrow[u] \arrow[dr, phantom, "\textit{Anti-commutes}"] & KK_{\mathcal{E}}(\mathsf{S}e_3, e) \arrow[l] \arrow[d, equal] & KK_{\mathcal{E}}(\mathsf{S}e_2, e) \arrow[l] \arrow[dl, phantom, "\textit{Anti-commutes}"] & {} \\
KK_{\mathcal{E}}(\mathsf{S}e_4, e) \arrow[uuu] \arrow[ur, "\widehat{\mathsf{S}}\mathbf{x}_{e_1}\times -", "\cong"'] & & KK_{\mathcal{E}}(\mathsf{S}e_3, e) \arrow[ll] & & KK_{\mathcal{E}}(\mathfrak{mc}\mathsf{S}e_1, e) \arrow[ll] \arrow[ul, "\cong", "KK_{\mathcal{E}}(\theta_{e_1})"']
\end{tikzcd}$$

where the inner and outer sequences are the cyclic six term exact sequences in $KK_{\mathcal{E}}$-theory induced by $\mathfrak{mc}(\mathfrak{i}(e_2)) \xhookrightarrow{\mathfrak{mc}(\mathfrak{i}_{e_2})} \mathfrak{mc}(e_2) \xtwoheadrightarrow{\mathfrak{mc}(\mathfrak{q}_{e_2})} \mathfrak{mc}(\mathfrak{q}(e_2))$ and $\mathfrak{mc}^2(\mathfrak{i}(e_2)) \xhookrightarrow{\mathfrak{mc}^2(\mathfrak{i}_{e_2})} \mathfrak{mc}^2(e_2) \xtwoheadrightarrow{\mathfrak{mc}^2(\mathfrak{q}_{e_2})} \mathfrak{mc}^2(\mathfrak{q}(e_2))$, resp.

Proof. First, we write out the short exact sequences $\mathsf{S}e_0 \xhookrightarrow{\mathfrak{i}_{e_2}} e_2 \xtwoheadrightarrow{\mathfrak{q}_{e_2}} e_1$, $\mathsf{S}e_1 \xhookrightarrow{\mathfrak{mc}(\mathfrak{i}_{e_2})} e_3 \xtwoheadrightarrow{\mathfrak{mc}(\mathfrak{q}_{e_2})} e_2$, and $\mathfrak{mc}\mathsf{S}e_1 \xhookrightarrow{\mathfrak{mc}^2(\mathfrak{i}_{e_2})} e_4 \xtwoheadrightarrow{\mathfrak{mc}^2(\mathfrak{q}_{e_2})} e_3$:

$$\begin{tikzcd}
\mathsf{S}\mathfrak{A} \arrow[r, equal] \arrow[d, equal] & \mathsf{S}\mathfrak{A} \arrow[r] \arrow[d, hook, "\iota"] & 0 \arrow[d, hook] \\
\mathsf{S}\mathfrak{A} \arrow[r, hook, "\iota"] \arrow[d] & \mathsf{C}\mathfrak{A} \arrow[r, two heads, "\pi"] \arrow[d, two heads, "\pi"] & \mathfrak{A} \arrow[d, equal] \\
0 \arrow[r] & \mathfrak{A} \arrow[r, equal] & \mathfrak{A}
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathsf{S}\mathfrak{A} \arrow[r, equal] \arrow[d, hook, "\iota_2"] & \mathsf{S}\mathfrak{A} \arrow[d, hook, "\iota"] \\
\mathsf{S}\mathfrak{A} \arrow[r, hook, "\iota_1"] \arrow[d, equal] & \mathsf{C}\mathfrak{A} \oplus_{\pi,\pi} \mathsf{C}\mathfrak{A} \arrow[r, two heads, "\pi_2"] \arrow[d, two heads, "\pi_1"] & \mathsf{C}\mathfrak{A} \arrow[d, two heads, "\pi"] \\
\mathsf{S}\mathfrak{A} \arrow[r, hook, "\iota"] & \mathsf{C}\mathfrak{A} \arrow[r, two heads, "\pi"] & \mathfrak{A}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathsf{S}\mathsf{S}\mathfrak{A} \arrow[r, hook, "\mathsf{S}\iota"] \arrow[d, hook, "\iota_{\mathsf{S}\mathfrak{A}}"] & \mathsf{S}\mathsf{C}\mathfrak{A} \arrow[r, two heads, "\mathsf{S}\pi"] \arrow[d, hook, "{(0,0,\iota_{\mathsf{C}\mathfrak{A}})}"] & \mathsf{S}\mathfrak{A} \arrow[d, hook, "\iota_2"] \\
\mathsf{C}\mathsf{S}\mathfrak{A} \arrow[r, hook, "{(\mathrm{ev}_1, 0, \mathsf{C}\iota)}"] \arrow[d, two heads, "\mathrm{ev}_1"] & (\mathsf{C}\mathfrak{A} \oplus_{\pi,\pi} \mathsf{C}\mathfrak{A}) \oplus_{\pi_1, \mathrm{ev}_1} \mathsf{C}\mathsf{C}\mathfrak{A} \arrow[r, two heads, "{(f,g,h)\mapsto(g,h(-,1))}"] \arrow[d, two heads, "{(f,g,h)\mapsto(f,g)}"] & \mathsf{C}\mathfrak{A} \oplus_{\pi,\pi} \mathsf{C}\mathfrak{A} \arrow[d, two heads, "\pi_1"] \\
\mathsf{S}\mathfrak{A} \arrow[r, hook, "\iota_1"] & \mathsf{C}\mathfrak{A} \oplus_{\pi,\pi} \mathsf{C}\mathfrak{A} \arrow[r, two heads, "\pi_2"] & \mathsf{C}\mathfrak{A}
\end{tikzcd}$$

Now, we write out the cyclic six term exact sequences of cyclic six term exact sequences corresponding to these three short exact sequences — where we horizontally use the KK-boundary maps and vertically use the K_{six}-boundary maps. For convenience we will identify K_1 with $K_0 \circ \mathsf{S}$. The diagrams are:

$$
\begin{tikzcd}
{} & {} \arrow[d] & {} \arrow[d, "\mathrm{id}"] & {} \arrow[d] & {} \arrow[d] & {} \arrow[d, "-\mathrm{id}"] & {} \arrow[d] & {} \\
{} \arrow[r] & K_0(\mathsf{S}\mathfrak{A}) \arrow[r, equal] \arrow[d, equal] & K_0(\mathsf{S}\mathfrak{A}) \arrow[r] \arrow[d, "K_0(\iota)"] & 0 \arrow[r] \arrow[d] & K_0(\mathsf{SS}\mathfrak{A}) \arrow[r, equal] \arrow[d, equal] & K_0(\mathsf{SS}\mathfrak{A}) \arrow[r] \arrow[d, "K_0(\mathsf{S}\iota)"] & 0 \arrow[r] \arrow[d] & {} \\
{} \arrow[r, "-\mathrm{id}"] & K_0(\mathsf{S}\mathfrak{A}) \arrow[r, "K_0(\iota)"] \arrow[d] & 0 \arrow[r, "K_0(\pi)"] \arrow[d, "K_0(\pi)"] & K_0(\mathfrak{A}) \arrow[r, "-\beta_{\mathfrak{A}}"] \arrow[d, equal] & K_0(\mathsf{SS}\mathfrak{A}) \arrow[r, "K_0(\mathsf{S}\iota)"] \arrow[d] & 0 \arrow[r, "K_0(\mathsf{S}\pi)"] \arrow[d, "K_0(\mathsf{S}\pi)"] & K_0(\mathsf{S}\mathfrak{A}) \arrow[r, "-\mathrm{id}"] \arrow[d, equal] & {} \\
{} \arrow[r] & 0 \arrow[r] \arrow[d] & K_0(\mathfrak{A}) \arrow[r, equal] \arrow[d, "-\beta_{\mathfrak{A}}"] & K_0(\mathfrak{A}) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & K_0(\mathsf{S}\mathfrak{A}) \arrow[r, equal] \arrow[d, "\beta_{\mathsf{S}\mathfrak{A}}"] & K_0(\mathsf{S}\mathfrak{A}) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & K_0(\mathsf{SS}\mathfrak{A}) \arrow[r, equal] \arrow[d, equal] & K_0(\mathsf{SS}\mathfrak{A}) \arrow[r] \arrow[d, "K_0(\mathsf{S}\iota)"] & 0 \arrow[r] \arrow[d] & K_0(\mathsf{SSS}\mathfrak{A}) \arrow[r, equal] \arrow[d, equal] & K_0(\mathsf{SSS}\mathfrak{A}) \arrow[r] \arrow[d, "K_0(\mathsf{SS}\iota)"] & 0 \arrow[r] \arrow[d] & {} \\
{} \arrow[r, "-\mathrm{id}"] & K_0(\mathsf{SS}\mathfrak{A}) \arrow[r, "K_0(\mathsf{S}\iota)"] \arrow[d] & 0 \arrow[r, "K_0(\mathsf{S}\pi)"] \arrow[d, "K_0(\mathsf{S}\pi)"] & K_0(\mathsf{S}\mathfrak{A}) \arrow[r, "-\beta_{\mathsf{S}\mathfrak{A}}"] \arrow[d, equal] & K_0(\mathsf{SSS}\mathfrak{A}) \arrow[r, "K_0(\mathsf{SS}\iota)"] \arrow[d] & 0 \arrow[r, "K_0(\mathsf{SS}\pi)"] \arrow[d, "K_0(\mathsf{SS}\pi)"] & K_0(\mathsf{SS}\mathfrak{A}) \arrow[r, "-\mathrm{id}"] \arrow[d, equal] & {} \\
{} \arrow[r] & 0 \arrow[r] \arrow[d] & K_0(\mathsf{S}\mathfrak{A}) \arrow[r, equal] \arrow[d, "\mathrm{id}"] & K_0(\mathsf{S}\mathfrak{A}) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & K_0(\mathsf{SS}\mathfrak{A}) \arrow[r, equal] \arrow[d, "-\mathrm{id}"] & K_0(\mathsf{SS}\mathfrak{A}) \arrow[r] \arrow[d] & {} \\
{} & {} & {} & {} & {} & {} & {} & {}
\end{tikzcd}
$$

$$
\begin{tikzcd}
{} & {} \arrow[d] & {} \arrow[d] & {} \arrow[d, "\mathrm{id}"] & {} \arrow[d] & {} \arrow[d] & {} \arrow[d, "-\mathrm{id}"] & {} \\
{} \arrow[r] & 0 \arrow[r] \arrow[d] & K_0(\mathsf{S}\mathfrak{A}) \arrow[r, equal] \arrow[d, "K_0(\iota_2)"] & K_0(\mathsf{S}\mathfrak{A}) \arrow[r] \arrow[d, "K_0(\iota)"] & 0 \arrow[r] \arrow[d] & K_0(\mathsf{SS}\mathfrak{A}) \arrow[r, equal] \arrow[d, "K_0(\mathsf{S}\iota_2)"] & K_0(\mathsf{SS}\mathfrak{A}) \arrow[r] \arrow[d, "K_0(\mathsf{S}\iota)"] & {} \\
{} \arrow[r] & K_0(\mathsf{S}\mathfrak{A}) \arrow[r, "K_0(\iota_1)"] \arrow[d, equal] & K_0(\mathsf{C}\mathfrak{A}\oplus_{\pi,\pi}\mathsf{C}\mathfrak{A}) \arrow[r, "K_0(\pi_2)"] \arrow[d, "K_0(\pi_1)"] & 0 \arrow[r] \arrow[d, "K_0(\pi)"] & K_0(\mathsf{SS}\mathfrak{A}) \arrow[r, "K_0(\mathsf{S}\iota_1)"] \arrow[d, equal] & K_0(\mathsf{S}(\mathsf{C}\mathfrak{A}\oplus_{\pi,\pi}\mathsf{C}\mathfrak{A})) \arrow[r, "K_0(\mathsf{S}\pi_2)"] \arrow[d, "K_0(\mathsf{S}\pi_1)"] & 0 \arrow[r] \arrow[d, "K_0(\mathsf{S}\pi)"] & {} \\
{} \arrow[r, "-\mathrm{id}"] & K_0(\mathsf{S}\mathfrak{A}) \arrow[r, "K_0(\iota)"] \arrow[d] & 0 \arrow[r, "K_0(\pi)"] \arrow[d] & K_0(\mathfrak{A}) \arrow[r, "-\beta_{\mathfrak{A}}"] \arrow[d, "-\beta_{\mathfrak{A}}"] & K_0(\mathsf{SS}\mathfrak{A}) \arrow[r, "K_0(\mathsf{S}\iota)"] \arrow[d] & 0 \arrow[r, "K_0(\mathsf{S}\pi)"] \arrow[d] & K_0(\mathsf{S}\mathfrak{A}) \arrow[r, "-\mathrm{id}"] \arrow[d, "\beta_{\mathsf{S}\mathfrak{A}}"] & {} \\
{} \arrow[r] & 0 \arrow[r] \arrow[d] & K_0(\mathsf{SS}\mathfrak{A}) \arrow[r, equal] \arrow[d, "K_0(\mathsf{S}\iota_2)"] & K_0(\mathsf{SS}\mathfrak{A}) \arrow[r] \arrow[d, "K_0(\mathsf{S}\iota)"] & 0 \arrow[r] \arrow[d] & K_0(\mathsf{SSS}\mathfrak{A}) \arrow[r, equal] \arrow[d, "K_0(\mathsf{SS}\iota_2)"] & K_0(\mathsf{SSS}\mathfrak{A}) \arrow[r] \arrow[d, "K_0(\mathsf{SS}\iota)"] & {} \\
{} \arrow[r] & K_0(\mathsf{SS}\mathfrak{A}) \arrow[r, "K_0(\mathsf{S}\iota_1)"] \arrow[d, equal] & K_0(\mathsf{S}(\mathsf{C}\mathfrak{A}\oplus_{\pi,\pi}\mathsf{C}\mathfrak{A})) \arrow[r, "K_0(\mathsf{S}\pi_2)"] \arrow[d, "K_0(\mathsf{S}\pi_1)"] & 0 \arrow[r] \arrow[d, "K_0(\mathsf{S}\pi)"] & K_0(\mathsf{SSS}\mathfrak{A}) \arrow[r, "K_0(\mathsf{SS}\iota_1)"] \arrow[d, equal] & K_0(\mathsf{SS}(\mathsf{C}\mathfrak{A}\oplus_{\pi,\pi}\mathsf{C}\mathfrak{A})) \arrow[r, "K_0(\mathsf{SS}\pi_2)"] \arrow[d, "K_0(\mathsf{SS}\pi_1)"] & 0 \arrow[r] \arrow[d, "K_0(\mathsf{SS}\pi)"] & {} \\
{} \arrow[r, "-\mathrm{id}"] & K_0(\mathsf{SS}\mathfrak{A}) \arrow[r, "K_0(\mathsf{S}\iota)"] \arrow[d] & 0 \arrow[r, "K_0(\mathsf{S}\pi)"] \arrow[d] & K_0(\mathsf{S}\mathfrak{A}) \arrow[r, "-\beta_{\mathsf{S}\mathfrak{A}}"] \arrow[d, "\mathrm{id}"] & K_0(\mathsf{SSS}\mathfrak{A}) \arrow[r, "K_0(\mathsf{SS}\iota)"] \arrow[d] & 0 \arrow[r, "K_0(\mathsf{SS}\pi)"] \arrow[d] & K_0(\mathsf{SS}\mathfrak{A}) \arrow[r, "-\mathrm{id}"] \arrow[d, "-\mathrm{id}"] & {} \\
{} & {} & {} & {} & {} & {} & {} & {}
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & {} \arrow[d, "\mathrm{id}"] & {} \arrow[d] & {} \arrow[d] & {} \arrow[d, "-\mathrm{id}"] & {} \arrow[d] & {} \arrow[d] & \\
{} \arrow[r, "\mathrm{id}"] & K_0(\mathsf{SS}\mathfrak{A}) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & K_0(\mathsf{S}\mathfrak{A}) \arrow[r, "\beta_{\mathsf{S}\mathfrak{A}}"] \arrow[d, "K_0(\iota_2)"] & K_0(\mathsf{SSS}\mathfrak{A}) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & K_0(\mathsf{SS}\mathfrak{A}) \arrow[r, "\mathrm{id}"] \arrow[d, "K_0(\mathsf{S}\iota_2)"] & {} \\
{} \arrow[r] & 0 \arrow[r] \arrow[d] & K_0((\mathsf{C}\mathfrak{A}\oplus_{\pi,\pi}\mathsf{C}\mathfrak{A})\oplus_{\pi_1,\mathrm{ev}_1}\mathsf{CC}\mathfrak{A}) \arrow[r] \arrow[d] & K_0(\mathsf{C}\mathfrak{A}\oplus_{\pi,\pi}\mathsf{C}\mathfrak{A}) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & K_0(\mathsf{S}((\mathsf{C}\mathfrak{A}\oplus_{\pi,\pi}\mathsf{C}\mathfrak{A})\oplus_{\pi_1,\mathrm{ev}_1}\mathsf{CC}\mathfrak{A})) \arrow[r] \arrow[d] & K_0(\mathsf{S}(\mathsf{C}\mathfrak{A}\oplus_{\pi,\pi}\mathsf{C}\mathfrak{A})) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & K_0(\mathsf{S}\mathfrak{A}) \arrow[r, "K_0(\iota_1)"] \arrow[d, "-\beta_{\mathsf{S}\mathfrak{A}}"] & K_0(\mathsf{C}\mathfrak{A}\oplus_{\pi,\pi}\mathsf{C}\mathfrak{A}) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & K_0(\mathsf{SS}\mathfrak{A}) \arrow[r, "K_0(\mathsf{S}\iota_1)"] \arrow[d, "\beta_{\mathsf{SS}\mathfrak{A}}"] & K_0(\mathsf{S}(\mathsf{C}\mathfrak{A}\oplus_{\pi,\pi}\mathsf{C}\mathfrak{A})) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & {} \\
{} \arrow[r, "\mathrm{id}"] & K_0(\mathsf{SSS}\mathfrak{A}) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & K_0(\mathsf{SS}\mathfrak{A}) \arrow[r, "\beta_{\mathsf{SS}\mathfrak{A}}"] \arrow[d, "K_0(\mathsf{S}\iota_2)"] & K_0(\mathsf{SSSS}\mathfrak{A}) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & K_0(\mathsf{SSS}\mathfrak{A}) \arrow[r, "\mathrm{id}"] \arrow[d, "K_0(\mathsf{SS}\iota_2)"] & {} \\
{} \arrow[r] & 0 \arrow[r] \arrow[d] & K_0(\mathsf{S}((\mathsf{C}\mathfrak{A}\oplus_{\pi,\pi}\mathsf{C}\mathfrak{A})\oplus_{\pi_1,\mathrm{ev}_1}\mathsf{CC}\mathfrak{A})) \arrow[r] \arrow[d] & K_0(\mathsf{S}(\mathsf{C}\mathfrak{A}\oplus_{\pi,\pi}\mathsf{C}\mathfrak{A})) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & K_0(\mathsf{SS}((\mathsf{C}\mathfrak{A}\oplus_{\pi,\pi}\mathsf{C}\mathfrak{A})\oplus_{\pi_1,\mathrm{ev}_1}\mathsf{CC}\mathfrak{A})) \arrow[r] \arrow[d] & K_0(\mathsf{SS}(\mathsf{C}\mathfrak{A}\oplus_{\pi,\pi}\mathsf{C}\mathfrak{A})) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & K_0(\mathsf{SS}\mathfrak{A}) \arrow[r, "K_0(\mathsf{S}\iota_1)"] \arrow[d, "\mathrm{id}"] & K_0(\mathsf{S}(\mathsf{C}\mathfrak{A}\oplus_{\pi,\pi}\mathsf{C}\mathfrak{A})) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & K_0(\mathsf{SSS}\mathfrak{A}) \arrow[r, "K_0(\mathsf{SS}\iota_1)"] \arrow[d, "-\mathrm{id}"] & K_0(\mathsf{SS}(\mathsf{C}\mathfrak{A}\oplus_{\pi,\pi}\mathsf{C}\mathfrak{A})) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & {} \\
 & {} & {} & {} & {} & {} & {} &
\end{tikzcd}
$$

We see that $\mathbf{x}_{e_0}$, $-\mathbf{x}_{e_1}$, and $KK_{\mathcal{E}}(\theta_{e_1})$ are induced by the morphisms

$$\begin{tikzcd}
\mathsf{S}\mathfrak{A} \arrow[r, equal] \arrow[d, equal] & \mathsf{S}\mathfrak{A} \arrow[d, hook, "\iota_2"] \\
\mathsf{S}\mathfrak{A} \arrow[r, "\iota_2"] \arrow[d] & \mathsf{C}\mathfrak{A} \oplus_{\pi,\pi} \mathsf{C}\mathfrak{A} \arrow[d, "\pi_1"] \\
0 \arrow[r] & \mathsf{C}\mathfrak{A}
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathsf{S}\mathsf{C}\mathfrak{A} \arrow[d, hook] \\
\mathsf{S}\mathfrak{A} \arrow[r, "{(0,\iota,0)}"] \arrow[d] & (\mathsf{C}\mathfrak{A} \oplus_{\pi,\pi} \mathsf{C}\mathfrak{A}) \oplus_{\pi_1,\mathrm{ev}_1} \mathsf{C}\mathsf{C}\mathfrak{A} \arrow[d, two heads] \\
\mathsf{S}\mathfrak{A} \arrow[r, "\iota_2"] & \mathsf{C}\mathfrak{A} \oplus_{\pi,\pi} \mathsf{C}\mathfrak{A}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{S}\mathsf{S}\mathfrak{A} \arrow[r, "\mathrm{flip}"] \arrow[d, hook] & \mathsf{S}\mathsf{S}\mathfrak{A} \arrow[d, hook] \\
\mathsf{C}\mathsf{S}\mathfrak{A} \arrow[r, "\mathrm{flip}"] \arrow[d, two heads] & \mathsf{S}\mathsf{C}\mathfrak{A} \arrow[d, two heads] \\
\mathsf{S}\mathfrak{A} \arrow[r, equal] & \mathsf{S}\mathfrak{A}
\end{tikzcd}$$

Using all these diagram, a long, tedious, straightforward verification shows the Proposition. ●●●

Remark 6.3.3. What we actually showed in the proof of the preceding proposition, is that the corresponding diagrams of morphisms in the category $\mathbf{KK}_{\mathcal{E}}$ (*i.e.*, before we apply $KK_{\mathcal{E}}(-,e)$) commute resp. anti-commute. This observation will be useful in the sequel.

Proof of the first part of Theorem 6.2.14. By definition, $F_{n,i-1} \xrightarrow{f_{n,i-1}} F_{n,i} \xrightarrow{f_{n,i}} F_{n,i+1}$ is exact for all $n \in \mathbb{N}$ and all $i = 0,1,2,3,4,5$.

We have a commuting square

$$\begin{tikzcd}
\mathbb{C} \arrow[r, equal] \arrow[d, equal] & \mathbb{C} \arrow[d] \\
\mathbb{C} \arrow[r] & \mathsf{M}_n,
\end{tikzcd}$$

where the maps $\mathbb{C} \to \mathsf{M}_n$ are the unital $*$-homomorphisms. By naturality of the mapping cone construction, this induces a morphism $\phi = (\phi_0, \phi_1, \phi_2)$ from $\mathfrak{f}_{1,2}$ to $\mathfrak{e}_{n,0}$. This gives a commuting diagram

$$\begin{tikzcd}
\mathfrak{i}(\mathfrak{f}_{1,2}) \arrow[r, hook] \arrow[d, "{(\phi_0,\phi_0,0)}"] & \mathfrak{f}_{1,2} \arrow[r, two heads] \arrow[d, "\phi"] & \mathfrak{q}(\mathfrak{f}_{1,2}) \arrow[d, equal] \\
\mathfrak{i}(\mathfrak{e}_{n,0}) \arrow[r, hook] & \mathfrak{e}_{n,0} \arrow[r, two heads] & \mathfrak{q}(\mathfrak{e}_{n,0})
\end{tikzcd}$$

of short exact sequences. If we apply $KK_{\mathcal{E}}(-,e)$ to this diagram we will get a morphism between two cyclic six term exact sequences in $KK_{\mathcal{E}}$-theory. Using the standard equivalences introduced so far, we arrive at the commuting diagram

$$\begin{tikzcd}
{} \arrow[r, "f_{1,0}"] & KK_{\mathcal{E}}(\mathfrak{f}_{1,1},e) \arrow[r, "f_{1,1}"] \arrow[d, equal] & KK_{\mathcal{E}}(\mathfrak{f}_{1,2},e) \arrow[r, "f_{1,2}"] \arrow[d, equal] & KK_{\mathcal{E}}(\mathfrak{f}_{1,3},e) \arrow[r, "f_{1,3}"] \arrow[d, "\mathbf{x}_{\mathfrak{f}_{1,0}}\times-", "\cong"'] & KK_{\mathcal{E}}(\mathfrak{f}_{1,4},e) \arrow[r, "f_{1,4}"] \arrow[d, "\mathbf{x}_{\mathfrak{f}_{1,1}}\times-", "\cong"'] & KK_{\mathcal{E}}(\mathfrak{f}_{1,5},e) \arrow[r, "f_{1,5}"] \arrow[d, "\mathbf{x}_{\mathfrak{f}_{1,2}}\times-", "\cong"'] & KK_{\mathcal{E}}(\mathfrak{f}_{1,0},e) \arrow[r, "f_{1,0}"] \arrow[d, "\mathbf{b}_{\mathfrak{f}_{1,0}}^{-1}\times-", "\cong"'] & {} \\
{} \arrow[r] & KK_{\mathcal{E}}(\mathfrak{f}_{1,1},e) \arrow[r] \arrow[d, equal] & KK_{\mathcal{E}}(\mathfrak{f}_{1,2},e) \arrow[r] & KK_{\mathcal{E}}(\mathsf{S}\mathfrak{f}_{1,0},e) \arrow[r] & KK_{\mathcal{E}}(\mathsf{S}\mathfrak{f}_{1,1},e) \arrow[r] \arrow[d, equal] & KK_{\mathcal{E}}(\mathsf{S}\mathfrak{f}_{1,2},e) \arrow[r] & KK_{\mathcal{E}}(\mathsf{SS}\mathfrak{f}_{1,0},e) \arrow[r] & {} \\
{} \arrow[r] & KK_{\mathcal{E}}(\mathfrak{f}_{1,1},e) \arrow[r] \arrow[d, equal] & KK_{\mathcal{E}}(\mathfrak{e}_{n,0},e) \arrow[r] \arrow[u, "KK_{\mathcal{E}}(\phi)\times-"'] \arrow[d, equal] & KK_{\mathcal{E}}(\mathsf{i}(\mathfrak{e}_{n,0}),e) \arrow[r] \arrow[u, "KK_{\mathcal{E}}((\phi_0,\phi_0,0))\times-"'] \arrow[d, "\mathbf{x}_{\mathfrak{f}_{1,0}}^{-1}\times\mathbf{z}_n\times-", "\cong"'] & KK_{\mathcal{E}}(\mathsf{S}\mathfrak{f}_{1,1},e) \arrow[r] \arrow[d, "\mathbf{x}_{\mathfrak{f}_{1,1}}^{-1}\times-", "\cong"'] & KK_{\mathcal{E}}(\mathsf{S}\mathfrak{e}_{n,0},e) \arrow[r] \arrow[u, "KK_{\mathcal{E}}(\mathsf{S}\phi)\times-"'] \arrow[d, "\mathbf{x}_{\mathfrak{e}_{n,0}}^{-1}\times-", "\cong"'] & KK_{\mathcal{E}}(\mathsf{Si}(\mathfrak{e}_{n,0}),e) \arrow[r] \arrow[u, "KK_{\mathcal{E}}(\mathsf{S}(\phi_0,\phi_0,0))\times-"'] \arrow[d, "\mathbf{b}_{\mathfrak{f}_{1,0}}\times\widehat{\mathsf{S}}\mathbf{z}_n\times-", "\cong"'] & {} \\
& KK_{\mathcal{E}}(\mathfrak{f}_{1,1},e) \arrow[r] \arrow[d, equal] & KK_{\mathcal{E}}(\mathfrak{e}_{n,0},e) \arrow[r] \arrow[d, equal] & KK_{\mathcal{E}}(\mathfrak{f}_{1,3},e) \arrow[d, equal] & KK_{\mathcal{E}}(\mathfrak{f}_{1,4},e) \arrow[r] \arrow[d, equal] & KK_{\mathcal{E}}(\mathfrak{e}_{n,3},e) \arrow[r] \arrow[d, equal] & KK_{\mathcal{E}}(\mathfrak{f}_{1,0},e) \arrow[d, equal] & \\
& F_{1,1} \arrow[r, "h_{n,0}^{1,1,in}"] & H_{n,0} \arrow[r, "h_{n,0}^{1,1,out}"] & F_{1,3} & F_{1,4} \arrow[r, "h_{n,3}^{1,1,in}"] & H_{n,3} \arrow[r, "h_{n,3}^{1,1,out}"] & F_{1,0} &
\end{tikzcd}$$

with exact rows. We use Lemma 6.2.5 for commutativity the two squares on the right hand side between row three and four — and we use that

$$\mathrm{Ext}^1_{\mathrm{six}}(K_{\mathrm{six}}(\mathfrak{e}_{n,0}), K_{\mathrm{six}}(\mathsf{S}\mathfrak{f}_{1,1})) = 0,$$

$$\mathrm{Ext}^1_{\mathrm{six}}(K_{\mathrm{six}}(\mathsf{S}\mathfrak{f}_{1,3}), K_{\mathrm{six}}(\mathsf{S}\mathfrak{e}_{n,3})) = 0.$$

This is easily verified using projective resolutions.

