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Projections and residues on manifolds with boundary

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Abstract

It is a well-known result that the noncommutative residue of a pseudodifferential projection is zero on a compact manifold without boundary. Equivalently, the value of the zeta-function of P at zero, $\zeta_\theta(P, 0)$, is independent of θ for any elliptic operator P. Here θ denotes the angle of a ray where the resolvent of P has minimal growth.

In this thesis, we consider the analogous questions on a compact manifold *with* boundary. We show that the noncommutative residue is zero for any projection in Boutet de Monvel's calculus of pseudodifferential boundary problems.

For an elliptic boundary problem $\{P_+ + G, T\}$, with the corresponding realization $B = (P + G)_T$, we define the sectorial projection $\Pi_{\theta,\varphi}(B)$ and the residue of this projection. We discuss whether this residue is always zero, through various analyses of the structure of the projection. The question is interesting since $\zeta_\theta(B, 0)$ is independent of θ exactly when the residues of the corresponding sectorial projections are zero; in particular this holds when the projections are in Boutet de Monvel's calculus. This happens in certain cases, but we also give examples where the projections lie outside the calculus.

Resumé

Det er et velkendt resultat at det ikke-kommutative residuum af en pseudodifferentiel projektion er nul på en kompakt mangfoldighed uden rand. Et hermed ækvivalent udsagn er, at zeta-værdien af P i nul, $\zeta_\theta(P, 0)$, er uafhængig af θ for enhver elliptisk operator P. Her betegner θ vinklen for en stråle hvor resolventen for P har minimal vækst.

I denne afhandling betragter vi de tilsvarende problemstillinger på en kompakt mangfoldighed *med* rand. Det vises at det ikke-kommutative residuum er nul for enhver projektion i Boutet de Monvels kalkyle af pseudodifferentielle randværdiproblemer.

For et elliptisk randværdiproblem $\{P_+ + G, T\}$, med den tilhørende realisation $B = (P + G)_T$, definerer vi den sektorielle projektion $\Pi_{\theta,\varphi}(B)$ og dennes residuum. Vi diskuterer hvorvidt residuet altid er nul, gennem forskellige analyser af projektionens struktur. Dette spørgsmål er interessant, da $\zeta_\theta(B, 0)$ er uafhængig af θ netop når residuerne af de tilhørende sektorielle projektioner er nul; specielt gælder dette altså når projektionerne ligger i Boutet de Monvels kalkyle. Det forekommer i visse tilfælde, men vi giver også eksempler hvor projektionerne ligger uden for kalkylen.

Contents

Preface

This text constitutes my thesis for the Ph.D. degree in mathematics at the University of Copenhagen.

My time as a Ph.D. student, November 2005 to October 2008, has primarily been spent focusing on the noncommutative residue of sectorial projections for boundary problems; the main ingredients of my work are the two included articles *Logarithms and sectorial projections for elliptic boundary problems* [GG08], coauthored with Gerd Grubb, and *Noncommutative residue of projections in Boutet de Monvel's algebra* [Gaa07].

The article [GG08] has been accepted for publication in Mathematica Scandinavica and appears here in its final form; [Gaa07] has been accepted by Journal of Noncommutative Geometry, but may be revised before publication.

In addition to the articles, quite some time has been used on the work described in the final chapter, unfortunately not reaching results of a sufficiently definitive form for publication.

I would like to thank, first and foremost, my advisor, Gerd Grubb, for her help and support and for the numerous valuably discussions we have had. Also, I would like to thank Rafe Mazzeo, the faculty and the graduate students of the Department of Mathematics at Stanford University, where I spent the spring of 2006, and Elmar Schrohe and the people of the Institut für Analysis at Leibniz Universität Hannover, where I spent part of the spring 2008. Finally, I thank the Faculty of Science and the SNF Center in Noncommutative Geometry for my Ph.D. stipend, my office mate Jonas B. Rasmussen for copy-editing the thesis, and the math students at the University of Copenhagen in general.

Copenhagen, October 2008

Anders Gaarde

Outline

As mentioned, the main work lies in the two articles [GG08] and [Gaa07]. They appear at the end of the present text. Besides these papers, the thesis is divided into three parts:

1. Chapters 1 and 2 give a short introduction to the subject at hand: zeta- and eta-functions of elliptic operators, noncommutative residue, and trace expansions. Chapter 1 concerns operators on closed manifolds, while Chapter 2 deals with the case of manifolds with boundary and in particular Boutet de Monvel's calculus of boundary value problems. None of this is my work, but it is a historical description — primarily with original references — and it partly serves as a motivation for my interest in the subject.

2. Chapters 3 and 4 basically recap the articles mentioned above, rewritten to help the "flow" of the text — to make the dissertation more easily readable — although Chapter 4 also contains some additional work on the noncommutative residue of a sectorial projection and some examples.

3. Finally, Chapter 5 is a description of the various further results I have achieved in light of the main results.

Almost no proofs are given in Chapters 1-4 (none until Section 4.2), and my goal was to make these chapters as non-technical as possible and hopefully in this way readable for non-experts. I have attempted to write this part so that graduate students with some knowledge of PDE theory and geometric analysis should be able to get an intuitive understanding of the subject.

Note that while the articles [GG08] and [Gaa07] appear in chronological order at the end of the text, I reversed the order in Chapters 3 and 4 as I felt the text as a whole would benefit from this.

Notation

We generally follow the notation conventions of Grubb [Gru96], except for a few instances. For example, we do not include zero in $\mathbb{N}$.

We shall often consider the halfspace

$$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x = (x', x_n) \text{ with } x_n > 0\}. \tag{P.1}$$

Whenever we consider a manifold with boundary, it is understood that a fixed atlas is chosen such that each chart intersecting the boundary of the manifold corresponds to a set in $\overline{\mathbb{R}^n_+}$ with the boundary at $x_n = 0$, as explained in the appendix of [Gru96]. As indicated above, tangential variables are primed while normal variables have a subscript n; in some instances, x_n can be seen as a boundary defining function on the manifold.

All manifolds are understood to be smooth and equipped with a smooth positive density dx; all vector bundles are understood to be smooth and Hermitian. In our terminology, a compact manifold without boundary is called closed. The fiberwise trace in a vector bundle — corresponding to the standard matrix trace in local trivializations — is denoted tr.

For function spaces on $\mathbb{R}^n$ and $\mathbb{R}^n_+$, respectively, r^+ restricts from $\mathbb{R}^n$ to $\mathbb{R}^n_+$ while e^+ extends from $\mathbb{R}^n_+$ to $\mathbb{R}^n$ by zero. Similarly, for a manifold X embedded in a larger manifold $\widetilde{X}$, r^+ restricts from $\widetilde{X}$ to X° while e^+ extends functions on X by zero to $\widetilde{X}$.

Let P be an operator on a Hilbert space $\mathcal{H}$ with domain $\mathcal{D}(P)$, and $\lambda \in \operatorname{sp}(P)$, the spectrum of P. We define the generalized eigenspace

$$E_\lambda(P) = \{u \in \mathcal{D}(P) \mid (P - \lambda)^N u = 0, \text{ for some } N \in \mathbb{N}\}. \tag{P.2}$$

We say that P has a complete set of root vectors, when the algebraic direct sum of all E_λ, $\lambda \in \operatorname{sp}(P)$, is dense in $\mathcal{H}$.

For $\theta < \varphi < \theta + 2\pi$ we define the sector

$$\Lambda_{\theta,\varphi} = \{\, \lambda \in \mathbb{C} \setminus \{0\} \mid \theta < \arg \lambda < \varphi \} = \{\, re^{i\omega} \mid r > 0,\, \theta < \omega < \varphi \}. \tag{P.3}$$

We define the algebraic direct sum of the generalized eigenspaces for all the eigenvalues in the sector $\Lambda_{\theta,\varphi}$:

$$E_{\theta,\varphi}(P) = \dot{+}_{\lambda \in \operatorname{sp}(P) \cap \Lambda_{\theta,\varphi}} E_\lambda. \tag{P.4}$$

Chapter 1

Closed manifolds

The subject of my thesis has mainly been noncommutative residues of sectorial projections of boundary value problems. This introductory chapter gives a brief historical description of the analogous results in the setting of manifolds without boundary.

The setup is the following: Let X denote a closed manifold of dimension n and E a vector bundle over X. The class of classical pseudodifferential operators (ψdos) of order m acting on the sections of E is denoted $\Psi^m(X,E)$.

1.1 Zeta- and eta-functions

We begin with a short description of the complex powers of an elliptic ψdo, as defined by Seeley [See67]: Let $P \in \Psi^m(X,E)$ be elliptic, and assume that the principal symbol $p_m(x,\xi)$ of P has no eigenvalues on the negative real line $\mathbb{R}_-$. Then P has (possibly after a small rotation) $\mathbb{R}_-$ as a *ray of minimal growth*, i.e., the spectrum of P is disjoint from $\mathbb{R}_-$ and $\|(P-\lambda)^{-1}\|$ is $O(\lambda^{-1})$, where the norm is the operator norm in $L_2(X,E)$.

Letting $\mathcal{C}$ be a Laurent loop, a contour in the complex plane going around the non-zero spectrum of P,

$$\mathcal{C} = \{re^{i\pi} \mid \infty > r > r_0\} \cup \{r_0 e^{i\omega} \mid \pi \geq \omega \geq -\pi\} \cup \{re^{-i\pi} \mid r_0 < r < \infty\}, \quad (1.1.1)$$

we define the *complex powers* of P by

$$\begin{aligned} &P^s = \frac{i}{2\pi}\int_{\mathcal{C}} \lambda^s (P-\lambda)^{-1}\, d\lambda, \quad \operatorname{Re} s < 0, \\ &P^s = P^k P^{s-k}, \quad \operatorname{Re} s \geq 0, \text{ where } k \in \mathbb{N} \text{ is such that } \operatorname{Re}(s-k) < 0. \end{aligned} \quad (1.1.2)$$

Seeley showed the following properties of the complex powers:

Theorem 1.1.1 ([See67]). *P^s is a ψdo of order ms, and is in particular trace-class for* $\operatorname{Re} s < -\frac{n}{m}$. *The* zeta-function *of P is defined as*

$$\zeta(P, s) = \operatorname{Tr} P^{-s}, \quad \operatorname{Re} s > \tfrac{n}{m}. \tag{1.1.3}$$

It is a holomorphic function of s which extends to a meromorphic function — also denoted $\zeta(P, s)$ — in the entire complex plane with at worst simple poles at $s = (n-j)/m$, $j \in \mathbb{N}_0$. The pole at $s = 0$ is a removable singularity.

Consider now a self-adjoint, elliptic ψdo $A \in \Psi^m(X, E)$. Then the spectrum $\operatorname{sp}(A)$ is a discrete set of real eigenvalues, and Atiyah, Patodi and Singer [APS75] defined the *eta-function* of A to be

$$\eta(A, s) = \sum_{\lambda \in \operatorname{sp}(A) \setminus \{0\}} \operatorname{sgn}(\lambda)\ |\lambda|^{-s}, \quad \operatorname{Re} s > \tfrac{n}{m}, \tag{1.1.4}$$

summing over the eigenvalues in accordance with multiplicity. Like the zeta-function, $\eta(A, s)$ is an analytic function of s, extendable to a meromorphic function in all of $\mathbb{C}$ with at worst simple poles at $s = (n-j)/m$, $j \in \mathbb{N}_0$.

Unlike the zeta-function, however, it was not clear that the pole at the origin is removable and $\eta(A, s)$ thus regular at $s = 0$: Atiyah-Patodi-Singer [APS76] themselves — connecting the question to the investigation of first order boundary problems — showed that the residue at the origin does indeed vanish when the dimension n is odd.

Whether the same held true in even dimensions was long an open problem. The question was answered in the affirmative by Gilkey [Gil81]:

Theorem 1.1.2 ([APS76, Gil81]). *The residue $R(A) = \operatorname{res}_{s=0} \eta(A, s)$ is zero, so $\eta(A, s)$ is regular at the origin $s = 0$. The value $\eta(A, 0)$ is called the eta-invariant of A and denoted $\eta(A)$.*

Gilkey used topological methods in his proof, making Wodzicki wonder if the local nature of $R(A)$ could possibly contribute to an analytic proof. Here, the term local refers to the fact that $R(A)$ can be computed as the integral of an explicit density on X, depending only on finitely many homogeneous terms of the symbol of A.

Wodzicki [Wod82, Wod84] began his search for an analytic proof with the observation by Shubin [Shu78, Problem 13.5] that there is a connection between $R(A)$ and the zeta-function(s) of A. To describe this, we generalize the definition of complex powers slightly to depend on a specific spectral cut: assume that, instead of $\mathbb{R}_-$, it is the ray $\Gamma_\theta = e^{i\theta}\mathbb{R}_+$ which is disjoint from the spectrum of $p_m(x,\xi)$, such that Γ_θ is a ray of minimal growth for P. Defining the contour

$$\mathcal{C}_\theta = \{re^{i\theta} \mid \infty > r > r_0\} \cup \{r_0 e^{i\omega} \mid \theta \geq \omega \geq \theta - 2\pi\} \cup \{re^{i(\theta-2\pi)} \mid r_0 < r < \infty\}, \tag{1.1.5}$$

we then define

$$\begin{aligned} P_\theta^s &= \frac{i}{2\pi}\int_{\mathcal{C}_\theta} \lambda_\theta^s (P-\lambda)^{-1}\, d\lambda, \quad \operatorname{Re} s < 0, \\ P_\theta^s &= P^k P_\theta^{s-k}, \quad \operatorname{Re} s \geq 0, \text{ where } k \in \mathbb{N} \text{ is such that } \operatorname{Re}(s-k) < 0. \end{aligned} \tag{1.1.6}$$

Here, the subscript θ indicates that we take the holomorphic branch of λ^s with a branch cut at Γ_θ. The properties of Theorem 1.1.1 hold for P_θ^s as well, and analogously to (1.1.3) we then define the *ray-dependent zeta-function*

$$\zeta_\theta(P,s) = \operatorname{Tr} P_\theta^{-s}, \quad \operatorname{Re} s > \tfrac{n}{m}, \tag{1.1.7}$$

meromorphically extended to $\mathbb{C}$. $\zeta_\theta(P,s)$ has the same pole structure as described above for $\zeta = \zeta_{-\pi}$.

For the self-adjoint operator A above, any ray Γ_θ in the complex plane with $\theta \neq 0, 2\pi$ will be a ray of minimal growth, and $\zeta_\theta(A,s)$ hence well-defined for such θ.

Proposition 1.1.3 ([Shu78])**.**

$$R(A) = \operatorname{res}_{s=0} \eta(A,s) = \frac{i}{\pi}\big(\zeta_\uparrow(A,0) - \zeta_\downarrow(A,0)\big), \tag{1.1.8}$$

where $\uparrow$, resp. $\downarrow$, refers to any angle θ in the upper, resp. lower, halfplane.

From this identity we immediately obtain that if $\zeta_\theta(A,0)$ is independent of θ then $\eta(A,s)$ is regular at the origin. Wodzicki proved the former statement and thus obtained a new proof of the regularity of the eta-invariant:

Theorem 1.1.4 ([Wod82, Wod84])**.** *$\zeta_\theta(P,0)$ is independent of θ.*

Unfortunately, a careless formulation in the introduction of [See67] caused an important flaw in the original proof of Theorem 1.1.4 in [Wod82]. Instead, Wodzicki had to rely on a rather complicated characterization of *local invariants of spectral asymmetry* [Wod84] to prove the result, and it could be argued that not much had been gained in relation to a simpler proof of Theorem 1.1.2.

However, Wodzicki had many other interesting observations, including the definition of the noncommutative residue which we describe in the next section.

1.2 The noncommutative residue

Consider a ψdo $P \in \Psi^m(X, E)$. In local trivializations of E, the symbol $p(x, \xi)$ of P has an asymptotic expansion

$$p(x,\xi) \sim \sum_{k=0}^{\infty} p_{m-k}(x,\xi), \tag{1.2.1}$$

with each p_{m-k} homogeneous in the sense that

$$p_{m-k}(x, t\xi) = t^{m-k} p_{m-k}(x,\xi), \text{ for } t \geq 1, |\xi| \geq 1. \tag{1.2.2}$$

Under changes of coordinates the homogeneous terms transform in a way such that the principal symbol $p_m(x, \xi)$ is the only term which is invariantly defined on X — or rather on S^*X, the cosphere bundle.

However, the term of degree $-n$, when integrated over the unit sphere $S^*_x X$, transforms such that the density $\operatorname{res}_x P \, dx$, with

$$\operatorname{res}_x P = \int_{|\xi|=1} p_{-n}(x,\xi) \, đS(\xi), \tag{1.2.3}$$

is in fact also invariant under coordinate changes. Here, $đS(\xi)$ denotes the surface measure on $S^*_x X$, divided by $(2\pi)^n$.

We call $\operatorname{res}_x P \, dx$ the *noncommutative residue density* of P. Its invariance under transformations allows the following definition:

Definition 1.2.1 ([Wod84]). *We define the noncommutative residue of P to be*

$$\operatorname{res} P = \int_X \operatorname{tr} \operatorname{res}_x P \, dx = \int_{S^*X} \operatorname{tr} \, p_{-n}(x,\xi) \, đS(\xi) \, dx. \tag{1.2.4}$$

Although we have mostly concerned ourselves with Wodzicki's work, it should be noted that the residue (and some of the theorems below) was independently discovered by Guillemin [Gui85] in a different setting.

The noncommutative residue has a number of interesting properties:

Theorem 1.2.2 ([Wod84]). res *is a trace on* $\Psi^\infty(X,E)$*, i.e., a linear functional which vanishes on commutators:* $\operatorname{res}([P,Q]) = 0$ *for* $P, Q \in \Psi^\infty(X,E)$*.*

When S^*X *is connected,* res *is the only trace on* $\Psi^\infty(X,E)$ *up to multiplication by a scalar.*

Requiring S^*X to be connected is equivalent to requiring that X is connected and of dimension n at least 2; if X has more than one component, all linear combinations of the component-wise residues will form a trace on $\Psi^\infty(X,E)$. Likewise, for $n = 1$ the integral over $S^*_x X$ becomes a sum $\int_{|\xi|=1} = \sum_{\xi=\pm 1}$ and there is an ambiguity in the choice of constants for each term.