It is easy to verify that, up to a sign, we have

$$KK_{\mathcal{E}}(\mathbf{z}_n^{-1} \times KK_{\mathcal{E}}((\phi_0, \phi_0, 0)), e) = n\mathrm{id} \quad \text{and} \quad KK_{\mathcal{E}}(\widehat{\mathsf{S}}\mathbf{z}_n^{-1} \times \widehat{\mathsf{S}}KK_{\mathcal{E}}((\phi_0, \phi_0, 0)), e) = n\mathrm{id}.$$

Consequently, $nf_{1,0}$ and $nf_{1,3}$ are exactly the connecting homomorphisms of the cyclic six term exact sequence in the bottom.

This proves exactness of the first of the four cyclic sequences in the theorem, for $i = 0, 3$.

This same result also works for $i = 1, 2, 4, 5$, by invoking Proposition 6.3.2 (remember that we do not care about the signs, because that does not change exactness).

We have a commuting square

$$\begin{tikzcd}
\mathbb{I}_{n,0} \arrow[r, equal] \arrow[d, equal] & \mathbb{I}_{n,0} \arrow[d] \\
\mathbb{I}_{n,0} \arrow[r] & \mathbb{C},
\end{tikzcd}$$

where the maps $\mathbb{I}_{n,0} \to \mathbb{C}$ are the canonical surjective $*$-homomorphisms. By naturality of the mapping cone construction, this induces a morphism $\phi = (\phi_0, \phi_1, \phi_2)$ from $\mathfrak{f}_{n,2}$ to $\mathfrak{e}_{n,1}$. This gives a commuting diagram

$$\begin{tikzcd}
\mathfrak{i}(\mathfrak{f}_{n,2}) \arrow[r, hook] \arrow[d, "(\phi_0,\phi_0,0)"] & \mathfrak{f}_{n,2} \arrow[r, two heads] \arrow[d, "\phi"] & \mathfrak{q}(\mathfrak{f}_{n,2}) \arrow[d, equal] \\
\mathfrak{i}(\mathfrak{e}_{n,1}) \arrow[r, hook] & \mathfrak{e}_{n,1} \arrow[r, two heads] & \mathfrak{q}(\mathfrak{e}_{n,1})
\end{tikzcd}$$

of short exact sequences. If we apply $KK_{\mathcal{E}}(-, e)$ to this diagram we will get a morphism between two cyclic six term exact sequences in $KK_{\mathcal{E}}$-theory. Using the standard equivalences introduced so far, we arrive at the commuting diagram

$$
\begin{tikzcd}[column sep=small]
{} \arrow[r, "f_{n,0}"] & KK_{\mathcal{E}}(\mathfrak{f}_{n,1},e) \arrow[r, "f_{n,1}"] \arrow[d, equal] & KK_{\mathcal{E}}(\mathfrak{f}_{n,2},e) \arrow[r, "f_{n,2}"] \arrow[d, equal] & KK_{\mathcal{E}}(\mathfrak{f}_{n,3},e) \arrow[r, "f_{n,3}"] \arrow[d, "\mathbf{x}_{\mathfrak{f}_{n,0}}\times-", "\cong"'] & KK_{\mathcal{E}}(\mathfrak{f}_{n,4},e) \arrow[r, "f_{n,4}"] \arrow[d, "\mathbf{x}_{\mathfrak{f}_{n,1}}\times-", "\cong"'] & KK_{\mathcal{E}}(\mathfrak{f}_{n,5},e) \arrow[r, "f_{n,5}"] \arrow[d, "\mathbf{x}_{\mathfrak{f}_{n,2}}\times-", "\cong"'] & KK_{\mathcal{E}}(\mathfrak{f}_{n,0},e) \arrow[r, "f_{n,0}"] \arrow[d, "\mathbf{b}^{-1}_{\mathfrak{f}_{n,0}}\times-", "\cong"'] & {} \\
{} \arrow[r] & KK_{\mathcal{E}}(\mathfrak{f}_{n,1},e) \arrow[r] \arrow[d, equal] & KK_{\mathcal{E}}(\mathfrak{f}_{n,2},e) \arrow[r] & KK_{\mathcal{E}}(\mathsf{S}\mathfrak{f}_{n,0},e) \arrow[r] & KK_{\mathcal{E}}(\mathsf{S}\mathfrak{f}_{n,1},e) \arrow[r] \arrow[d, equal] & KK_{\mathcal{E}}(\mathsf{S}\mathfrak{f}_{n,2},e) \arrow[r] & KK_{\mathcal{E}}(\mathsf{SS}\mathfrak{f}_{n,0},e) \arrow[r] & {} \\
{} \arrow[r] & KK_{\mathcal{E}}(\mathfrak{f}_{n,1},e) \arrow[r] \arrow[d, equal] & KK_{\mathcal{E}}(\mathfrak{e}_{n,1},e) \arrow[r] \arrow[u, "KK_{\mathcal{E}}(\phi)\times-"'] \arrow[d, equal] & KK_{\mathcal{E}}(\mathsf{S}\mathfrak{f}_{1,0},e) \arrow[r] \arrow[u, "KK_{\mathcal{E}}((\phi_0,\phi_0,0))\times-"'] \arrow[d, "\mathbf{x}^{-1}_{\mathfrak{f}_{1,0}}\times-", "\cong"'] & KK_{\mathcal{E}}(\mathsf{S}\mathfrak{f}_{n,1},e) \arrow[r] \arrow[d, "\mathbf{x}^{-1}_{\mathfrak{f}_{n,1}}\times-", "\cong"'] & KK_{\mathcal{E}}(\mathsf{S}\mathfrak{e}_{n,1},e) \arrow[r] \arrow[u, "KK_{\mathcal{E}}(\mathsf{S}\phi)\times-"'] \arrow[d, "\mathbf{x}^{-1}_{\mathfrak{e}_{n,1}}\times-", "\cong"'] & KK_{\mathcal{E}}(\mathsf{SS}\mathfrak{f}_{1,0},e) \arrow[r] \arrow[u, "KK_{\mathcal{E}}(\mathsf{S}(\phi_0,\phi_0,0))\times-"'] \arrow[d, "\mathbf{b}_{\mathfrak{f}_{1,0}}\times-", "\cong"'] & {} \\
 & KK_{\mathcal{E}}(\mathfrak{f}_{n,1},e) \arrow[r] \arrow[d, equal] & KK_{\mathcal{E}}(\mathfrak{e}_{n,1},e) \arrow[r] \arrow[d, equal] & KK_{\mathcal{E}}(\mathfrak{f}_{1,3},e) \arrow[d, equal] & KK_{\mathcal{E}}(\mathfrak{f}_{n,4},e) \arrow[r] \arrow[d, equal] & KK_{\mathcal{E}}(\mathfrak{e}_{n,4},e) \arrow[r] \arrow[d, equal] & KK_{\mathcal{E}}(\mathfrak{f}_{1,0},e) \arrow[d, equal] & \\
 & F_{n,1} \arrow[r, "h^{n,1,in}_{n,1}"] & H_{n,1} \arrow[r, "h^{n,1,out}_{n,1}"] & F_{1,3} & F_{n,4} \arrow[r, "h^{n,1,in}_{n,4}"] & H_{n,4} \arrow[r, "h^{n,1,out}_{n,4}"] & F_{1,0} &
\end{tikzcd}
$$

with exact rows. We use Lemma 6.2.5 for commutativity the two squares on the right hand side between row three and four — and we use that

$$\mathrm{Ext}^1_{\mathrm{six}}(K_{\mathrm{six}}(\mathsf{S}\mathfrak{e}_{n,1}), K_{\mathrm{six}}(\mathsf{S}\mathfrak{f}_{n,4})) = 0,$$

$$\mathrm{Ext}^1_{\mathrm{six}}(K_{\mathrm{six}}(\mathsf{S}\mathfrak{f}_{1,3}), K_{\mathrm{six}}(\mathsf{S}\mathfrak{e}_{n,4})) = 0.$$

This is easily verified using projective resolutions.

Using naturality of $\mathbf{b}_{-}$ and Lemma 6.2.5, it is easy to see that

$$KK_{\mathcal{E}}(\mathbf{x}^{-1}_{\mathfrak{f}_{n,0}} \times KK_{\mathcal{E}}((\phi_0,\phi_0,0)) \times \mathbf{x}_{\mathfrak{f}_{1,0}}, e) = \rho_{n,3}, \quad \text{and}$$

$$KK_{\mathcal{E}}(\mathbf{b}_{\mathfrak{f}_{n,0}} \times KK_{\mathcal{E}}(\mathsf{S}(\phi_0,\phi_0,0)) \times \mathbf{b}^{-1}_{\mathfrak{f}_{1,0}}, e) = \rho_{n,0}.$$

Consequently, $f_{n,3} \circ \rho_{n,3}$ and $f_{n,0} \circ \rho_{n,0}$ are exactly the connecting homomorphisms of the cyclic six term exact sequence in the bottom.

This proves exactness of the second of the four cyclic sequences in the theorem, for $i = 1, 4$.

This same result also works for $i = 0, 2, 3, 5$, by invoking Proposition 6.3.2.

We have a commuting square

$$\begin{tikzcd}
\mathbb{I}_{n,1} \arrow[r] \arrow[d] & \mathbb{I}_{n,0} \arrow[d, equal] \\
\mathbb{I}_{n,0} \arrow[r, equal] & \mathbb{I}_{n,0},
\end{tikzcd}$$

where the maps $\mathbb{I}_{n,1} \to \mathbb{I}_{n,0}$ are the canonical surjective $*$-homomorphisms. By naturality of the mapping cone construction, this induces a morphism $\phi = (\phi_0, \phi_1, \phi_2)$ from $\mathfrak{e}_{n,2}$ to $\mathfrak{f}_{n,2}$. This gives a commuting diagram

$$\begin{tikzcd}
\mathfrak{i}(\mathfrak{e}_{n,2}) \arrow[r, hook] \arrow[d, equal] & \mathfrak{e}_{n,2} \arrow[r, two heads] \arrow[d, "\phi"] & \mathfrak{q}(\mathfrak{e}_{n,2}) \arrow[d, "{(0,\phi_2,\phi_2)}"] \\
\mathfrak{i}(\mathfrak{f}_{n,2}) \arrow[r, hook] & \mathfrak{f}_{n,2} \arrow[r, two heads] & \mathfrak{q}(\mathfrak{f}_{n,2})
\end{tikzcd}$$

of short exact sequences. If we apply $KK_{\mathcal{E}}(-, e)$ to this diagram we will get a morphism between two cyclic six term exact sequences in $KK_{\mathcal{E}}$-theory. Using the standard equivalences introduced so far, we arrive at the commuting diagram

$$
\begin{tikzcd}[column sep=small]
& F_{1,4} \arrow[r, "h^{1,n,in}_{n,2}"] \arrow[d, equal] & H_{n,2} \arrow[r, "h^{1,n,out}_{n,2}"] \arrow[d, equal] & F_{n,3} \arrow[d, equal] & F_{1,1} \arrow[r, "h^{1,n,in}_{n,5}"] \arrow[d, equal] & H_{n,5} \arrow[r, "h^{1,n,out}_{n,5}"] \arrow[d, equal] & F_{n,0} \arrow[d, equal] & \\
& KK_{\mathcal{E}}(\mathfrak{f}_{1,4},e) \arrow[r] \arrow[d, dashed, "\mathbf{w}_n^{-1}\times\widehat{\mathfrak{mc}}(\mathbf{z}_n^{-1})\times\mathbf{x}_{\mathfrak{f}_{1,1}}\times-"'] & KK_{\mathcal{E}}(\mathfrak{e}_{n,2},e) \arrow[r] \arrow[d, equal] & KK_{\mathcal{E}}(\mathfrak{f}_{n,3},e) \arrow[d, "\mathbf{x}_{\mathfrak{f}_{n,0}}\times-", "\cong"'] & KK_{\mathcal{E}}(\mathfrak{f}_{1,1},e) \arrow[r] \arrow[d, dashed, "\widehat{\mathsf{S}}(\mathbf{w}_n^{-1}\times\widehat{\mathfrak{mc}}(\mathbf{z}_n^{-1}))\times\mathbf{b}^{-1}_{\mathfrak{f}_{1,1}}\times-"'] & KK_{\mathcal{E}}(\mathfrak{e}_{n,5},e) \arrow[r] \arrow[d, "\mathbf{x}_{\mathfrak{e}_{n,2}}\times-", "\cong"'] & KK_{\mathcal{E}}(\mathfrak{f}_{n,0},e) \arrow[d, "\mathbf{b}^{-1}_{\mathfrak{f}_{n,0}}\times-", "\cong"'] & \\
\arrow[r] & KK_{\mathcal{E}}(\mathsf{q}(\mathfrak{e}_{n,2}),e) \arrow[r] & KK_{\mathcal{E}}(\mathfrak{e}_{n,2},e) \arrow[r] & KK_{\mathcal{E}}(\mathsf{S}\mathfrak{f}_{n,0},e) \arrow[r] & KK_{\mathcal{E}}(\mathsf{S}\mathsf{q}(\mathfrak{e}_{n,2}),e) \arrow[r] & KK_{\mathcal{E}}(\mathsf{S}\mathfrak{e}_{n,2},e) \arrow[r] & KK_{\mathcal{E}}(\mathsf{S}\mathsf{S}\mathfrak{f}_{n,0},e) \arrow[r] & {} \\
\arrow[r] & KK_{\mathcal{E}}(\mathfrak{f}_{n,1},e) \arrow[r] \arrow[u, "KK_{\mathcal{E}}((0,\phi_2,\phi_2))\times-"'] & KK_{\mathcal{E}}(\mathfrak{f}_{n,2},e) \arrow[r] \arrow[u, "KK_{\mathcal{E}}(\phi)\times-"'] & KK_{\mathcal{E}}(\mathsf{S}\mathfrak{f}_{n,0},e) \arrow[r] \arrow[u, equal] & KK_{\mathcal{E}}(\mathsf{S}\mathfrak{f}_{n,1},e) \arrow[r] \arrow[u, "KK_{\mathcal{E}}(\mathsf{S}(0,\phi_2,\phi_2))\times-"'] & KK_{\mathcal{E}}(\mathsf{S}\mathfrak{f}_{n,2},e) \arrow[r] \arrow[u, "KK_{\mathcal{E}}(\mathsf{S}\phi)\times-"'] & KK_{\mathcal{E}}(\mathsf{S}\mathsf{S}\mathfrak{f}_{n,0},e) \arrow[r] \arrow[u, equal] & {} \\
\arrow[r, "f_{n,0}"] & KK_{\mathcal{E}}(\mathfrak{f}_{n,1},e) \arrow[r, "f_{n,1}"] \arrow[u, equal] & KK_{\mathcal{E}}(\mathfrak{f}_{n,2},e) \arrow[r, "f_{n,2}"] \arrow[u, equal] & KK_{\mathcal{E}}(\mathfrak{f}_{n,3},e) \arrow[r, "f_{n,3}"] \arrow[u, "\mathbf{x}_{\mathfrak{f}_{n,0}}\times-"', "\cong"] & KK_{\mathcal{E}}(\mathfrak{f}_{n,4},e) \arrow[r, "f_{n,4}"] \arrow[u, "\mathbf{x}_{\mathfrak{f}_{n,1}}\times-"', "\cong"] & KK_{\mathcal{E}}(\mathfrak{f}_{n,5},e) \arrow[r, "f_{n,5}"] \arrow[u, "\mathbf{x}_{\mathfrak{f}_{n,2}}\times-"', "\cong"] & KK_{\mathcal{E}}(\mathfrak{f}_{n,0},e) \arrow[r, "f_{n,0}"] \arrow[u, "\mathbf{b}^{-1}_{\mathfrak{f}_{n,0}}\times-"', "\cong"] & {}
\end{tikzcd}
$$

with exact rows. We use Lemma 6.2.5 for commutativity the two squares on the right hand side between row two and three — and we use that

$$\operatorname{Ext}^1_{\mathrm{six}}(K_{\mathrm{six}}(\mathsf{S}\mathfrak{e}_{n,2}), K_{\mathrm{six}}(\mathsf{S}\mathfrak{f}_{1,7})) = 0,$$

$$\operatorname{Ext}^1_{\mathrm{six}}(K_{\mathrm{six}}(\mathfrak{f}_{n,0}), K_{\mathrm{six}}(\mathsf{SS}\mathfrak{e}_{n,2})) = 0.$$

This is easily verified using projective resolutions.

Using naturality of $\mathbf{b}_-$ and Lemma 6.2.5, it is easy to see that

$$KK_{\mathcal{E}}(\mathbf{x}_{\mathfrak{f}_{1,1}} \times \widehat{\mathfrak{mc}}(\mathbf{z}_n) \times \mathbf{w}_n \times KK_{\mathcal{E}}((0, \phi_2, \phi_2)), e) = -\beta_{n,1}, \quad \text{and}$$

$$KK_{\mathcal{E}}(\mathbf{b}_{\mathfrak{f}_{1,1}} \times \widehat{\mathsf{S}}\widehat{\mathfrak{mc}}(\mathbf{z}_n) \times \widehat{\mathsf{S}}\mathbf{w}_n \times KK_{\mathcal{E}}(\mathsf{S}(0, \phi_2, \phi_2)), e) = -\beta_{n,3}.$$

Consequently, $\beta_{n,4} \circ f_{n,3}$ and $\beta_{n,1} \circ f_{n,0}$ are exactly the connecting homomorphisms of the cyclic six term exact sequence in the top (up to a sign, of course).

This proves exactness of the third of the four cyclic sequences in the theorem, for $i = 2, 5$.

This same result also works for $i = 0, 1, 3, 4$, by invoking Proposition 6.3.2.

That the last one of the sequences is exact for all $i = 0, 1, 2$ is straightforward to check. ●●●

Proof of the second part of Theorem 6.2.14. Diagram (6.1). First we prove it for $i = 1$. We have a commuting diagram

$$\begin{tikzcd}
0 \arrow[r, hook] \arrow[d, hook] & \mathfrak{mc}(\mathfrak{i}(\mathfrak{e}_{n,0})) \arrow[r, equal] \arrow[d, hook, "\mathfrak{mc}(\mathfrak{i}_{\mathfrak{e}_{n,0}})"] & \mathfrak{mc}(\mathfrak{i}(\mathfrak{e}_{n,0})) \arrow[d, hook] \\
\mathfrak{i}(\mathfrak{e}_{n,1}) \arrow[r, hook, "\mathfrak{i}_{\mathfrak{e}_{n,1}}"] \arrow[d, equal] & \mathfrak{e}_{n,1} \arrow[r, two heads, "\mathfrak{q}_{\mathfrak{e}_{n,1}}"] \arrow[d, "\mathfrak{mc}(\mathfrak{q}_{\mathfrak{e}_{n,0}})"] & \mathfrak{q}(\mathfrak{e}_{n,1}) \arrow[d] \\
\mathfrak{i}(\mathfrak{e}_{n,1}) \arrow[r, hook] & \mathfrak{mc}(\mathfrak{q}(\mathfrak{e}_{n,0})) \arrow[r, two heads] & \mathfrak{f}_{1,1}
\end{tikzcd}$$

of objects from $S\mathcal{E}$ with short exact rows and short exact columns. Note that the short exact sequences $\mathfrak{i}(\mathfrak{e}_{n,1}) \hookrightarrow \mathfrak{mc}(\mathfrak{q}(\mathfrak{e}_{n,0})) \twoheadrightarrow \mathfrak{f}_{1,1}$ and $\mathfrak{mc}(\mathfrak{i}(\mathfrak{e}_{n,0})) \hookrightarrow \mathfrak{q}(\mathfrak{e}_{n,1}) \twoheadrightarrow \mathfrak{f}_{1,1}$ are exactly the short exact sequences $\mathfrak{i}(\mathfrak{f}_{1,2}) \xhookrightarrow{\mathfrak{i}_{\mathfrak{f}_{1,2}}} \mathfrak{f}_{1,2} \xrightarrow{\mathfrak{q}_{\mathfrak{f}_{1,2}}} \mathfrak{q}(\mathfrak{f}_{1,2})$ and $\mathfrak{mc}$ applied to the sequence $\mathfrak{i}(\mathfrak{e}_{n,0})) \xhookrightarrow{x} \mathfrak{f}_{n,0} \xrightarrow{y} \mathfrak{f}_{1,0}$ from Definition 6.2.12, resp. Now apply $KK_{\mathcal{E}}(-, e)$, then one easily shows the commutativity of the diagram (using the definitions of the different maps)

$$\begin{tikzcd}
F_{1,1} \arrow[r, "\tilde{f}_{1,1}"] \arrow[d, "\rho_{n,1}"'] & F_{1,2} \arrow[d, "h_{n,1}^{1,1,in}"'] \arrow[dr, "\tilde{f}_{1,2}"] & \\
F_{n,1} \arrow[r, "h_{n,1}^{n,1,in}"] \arrow[dr, "\beta_{n,1}"'] & H_{n,1} \arrow[r, "h_{n,1}^{n,1,out}"'] \arrow[d, "h_{n,1}^{1,1,out}"] & F_{1,3} \\
& F_{1,4} &
\end{tikzcd}$$

If we apply $\mathfrak{mc}^k$ to the diagram, for $k = 1, 2, 3, 4, 5$, we obtain commutativity of the corresponding part of Diagram (6.1), for $i = 2, 3, 4, 5, 0$, resp. — this is, indeed, a *very long* and tedious verification using the identifications and results above.

Now we prove commutativity of the remaining square in Diagram (6.1) for $i = 1$. We have a commuting diagram

$$\begin{tikzcd}
\mathsf{S}\mathfrak{f}_{1,1} \arrow[r] \arrow[d] & \mathsf{S}\mathfrak{f}_{1,0} \arrow[d, "\mathfrak{i}_{\mathfrak{e}_{n,1}}"] \\
\mathfrak{q}(\mathfrak{e}_{n,2}) \arrow[r, "\phi"'] & \mathfrak{e}_{n,1},
\end{tikzcd}$$

where the horizontal morphisms are the unique morphism which are the identity on the extension algebra, and the vertical morphism from $\mathsf{S}\mathfrak{f}_{1,1}$ to $\mathfrak{q}(\mathfrak{e}_{n,2})$ is the morphism induced by the $*$-homomorphism $\mathsf{S}\mathbb{C} \to \mathbb{I}_{n,1}$ in the extension $\mathfrak{e}_{n,1}$. It is easy to see that $\mathfrak{mc}(\mathfrak{i}_{\mathfrak{e}_{n,0}})$ is exactly $\phi \circ w_n$, where w_n is the morphism inducing $\mathbf{w}_n$. Now we get commutativity of

$$\begin{tikzcd}
H_{n,1} \arrow[r, "h_{n,1}^{n,1,out}"] \arrow[d, "h_{n,1}^{1,1,out}"] & F_{1,3} \arrow[d, "\tilde{f}_{1,3}"] \\
F_{1,4} \arrow[r, "\times n"'] & F_{1,4}
\end{tikzcd}$$

by applying $KK_{\mathcal{E}}(-, e)$ to the above diagram. If we first apply $\mathfrak{mc}^k$ to the diagram, for $k = 1, 2, 3, 4, 5$, we obtain commutativity of the corresponding square of Diagram (6.1), for $i = 2, 3, 4, 5, 0$, resp.

Diagram (6.2). We first prove it for $i = 2$. We have a commutative diagram

$$\begin{tikzcd}
\mathfrak{i}(\mathfrak{mc}^2(\mathsf{S}\mathfrak{f}_{1,0})) \arrow[r, "\mathfrak{i}_{\mathfrak{mc}^2(\mathsf{S}\mathfrak{f}_{1,0})}"] \arrow[d, "\mathsf{S}(x \circ z_n)"] & \mathfrak{mc}^2(\mathsf{S}\mathfrak{f}_{1,0}) \arrow[r, "\mathfrak{q}_{\mathfrak{mc}^2(\mathsf{S}\mathfrak{f}_{1,0})}"] \arrow[d, "\mathfrak{mc}^2(\mathfrak{i}_{\mathfrak{e}_{n,0}} \circ z_n)"] & \mathfrak{q}(\mathfrak{mc}^2(\mathsf{S}\mathfrak{f}_{1,0})) \arrow[d, "w_n \circ \mathfrak{mc}(z_n)"] \\
\mathfrak{i}(\mathfrak{e}_{n,2}) \arrow[r, "\mathfrak{i}_{\mathfrak{e}_{n,2}}"] \arrow[d, "\mathsf{S}y"] & \mathfrak{e}_{n,2} \arrow[r, "\mathfrak{q}_{\mathfrak{e}_{n,2}}"] \arrow[d, "\mathfrak{mc}^2(\mathfrak{q}_{\mathfrak{e}_{n,0}})"] & \mathfrak{q}(\mathfrak{e}_{n,2}) \\
\mathsf{S}\mathfrak{f}_{1,0} \arrow[r, "x_{\mathfrak{f}_{1,0}}"] & \mathfrak{f}_{1,3} &
\end{tikzcd}$$

where $x_{\mathfrak{f}_{1,0}}$, z_n and w_n denote the morphisms inducing $\mathbf{x}_{\mathfrak{f}_{1,0}}$, $\mathbf{z}_n$ and $\mathbf{w}_n$, resp., and the first column is the suspension of the short exact sequence introduced in Definition 6.2.12 (note that we do not claim the columns and rows to be exact).

A computation shows that this gives rise to a commutative diagram (by applying $KK_{\mathcal{E}}(-, e)$)

$$\begin{tikzcd}
& F_{1,3} \arrow[d, "h_{n,2}^{1,1,in}"'] \arrow[dr, "\rho_{n,3}"] & \\
F_{1,4} \arrow[r, "h_{n,2}^{1,n,in}"] \arrow[dr, "\tilde{f}_{1,4}"'] & H_{n,2} \arrow[r, "h_{n,2}^{1,n,out}"'] \arrow[d, "h_{n,2}^{1,1,out}"] & F_{n,3} \arrow[d, "-\beta_{n,3}"] \\
& F_{1,5} \arrow[r, "\tilde{f}_{1,5}"'] & F_{1,0}.
\end{tikzcd}$$

If we apply $\mathfrak{mc}^k$ to the diagram, for $k = 1, 2, 3, 4, 5$, we obtain commutativity of the corresponding part of Diagram (6.3), for $i = 3, 4, 5, 0, 1$, resp. — this is, indeed, a *very long* and tedious verification using the identifications and results above.

Now we prove commutativity of the remaining square in Diagram (6.2) for $i = 2$. We have a commuting diagram

$$\begin{tikzcd}
\mathfrak{e}_{n,2} \arrow[r, "\mathfrak{q}_{\mathfrak{e}_{n,2}}"] \arrow[d, "(*,\mathrm{id},0)"] & \mathfrak{q}(\mathfrak{e}_{n,2}) \arrow[d, "(0,\mathrm{id},0)"] \\
{[\mathbb{I}_{n,2} \hookrightarrow \mathbb{I}_{n,2} \twoheadrightarrow 0]} \arrow[r] & {[\mathbb{I}_{n,1} \hookrightarrow \mathbb{I}_{n,1} \twoheadrightarrow 0]},
\end{tikzcd}$$

where the bottom horizontal morphism is the morphism induced by the $*$-homomorphism from $\mathbb{I}_{n,2}$

to $\mathbb{I}_{n,1}$ in the extension $\mathfrak{e}_{n,2}$. It is easy to see that we have a commuting square

$$\begin{tikzcd}
\mathsf{S}\mathfrak{f}_{1,1} \arrow[r, "\mathbf{x}_{\mathfrak{f}_{1,1}}"] \arrow[d, "KK_{\mathcal{E}}(\mathsf{S}(0,\mathrm{id}_{\mathbb{C}},0))"'] & \mathfrak{f}_{1,4} \arrow[d, "KK_{\mathcal{E}}(\mathfrak{mc}^3(0,\mathrm{id}_{\mathbb{C}},0))=KK_{\mathcal{E}}(\mathfrak{mc}^2(\mathfrak{q}_{\mathfrak{f}_{1,2}}))"] \\
\mathsf{S}\mathfrak{f}_{1,0} \arrow[r, "\mathbf{x}_{\mathfrak{f}_{1,0}}"'] & \mathfrak{f}_{1,3}
\end{tikzcd}$$

in $\mathbf{KK}_{\mathcal{E}}$. Using that $KK_{\mathcal{E}}(\mathfrak{e}_{n,2}, \mathsf{S}\mathfrak{f}_{1,0})$ is naturally isomorphic to $\mathrm{Hom}_{\mathrm{six}}(K_{\mathrm{six}}(\mathfrak{e}_{n,2}), K_{\mathrm{six}}(\mathsf{S}\mathfrak{f}_{1,0}))$ (since $\mathrm{Ext}^1_{\mathrm{six}}(K_{\mathrm{six}}(\mathfrak{e}_{n,2}), K_{\mathrm{six}}(\mathsf{SS}\mathfrak{f}_{1,0})) = 0$), we can show that the square

$$\begin{tikzcd}
\mathfrak{e}_{n,2} \arrow[r, "KK_{\mathcal{E}}(\mathfrak{mc}^2(\mathfrak{q}_{\mathfrak{e}_{n,0}}))"] \arrow[d, "KK_{\mathcal{E}}((*,\mathrm{id},0))"'] & \mathfrak{mc}^2(\mathfrak{q}(\mathfrak{e}_{n,0})) \arrow[d, "\mathbf{x}^{-1}_{\mathfrak{f}_{1,0}}"] \\
{[\mathbb{I}_{n,2} \hookrightarrow \mathbb{I}_{n,2} \twoheadrightarrow 0]} \arrow[r] & \mathsf{S}\mathfrak{f}_{1,0}
\end{tikzcd}$$

anti-commutes in $\mathbf{KK}_{\mathcal{E}}$, where the bottom horizontal map is the canonical identification. Using all this, we can show that we have a commuting diagram

$$\begin{tikzcd}
F_{1,3} \arrow[r, "\times n"] \arrow[d, "\tilde{f}_{1,3}"'] & F_{1,3} \arrow[d, "h^{1,1,in}_{n,2}"] \\
F_{1,4} \arrow[r, "h^{1,n,in}_{n,2}"'] & H_{n,2}.
\end{tikzcd}$$

If we first apply $\mathfrak{mc}^k$ to the diagrams, for $k = 1, 2, 3, 4, 5$, we obtain commutativity of the corresponding square of Diagram (6.2), for $i = 3, 4, 5, 0, 1$, resp.

Diagram (6.3). First we prove it for $i = 2$. We have a commuting diagram of objects from $S\mathcal{E}$

$$\begin{tikzcd}
0 \arrow[r, hook] \arrow[d, hook] & \mathfrak{mc}(\mathfrak{i}(\mathfrak{e}_{n,1})) \arrow[r, equal] \arrow[d, hook, "\mathfrak{mc}(\mathfrak{i}_{\mathfrak{e}_{n,1}})"] & \mathfrak{mc}(\mathfrak{i}(\mathfrak{e}_{n,1})) \arrow[d, hook] \\
\mathfrak{i}(\mathfrak{e}_{n,2}) \arrow[r, hook, "\mathfrak{i}_{\mathfrak{e}_{n,2}}"] \arrow[d, equal] & \mathfrak{e}_{n,2} \arrow[r, two heads, "\mathfrak{q}_{\mathfrak{e}_{n,2}}"] \arrow[d, two heads, "\mathfrak{mc}(\mathfrak{q}_{\mathfrak{e}_{n,1}})"] & \mathfrak{q}(\mathfrak{e}_{n,2}) \arrow[d, two heads] \\
\mathfrak{i}(\mathfrak{e}_{n,2}) \arrow[r, hook] & \mathfrak{mc}(\mathfrak{q}(\mathfrak{e}_{n,1})) \arrow[r, two heads] & \mathfrak{f}_{n,1}
\end{tikzcd}$$

with short exact rows and columns. The short exact sequence $\mathfrak{i}(\mathfrak{e}_{n,2}) \hookrightarrow \mathfrak{mc}(\mathfrak{q}(\mathfrak{e}_{n,1})) \twoheadrightarrow \mathfrak{f}_{n,1}$ is exactly the short exact sequence $\mathfrak{i}(\mathfrak{f}_{n,2}) \xhookrightarrow{\mathfrak{i}_{\mathfrak{f}_{n,2}}} \mathfrak{f}_{n,2} \xrightarrow{\mathfrak{q}_{\mathfrak{f}_{n,2}}} \mathfrak{q}(\mathfrak{f}_{n,2})$. Moreover, the short exact sequence $\mathfrak{mc}(\mathfrak{i}(\mathfrak{e}_{n,1})) \hookrightarrow \mathfrak{q}(\mathfrak{e}_{n,2}) \twoheadrightarrow \mathfrak{f}_{n,1}$ is exactly the short exact sequence $\mathsf{S}\mathfrak{f}_{1,1} \hookrightarrow e \twoheadrightarrow \mathfrak{f}_{n,1}$ induced by the extension $\mathfrak{e}_{n,1} \colon \mathsf{S}\mathbb{C} \hookrightarrow \mathbb{I}_{n,1} \twoheadrightarrow \mathbb{I}_{n,0}$, where e is $0 \hookrightarrow \mathbb{I}_{n,1} \twoheadrightarrow \mathbb{I}_{n,1}$.

A computation shows that this gives rise to a commutative diagram (by applying $KK_{\mathcal{E}}(-, e)$)

$$\begin{tikzcd}
F_{n,1} \arrow[r, "\tilde{f}_{n,1}"] \arrow[d, "-\beta_{n,1}"'] & F_{n,2} \arrow[d, "h^{n,1,in}_{n,2}"'] \arrow[dr, "\tilde{f}_{n,2}"] & \\
F_{1,4} \arrow[r, "h^{1,n,in}_{n,2}"] \arrow[dr, "\times n"'] & H_{n,2} \arrow[r, "h^{1,n,out}_{n,2}"'] \arrow[d, "h^{n,1,out}_{n,2}"] & F_{n,3} \\
& F_{1,4} &
\end{tikzcd}$$

If we apply $\mathfrak{mc}^k$ to the diagram, for $k = 1, 2, 3, 4, 5$, we obtain commutativity of the corresponding part of Diagram (6.3), for $i = 3, 4, 5, 0, 1$, resp. — this is, indeed, a *very long* and tedious verification using the identifications and results above.

Now we prove commutativity of the remaining square in Diagram (6.3) for $i = 2$. We have a commuting diagram

$$\begin{tikzcd} \mathsf{S}\mathfrak{f}_{n,1} \arrow[r] \arrow[d] & e \arrow[d] \\ \mathfrak{i}(\mathfrak{e}_{n,2}) \arrow[r, "\mathfrak{i}_{\mathfrak{e}_{n,2}}"] & \mathfrak{e}_{n,2}, \end{tikzcd}$$

where e is the extension $0 \hookrightarrow \mathbb{I}_{n,2} \xrightarrow{\mathrm{id}} \mathbb{I}_{n,2}$, the map from $\mathsf{S}\mathfrak{f}_{n,1}$ to e is the one induced by the map $\mathsf{S}\mathbb{I}_{n,0} \to \mathbb{I}_{n,2}$ in $\mathfrak{e}_{n,2}$, the map from $\mathsf{S}\mathfrak{f}_{n,1}$ to $\mathsf{S}\mathfrak{f}_{n,0} = \mathfrak{i}(\mathfrak{e}_{n,2})$ is the unique morphism which is the identity on the extension algebra, and the morphism from e to $\mathfrak{e}_{n,2}$ is the unique morphism which is the identity on the extension algebra. It is elementary to see that if we compose the morphism from e to $\mathfrak{e}_{n,2}$ with the canonical identification of e with $\mathsf{S}\mathfrak{f}_{1,1}$, we get exactly the morphism $\mathfrak{mc}(\mathfrak{i}_{\mathfrak{e}_{n,1}})$. If ϕ denotes the obvious morphism from $\mathfrak{f}_{n,1}$ to $\mathfrak{f}_{n,0}$, it is elementary to show that $\mathfrak{mc}^3(\phi)$ is $\mathfrak{mc}^2(\mathfrak{q}_{\mathfrak{f}_{n,2}})$. Using all this, we see that this gives rise to a commuting square

$$\begin{tikzcd} H_{n,2} \arrow[r, "h^{1,n,out}_{n,2}"'] \arrow[d, "h^{n,1,out}_{n,2}"] & F_{n,3} \arrow[d, "\tilde{f}_{n,3}"] \\ F_{1,4} \arrow[r, "\rho_{n,4}"'] & F_{n,4}. \end{tikzcd}$$

If we first apply $\mathfrak{mc}^k$ to the diagrams, for $k = 1, 2, 3, 4, 5$, we obtain commutativity of the corresponding square of Diagram (6.3), for $i = 3, 4, 5, 0, 1$, resp. ●●●

Appendix A

On Rørdam's classification of certain C^*-algebras with one non-trivial ideal

This article is a follow up on a talk given by the first named author of the article, Søren Eilers, at the first Abel Symposium, which was held in Oslo in 2004. It has been published in the proceedings of this symposium, *cf.* [ER06]. The article is followed up by the article in Appendix B.

For copyright reasons, this paper has been left out

Appendix B

On Rørdam's classification of certain C^*-algebras with one non-trivial ideal, II

This article is a follow up on the article from Appendix A, and has been published in Mathematica Scandinavica ([RR07]). Partly it generalizes some of the arguments of the that article, and it solves in a very satisfactory way the classification problem of essential extensions of Kirchberg algebras, which was initiated by Rørdam in his seminal paper [Rør97].

For copyright reasons, this paper has been left out

Appendix C

Classification of extensions of classifiable C^*-algebras

The project of this paper was initiated by a question from the first named author, Søren Eilers, and Toke Meier Carlsen concerning the completeness of the Matsumoto algebras as an invariant of flow equivalence of shift spaces. It answered the question in the negative, and, moreover, it turned out that the methods could be formulated in much more generality. We include an application of our results to graph algebras (with the help of Mark Tomforde). The included paper in this appendix, is an unpublished preprint. Most likely, it will be reorganized before submission (to make it shorter and more concise).

CLASSIFICATION OF EXTENSIONS OF CLASSIFIABLE C^*-ALGEBRAS

SØREN EILERS, GUNNAR RESTORFF, AND EFREN RUIZ

Abstract. For a certain class of extensions $\mathfrak{e} : 0 \to B \to E \to A \to 0$ of C^*-algebras in which B and A belong to a classifiable class of C^*-algebras, we show that the functor which sends $\mathfrak{e}$ to its associated six term exact sequence in K-theory and the positive cones of $K_0(B)$ and $K_0(A)$ is a classification functor. We give two independent applications addressing the classification of a class of C^*-algebras arising from substitutional shift spaces on one hand and of graph algebras on the other. The former application leads to the answer of a question of Carlsen and the first named author concerning the completeness of stabilized Matsumoto algebras as an invariant of flow equivalence.

Introduction

Associated to every extension $0 \to B \to E \to A \to 0$ of nonzero C^*-algebras is a six term exact sequence of K-groups

$$\begin{tikzcd} K_0(B) \arrow[r] & K_0(E) \arrow[r] & K_0(A) \arrow[d] \\ K_1(A) \arrow[u] & K_1(E) \arrow[l] & K_1(B) \arrow[l] \end{tikzcd}$$

This six term exact sequence of K-groups provides a necessary condition for two extensions to be isomorphic, which leads one to wonder if the above exact sequence is sufficient to distinguish certain extensions of C^*-algebras.

For examples of classification results involving the six term exact sequence of K-groups see [29], [24], [23], and [40]. In each case, the extensions considered were extensions that can be expressed as inductive limit of simpler extensions. The classification results were achieved by using the standard intertwining argument.

In [37], Rørdam used a completely different technique to classify a certain class of extensions. He considered essential extensions of separable nuclear purely infinite simple C^*-algebras in $\mathcal{N}$, where $\mathcal{N}$ is the bootstrap category of

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Rosenberg and Schochet [39]. Using the fact that every invertible element of $KK(A,B)$ (where A and B are separable nuclear stable purely infinite simple C^*-algebras) lifts to a $*$-isomorphism from A to B and that every essential extension of A by B is absorbing, Rørdam showed the following:

Suppose A_1, A_2, B_1, and B_2 are separable nuclear stable purely infinite simple C^*-algebras in $\mathcal{N}$. Two essential extensions

$$\begin{aligned} \mathfrak{e}_1 : \quad & 0 \to B_1 \to E_1 \to A_1 \to 0 \\ \mathfrak{e}_2 : \quad & 0 \to B_2 \to E_2 \to A_2 \to 0 \end{aligned}$$

are isomorphic if and only if the six term exact sequences of K-groups of $\mathfrak{e}_1$ and $\mathfrak{e}_2$ are isomorphic. Moreover, E_1 is isomorphic to E_2 if and only if $\mathfrak{e}_1$ is isomorphic to $\mathfrak{e}_2$ since A_1, A_2, B_1, and B_2 are simple C^*-algebras.

The purpose of this paper is to extend the above result to other classes of C^*-algebras that are classified via K-theoretical invariants. We will show that certain classes of essential extensions of classifiable C^*-algebras are classified by their six term exact sequence of K-groups together with the positive cone of the K_0-groups of the distinguished ideal and quotient. This class of extensions includes essential extensions of $A \otimes \mathcal{K}$ by $B \otimes \mathcal{K}$ which satisfy a certain fullness condition, where A and B can be unital separable nuclear purely infinite simple C^*-algebras satisfying the Universal Coefficient Theorem or unital simple AT-algebras with real rank zero.

The motivation of our work was an application to a class of C^*-algebras introduced in the work of Matsumoto. Carlsen has in recent work (see [4]) showed for each minimal shift space $\underline{\mathsf{X}}$ with a certain technical property $(**)$ introduced in [8] that the Matsumoto algebra $\mathcal{O}_{\underline{\mathsf{X}}}$ fits in a short exact sequence of the form

$$0 \longrightarrow \mathcal{K}^n \longrightarrow \mathcal{O}_{\underline{\mathsf{X}}} \longrightarrow C(\underline{\mathsf{X}}) \rtimes_\sigma \mathbb{Z} \longrightarrow 0$$

where n is an integer determined by the structure of the so-called *special words* of $\underline{\mathsf{X}}$. It turns out that $C(\underline{\mathsf{X}}) \rtimes_\sigma \mathbb{Z}$ is a unital simple AT-algebra with real rank zero. Using our results in Section 3, we show that two such C^*-algebras are stably isomorphic if and only if their six term exact sequences of K-groups are isomorphic and the isomorphism between the K_0-groups of the distinguished ideals and the isomorphism between the K_0-groups of the distinguished quotients are order isomorphisms.

The paper is organized as follows. In Section 1, we give basic properties and develop some notation concerning extensions of C^*-algebras. Section 2 gives notation concerning the six term exact sequence of K-groups. Most of the notations were introduced in [37]. Section 3 contains our main results (Theorem 3.11 and Theorem 3.16). In the last section we use these results to classify the C^*-algebras described in the previous paragraph. We also present an alternative application to *graph algebras*.

1. Extensions

We first develop some notation concerning extensions that will appear in the sequel. We also give some basic facts about extensions all of which are taken from [37].

For a stable C^*-algebra B and a C^*-algebra A, we will denote the class of essential extensions

$$0 \to B \xrightarrow{\varphi} E \xrightarrow{\psi} A \to 0$$

by $\mathcal{E}\mathrm{xt}(A, B)$.

Since the goal of this paper is to classify extensions of separable nuclear C^*-algebras, throughout the rest of the paper we will only consider C^*-algebras that are separable and nuclear.

Assumption 1.1. *In the rest of the paper (unless stated otherwise) all C^*-algebras considered are assumed to be separable and nuclear.*

Under the above assumption, if B is a stable C^*-algebra, then we may identify $\mathrm{Ext}(A, B)$ with $KK^1(A, B)$ (for the definition of $\mathrm{Ext}(A, B)$ and $KK^i(A, B)$ see Chapter 7 and Chapter 8 in [1]). Using this identification, for x in $\mathrm{Ext}(A, B)$ and y in $KK^i(B, C)$ it makes sense to consider the Kasparov product of x and y, which we denote by $x \times y$. Note that $x \times y$ is an element of $KK^{i+1}(A, C)$.

Definition 1.2. Suppose $\mathfrak{e}_1(\varphi_1, E_1, \psi_1)$ is in $\mathcal{E}\mathrm{xt}(A_1, B_1)$ and $\mathfrak{e}_2(\varphi_2, E_2, \psi_2)$ is in $\mathcal{E}\mathrm{xt}(A_2, B_2)$. A homomorphism from $\mathfrak{e}_1(\varphi_1, E_1, \psi_1)$ to $\mathfrak{e}_2(\varphi_2, E_2, \psi_2)$ is a triple (β, η, α) where α from A_1 to A_2, η from E_1 to E_2, and β from B_1 to B_2 are $*$-homomorphisms which make the diagram

$$\begin{tikzcd}
0 \arrow[r] & B_1 \arrow[r, "\varphi_1"] \arrow[d, "\beta"'] & E_1 \arrow[r, "\psi_1"] \arrow[d, "\eta"'] & A_1 \arrow[r] \arrow[d, "\alpha"'] & 0 \\
0 \arrow[r] & B_2 \arrow[r, "\varphi_2"'] & E_2 \arrow[r, "\psi_2"'] & A_2 \arrow[r] & 0
\end{tikzcd}$$

commutative. We define the composition of two homomorphisms and the notion of isomorphisms between two extensions in the obvious way. If $\mathfrak{e}_1(\varphi_1, E_1, \psi_1)$ and $\mathfrak{e}_2(\varphi_2, E_2, \psi_2)$ in $\mathcal{E}\mathrm{xt}(A, B)$ are isomorphic via $(\mathrm{id}_B, \eta, \mathrm{id}_A)$ for some $*$-isomorphism η from E_1 to E_2, then we say that the extensions are *congruent.* For notational convenience we will sometimes refer to $\mathfrak{e}(\varphi, E, \psi)$ in $\mathcal{E}\mathrm{xt}(A, B)$ by just $\mathfrak{e}$.

Denote the multiplier algebra of B by $\mathcal{M}(B)$ and denote the corona algebra $\mathcal{M}(B)/B$ of B by $\mathcal{Q}(B)$. For every element $\mathfrak{e}(\varphi, E, \psi)$ of $\mathcal{E}\mathrm{xt}(A, B)$, there exist unique injective $*$-homomorphisms $\eta_{\mathfrak{e}}$ from E to $\mathcal{M}(B)$ and $\tau_{\mathfrak{e}}$ from A to $\mathcal{Q}(B)$

which make the diagram

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r, "\varphi"] \arrow[d, equal] & E \arrow[r, "\psi"] \arrow[d, "\eta_{\mathfrak{e}}"'] & A \arrow[r] \arrow[d, "\tau_{\mathfrak{e}}"'] & 0 \\
0 \arrow[r] & B \arrow[r, hook] & \mathcal{M}(B) \arrow[r, "\pi"'] & \mathcal{Q}(B) \arrow[r] & 0
\end{tikzcd}$$

commutative, where π from $\mathcal{M}(B)$ to $\mathcal{Q}(B)$ is the canonical projection. The $*$-homomorphism $\tau_{\mathfrak{e}}$ is called the *Busby invariant* of $\mathfrak{e}$. Note that there exists a unique sub-C^*-algebra E_1 of $\mathcal{M}(B)$ such that B is an ideal of E_1 and $\mathfrak{e}(\varphi, E, \psi)$ is isomorphic to $\mathfrak{e}_1(\iota, E_1, \pi)$ via the isomorphism $(\mathrm{id}_B, \eta_{\mathfrak{e}}, \tau_{\mathfrak{e}})$, where ι is the canonical embedding from B to $\mathcal{M}(B)$.

Note that each element of $\mathcal{E}\mathrm{xt}(A, B)$ represents an element of $\mathrm{Ext}(A, B)$. For every element $\mathfrak{e}$ of $\mathcal{E}\mathrm{xt}(A, B)$, we use $x_{A,B}(\mathfrak{e})$ to denote the element of $\mathrm{Ext}(A, B)$ that is represented by $\mathfrak{e}$.

For each injective $*$-homomorphism α from A_1 to A_2 and for each $\mathfrak{e}$ in $\mathcal{E}\mathrm{xt}(A_2, B)$, there exists a unique extension $\alpha \cdot \mathfrak{e}$ in $\mathcal{E}\mathrm{xt}(A_1, B)$ such that the diagram

$$\begin{tikzcd}
\alpha \cdot \mathfrak{e} : & 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & \widetilde{E} \arrow[r] \arrow[d, hook] & A_1 \arrow[r] \arrow[d, "\alpha"'] & 0 \\
\mathfrak{e} : & 0 \arrow[r] & B \arrow[r] & E \arrow[r] & A_2 \arrow[r] & 0
\end{tikzcd}$$

is commutative. For each $*$-isomorphism β from B_1 to B_2 and for each $\mathfrak{e}$ in $\mathcal{E}\mathrm{xt}(A, B_1)$, there exists a unique extension $\mathfrak{e} \cdot \beta$ in $\mathcal{E}\mathrm{xt}(A, B_2)$ such that the diagram

$$\begin{tikzcd}
\mathfrak{e} : & 0 \arrow[r] & B_1 \arrow[r] \arrow[d, "\beta"'] & E \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[d, equal] & 0 \\
\mathfrak{e} \cdot \beta : & 0 \arrow[r] & B_2 \arrow[r] & E \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

is commutative.

The following propositions are Proposition 1.1 and Proposition 1.2 in [37].