We consider now the case of a ψdo projection Π, i.e., $\Pi \in \Psi^\infty(X,E)$ satisfying $\Pi^2 = \Pi$. We follow the terminology from functional analysis, which does not require a projection to be self-adjoint (as opposed to the terminology of operator algebraists). Obviously, Π must have order 0 or $-\infty$.

Theorem 1.2.3 ([Wod84]). *Let* Π *be a* ψ*do projection on* X*. Then* $\operatorname{res}\Pi = 0$*.*

Wodzicki proved this theorem by showing the statement to be equivalent to his earlier Theorem 1.1.4; see also Proposition 1.3.2 below. Due to an observation by Brüning and Lesch [BL99, Lemma 2.7], Theorem 1.2.3 can in fact also be deduced from Theorem 1.1.2.

1.3 Sectorial projection

Assume now that P has two rays of minimal growth Γ_θ and Γ_φ, with $\theta < \varphi < \theta + 2\pi$.

For λ on either ray and $u \in \mathcal{D}(P) = H^m(X,E)$, the m'th Sobolev space of sections of E, we then have

$$\|\lambda^{-1}P(P-\lambda)^{-1}u\| \le \|\lambda^{-1}(P-\lambda)^{-1}\| \cdot \|Pu\| = O(\lambda^{-2}). \tag{1.3.1}$$

This permits the definition of the sectorial projection

$$\Pi_{\theta,\varphi}(P)u = \frac{i}{2\pi}\int_{\Gamma_{\theta,\varphi}} \lambda^{-1}P(P-\lambda)^{-1}\,u\,d\lambda, \quad u \in H^m(X,E), \tag{1.3.2}$$

where $\Gamma_{\theta,\varphi}$ is the integration contour

$$\{re^{i\varphi} \mid \infty > r > r_0\} \cup \{r_0 e^{i\omega} \mid \varphi \geq \omega \geq \theta\} \cup \{re^{i\theta} \mid r_0 < r < \infty\}. \tag{1.3.3}$$

Wodzicki showed interest for this projection — Burak considered it earlier, but in a different setting — in connection with his here mentioned results, but gave few details. Ponge [Pon06] filled in the details:

Proposition 1.3.1 ([Pon06])**.** *$\Pi_{\theta,\varphi}(P)$ is a ψdo projection of order 0, in particular it extends to a bounded projection on $L_2(X, E)$.*

Its image contains the closure of $E_{\theta,\varphi}(P)$, the algebraic sum of the generalized eigenspaces for the eigenvalues of P in the sector $\Lambda_{\theta,\varphi}$. Its kernel contains the closure of $E_{\varphi,\theta+2\pi}(P) \dotplus E_0(P)$, the corresponding space for the eigenvalues in $\Lambda_{\varphi,\theta+2\pi} \cup \{0\}$.

(Cf. equations (P.2) and (P.4) for the $E(P)$-spaces.)

We call $\Pi_{\theta,\varphi}$ the sectorial projection since it is "essentially" the spectral projection onto the spectrum of P in the sector $\Lambda_{\theta,\varphi} = \{\theta < \arg\lambda < \varphi\}$. In certain cases, $\Pi_{\theta,\varphi}(P)$ *equals* this spectral projection, for example when P is a normal operator (P commutes with its adjoint P^*).

The polyhomogeneous symbol (in local trivializations) of $\Pi_{\theta,\varphi}(P)$ is given by

$$\pi_{\theta,\varphi} \sim \sum_{j=0}^{\infty} \pi_{\theta,\varphi,-j}, \quad \pi_{\theta,\varphi,-j}(x,\xi) = \frac{i}{2\pi} \int_{\mathcal{C}_{(x,\xi)}} q_{-m-j}(x,\xi,\lambda)\, d\lambda, \tag{1.3.4}$$

where $q \sim \sum_{j=0}^{\infty} q_{-m-j}$ is the symbol of the resolvent $(P-\lambda)^{-1}$ and $\mathcal{C}_{(x,\xi)}$ is a closed contour in the complex λ plane going around (once, in the positive direction) the zeroes of the λ polynomial $\det(p_m(x,\xi) - \lambda)$ lying in the sector $\Lambda_{\theta,\varphi}$.

The connection with the previous sections is given by the following proposition, which "completes the circle" (well, not quite) between Theorems 1.1.2, 1.1.4 and 1.2.3:

Proposition 1.3.2 ([Wod84])**.**

$$\zeta_\theta(P,0) - \zeta_\varphi(P,0) = \frac{2\pi i}{m} \operatorname{res} \Pi_{\theta,\varphi}(P). \tag{1.3.5}$$

The proof of this proposition is actually quite easy and will appear as an intermediate result in Chapter 4.

1.4 Trace expansions

As we saw above, the zeta-function is defined with the use of a holomorphic family of complex powers $(P_\theta^s)_{s\in\mathbb{C}}$. We wish to consider the notion of zeta-functions in the setting of manifolds with boundary, and since complex powers of boundary problems are in general difficult to work with, we mention here a different approach using resolvent trace expansions.

We consider once again $P \in \Psi^m(X, E)$ with Γ_θ as a ray of minimal growth. For convenience, we temporarily add the assumption $m > n$, such that the resolvent $(P-\lambda)^{-1}$ is trace-class.

Theorem 1.4.1 ([Gru96, 3.3.5])**.** *There is a trace expansion*

$$\operatorname{Tr}(P - e^{i\theta}\lambda)^{-1} = \sum_{k=0}^{n} c_{k,\theta}(-e^{i\theta}\lambda)^{\frac{n-k}{m}-1} + O(\lambda^{-1-\frac{1}{4m}}), \tag{1.4.1}$$

for λ going to ∞ in a small sector $\Lambda_{-\varepsilon,\varepsilon}$ of $\mathbb{C}$ containing $\mathbb{R}_+$. The coefficients of the expansion are given by

$$c_{k,\theta} = \int_X c_{k,\theta}(x)dx, \quad c_{k,\theta}(x) = -e^{i\theta}\int_{\mathbb{R}^n} q^h_{-m-k}(x,\xi,e^{i\theta})đ\xi, \tag{1.4.2}$$

where q^h_{-m-k} are the strictly homogeneous terms of the symbol $q \sim \sum_{k=0}^\infty q_{-m-k}$ of the resolvent $(P-\lambda)^{-1}$.

Note that λ need not be a real parameter in (1.4.1), but will have relatively small imaginary part. The branch λ_θ^s is used for the fractional powers. The strictly homogeneous version of the term q_{-m-k} is obtained by extending the domain of homogeneity inside the unit ball $|\xi| \le 1$, i.e., it satisfies

$$\begin{aligned} q^h_{-m-k}(x,\xi,\lambda) &= q_{-m-k}(x,\xi,\lambda), && |\xi| \ge 1, \\ q^h_{-m-k}(x,t\xi,t^m\lambda) &= t^{-m-k}q^h_{-m-k}(x,\xi,\lambda), && t>0, \xi \ne 0. \end{aligned} \tag{1.4.3}$$

Remark 1.4.2. Determining the "correct" reference is quite difficult in this case; the result was possibly known by Seeley in connection with his work on complex powers [See67], but the first explicit references seem to be Agmon and Kannai [AK67] in the case of self-adjoint differential operators and Grubb [Gru78] in the case of self-adjoint ψdos. We have listed [Gru96] as the reference here, since the first edition of this monograph (from 1986) is apparently the first source with a complete description.

We have chosen to present the trace expansion slightly different than usually done: Traditionally, the dependence upon θ is not made explicit and one considers the trace of $(P-\lambda)^{-1}$ for λ going to infinity in a small sector containing Γ_θ. We need to be able to handle coefficients for different rays simultaneously, cf. Definition 4.2.3, and thus need λ to run in a particular sector independent of θ, here chosen to be close to the real line.

The reason for the particular "sign" convention in (1.4.1) two-fold: we ensure that (1.4.4) below is satisfied without having to insert a phase factor, and at the same time the expression in (1.4.1) is comparable with the standard convention for $\theta=\pi$, i.e., for $\theta=\pi$ the coefficients $c_{k,\pi}$ equal the coefficients of e.g. [Gru96, eq. (3.3.33)].

The coeffient of λ^{-1} in the above expansion is of particular interest to us, for it is essentially equal to the zeta-function at the origin:

Theorem 1.4.3 ([See67]). *We have an equality*

$$c_{n,\theta}=\zeta_\theta(P,0)+\nu_0, \tag{1.4.4}$$

where ν_0 is the algebraic multiplicity of 0 as an eigenvalue of P.

According to the definition (P.2) of $E_\lambda(P)$, ν_0 can also be characterized as the dimension of the generalized nullspace $E_0(P)$.

If $m\le n$, one can instead consider the N'th power of the resolvent with $N\in\mathbb{N}$ so large that $(P-e^{i\theta}\lambda)^{-N}$ is trace-class (satisfied for $N>n/m$). Then the trace expansion has the form

$$\operatorname{Tr}(P-e^{i\theta}\lambda)^{-N}=\sum_{k=0}^{n}c_{k,\theta}^{(N)}(-e^{i\theta}\lambda)^{\frac{n-k}{m}-N}+O(\lambda^{-N-\frac{1}{4m}}). \tag{1.4.5}$$

In this case, it is the coefficient of λ^{-N} which interests us: as explained in e.g. [Gru05, Remark 3.12] one can use the identity $(P-\lambda)^{-N}=\frac{\partial_\lambda^{N-1}}{(N-1)!}(P-\lambda)^{-1}$ to turn the iterates into λ-derivatives and show that in fact $c_{n,\theta}^{(N)}$ is independent of N and can be inserted into equations (1.4.4) and (1.4.2):

Proposition 1.4.4. *$c_{n,\theta}^{(N)}$ is independent of N and satifies*

$$c_{n,\theta}^{(N)}=\zeta_\theta(P,0)+\nu_0, \tag{1.4.6}$$

$$c_{n,\theta}^{(N)}=\int_X c_{n,\theta}^{(N)}(x)dx,\ \text{where } c_{n,\theta}^{(N)}(x)=-e^{i\theta}\int_{\mathbb{R}^n}q_{-m-n}^h(x,\xi,e^{i\theta})\,đ\xi. \tag{1.4.7}$$

Chapter 2

Manifolds with boundary

This chapter begins with a description of Boutet de Monvel's calculus of boundary value problems. Then the results from Chapter 1 are transferred to the setting of manifolds with boundary, that is, we discuss noncommutative residues of boundary operators, and zeta-functions and trace expansions for realizations of boundary problems.

In this chapter, X denotes a compact n-dimensional manifold with boundary ∂X. We assume that X is embedded in a closed manifold $\widetilde{X}$; for instance, we could choose $\widetilde{X}$ as the double manifold $2X$.

2.1 Boutet de Monvel's calculus

Boutet de Monvel [BdM71] constructed a calculus with the aim that it should contain all differential boundary value problems, and the parametrices of the elliptic ones.

A standard example of such a boundary problem is the Dirichlet problem for the Laplacian (plus the identity in this case, for technical reasons):

$$\begin{aligned} (1-\Delta)u &= f \qquad \text{on } X, \\ u|_{\partial X} &= \varphi \qquad \text{on } \partial X. \end{aligned} \tag{2.1.1}$$

The solution of this problem is well-known to be

$$u = (Q_+ + G)f + K\varphi, \tag{2.1.2}$$

where $Q_+ = r^+ Q e^+$ is the truncation of $Q = (1-\Delta)^{-1}$, the inverse of $1-\Delta$ on $\widetilde{X}$, K is a Poisson operator, going from the boundary to X, while G is a so-called *singular Green operator* with the intuitive description of Gf as a "boundary correction term".

We can describe this problem as a column vector of operators

$$\mathcal{A} = \begin{pmatrix} -\Delta_+ \\ \gamma_0 \end{pmatrix} : C^\infty(X) \to \begin{matrix} C^\infty(X) \\ \times \\ C^\infty(\partial X), \end{matrix} \tag{2.1.3}$$

which maps u to (f, φ). Here γ_0 is the standard trace operator of order 0, which is merely restriction to the boundary.

The solution in (2.1.2) can likewise be described as the operator row vector

$$\mathcal{A}^{-1} = \begin{pmatrix} Q_+ + G & K \end{pmatrix}, \tag{2.1.4}$$

which maps (f, φ) to u.

Let $\widetilde{E}_1, \widetilde{E}_2$ be vector bundles over $\widetilde{X}$; we denote their restrictions to X by E_1, E_2. Let F_1, F_2 be vector bundles over ∂X. Boutet de Monvel combined the "row" and "column" operators in (2.1.3) and (2.1.4) in a larger setting of matrices of operators:

A *Green operator*, or *pseudodifferential boundary operator* (ψdbo), is an operator of the form

$$\mathcal{A} = \begin{pmatrix} P_+ + G & K \\ T & S \end{pmatrix} : \begin{matrix} C^\infty(X, E_1) \\ \times \\ C^\infty(\partial X, F_1) \end{matrix} \to \begin{matrix} C^\infty(X, E_2) \\ \times \\ C^\infty(\partial X, F_2), \end{matrix} \tag{2.1.5}$$

where $P_+ = r^+ P e^+$ is the truncation to X of a ψdo P on $\widetilde{X}$, going from $\widetilde{E}_1$ to $\widetilde{E}_2$ and satisfying the *transmission condition* with respect to ∂X, i.e., P_+ maps $C^\infty(X, E_1)$ into $C^\infty(X, E_2)$; G is a singular Green operator from E_1 to E_2; T is a trace operator from E_1 to F_2 (going from the manifold X to its boundary); K is a Poisson operator from F_1 to E_2 (going from the boundary to X); and S is a ψdo on the closed manifold ∂X from F_1 to F_2.

A more thorough introduction to Green operators can be found in [Gru96] or [Sch01], in particular a detailed description of the different types of operators mentioned above.

The following important property allows us to speak of the set of Green operators as a calculus of pseudodifferential boundary problems, called Boutet de Monvel's calculus (or the Boutet de Monvel calculus):

Theorem 2.1.1 ([BdM71]). *The composition of two Green operator (with matching vector bundles) is a Green operator.*

The different components of a Green operator each have an order and class assigned to them, they have symbols with polyhomogeneous expansions and can be thought of as being operatorvalued ψdos along the boundary. This is most easily described in local coordinates $(x', x_n, \xi', \xi_n) \in T^*X$ in a neighborhood of the boundary. Here, the action of G, a singular Green operator of order m and class 0, is given by a *symbol-kernel* $\widetilde{g}(x', x_n, y_n, \xi')$:

$$Gu(x', x_n) = \int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'} \int_0^\infty \widetilde{g}(x', x_n, y_n, \xi')\, \acute{u}(\xi', x_n)\, dy_n\, đ\xi', \tag{2.1.6}$$

where $\acute{u}$ denotes the partial Fourier transform $\acute{u}(\xi', x_n) = \mathcal{F}_{x'\to\xi'} u(x', x_n)$.

The symbol-kernel satisfies estimates combining the usual $S_{1,0}$ estimates in the (x', ξ')-variables with rapid decay estimates in the (x_n, y_n)-variables:

$$\sup_{x_n, y_n \in \mathbb{R}_+} |x_n^k D_{x_n}^{k'} y_n^l D_{y_n}^{l'} D_{x'}^\beta D_{\xi'}^\alpha \widetilde{g}(x', x_n, y_n, \xi')| \le c(x') \langle \xi' \rangle^{m+1-k+k'-l+l'-|\alpha|}. \tag{2.1.7}$$

The *boundary symbol operator* of G on $L_2(\mathbb{R}_+)$ is the compact integral operator

$$g(x', \xi', D_n) v(x_n) = \int_0^\infty \widetilde{g}(x', x_n, y_n, \xi')\, v(y_n)\, dy_n, \tag{2.1.8}$$

and the action of G is then

$$Gu(x', x_n) = \int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'} g(x', \xi', D_n)\, \acute{u}(\xi', y_n)\, dy_n\, đ\xi', \tag{2.1.9}$$

where we see that G behaves like a pseudodifferential operator on $\mathbb{R}^{n-1}$ with symbol $g(x', \xi', D_n)$, an operator on $L_2(\mathbb{R}_+)$ for each fixed (x', ξ'). The symbol of G is obtained by Fourier (and conjugate Fourier) transforming the symbol-kernel

$$g(x', \xi', \xi_n, \eta_n) = \mathcal{F}_{x_n\to\xi_n} \overline{\mathcal{F}}_{y_n\to\eta_n} \widetilde{g}(x', x_n, y_n, \xi'). \tag{2.1.10}$$

As mentioned, g has an expansion in homogenous terms; we will denote by g^0 the principal term, homogeneous in (ξ, η_n) of degree $m-1$ when G is of order m. Inverting the relation between g and $\widetilde{g}$ above, we get a principal symbol-kernel $\widetilde{g}^0$ which satisfies the following quasi-homogeneity:

$$\widetilde{g}^0(x', \tfrac{x_n}{t}, \tfrac{y_n}{t}, t\xi') = t^{m+1}\, \widetilde{g}^0(x', x_n, y_n, \xi'), \quad t \ge 1, |\xi'| \ge 1. \tag{2.1.11}$$

The corresponding integral operator on $L_2(\mathbb{R}_+)$ with this kernel is then denoted $g^0(x', \xi', D_n)$, the *principal boundary symbol operator* (of G).

Similarly, K and T can be ascribed symbols and principal boundary symbol operators, $k^0(x', \xi', D_n) : \mathbb{C} \to L_2(\mathbb{R}_+)$, the multiplication by the principal symbol-kernel $\widetilde{k}^0(x', x_n, \xi')$, and $t^0(x', \xi', D_n) : L_2(\mathbb{R}_+) \to \mathbb{C}$, the operator $v \mapsto \int_0^\infty \widetilde{t}^0(x', y_n, \xi') v(y_n) dy_n$. The analogous notions for P and Q should be obvious, and we can define the principal boundary symbol operator of $\mathcal{A}$:

$$a^0(x', \xi', D_n) = \begin{pmatrix} p^0(x', 0, \xi', D_n)_+ + g^0(x', \xi', D_n) & k^0(x', \xi', D_n) \\ t^0(x', \xi', D_n) & s^0(x', \xi') \end{pmatrix}. \tag{2.1.12}$$

Occasionally this is denoted $\gamma(\mathcal{A})$ and simply called the *boundary symbol* (in comparison with the *interior symbol* $\sigma(\mathcal{A})$, the principal symbol of P). For simplicity, we have sketched only the case of class 0 operators here.