Proposition 1.3. *Suppose α from A_1 to A_2 is an injective $*$-homomorphism and suppose β from B_1 to B_2 is a $*$-isomorphism. If $\mathfrak{e}_1$ is in $\mathcal{E}\mathrm{xt}(A, B_1)$ and $\mathfrak{e}_2$ is in $\mathcal{E}\mathrm{xt}(A_2, B)$, then*

$$x_{A,B_2}(\mathfrak{e}_1 \cdot \beta) = x_{A,B_1}(\mathfrak{e}_1) \times KK(\beta)$$
$$x_{A_1,B}(\alpha \cdot \mathfrak{e}_2) = KK(\alpha) \times x_{A_2,B}(\mathfrak{e}_2).$$

Proposition 1.4. *Let $\mathfrak{e}_j(\varphi_j, E_j, \psi_j)$ be an element of $\mathcal{E}\mathrm{xt}(A_j, B_j)$ for $j = 1, 2$.*

(1) *If α from A_1 to A_2 and β from B_1 to B_2 are $*$-isomorphisms, then $\mathfrak{e}_1$ is isomorphic to $\mathfrak{e}_1 \cdot \beta$ and $\mathfrak{e}_2$ is isomorphic to $\alpha \cdot \mathfrak{e}_2$.*

(2) *$\mathfrak{e}_1$ is isomorphic to $\mathfrak{e}_2$ if and only if $\mathfrak{e}_1 \cdot \beta$ is congruent to $\alpha \cdot \mathfrak{e}_2$ for some $*$-isomorphisms α from A_1 to A_2 and β from B_1 to B_2.*

(3) *If $\mathfrak{e}_1$ is isomorphic to $\mathfrak{e}_2$, then E_1 is isomorphic to E_2, and if each A_j and B_j are simple, then E_1 is isomorphic to E_2 implies that $\mathfrak{e}_1$ is isomorphic to $\mathfrak{e}_2$.*

2. Six term exact sequence in K-theory

This section contains basic facts and notation concerning extensions, the corresponding six term exact sequence of K-groups, and their mutual interaction. Most of the notation was introduced by Rørdam in [37].

2.1. We will start by introducing several graded groups.

Definition 2.1. Suppose A and B are C^*-algebras and suppose G_0, G_1, H_0, and H_1 are abelian groups. Then

(1) The graded group $K_0(A) \oplus K_1(A)$ will be denoted by $K_*(A)$.
(2) The graded group $\mathrm{Ext}^1_{\mathbb{Z}}(G_0, H_0) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(G_1, H_1)$ will be denoted by $\mathrm{Ext}^1_{\mathbb{Z}}(G_*, H_*)$.
(3) The graded group $\mathrm{Ext}^1_{\mathbb{Z}}(G_0, H_1) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(G_1, H_0)$ will be denoted by $\mathrm{Ext}^1_{\mathbb{Z}}(G_*, H_{*+1})$.
(4) The graded group $\mathrm{Hom}(G_0, H_0) \oplus \mathrm{Hom}(G_1, H_1)$ will be denoted by $\mathrm{Hom}(G_*, H_*)$.
(5) The graded group $\mathrm{Hom}(G_0, H_1) \oplus \mathrm{Hom}(G_1, H_0)$ will be denoted by $\mathrm{Hom}(G_*, H_{*+1})$.

In all cases, by a homomorphism between two graded groups we mean two group homomorphisms respecting the grading. For example, a homomorphism α_* from $K_*(A)$ to $K_*(B)$ consists of two group homomorphisms α_0 from $K_0(A)$ to $K_0(B)$ and α_1 from $K_1(A)$ to $K_1(B)$.

We say that A *satisfies the Universal Coefficient Theorem* if for every C^*-algebra B with a countable approximate identity, the sequence

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(K_*(A), K_{*+1}(B)) \xrightarrow{\epsilon} KK^0(A,B) \xrightarrow{K_*} \mathrm{Hom}(K_*(A), K_*(B)) \longrightarrow 0$$

is exact. Hence, if A satisfies the Universal Coefficient Theorem, then for every C^*-algebra B with a countable approximate identity, the sequence

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(B)) \xrightarrow{\epsilon} KK^1(A,B) \xrightarrow{K_*} \mathrm{Hom}(K_*(A), K_{*+1}(B)) \longrightarrow 0$$

is exact. Rosenberg and Schochet in [39] showed that every separable nuclear C^*-algebra in $\mathcal{N}$ satisfies the Universal Coefficient Theorem.

2.2. Suppose $\mathfrak{e}(\varphi, E, \psi)$ is an element of $\mathcal{E}\mathrm{xt}(A, B)$. Associated to $\mathfrak{e}(\varphi, E, \psi)$ is the following six term exact sequence of K-groups

$$\begin{tikzcd} K_0(B) \arrow[r, "K_0(\varphi)"] & K_0(E) \arrow[r, "K_0(\psi)"] & K_0(A) \arrow[d, "\delta_0^E"] \\ K_1(A) \arrow[u, "\delta_1^E"] & K_1(E) \arrow[l, "K_1(\psi)"] & K_1(B) \arrow[l, "K_1(\varphi)"] \end{tikzcd}$$

The homomorphism δ_0^E is called the *exponential map* and δ_1^E is called the *index map.* For every $\mathfrak{e} = \mathfrak{e}(\phi, E, \psi)$ in $\mathcal{E}\mathrm{xt}(A, B)$, denote the six term exact sequence associated to $\mathfrak{e}$ by $K_{\mathrm{six}}(\mathfrak{e})$.

Let $\mathcal{H}\mathrm{ext}(A, B)$ denote the class of all six term exact sequences of K-groups arising from extensions in $\mathcal{E}\mathrm{xt}(A, B)$. A homomorphism from an element of $\mathcal{H}\mathrm{ext}(A_1, B_1)$ to an element of $\mathcal{H}\mathrm{ext}(A_2, B_2)$ is a triple $(\beta_*, \eta_*, \alpha_*)$, where β_* from $K_*(B_1)$ to $K_*(B_2)$, η_* from $K_*(E_1)$ to $K_*(E_2)$, and α_* from $K_*(A_1)$ to $K_*(A_2)$ are homomorphisms making the obvious diagrams commute. Isomorphisms are defined in the obvious way.

Suppose h_1 and h_2 are elements of $\mathcal{H}\mathrm{ext}(A, B)$. We say that h_1 and h_2 are *congruent* if h_1 is isomorphic to h_2 via an isomorphism $(\mathrm{id}_{K_*(B)}, \eta_*, \mathrm{id}_{K_*(A)})$ and we write $h_1 \equiv h_2$. Let $\mathrm{Hext}(A, B)$ be the set of all congruence classes of $\mathcal{H}\mathrm{ext}(A, B)$. For every element h of $\mathcal{H}\mathrm{ext}(A, B)$, we use $\mathbf{x}_{A,B}(h)$ to denote the element of $\mathrm{Hext}(A, B)$ that is represented by h.

The following proposition is Proposition 2.1 in [37].

Proposition 2.2. *For every pair of C^*-algebras A and B with B stable, there is a unique map $\mathbf{K}_{\mathrm{six}}$ from $\mathrm{Ext}(A, B)$ to $\mathrm{Hext}(A, B)$ such that the following statements hold:*

(1) *If $\mathfrak{e}_1$ and $\mathfrak{e}_2$ are elements of $\mathcal{E}\mathrm{xt}(A, B)$ that represent the same element of $\mathrm{Ext}(A, B)$, then $K_{\mathrm{six}}(e_1)$ is congruent to $K_{\mathrm{six}}(e_2)$.*

(2) *For every element $\mathfrak{e}$ of $\mathcal{E}\mathrm{xt}(A, B)$, we have that*

$$\mathbf{x}_{A,B}(K_{\mathrm{six}}(\mathfrak{e})) = \mathbf{K}_{\mathrm{six}}(x_{A,B}(\mathfrak{e})).$$

Suppose z is in $\mathrm{Hext}(A, B)$. If h_1 and h_2 are elements of $\mathcal{H}\mathrm{ext}(A, B)$ that represent z, then the exponential map of h_1 is equal to the exponential map of h_2 and the index map of h_1 is equal to the index map of h_2. Hence, for each z in $\mathrm{Hext}(A, B)$ it makes sense to say that the *exponential map* of z is the exponential map of h in $\mathcal{H}\mathrm{ext}(A, B)$ for any representative h of z. We denote this map by $K_0(z)$. Similarly, the *index map* of z will be the index map of any representative h in $\mathcal{H}\mathrm{ext}(A, B)$ of z and we denote this map by $K_1(z)$.

Let δ_* be an element of $\mathrm{Hom}(K_*(A), K_{*+1}(B))$. We will denote the set of all z in $\mathrm{Hext}(A, B)$ that satisfy $K_j(z) = \delta_j$ for $j = 0, 1$ by $\mathrm{Hext}(A, B; \delta_*)$. Define a map

$$(\sigma_{\delta_*} =)\sigma_{A,B,\delta_*} : \mathrm{Hext}(A, B; \delta_*) \to \mathrm{Ext}^1_{\mathbb{Z}}(\ker(\delta_*), \mathrm{coker}(\delta_{*+1}))$$

as follows. Suppose z is represented by the following six term exact sequence:

$$\begin{tikzcd} K_0(B) \arrow[r, "K_0(\varphi)"] & K_0(E) \arrow[r, "K_0(\psi)"] & K_0(A) \arrow[d, "\delta_0"] \\ K_1(A) \arrow[u, "\delta_1"] & K_1(E) \arrow[l, "K_1(\psi)"] & K_1(B) \arrow[l, "K_1(\varphi)"] \end{tikzcd}$$

Then $\sigma_{\delta_*}(z)$ in $\mathrm{Ext}^1_{\mathbb{Z}}(\ker(\delta_*), \mathrm{coker}(\delta_{*+1}))$ is the element represented by the two short exact sequences in the top and bottom rows in the diagram below.

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{coker}(\delta_1) \arrow[r] & K_0(E) \arrow[r] & \ker(\delta_0) \arrow[r] \arrow[d, hook] & 0 \\ & K_0(B) \arrow[r] \arrow[u, "\pi"] & K_0(E) \arrow[r] \arrow[u, equal] & K_0(A) \arrow[d, "\delta_0"] & \\ & K_1(A) \arrow[u, "\delta_1"] & K_1(E) \arrow[l] & K_1(B) \arrow[l] \arrow[d, "\pi"] & \\ 0 & \ker\delta_1 \arrow[l] \arrow[u, hook] & K_1(E) \arrow[l] \arrow[u, equal] & \mathrm{coker}(\delta_0) \arrow[l] & 0 \arrow[l] \end{tikzcd}$$

It is straightforward to check that σ_{δ_*} is a well-defined map.

For each δ_* in $\mathrm{Hom}(K_*(A), K_{*+1}(B))$, we denote the subgroup of $\mathrm{Ext}(A, B)$ consisting of all x in $\mathrm{Ext}(A, B)$ satisfying

(1) $\ker(\delta_j) \subset \ker(K_j(x))$ for $j = 0, 1$ and
(2) $\mathrm{image}(K_j(x)) \subset \mathrm{image}(\delta_j)$ for $j = 0, 1$

by $\mathrm{Ext}_{\delta_*}(A, B)$. We now define

$$(s_{\delta_*} =)\ s_{A,B,\delta_*} : \mathrm{Ext}_{\delta_*}(A, B) \to \mathrm{Ext}^1_{\mathbb{Z}}(\ker(\delta_*), \mathrm{coker}(\delta_{*+1}))$$

to be the map given by

$$\begin{aligned} x \in \mathrm{Ext}_{\delta_*}(A, B) &\mapsto \mathbf{K}_{\mathrm{six}}(x) \in \mathrm{Hext}(A, B; K_*(x)) \\ &\mapsto \sigma_{K_*(x)}(\mathbf{K}_{\mathrm{six}}(x)) \in \mathrm{Ext}^1_{\mathbb{Z}}(\ker(K_*(x)), \mathrm{coker}(K_{*+1}(x))) \\ &\mapsto s_{\delta_*}(x) \in \mathrm{Ext}^1_{\mathbb{Z}}(\ker(\delta_*), \mathrm{coker}(\delta_{*+1})), \end{aligned}$$

where the last map is induced by the maps

$$\ker(\delta_*) \hookrightarrow \ker(K_*(x)) \text{ and } \mathrm{coker}(K_*(x)) \twoheadrightarrow \mathrm{coker}(\delta_*).$$

Note that the maps $\ker(\delta_*) \hookrightarrow K_*(A)$ and $K_*(B) \twoheadrightarrow \mathrm{coker}(\delta_*)$ induce a surjective homomorphism from $\mathrm{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(B))$ to $\mathrm{Ext}^1_{\mathbb{Z}}(\ker(\delta_*), \mathrm{coker}(\delta_{*+1}))$. We will denote this homomorphism by ζ_{δ_*}.

Lemma 2.3. *Let A, B, and C be separable nuclear C^*-algebras with B stable and let δ_* be an element of $\operatorname{Hom}(K_*(C), K_{*+1}(B))$. Suppose C is in $\mathcal{N}$ and suppose x in $KK(A, C)$ is a KK-equivalence.*

Set $\lambda_* = \delta_* \circ K_*(x)$ in $\operatorname{Hom}(K_*(A), K_{*+1}(B))$. Then

(1) *$x \times (\cdot)$ is an isomorphism from $\operatorname{Ext}_{\delta_*}(C, B)$ onto $\operatorname{Ext}_{\lambda_*}(A, B)$.*
(2) *x induces an isomorphism $[K_*(x)]$ from $\operatorname{Ext}^1_{\mathbb{Z}}(\ker(\delta_*), \operatorname{coker}(\delta_{*+1}))$ onto $\operatorname{Ext}^1_{\mathbb{Z}}(\ker(\lambda_*), \operatorname{coker}(\lambda_{*+1}))$.*
(3) *Moreover, if A and B are in $\mathcal{N}$ and if $x = KK(\alpha)$ for some injective $*$-homomorphism α from A to C, then the diagram*

$$\begin{tikzcd}
\operatorname{Ext}_{\delta_*}(C,B) \arrow[r, "x\times(\cdot)", "\cong"'] \arrow[d, "s_{\delta_*}"'] & \operatorname{Ext}_{\lambda_*}(A,B) \arrow[d, "s_{\lambda_*}"] \\
\operatorname{Ext}^1_{\mathbb{Z}}(\ker(\delta_*), \operatorname{coker}(\delta_{*+1})) \arrow[r, "\cong", "{[K_*(x)]}"'] & \operatorname{Ext}^1_{\mathbb{Z}}(\ker(\lambda_*), \operatorname{coker}(\lambda_{*+1}))
\end{tikzcd}$$

is commutative.

Proof. Since x is a KK-equivalence, $x \times (\cdot)$ is an isomorphism from $\operatorname{Ext}(C, B)$ onto $\operatorname{Ext}(A, B)$. Therefore, to prove (1) it is enough to show that $x \times (\cdot)$ maps $\operatorname{Ext}_{\delta_*}(C, B)$ to $\operatorname{Ext}_{\lambda_*}(A, B)$ and $x^{-1} \times (\cdot)$ maps $\operatorname{Ext}_{\lambda_*}(A, B)$ to $\operatorname{Ext}_{\delta_*}(C, B)$.

Note that $K_*(x)$ is an isomorphism and $K_j(x \times z) = K_j(z) \circ K_j(x)$ for $j = 0, 1$ and z in $\operatorname{Ext}(C, B)$. Hence, $\operatorname{image}(K_j(z) \circ K_j(x)) = \operatorname{image}(K_j(z))$ and $\operatorname{image}(\delta_j) = \operatorname{image}(\delta_j \circ K_j(x))$. By definition, if z is in $\operatorname{Ext}_{\delta_*}(C, B)$, then $\operatorname{image}(K_j(z)) \subset \operatorname{image}(\delta_j)$ for $j = 0, 1$. Therefore, for $j = 0, 1$,

$$\operatorname{image}(K_j(z) \circ K_j(x)) \subset \operatorname{image}(\delta_j \circ K_j(x)) = \operatorname{image}(\lambda_j).$$

A straightforward computation shows that $\ker(\lambda_j) \subset \ker(K_j(z) \circ K_j(x))$. Hence, $x \times z$ is an element of $\operatorname{Ext}_{\lambda_*}(A, B)$ for all z in $\operatorname{Ext}_{\delta_*}(C, B)$. A similar computation shows that $x^{-1} \times (\cdot)$ maps $\operatorname{Ext}_{\lambda_*}(A, B)$ to $\operatorname{Ext}_{\delta_*}(C, B)$. We have just proved (1).

Since $K_*(x)$ is an isomorphism, $\operatorname{image}(\lambda_*) = \operatorname{image}(\delta_*)$. It is straightforward to show that $K_j(x)$ is an isomorphism from $\ker(\lambda_j)$ onto $\ker(\delta_j)$ for $j = 0, 1$. Therefore, $[K_*(x)]$ is the isomorphism induced by $K_*(x)$. This proves (2).

We now prove (3). Let z be in $\operatorname{Ext}_{\lambda_*}(C, B)$. Let $\mathfrak{e}$ be an element of $\mathcal{E}\mathrm{xt}(C, B)$ such that $x_{C,B}(\mathfrak{e}) = z$. Since $x = KK(\alpha)$ for some injective $*$-homomorphism α from A to C, there exist $\alpha \cdot \mathfrak{e}$ in $\mathcal{E}\mathrm{xt}(A, B)$ and a homomorphism $(\mathrm{id}_B, \eta, \alpha)$ from $\alpha \cdot \mathfrak{e}$ to $\mathfrak{e}$. By Proposition 1.3

$$x_{A,B}(\alpha \cdot \mathfrak{e}) = KK(\alpha) \times x_{C,B}(\mathfrak{e}) = x \times z.$$

By the Five Lemma, $(K_*(\mathrm{id}_B), K_*(\eta), K_*(\alpha))$ is an isomorphism from $\mathbf{K}_{\mathrm{six}}(z)$ onto $\mathbf{K}_{\mathrm{six}}(x \times z)$ since $K_*(x)$ and $K_*(\mathrm{id}_B)$ are isomorphisms.

It is clear from the observations made in the previous paragraph and from the definition of s_{δ_*}, s_{λ_*}, $x \times (\cdot)$, and $[K_*(x)]$ that the above diagram is commutative. $\square$

The next two theorems were proved by Rørdam for the case that A and B are separable nuclear purely infinite simple C^*-algebras in $\mathcal{N}$ (Proposition 3.1 and Theorem 3.2 in [37]). Rørdam used these results to show that if $\mathfrak{e}_1$ is in $\mathcal{E}\mathrm{xt}(A_1, B_1)$ and $\mathfrak{e}_2$ is in $\mathcal{E}\mathrm{xt}(A_2, B_2)$ where A_1, A_2, B_1, and B_2 are stable separable nuclear purely infinite simple C^*-algebras in $\mathcal{N}$ with $K_{\mathrm{six}}(\mathfrak{e}_1)$ isomorphic to $K_{\mathrm{six}}(\mathfrak{e}_2)$, then $\mathfrak{e}_1$ is isomorphic to an element $\widetilde{\mathfrak{e}}_1$ of $\mathcal{E}\mathrm{xt}(A_1, B_2)$ and $\mathfrak{e}_2$ is isomorphic to an element $\widetilde{\mathfrak{e}}_2$ of $\mathcal{E}\mathrm{xt}(A_1, B_2)$ such that $x_{A_1,B_2}(\widetilde{\mathfrak{e}}_1) = x_{A_1,B_2}(\widetilde{\mathfrak{e}}_2)$. He then used Kirchberg's absorption theorem to show that $\widetilde{\mathfrak{e}}_1$ is isomorphic to $\widetilde{\mathfrak{e}}_2$. Rørdam conjectured that Proposition 3.1 and Theorem 3.2 in [37] are true for all separable nuclear C^*-algebras in $\mathcal{N}$.

Theorem 2.4. *Let A and B be separable nuclear C^*-algebras in $\mathcal{N}$ with B stable. Let $\delta_* = (\delta_0, \delta_1)$ be an element of* $\mathrm{Hom}(K_*(A), K_{*+1}(B))$.

(1) *The map*
$$s_{\delta_*} = s_{A,B,\delta_*} : \mathrm{Ext}_{\delta_*}(A, B) \to \mathrm{Ext}^1_{\mathbb{Z}}(\ker(\delta_*), \mathrm{coker}(\delta_{*+1}))$$
is a group homomorphism.

(2) *If x is in* $\mathrm{Ext}(A, B)$ *and if $K_*(x) = \delta_*$, then $s_{\delta_*}(x) = \sigma_{\delta_*}(\mathbf{K}_{\mathrm{six}}(x))$.*

(3) *If z is in* $\mathrm{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(B))$*, then $s_{\delta_*}(\epsilon(z)) = \zeta_{\delta_*}(z)$, where ϵ is the canonical embedding of* $\mathrm{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(B))$ *into* $\mathrm{Ext}(A, B)$.

Proof. (2) and (3) are clear from the definition of s_{δ_*} and ζ_{δ_*}.

We now prove (1). We claim that it is enough to prove (1) for the case that A is a unital separable nuclear purely infinite simple C^*-algebra in $\mathcal{N}$. Indeed, by the range results in [36] and [18], there exists a unital separable nuclear purely infinite simple C^*-algebra A_0 in $\mathcal{N}$ such that $K_i(A)$ is isomorphic to $K_i(A_0)$. Denote this isomorphism by λ_i. Suppose A is unital. Then, by Theorem 6.7 in [26], there exists an injective $*$-homomorphism ψ from A to A_0 which induces λ_*. Suppose A is not unital. Let ε be the embedding of A into the unitization of A, which we denote by $\widetilde{A}$. It is easy to find a homomorphism $\widetilde{\lambda}_i$ from $K_i(\widetilde{A})$ to $K_i(A_0)$ such that $\widetilde{\lambda}_i \circ K_i(\varepsilon) = \lambda_i$. Note that $\widetilde{A}$ is a separable unital C^*-algebra in $\mathcal{N}$. By Theorem 6.7 in [26], there exists an injective $*$-homomorphism $\widetilde{\psi}$ from $\widetilde{A}$ to A_0 which induces $\widetilde{\lambda}_*$. Hence, $\psi = \widetilde{\psi} \circ \varepsilon$ is an injective $*$-homomorphism from A to A_0 which induces λ_*. Therefore, in both the unital or the non-unital case, we have an injective $*$-homomorphism ψ which induces an isomorphism from $K_i(A)$ to $K_i(A_0)$. An easy consequence of the Universal Coefficient Theorem [39] and the Five Lemma shows that $KK(\psi)$ is a KK-equivalence. Therefore by Lemma 2.3 our claim is true.

Let A be a unital separable nuclear purely infinite simple C^*-algebra in $\mathcal{N}$. By the range results of [36] and [18], there exist separable nuclear purely infinite simple C^*-algebras A_0 and B_0 in $\mathcal{N}$ such that A_0 is unital, B_0 is stable, and

$$\alpha_j : K_j(A_0) \cong \ker(\delta_j : K_j(A) \to K_{j+1}(B))$$
$$\beta_j : K_j(B_0) \cong \operatorname{coker}(\delta_{j+1} : K_{j+1}(A) \to K_j(B))$$

for $j = 0, 1$. Since A and A_0 are unital separable nuclear purely infinite simple C^*-algebras satisfying the Universal Coefficient Theorem, by Theorem 6.7 in [26] there exists an injective $*$-homomorphism φ from A_0 to A such that for $j = 0, 1$ the map $K_j(A_0) \xrightarrow{\alpha_j} \ker(\delta_j) \hookrightarrow K_j(A)$ is equal to $K_j(\varphi)$. Choose b in $KK(B, B_0)$ such that for $j = 0, 1$ the map from $K_j(B)$ to $\operatorname{coker}(\delta_{j+1})$ is equal to $\beta_j \circ K_j(b)$. Now, using the same argument as Proposition 3.1 in [37], we have that the map s_{δ_*} is a group homomorphism. □

Replacing Proposition 3.1 in [37] by the above theorem and arguing as in Theorem 3.2 in [37], we get the following result.

Theorem 2.5. *Let A and B be separable nuclear C^*-algebras in $\mathcal{N}$ with B stable. Suppose x_1 and x_2 are elements of $\operatorname{Ext}(A, B)$. Then $\mathbf{K}_{\mathrm{six}}(x_1) = \mathbf{K}_{\mathrm{six}}(x_2)$ in $\operatorname{Hext}(A, B)$ if and only if there exist elements a of $KK(A, A)$ and b of $KK(B, B)$ with $K_*(a) = K_*(\mathrm{id}_A)$ and $K_*(b) = K_*(\mathrm{id}_B)$ such that $x_1 \times b = a \times x_2$.*

3. Classification results

We will now use the results of the previous sections to generalize Rørdam's results in [37].

Since in the sequel we will be mostly interested in C^*-algebras that are classified by $(K_0(A), K_0(A)_+, K_1(A))$, we will not state the Elliott invariant in its full generality.

Definition 3.1. For a C^*-algebra A, the *Elliott invariant* (which we denote by $\operatorname{Ell}(A)$) consists of the triple

$$\operatorname{Ell}(A) = (K_0(A), K_0(A)_+, K_1(A)).$$

A homomorphism α_* from $\operatorname{Ell}(A)$ to $\operatorname{Ell}(B)$ consists of a group homomorphism α_0 from $K_0(A)$ to $K_0(B)$, which maps $K_0(A)_+$ to $K_0(B)_+$ and a group homomorphism α_1 from $K_1(A)$ to $K_1(B)$.

If A and B are unital, then a homomorphism α_* from $(\operatorname{Ell}(A), [1_A])$ to $(\operatorname{Ell}(A), [1_B])$ is a homomorphism α_* from $\operatorname{Ell}(A)$ to $\operatorname{Ell}(B)$ such that $\alpha_0([1_A]) = [1_B]$. Isomorphisms are defined in the obvious way. It is well-known that the canonical embedding of a C^*-algebra A into its stabilization $A \otimes \mathcal{K}$ induces an isomorphism from $\operatorname{Ell}(A)$ to $\operatorname{Ell}(A \otimes \mathcal{K})$ (this follows easily from Theorem 6.3.2 and the proof of Proposition 4.3.8 in [38]).