The homogeneous terms of the symbols provide us with a countable family of seminorms, which allows us to define a Fréchet topology on the set of Green operators.

The symbol is also important for the definition of ellipticity: for $\mathcal{A}$ to be elliptic, we require that both symbols — the interior symbol $\sigma(\mathcal{A})$ and the boundary symbol $\gamma(\mathcal{A})$ — are invertible. Since $\sigma(\mathcal{A}) = p^0(x, \xi)$, the first requirement is just that P is elliptic in the usual sense.

In the remainder of this chapter, we consider only the case with $E_1 = E_2 = E$ and $F_1 = F_2 = F$. The Green operators then form an algebra, occasionally called Boutet de Monvel's algebra. With the topology mentioned above it becomes a Fréchet algebra. We denote it $\mathscr{A}^\infty_{EF}$ or, more often, just $\mathscr{A}^\infty$ when the specific choice of vector bundles is understood (or irrelevant).

Remark 2.1.2. Regarding the assumption that X is embedded in a closed manifold, there are in fact examples of elliptic operators on X that do not extend to elliptic operators on $\widetilde{X}$. However, in our applications there is no loss of generality in this assumption since we consider only operators with a ray of minimal growth, and such operators do indeed have an elliptic extension to $\widetilde{X}$. See e.g. [Gru99, Theorem 7.4] for details.

2.2 The noncommutative residue

The notion of noncommutative residue of a Green operator was defined by Fedosov, Golse, Leichtnam and Schrohe [FGLS96]:

For the ψdbo $\mathcal{A}$ in (2.1.5), since P is a classical ψdo on the closed manifold $\widetilde{X}$, its residue density $\operatorname{res}_x P\,dx$ is well-defined for all $x \in \widetilde{X}$, in particular for $x \in X$. Similarly, S is a ψdo on the closed manifold ∂X and has a residue density $\operatorname{res}_{x'} S\,dx'$. Finally, from the singular Green operator G with symbol-kernel $\widetilde{g}(x', x_n, y_n, \xi')$, we define its normal trace $\operatorname{tr}_n G$ with symbol

$$\operatorname{tr}_n g(x', \xi') = \int_0^\infty \widetilde{g}(x', x_n, x_n, \xi') dx_n \tag{2.2.1}$$

in local coordinates. This is a ψdo on ∂X as well, and hence also the density $\operatorname{res}_{x'}(\operatorname{tr}_n G)\,dx'$ makes sense.

Definition 2.2.1. *The noncommutative residue of the operator $\mathcal{A}$ is defined as*

$$\begin{aligned} \operatorname{res}\mathcal{A} &= \int_X \operatorname{tr}\operatorname{res}_x P\,dx + \int_{\partial X} \operatorname{tr}\operatorname{res}_{x'}(\operatorname{tr}_n G)\,dx' + \int_{\partial X} \operatorname{tr}\operatorname{res}_{x'} S\,dx' \\ &= \int_{S^*X} \operatorname{tr}\, p_{-n}(x,\xi) đS(\xi) dx + \int_{S^*\partial X} \operatorname{tr}\,\operatorname{tr}_n g_{1-n}(x',\xi') đS(\xi') dx' \\ &\qquad + \int_{S^*\partial X} \operatorname{tr}\, s_{1-n}(x',\xi') đS(\xi') dx'. \end{aligned} \tag{2.2.2}$$

(Here, tr denotes the bundletrace in E, $E|_{\partial X}$ and F, respectively. Also, a sign error in [FGLS96] has been corrected, cf. Grubb and Schrohe [GSc01, eq. (1.5)].) The term $\int_X \operatorname{tr}\operatorname{res}_x P\,dx$ is occasionally denoted $\operatorname{res}_+ P$ to emphasize that one only integrates over X, cf. the plus symbol in r^+, restriction to X.

Remark 2.2.2. Although $\operatorname{tr}_n G$ (and thus $\operatorname{res}_{x'}(\operatorname{tr}_n G)\,dx'$) can depend on the specific choice of boundary charts, the noncommutative residue is independent of this choice.

The noncommutative residue is a continuous trace on the Fréchet algebra $\mathscr{A}^\infty$, in fact the unique such trace:

Theorem 2.2.3 ([FGLS96])**.** $\operatorname{res} : \mathscr{A}^\infty \to \mathbb{C}$ *is continuous and vanishes on commutators. When X and $S^*\partial X$ are connected,* res *is — up to scalar multiplication — the only continuous trace on $\mathscr{A}^\infty$.*

We see that the central properties (being tracial, uniqueness) of Wodzicki's noncommutative residue are preserved for the noncommutative residue of Fedosov et al. An open question has been whether Theorem 1.2.3 does as well: is the noncommutative residue of a projection in $\mathscr{A}^\infty$ zero? This question has been central in my thesis and was answered in the affirmative, see Chapter 3 and/or [Gaa07].

Regarding Theorem 1.1.4 (that $\zeta_\theta(P,0)$ is independent of θ) in the setting of boundary problems, I was not quite so fortunate and only reached a final conclusion in the cases where the corresponding sectorial projection belongs to Boutet de Monvel's calculus. The investigations in this matter are described in Chapter 5, but first we must dedicate the next couple of pages to introduce the notion of zeta-functions in the setting of boundary value problems.

2.3 Zeta-functions and trace expansions

We consider now a boundary value problem$\{P_+ + G, T\}$ of order $m \in \mathbb{N}$ in Boutet de Monvel's calculus, that is, a Green operator of the form

$$\mathcal{A} = \begin{pmatrix} P_+ + G \\ T \end{pmatrix}. \tag{2.3.1}$$

Here T is a *system of trace operators*, a column vector $T = \{T_0, \dots, T_{m-1}\}$ with each T_i a trace operator, cf. [Gru96, Section 1.4]. We define the realization $B = (P+G)_T$ to be the operator on $L_2(X,E)$ with domain

$$\mathcal{D}(B) = \{u \in H^m(X,E) \mid Tu = 0\}, \tag{2.3.2}$$

acting on $\mathcal{D}(B)$ as $P_+ + G$, in the distributional sense.

Assume now that $\{P_+ + G - \lambda, T\}$ is parameter-elliptic for λ on the rays in a small sector around Γ_θ, as defined in [Gru96, Definition 1.5.5].

Then — cf. [Gru96, Section 3.3] — the realization $B - \lambda$ will be invertible for $\lambda \in \Gamma_\theta$ when $|\lambda|$ is sufficiently large, and the resolvent there has the form

$$R_\lambda = (B-\lambda)^{-1} = Q_{\lambda,+} + G_\lambda. \tag{2.3.3}$$

Here $Q_\lambda = (P-\lambda)^{-1}$ is the parameter-dependent resolvent of P on $\widetilde{X}$, with symbol $q(x,\xi,\lambda) \sim \sum_{j=0}^{\infty} q_{-m-j}(x,\xi,\lambda)$. G_λ is a parameter-dependent singular Green operator with symbol-kernel $\widetilde{g}(x', x_n, y_n, \xi', \lambda) \sim \sum_{j=0}^{\infty} \widetilde{g}_{-m-j}(x', x_n, y_n, \xi', \lambda)$.

Furthermore, Γ_θ is a ray of minimal growth, i.e., the spectrum of B is disjoint from Γ_θ and the $L_2(X,E)$ operator norm of R_λ satisfies

$$\|(B-\lambda)^{-1}\| = O(\lambda^{-1}). \tag{2.3.4}$$

This is clearly similar to the situation in Chapter 1 that allowed the definition of P^s. Like there, we are interested in the zeta-function $\zeta(B,s)$ at $s=0$.

That Γ_θ is a ray of minimal growth allows for the definition of complex powers B^s_θ for $\operatorname{Re} s < 0$ as in the case of ψdos on closed manifolds, cf. (1.1.6). For $\operatorname{Re} s > \frac{n}{m}$ the operator B^{-s}_θ is trace-class [Gru96, Corollary 4.5.11] and one can then analogously define the zeta-function $\zeta_\theta(B,s)$ as the trace of B^{-s}_θ for large $\operatorname{Re} s$.

Seeley [See69a, See69b] used this approach for the case of a differential boundary problem P_T (P and T differential, $G=0$), and managed to show that $\zeta(P_T,s)$ extends to a meromorphic function in $\mathbb{C}$ with a pole-structure comparable to (although not quite as nice as) the one in Theorem 1.1.1. In particular the pole at $s=0$ is removable, so $\zeta(P_T,0)$ is defined.

However, Seeley specifically made use of the symbol-structure of Q_λ and G_λ in (2.3.3), which is much more explicit and easier to work with in the differential case than in the more general case of pseudodifferential boundary problems, and his methods cannot be directly applied in general.

In fact, the complex powers B^s_θ do not lie in Boutet de Monvel's calculus [Gru96, Section 4.4], and hence a more appealing approach is to take the route via trace expansions. Here, one can work in a parameter-dependent version of Boutet de Monvel's calculus, developed by Grubb [Gru96].

If $m > n$ we have a trace expansion [Gru96, Theorems 3.3.5, 3.3.10]

$$\operatorname{Tr}(B - e^{i\theta}\lambda)^{-1} = \sum_{j=0}^{n} c_{j,\theta}(-e^{i\theta}\lambda)^{\frac{n-j}{m}-1} + O(\lambda^{-1-\frac{1}{4m}}). \tag{2.3.5}$$

We state the more general statement, valid also for $m \le n$, in the following theorem:

Theorem 2.3.1. *For $N \in \mathbb{N}$ such that $N > n/m$, we have*

$$\operatorname{Tr}(B - e^{i\theta}\lambda)^{-N} = \sum_{j=0}^{n} c^{(N)}_{j,\theta}(-e^{i\theta}\lambda)^{\frac{n-j}{m}-N} + O(\lambda^{-N-\frac{1}{4m}}), \tag{2.3.6}$$

for $|\lambda| \to \infty$ in the sector V_+.

Like in Chapter 1, original references are somewhat unclear. This theorem, and the statements in the rest of this section, can essentially be found in [Gru96] (in fact, in the 1986 edition).

As in Proposition 1.4.4, the coefficient $c_{n,\theta}^{(N)}$ is independent of N, and we define it to be the *basic zeta coefficient* of B

$$C_{0,\theta}(B) = C_{0,\theta}(P+G)_T = c_{n,\theta}^{(N)}. \tag{2.3.7}$$

According to e.g. [Gru05, eq. (5.5) and Remark 3.12] we have

$$c_{n,\theta}^{(N)} = c_{n,\theta,+}^{Q} + c_{n,\theta}^{G} = \int_X \operatorname{tr} c_{n,\theta}^{Q}(x)dx + \int_{\partial X} \operatorname{tr} c_{n,\theta}^{G}(x')dx', \tag{2.3.8}$$

where, in local coordinates, $c_{n,\theta}^{Q}(x)$ is given by (1.4.7)

$$c_{n,\theta}^{Q}(x) = -e^{i\theta} \int_{\mathbb{R}^n} q_{-m-n}^{h}(x,\xi,e^{i\theta}) đ\xi, \tag{2.3.9}$$

while $c_{n,\theta}^{G}(x')$ is given by

$$c_{n,\theta}^{G}(x') = -e^{i\theta} \int_{\mathbb{R}^{n-1}} s_{-m+1-n}^{h}(x',\xi',e^{i\theta}) đ\xi', \tag{2.3.10}$$

with s_{-m+1-n}^{h} the strictly homogeneous version of s_{-m+1-n}, where $s = \operatorname{tr}_n g$ is the symbol of the normal trace $\operatorname{tr}_n G_\lambda$:

$$s_{-m+1-n}(x',\xi',\lambda) = \int_0^\infty \widetilde{g}_{-m+1-n}(x',x_n,x_n,\xi',\lambda)dx_n. \tag{2.3.11}$$

We recap this in the following proposition:

Proposition 2.3.2. *The basic zeta coefficient satisfies the equation*

$$C_{0,\theta}(B) = \int_X \operatorname{tr} c_{n,\theta}^{Q}(x)dx + \int_{\partial X} \operatorname{tr} c_{n,\theta}^{G}(x')dx'. \tag{2.3.12}$$

As shown in e.g. [GSe96], the existence of the asymptotic trace expansion (2.3.6) implies that the zeta-function $\zeta(B,s)$ does indeed have a meromorphic extension beyond the set $\{\operatorname{Re} s > \frac{n}{m}\}$ (although not to all of $\mathbb{C}$) with a pole structure resembling that in Chapter 1. Moreover, the pole at the origin is removable and the value there identifies with the trace expansion coefficient in a way similar to (1.4.6):

$$\zeta_\theta(B,0) + \nu_0 = c_{n,\theta}^{(N)} = C_{0,\theta}, \tag{2.3.13}$$

where ν_0 is the algebraic multiplicity of zero as an eigenvalue of $B = (P+G)_T$. We see here the motivation for calling $C_{0,\theta}(B)$ the basic zeta value.

A direct proof of (2.3.13) can be found in [Gru96, Theorem 4.4.8].

Chapter 3

Noncommutative residue of Green operator projections

This chapter recaps the article [Gaa07] — the second article in this text — and essentially contains only one result, Theorem 3.1.1 below, which is one of the main theorems of the present thesis.

We attempt to give an intuitive understanding of the proof, which is based on the work of Melo, Nest and Schrohe [MNS03], and Melo, Schick and Schrohe [MSS06].

3.1 Theorem and idea of proof

In Theorem 1.2.3 we saw that Wodzicki's noncommutative residue vanishes when applied to (classical) ψdo projections. As mentioned, a central part of my thesis has been to investigate whether this holds true for the noncommutative residue of Fedosov et al. It does:

Theorem 3.1.1 ([Gaa07, Theorem 1.1])**.** *The noncommutative residue of a projection in Boutet de Monvel's calculus is zero.*

The idea of the proof is to use K-theoretic methods to reduce the question to the well-known case of projections on closed manifolds. The proof is based on a number of results on the K-theory of the Boutet de Monvel algebra shown in [MNS03, MSS06]. We will not give specific references below, they can be found in [Gaa07].

We consider the Boutet de Monvel algebra $\mathscr{A}^\infty$, and in particular the subalgebra of all operators of order and class zero, which we denote $\mathscr{A}$. Being of order and class zero ensures that the Green operators are bounded operators on the Hilbert space $\mathcal{H} = L_2(X, E) \oplus H^{-1/2}(\partial X, F)$, and that $\mathscr{A}$ is closed under taking adjoints. Also, all projections in Boutet de Monvel's calculus lie in $\mathscr{A}$, since a projection must have order and class zero.

$\mathscr{A}$ is a Fréchet $*$-algebra with the topology from Section 2.1. Moreover, it is contained in $\mathcal{B}(\mathcal{H})$, the C^*-algebra of bounded operators on $\mathcal{H}$; we denote the C^*-closure of $\mathscr{A}$ by $\mathfrak{A}$. $\mathscr{A}$ is closed under holomorphic function calculus, which implies that $\mathscr{A}$ is local in the C^*-algebra $\mathfrak{A}$ in the sense of Blackadar [Bla98]. Thus the K-theory of $\mathscr{A}$ equals the K-theory of $\mathfrak{A}$, and in particular $K_0(\mathscr{A}) \cong K_0(\mathfrak{A})$.

The noncommutative residue

$$\mathrm{res} : \mathscr{A} \to \mathbb{C} \tag{3.1.1}$$

is a continuous trace in the Fréchet-topology of $\mathscr{A}$ (although not continuous with respect to the coarser norm-topology in $\mathfrak{A}$) and induces a map

$$\mathrm{res}_* : K_0(\mathscr{A}) \to \mathbb{C}. \tag{3.1.2}$$

The K_0-classes of $K_0(\mathscr{A})$ are given by projections — synonymous to idempotents in our terminology — in $M_n(\mathscr{A})$, the set of $n \times n$ matrices with entries from $\mathscr{A}$. Every ψdbo projection A thus defines a class $[A]_0 \in K_0(\mathscr{A})$ such that

$$\mathrm{res}(A) = \mathrm{res}_*[A]_0. \tag{3.1.3}$$

We show that the linear map res_* is the zero map, and hence $\mathrm{res}(A) = 0$ for any idempotent A.

To prove this, we first show that $K_0(\mathscr{A})$ is the sum of two parts:

$$K_0(\mathscr{A}) \cong m_* K_0(C^\infty(X)) + (\sigma_*)^{-1} K_0(C_0^\infty(S^*X^\circ)). \tag{3.1.4}$$

Here m_* and σ_* are the K_0-induced maps of $m : C^\infty(X) \to \mathscr{A}$, which maps a function f to the multiplication operator

$$m(f) = \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}, \tag{3.1.5}$$

and $\sigma : \mathscr{A} \to C^\infty(S^*X)$, which maps a Green operator $\begin{pmatrix} P_+ + G & K \\ T & S \end{pmatrix}$ to the principal symbol $p^0(x, \xi)$ of its pseudodifferential part P.

While σ_* is not an isomorphism on $K_0(\mathscr{A})$, it does restrict to an isomorphism with range $K_0(C_v(S^*X^\circ)) \cong K_0(C_0^\infty(S^*X^\circ))$ such that $(\sigma_*)^{-1}$ is defined there. (The domain of this isomorphism is $K_0(\mathfrak{J}/\mathfrak{K})$, details can be found in [Gaa07].)

So (3.1.4) gives us that, intuitively, (the K_0-class of) each ψdbo projection is the sum of (the K_0-class of) a multiplication operator and (the K_0-class of) an operator supported in the interior of X, away from the boundary ∂X. This decomposition was essentially done in [MSS06], and we just had to improve slightly on their result.

It only remains to show that res_* vanishes for each of the two parts in (3.1.4). For the multiplication operators it is fairly obvious [Gaa07, Lemma 3.1], while the other is more difficult. Here, we had to rely on a technical lemma [Gaa07, Lemma 3.2] to show that we only need to investigate the residue of a certain kind of ψdbo projection Π_+; the advantage is that the residue $\mathrm{res}_X(\Pi_+)$ of this particular kind of operator in fact equals the residue $\mathrm{res}_{\widetilde{X}}\,\Pi$ of a ψdo projection Π on the closed manifold $\widetilde{X}$.