Suppose A and B are separable nuclear C^*-algebras in $\mathcal{N}$. Let x be an element of $KK(A, B)$. We say that x induces a homomorphism from $\operatorname{Ell}(A)$

to $\mathrm{Ell}(B)$ if $K_*(x)$ is a homomorphism from $\mathrm{Ell}(A)$ to $\mathrm{Ell}(B)$. If, moreover $K_0(x)([1_A]) = [1_B]$, then we say x induces a homomorphism from $(\mathrm{Ell}(A), [1_A])$ to $(\mathrm{Ell}(B), [1_B])$.

Definition 3.2. We will be interested in classes $\mathcal{C}$ of separable nuclear unital simple C^*-algebras in $\mathcal{N}$ satisfying the following properties:

(1) Any element of $\mathcal{C}$ is either purely infinite or stably finite.
(2) $\mathcal{C}$ is closed under tensoring with $\mathbf{M}_n$, where $\mathbf{M}_n$ is the C^*-algebra of n by n matrices over $\mathbb{C}$.
(3) If A is in $\mathcal{C}$, then any unital hereditary sub-C^*-algebra of A is in $\mathcal{C}$.
(4) For all A and B in $\mathcal{C}$ and for all x in $KK(A, B)$ which induce an isomorphism from $(\mathrm{Ell}(A), [1_A])$ to $(\mathrm{Ell}(A), [1_B])$, there exists a $*$-isomorphism α from A to B such that $KK(\alpha) = x$.

Remark 3.3. (1) The class of all unital separable nuclear purely infinite simple C^*-algebras satisfying the Universal Coefficient Theorem satisfies the properties in Definition 3.2 (see [19] and [33]).
(2) The class of all unital simple AT-algebras with real rank zero satisfies the properties in Definition 3.2 (see Corollary 3.13 in [20]).
(3) The class of all unital separable nuclear simple C^*-algebras satisfying the Universal Coefficient Theorem with tracial topological rank zero and finitely generated K-theory satisfies the properties in Definition 3.2 (see Theorem 1.1 in [11]). In recent work by Lin and Niu they are able to remove the assumption that the K-theory is finitely generated (see Corollary 3.26 in [28]).

Notation 3.4. For the C^*-algebra of compact operators $\mathcal{K}$ on a separable Hilbert space, we will denote the canonical system of matrix units of $\mathcal{K}$ by $\{e_{ij}\}_{i,j\in\mathbb{N}}$.

Lemma 3.5. *Let $\mathcal{C}$ be a class of separable nuclear unital simple C^*-algebras in $\mathcal{N}$ satisfying the properties in Definition 3.2. Let A and B be in $\mathcal{C}$. Suppose there exists x in $KK(A\otimes\mathcal{K}, B\otimes\mathcal{K})$ such that x induces an isomorphism from* $\mathrm{Ell}(A\otimes\mathcal{K})$ *onto* $\mathrm{Ell}(B\otimes\mathcal{K})$ *and* $K_0(x)([1_A\otimes e_{11}]) = [1_B\otimes e_{11}]$*. Then there exists a $*$-isomorphism α from $A\otimes\mathcal{K}$ onto $B\otimes\mathcal{K}$ such that $KK(\alpha) = x$.*

Proof. Let ι from A to $A\otimes\mathcal{K}$ be the embedding $\iota(a) = a\otimes e_{11}$. Note that $KK(\iota)$ is a KK-equivalence. Since x induces an isomorphism from $\mathrm{Ell}(A\otimes\mathcal{K})$ onto $\mathrm{Ell}(B\otimes\mathcal{K})$ and $K_0(x)([1_A\otimes e_{11}]) = [1_B\otimes e_{11}]$, we have that $KK(\iota)\times x\times KK(\iota)^{-1}$ induces an isomorphism from $(\mathrm{Ell}(A), [1_A])$ onto $(\mathrm{Ell}(B), [1_B])$. By the definition of $\mathcal{C}$, there exists a $*$-isomorphism φ from A to B such that $KK(\varphi) = KK(\iota)\times x\times KK(\iota)^{-1}$. Define α from $A\otimes\mathcal{K}$ to $B\otimes\mathcal{K}$ by $\alpha = \varphi\otimes\mathrm{id}_{\mathcal{K}}$. Then $KK(\alpha) = x$. □

Let a be an element of a C^*-algebra A. We say that *a is norm-full in A* if a is not contained in any norm-closed proper ideal of A. The word "full" is also widely used, but since we will often work in multiplier algebras, we emphasis that it is the norm topology we are using, rather than the strict topology. The next lemma is a consequence of a result of L.G. Brown (see Corollary 2.6 in [2]).

Lemma 3.6. *Let A be a separable C^*-algebra. If p is a norm-full projection in $A \otimes \mathbf{M}_n \subset A \otimes \mathcal{K}$, then there exists a $*$-isomorphism φ from $A \otimes \mathcal{K}$ onto $p(A \otimes \mathcal{K})p \otimes \mathcal{K}$ such that $[\varphi(p)] = [p \otimes e_{11}]$.*

Proof. Using Corollary 2.6 in [2], we get a $*$-isomorphism φ_0 from $A \otimes \mathcal{K} \otimes \mathcal{K}$ onto $p(A \otimes \mathcal{K})p \otimes \mathcal{K}$ that is induced by a partial isometry v in $\mathcal{M}(A \otimes \mathcal{K} \otimes \mathcal{K})$ with the property that $v^*v = 1_{\mathcal{M}(A \otimes \mathcal{K} \otimes \mathcal{K})}$ and $vv^* = p \otimes 1_{\mathcal{M}(\mathcal{K})}$.

Let $\iota_{\mathcal{K}}$ from $\mathcal{K}$ to $\mathcal{K} \otimes \mathcal{K}$ be the canonical embedding. By the classification of AF-algebras, there exists a $*$-isomorphism λ from $\mathcal{K}$ to $\mathcal{K} \otimes \mathcal{K}$ such that $K_0(\lambda) = K_0(\iota_{\mathcal{K}})$, and hence λ is approximately unitarily equivalent to $\iota_{\mathcal{K}}$ with the implementing unitaries in the multiplier algebra of $\mathcal{K} \otimes \mathcal{K}$. Consequently, $\mathrm{id}_A \otimes \lambda$ is an isomorphism from $A \otimes \mathcal{K}$ to $A \otimes \mathcal{K} \otimes \mathcal{K}$ which is approximately unitarily equivalent to $\mathrm{id}_A \otimes \iota_{\mathcal{K}}$ with the implementing unitaries in the multiplier algebra of $A \otimes \mathcal{K}$. Hence, $[(\mathrm{id}_A \otimes \lambda)(p)] = [(\mathrm{id}_A \otimes \iota_{\mathcal{K}})(p)] = [p \otimes e_{11}]$.

Define φ from $A \otimes \mathcal{K}$ to $p(A \otimes \mathcal{K})p \otimes \mathcal{K}$ by $\varphi_0 \circ (\mathrm{id}_A \otimes \lambda)$. Then φ is a $*$-isomorphism and

$$[\varphi(p)] = [\varphi_0(p \otimes e_{11})] = [v(p \otimes e_{11})v^*] = [p \otimes e_{11}]. \qquad \square$$

Lemma 3.7. *Let A_1, A_2, B_1, and B_2 be unital separable nuclear C^*-algebras and let*

$$\mathfrak{e} : \ 0 \to B_1 \otimes \mathcal{K} \to E_1 \to A_1 \otimes \mathcal{K} \to 0$$

be an essential extension. Let α_* from $\mathrm{Ell}(A_1 \otimes \mathcal{K})$ to $\mathrm{Ell}(A_2 \otimes \mathcal{K})$ and β_* from $\mathrm{Ell}(B_1 \otimes \mathcal{K})$ to $\mathrm{Ell}(B_2 \otimes \mathcal{K})$ be isomorphisms. Suppose there exist a norm-full projection p in $\mathbf{M}_n(A_1)$ and a norm-full projection q in $\mathbf{M}_r(B_1)$ such that $\alpha_0([p]) = [1_{A_2} \otimes e_{11}]$, and $\beta_0([q]) = [1_{B_2} \otimes e_{11}]$.

Then there exist $*$-isomorphisms φ from $p\mathbf{M}_n(A_1)p \otimes \mathcal{K}$ to $A_1 \otimes \mathcal{K}$ and ψ from $q\mathbf{M}_r(B_1)q \otimes \mathcal{K}$ to $B_1 \otimes \mathcal{K}$ such that $\varphi \cdot \mathfrak{e}$ is isomorphic to $\mathfrak{e}$ via the isomorphism $(\mathrm{id}_{B_1 \otimes \mathcal{K}}, \mathrm{id}_{E_1}, \varphi)$ with $(\alpha_0 \circ K_0(\varphi))([p \otimes e_{11}]) = [1_{A_2} \otimes e_{11}]$ and $\mathfrak{e}$ is isomorphic to $\mathfrak{e} \cdot \psi^{-1}$ via the isomorphism $(\psi^{-1}, \mathrm{id}_{E_1}, \mathrm{id}_{A_1})$ with $(\beta_0 \circ K_0(\psi))([q \otimes e_{11}]) = [1_{B_2} \otimes e_{11}]$.

Moreover, $\mathfrak{e}$ is isomorphic to $\varphi \cdot \mathfrak{e} \cdot \psi^{-1}$ via the isomorphism $(\psi^{-1}, \mathrm{id}_{E_1}, \varphi)$.

Proof. By Lemma 3.6, there exists a $*$-isomorphism φ from $p(A_1 \otimes \mathcal{K})p \otimes \mathcal{K}$ to $A_1 \otimes \mathcal{K}$ such that $[\varphi(p \otimes e_{11})] = [p]$. By the definition of $\varphi \cdot \mathfrak{e}$, we have that $\varphi \cdot \mathfrak{e}$ is isomorphic to $\mathfrak{e}$ via the isomorphism $(\mathrm{id}_{B_1 \otimes \mathcal{K}}, \mathrm{id}_{E_1}, \varphi)$. Also note that $(\alpha_0 \circ K_0(\varphi))([p \otimes e_{11}]) = \alpha_0([p]) = [1_{A_2} \otimes e_{11}]$.

Using Lemma 3.6 again, there exists a $*$-isomorphism ψ from $q(B_1 \otimes \mathcal{K})q \otimes \mathcal{K}$ to $B_1 \otimes \mathcal{K}$ such that $[\psi(q \otimes e_{11})] = [q]$. By the definition of $\mathfrak{e} \cdot \psi^{-1}$, we have that $\mathfrak{e}$ is isomorphic to $\mathfrak{e} \cdot \psi^{-1}$ via the isomorphism $(\psi^{-1}, \mathrm{id}_{E_1}, \mathrm{id}_{A_1 \otimes \mathcal{K}})$. Note that $(\beta_0 \circ K_0(\psi))([q \otimes e_{11}]) = \beta_0([q]) = [1_{B_2} \otimes e_{11}]$.

Note that the composition of $(\mathrm{id}_{B_1 \otimes \mathcal{K}}, \mathrm{id}_{E_1}, \varphi)$ with $(\psi^{-1}, \mathrm{id}_{E_1}, \mathrm{id}_{A_1 \otimes \mathcal{K}})$ gives an isomorphism $(\psi^{-1}, \mathrm{id}_{E_1}, \varphi)$ from $\mathfrak{e}$ onto $\varphi \cdot \mathfrak{e} \cdot \psi^{-1}$. □

The next lemma is well-known and we omit the proof.

Lemma 3.8. *Let $\mathfrak{e}_1$ and $\mathfrak{e}_2$ be in $\mathcal{E}\mathrm{xt}(A, B)$ and let τ_1 and τ_2 be the Busby invariant of $\mathfrak{e}_1$ and $\mathfrak{e}_2$ respectively. If τ_1 is unitarily equivalent to τ_2 with implementing unitary coming from the multiplier algebra of B, then $\mathfrak{e}_1$ is isomorphic to $\mathfrak{e}_2$.*

A key component used by Rørdam in [37] was Kirchberg's absorption theorem. Elliott and Kucerovsky in [17] give a criterion for when extensions are absorbing. They call such extensions purely large. By Kirchberg, every essential extension of separable nuclear C^*-algebras by stable purely infinite simple C^*-algebras are purely large. Kucerovsky and Ng (see [30] and [21]) studied C^*-algebras satisfying the corona factorization property. They proved the following result: Suppose $B \otimes \mathcal{K}$ satisfies the corona factorization property and suppose τ from A to $\mathcal{Q}(B \otimes \mathcal{K})$ is an essential extension of a separable C^*-algebra A with the property that for every nonzero element a of A, $\tau(a)$ is norm-full in $\mathcal{Q}(B \otimes \mathcal{K})$. Then τ is a purely large extension. Properties similar to the corona factorization property were also studied by Lin [25].

Definition 3.9. Let B be a separable stable C^*-algebra. Then B is said to have the *corona factorization property* if every norm-full projection in $\mathcal{M}(B)$ is Murray-von Neumann equivalent to $1_{\mathcal{M}(B)}$.

A result of Kucerovsky and Ng shows that many simple stable separable nuclear C^*-algebras, which have been successfully classified using K-theoretical data, have the corona factorization property. We quote some of their results here (see [30] and [21]).

Theorem 3.10. *Let A be a unital separable simple C^*-algebra.*

(1) *If A is exact, A has real rank zero and stable rank one, and $K_0(A)$ is weakly unperforated, then $A \otimes \mathcal{K}$ has the corona factorization property.*
(2) *If A is purely infinite, then $A \otimes \mathcal{K}$ has the corona factorization property.*

The following theorem is one of two main results in this paper. Using terminology introduced by Elliott in [14], the next result shows that the six term exact sequence together with certain positive cones is a classification functor for certain essential extensions of simple strongly classifiable C^*-algebras.

Theorem 3.11. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be classes of unital nuclear separable simple C^*-algebras in $\mathcal{N}$ satisfying the properties of Definition 3.2. Let A_1 and A_2 be* *in $\mathcal{C}_1$ and let B_1 and B_2 be in $\mathcal{C}_2$ with $B_1 \otimes \mathcal{K}$ and $B_2 \otimes \mathcal{K}$ satisfying the corona factorization property. Let*

$$\begin{aligned} \mathfrak{e}_1 : \quad & 0 \to B_1 \otimes \mathcal{K} \to E_1 \to A_1 \otimes \mathcal{K} \to 0 \\ \mathfrak{e}_2 : \quad & 0 \to B_2 \otimes \mathcal{K} \to E_2 \to A_2 \otimes \mathcal{K} \to 0 \end{aligned}$$

be essential extensions. Let $\tau_{\mathfrak{e}_1}$ and $\tau_{\mathfrak{e}_2}$ be the Busby invariants of $\mathfrak{e}_1$ and $\mathfrak{e}_2$ respectively. Suppose for every nonzero a_i in $A_i \otimes \mathcal{K}$, we have that $\tau_{\mathfrak{e}_i}(a_i)$ is norm-full in $\mathcal{Q}(B_i \otimes \mathcal{K})$ for $i = 1, 2$. Then the following are equivalent:

1. *There exists a $*$-isomorphism η from E_1 to E_2.*
2. *There exists an isomorphism (β, η, α) from $\mathfrak{e}_1$ to $\mathfrak{e}_2$.*
3. *There exists an isomorphism $(\beta_*, \eta_*, \alpha_*)$ from $K_{\mathrm{six}}(e_1)$ to $K_{\mathrm{six}}(e_2)$ such that β_* is an isomorphism from $\mathrm{Ell}(B_1 \otimes \mathcal{K})$ onto $\mathrm{Ell}(B_2 \otimes \mathcal{K})$ and α_* is an isomorphism from $\mathrm{Ell}(A_1 \otimes \mathcal{K})$ onto $\mathrm{Ell}(A_2 \otimes \mathcal{K})$.*

Proof. Since A_1, A_2, B_1, and B_2 are simple C^*-algebras, by Proposition 1.4 E_1 is isomorphic to E_2 if and only if $\mathfrak{e}_1$ is isomorphic to $\mathfrak{e}_2$. It is clear that an isomorphism from $\mathfrak{e}_1$ onto $\mathfrak{e}_2$ induces an isomorphism $(\beta_*, \eta_*, \alpha_*)$ from $K_{\mathrm{six}}(\mathfrak{e}_1)$ onto $K_{\mathrm{six}}(\mathfrak{e}_2)$ such that β_* is an isomorphism from $\mathrm{Ell}(B_1 \otimes \mathcal{K})$ onto $\mathrm{Ell}(B_2 \otimes \mathcal{K})$ and α_* is an isomorphism from $\mathrm{Ell}(A_1 \otimes \mathcal{K})$ onto $\mathrm{Ell}(A_2 \otimes \mathcal{K})$.

So we only need to prove (3) implies (2). Using the fact that the canonical embedding of A_i into $A_i \otimes \mathcal{K}$ induces an isomorphism between $K_j(A_i)$ and $K_j(A \otimes \mathcal{K})$ and since A_i is simple, by Lemma 3.7 we may assume $\beta_0([1_{B_1} \otimes e_{11}]) = [1_{B_2} \otimes e_{11}]$ and $\alpha_0([1_{A_1} \otimes e_{11}]) = [1_{A_2} \otimes e_{11}]$. Hence, by Lemma 3.5 and the Universal Coefficient Theorem, there exist $*$-isomorphisms β from $B_1 \otimes \mathcal{K}$ to $B_2 \otimes \mathcal{K}$ and α from $A_1 \otimes \mathcal{K}$ to $A_2 \otimes \mathcal{K}$ such that $K_*(\beta) = \beta_*$ and $K_*(\alpha) = \alpha_*$.

By Proposition 1.4, $\mathfrak{e}_1$ is isomorphic to $\mathfrak{e}_1 \cdot \beta$ and $\mathfrak{e}_2$ is isomorphic to $\alpha \cdot \mathfrak{e}_2$. It is straightforward to check that $(K_*(\mathrm{id}_{B_2 \otimes \mathcal{K}}), \eta_*, K_*(\mathrm{id}_{A_1 \otimes \mathcal{K}}))$ gives a congruence between $K_{\mathrm{six}}(\mathfrak{e}_1 \cdot \beta)$ and $K_{\mathrm{six}}(\alpha \cdot \mathfrak{e}_2)$. Therefore, by Proposition 2.2,

$$\mathbf{K}_{\mathrm{six}}(x_{A_1 \otimes \mathcal{K}, B_2 \otimes \mathcal{K}}(\mathfrak{e}_1 \cdot \beta)) = \mathbf{K}_{\mathrm{six}}(x_{A_1 \otimes \mathcal{K}, B_2 \otimes \mathcal{K}}(\alpha \cdot \mathfrak{e}_2)).$$

Let $x_j = x_{A_j \otimes \mathcal{K}, B_j \otimes \mathcal{K}}(\mathfrak{e}_j)$ for $j = 1, 2$. By Proposition 1.3,

$$\begin{aligned} \mathbf{K}_{\mathrm{six}}(x_1 \times KK(\beta)) &= \mathbf{K}_{\mathrm{six}}(x_{A_1 \otimes \mathcal{K}, B_2 \otimes \mathcal{K}}(\mathfrak{e}_1 \cdot \beta)) \\ &= \mathbf{K}_{\mathrm{six}}(x_{A_1 \otimes \mathcal{K}, B_2 \otimes \mathcal{K}}(\alpha \cdot \mathfrak{e}_2)) \\ &= \mathbf{K}_{\mathrm{six}}(KK(\alpha) \times x_2). \end{aligned}$$

By Theorem 2.5, there exist invertible elements a of $KK(A_1 \otimes \mathcal{K}, A_1 \otimes \mathcal{K})$ and b of $KK(B_2 \otimes \mathcal{K}, B_2 \otimes \mathcal{K})$ such that

1. $K_*(a) = K_*(\mathrm{id}_{A_1 \otimes \mathcal{K}})$ and $K_*(b) = K_*(\mathrm{id}_{B_2 \otimes \mathcal{K}})$ and
2. $x_1 \times KK(\beta) \times b = a \times KK(\alpha) \times x_2$.

Since A_1 is in $\mathcal{C}_1$ and B_2 is in $\mathcal{C}_2$, by Lemma 3.5 there exist $*$-isomorphisms ρ from $A_1 \otimes \mathcal{K}$ to $A_1 \otimes \mathcal{K}$ and γ from $B_2 \otimes \mathcal{K}$ to $B_2 \otimes \mathcal{K}$ such that $KK(\rho) = a$ and $KK(\gamma) = b$

Using Proposition 1.4 once again, $\mathfrak{e}_1 \cdot \beta$ is isomorphic to $\mathfrak{e}_1 \cdot \beta \cdot \gamma$ and $\alpha \cdot \mathfrak{e}_2$ is isomorphic to $\rho \cdot \alpha \cdot \mathfrak{e}_2$. By Proposition 1.3,

$$\begin{aligned} x_{A_1 \otimes \mathcal{K}, B_2 \otimes \mathcal{K}}(\mathfrak{e}_1 \cdot \beta \cdot \gamma) &= x_1 \times KK(\beta) \times KK(\gamma) = x_1 \times KK(\beta) \times b \\ &= a \times KK(\alpha) \times x_2 \qquad = KK(\rho) \times KK(\alpha) \times x_2 \\ &= x_{A_1 \otimes \mathcal{K}, B_2 \otimes \mathcal{K}}(\rho \cdot \alpha \cdot \mathfrak{e}_2) \ . \end{aligned}$$

Let τ_1 be the Busby invariant of $\mathfrak{e}_1 \cdot \beta \cdot \gamma$ and let τ_2 be the Busby invariant of $\rho \cdot \alpha \cdot \mathfrak{e}_2$. By the above equation, $[\tau_1] = [\tau_2]$ in $\mathrm{Ext}(A_1 \otimes \mathcal{K}, B_2 \otimes \mathcal{K})$. By our assumption, $B_2 \otimes \mathcal{K}$ satisfies the corona factorization property and $\tau_i(a)$ is norm-full in $\mathcal{Q}(B_2 \otimes \mathcal{K})$ for all nonzero element a of $A_1 \otimes \mathcal{K}$. Therefore, by Theorem 3.2(2) in [30] (also see Corollary 1.9 in [21]), there exists a unitary u in $\mathcal{M}(B_2 \otimes \mathcal{K})$ such that $\pi(u)\tau_1(a)\pi(u)^* = \tau_2(a)$ for all a in A_1. By Lemma 3.8, $\mathfrak{e}_1 \cdot \beta \cdot \gamma$ is isomorphic to $\rho \cdot \alpha \cdot \mathfrak{e}_2$. Hence, $\mathfrak{e}_1$ is isomorphic to $\mathfrak{e}_2$. □

Remark 3.12. In the above theorem, if $\mathcal{Q}(B_1 \otimes \mathcal{K})$ is simple, then for every nonzero element a_1 of A_1, we have that $\tau_{\mathfrak{e}_1}(a_1)$ is norm-full in $\mathcal{Q}(B_1 \otimes \mathcal{K})$. This is the case when $B_1 \otimes \mathcal{K}$ is a purely infinite simple C^*-algebra.

Using similar techniques as above, we will show that a class of extensions coming from substitutional dynamical systems are classified (up to stable isomorphism) by their six term exact sequence in K-theory together with the order from the K_0-groups of the distinguished ideal and quotient.

Lemma 3.13. *Let A be a unital AF-algebra. Then $A \otimes \mathcal{K}$ has the corona factorization property.*

Proof. Suppose p is a norm-full projection in $\mathcal{M}(A \otimes \mathcal{K})$. Then, by Corollary 3.6 in [25], there exists z in $\mathcal{M}(A \otimes \mathcal{K})$ such that $zpz^* = 1_{\mathcal{M}(A \otimes \mathcal{K})}$. Therefore, $1_{\mathcal{M}(A \otimes \mathcal{K})}$ is Murray-von Neumann equivalent to a sub-projection of p. Since $1_{\mathcal{M}(A \otimes \mathcal{K})}$ is a properly infinite projection, p is a properly infinite projection. By the results of Cuntz in [10] and the fact that $K_0(\mathcal{M}(A \otimes \mathcal{K})) = 0$, we have that $1_{\mathcal{M}(A \otimes \mathcal{K})}$ is Murray-von Neumann equivalent to p. □

Lemma 3.14. *Let A be a separable stable C^*-algebra satisfying the corona factorization property. Let q be a norm-full projection in $\mathcal{M}(A)$. Then qAq is isomorphic to A and hence qAq is stable.*

Proof. Since q is norm-full in $\mathcal{M}(A)$ and since A has the corona factorization property, there exists a partial isomerty v in $\mathcal{M}(A)$ such that $v^*v = 1_{\mathcal{M}(A)}$ and $vv^* = q$. Therefore v induces a $*$-isomorphism from A onto qAq. Since A is stable, qAq is stable. □

Note that one of the key ingredients of the proof of Theorem 3.11 was that the Busby invariant of the extension

$$0 \to B_1 \otimes \mathcal{K} \to E_1 \to A_1 \otimes \mathcal{K} \to 0$$

took every nonzero element of $A_1 \otimes \mathcal{K}$ to a norm-full element of $\mathcal{Q}(B_1 \otimes \mathcal{K})$. When we replace $A_1 \otimes \mathcal{K}$ by a unital simple C^*-algebra and assume that the Busby invariant is unital, then the Busby invariant always satisfies this fullness condition.

In Theorem 3.16, we will be considering extensions

$$0 \to B \to E \to A \to 0$$

where A is a unital simple C^*-algebra and the Busby invariant is unital. We will classify a certain class of extensions of this form up to stable isomorphism. Hence, as in the proof of Theorem 3.11, will need to know that the Busby invariant of the stabilized extension

$$0 \to B \otimes \mathcal{K} \to E \otimes \mathcal{K} \to A \otimes \mathcal{K} \to 0$$

has the fullness condition stated in the previous paragraph. This may be a well-known result but we have not been able to find a reference so we prove it here.

Proposition 3.15. *Let $\mathfrak{e} : \ 0 \to B \xrightarrow{\iota} E \xrightarrow{\pi} A \to 0$ be an essential extension where B is a separable, stable C^*-algebra. Denote the Busby invariant of this extension by $\tau_{\mathfrak{e}}$ and denote the Busby invariant of the essential extension*

$$\mathfrak{e}^s : \qquad 0 \longrightarrow B \otimes \mathcal{K} \xrightarrow{\iota \otimes \mathrm{id}_{\mathcal{K}}} E \otimes \mathcal{K} \xrightarrow{\pi \otimes \mathrm{id}_{\mathcal{K}}} A \otimes \mathcal{K} \longrightarrow 0$$

by $\tau_{\mathfrak{e}^s}$. Suppose for every nonzero element a of A, $\tau_{\mathfrak{e}}(a)$ is norm-full in $\mathcal{Q}(B)$. Then for every nonzero element x of $A \otimes \mathcal{K}$, we have that $\tau_{\mathfrak{e}^s}(x)$ is norm-full in $\mathcal{Q}(B \otimes \mathcal{K})$.