According to the well-known result by Wodzicki (Theorem 1.2.3) the residue $\mathrm{res}_{\widetilde{X}}\,\Pi$ vanishes, and we can conclude the proof, having essentially reduced the question to the known (but non-trivial) case of closed manifolds.

Remark 3.1.2. Our notation here differs slightly from that in [Gaa07]: Here C_0^∞ denotes “smooth, with compact support”, corresponding to C_c^∞ there; C_v here means “vanishing at infinity”, corresponding to C_0 there.

Remark 3.1.3. Since $K_0(\mathscr{A}) \cong K_0(\mathfrak{A})$, we can in fact extend the residue to idempotents in $\mathfrak{A}$: For any idempotent $\mathsf{A} \in \mathfrak{A}$ there is a corresponding $[A]_0$ in $K_0(\mathscr{A})$, and we can define $\overline{\mathrm{res}}(\mathsf{A}) = \mathrm{res}_*[A]_0$. In any case, this is zero.

Note that res does not have a continuous extension to $\mathfrak{A}$: $\mathrm{res}(R)$ vanishes for all smoothing operators R and, in the topology of $\mathfrak{A}$, the set of smoothing operators is dense in the set of compact operators, which includes the operators of order $-n$. The definition of $\overline{\mathrm{res}}$ above is valid only for idempotents in $\mathfrak{A}$.

Chapter 4

The sectorial projection

This chapter takes up the results regarding the sectorial projection of an elliptic boundary value problem from the article [GG08].

We define the operator and name some of its properties, both the general functional analytic properties as well as the more specific structural properties in our case. We discuss how it fits into Boutet de Monvel's calculus and its connection to the logarithms, and we show that the noncommutative residue in a natural way extends to the set of sectorial projections.

Finally, we look at a few interesting examples.

4.1 The sectorial projection

We consider a boundary value problem $\{P_+ + G, T\}$ in the Boutet de Monvel calculus of order $m > 0$, where P_+ is the truncation of a ψdo P on $\widetilde{X}$, G is a singular Green operator, and $T = \{T_0, \dots, T_{m-1}\}$ a system of trace operators. $B = (P + G)_T$ is the realization as described in Section 2.3.

Let $\theta < \varphi < \theta + 2\pi$, and assume that $\{P_+ + G - \lambda, T\}$ is parameter-elliptic for λ in two sectors V_θ and V_φ around Γ_θ and Γ_φ, respectively, such that both are rays of minimal growth.

Like in (1.3.2) we can then define the sectorial projection

$$\Pi_{\theta,\varphi}(B)u = \frac{i}{2\pi} \int_{\Gamma_{\theta,\varphi}} \lambda^{-1} B(B - \lambda)^{-1}\, u\, d\lambda, \quad u \in \mathcal{D}(B). \tag{4.1.1}$$

Unlike the case of Proposition 1.3.1, $\Pi_{\theta,\varphi}(B)$ is not necessarily a bounded operator and, a priori, its domain is only $\mathcal{D}(B)$. In some cases it is bounded, cf.

the proposition below, and we then extend the domain to all of $L_2(X,E)$ by continuity.

There are parallels to Proposition 1.3.1 in any case:

Proposition 4.1.1 ([GG08, Proposition 4.1]). *The operator $\Pi_{\theta,\varphi}(B)$ is a projection in $L_2(X,E)$: $\Pi_{\theta,\varphi}(B)^2 = \Pi_{\theta,\varphi}(B)$. Its range contains $E_{\theta,\varphi}(B)$ and its kernel contains $E_{\varphi,\theta+2\pi}(B) \dotplus E_0(B)$.*

(a) If B has a complete set of root vectors, then $\Pi_{\theta,\varphi}(B)$ is the bounded projection onto $\overline{E_{\theta,\varphi}(B)}$ along $\overline{E_{\theta,\varphi}(B)} \dotplus E_0(B)$.

(b) If B is normal (commutes with B^*), then $\Pi_{\theta,\varphi}(B)$ is the bounded orthogonal projection onto $\oplus_{\lambda\in sp(B)\cap\Lambda_{\theta,\varphi}} \ker(B-\lambda)$ along $\oplus_{\lambda\in sp(B)\setminus\Lambda_{\theta,\varphi}} \ker(B-\lambda)$.

The above proposition is essentially just functional analysis. In our case, we have additional information on the structure of the resolvent from (2.3.3), namely that

$$(B-\lambda)^{-1} = Q_{\lambda,+} + G_\lambda. \tag{4.1.2}$$

We use this to decompose $\Pi_{\theta,\varphi}(B)$ into two parts:

Theorem 4.1.2 ([GG08, Theorem 4.5]). *The sectorial projection satisfies*

$$\Pi_{\theta,\varphi}(B) = \Pi_{\theta,\varphi}(P)_+ + G_{\theta,\varphi} \tag{4.1.3}$$

where $\Pi_{\theta,\varphi}(P)_+$ is the truncation of the bounded ψdo $\Pi_{\theta,\varphi}(P)$ on $\widetilde{X}$ (from Section 1.3), while $G_{\theta,\varphi}$ is a generalized singular Green operator, given by

$$G_{\theta,\varphi} = \frac{i}{2\pi}\int_{\Gamma_{\theta,\varphi}} G_\lambda\, d\lambda, \tag{4.1.4}$$

bounded from $L_2(X,E)$ to $H^{-\varepsilon}(X,E)$.

For the specific (and rather technical) meaning of the term generalized singular Green operator, we refer to [GG08, Theorem 2.6]. The intuitive understanding is that $G_{\theta,\varphi}$ acts like a singular Green operator, cf. (2.1.6),

$$G_{\theta,\varphi}u(x) = \int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'} \int_0^\infty \widetilde{g}_{\theta,\varphi}(x',x_n,y_n,\xi')\, \acute{u}(\xi',y_n)\, dy_n\, d\!\!{}^-\xi', \tag{4.1.5}$$

but the symbol-kernel $\widetilde{g}_{\theta,\varphi}$ does not satisfy all the required decay-estimates. However, it does have an asymptotic expansion $\widetilde{g}_{\theta,\varphi} \sim \sum_{j=0}^\infty \widetilde{g}_{\theta,\varphi,-j}$ with terms given by

$$\widetilde{g}_{\theta,\varphi,-j}(x',x_n,y_n,\xi') = \frac{i}{2\pi}\int_{\Gamma_{\theta,\varphi}} \widetilde{g}_{-m-j}(x',x_n,y_n,\xi',\lambda)\, d\lambda, \tag{4.1.6}$$

where $\widetilde{g} \sim \sum_{j=0}^{\infty} \widetilde{g}_{-m-j}$ is the symbol-kernel of G_λ and $\Gamma_{\theta,\varphi}$ is the integration contour from (1.3.3). For $j > 0$, $\widetilde{g}_{\theta,\varphi,-j}$ is quasi-homogeneous in the sense

$$\widetilde{g}_{\theta,\varphi,-j}(x', \tfrac{x_n}{t}, \tfrac{y_n}{t}, t\xi') = t^{1-j}\, \widetilde{g}_{\theta,\varphi,-j}(x', x_n, y_n, \xi'), \quad t \geq 1, |\xi'| \geq 1. \tag{4.1.7}$$

We now consider the case where the system $\{P_+ + G, T\}$ is differential, that is, the operators P, $T_0, \dots, T_{m-1}$ are differential and $G = 0$.

Theorem 4.1.3 ([GG08, Theorem 4.6]). *Assume that $B = P_T$ is differential. Then $G_{\theta,\varphi}$ is bounded and $\Pi_{\theta,\varphi}(B)$ is thus a bounded projection in $L_2(X, E)$.*

The reason for $G_{\theta,\varphi}$ being bounded in this case is that the homogeneous terms $\widetilde{g}_{-m-j}$ of the symbol-kernel of G_λ have a certain exponential decay when B is differential, cf. [See67]. This also ensures that the quasi-homogeneity described in (4.1.7) applies for $j = 0$ (the principal part of the symbol-kernel) as well.

The proof of Theorem 4.1.2 cannot be improved in order to drop the term "generalized" in the description of the singular Green part. We show this in Example 4.4.1 where we consider a differential problem $B = P_T$ and show explicitly that $G_{\theta,\varphi}$ is not a true singular Green operator.

So the singular Green part of $\Pi_{\theta,\varphi}(B)$ is in general not in the Boutet de Monvel calculus, but how about the ψdo part $\Pi_{\theta,\varphi}(B)$? This question is determined by whether or not its symbol $\pi_{\theta,\varphi}$, given in (1.3.4), satisfies the transmission condition. The answer is the following:

Proposition 4.1.4 ([GG08, Lemma 4.7]). *$\pi_{\theta,\varphi}(x, \xi)$ satisfies the transmission condition when the order m of P is even. Hence $\Pi_{\theta,\varphi}(P)_+$ is in the Boutet de Monvel calculus for even m.*

Also in this case, the proof cannot be improved to general m: Example 4.4.1 — the same as mentioned above — gives a concrete case of an odd order operator where $\Pi_{\theta,\varphi}(P)$ does not satisfy the transmission condition.

4.2 Residue of the sectorial projection

As we have just mentioned, $\Pi_{\theta,\varphi}(B)$ is not always in Boutet de Monvel's calculus and hence the residue definition does not apply in general. However, in case it is defined, it satisfies an identity similar to Proposition 1.3.2:

Proposition 4.2.1. *Assume that $\Pi_{\theta,\varphi}(B)$ is in the Boutet de Monvel calculus. Then the noncommutative residue satisfies the identity*

$$\operatorname{res}\Pi_{\theta,\varphi}(B) = \frac{m}{2\pi i}\big(C_{0,\theta}(B) - C_{0,\varphi}(B)\big). \tag{4.2.1}$$

Proof. We begin with the right hand side of (4.2.1). Recall from (2.3.12) that

$$C_{0,\theta}(B) - C_{0,\varphi}(B) = \int_X \operatorname{tr}\left[c^Q_{n,\theta}(x) - c^Q_{n,\varphi}(x)\right]dx + \int_{X'} \operatorname{tr}\left[c^G_{n,\theta}(x') - c^G_{n,\varphi}(x')\right]dx'. \tag{4.2.2}$$

with

$$\begin{aligned} c^Q_{n,\theta}(x) - c^Q_{n,\varphi}(x) &= -\int_{\mathbb{R}^n}\left[e^{i\theta}q^h_{-m-n}(x,\xi,e^{i\theta}) - e^{i\varphi}q^h_{-m-n}(x,\xi,e^{i\varphi})\right]đ\xi,\\ c^G_{n,\theta}(x) - c^G_{n,\varphi}(x) &= -\int_{\mathbb{R}^{n-1}}\left[e^{i\theta}s^h_{-m+1-n}(x',\xi',e^{i\theta}) - e^{i\varphi}s^h_{-m+1-n}(x',\xi',e^{i\varphi})\right]đ\xi'. \end{aligned} \tag{4.2.3}$$

By Lemma 4.2.2 below, this reduces to

$$\begin{aligned} c^Q_{n,\theta}(x) - c^Q_{n,\varphi}(x) &= -\frac{1}{m}\int_{|\xi|=1}\int_{\Gamma^h_{\theta,\varphi}} q^h_{-m-n}(x,\xi,\lambda)\,d\lambda\,đS(\xi),\\ c^G_{n,\theta}(x) - c^G_{n,\varphi}(x) &= -\frac{1}{m}\int_{|\xi'|=1}\int_{\Gamma^h_{\theta,\varphi}} s^h_{-m+1-n}(x',\xi',\lambda)\,d\lambda\,đS(\xi'), \end{aligned} \tag{4.2.4}$$

where $\Gamma^h_{\theta,\varphi}$ is the integration contour

$$\Gamma^h_{\theta,\varphi} = \{te^{i\varphi} \mid \infty > t \geq 0\} \cup \{te^{i\theta} \mid 0 \leq t < \infty\}. \tag{4.2.5}$$

Now, for each $|\xi| = 1$ we have

$$\int_{\Gamma^h_{\theta,\varphi}} q^h_{-m-n}(x,\xi,\lambda)\,d\lambda = \int_{\mathcal{C}_{(x,\xi)}} q_{-m-n}(x,\xi,\lambda)\,d\lambda, \tag{4.2.6}$$

where $\mathcal{C}_{(x,\xi)}$ is the curve from (1.3.4). To see this, let $\mu = |\lambda|^{1/m}$. From [Gru96, eqs. (3.3.35-36)], we have

$$|q^h_{-m-n}(x,\xi,\lambda)| \leq C\begin{cases} |\xi,\mu|^{-m-n}, & \text{if } n < m,\\ |\xi|^{m-n}|\xi,\mu|^{-2m} & \text{if } n \leq m, \end{cases} \tag{4.2.7}$$

so that in any case, for $|\xi| = 1$ we have

$$|q^h_{-m-n}(x,\xi,\lambda)| \leq C|1,\mu|^{-m-\varepsilon},\ \text{which is } O(\langle\lambda\rangle^{-1-\delta}). \tag{4.2.8}$$

Furthermore, for $|\xi| = 1$ we have $q^h_{-m-n}(x,\xi,\lambda) = q_{-m-n}(x,\xi,\lambda)$, which is a meromorphic function in $\lambda \in \Lambda_{\theta,\varphi}$ with poles at the eigenvalues of $p_m(x,\xi)$, i.e., inside $\mathcal{C}_{(x,\xi)}$. As it is $O(|\lambda|^{-1-\delta})$ for $\lambda \to \infty$, it follows that the infinite contour $\Gamma^h_{\theta,\varphi}$ can be deformed into the closed contour $\mathcal{C}_{(x,\xi)}$, since the two contours enclose the same poles. Hence, (4.2.6) is proved.

By (1.3.4), the right hand side of (4.2.6) is $(-2\pi i)\,\pi_{\theta,\varphi,-n}(x,\xi)$, the symbol of $\Pi_{\theta,\varphi}(P)$ of degree $-n$. So combining this with (4.2.4) we get

$$\frac{m}{2\pi i}\left(c^Q_{n,\theta}(x) - c^Q_{n,\varphi}(x)\right) = \int_{|\xi|=1} \pi_{-n}(x,\xi)\, đS(\xi), \tag{4.2.9}$$

which, by definition, is $\operatorname{res}_x \Pi_{\theta,\varphi}(P)$.

For the singular Green part of (4.2.4) we have, for $|\xi'| = 1$,

$$\int_{\Gamma^h_{\theta,\varphi}} s^h_{-m+1-n}(x',\xi',\lambda)\,d\lambda = \int_0^\infty \int_{\Gamma_{\theta,\varphi}} \widetilde{g}_{-m+1-n}(x',x_n,x_n,\xi',\lambda)\,d\lambda\,dx_n, \tag{4.2.10}$$

where the right hand side is $(-2\pi i)\ \operatorname{tr}_n g_{\theta,\varphi,1-n}(x',\xi')$, cf. (4.1.6). To show this, note that s^h_{-m+1-n} equals s_{-m+1-n} for $|\xi'| = 1$, and is holomorphic in λ in a neighborhood of $\Gamma^h_{\theta,\varphi}$, so the integration contour $\Gamma^h_{\theta,\varphi}$ can be deformed to $\Gamma_{\theta,\varphi}$. The left hand side of (4.2.9) thus equals

$$\int_{\Gamma_{\theta,\varphi}} s_{-m+1-n}(x',\xi',\lambda)\,d\lambda = \int_{\Gamma_{\theta,\varphi}} \int_0^\infty \widetilde{g}_{-m+1-n}(x',x_n,x_n,\xi',\lambda)\,dx_n\,d\lambda. \tag{4.2.11}$$

To arrive at the right hand side of (4.2.10), we only need to change the order of integration in x_n and λ; this is easily achieved using Fubini's theorem ($\widetilde{g}_{-m+1-n}$ is $O(\langle x_n\rangle^{-N}\langle\lambda\rangle^{-1-\delta})$ for $|\xi'| = 1$, any $N \in \mathbb{N}$).

Combining (4.2.10) with (4.2.4) now gives us

$$\frac{m}{2\pi i}\left(c^G_{n,\theta}(x) - c^G_{n,\varphi}(x)\right) = \int_{|\xi'|=1} \operatorname{tr}_n g_{\theta,\varphi,1-n}(x',\xi')\, đS(\xi'), \tag{4.2.12}$$

which equals $\operatorname{res}_{x'} \operatorname{tr}_n G_{\theta,\varphi}$. Inserting this into (4.2.2) we obtain

$$\frac{m}{2\pi i}\left(C_{0,\theta}(B) - C_{0,\varphi}(B)\right) = \int_X \operatorname{tr}\ \operatorname{res}_x \Pi_{\theta,\varphi}(P)\,dx + \int_{\partial X} \operatorname{tr}\ \operatorname{res}_{x'} \operatorname{tr}_n G_{\theta,\varphi}\,dx', \tag{4.2.13}$$

the residue of $\Pi_{\theta,\varphi}(B) = \Pi_{\theta,\varphi}(P)_+ + G_{\theta,\varphi}$. □

In the proof we used the following lemma, where Γ is the set

$$\Gamma = \{re^{i\omega} \mid r \geq 0,\ \omega \in [\theta-\varepsilon,\theta+\varepsilon] \cup [\varphi-\varepsilon,\varphi+\varepsilon]\}, \tag{4.2.14}$$

for some sufficiently small $\varepsilon > 0$.

Lemma 4.2.2. *Let $m > 0$. Let $f(\xi, \lambda)$ be continuous for $(\xi, \lambda) \in (\mathbb{R}^k \setminus \{0\}) \times \Gamma$ and quasi-homogeneous there in the sense that $f(s\xi, s^m\lambda) = s^{-m-k} f(\xi, \lambda)$ for all $s > 0$, and integrable at $\xi = 0$ for each $\lambda \neq 0$. Then*

$$\int_{\mathbb{R}^k} \left[e^{i\theta} f(\xi, e^{i\theta}) - e^{i\varphi} f(\xi, e^{i\varphi})\right] đ\xi = \frac{1}{m} \int_{|\xi|=1} \int_{\Gamma^h_{\theta,\varphi}} f(\xi, \lambda)\, d\lambda\, đS(\xi). \tag{4.2.15}$$

Proof. The proof is completely analogous to the proof of [Gru05, Lemma 1.3], except for the phases $e^{i\theta}$ and $e^{i\varphi}$ which enter the contour integral. □

Since $C_{0,\theta}(B)$ and $C_{0,\varphi}(B)$ are well-defined in any case, we can now use Proposition 4.2.1 to define the noncommutative residue of $\Pi_{\theta,\varphi}(B)$ even when it is not a Boutet de Monvel operator:

Definition 4.2.3. *We define the noncommutative residue of $\Pi_{\theta,\varphi}(B)$ to be*

$$\operatorname{res} \Pi_{\theta,\varphi}(B) = \frac{m}{2\pi i}\big(C_{0,\theta}(B) - C_{0,\varphi}(B)\big). \tag{4.2.16}$$

As shown in the proposition above, this is consistent with the definition of [FGLS96] when the latter is applicable.

In (4.2.9), we saw that

$$\operatorname{res}_x \Pi_{\theta,\varphi}(P) = \frac{m}{2\pi i}\big(c^Q_{n,\theta}(x) - c^Q_{n,\varphi}(x)\big). \tag{4.2.17}$$

Together with Theorems 1.4.1 and 1.4.3 (or Proposition 1.4.4), this proves Proposition 1.3.2 as we promised in Chapter 1.

Moreover,

$$\operatorname{res} \Pi_{\theta,\varphi}(B) = \int_X \operatorname{tr} \operatorname{res}_x \Pi_{\theta,\varphi}(P) dx + \frac{m}{2\pi i} \int_{X'} \operatorname{tr} \left[c^G_{n,\theta}(x') - c^G_{n,\varphi}(x')\right] dx', \tag{4.2.18}$$

where we recognize the X-integral as $\operatorname{res}_+ \Pi_{\theta,\varphi}(P)$, well-defined whether or not $\Pi_{\theta,\varphi}(P)$ satisfies the transmission condition. This prompts us to define

$$\operatorname{res}_{x'} G_{\theta,\varphi} = \frac{m}{2\pi i}\big(c^G_{n,\theta}(x') - c^G_{n,\varphi}(x')\big), \quad \operatorname{res} G_{\theta,\varphi} = \int_{X'} \operatorname{tr} \operatorname{res}_{x'} G_{\theta,\varphi}\, dx', \tag{4.2.19}$$

such that we obtain the "usual" formula

$$\operatorname{res} \Pi_{\theta,\varphi}(B) = \operatorname{res}(\Pi_{\theta,\varphi}(P)_+ + G_{\theta,\varphi}) = \operatorname{res}_+ \Pi_{\theta,\varphi}(P) + \operatorname{res} G_{\theta,\varphi}. \tag{4.2.20}$$

As shown above, whenever $G_{\theta,\varphi}$ is a singular Green operator, the definition (4.2.19) is indeed in agreement with the usual definition by [FGLS96].

Remark 4.2.4. We have not investigated if the noncommutative residue, as defined above, is in fact a trace (in the sense that it vanishes on commutators). Partial results in this direction have been obtained by Grubb [Gru08]; for instance, if B is of even order m and $A = P'_+ + G'$ has order and class 0, the residue of $[A, \Pi_{\theta,\varphi}(B)]$ vanishes [Gru08, Theorem 7.5].

4.2.1 Spectral asymmetry

Combining (2.3.13) and the above Definition 4.2.3 we easily get

$$\zeta_\theta(B,0) - \zeta_\varphi(B,0) = \frac{2\pi i}{m} \operatorname{res} \Pi_{\theta,\varphi}(B), \tag{4.2.21}$$

which shows that — in this case as well, cf. Proposition 1.3.2 — the sectorial projection encodes the dependence of $\zeta_\theta(B,0)$ upon θ. Occasionally, this dependence is somewhat vaguely referred to as the (sectorial) *spectral asymmetry* of B, although, for self-adjoint B, the phrase spectral asymmetry traditionally refers to $\eta(B)$.

The statement in Theorem 1.1.4, in the setting of boundary problems, has been a central question in our work: is $\zeta_\theta(B,0)$ independent of θ? Or, equivalently, is the residue of $\Pi_{\theta,\varphi}(B)$ zero?

Unfortunately, we have not obtained a final result in this matter. From Chapter 3, we see that the answer is yes when $\Pi_{\theta,\varphi}(B)$ is a Boutet de Monvel operator; Example 4.4.2 gives an example of a sectorial projection not in Boutet de Monvel's calculus which nevertheless has residue zero. But otherwise it is still an open question: our work in this direction is explained in Chapter 5.

If we assume B to be self-adjoint, the dependence of $\zeta_\theta(B,0)$ on θ is given in terms of the residue of the eta-function:

$$\zeta_\downarrow(B,0) - \zeta_\uparrow(B,0) = i\pi\, R(B) = i\pi \operatorname{res}_{s=0} \eta(B,s). \tag{4.2.22}$$

This follows from Proposition 1.1.3, the proof of which only requires that the operator is self-adjoint; $\downarrow$ ($\uparrow$) refers to any angle in the lower (upper) half-plane. The projection $\Pi_{\downarrow\uparrow}(B)$ is the projection onto the eigenspaces for the eigenvalues with positive real part, i.e., it equals the projection $\Pi_>$ onto the eigenspaces corresponding to eigenvalues in $\mathbb{R}_+$. This proves the following proposition:

Proposition 4.2.5. *When the realization B is self-adjoint, we have*

$$\operatorname{res} \Pi_> = \operatorname{res} \Pi_{\downarrow\uparrow}(B) = \frac{m}{2}\, R(B) = \frac{m}{2} \operatorname{res}_{s=0} \eta(B,s). \tag{4.2.23}$$

However, not much is known about $R(B)$ for the boundary problems we consider — as opposed to the boundaryless case where it has been studied intensively (and is known to vanish). See also Section 5.4.

4.3 Logarithms

The results presented in Section 4.1 are actually shown in [GG08] as corollaries to similar results for $\log B$, the logarithm of B. Since these logarithms were not central in my thesis work, I have chosen to downplay their role in the presentation of the material here. Although logarithms were used in the proofs in [GG08], one could easily have avoided this: working directly with the symbol-structure of $\Pi_{\theta,\varphi}(B)$ one can show the results above, often in a manner very similar to the methods used in [GG08, Sections 2 and 3].

However, let us quickly discuss the connection between $\log B$ and $\Pi_{\theta,\varphi}(B)$: Since B has Γ_θ as a ray of minimal growth, we can define an operator

$$\log_\theta B\, u = \frac{i}{2\pi} \lim_{s\to 0^+} \int_{\mathcal{C}_\theta} \lambda_\theta^{-s} \log_\theta \lambda\, (B-\lambda)^{-1}\, u\, d\lambda, \quad u \in \mathcal{D}(B). \tag{4.3.1}$$

The subscript θ once again indicates that we take the holomorphic branch of $\lambda^{-s}\log\lambda$ with a branch cut at Γ_θ.

The logarithm has a "Boutet de Monvel-like" structure [GG08, Theorem 2.2]

$$\log_\theta B = \log_\theta (P+G)_T = (\log_\theta P)_+ + G^{\log_\theta}, \tag{4.3.2}$$

where $\log_\theta P$ is the logarithm (with a branch cut at θ) of P on $\widetilde{X}$ and $G^{\log_\theta}$ is a generalized singular Green operator. $\log_\theta B$ is bounded from $L_2(X,E)$ to $H^{-\varepsilon}(X,E)$ for any $\varepsilon > 0$.

The basic zeta value of B can be interpreted as a noncommutative residue

$$C_{0,\theta}(B) = -\frac{1}{m}\operatorname{res}_+(\log_\theta P) - \frac{1}{m}\operatorname{res} S_{\mathrm{sub}}^{\log_\theta}, \tag{4.3.3}$$

[GG08, Theorem 3.2]. Here $\operatorname{res}_+(\log_\theta P)$ is the integration over X of the residue density $\operatorname{res}_{x,1}\log_\theta P$ on $\widetilde{X}$, a generalization by Lesch [Les99] of Wodzicki's noncommutative residue to log-polyhomogeneous operators; $\operatorname{res}(S_{\mathrm{sub}}^{\log_\theta})$ is intuitively the residue of the (generalized) normal trace of $G^{\log_\theta}$ from above, subtracted the principal part for technical reasons.

For this reason — and because this would be an analogy to the boundaryless case, cf. Scott [Sco05] — we define ([GG08, Definition 3.3]) the noncommutative residue of the logarithm to be

$$\operatorname{res}(\log_\theta B) = -m\, C_{0,\theta}(B). \tag{4.3.4}$$

The dependence of the choice of θ in the definition of the logarithm is given by the sectorial projection:

Proposition 4.3.1 ([GG08, Proposition 4.4])**.** *For $u \in \mathcal{D}(B)$,*

$$\log_\theta B\, u - \log_\varphi B\, u = -2\pi i\ \Pi_{\theta,\varphi}(B)\, u. \tag{4.3.5}$$

The structure of $\log_\theta B$ from (4.3.2) is consistent with this, in the sense that, when the operators below are applied to $u \in \mathcal{D}(B)$, we have

$$(\log_\theta P)_+ - (\log_\varphi P)_+ = -2\pi i\ \Pi_{\theta,\varphi}(P)_+, \quad G^{\log_\theta} - G^{\log_\varphi} = -2\pi i\, G_{\theta,\varphi}. \tag{4.3.6}$$

The residue definitions (4.2.16) and (4.3.4) are consistent with the proposition:

$$\operatorname{res}(\log_\theta B) - \operatorname{res}(\log_\varphi B) = -2\pi i\ \operatorname{res}\Pi_{\theta,\varphi}(B). \tag{4.3.7}$$

Obviously, this is no surprise, since we used the identity in the proposition to define the residue of $\Pi_{\theta,\varphi}(B)$ from the residues of the logarithms.

4.4 Examples

We conclude this chapter with two concrete examples of sectorial projections. One is the abovementioned example showing that the sectorial projection need not be in Boutet de Monvel's calculus. The other is interesting because it gives an example of a sectorial projection which is not in Boutet de Monvel's algebra, but nevertheless has vanishing residue — even though its residue densities are locally non-vanishing.

Example 4.4.1. This example is given in detail in [GG08, Example 4.8], so we skip the intermediate computations here. We consider the differential operators A and P on $\mathbb{R}^4_+$ given by

$$A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} D_1 + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} D_2 + \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} D_3 + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} D_4, \tag{4.4.1}$$

and

$$P = \begin{pmatrix} 0 & -A^* \\ A & 0 \end{pmatrix}, \tag{4.4.2}$$

where A^* is the formal adjoint of A. The trace operator $T = B\gamma_0$, with

$$B = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix},$$

is then added to obtain the realization $B = P_T$. The system $\{P - \lambda, T\}$ is parameter-elliptic for λ on any ray in $\mathbb{C} \setminus i\mathbb{R}$, so we can consider the sectorial projection $\Pi_{\theta,\varphi}(B)$ with $\theta = 0$ and $\varphi = \pi$, that is, the projection on the eigenspaces corresponding to eigenvalues with positive imaginary part. (We regard the whole thing as a localization of an operator on a compact manifold.)

First, the symbol of the ψdo part $\Pi_{\theta,\varphi}(P)$ is

$$\pi_{\theta,\varphi}(\xi) = \frac{1}{2|\xi|} \begin{pmatrix} |\xi| & 0 & \xi_1 + i\xi_4 & -i\xi_2 + \xi_3 \\ 0 & |\xi| & i\xi_2 + \xi_3 & -\xi_1 + i\xi_4 \\ \xi_1 - i\xi_4 & -i\xi_2 + \xi_3 & |\xi| & 0 \\ i\xi_2 + \xi_3 & -\xi_1 - i\xi_4 & 0 & |\xi| \end{pmatrix}, \tag{4.4.3}$$

which is easily seen not to satisfy the transmission condition: In order for a polyhomogeneous symbol $p(x,\xi) \sim \sum_{j=0}^{\infty} p_{m-j}(x,\xi)$ of order $m \in \mathbb{Z}$ to satisfy the transmission condition, it has to obey

$$D_x^\beta D_\xi^\alpha p_{m-j}(x', 0, 0, \xi_n) = (-1)^{m-j-|\alpha|} D_x^\beta D_\xi^\alpha p_{m-j}(x', 0, 0, -\xi_n). \tag{4.4.4}$$

In particular, $\pi_{\theta,\varphi}$ would have to be even in ξ_n for $\xi' = 0$, which it clearly is not.

Turning to the singular Green part $G_{\theta,\varphi}$, its symbol-kernel is given by

$$\widetilde{g}_{\theta,\varphi}(x_n, y_n, \xi') = \frac{i}{2\pi} \int_{-\infty}^{\infty} \widetilde{g}(x_n, y_n, \xi', \lambda)\, d\lambda, \tag{4.4.5}$$

where the integrand is the parameter-dependent symbol-kernel of G_λ (the singular Green part of the resolvent):

$$\widetilde{g}(x_n, y_n, \xi', \lambda) = \frac{1}{2\sigma} \begin{pmatrix} -i\xi_1 + i\sigma & -\xi_2 - i\xi_3 & -\lambda & 0 \\ \xi_2 - i\xi_3 & i\xi_1 + i\sigma & 0 & -\lambda \\ -\lambda & 0 & -i\xi_1 - i\sigma & -\xi_2 - i\xi_3 \\ 0 & -\lambda & \xi_2 - i\xi_3 & i\xi_1 - i\sigma \end{pmatrix} e^{-\sigma(x_n + y_n)}, \tag{4.4.6}$$

with $\sigma = (|\xi'|^2 + \lambda^2)^{1/2}$.

A necessary condition for $\widetilde{g}_{\theta,\varphi}$ to be the symbol-kernel of a true singular Green operator, is that, for each (x', ξ'), it lies in $\mathcal{S}(\overline{\mathbb{R}}^2_{++})$, i.e., it can be described as the restriction to $\mathbb{R}^2_{++} = \mathbb{R}_+ \times \mathbb{R}_+$ of a Schwartz function in $(x_n, y_n) \in \mathbb{R}^2$. However, $\widetilde{g}_{\theta,\varphi}$ is in fact unbounded for $(x_n, y_n) \to 0$. To see this, note that the diagonal contains terms of the form

$$\int_{-\infty}^{\infty} e^{-(|\xi'|^2+t^2)^{1/2}(x_n+y_n)}\, dt \geq \int_{-\infty}^{\infty} e^{-(|\xi'|+|t|)(x_n+y_n)}\, dt = 2\,\frac{e^{-|\xi'|(x_n+y_n)}}{x_n+y_n}. \tag{4.4.7}$$

So $\widetilde{g}_{\theta,\varphi}$ cannot be the restriction of a smooth function on $\mathbb{R}^2$, and is not the symbol-kernel of a singular Green operator.

Although we discussed this example on $\mathbb{R}^4_+$, it can easily be carried over to a compact manifold, e.g. $S^1 \times S^1 \times S^1 \times [0, 1]$.

Let us now look at another interesting example, namely a differential realization $B = P_T$ where, in the calculation of $\operatorname{res} \Pi_{\theta,\varphi}(B)$, we find that both $\operatorname{res}_+ \Pi_{\theta,\varphi}(P)$ and $\operatorname{res} G_{\theta,\varphi}$ are non-zero — but of same magnitude and opposite sign such that they cancel each other out.

The present example is inspired by Wodzicki's example of a nonvanishing residue density [Wod82].

Example 4.4.2. We consider the Dirichlet realization $B = P_{\gamma_0}$ of the differential operator P on $\mathbb{R}^2_+$ given by

$$P = \begin{pmatrix} -\Delta & D_1^2 \\ \varphi(x_2) D_2 & \Delta \end{pmatrix}, \tag{4.4.8}$$

where φ is a smooth function on $\overline{\mathbb{R}}_+$. We will use the notions (x_1, x_2) and (x', x_n) interchangeably. We proceed to find Q_λ and G_λ in the usual manner:

P has principal symbol

$$p_2(x, \xi) = \begin{pmatrix} \xi_1^2 + \xi_2^2 & \xi_1^2 \\ 0 & -\xi_1^2 - \xi_2^2 \end{pmatrix}. \tag{4.4.9}$$

The eigenvalues of the principal symbol are $\pm|\xi|^2 = \pm(\xi_1^2 + \xi_2^2)$. The principal symbol of the parametrix with parameter Q_λ thus becomes

$$q_{-2}(x, \xi, \lambda) = (p_2(x, \xi) - \lambda)^{-1} = \frac{1}{|\xi|^4 - \lambda^2} \begin{pmatrix} |\xi|^2 + \lambda & \xi_1^2 \\ 0 & -|\xi|^2 + \lambda \end{pmatrix}. \tag{4.4.10}$$

Working out the singular Green part, G_λ, one finds that $\{P-\lambda,\gamma_0\}$ is parameter-elliptic for λ on all rays in $\mathbb{C}\setminus\mathbb{R}$. For such λ the principal part of the symbol-kernel $\widetilde{g}$ is

$$\widetilde{g}_{-2}(x',x_n,y_n,\xi',\lambda) = -\begin{pmatrix} \frac{e^{-\sqrt{\xi_1^2-\lambda}(x_n+y_n)}}{2\sqrt{\xi_1^2-\lambda}} & \frac{\xi_1^2}{4\lambda}\Big(\frac{e^{-\sqrt{\xi_1^2-\lambda}(x_n+y_n)}}{\sqrt{\xi_1^2-\lambda}} - \frac{e^{-\sqrt{\xi_1^2+\lambda}(x_n+y_n)}}{\sqrt{\xi_1^2+\lambda}}\Big) \\ 0 & -\frac{e^{-\sqrt{\xi_1^2+\lambda}(x_n+y_n)}}{2\sqrt{\xi_1^2+\lambda}} \end{pmatrix} \tag{4.4.11}$$

We seek the operator $\Pi_{\theta,\varphi}(P_{\gamma_0})$ with $\theta=-\frac{\pi}{2}$ and $\varphi=\frac{\pi}{2}$, the projection on the eigenspaces corresponding to eigenvalues with positive real part.