Proof. For any C^*-algebra C, denote the embedding of C into $C \otimes \mathcal{K}$ which sends c into $c \otimes e_{11}$ by ι_C and denote the canonical embedding of C as an essential ideal of the multiplier algebra $\mathcal{M}(C)$ of C by θ_C. We will first show that ι_B satisfies the following properties:

(1) ι_B has an extension $\widetilde{\iota}_B$ from $\mathcal{M}(B)$ to $\mathcal{M}(B \otimes \mathcal{K})$ (i.e. $\theta_{B \otimes \mathcal{K}} \circ \iota_B = \widetilde{\iota}_B \circ \theta_B$), which maps $1_{\mathcal{M}(B)}$ to a norm-full projection in $\mathcal{M}(B \otimes \mathcal{K})$ and
(2) the map $\overline{\iota}_B$ from $\mathcal{Q}(B)$ to $\mathcal{Q}(B \otimes \mathcal{K})$ induce by $\widetilde{\iota}_B$ intertwines the Busby invariants of $\mathfrak{e}$ and $\mathfrak{e}^s$, and the $*$-homomorphism ι_A (i.e. $\tau_{\mathfrak{e}^s} \circ \iota_A = \overline{\iota}_B \circ \tau_{\mathfrak{e}}$).

First note that there exist unique injective $*$-homomorphisms σ from E to $\mathcal{M}(B)$ and σ^s from $E \otimes \mathcal{K}$ to $\mathcal{M}(B \otimes \mathcal{K})$ such that $\theta_B = \sigma \circ \iota$ and $\theta_{B \otimes \mathcal{K}} = \sigma^s \circ (\iota \otimes \mathrm{id}_{\mathcal{K}})$. It is well-known that we have a unique $*$-homomorphism ρ from $\mathcal{M}(B) \otimes \mathcal{M}(\mathcal{K})$ to $\mathcal{M}(B \otimes \mathcal{K})$ such that $\theta_{B \otimes \mathcal{K}} = \rho \circ (\theta_B \otimes \theta_{\mathcal{K}})$ and that this map is injective and unital (see Lemma 11.12 in [32]).

In the following diagram, all the maps are injective $*$-homomorphisms

$$\begin{tikzcd}
 & B \otimes \mathcal{K} \arrow[ddl, bend right, "\theta_B \otimes \mathrm{id}_{\mathcal{K}}"'] \arrow[d, "\iota \otimes \mathrm{id}_{\mathcal{K}}"'] \arrow[ddr, bend left, "\theta_{B \otimes \mathcal{K}}"] & \\
 & E \otimes \mathcal{K} \arrow[dl, "\sigma \otimes \mathrm{id}_{\mathcal{K}}"] \arrow[dr, "\sigma^s"'] & \\
\mathcal{M}(B) \otimes \mathcal{K} \arrow[r, "\mathrm{id}_{\mathcal{M}(B)} \otimes \theta_{\mathcal{K}}"'] & \mathcal{M}(B) \otimes \mathcal{M}(K) \arrow[r, "\rho"'] & \mathcal{M}(B \otimes \mathcal{K})
\end{tikzcd}$$

Everything commutes except possibly the bottom triangle but by the uniqueness of σ^s this triangle commutes.

Now let $\widetilde{\iota}_B = \rho \circ (\mathrm{id}_{\mathcal{M}(B)} \otimes \theta_{\mathcal{K}}) \circ \iota_{\mathcal{M}(B)}$. Clearly, $\theta_{B \otimes \mathcal{K}} \circ \iota_B = \widetilde{\iota}_B \circ \theta_B$ and $p = \widetilde{\iota}_B(1_{\mathcal{M}(B)})$ is a projection in $\mathcal{M}(B \otimes \mathcal{K})$. Note that $\iota_B(B) = B \otimes e_{11} \cong p\theta_{B \otimes \mathcal{K}}(B \otimes \mathcal{K})p$. Therefore, $p\theta_{B \otimes \mathcal{K}}(B \otimes \mathcal{K})p$ is a stable, hereditary, sub-C^*-algebra of $\theta_{B \otimes \mathcal{K}}(B \otimes \mathcal{K})$ which is not contained in any proper ideal of $\theta_{B \otimes \mathcal{K}}(B \otimes \mathcal{K})$. By Theorem 4.23 in [3], p is Murray-von Neumann equivalent to $1_{\mathcal{M}(B \otimes \mathcal{K})}$. Hence, $p = \widetilde{\iota}_B(1_{\mathcal{M}(B)})$ is norm-full in $\mathcal{M}(B \otimes \mathcal{K})$.

Now we see that $\widetilde{\iota}_B \circ \sigma = \sigma^s \circ \iota_E$ since the following diagram is commutative:

$$\begin{tikzcd}
E \arrow[rrr, "\iota_E"] \arrow[d, "\sigma"'] & & & E \otimes \mathcal{K} \arrow[dll, "\sigma \otimes \mathrm{id}_{\mathcal{K}}"'] \arrow[d, "\sigma^s"] \\
\mathcal{M}(B) \arrow[r, "\iota_{\mathcal{M}(B)}"'] & \mathcal{M}(B) \otimes \mathcal{K} \arrow[r, "\mathrm{id}_{\mathcal{M}(B)} \otimes \theta_{\mathcal{K}}"'] & \mathcal{M}(B) \otimes \mathcal{M}(\mathcal{K}) \arrow[r, "\rho"'] & \mathcal{M}(B \otimes \mathcal{K})
\end{tikzcd}$$

Let $\overline{\iota}_B$ denote the $*$-homomorphism from $\mathcal{Q}(B)$ to $\mathcal{Q}(B \otimes \mathcal{K})$ which is induced by $\widetilde{\iota}_B$. Arguing as in the proof of Theorem 2.2 in [12], we have that the diagram

$$\begin{tikzcd}
A \arrow[r, "\tau_{\mathfrak{e}}"] \arrow[d, "\iota_A"'] & \mathcal{Q}(B) \arrow[d, "\overline{\iota}_B"] \\
A \otimes \mathcal{K} \arrow[r, "\tau_{\mathfrak{e}^s}"'] & \mathcal{Q}(B \otimes \mathcal{K})
\end{tikzcd} \tag{3.1}$$

is commutative since $(\iota_B, \iota_E, \iota_A)$ is a morphism from $\mathfrak{e}$ to $\mathfrak{e}^s$. This finishes the proof of the two claims (1) and (2) above.

We are now ready to prove the proposition. Let x be a nonzero positive element of $A \otimes \mathcal{K}$. Then there exist t and s in $A \otimes \mathcal{K}$ such that $sx^{\frac{1}{2}}t = \iota_A(y)$ for some nonzero positive element y of A. Let ϵ be a strictly positive number. From (1) of our claim, there exist $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ in $\mathcal{Q}(B \otimes \mathcal{K})$ such that

$$\left\| 1_{\mathcal{Q}(B \otimes \mathcal{K})} - \sum_{i=1}^{n} x_i \overline{\iota}_B(1_{\mathcal{Q}(B)}) y_i \right\| < \frac{\epsilon}{2}.$$

From our assumption on $\tau_{\mathfrak{e}}$, there exist $t_1, \ldots, t_m$, $s_1, \ldots, s_m$ in $\mathcal{Q}(B)$ such that

$$\left\| 1_{\mathcal{Q}(B)} - \sum_{j=1}^{m} s_j \tau_{\mathfrak{e}}(y) t_j \right\| < \frac{\epsilon}{2(\sum_{i=1}^{n} \|x_i\| \, \|y_i\| + 1)}.$$

An easy computation shows that

$$\left\| 1_{\mathcal{Q}(B\otimes\mathcal{K})} - \sum_{i=1}^{n} x_i \left(\sum_{j=1}^{m} \bar{\iota}_B(s_j \tau_{\mathfrak{e}}(y) t_j) \right) y_i \right\| < \epsilon.$$

By the commutativity of Diagram (3.1), we have that

$$\left\| 1_{\mathcal{Q}(B\otimes\mathcal{K})} - \sum_{i=1}^{n} x_i \left(\sum_{j=1}^{m} \bar{\iota}_B(s_j) \tau_{\mathfrak{e}}^{s}(s x^{\frac{1}{2}} t) \bar{\iota}_B(t_j) \right) y_i \right\| < \epsilon.$$

Therefore, the ideal of $\mathcal{Q}(B\otimes\mathcal{K})$ generated by $\tau_{\mathfrak{e}^s}(x^{\frac{1}{2}})$ is equal to $\mathcal{Q}(B\otimes\mathcal{K})$. Since $x^{\frac{1}{2}}$ is contained in the ideal of $A\otimes\mathcal{K}$ generated by x, we have that x is norm-full in $\mathcal{Q}(B\otimes\mathcal{K})$.

For an arbitrary nonzero element x of $A\otimes\mathcal{K}$, consider the positive nonzero element x^*x of $A\otimes\mathcal{K}$ and apply the result on positive elements to conclude that $\tau_{\mathfrak{e}^s}(x^*x)$ is norm-full in $\mathcal{Q}(B\otimes\mathcal{K})$. Therefore, $\tau_{\mathfrak{e}^s}(x)$ is norm-full in $\mathcal{Q}(B\otimes\mathcal{K})$ since x^*x is contained in the ideal of $A\otimes\mathcal{K}$ generated by x. □

The next theorem will be used to classify a class of C^*-algebras associated to certain minimal shift spaces. We will then use this result in the next section to show that stable isomorphism of these C^*-algebras associated to minimal shift spaces arising from basic substitutional dynamical systems must be a strictly coarser relation than flow equivalence.

Theorem 3.16. *Let A_1 and A_2 be unital simple AT-algebras with real rank zero such that $K_1(A_1)$ and $K_1(A_2)$ are non-trivial abelian groups. Let B_1 and B_2 be unital AF-algebras. Suppose*

$$\mathfrak{e}_1 : \qquad 0 \to B_1 \otimes \mathcal{K} \xrightarrow{\varphi_1} E_1 \xrightarrow{\psi_1} A_1 \to 0$$

$$\mathfrak{e}_2 : \qquad 0 \to B_2 \otimes \mathcal{K} \xrightarrow{\varphi_2} E_2 \xrightarrow{\psi_2} A_2 \to 0$$

are unital essential extensions. Let $\mathfrak{e}_1^s$ and $\mathfrak{e}_2^s$ be the extensions obtained by tensoring $\mathfrak{e}_1$ and $\mathfrak{e}_2$ with the compact operators. Then the following are equivalent:

1. *$E_1 \otimes \mathcal{K}$ is isomorphic to $E_2 \otimes \mathcal{K}$.*
2. *$\mathfrak{e}_1^s$ is isomorphic to $\mathfrak{e}_2^s$.*
3. *there exists an isomorphism $(\beta_*, \eta_*, \alpha_*)$ from $K_{\mathrm{six}}(\mathfrak{e}_1^s)$ onto $K_{\mathrm{six}}(\mathfrak{e}_2^s)$ such that β_* is an isomorphism from $\mathrm{Ell}(B_1\otimes\mathcal{K}\otimes\mathcal{K})$ onto $\mathrm{Ell}(B_2\otimes\mathcal{K}\otimes\mathcal{K})$ and α_* is an isomorphism from $\mathrm{Ell}(A_1\otimes\mathcal{K})$ onto $\mathrm{Ell}(A_2\otimes\mathcal{K})$.*

(4) *there exists an isomorphism* $(\beta_*, \eta_*, \alpha_*)$ *from* $K_{\mathrm{six}}(\mathfrak{e}_1)$ *to* $K_{\mathrm{six}}(\mathfrak{e}_2)$ *such that* β_* *is an isomorphism from* $\mathrm{Ell}(B_1 \otimes \mathcal{K})$ *to* $\mathrm{Ell}(B_2 \otimes \mathcal{K})$ *and* α_* *is an isomorphism from* $\mathrm{Ell}(A_1)$ *to* $\mathrm{Ell}(A_2)$.

Proof. First we show that (1) implies (2). Suppose that there exists a $*$-isomorphism η from $E_1 \otimes \mathcal{K}$ onto $E_2 \otimes \mathcal{K}$. Note that for $i = 1, 2$, $A_i \otimes \mathcal{K}$ is not an AF-algebra since $K_1(A_i \otimes \mathcal{K}) \neq 0$. Since $[(\psi_2 \otimes \mathrm{id}_{\mathcal{K}}) \circ \eta \circ (\varphi_1 \otimes \mathrm{id}_{\mathcal{K}})](B_1 \otimes \mathcal{K} \otimes \mathcal{K})$ is an ideal of $A_1 \otimes \mathcal{K}$ and $A_1 \otimes \mathcal{K}$ is a simple C^*-algebra, $[(\psi_2 \otimes \mathrm{id}_{\mathcal{K}}) \circ \eta \circ (\varphi_1 \otimes \mathrm{id}_{\mathcal{K}})](B_1 \otimes \mathcal{K} \otimes \mathcal{K})$ is either zero or $A_1 \otimes \mathcal{K}$. Since the image of an AF-algebra is again an AF-algebra, $[(\psi_2 \otimes \mathrm{id}_{\mathcal{K}}) \circ \eta \circ (\varphi_1 \otimes \mathrm{id}_{\mathcal{K}})](B_1 \otimes \mathcal{K} \otimes \mathcal{K}) = 0$. Hence, η induces an isomorphism from $\mathfrak{e}_1^s$ onto $\mathfrak{e}_2^s$.

Clearly (2) implies both (1) and (3). As noted in Definition 3.1, (3) implies (4). We now prove (3) implies (2). By Lemma 3.7, we may assume that $\alpha_0([1_{A_1} \otimes e_{11}]) = [1_{A_2} \otimes e_{11}]$. Using Elliott's classification theorems for AF-algebras and AT-algebras (see [15] and [16]), we get $*$-isomorphisms α from $A_1 \otimes \mathcal{K}$ to $A_2 \otimes \mathcal{K}$ and β from $B_1 \otimes \mathcal{K} \otimes \mathcal{K}$ to $B_2 \otimes \mathcal{K} \otimes \mathcal{K}$ such that $K_*(\alpha) = \alpha_*$ and $K_*(\beta) = \beta_*$.

By Proposition 1.4, $\mathfrak{e}_1^s$ is isomorphic to $\mathfrak{e}_1^s \cdot \beta$ and $\mathfrak{e}_2^s$ is isomorphic to $\alpha \cdot \mathfrak{e}_2^s$. It is straightforward to check that $K_{\mathrm{six}}(\mathfrak{e}_1^s \cdot \beta)$ is congruent to $K_{\mathrm{six}}(\alpha \cdot \mathfrak{e}_2^s)$. Hence, by Theorem 2.5 there exist invertible elements a of $KK(A_1 \otimes \mathcal{K}, A_1 \otimes \mathcal{K})$ and b of $KK(B_2 \otimes \mathcal{K} \otimes \mathcal{K}, B_2 \otimes \mathcal{K} \otimes \mathcal{K})$ such that

(1) $K_*(a) = K_*(\mathrm{id}_{A_1 \otimes \mathcal{K}})$ and $K_*(b) = K_*(\mathrm{id}_{B_2 \otimes \mathcal{K} \otimes \mathcal{K}})$; and
(2) $x_{A_1 \otimes \mathcal{K}, B_2 \otimes \mathcal{K} \otimes \mathcal{K}}(\mathfrak{e}_1^s \cdot \beta) \times b = a \times x_{A_1 \otimes \mathcal{K}, B_2 \otimes \mathcal{K} \otimes \mathcal{K}}(\alpha \cdot \mathfrak{e}_2^s)$.

By the Universal Coefficient Theorem, $b = KK(\mathrm{id}_{B_2 \otimes \mathcal{K} \otimes \mathcal{K}})$ since $B_2 \otimes \mathcal{K} \otimes \mathcal{K}$ is an AF-algebra. Since A_1 is a unital simple AT-algebra with real rank zero and $K_*(a) = \mathrm{id}_{A_1 \otimes \mathcal{K}}$, by Corollary 3.13 in [20] and Lemma 3.5 there exists a $*$-isomorphism ρ from $A_1 \otimes \mathcal{K}$ to $A_1 \otimes \mathcal{K}$ such that $KK(\rho) = a$.

By Proposition 1.3 and Proposition 1.4, $\rho \cdot \alpha \cdot \mathfrak{e}_2^s$ is isomorphic to $\alpha \cdot \mathfrak{e}_2^s$ and

$$\begin{aligned} x_{A_1 \otimes \mathcal{K}, B_2 \otimes \mathcal{K} \otimes \mathcal{K}}(\mathfrak{e}_1^s \cdot \beta) &= x_1 \times KK(\beta) \\ &= KK(\rho) \times KK(\alpha) \times x_2 \\ &= x_{A_1 \otimes \mathcal{K}, B_2 \otimes \mathcal{K} \otimes \mathcal{K}}(\rho \cdot \alpha \cdot \mathfrak{e}_2^s), \end{aligned}$$

where $x_i = x_{A_i \otimes \mathcal{K}, B_i \otimes \mathcal{K} \otimes \mathcal{K}}(\mathfrak{e}_i^s)$.

Let τ_1 be the Busby invariant of $\mathfrak{e}_1^s \cdot \beta$ and let τ_2 be the Busby invariant of $\rho \cdot \alpha \cdot \mathfrak{e}_2^s$. Then, $[\tau_1] = [\tau_2]$ in $\mathrm{Ext}(A_1 \otimes \mathcal{K}, B_2 \otimes \mathcal{K} \otimes \mathcal{K})$. Note that since A_i is a simple unital C^*-algebra and $\mathfrak{e}_i$ is a unital essential extension, we have that for every nonzero element a of A_i, $\tau_{\mathfrak{e}_i}(a)$ is norm-full in $\mathcal{Q}(B_i \otimes \mathcal{K})$. If $\tau_{\mathfrak{e}_i^s}$ denotes the Busby invariant for the extension $\mathfrak{e}_i^s$, then by Proposition 3.15 we have that $\tau_{\mathfrak{e}_i^s}(x)$ is norm-full in $\mathcal{Q}(B_i \otimes \mathcal{K} \otimes \mathcal{K})$ for any nonzero x in $A_i \otimes \mathcal{K}$. Using this observation and the fact that β, α, and ρ are $*$-isomorphisms, it is clear that $\tau_i(x)$ is norm-full in $\mathcal{Q}(B_2 \otimes \mathcal{K})$ for $i = 1, 2$ and for any nonzero element x of $A_1 \otimes \mathcal{K}$.

Note that by Lemma 3.13, $B_2 \otimes \mathcal{K} \otimes \mathcal{K}$ has the corona factorization property. Therefore, by the observations made in the previous paragraph one can apply Theorem 3.2(2) in [30] to get a unitary u in $\mathcal{M}(B_2 \otimes \mathcal{K} \otimes \mathcal{K})$ such that

$$\pi(u)\tau_1(x)\pi(u)^* = \tau_2(x)$$

for all a in $A_1 \otimes \mathcal{K}$. Hence, by Lemma 3.8, $\mathfrak{e}_1^s \cdot \beta$ and $\rho \circ \alpha \cdot \mathfrak{e}_2^s$ are isomorphic. Therefore, $\mathfrak{e}_1^s$ is isomorphic to $\mathfrak{e}_2^s$. □

4. Examples

Clearly, Theorem 3.11 applies to essential extensions of separable nuclear purely infinite simple stable C^*-algebras in $\mathcal{N}$ (and gives us the classification obtained by Rørdam in [37]). We present here two other examples of classes of special interest, to which our results apply.

4.1. Matsumoto algebras. The results of the previous section apply to a class of C^*-algebras introduced in the work by Matsumoto which was investigated in recent work by the first named author and Carlsen ([4],[5],[7],[8],[9]). Indeed, as seen in [4] we have for each minimal shift space $\underline{\mathsf{X}}$ with a certain technical property $(**)$ introduced in Definition 3.2 in [8] that the Matsumoto algebra $\mathcal{O}_{\underline{\mathsf{X}}}$ fits in a short exact sequence of the form

$$0 \longrightarrow \mathcal{K}^n \longrightarrow \mathcal{O}_{\underline{\mathsf{X}}} \longrightarrow C(\underline{\mathsf{X}}) \rtimes_\sigma \mathbb{Z} \longrightarrow 0 \tag{4.1}$$

where n is an integer determined by the structure of the so-called *special words* of $\underline{\mathsf{X}}$. Clearly the ideal is an AF-algebra and by the work of Putnam [34] the quotient is a unital simple AT-algebra with real rank zero. Let us record a couple of consequences of this:

Corollary 4.1. *Let $\underline{\mathsf{X}}_\alpha$ denote the Sturmian shift space associated to the parameter α in $[0,1]\backslash\mathbb{Q}$ and $\mathcal{O}_{\underline{\mathsf{X}}_\alpha}$ the Matsumoto algebra associated to $\underline{\mathsf{X}}_\alpha$. If α and β are elements of $[0,1]\backslash\mathbb{Q}$, then*

$$\mathcal{O}_{\underline{\mathsf{X}}_\alpha} \otimes \mathcal{K} \cong \mathcal{O}_{\underline{\mathsf{X}}_\beta} \otimes \mathcal{K}$$

if and only if $\mathbb{Z} + \alpha\mathbb{Z} \cong \mathbb{Z} + \beta\mathbb{Z}$ as ordered groups.

Proof. The extension (4.1) has the six term exact sequence

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "0"] \arrow[d, equal] & \mathbb{Z} + \alpha\mathbb{Z} \arrow[r, equal] & \mathbb{Z} + \alpha\mathbb{Z} \arrow[d] \\ \mathbb{Z} & 0 \arrow[l] & 0 \arrow[l] \end{tikzcd}$$

by Example 5.3 in [9]. Now apply Theorem 3.16. □

The bulk of the work in the papers [5]–[9] is devoted to the case of shift spaces associated to primitive, aperiodic substitutions. As a main result, an algorithm is devised to compute the ordered group $K_0(\mathcal{O}_{\underline{\mathsf{X}}_\tau})$ for any such substitution τ, thus providing new invariants for such dynamical systems up to flow equivalence (see [31]). The structure result of [4] applies in this case as well, and in fact, as noted in Section 6.4 of [7], the algorithm provides all the data in the six term exact sequence associated to the extension (4.1). This is based on computable objects $\mathsf{n}_\tau, \mathsf{p}_\tau, \mathsf{A}_\tau, \widetilde{\mathsf{A}}_\tau$ of which the latter two are square matrices with integer entries. For each such matrix, say A in $\mathbf{M}_n(\mathbb{Z})$, we define a group

$$DG(A) = \varinjlim \left(\mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^n \xrightarrow{A} \cdots \right)$$

which, when A has only nonnegative entries, may be considered as an ordered group which will be a dimension group. We get:

Theorem 4.2. *Let τ_1 and τ_2 be basic substitutions, see* [8]*, over the alphabets $\mathfrak{a}_1$ and $\mathfrak{a}_2$, respectively. Then*

$$\mathcal{O}_{\underline{\mathsf{X}}_{\tau_1}} \otimes \mathcal{K} \cong \mathcal{O}_{\underline{\mathsf{X}}_{\tau_2}} \otimes \mathcal{K}$$

if and only if there exist group isomorphisms ϕ_1, ϕ_2, ϕ_3 with ϕ_1 and ϕ_3 order isomorphisms, making the diagram

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "\mathsf{p}_{\tau_1}"] \arrow[d, equal] & \mathbb{Z}^{\mathsf{n}_{\tau_1}} \arrow[r, "Q_1"] \arrow[d, "\phi_1"'] & DG(\widetilde{\mathsf{A}}_{\tau_1}) \arrow[r, "R_\infty"] \arrow[d, "\phi_2"'] & DG(\mathsf{A}_{\tau_1}) \arrow[d, "\phi_3"'] \\
\mathbb{Z} \arrow[r, "\mathsf{p}_{\tau_2}"'] & \mathbb{Z}^{\mathsf{n}_{\tau_2}} \arrow[r, "Q_2"'] & DG(\widetilde{\mathsf{A}}_{\tau_2}) \arrow[r, "R_\infty"'] & DG(\mathsf{A}_{\tau_2})
\end{tikzcd}$$

commutative. Here the finite data n_{τ_i} in $\mathbb{N}$, p_{τ_i} in $\mathbb{Z}^{\mathsf{n}_{\tau_i}}$, A_{τ_i} in $\mathbf{M}_{|\mathfrak{a}_i|}(\mathbb{N}_0)$, $\widetilde{\mathsf{A}}_{\tau_i}$ in $\mathbf{M}_{|\mathfrak{a}_i|+\mathsf{n}_{\tau_i}}(\mathbb{Z})$ are as described in [7]*, the Q_i are defined by the canonical map to the first occurrence of $\mathbb{Z}^{\mathsf{n}_{\tau_i}}$ in the inductive limit, and R_∞ are induced by the canonical map from $\mathbb{Z}^{|\mathfrak{a}_i|+\mathsf{n}_{\tau_i}}$ to $\mathbb{Z}^{|\mathfrak{a}_i|}$.*

Proof. We have already noted above that Theorem 3.16 applies, proving "if". For "only if", we use that any $*$-isomorphism between $\mathcal{O}_{\underline{\mathsf{X}}_{\tau_1}} \otimes \mathcal{K}$ and $\mathcal{O}_{\underline{\mathsf{X}}_{\tau_2}} \otimes \mathcal{K}$ must preserve the ideal in (4.1) and hence induce isomorphisms on the corresponding six term exact sequence which are intertwined by the maps of this sequence as indicated. And since the vectors p_{τ_i} both have all entries positive, the isomorphism $x \mapsto -x$ between $\mathbb{Z}$ and $\mathbb{Z}$ can be ruled out by positivity of ϕ_1. $\square$

This result shows, essentially, that the information stored in the stabilized C^*-algebras is the same as the information stored in the six term exact sequence, hence putting further emphasis on the question raised in Section 6.4 in [7] of what relation stable isomorphism of the C^*-algebras induces on the shift spaces. We note here that that relation must be strictly coarser than flow equivalence:

Example 4.3. Consider the substitutions

$$\tau(0) = 10101000 \qquad \tau(1) = 10100$$

and

$$\upsilon(0) = 10100100 \qquad \upsilon(1) = 10100$$

We have that $\mathcal{O}_{\underline{\mathsf{X}}_\tau} \otimes \mathcal{K} \cong \mathcal{O}_{\underline{\mathsf{X}}_\upsilon} \otimes \mathcal{K}$ although $\underline{\mathsf{X}}_\tau$ and $\underline{\mathsf{X}}_\upsilon$ are not flow equivalent.