The principal symbol, resp. symbol-kernel, of $\Pi_{\theta,\varphi}(P)$, resp. $G_{\theta,\varphi}$, is

$$\pi_{\theta,\varphi,0}(x,\xi) = \frac{i}{2\pi}\int_{\mathcal{C}_\xi} q_{-2}(x,\xi,\lambda)d\lambda = \begin{pmatrix} 1 & \frac{\xi_2^2}{2|\xi|^2} \\ 0 & 0 \end{pmatrix} \tag{4.4.12}$$

$$\widetilde{g}_{\theta,\varphi,0}(x',x_n,y_n,\xi') = \frac{i}{2\pi}\int_{\Gamma_{\theta,\varphi}} \widetilde{g}_{-2}(x',x_n,y_n,\xi',\lambda)d\lambda = \begin{pmatrix} 0 & -\frac{1}{4}|\xi_1|e^{-|\xi_1|(x_n+y_n)} \\ 0 & 0 \end{pmatrix},$$

where $\mathcal{C}_\xi$ is a closed curve encircling the pole $\lambda=|\xi|^2$ exactly once.

We see that $\widetilde{g}_{\theta,\varphi,0}$ is a true singular Green symbol-kernel, and hence the "principal part" of $\Pi_{\theta,\varphi}(B)$ is in the Boutet de Monvel calculus. (The "full" operator is not a Boutet de Monvel operator, however. To see this, one can show that the second term $\widetilde{g}_{\theta,\varphi,-1}$ — although bounded in (x_n,y_n) — has unbounded x_n and y_n derivatives near the origin and $\widetilde{g}_{\theta,\varphi}$ cannot be a true singular Green symbol-kernel.)

To calculate the non-commutative residue of $\Pi_{\theta,\varphi}(B)$, we need to find the lower order terms of the parametrix. Since n is 2, the contribution to the residue from the pseudodifferential part stems from $\pi_{-2}(x,\xi)$.

The lower order terms of the symbol $p(x,\xi)$ are

$$p_1(x,\xi) = \begin{pmatrix} 0 & 0 \\ \varphi(x_2)\xi_2 & 0 \end{pmatrix} \quad \text{and} \quad p_0(x,\xi)=0, \tag{4.4.13}$$

and the next terms of our parametrix then become

$$q_{-3} = -q_{-2}p_1q_{-2}, \quad q_{-4} = -q_{-2}p_1q_{-3} - q_{-2}\sum_{|\alpha|=1} D_\xi^\alpha p_2 \partial_x^\alpha q_{-3} - q_{-2}p_0q_{-2}. \tag{4.4.14}$$

For us, q_{-4} is the relevant term:

$$q_{-4}(x,\xi,\lambda) = \begin{pmatrix} -\frac{\xi_1^2\xi_2^2\left(4i|\xi|^2\varphi'(x_2)-\xi_1^2\varphi(x_2)^2\right)}{(|\xi|^2-\lambda)(|\xi|^4-\lambda^2)^2} & -\frac{\xi_1^4\xi_2^2\left(4i|\xi|^2\varphi'(x_2)-\xi_1^2\varphi(x_2)^2\right)}{(|\xi|^4-\lambda^2)^3} \\ -\frac{i\xi_2^2\left(i\xi_1^2\varphi(x_2)^2+2(|\xi|^2-\lambda)\varphi'(x_2)\right)}{(|\xi|^4-\lambda^2)^2} & -\frac{\xi_1^2\xi_2^2\left(\xi_1^2\varphi(x_2)^2-2i(|\xi|^2-\lambda)\varphi'(x_2)\right)}{(|\xi|^2+\lambda)(|\xi|^4-\lambda^2)^2} \end{pmatrix}, \tag{4.4.15}$$

which leads to the following result for the projection's symbol

$$\begin{aligned} \pi_{-2}(x,\xi) &= \frac{i}{2\pi}\int_{\mathcal{C}_\xi} q_{-4}(x,\xi,\lambda)\,d\lambda = -\operatorname{Res}_{\lambda=|\xi|^2}\ q_{-4}(x,\xi,\lambda) \\ &= \begin{pmatrix} \frac{3\xi_1^2\xi_2^2\left(\xi_1^2\varphi(x_2)^2-4i|\xi|^2\varphi'(x_2)\right)}{16|\xi|^8} & \frac{3\xi_1^4\xi_2^2\left(\xi_1^2\varphi(x_2)^2-4i|\xi|^2\varphi'(x_2)\right)}{16|\xi|^{10}} \\ \frac{i\xi_2^2\left(i\xi_1^2\varphi(x_2)^2+2|\xi|^2\varphi'(x_2)\right)}{4|\xi|^6} & \frac{\xi_1^2\xi_2^2\left(4i|\xi|^2\varphi'(x_2)-3\xi_1^2\varphi(x_2)^2\right)}{16|\xi|^8} \end{pmatrix}. \end{aligned} \tag{4.4.16}$$

For the trace we find

$$\operatorname{tr}\pi_{-2}(x',\xi') = -\frac{i\xi_1^2\xi_2^2}{2|\xi|^6}\ \varphi'(x_n). \tag{4.4.17}$$

Integrating over the unit sphere in ξ-space, in polar coordinates, then gives us

$$\begin{aligned} \operatorname{res}_x \Pi_{\theta,\varphi}(P) &= \int_{|\xi|=1} \operatorname{tr}\pi_{-2}(x,\xi)\,{\it đ}S(\xi) = -\int_{|\xi|=1}\frac{i\xi_1^2\xi_2^2\varphi'(x_n)}{2|\xi|^6}\,{\it đ}S(\xi) \\ &= -\frac{i\varphi'(x_n)}{8\pi^2}\int_0^{2\pi}\cos(\omega)^2\sin(\omega)^2 d\omega = \frac{\varphi'(x_n)}{32\pi i}. \end{aligned} \tag{4.4.18}$$

Remark 4.4.3. It should be noted that we have chosen P with "minimal" lower order terms (4.4.13), in the sense that another operator P' with the same principal part as P, but lower order terms given by

$$p_1'(x,\xi) = \left(a_{ij}(x)\xi_1 + b_{ij}(x)\xi_2\right)_{i,j=1,2}, \quad p_0'(x,\xi) = \left(c_{ij}(x)\right)_{i,j=1,2}, \tag{4.4.19}$$

will have

$$\operatorname{res}_x \Pi_{\theta,\varphi}(P') = \frac{\partial_{x_n} b_{21}(x)}{32\pi i}, \tag{4.4.20}$$

i.e., only the coefficient of ξ_2 in the $(2,1)$-entry of the matrix p_1' will contribute.

We now turn our attention to the contribution from the singular Green part. Actually calculating $\widetilde{g}_{\theta,\varphi,-1}$ is quite difficult, but we do not need this. Recall that the noncommutative residue density of $G_{\theta,\varphi}$ is given by

$$\operatorname{res}_x G_{\theta,\varphi} = \frac{m}{2\pi i}\left(\operatorname{tr} c_{n,\theta}^G(x') - \operatorname{tr} c_{n,\varphi}^G(x')\right), \tag{4.4.21}$$

where

$$c^G_{n,\theta}(x') = -e^{i\theta} \int_{\mathbb{R}^{n-1}} s^h_{-m+1-n}(x', \xi', e^{i\theta}) đ\xi', \tag{4.4.22}$$

and s^h_{-m+1-n} is the strictly homogeneous version of

$$s_{-m+1-n}(x', \xi', \lambda) = \int_0^\infty \widetilde{g}_{-m-n}(x', x_n, x_n, \xi', \lambda) dx_n. \tag{4.4.23}$$

The expressions for g_{-3} and s_{-3} are quite complicated, but we end up with

$$\operatorname{tr} s^h_{-3}(x', \xi', \lambda) = \frac{i\xi_1^2}{8\lambda^3}\Big(2\sqrt{\xi_1^2+\lambda} - 2\sqrt{\xi_1^2-\lambda} - \frac{\lambda}{\sqrt{\xi_1^2-\lambda}} - \frac{\lambda}{\sqrt{\xi_1^2+\lambda}}\Big)\varphi(0). \tag{4.4.24}$$

(At first sight, this expression appears non-integrable in $\xi' = \xi_1$, but cancellations ensure a $O(|\xi'|^{-3})$ decay. Which we know it has from homogeneity in any case.) We obtain then, for $\lambda \in \mathbb{C} \setminus \mathbb{R}$,

$$\int_{\mathbb{R}} \operatorname{tr} s^h_{-3}(x', \xi', \lambda) đ\xi' = \frac{i\varphi(0)}{64\pi} \, \frac{\log(-\lambda) - \log(\lambda)}{\lambda}. \tag{4.4.25}$$

Hence, inserting $e^{i\theta} = -i$ in (4.4.22), we find

$$\operatorname{tr} c^G_{n,\theta}(x') = -(-i)\frac{i\varphi(0)}{64\pi}\frac{\log(i) - \log(-i)}{-i} = \frac{\varphi(0)}{64}. \tag{4.4.26}$$

Likewise for $e^{i\varphi} = i$, where we get $\operatorname{tr} c^G_{n,\varphi} = -\varphi(0)/64$. We arrive at

$$\operatorname{res}_{x'} G_{\theta,\varphi} = \frac{m}{2\pi i}\big[\operatorname{tr} c^G_{n,\theta}(x') - \operatorname{tr} c^G_{n,\varphi}(x')\big] = \frac{\varphi(0)}{32\pi i}. \tag{4.4.27}$$

Now, let us consider this problem on the compact manifold $X = S^1 \times [0, a]$. The operator P from (4.4.8) has an obvious analogue on X if we consider $D_2 = D_n$ as the usual differentiaion on $[0, a]$ and $D_1 = D'$ as the usual invariant vector field on S^1. We take $\varphi \in C^\infty([0, a])$.

Carrying the local calculations above over to X, we obtain the following results: The noncommutative residue of the ψdo $\Pi_{\theta,\varphi}(P)_+$ is

$$\begin{aligned}\operatorname{res}_+ \Pi_{\theta,\varphi}(P) &= \int_X \operatorname{res}_x \Pi_{\theta,\varphi}(P)\, dx \\ &= \frac{\operatorname{vol}(S^1)}{32\pi i}\int_0^a \varphi'(x_n) dx_n = \frac{\varphi(a) - \varphi(0)}{16i},\end{aligned} \tag{4.4.28}$$

while the residue of the singular Green part is

$$\operatorname{res} G_{\theta,\varphi} = \int_{\partial X} \operatorname{res}_{x'} G_{\theta,\varphi}\, dx'. \tag{4.4.29}$$

The boundary is the disjoint union of two circles

$$\partial X = S^1 \times \{0\} \cup S^1 \times \{a\}, \tag{4.4.30}$$

and hence the integral over ∂X (4.4.21) splits into two:

$$\int_{\partial X} (\cdots)\, dx' = \int_{S^1 \times \{0\}} (\cdots)\, dx' - \int_{S^1 \times \{a\}} (\cdots)\, dx', \tag{4.4.31}$$

where the sign is due to the opposite orientation of the boundaries at $x_2 = 0$ and $x_2 = a$, respectively. In comparison with (4.4.27), we find at $x_2 = a$ that

$$\operatorname{res}_{x'} G_{\theta,\varphi} = \frac{\varphi(a)}{32\pi i}, \tag{4.4.32}$$

such that the residue of $G_{\theta,\varphi}$ becomes

$$\begin{aligned} \operatorname{res} G_{\theta,\varphi} &= \frac{1}{32\pi i}\Big(\int_{S^1 \times \{0\}} \varphi(0)\, dx' - \int_{S^1 \times \{a\}} \varphi(a)\, dx' \Big) \\ &= \frac{\operatorname{vol}(S^1)}{32\pi i}\big(\varphi(0) - \varphi(a)\big) = \frac{\varphi(0) - \varphi(a)}{16i} = -\operatorname{res}_+(\Pi_{\theta,\varphi}(P)). \end{aligned} \tag{4.4.33}$$

The residue of the projection is then

$$\operatorname{res} \Pi_{\theta,\varphi}(B) = \operatorname{res}_+ \Pi_{\theta,\varphi}(P) + \operatorname{res} G_{\theta,\varphi} = 0. \tag{4.4.34}$$

So the residue *does* indeed vanish, through a somewhat miraculous cancellation.

One might think that this vanishing has nothing to do with the nature of the projection, and that in fact the contribution from each ray vanishes, i.e, that $C_{0,\theta}$ and $C_{0,\varphi}$ are both zero. However this is not the case, as we find

$$C_{0,\theta}(B) = C_{0,\varphi}(B) = \frac{1}{64} \int_0^a \varphi(x_n)^2\, dx_n + \frac{\varphi(a) - \varphi(0)}{8i}. \tag{4.4.35}$$

So it seems the property of being a projection is important for the vanishing.

Note that the residue of $\log_\theta B$ is $-mC_{0,\theta}(B)$, which is apparently non-zero (in general) and independent of θ in this case.

Although — as noted above — $\Pi_{\theta,\varphi}(B)$ is not in $\mathscr{A}$, its principal part is; so the operator is of the form ($\mathscr{A}$ + compact) and therefore lies in $\mathfrak{A}$, the C^*-closure of $\mathscr{A}$, and its residue should indeed vanish, cf. the discussions in Remarks 3.1.3 and 5.1.3.

Chapter 5

Is it zero?

It was shown in Chapter 3 that the noncommutative residue of [FGLS96] always vanishes on projections in $\mathscr{A}$, the algebra of Boutet de Monvel boundary operators of order and class zero. So $\operatorname{res}\Pi_{\theta,\varphi}(B)$ is zero whenever the sectorial projection lies in $\mathscr{A}$. Further, we saw in Chapter 4 that we can define a residue of $\Pi_{\theta,\varphi}(B)$ even when this operator is not a Boutet de Monvel operator.

Basically, our work in the final year of the thesis project was spent on investigating whether the vanishing can be extended to the latter case:

Is the noncommutative residue of $\Pi_{\theta,\varphi}(B)$ *always zero?*

For simplicity, we considered only differential problems $B = P_T$, which have two important properties compared to the general pseudodifferential case: the operator $\Pi_{\theta,\varphi}(B)$ is then known to be bounded (Theorem 4.1.3) and we have the exponential decay of [See69a] at our disposal (see below).

We did not arrive at a final conclusion, but achieved partial results. This chapter is a discussion of those results, as well as the outline of a few related ideas which we did not work on in detail.

5.1 C^*-closure

$\Pi_{\theta,\varphi}(B)$ is not always in $\mathscr{A}$, but since the K-theoretic arguments of [Gaa07] actually show that the residue of [FGLS96] is extendable to projections — and vanishes for these also — in the C^*-closure $\mathfrak{A}$ of $\mathscr{A}$, cf. Remark 3.1.3, our first aim was to examine if $\Pi_{\theta,\varphi}(B)$ lies in $\mathfrak{A}$. Observe that we only consider $\mathscr{A}$ where the vector bundle F over ∂X is empty, since $\Pi_{\theta,\varphi}(B)$ is an operator on $\mathcal{H} = L_2(X, E)$.

That is, $\mathscr{A}$ here is the set of Green operators $\mathcal{A} = P_+ + G$ of order and class zero, with $\mathfrak{A}$ its C^*-closure in $\mathcal{B}(L_2(X, E))$.

The pseudodifferential part $\Pi_{\theta,\varphi}(P)_+$ is the truncation of a ψdo on $\widetilde{X}$, and is thus in $\mathscr{A}$ exactly when its symbol $\pi_{\theta,\varphi}(x, \xi)$ satisfies the transmission property. For example, this holds when the order m of P is even, cf. Proposition 4.1.4.

The principal symbol map $\sigma : \mathscr{A} \to C^\infty(S^*X)$ — mapping $\mathcal{A} = P_+ + G$ to $p^0(x, \xi)$, the principal symbol of P — extends to a map $\overline{\sigma} : \mathfrak{A} \to C(S^*X)$, by [MNS03, Theorem 5]. The range of this map is given by the functions in $C(S^*X)$ which, over each point of the boundary, take the same value at the two covectors that vanish on the tangent space of ∂X, that is,

$$\overline{\sigma}(\mathfrak{A}) = \{ f \in C(S^*X) \mid f(x', 0, 0, 1) = f(x', 0, 0, -1) \}. \tag{5.1.1}$$

In other words, $\Pi_{\theta,\varphi}(P)_+$ can lie in $\mathfrak{A}$ only if its principal symbol $\pi^0_{\theta,\varphi}(x, \xi)$ satisfies

$$\pi^0_{\theta,\varphi}(x', 0, 0, 1) = \pi^0_{\theta,\varphi}(x', 0, 0, -1). \tag{5.1.2}$$

This is also known as the *weak* transmission property. For odd m, we can find examples where this does not hold, for instance our Example 4.4.1 where the (principal) symbol is given by (4.4.3), such that

$$\pi^0_{\theta,\varphi}(x', 0, 0, t) = \frac{1}{2|t|} \begin{pmatrix} |t| & 0 & it & 0 \\ 0 & |t| & 0 & it \\ -it & 0 & |t| & 0 \\ 0 & -it & 0 & |t| \end{pmatrix}. \tag{5.1.3}$$

This clearly does not satisfy (5.1.2). The conclusion is that $\Pi_{\theta,\varphi}(P)_+$ is in $\mathfrak{A}$ for even m, but not in general.

We now look into whether $G_{\theta,\varphi}$ lies in $\mathfrak{A}$. First, we consider if $G_{\theta,\varphi}$ is the limit in $\mathfrak{A}$ of a sequence of singular Green operators. We have the following composition sequence, cf. [MNS03, eq. (11)],

$$0 \subset \mathfrak{K} \subset \mathfrak{G} \subset \mathfrak{A}, \tag{5.1.4}$$

where $\mathfrak{K}$ denotes the ideal of compact operators in $\mathcal{B}(\mathcal{H})$, and $\mathfrak{G}$ is the closure of $\mathscr{G}$, the algebra of all Green operators $\mathcal{A}$ in $\mathscr{A}$ with P of negative order. In other words, we investigate if $G_{\theta,\varphi}$ is in $\mathfrak{G}$.

The boundary symbol map γ, cf. (2.1.12),

$$\gamma(\mathcal{A}) = a^0(x', \xi', D_n) = p^0(x', 0, \xi', D_n)_+ + g^0(x', \xi', D_n), \tag{5.1.5}$$

induces an isometry from $\mathfrak{G}/\mathfrak{K}$ to $C(S^*\partial X) \otimes \mathfrak{K}_{\mathbb{R}_+}$ [MNS03, Theorem 6], where $\mathfrak{K}_{\mathbb{R}_+}$ is the ideal of compact operators in $L_2(\mathbb{R}_+)$.

In particular, for $G_{\theta,\varphi}$ to be the limit of a sequence of singular Green operators, the corresponding boundary principal symbol operator $\gamma(G_{\theta,\varphi}) = g^0_{\theta,\varphi}(x', \xi', D_n)$ would have to be a compact operator in $L_2(\mathbb{R}_+)$ for all (x', ξ'). Recall that the boundary symbol operator is the integral operator with kernel $\widetilde{g}_{\theta,\varphi,0}(x', x_n, y_n, \xi')$.

Like above, we have to look no further than Example 4.4.1 to find a counterexample where $g^0_{\theta,\varphi}(x', \xi', D_n)$ is not compact in $L_2(\mathbb{R}_+)$, and so, $G_{\theta,\varphi}$ is not in $\mathfrak{G}$. Here, the parameter-dependent singular Green kernel is given by (4.4.6):

$$\widetilde{g}(x_n, y_n, \xi', \lambda) = \frac{1}{2\sigma} \begin{pmatrix} -i\xi_1 + i\sigma & -\xi_2 - i\xi_3 & -\lambda & 0 \\ \xi_2 - i\xi_3 & i\xi_1 + i\sigma & 0 & -\lambda \\ -\lambda & 0 & -i\xi_1 - i\sigma & -\xi_2 - i\xi_3 \\ 0 & -\lambda & \xi_2 - i\xi_3 & i\xi_1 - i\sigma \end{pmatrix} e^{-\sigma(x_n+y_n)}, \tag{5.1.6}$$

where $\sigma = (|\xi'|^2 + \lambda^2)^{1/2}$. For $\theta = 0$ and $\varphi = \pi$ we obtain, by holomorphy, that

$$\widetilde{g}_{\theta,\varphi}(x_n, y_n, \xi') = \frac{i}{2\pi} \int_{\Gamma_{\theta,\varphi}} \widetilde{g}(x_n, y_n, \xi', \lambda)\, d\lambda = \frac{i}{2\pi} \int_{-\infty}^{\infty} \widetilde{g}(x_n, y_n, \xi', t)\, dt. \tag{5.1.7}$$

The action of $\gamma(G_{\theta,\varphi})$ on $u \in L_2(\mathbb{R}_+)$ is then

$$g^0_{\theta,\varphi}(\xi', D_n)u(x_n) = \int_0^{\infty} \widetilde{g}_{\theta,\varphi}(x_n, y_n, \xi')\, u(y_n)\, dy_n. \tag{5.1.8}$$

Proposition 5.1.1. *$g^0_{\theta,\varphi}(\xi', D_n)$ is not compact.*

Proof. There are three types of entries in the "matrix-part" of $\widetilde{g}$: those with only ξ_2 and ξ_3, those with only λ, and those with ξ_1 and σ.

For the first type, the corresponding symbol-kernel $\widetilde{g}^{(ij)}_{\theta,\varphi}(x_n, y_n, \xi')$ is in fact in $L_{2,x_n,y_n}(\mathbb{R}^2_{++})$ for each ξ' (we ignore the factors of $\xi_2 \pm i\xi_3$ here):

$$\begin{aligned} \Big(\int_{\mathbb{R}_+\times\mathbb{R}_+} \Big| \int_{-\infty}^{\infty} \frac{e^{-\sigma(x_n+y_n)}}{\sigma} dt \Big|^2 dx_n dy_n \Big)^{1/2} \\ \le \int_{-\infty}^{\infty} \Big(\int_{\mathbb{R}_+\times\mathbb{R}_+} \Big| \frac{e^{-\sigma(x_n+y_n)}}{\sigma} \Big|^2 dx_n dy_n \Big)^{1/2} dt \\ = \int_{-\infty}^{\infty} \frac{1}{\sigma} \Big(\int_0^{\infty} e^{-2\sigma x_n} dx_n \int_0^{\infty} e^{-2\sigma y_n} dy_n \Big)^{1/2} dt \\ = \int_{-\infty}^{\infty} \frac{1}{2\sigma^2} dt = \int_{-\infty}^{\infty} \frac{1}{2(|\xi'|^2 + t^2)} dt = \frac{\pi}{2|\xi'|} < \infty, \end{aligned} \tag{5.1.9}$$

where a form of Minkowski's inequality (Lieb and Loss [LL01, Theorem 2.4]) was employed at the inequality sign. These matrix entries become Hilbert-Schmidt operators in $L_2(\mathbb{R}_+)$.

For the entries of the second type we get

$$\widetilde{g}^{(ij)}_{\theta,\varphi}(x_n, y_n, \xi') = -\frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{t}{(|\xi'|^2+t^2)^{1/2}}\,e^{-(|\xi'|^2+t^2)^{1/2}(x_n+y_n)}\,dt = 0 \qquad (5.1.10)$$

since the integrand is odd in t.

The entries of the third type split into terms with ξ_1 and σ, respectively. For the former, the story is the same as in (5.1.9), i.e., a Hilbert-Schmidt operator. For the terms with σ, we get (since the σ cancels out the $\frac{1}{\sigma}$)

$$\begin{aligned}\frac{4\pi}{i}\widetilde{g}^{(ij)}_{\theta,\varphi}(x_n, y_n, \xi') &= \int_{-\infty}^{\infty} e^{-(|\xi'|^2+t^2)^{1/2}(x_n+y_n)}dt = 2\int_0^{\infty} e^{-(|\xi'|^2+t^2)^{1/2}(x_n+y_n)}dt\\ &\geq 2\int_0^{\infty} e^{-(|\xi'|+t)(x_n+y_n)}dt = \frac{2\,e^{-|\xi'|(x_n+y_n)}}{x_n+y_n}. \qquad (5.1.11)\end{aligned}$$

So we get a singularity of the type $(x_n + y_n)^{-1}$ and by Lemma 5.1.4 below, $g^0_{\theta,\varphi}(\xi', D_n)$ cannot be compact in $L_2(\mathbb{R}_+)$. □

So $\Pi_{\theta,\varphi}(P)_+$ is not in $\mathfrak{A}$ and $G_{\theta,\varphi}$ is not in $\mathfrak{G}$, but this does not rule out that their sum $\Pi_{\theta,\varphi}(B)$ could be in $\mathfrak{A}$. However, the matrix $M^{ij} \in \mathcal{M}_4(\mathbb{C})$ with a 1 in the (i, j)-entry and zeroes elsewhere is clearly in $\mathscr{A}$, so $M^{ii}\mathcal{A}M^{jj}$ — essentially the (i, j)-entry of $\mathcal{A}$ — is in $\mathscr{A}$ for any $\mathcal{A} \in \mathscr{A}$.

Now, the $(1, 3)$-entry of $\Pi_{\theta,\varphi}(B)$ equals the corresponding entry of $\Pi_{\theta,\varphi}(P)_+$, since the $(1, 3)$-entry in $G_{\theta,\varphi}$ vanishes, by (5.1.10). So the sum $\Pi_{\theta,\varphi}(B)$ cannot lie in $\mathfrak{A}$ either. (According to (5.1.3), this entry does not satisfy (5.1.2).) We summarize our findings in the following theorem:

Theorem 5.1.2. *The operator $\Pi_{\theta,\varphi}(B)$ from Example 4.4.1 is not in $\mathfrak{A}$, so we cannot draw a final conclusion on the residue just from our results on $\mathfrak{A}$.*

Remark 5.1.3. It should be noted that it remains to be proven that the K-theoretic extension to idempotents in $\mathfrak{A}$, from Remark 3.1.3, actually agrees with the definition given in 4.2.3. It would seem counter-intuitive if this is not the case, but we have not spent time on this question since it would not help us much, in light of the above.

We conclude this section with the lemma used above and a remark on the normal trace of $G_{\theta,\varphi}$.

Lemma 5.1.4. *The integral operator K with kernel $k(x,y) = \frac{1}{x+y}$ is not a compact operator on $L_2(\mathbb{R}_+)$.*

Proof. Let, for $n \in \mathbb{N}$, g_n be the characteristic function of the interval $(0, \frac{1}{n})$ and $f_n = \sqrt{n}\, g_n$. Then $\|f_n\| = 1$ and

$$Kf_n(x) = \sqrt{n} \int_0^{\frac{1}{n}} \frac{1}{x+y} dy. \tag{5.1.12}$$

For $0 < x < \frac{1}{n}$ we have

$$Kf_n(x) = \sqrt{n} \int_0^{\frac{1}{n}} \frac{1}{x+y} dy \geq \sqrt{n} \int_0^{x} \frac{1}{x+y} dy \geq \frac{\sqrt{n}}{2x} \int_0^{x} dy = \frac{\sqrt{n}}{2}. \tag{5.1.13}$$

So $Kf_n \geq \frac{1}{2} f_n \geq 0$, and hence $\|Kf_n\| \geq \frac{1}{2}$ for all n. In particular $\|Kf_n\| \not\to 0$.

But f_n converges weakly to zero: Since

$$\langle f_n, v \rangle = \sqrt{n} \int_0^{\frac{1}{n}} \overline{v(x)} dx \tag{5.1.14}$$

we obviously have $\lim_{n\to\infty} \langle f_n, v_0 \rangle = 0$ for all $v_0 \in C_0^\infty(\mathbb{R}_+)$.

Let $v \in L_2(\mathbb{R}_+)$ and let $\varepsilon > 0$. Find $v_0 \in C_0^\infty(\mathbb{R}_+)$ such that $\|v - v_0\| < \frac{\varepsilon}{2}$, and find N such that $|\langle f_n, v_0 \rangle| < \frac{\varepsilon}{2}$ for $n \geq N$.

Then, for $n \geq N$,

$$|\langle f_n, v \rangle| \leq |\langle f_n, v - v_0 \rangle| + |\langle f_n, v_0 \rangle| \leq \|f_n\| \cdot \|v - v_0\| + |\langle f_n, v_0 \rangle| < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon. \tag{5.1.15}$$

So (f_n) is a sequence of L_2-functions converging weakly to 0. Compact operators map weakly convergent sequences to norm convergent sequences, so K is not compact since $\|Kf_n\|$ does not converge to 0. □

With the arguments from the proof of Proposition 5.1.1 at our disposal, we can easily show that the normal trace of $G_{\theta,\varphi}$ is in general not defined. Again, we consider $G_{\theta,\varphi}$ from Example 4.4.1:

Proposition 5.1.5. $\operatorname{tr}_n g_{\theta,\varphi}(\xi')$ *does not converge.*

Proof. The normal trace of $g_{\theta,\varphi}$ is given by

$$\operatorname{tr}_n g_{\theta,\varphi}(\xi') = \int_0^\infty \widetilde{g}_{\theta,\varphi}(x_n, x_n, \xi')dx_n = \frac{i}{2\pi}\int_0^\infty \int_{-\infty}^\infty \widetilde{g}(x_n, x_n, \xi', t)dt\,dx_n. \tag{5.1.16}$$

For the terms with σ — as in (5.1.11) above — this becomes

$$\int_0^\infty \int_{-\infty}^\infty e^{-2(|\xi'|^2+t^2)^{1/2}x_n}dt\,dx_n \tag{5.1.17}$$
$$= \int_{-\infty}^\infty \int_0^\infty e^{-2(|\xi'|^2+t^2)^{1/2}x_n}dx_n\,dt = \int_{-\infty}^\infty \frac{1}{2(|\xi'|^2+t^2)^{1/2}}dt = \infty,$$

using Tonelli's theorem. □

This proves that the normal trace of the principal part is not in general well-defined for $G_{\theta,\varphi}$, just as in the case of $G^{\log}$ [GG08, Section 3].

5.2 Larger algebras

Another approach to the question at hand, is to find an algebra containing $\mathscr{A}$ as well as all the sectorial projections. Of course, this is an interesting problem in its own right, but moreover, one could then hope to extend the definition of non-commutative residue to this algebra, and maybe even use K-theoretic arguments to show that the map induced by res in K_0 vanishes, as it does in the case of $\mathscr{A}$ itself.

The algebra in question should contain $\mathscr{A}$ as well as the set of ψdos without the transmission property — because of $\Pi_{\theta,\varphi}(P)_+$ — and the generalized singular Green operators of type $G_{\theta,\varphi}$. One suggestion is the algebra $\mathfrak{B}^0$ by Rempel and Schulze [RS82]; we show why this is a good suggestion further below.

5.2.1 Description of $G_{\theta,\varphi}$

Let us first give a more thorough description of $G_{\theta,\varphi}$, the operator which poses the greater questions in this connection. Recall from Chapter 4 that it is given by

$$G_{\theta,\varphi}u(x) = \frac{i}{2\pi}\int_{\Gamma_{\theta,\varphi}} G_\lambda u(x)\,d\lambda, \tag{5.2.1}$$

where G_λ is the singular Green part of the resolvent $(B-\lambda)^{-1} = Q_{\lambda,+} + G_\lambda$. Since we assume B to be differential, we know from Seeley [See69a] that G_λ is given (in local coordinates) by a parameter-dependent singular Green symbol-kernel $\widetilde{g} \sim \sum_{j=0}^{\infty} \widetilde{g}_{-m-j}$:

$$G_\lambda u(x) = \int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'} \int_0^\infty \widetilde{g}(x', x_n, y_n, \xi', \lambda)\, \acute{u}(\xi', y_n)\, dy_n\, đ\xi', \tag{5.2.2}$$

where the terms are quasi-homogeneous

$$\widetilde{g}_{-m-j}(x', \tfrac{x_n}{t}, \tfrac{y_n}{t}, t\xi', t^m\lambda) = t^{1-m-j}\, \widetilde{g}_{-m-j}(x', x_n, y_n, \xi', \lambda), \quad \text{for } t \geq 1, |\xi'| \geq 1, \tag{5.2.3}$$

and satisfy estimates

$$|D_{x'}^\beta D_{\xi'}^\alpha x_n^k D_{x_n}^{k'} y_n^\ell D_{y_n}^{\ell'} D_\lambda^p\, \widetilde{g}_{-m-j}| \leq C\, \kappa^{1-m-j-|\alpha|-k+k'-\ell+\ell'-mp} e^{-c\,\kappa(x_n+y_n)}, \tag{5.2.4}$$

for all indices, when $\kappa = |\xi'| + |\lambda|^{1/m} \geq \varepsilon$. It is here we see the previously mentioned exponential decay. We can then write $G_{\theta,\varphi}$ as

$$G_{\theta,\varphi} u(x) = \int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'} \int_0^\infty \widetilde{g}_{\theta,\varphi}(x', x_n, y_n, \xi')\, \acute{u}(\xi', y_n)\, dy_n\, đ\xi', \tag{5.2.5}$$

a generalized singular Green operator with symbol-kernel $\widetilde{g}_{\theta,\varphi} \sim \sum_{j=0}^{\infty} \widetilde{g}_{\theta,\varphi,-j}$ given by

$$\widetilde{g}_{\theta,\varphi}(x', x_n, y_n, \xi') = \frac{i}{2\pi} \int_{\Gamma_{\theta,\varphi}} \widetilde{g}(x', x_n, y_n, \xi', \lambda)\, d\lambda, \tag{5.2.6}$$

where the terms $\widetilde{g}_{\theta,\varphi,-j}$ are given by similar integrals of $\widetilde{g}_{-m-j}$, cf. (4.1.6). They are quasi-homogeneous

$$\widetilde{g}_{\theta,\varphi,-j}(x', \tfrac{x_n}{t}, \tfrac{y_n}{t}, t\xi') = t^{1-j}\, \widetilde{g}_{\theta,\varphi,-j}(x', x_n, y_n, \xi'), \quad \text{for } t \geq 1, |\xi'| \geq 1, \tag{5.2.7}$$

and satisfy estimates

$$|D_{x'}^\beta D_{\xi'}^\alpha x_n^k D_{x_n}^{k'} y_n^\ell D_{y_n}^{\ell'}\, \widetilde{g}_{\theta,\varphi,-j}| \leq C|\xi'|^{-j-|\alpha|-k+k'-\ell+\ell'}\, \frac{e^{-c|\xi'|(x_n+y_n)}}{x_n+y_n}, \quad \text{for } |\xi'| \geq \varepsilon. \tag{5.2.8}$$

The claims here are proven in [GG08, Theorem 2.4]. Further remarks on the (x_n, y_n)-singularities for $j > 0$ are found in [GG08, Remark 2.5].

In short, we seek an algebra containing operators $G_{\theta,\varphi}$ as described here.

5.2.2 Model operator

Our investigations initially centered on finding a suitable algebra containing the generalized singular Green operator with symbol-kernel

$$\widetilde{g}(x_n, y_n, \xi') = \frac{e^{-|\xi'|(x_n+y_n)}}{x_n+y_n}. \tag{5.2.9}$$

Here in Section 5.2 we will call the operator $g(\xi', D_n)$ — acting on $L_2(\mathbb{R}_+)$ with this $\widetilde{g}$ as its integration-kernel — our *model operator*.

The reason for this particular choice should be obvious from the estimate (5.2.8) above, but let us elaborate in any case: In Example 4.4.1 we saw an example of an operator $\Pi_{\theta,\varphi}(B)$ where the "troublesome" terms of the singular Green part had a symbol-kernel

$$\widetilde{g}_{\theta,\varphi}(x_n, y_n, \xi') = \int_{-\infty}^{\infty} e^{-(|\xi'|^2+t^2)^{1/2}(x_n+y_n)} dt, \tag{5.2.10}$$

which is $O(\frac{1}{x_n+y_n})$ for $x_n + y_n \to 0$ and $O(e^{-|\xi'|(x_n+y_n)})$ for $x_n + y_n \to \infty$, just like (5.2.9). Moreover, the generalized singular Green operators we wish to investigate can be written as the difference between two logarithmic terms, cf. (4.3.6),

$$G_{\theta,\varphi} = \frac{i}{2\pi}\big(G^{\log_\theta} - G^{\log_\varphi}\big), \tag{5.2.11}$$

so it seems fair to assume that $G_{\theta,\varphi}$ will resemble operators of type $G^{\log}$ in general; for the Dirichlet realization of the Laplacian in $\mathbb{R}^n_+$, the logarithm is given by [GG08, Example 2.3]

$$\log(-\Delta_{\gamma_0}) = \log(-\Delta)_+ + G^{\log} = \mathrm{OP}(2\log|\xi|)_+ + G^{\log}, \tag{5.2.12}$$

where $G^{\log}$ is the generalized singular Green operator with symbol-kernel

$$\widetilde{g}^{\log}(x_n, y_n, \xi') = \frac{e^{-|\xi'|(x_n+y_n)}}{x_n + y_n}. \tag{5.2.13}$$

So although we might not actually obtain a sectorial projection with (5.2.9) as its singular Green symbol-kernel (e.g., $G_{\theta,\varphi}$ is zero for the Laplacian), we will definitely get operators very similar to it and we study this particular choice of operator for simplicity.