Proof. Since both substitutions are chosen to be basic, computations using the algorithm from [5] (for instance using the program [6]) show that the six term exact sequence degenerates to

$$\mathbb{Z} = \mathbb{Z} \xrightarrow{0} DG\left(\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}\right) = DG\left(\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}\right)$$

for both substitutions (see Corollary 5.20 in [8]). Hence by Theorem 4.2, the C^*-algebras $\mathcal{O}_{\underline{\mathsf{X}}_\tau}$ and $\mathcal{O}_{\underline{\mathsf{X}}_\upsilon}$ are stably isomorphic. However, the *configuration data* (see [5]) are different, namely

respectively, and since this is a flow invariant, the shift spaces $\underline{\mathsf{X}}_\tau$ and $\underline{\mathsf{X}}_\upsilon$ are not flow equivalent. ☐

4.2. Graph algebras. A completely independent application is presented by the first named author and Tomforde in a forthcoming paper ([13]) and we sketch it here. By the work of many hands (see [35] and the references therein) a *graph C^*-algebra* may be associated to any directed graph (countable, but possibly infinite). When such C^*-algebras are simple, they are always nuclear and in the bootstrap class $\mathcal{N}$, and either purely infinite or AF. They are hence, by appealing to either [19] or [15], classifiable by the Elliott invariant. Our first main result Theorem 3.11 applies to prove the following:

Theorem 4.4. *([13]) Let A and A' be unital graph algebras with exactly one nontrivial ideal I and I', respectively. Then $A \otimes \mathcal{K} \cong A' \otimes \mathcal{K}$ if and only if there exists an isomorphism $(\eta_*, \alpha_*, \beta_*)$ between the six term exact sequences associated with A and A' such that η_0 and α_0 are positive.*

*S*ketch of proof. Known structure results for graph C^*-algebras establish that all of $I, I', A/I$ and A'/I' are themselves graph C^*-algebras, but to invoke Theorem 3.11 we furthermore need to know that I and I' are stable and of the form $J \otimes \mathcal{K}$ for J a unital graph algebra. This is a nontrivial result which is established in [13].

With this we can choose as $\mathcal{C}$ in Theorem 3.11 the union of the set of unital Kirchberg algebras with UCT and the unital simple AF-algebras. Then it is easy to check that properties (1)-(4) in Definition 3.2 are satisfied, as is the corona factorization property. Since A/I is simple, and A is unital, the Busby invariant of $0 \to I \to A \to A/I \to 0$ will satisfy the fullness condition. By Proposition 3.15, it follows that $0 \to I \otimes \mathcal{K} \to A \otimes \mathcal{K} \to (A/I) \otimes \mathcal{K} \to 0$ also satisfies the needed fullness condition (and likewise for A'). □

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Appendix D

Non-splitting in Kirchberg's ideal-related KK-theory

The article in this appendix has been accepted for publication in Canadian Mathematical Bulletin. It is very closely related to Chapter 6 of the thesis. In principle Chapter 6 and this article can be read independently, but — since this article is the main motivation for Chapter 6 — it is a good idea to read the article before one reads Chapter 6.

NON-SPLITTING IN KIRCHBERG'S IDEAL-RELATED KK-THEORY

SØREN EILERS, GUNNAR RESTORFF, AND EFREN RUIZ

Abstract. A universal coefficient theorem in the setting of Kirchberg's ideal-related KK-theory was obtained in the fundamental case of a C^*-algebra with one specified ideal by Bonkat in [1] and proved there to split, unnaturally, under certain conditions. Employing certain K-theoretical information derivable from the given operator algebras in a way introduced here, we shall demonstrate that Bonkat's UCT does not split in general. Related methods lead to information on the complexity of the K-theory which must be used to classify $*$-isomorphisms for purely infinite C^*-algebras with one non-trivial ideal.

1. Introduction

The KK-theory introduced by Kasparov ([9]) is one of the most important tools in the theory of classification of C^*-algebras, of use especially for simple C^*-algebras. Recently, Kirchberg has developed the socalled ideal-related KK-theory — a generalisation of Kasparov's KK-theory which takes into account the ideal structure of the algebras considered — and obtained strong isomorphism theorems for stable, nuclear, separable, strongly purely infinite C^*-algebras ([10]). The results obtained by Kirchberg establish ideal-related KK-theory as an essential tool in the classification theory of non-simple C^*-algebras.

KK-theory is a bivariant functor; to obtain a real classification result one needs a univariant classification functor instead. For ordinary KK-theory this is obtained (within the bootstrap category) by invoking the Universal Coefficient Theorem (UCT) of Rosenberg and Schochet:

Theorem 1 (Rosenberg-Schochet's UCT, [15]). *Let A and B be separable C^*-algebras in the bootstrap category $\mathcal{N}$. Then there is a short exact sequence*

$$\mathrm{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(SB)) \hookrightarrow KK(A,B) \overset{\gamma}{\twoheadrightarrow} \mathrm{Hom}_{\mathbb{Z}}(K_*(A), K_*(B))$$

(here $K_*(-)$ denotes the graded group $K_0(-)\oplus K_1(-)$). The sequence is natural in both A and B, and splits (unnaturally, in general). Moreover, an element x in $KK(A,B)$ is invertible if and only if $\gamma(x)$ is an isomorphism.

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This UCT allows us to turn isomorphism results (such as Kirchberg-Phillips' theorem [11]) into strong classification theorems. Moreover, using the splitting, it allows us to determine completely the additive structure of the KK-groups.

To transform Kirchberg's general result into a strong classification theorem, one would need a UCT for ideal-related KK-theory. This was achieved by Bonkat ([1]) in the special case where the specified ideal structure is just a single ideal. Progress into more general cases with finitely many ideals has recently been announced by Mayer-Nest and by the second named author, but in this paper we will only consider the case with one specified ideal:

Theorem 2 (Bonkat's UCT, [1, Satz 7.5.3, Satz 7.7.1, and Proposition 7.7.2])**.** *Let e_1 and e_2 be extensions of separable, nuclear C^*-algebras in the bootstrap category $\mathcal{N}$. Then there is a short exact sequence*

$$\mathrm{Ext}^1_{\mathcal{Z}_6}(K_{\mathrm{six}}(e_1), K_{\mathrm{six}}(Se_2)) \hookrightarrow KK_{\mathcal{E}}(e_1, e_2) \xrightarrow{\Gamma} \mathrm{Hom}_{\mathcal{Z}_6}(K_{\mathrm{six}}(e_1), K_{\mathrm{six}}(e_2))$$

(here $K_{\mathrm{six}}(-)$ is the standard cyclic six term exact sequence, $\mathcal{Z}_6$ is the category of cyclic six term chain complexes, and Se denotes the extension obtained by tensoring all the C^*-algebras in the extension e with $C_0(0,1)$). The sequence is natural in both e_1 and e_2. Moreover, an element $x \in KK_{\mathcal{E}}(e_1, e_2)$ is invertible if and only if $\Gamma(x)$ is an isomorphism.

Bonkat leaves open the question of whether this UCT splits in general. We prove here that this is not always the case, even in the fundamental case considered by Bonkat (see Proposition 6(1) below).

This observation tells us — in contrast to the ordinary KK-theory — that we cannot, in general, completely determine the additive structure of $KK_{\mathcal{E}}$ just by using the UCT. It is comforting to note, as may be inferred from the results in [14], [6] and [13], that this has only marginal impact on the usefulness of Bonkat's result in the context of classification of e.g. the C^*-algebras considered by Kirchberg. But as we shall see it has several repercussions concerning the classification of homomorphisms and automorphisms of such C^*-algebras, and opens an intriguing discussion — which it is our ambition to close elsewhere ([7]) in the important special case of Cuntz-Krieger algebras satisfying condition (II) — on the nature of an invariant classifying such morphisms.

Indeed, examples abound in classification theory in which the invariant needed to classify automorphims up to approximate unitary equivalence on a certain class of C^*-algebras is more complicated than the classifying invariant for the algebras themselves. For instance, even though $K_*(-)$ is a classifying invariant for stable Kirchberg algebras (i.e. nuclear, separable, simple, purely infinite C^*-algebras) one needs to turn to *total* K-theory — the collection of $K_*(-)$ and all torsion coefficient groups $K_*(-;\mathbb{Z}_n)$ — in order to obtain exactness of

$$\{1\} \to \overline{\mathrm{Inn}}(A) \to \mathrm{Aut}(A) \to \mathrm{Aut}_\Lambda(\underline{K}(A)) \to \{1\}, \tag{1.1}$$

where $\overline{\mathrm{Inn}}(A)$ is the group of automorphisms of A that are approximately unitarily equivalent to id_A and the subscript Λ indicates that the group isomorphism on $\underline{K}(A)$ must commute with all the natural *Bockstein* operations.

The appearance of total K-theory in (1.1) is explained by the Universal Multicoefficient Theorem obtained by Dadarlat and Loring in [4]:

Theorem 3 (Dadarlat-Loring's UMCT, [4])**.** *Let A and B be separable C^*-algebras in the bootstrap category $\mathcal{N}$. Then there is a short exact sequence*

$$\mathrm{Pext}^1_{\mathbb{Z}}(K_*(A), K_*(SB)) \hookrightarrow KK(A,B) \twoheadrightarrow \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$$

(here $\mathrm{Pext}^1_{\mathbb{Z}}$ denotes the subgroup of $\mathrm{Ext}^1_{\mathbb{Z}}$ consisting of pure extensions, and Hom_Λ denotes the group of homomorphisms respecting the Bockstein operations). The sequence is natural in both A and B, and an element x in $KK(A,B)$ is invertible if and only if the induced element is an isomorphism. Moreover, $\mathrm{Pext}^1_{\mathbb{Z}}(K_*(A), K_*(SB))$ is zero whenever the K-theory of A is finitely generated.

Dadarlat has pointed out to us that although [4] states that the UMCT splits in general, this is not true. The problem can be traced to one in [16], cf. [17] and [18].

In the stably finite case, as exemplified by stable real rank zero AD algebras, the UMCT leads to exactness of

$$\{1\} \to \overline{\mathrm{Inn}}(A) \to \mathrm{Aut}(A) \to \mathrm{Aut}_{\Lambda,+}(\underline{K}(A)) \to \{1\}, \tag{1.2}$$

in which the subscript "+" indicates the presence of positivity conditions (see [4] for details). Noting the way the usage of a six term exact sequence in [14] parallels the usage of positivity in the stably finite case (cf. [3]) it is natural to speculate (as indeed the first named author did at The First Abel Symposium, cf. [6]) that by combining all coefficient six term exact sequences into an invariant $\underline{K}_{\mathrm{six}}(-)$ one obtains an exact sequence of the form

$$\{1\} \longrightarrow \overline{\mathrm{Inn}}(e) \longrightarrow \mathrm{Aut}(e) \longrightarrow \mathrm{Aut}_\Lambda(\underline{K}_{\mathrm{six}}(e)) \longrightarrow \{1\}, \tag{1.3}$$

and to search for a corresponding UMCT along the lines of Theorem 3.

This sequence is clearly a chain complex, but as we will see, the natural map from $KK_{\mathcal{E}}(e_1, e_2)$ to $\mathrm{Hom}_\Lambda(\underline{K}_{\mathrm{six}}(e_1), \underline{K}_{\mathrm{six}}(e_2))$ is not injective nor is it surjective in general for extensions e_1 and e_2 with finitely generated K-theory (see Proposition 6(2),(3)), and we will give an example of an extension of stable Kirchberg algebras in the bootstrap category $\mathcal{N}$ with finitely generated K-theory, such that (1.3) is only exact at $\overline{\mathrm{Inn}}(e)$, telling us in unmistakable terms that this is the wrong invariant to use.

Our methods are based on computations related to a class of extensions which, we believe, should be thought of as a substitute for the total K-theory of relevance in the classification of, e.g., non-simple, stably finite C^*-algebras with real rank zero. We shall undertake a more systematic study of these objects elsewhere, and show there how they may be employed to the task of computing Kirchberg's groups $KK_{\mathcal{E}}(-,-)$.

2. Preliminaries

We first set up some notation that will be used throughout.

Definition 4. Let $n \geq 2$ be an integer and denote the non-unital dimension drop algebra by $\mathbb{I}_n^0 = \{f \in C_0((0,1], \mathsf{M}_n) : f(1) \in \mathbb{C}1_{\mathsf{M}_n}\}$. Then $\mathbb{I}_n^0$ fits into the short exact sequence

$$\mathfrak{e}_{n,0} : S\mathsf{M}_n \hookrightarrow \mathbb{I}_n^0 \twoheadrightarrow \mathbb{C}.$$

It is well known that $K_0(\mathbb{I}_n^0) = 0$ and $K_1(\mathbb{I}_n^0) = \mathbb{Z}_n$, where $\mathbb{Z}_n$ denotes the cyclic abelian group with n elements.

Let $\mathfrak{e}_{n,1} : S\mathbb{C} \hookrightarrow \mathbb{I}_n^1 \twoheadrightarrow \mathbb{I}_n^0$ be the extension obtained from the mapping cone of the map $\mathbb{I}_n^0 \twoheadrightarrow \mathbb{C}$. The diagram

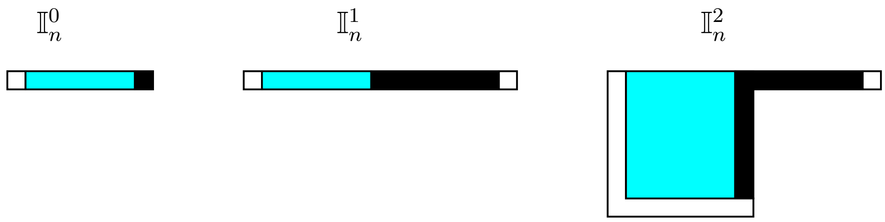
$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & S\mathbb{C} \arrow[r, equal] \arrow[d, hook] & S\mathbb{C} \arrow[d, hook] \\
S\mathsf{M}_n \arrow[r, hook] \arrow[d, equal] & \mathbb{I}_n^1 \arrow[r, two heads] \arrow[d, two heads] & C\mathbb{C} \arrow[d, two heads] \\
S\mathsf{M}_n \arrow[r, hook] & \mathbb{I}_n^0 \arrow[r, two heads] & \mathbb{C}
\end{tikzcd} \tag{2.1}$$

is commutative and the columns and rows are short exact sequences. Note that the $*$-homomorphism from $S\mathsf{M}_n$ to $\mathbb{I}_n^1$ induces a KK-equivalence.

Let $\mathfrak{e}_{n,2} : S\mathbb{I}_n^0 \hookrightarrow \mathbb{I}_n^2 \twoheadrightarrow \mathbb{I}_n^1$ be the extension obtained from the mapping cone of the canonical map $\mathbb{I}_n^1 \twoheadrightarrow \mathbb{I}_n^0$. Then the diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & S\mathbb{I}_n^0 \arrow[r, equal] \arrow[d, hook] & S\mathbb{I}_n^0 \arrow[d, hook] \\
S\mathbb{C} \arrow[r, hook] \arrow[d, equal] & \mathbb{I}_n^2 \arrow[r, two heads] \arrow[d, two heads] & C\mathbb{I}_n^0 \arrow[d, two heads] \\
S\mathbb{C} \arrow[r, hook] & \mathbb{I}_n^1 \arrow[r, two heads] & \mathbb{I}_n^0
\end{tikzcd} \tag{2.2}$$

is commutative and the columns and rows are short exact sequences. Note that the $*$-homomorphism from $S\mathbb{C}$ to $\mathbb{I}_n^2$ induces a KK-equivalence. This implies, with a little more work, that we get no new K-theoretical information from considering objects $\mathbb{I}_n^k$ or $\mathfrak{e}_{n,k}$ for $k > 2$. Note also that the C^*-algebras $\mathbb{I}_n^0, \mathbb{I}_n^1$ and $\mathbb{I}_n^2$ are NCCW complexes of dimension 1, 1, and 2, respectively, in the sense of [5]. See Figure 2.1.



[Fibre legends: □=0, ■=M_n, ■=$\mathbb{C}$]

FIGURE 2.1. NCCW structure of $\mathbb{I}_n^i$

Let $e : A_0 \hookrightarrow A_1 \twoheadrightarrow A_2$ be an extension of C^*-algebras. We have an "ideal-related K-theory with $\mathbb{Z}_n$-coefficients" denoted by $K_{\mathrm{six}}(e;\mathbb{Z}_n)$. More precisely, $K_{\mathrm{six}}(e;\mathbb{Z}_n)$ denotes the six term exact sequence

$$\begin{tikzcd} K_0(A_0;\mathbb{Z}_n) \arrow[r] & K_0(A_1;\mathbb{Z}_n) \arrow[r] & K_0(A_2;\mathbb{Z}_n) \arrow[d] \\ K_1(A_2;\mathbb{Z}_n) \arrow[u] & K_1(A_1;\mathbb{Z}_n) \arrow[l] & K_1(A_0;\mathbb{Z}_n) \arrow[l] \end{tikzcd}$$

obtained by applying the covariant functor $KK^*(\mathbb{I}_n^0, -)$ to the extension e.

Let us denote the standard six term exact sequence in K-theory by $K_{\mathrm{six}}(e)$. The collection consisting of $K_{\mathrm{six}}(e)$ and $K_{\mathrm{six}}(e;\mathbb{Z}_n)$ for all $n \geq 2$ will be denoted by $\underline{K}_{\mathrm{six}}(e)$. A homomorphism from $\underline{K}_{\mathrm{six}}(e_1)$ to $\underline{K}_{\mathrm{six}}(e_2)$ consists of a morphism from $K_{\mathrm{six}}(e_1)$ to $K_{\mathrm{six}}(e_2)$ along with an infinite family of morphisms from $K_{\mathrm{six}}(e_1;\mathbb{Z}_n)$ to $K_{\mathrm{six}}(e_2;\mathbb{Z}_n)$ respecting the Bockstein operations in Λ. We will denote the group of homomorphisms from $\underline{K}_{\mathrm{six}}(e_1)$ to $\underline{K}_{\mathrm{six}}(e_2)$ by $\mathrm{Hom}_\Lambda(\underline{K}_{\mathrm{six}}(e_1), \underline{K}_{\mathrm{six}}(e_2))$. We turn $\underline{K}_{\mathrm{six}}$ into a functor in the obvious way.

Lemma 5. *There is a natural homomorphism*

$$\Gamma_{e_1,e_2} : KK_{\mathcal{E}}(e_1, e_2) \longrightarrow \mathrm{Hom}_\Lambda(\underline{K}_{\mathrm{six}}(e_1), \underline{K}_{\mathrm{six}}(e_2)).$$

Proof. A computation shows that $K_{\mathrm{six}}(-;\mathbb{Z}_n)$ is a stable, homotopy invariant, split exact functor since KK satisfies these properties. Therefore, $\underline{K}_{\mathrm{six}}(-)$ is a stable, homotopy invariant, split exact functor. Hence, for every fixed extension e_1 of C^*-algebras, $\mathrm{Hom}_\Lambda(\underline{K}_{\mathrm{six}}(e_1), \underline{K}_{\mathrm{six}}(-))$ is a stable, homotopy invariant, split exact functor. By Satz 3.5.9 of [1], we have a natural transformation $\Gamma_{e_1,-}$ from $KK_{\mathcal{E}}(e_1,-)$ to $\mathrm{Hom}_\Lambda(\underline{K}_{\mathrm{six}}(e_1), \underline{K}_{\mathrm{six}}(-))$ such that $\Gamma_{e_1,-}$ sends $[\mathrm{id}_{e_1}]$ to $\underline{K}_{\mathrm{six}}(\mathrm{id}_{e_1})$. Arguing as in the proof of Lemma 3.2 of [8], we have that

$$\Gamma_{e_1,e_2} : KK_{\mathcal{E}}(e_1, e_2) \longrightarrow \mathrm{Hom}_\Lambda(\underline{K}_{\mathrm{six}}(e_1), \underline{K}_{\mathrm{six}}(e_2))$$

is a group homomorphism. □

Another collection of groups that we will use in this paper is the following: for each $n \geq 2$, set

$$K_{\mathcal{E}}(e; \mathbb{Z}_n) = \bigoplus_{i=0}^{2} \left(KK^*_{\mathcal{E}}(\mathfrak{e}_{n,i}, e) \oplus KK^*(\mathbb{I}^0_n, A_i) \oplus KK^*(\mathbb{C}, A_i) \right).$$

3. Examples

Accompanied with the groups $KK^*_{\mathcal{E}}(\mathfrak{e}_{n,i}, e)$ are naturally defined diagrams, which will be systematically described in a forthcoming paper. For now, we will use these groups to show the following:

Proposition 6. *(1) The UCT of Bonkat (Theorem 2) does not split in general.*

(2) There exist e_1 and e_2 extensions of separable, nuclear C^*-algebras in the bootstrap category $\mathcal{N}$ of Rosenberg and Schochet [15] *such that the six term exact sequence of K-groups associated to e_1 is finitely generated and*

$$\Gamma_{e_1,e_2} : KK_{\mathcal{E}}(e_1, e_2) \longrightarrow \mathrm{Hom}_{\Lambda}(\underline{K}_{\mathrm{six}}(e_1), \underline{K}_{\mathrm{six}}(e_2))$$

is not injective.

(3) There exist e_1 and e_2 extensions of separable, nuclear C^*-algebras in the bootstrap category $\mathcal{N}$ of Rosenberg and Schochet [15] *such that the six term exact sequence of K-groups associated to e_1 is finitely generated and*

$$\Gamma_{e_1,e_2} : KK_{\mathcal{E}}(e_1, e_2) \longrightarrow \mathrm{Hom}_{\Lambda}(\underline{K}_{\mathrm{six}}(e_1), \underline{K}_{\mathrm{six}}(e_2))$$

is not surjective.

The proposition will be proved through a series of examples. The following example shows that the UCT of Bonkat does not split in general. Also it shows that there exist extensions e_1 and e_2 of separable, nuclear C^*-algebras in $\mathcal{N}$ with finitely generated K-theory, such that Γ_{e_1,e_2} is not injective.

Example 7. Let n be a prime number. By Korollar 7.1.6 of [1], we have that

$$\mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow KK^1_{\mathcal{E}}(\mathfrak{e}_{n,0}, \mathfrak{e}_{n,1}) \longrightarrow 0$$

is an exact sequence. Therefore, $KK^1_{\mathcal{E}}(\mathfrak{e}_{n,0}, \mathfrak{e}_{n,1})$ is a cyclic group. By Korollar 7.1.6 of [1], $KK^1_{\mathcal{E}}(\mathfrak{e}_{n,0}, \mathfrak{e}_{n,1})$ fits into the following exact sequence

$$0 \longrightarrow \mathbb{Z}_n \longrightarrow KK^1_{\mathcal{E}}(\mathfrak{e}_{n,0}, \mathfrak{e}_{n,1}) \longrightarrow \mathbb{Z}_n \longrightarrow 0.$$

So, $KK^1_{\mathcal{E}}(\mathfrak{e}_{n,0}, \mathfrak{e}_{n,1})$ is isomorphic to $\mathbb{Z}_{n^2}$.

An easy computation shows that $\mathrm{Hom}(K_{\mathrm{six}}(\mathfrak{e}_{n,0}), K_{\mathrm{six}}(S\mathfrak{e}_{n,1}))$ is isomorphic to $\mathbb{Z}_n$. Using this fact and the fact that $KK_{\mathcal{E}}(\mathfrak{e}_{n,0}, S\mathfrak{e}_{n,1}) \cong KK^1_{\mathcal{E}}(\mathfrak{e}_{n,0}, \mathfrak{e}_{n,1})$ is $\mathbb{Z}_{n^2}$, we immediately see that the UCT of Bonkat does not split in this case.

We would like to also point out another consequence of this example. Since n is prime and $\mathrm{Ext}^1_{\mathcal{Z}_6}(K_{\mathrm{six}}(\mathfrak{e}_{n,0}), K_{\mathrm{six}}(\mathfrak{e}_{n,1}))$ injects into a proper subgroup of

$KK^1_{\mathcal{E}}(\mathfrak{e}_{n,0}, \mathfrak{e}_{n,1})$, we have that $\operatorname{Ext}^1_{\mathcal{Z}_6}(K_{\mathrm{six}}(\mathfrak{e}_{n,0}), K_{\mathrm{six}}(\mathfrak{e}_{n,1}))$ is isomorphic to $\mathbb{Z}_n$. Therefore, n annihilates all torsional K-theory information but n does not annihilate the torsion group $KK^1_{\mathcal{E}}(\mathfrak{e}_{n,0}, \mathfrak{e}_{n,1})$.

We will now show that the natural map $\Gamma_{\mathfrak{e}_{n,0}, S\mathfrak{e}_{n,1}}$ from $KK_{\mathcal{E}}(\mathfrak{e}_{n,0}, S\mathfrak{e}_{n,1})$ to $\operatorname{Hom}_\Lambda(\underline{K}_{\mathrm{six}}(\mathfrak{e}_{n,0}), \underline{K}_{\mathrm{six}}(S\mathfrak{e}_{n,1}))$ is not injective. Let $A_0 \hookrightarrow A_1 \twoheadrightarrow A_2$ and $B_0 \hookrightarrow B_1 \twoheadrightarrow B_2$ denote the extensions $\mathfrak{e}_{n,0}$ and $S\mathfrak{e}_{n,1}$, respectively. Note that the corresponding six term exact sequences are (isomorphic to)

$$
\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[d] \\
0 \arrow[u] & \mathbb{Z}_n \arrow[l] & \mathbb{Z} \arrow[l]
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z}_n \arrow[d] \\
0 \arrow[u] & 0 \arrow[l] & 0 \arrow[l]
\end{tikzcd},
$$

respectively. Using the UCT of Rosenberg and Schochet, a short computation shows that $n \prod_{i=0}^2 KK(A_i, B_i) = 0$. Since all the K-theory is finitely generated, we have by Dadarlat and Loring's UMCT that $\operatorname{Hom}_\Lambda(\underline{K}_{\mathrm{six}}(\mathfrak{e}_{n,0}), \underline{K}_{\mathrm{six}}(S\mathfrak{e}_{n,1}))$ is isomorphic to a subgroup of $\prod_{i=0}^2 KK(A_i, B_i)$. Since the latter group has no element of order n^2 and $KK_{\mathcal{E}}(\mathfrak{e}_{n,0}, S\mathfrak{e}_{n,1})$ is isomorphic to $\mathbb{Z}_{n^2}$, we have that $\Gamma_{\mathfrak{e}_{n,0}, S\mathfrak{e}_{n,1}}$ is not injective.

The above example also provides a counterexample to Satz 7.7.6 of [1]. The arguments in the proof of Satz 7.7.6 are correct but it appears that Bonkat overlooked the case were the six term exact sequences are of the form:

$$
\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & * \arrow[d] \\
0 \arrow[u] & * \arrow[l] & * \arrow[l]
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & 0 \arrow[d] \\
* \arrow[u] & * \arrow[l] & * \arrow[l]
\end{tikzcd}
$$

Our next example shows that there exist extensions e_1 and e_2 of separable, nuclear C^*-algebras in $\mathcal{N}$ with finitely generated K-groups, such that Γ_{e_1,e_2} is not surjective.

Example 8. Let n be a prime number. Consider the following short exact sequences of extensions:

$$
\begin{tikzcd}
S\mathbb{C} \arrow[r, equal] \arrow[d, equal] & S\mathbb{C} \arrow[r] \arrow[d, hook] & 0 \arrow[d] \\
S\mathbb{C} \arrow[r, hook] \arrow[d] & \mathbb{I}_n^1 \arrow[r, two heads] \arrow[d, two heads] & \mathbb{I}_n^0 \arrow[d, equal] \\
0 \arrow[r] & \mathbb{I}_n^0 \arrow[r, equal] & \mathbb{I}_n^0
\end{tikzcd}
\tag{3.1}
$$

and

$$
\begin{tikzcd}
S\mathsf{M}_n \arrow[r, equal] \arrow[d, equal] & S\mathsf{M}_n \arrow[r] \arrow[d, hook] & 0 \arrow[d] \\
S\mathsf{M}_n \arrow[r, hook] \arrow[d] & \mathbb{I}_n^0 \arrow[r, two heads] \arrow[d, two heads] & \mathbb{C} \arrow[d, equal] \\
0 \arrow[r] & \mathbb{C} \arrow[r, equal] & \mathbb{C}
\end{tikzcd}
\tag{3.2}
$$

By applying the bivariant functor $KK^*_{\mathcal{E}}(-,-)$ to the above exact sequences of extensions with (3.1) in the first variable and (3.2) in the second variable and by Lemma 7.1.5 of [1], we get that the diagram

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z}_n \arrow[r] \arrow[d] & \mathbb{Z}_n \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, "n"] & KK_{\mathcal{E}}(\mathfrak{e}_{n,1}, \mathfrak{e}_{n,0}) \arrow[r] \arrow[d] & \mathbb{Z}_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, equal] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z}_n \arrow[r] \arrow[d] & \mathbb{Z}_n \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

is commutative. By Korollar 3.4.6 of [1] the columns and rows of the above diagram are exact sequences. Therefore, we have that $KK_{\mathcal{E}}(\mathfrak{e}_{n,1}, \mathfrak{e}_{n,0})$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_n$.

A straightforward computation gives that $K_{\mathrm{six}}(\mathfrak{e}_{n,0})$ and $K_{\mathrm{six}}(\mathfrak{e}_{n,0}; \mathbb{Z}_m)$ are given by

$$
\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[d] \\
0 \arrow[u] & \mathbb{Z}_n \arrow[l] & \mathbb{Z} \arrow[l]
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}_{(m,n)} \arrow[r] & \mathbb{Z}_n \arrow[d] \\
0 \arrow[u] & \mathbb{Z}_{n/(m,n)} \arrow[l] & \mathbb{Z}_n \arrow[l]
\end{tikzcd}
$$

and similarly, for $\mathfrak{e}_{n,1}$

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[d] \\ \mathbb{Z}_n \arrow[u] & \mathbb{Z} \arrow[l] & \mathbb{Z} \arrow[l] \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}_{(n,m)} \arrow[d] \\ \mathbb{Z}_{n/(n,m)} \arrow[u] & \mathbb{Z}_n \arrow[l] & \mathbb{Z}_n \arrow[l] \end{tikzcd}$$

A map $K_{\mathrm{six}}(\mathfrak{e}_{n,1}) \oplus K_{\mathrm{six}}(\mathfrak{e}_{n,1};\mathbb{Z}_n) \to K_{\mathrm{six}}(\mathfrak{e}_{n,0}) \oplus K_{\mathrm{six}}(\mathfrak{e}_{n,0};\mathbb{Z}_n)$ is given by a 12-tuple

$$((0,0,0,x,a,0),(0,0,b,c,d,0))$$

where $x \in \mathbb{Z}$ and $a,b,c,d \in \mathbb{Z}_n$. To commute with the maps in the diagrams as well as the Bockstein maps of type ρ and β, we must have $d = a$ and $c = \overline{x}$, and straightforward computations show that this tuple extends uniquely to an element of $\mathrm{Hom}_\Lambda(\underline{K}_{\mathrm{six}}(\mathfrak{e}_{n,1}), \underline{K}_{\mathrm{six}}(\mathfrak{e}_{n,0}))$. Hence this group is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_n \oplus \mathbb{Z}_n$. Finally, note that no surjection $\mathbb{Z} \oplus \mathbb{Z}_n \to \mathbb{Z} \oplus \mathbb{Z}_n \oplus \mathbb{Z}_n$ exists.

Remark 9. The matrices

$$A = \begin{pmatrix} 1&1&0&0&0&0 \\ 1&1&1&0&0&0 \\ 0&1&1&0&0&0 \\ 0&0&0&1&1&1 \\ 0&0&0&1&1&1 \\ 1&0&0&1&1&1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1&1&1&0&0&0 \\ 1&1&1&0&0&0 \\ 1&1&1&0&0&0 \\ 0&0&0&1&1&0 \\ 0&0&0&1&1&1 \\ 1&0&0&0&1&1 \end{pmatrix}$$

satisfy condition (II) of Cuntz ([2]). Hence, the Cuntz-Krieger algebras $\mathcal{O}_A$ and $\mathcal{O}_B$ are purely infinite C^*-algebras and have exactly one non-trivial ideal. Using the Smith normal form and [12, Proposition 3.4] we see that the six term exact sequence corresponding to $\mathcal{O}_A$ and $\mathcal{O}_B$ are (isomorphic to) the sequences

$$\begin{tikzcd} \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z}_2 \arrow[d, "0"] \\ 0 \arrow[u] & \mathbb{Z} \arrow[l] & \mathbb{Z} \arrow[l, "\cong"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathbb{Z}_2 \arrow[r, "0"] & \mathbb{Z} \arrow[r, "\cong"] & \mathbb{Z} \arrow[d] \\ \mathbb{Z} \arrow[u] & \mathbb{Z} \arrow[l] & 0 \arrow[l] \end{tikzcd},$$

respectively. Using $KK_{\mathcal{E}}$-equivalent extensions, that $KK_{\mathcal{E}}$ is split exact, and arguments similar to Example 7, one easily shows that the natural map Γ_{e_1,e_2} is not injective for the extensions e_1 and e_2 corresponding to the Cuntz-Krieger algebras $\mathcal{O}_A$ and $\mathcal{O}_B$, respectively. Similar considerations on

$$C = \begin{pmatrix} 1&1&0&0&0&0 \\ 1&1&1&0&0&0 \\ 0&1&1&0&0&0 \\ 0&0&0&1&1&1 \\ 0&0&0&1&1&1 \\ 1&0&0&1&1&1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 1&1&0&0&0&0 \\ 1&1&1&0&0&0 \\ 0&1&1&0&0&0 \\ 0&0&0&1&1&0 \\ 0&0&0&1&1&1 \\ 1&0&0&0&1&1 \end{pmatrix}$$

yield a version of Example 8 in the realm of Cuntz-Krieger algebras.

One may ask if Γ_{e_1,e_2} is ever surjective and the answer is yes. If e_1 is an extension of separable, nuclear C^*-algebra in $\mathcal{N}$ such that the K-groups of $K_{\mathrm{six}}(e_1)$ are torsion free, then $\mathrm{Hom}_\Lambda(\underline{K}_{\mathrm{six}}(e_1), \underline{K}_{\mathrm{six}}(e_2))$ is naturally isomorphic to $\mathrm{Hom}_{\mathcal{Z}_6}(K_{\mathrm{six}}(e_1), K_{\mathrm{six}}(e_2))$ such that the composition of Γ_{e_1,e_2} with this natural isomorphism is the natural map from $KK_{\mathcal{E}}(e_1, e_2)$ to $\mathrm{Hom}_{\mathcal{Z}_6}(K_{\mathrm{six}}(e_1), K_{\mathrm{six}}(e_2))$. Hence, by the UCT of Bonkat, we have that Γ_{e_1,e_2} is surjective.

4. Automorphisms of extensions of Kirchberg algebras

The class $\mathcal{R}$ of C^*-algebras considered by Rørdam in [14] consists of all C^*-algebras A_1 fitting in an essential extension $e\colon A_0 \hookrightarrow A_1 \twoheadrightarrow A_2$ where A_0 and A_2 are Kirchberg algebras in $\mathcal{N}$ (with A_0 necessarily being stable). For convenience we shall often identify e and A_1 in this setting, as indeed we can without risk of confusion. As explained in [14] one needs to consider three distinct cases: (1) A_1 is stable; (2) A_1 is unital; and (3) A_1 is neither stable nor unital.

A functor F is called a *classification functor*, if $A \cong B \Leftrightarrow F(A) \cong F(B)$ (for all algebras A and B in the class considered). Such a functor F is called a *strong classification functor* if every isomorphism from $F(A)$ to $F(B)$ is induced by an isomorphism from A to B (for all algebras A and B in the class considered).

Rørdam in [14] showed K_{six} to be a classification functor for stable algebras in $\mathcal{R}$. More recently, the authors in [6] and [13] showed that K_{six} (respectively K_{six} together with the class of the unit) is a strong classification functor for stable (respectively unital) algebras in $\mathcal{R}$. Moreover, they also showed that K_{six} is a classification functor for non-stable, non-unital algebras in $\mathcal{R}$.

In this section we will address some questions regarding the automorphism group of e, where e is in $\mathcal{R}$. If $e : A_0 \hookrightarrow A_1 \twoheadrightarrow A_2$ is an essential extension of separable C^*-algebas, then an automorphism of e is a triple (ϕ_0, ϕ_1, ϕ_2) such that ϕ_i is an automorphism of A_i and the diagram

$$\begin{tikzcd} A_0 \arrow[r, hook] \arrow[d, "\phi_0"] & A_1 \arrow[r, two heads] \arrow[d, "\phi_1"] & A_2 \arrow[d, "\phi_2"] \\ A_0 \arrow[r, hook] & A_1 \arrow[r, two heads] & A_2 \end{tikzcd}$$

is commutative. We denote the group of automorphisms of e by $\mathrm{Aut}(e)$. If A_0 and A_2 are simple C^*-algebras, then $\mathrm{Aut}(e)$ and $\mathrm{Aut}(A_1)$ are canonically isomorphic. Two automorphisms (ϕ_0, ϕ_1, ϕ_2), (ψ_0, ψ_1, ψ_2) of e are said to be approximately unitarily equivalent if ϕ_1 and ψ_1 are approximately unitarily equivalent. A consequence of Kirchberg's results [10] is that $KK_{\mathcal{E}}(e, e)$ classifies automorphisms of stable algebras in $\mathcal{R}$.

In [6] the first and second named authors asked whether the canonical map from $\mathrm{Aut}(e)$ to $\mathrm{Aut}_\Lambda(\underline{K}_{\mathrm{six}}(e))$ was surjective, cf. (1.3). We answer this in the negative as follows:

Proposition 10. *There is a C^*-algebra $e \in \mathcal{R}$ with finitely generated K-theory such that (1.3) is exact only at*

$$\{1\} \longrightarrow \overline{\mathrm{Inn}}(e) \longrightarrow \mathrm{Aut}(e)$$

Before proving the above proposition we first need to set up some notation. For ϕ in $\mathrm{Aut}(e)$, the element in $KK_{\mathcal{E}}(e,e)$ induced by ϕ will be denoted by $KK_{\mathcal{E}}(\phi)$ and the element in $\mathrm{Hom}_\Lambda(\underline{K}_{\mathrm{six}}(e), \underline{K}_{\mathrm{six}}(e))$ induced by ϕ will be denoted by $\underline{K}_{\mathrm{six}}(\phi)$. We will also need the following result.

Proposition 11. *Let e be any extension of separable C^*-algebras. Define*

$$\Lambda_{\mathfrak{e}_{n,i},e} : KK_{\mathcal{E}}(\mathfrak{e}_{n,i}, e) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(KK_{\mathcal{E}}(\mathfrak{e}_{n,i}, \mathfrak{e}_{n,i}), KK_{\mathcal{E}}(\mathfrak{e}_{n,i}, e))$$

by $\Lambda_{\mathfrak{e}_{n,i}}(x)(y) = y \times x$, where $y \times x$ is the generalized Kasparov product (see [1]*). Then $\Lambda_{\mathfrak{e}_{n,i},e}$ is an isomorphism for $i = 0, 1, 2$.*

Proof. We will only prove the case when $i = 0$, the other cases are similar. By the UCT of Bonkat one shows that $KK_{\mathcal{E}}(\mathfrak{e}_{n,0}, \mathfrak{e}_{n,0})$ is isomorphic to $\mathbb{Z}$ and is generated by $KK_{\mathcal{E}}(\mathrm{id}_{\mathfrak{e}_{n,0}})$. Therefore, if $\Lambda_{\mathfrak{e}_{n,0},e}(x) = 0$, then

$$x = KK_{\mathcal{E}}(\mathrm{id}_{\mathfrak{e}_{n,0}}) \times x = \Lambda_{\mathfrak{e}_{n,0},e}(x)(KK_{\mathcal{E}}(\mathrm{id}_{\mathfrak{e}_{n,0}})) = 0.$$

Hence, $\Lambda_{\mathfrak{e}_{n,0}}$ is injective. Suppose α is a homomorphism from $KK_{\mathcal{E}}(\mathfrak{e}_{n,0}, \mathfrak{e}_{n,0})$ to $KK_{\mathcal{E}}(\mathfrak{e}_{n,0}, e)$. Set $x = \alpha(KK_{\mathcal{E}}(\mathrm{id}_{\mathfrak{e}_{n,0}}))$. Then

$$\Lambda_{\mathfrak{e}_{n,0},e}(x)(KK_{\mathcal{E}}(\mathrm{id}_{\mathfrak{e}_{n,0}})) = x = \alpha(KK_{\mathcal{E}}(\mathrm{id}_{\mathfrak{e}_{n,0}})).$$

Therefore, $\Lambda_{\mathfrak{e}_{n,0},e}$ is surjective. $\square$

Proof of Proposition 10:

Set $e_1 = S\mathfrak{e}_{p,1} \oplus \mathfrak{e}_{p,1} \oplus \mathfrak{e}_{p,0}$ where p is a prime number. Let ι_1 be the embedding of $S\mathfrak{e}_{p,1}$ to e_1 and π_1 be the projection from e_1 to $\mathfrak{e}_{p,0}$. Note that

$$KK_{\mathcal{E}}(\iota_1) \times (-) : KK_{\mathcal{E}}(\mathfrak{e}_{p,0}, S\mathfrak{e}_{p,1}) \longrightarrow KK_{\mathcal{E}}(\mathfrak{e}_{p,0}, e_1)$$

and

$$(-) \times KK_{\mathcal{E}}(\pi_1) : KK_{\mathcal{E}}(\mathfrak{e}_{p,0}, e_1) \longrightarrow KK_{\mathcal{E}}(e_1, e_1)$$

are injective homomorphisms. Hence

$$\eta_1 = ((-) \times KK_{\mathcal{E}}(\pi_1)) \circ (KK_{\mathcal{E}}(\iota_1) \times (-))$$

is injective. Since $\Gamma_{-,-}$ is natural

$$\begin{array}{ccc} KK_{\mathcal{E}}(\mathfrak{e}_{p,0}, S\mathfrak{e}_{p,1}) & \xrightarrow{\eta_1} & KK_{\mathcal{E}}(e_1, e_1) \\ \downarrow{\scriptstyle \Gamma_{\mathfrak{e}_{p,0}, S\mathfrak{e}_{p,1}}} & & \downarrow{\scriptstyle \Gamma_{e_1,e_1}} \\ \mathrm{Hom}_\Lambda(\underline{K}_{\mathrm{six}}(\mathfrak{e}_{p,0}), \underline{K}_{\mathrm{six}}(S\mathfrak{e}_{p,1})) & \xrightarrow[\theta_1]{} & \mathrm{Hom}_\Lambda(\underline{K}_{\mathrm{six}}(e_1), \underline{K}_{\mathrm{six}}(e_1)) \end{array}$$

is commutative. By Example 7, $\Gamma_{\mathfrak{e}_{p,0}, S\mathfrak{e}_{p,1}}$ is not injective. Therefore, Γ_{e_1,e_1} is not injective.

Let π_2 be the projection of e_1 to $\mathfrak{e}_{p,0}$ and let ι_2 be the embedding of $\mathfrak{e}_{p,1}$ to $\mathfrak{e}_1$. Note that

$$KK_{\mathcal{E}}(\pi_2) \times (-) : KK_{\mathcal{E}}(e_1, e_1) \longrightarrow KK_{\mathcal{E}}(e_1, \mathfrak{e}_{p,0})$$

and

$$(-) \times KK_{\mathcal{E}}(\iota_2) : KK_{\mathcal{E}}(e_1, \mathfrak{e}_{p,0}) \longrightarrow KK_{\mathcal{E}}(\mathfrak{e}_{p,1}, \mathfrak{e}_{p,0})$$

are surjective homomorphisms. Therefore,

$$\eta_2 = ((-) \times KK_{\mathcal{E}}(\iota_2)) \circ (KK_{\mathcal{E}}(\pi_2) \times (-))$$

is surjective. Similarly, $\theta_2 = \underline{K}_{\mathrm{six}}(\iota_2) \circ \underline{K}_{\mathrm{six}}(\pi_2)$ is surjective. Since $\Gamma_{-,-}$ is natural,

$$\begin{tikzcd}
KK_{\mathcal{E}}(e_1, e_1) \arrow[r, "\eta_2"] \arrow[d, "\Gamma_{e_1,e_1}"'] & KK_{\mathcal{E}}(\mathfrak{e}_{p,1}, \mathfrak{e}_{p,0}) \arrow[d, "\Gamma_{\mathfrak{e}_{p,1},\mathfrak{e}_{p,0}}"'] \\
\mathrm{Hom}_{\Lambda}(\underline{K}_{\mathrm{six}}(e_1), \underline{K}_{\mathrm{six}}(e_1)) \arrow[r, "\theta_2"'] & \mathrm{Hom}_{\Lambda}(\underline{K}_{\mathrm{six}}(\mathfrak{e}_{p,1}), \underline{K}_{\mathrm{six}}(\mathfrak{e}_{p,0}))
\end{tikzcd}$$

is commutative. By Example 8, $\Gamma_{\mathfrak{e}_{p,1},\mathfrak{e}_{p,0}}$ is not surjective. Hence, Γ_{e_1,e_1} is not surjective.

We have just shown that Γ_{e_1,e_1} is neither surjective nor injective. By Proposition 5.4 of [14] there is a stable extension $e : A_0 \hookrightarrow A_1 \twoheadrightarrow A_2$ in $\mathcal{R}$ such that $K_{\mathrm{six}}(e) \cong K_{\mathrm{six}}(e_1)$. By the UCT of Bonkat, Theorem 2, we are able to lift this isomorphism to a $KK_{\mathcal{E}}$-equivalence. Therefore,

$$\begin{tikzcd}
KK_{\mathcal{E}}(e, e) \arrow[r, "\cong"] \arrow[d, "\Gamma_{e,e}"] & KK_{\mathcal{E}}(e_1, e_1) \arrow[d, "\Gamma_{e_1,e_1}"] \\
\mathrm{Hom}_{\Lambda}(\underline{K}_{\mathrm{six}}(e), \underline{K}_{\mathrm{six}}(e)) \arrow[r, "\cong"'] & \mathrm{Hom}_{\Lambda}(\underline{K}_{\mathrm{six}}(e_1), \underline{K}_{\mathrm{six}}(e_1))
\end{tikzcd}$$

is commutative. Hence, $\Gamma_{e,e}$ is neither injective nor surjective.

Denote the kernel of the surjective map from

$$\mathrm{Hom}_{\Lambda}(\underline{K}_{\mathrm{six}}(e), \underline{K}_{\mathrm{six}}(e)) \text{ to } \mathrm{Hom}_{\mathcal{Z}_6}(K_{\mathrm{six}}(e), K_{\mathrm{six}}(e))$$

by $\mathrm{Ext}_{\mathrm{six}}(K_{\mathrm{six}}(e), K_{\mathrm{six}}(Se))$. Note that if α is an element of $\mathrm{Hom}_{\Lambda}(\underline{K}_{\mathrm{six}}(e), \underline{K}_{\mathrm{six}}(e))$ such that $\alpha|_{K_{\mathrm{six}}(e)}$ is an isomorphism, then α is an isomorphism. Since $\Gamma_{e,e}$ is not surjective and

$$\begin{tikzcd}
\mathrm{Ext}_{\mathcal{Z}_6}(K_{\mathrm{six}}(e), K_{\mathrm{six}}(Se)) \arrow[r, hook] \arrow[d, "\Gamma_{e,e}"] & KK_{\mathcal{E}}(e, e) \arrow[r, two heads] \arrow[d, "\Gamma_{e,e}"] & \mathrm{Hom}_{\mathcal{Z}_6}(K_{\mathrm{six}}(e), K_{\mathrm{six}}(e)) \arrow[d, equal] \\
\mathrm{Ext}_{\mathrm{six}}(K_{\mathrm{six}}(e), K_{\mathrm{six}}(Se)) \arrow[r, hook] & \mathrm{Hom}_{\Lambda}(\underline{K}_{\mathrm{six}}(e), \underline{K}_{\mathrm{six}}(e)) \arrow[r, two heads] & \mathrm{Hom}_{\mathcal{Z}_6}(K_{\mathrm{six}}(e), K_{\mathrm{six}}(e))
\end{tikzcd} \tag{4.1}$$

is commutative, there exists β_1 in $\mathrm{Ext}_{\mathrm{six}}(K_{\mathrm{six}}(e), K_{\mathrm{six}}(Se))$ which is not in the image of $\Gamma_{e,e}$. Since $(\underline{K}_{\mathrm{six}}(\mathrm{id}_e) + \beta_1)|_{K_{\mathrm{six}}(e)} = \underline{K}_{\mathrm{six}}(\mathrm{id}_e)|_{K_{\mathrm{six}}(e)}$, we have that $\underline{K}_{\mathrm{six}}(\mathrm{id}_e) + \beta_1$ is an automorphism of $\underline{K}_{\mathrm{six}}(e)$. Since β_1 is not in the image of $\Gamma_{e,e}$, $\underline{K}_{\mathrm{six}}(\mathrm{id}_e) + \beta_1$ is not in the image of $\Gamma_{e,e}$. Hence, $\underline{K}_{\mathrm{six}}(\mathrm{id}_e) + \beta_1$ is an automorphism of $\underline{K}_{\mathrm{six}}(e)$ which does not lift to an automorphism of e. Consequently,

$$\mathrm{Aut}(e) \longrightarrow \mathrm{Aut}_\Lambda(\underline{K}_{\mathrm{six}}(e)) \longrightarrow \{1\}$$

is not exact.

Since the diagram in (4.1) is commutative and $\Gamma_{e,e}$ is not injective, there exists a nonzero element β_2 of $\mathrm{Ext}_{\mathcal{Z}_6}(K_{\mathrm{six}}(e), K_{\mathrm{six}}(Se))$ such that $\Gamma_{e,e}(\beta_2) = 0$. Therefore, $\beta_2 + KK_{\mathcal{E}}(\mathrm{id}_e)$ is an invertible element in $KK_{\mathcal{E}}(e,e)$ such that $\Gamma_{e,e}(\beta_2) + \underline{K}_{\mathrm{six}}(\mathrm{id}_e) = \underline{K}_{\mathrm{six}}(\mathrm{id}_e)$. By Folgerung 4.3 of [10], $\beta_2 + KK_{\mathcal{E}}(\mathrm{id}_e)$ lifts to an automorphism ϕ of e. So $\underline{K}_{\mathrm{six}}(\phi) = \underline{K}_{\mathrm{six}}(\mathrm{id}_e)$ in $\mathrm{Aut}_\Lambda(\underline{K}_{\mathrm{six}}(e))$.

Set

$$\begin{aligned} G &= \mathrm{Hom}(KK_{\mathcal{E}}(S\mathfrak{e}_{p,1}, e_1), KK_{\mathcal{E}}(S\mathfrak{e}_{p,1}, e_1)) \\ &\quad \oplus \left(\bigoplus_{i=0}^{2} \mathrm{Hom}(KK_{\mathcal{E}}(\mathfrak{e}_{p,i}, e_1), KK_{\mathcal{E}}(\mathfrak{e}_{p,i}, e_1)) \right) \\ H &= \mathrm{Hom}(KK_{\mathcal{E}}(S\mathfrak{e}_{p,1}, e), KK_{\mathcal{E}}(S\mathfrak{e}_{p,1}, e)) \\ &\quad \oplus \left(\bigoplus_{i=0}^{2} \mathrm{Hom}(KK_{\mathcal{E}}(\mathfrak{e}_{p,i}, e), KK_{\mathcal{E}}(\mathfrak{e}_{p,i}, e)) \right) \end{aligned}$$

Since e_1 is equal to $S\mathfrak{e}_{p,1} \oplus \mathfrak{e}_{p,1} \oplus \mathfrak{e}_{p,0}$, by Proposition 11 the map from $KK_{\mathcal{E}}(e_1, e_1)$ to G given by $x \mapsto (-) \times x$ is an isomorphism. Hence, the map from $KK_{\mathcal{E}}(e,e)$ to H given by $x \mapsto (-) \times x$ is an isomorphism. A computation shows that if ϕ is in $\overline{\mathrm{Inn}}(e)$, then ϕ induces the identity element in H. Therefore, ϕ is not approximately inner. We have just shown that

$$\overline{\mathrm{Inn}}(e) \longrightarrow \mathrm{Aut}(e) \longrightarrow \mathrm{Aut}_\Lambda(\underline{K}_{\mathrm{six}}(e))$$

is not exact. □

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