5.2.3 Èskin's algebra

The first step is to find an algebra of pseudodifferential operators on the half-line, which contains our model operator $g(\xi', D_n)$ for fixed ξ'. We consider an algebra $\mathfrak{E}$ of operators on $L_2(\mathbb{R}_+)$ constructed by Èskin [Èsk73, Chapter 15]. The operators in $\mathfrak{E}$ are of the form

$$p(\xi', D_n)_+ + \omega M + T. \tag{5.2.14}$$

Here $p(\xi) = p(\xi', \xi_n)$ is a symbol of order 0, which is independent of x and does not necessarily satisfy the transmission property; ω is (multiplication with) a fixed cutoff function

$$\omega \in C_0^\infty(\overline{\mathbb{R}}_+), \quad 0 \le \omega \le 1, \quad \omega(t) = 1 \text{ for } t \le 1; \tag{5.2.15}$$

T is a compact operator on $L_2(\mathbb{R}_+)$; and M is of the type $\mathfrak{M}$ described below. Èskin's purpose was to find an algebra that contained the parametrices of elliptic $p(\xi', D_n)_+$ in the case that $p(\xi)$ does not satisfy the transmission property.

We denote by $\mathfrak{M}$ the class of integral operators on $\mathbb{R}_+$ of the form

$$M\varphi(x_n) = \int_0^\infty \tau^{-1}\, m\Big(\frac{x_n}{\tau}\Big)\, \varphi(\tau)\, d\tau, \quad \varphi \in C_0^\infty(\mathbb{R}_+), \tag{5.2.16}$$

where m is in $\mathscr{M}^\infty$, the set of functions $m \in C^\infty(\mathbb{R}_+)$ which satisfy

$$|t^k \partial_t^k m(t)| = O(t^{-\delta-\varepsilon}),\ t \to 0, \qquad |t^k \partial_t^k m(t)| = O(t^{-1+\delta+\varepsilon}),\ t \to \infty, \tag{5.2.17}$$

for some $\delta \in [0, \frac{1}{2})$, dependent of m, and all $k \in \mathbb{N}_0$, $\varepsilon > 0$.

Applying the Mellin transform

$$\widetilde{u}(z) = \mathcal{M}u(z) = \int_0^\infty t^{z-1}\, u(t)\, dt \tag{5.2.18}$$

to (5.2.16), we obtain

$$\begin{aligned} \widetilde{M\varphi}(z) &= \int_0^\infty x_n^{z-1} M\varphi(x_n)\, dx_n = \int_0^\infty x_n^{z-1} \int_0^\infty \tau^{-1}\, m\Big(\frac{x_n}{\tau}\Big)\, \varphi(\tau)\, d\tau\, dx_n \\ &= \int_0^\infty \int_0^\infty t^{z-1}\, m(t)\, \tau^{z-1}\, \varphi(\tau)\, d\tau\, dt = \widetilde{m}(z)\, \widetilde{\varphi}(z), \end{aligned} \tag{5.2.19}$$

where the transformation $x_n = t\tau$ was used. We call $\widetilde{m}$ the *symbol* of M; it is the Mellin transform of m. From the estimates (5.2.17) it can be shown that $\widetilde{m}(z)$ is analytic in the strip $\delta < \operatorname{Re} z < 1 - \delta$ and

$$|z^k \widetilde{m}(z)| \le C_{\varepsilon,k}, \quad \delta + \varepsilon \le \operatorname{Re} z \le 1 - \delta - \varepsilon, \text{ for all } k \in \mathbb{N}_0,\ \varepsilon > 0. \tag{5.2.20}$$

Conversely, if $\widetilde{m}$ is analytic in $\delta < \operatorname{Re} z < 1 - \delta$ and satisfies (5.2.20) then m is in $\mathscr{M}^\infty$. In this manner, one could say that $\mathfrak{M}$ is the space of *Mellin multipliers*, whose symbols are in $\mathcal{M}(\mathscr{M}^\infty)$, i.e., satisfy (5.2.20).

The operators in $\mathfrak{M}$ (and hence the operators in $\mathfrak{E}$) are bounded and thus extend to all of $L_2(\mathbb{R}_+)$.

An important example in $\mathfrak{M}$ is the operator M_0 with $m_0(t) = \frac{1}{1+t}$ and symbol

$$\widetilde{m}_0(z) = -2\pi i \, \frac{e^{i\pi z}}{1 - e^{2\pi iz}} = \frac{\pi}{\sin(\pi z)}. \tag{5.2.21}$$

Its action can be written as

$$M_0\varphi(x_n) = \int_0^\infty \tau^{-1} \frac{1}{1 + \frac{x_n}{\tau}} \varphi(\tau) \, d\tau = \int_0^\infty \frac{\varphi(\tau)}{x_n + \tau} d\tau = \int_0^\infty \frac{\varphi(y_n)}{x_n + y_n} dy_n, \tag{5.2.22}$$

which evidently resembles our model operator $g(\xi', D_n)$ from (5.2.9) for small $x_n + y_n$; in fact, M_0 equals $g(0, D_n)$. This observation was one of the main reasons we became interested in Èskin's algebra.

However, the algebra does not allow M to depend on ξ' as $g(\xi', D_n)$ does. Instead, the idea is to subtract the ξ'-independent singularity for small (x_n, y_n):

We have an expansion

$$\frac{e^{-|\xi'|(x_n+y_n)}}{x_n + y_n} = \frac{1}{x_n + y_n} + |\xi'| + O(|\xi'|^2(x_n + y_n)). \tag{5.2.23}$$

Subtracting $1/(x_n + y_n)$ turns this into a bounded function of (x_n, y_n), which is in fact smooth up to the boundary on $\mathbb{R}^2_{++}$. Now,

$$h(x_n, y_n, \xi') = \frac{e^{-|\xi'|(x_n+y_n)}}{x_n + y_n} - \frac{\omega(x_n)}{x_n + y_n} \tag{5.2.24}$$

is in $L_{2,x_n,y_n}(\mathbb{R}^2_{++})$ and hence the operator with h as its integral-kernel is Hilbert-Schmidt. So our model kernel becomes

$$\widetilde{g}(x_n, y_n, \xi') = \omega(x_n)\frac{1}{x_n + y_n} + h(x_n, y_n, \xi'), \tag{5.2.25}$$

and in this way the model operator can be written

$$g(\xi', D_n) = \omega M_0 + T \in \mathfrak{E}, \tag{5.2.26}$$

where T is the compact operator with h as its integral kernel. Going on from the model operator, let us consider the concrete example from earlier:

Proposition 5.2.1. *The boundary symbol operator from Example 4.4.1,*

$$\pi_{\theta,\varphi}(\xi', D_n)_+ + g_{\theta,\varphi}(\xi', D_n), \tag{5.2.27}$$

is in $\mathfrak{E}$, with M_0 (times a coefficient matrix) as its Mellin multiplier.

Proof. Since $\pi_{\theta,\varphi}$ is a symbol of order 0, we only need to show that $g_{\theta,\varphi}(\xi', D_n)$ is the sum of a Mellin multiplier and a compact operator.

Recall its generalized singular Green symbol-kernel $\widetilde{g}_{\theta,\varphi}$ from (5.1.6), (5.1.7), and the description of its structure from the proof of Proposition 5.1.1. As described there, most of the matrix entries are Hilbert-Schmidt kernels and to prove that $g_{\theta,\varphi}(\xi', D_n)$ is in $\mathfrak{E}$, we only need to account for the entries of the form

$$\int_{-\infty}^{\infty} e^{-(|\xi'|^2+t^2)^{1/2}(x_n+y_n)} dt. \tag{5.2.28}$$

Subtracting the model kernel, we obtain

$$f(x_n, y_n) = \int_{-\infty}^{\infty} e^{-(|\xi'|^2+t^2)^{1/2}(x_n+y_n)} dt - 2\,\frac{e^{-|\xi'|(x_n+y_n)}}{x_n+y_n}, \tag{5.2.29}$$

which is in fact a Hilbert-Schmidt kernel as well. To see this, we rewrite it as

$$f(x_n, y_n) = 2\int_{0}^{\infty} \big[e^{-(|\xi'|^2+t^2)^{1/2}(x_n+y_n)} - e^{-(|\xi'|+t)(x_n+y_n)}\big] dt. \tag{5.2.30}$$

We can then bound its L_2 norm, again using the inequality [LL01, Theorem 2.4]:

$$\begin{aligned}\Big(\int_{\mathbb{R}^2_{++}} &|f(x_n,y_n)|^2 dx_n dy_n\Big)^{1/2} \\ &\leq 2\int_0^\infty \Big(\int_{\mathbb{R}^2_{++}} \big|e^{-(|\xi'|^2+t^2)^{1/2}(x_n+y_n)} - e^{-(|\xi'|+t)(x_n+y_n)}\big|^2 dx_n dy_n\Big)^{1/2} dt \\ &= \frac{1}{2}\int_0^\infty \Big(\frac{1}{(|\xi'|+t)^2} + \frac{1}{|\xi'|^2+t^2} - \frac{8}{(|\xi'|+t+\sqrt{|\xi'|^2+t^2})^2}\Big)^{1/2} dt.\end{aligned} \tag{5.2.31}$$

Putting the terms on a common denominator, cancellations will ensure that the contents of the square root is $O(t^{-4})$ and hence the integrand is $O(\langle t\rangle^{-2})$ and integrable for $\xi' \neq 0$, so $f(x_n, y_n)$ is square integrable.

From (5.2.29) we now see that the operator with the kernel in (5.2.28) is the sum of the compact operator with kernel f and (2 times) our model operator $g(\xi', D_n)$ from (5.2.26); the latter we proved to be in $\mathfrak{E}$ with M_0 as its Mellin multiplier. □

Obviously, the next interesting question here is whether the boundary symbol operator of a general sectorial projection is in $\mathfrak{E}$ for fixed (x', ξ'), i.e., whether the operator $g_{\theta,\varphi}(x', \xi', D_n)$, with $\widetilde{g}_{\theta,\varphi}$ as in Section 5.2.1, can be written as the sum of a Mellin multiplier and a compact operator, as in the example above.

Possibly, one could succeed with a proof similar to above, where we rewrote (5.2.29) as (5.2.30), but unfortunately, we discovered the idea of subtracting the ξ'-independent singularity (i.e., that the model operator is in fact in $\mathfrak{E}$) so late that there was no time left to really dig into the general question.

Nevertheless, let us continue with the results above.

5.2.4 Rempel and Schulze's algebra

Based on the works of Višik and Èskin [VÈ67, Èsk73], Rempel and Schulze [RS82] constructed an algebra of boundary operators $\mathfrak{B}^0$, which includes problems that do not satisfy the transmission condition. In apperance, the operators here resemble the Green operators in (2.1.5): an operator in $\mathfrak{B}^0$ has the form

$$\mathcal{B} = \begin{pmatrix} A' & K \\ T & S \end{pmatrix} : \begin{matrix} L_2(X, E_1) \\ \times \\ L_2(\partial X, F_1) \end{matrix} \rightarrow \begin{matrix} L_2(X, E_2) \\ \times \\ L_2(\partial X, F_2), \end{matrix} \tag{5.2.32}$$

where K, T, and S are just slightly more general than the corresponding operators in Boutet de Monvel's calculus. The big difference lies in the entry A' which, in local coordinates, can be described as pseudodifferential along the boundary with operator-valued symbols, where, for each (x', ξ'), the symbol belongs to a variant of Èskin's algebra $\mathfrak{E}$.

For our purposes — cf. the first paragraph of Section 5.1 — we need only consider the entry A'. (We can assume that $E = E_1 = E_2$ and that the F_i bundles are empty.) In local coordinates, the operator A' is given by

$$A' = P_+ + W + G, \tag{5.2.33}$$

modulo compact operators; P is a ψdo with no requirement of the transmission condition, G is a *B-singular Green operator*, and W is an *M-operator*.

A B-singular Green operator G is quite similar to Boutet de Monvel's notion of a singular Green operator (of order and class zero) and its action is also given by (2.1.6) with a symbol-kernel $\tilde{g}$. The difference is that $\tilde{g}$ is only required to be in $L_2(\mathbb{R}^2_{++})$ — as opposed to $\mathcal{S}(\overline{\mathbb{R}}^2_{++})$ — with respect to (x_n, y_n). (The term B-singular is our convention to avoid confusion; [RS82] simply calls them Green operators.) By working modulo compact operators, we disregard lower order terms; in particular, the symbol-kernel $\tilde{g}$ is assumed quasi-homogeneous of degree 1.

An M-operator W is of the form

$$Wu(x',x_n) = \int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'}\omega(|\xi'|x_n)\int_0^\infty y_n^{-1}\,\mu\Big(x',\frac{x_n}{y_n}\Big)\,\acute{u}(\xi',y_n)\,dy_n\,đ\xi', \quad (5.2.34)$$

where ω is a cutoff function as in (5.2.15), and μ is a smooth function of x' with values in $\mathscr{M}^\infty$. For fixed (x',ξ') we recognize the inner integral as an operator in $\mathfrak{M}$.

Now we go back to our model operator G and show that it lies in $\mathfrak{B}^0$. For this, we modify (5.2.24) and (5.2.25) slightly: Let

$$h(x',x_n,y_n,\xi') = \frac{e^{-|\xi'|(x_n+y_n)}}{x_n+y_n} - \frac{\omega(|\xi'|x_n)}{x_n+y_n}, \quad \mu_0(x',t) = \frac{1}{1+t}, \quad (5.2.35)$$

such that

$$\widetilde{g}(x_n,y_n,\xi') = \omega(|\xi'|x_n)\,y_n^{-1}\,\mu_0\Big(x',\frac{x_n}{y_n}\Big) + h(x',x_n,y_n,\xi'). \quad (5.2.36)$$

Then G, with

$$Gu(x',x_n) = \int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'}\int_0^\infty \widetilde{g}(x_n,y_n,\xi')\,\acute{u}(\xi',y_n)\,dy_n\,đ\xi', \quad (5.2.37)$$

equals W_0+H, where W_0 is the M-operator with $\mu=\mu_0$ and H is the B-singular Green operator with symbol-kernel h; h is in $L_{2,x_n,y_n}(\mathbb{R}^2_{++})$, as stated previously, and the quasi-homogeneity

$$h(x',\tfrac{x_n}{t},\tfrac{y_n}{t},t\xi') = t\,h(x',x_n,y_n,\xi') \quad (5.2.38)$$

is satisfied due to the factor $|\xi'|$ in the nonhomogeneous function ω.

So the model operator G is in $\mathfrak{B}^0$. Similarly to the proof of 5.2.1, one then shows the following statement:

Proposition 5.2.2. *The operator $\Pi_{\theta,\varphi}(B)$ from Example 4.4.1 is in $\mathfrak{B}^0$.*

Since this was our prime example of a sectorial projection not in $\mathscr{A}$, it is an obvious question whether all sectorial projections lie in $\mathfrak{B}^0$. This question is clearly related to the question at the end of the previous section; as explained there, we made the observations here too late to reach more general results.

Regarding the question of residues, $\mathfrak{B}^0$ actually seems to offer some promise since [RS82] states results on the C^*-closure of $\mathfrak{B}^0$, which might be needed for a K-theoretic proof in the spirit of Chapter 3.

5.2.5 Mellin operators

Before we realized that $\mathfrak{B}^0$ may suffice, we had spent some time looking at more modern versions of Rempel and Schulze's algebra. As mentioned, the $\mathfrak{M}$ operators in Èskin's class do not permit symbols that depend on ξ', and the same holds for the M-operators in [RS82]. Also, those algebras are quite crude in the sense that all lower order terms only appear as compact operators.

The concepts from [RS82] were later further developed in a wide range of applications, cf. e.g. the collaborations of Schulze with Schrohe and Seiler, respectively, [SS94, SS95, SSe02]. Here the algebras have a more detailed structure, in particular the M-operators are allowed to depend on ξ' as we describe below.

Remark 5.2.3. In this section, (x_n, y_n) will be denoted by (t, t'), since this is customary when working with Mellin operators. In particular, the prime on t' should not be understood as a "tangential prime" in the sense of x' and ξ'.

The Mellin transform is invertible in the sense that

$$f(t) = \mathcal{M}^{-1}_{z\to t}\big(\widetilde{f}(z)\big) = \frac{1}{2\pi i}\int_{L_\gamma} t^{-z} \int_0^\infty r^{z-1} f(r)\, dr\, dz, \tag{5.2.39}$$

for "suitable" f, where $L_\gamma = \{z \in \mathbb{C} \mid \mathrm{Re}\,(z) = 1/2 + \gamma\}$. We apply this to (5.2.19) and obtain

$$\begin{aligned} M\varphi(t) &= \mathcal{M}^{-1}_{z\to t}\big(\widetilde{m}(z)\,\widetilde{\varphi}(z)\big) \\ &= \frac{1}{2\pi i}\int_{L_\gamma} t^{-z}\ \widetilde{m}(z)\ \Big(\int_0^\infty (t')^{z-1}\,\varphi(t')\,dt'\Big) dz \\ &= \frac{1}{2\pi i}\int_{L_\gamma}\int_0^\infty (t/t')^{-z}\ \widetilde{m}(z)\ \varphi(t')\,\frac{dt'}{t'}\ dz. \end{aligned} \tag{5.2.40}$$

Now, just like we obtain the pseudodifferential operators from the Fourier multipliers by allowing the symbol to depend on (x, y), we allow the symbol $\widetilde{m}(z)$ of the Mellin multiplier in (5.2.40) to depend on (t, t') to get the *Mellin operator* $\mathrm{op}^\gamma_M f$ with symbol $f(t, t', z)$:

$$[\mathrm{op}^\gamma_M f]u(t) = \frac{1}{2\pi i}\int_{L_\gamma}\int_0^\infty (t/t')^{-z}\ f(t, t', z)\, u(t')\,\frac{dt'}{t'}\ dz. \tag{5.2.41}$$

If f is independent of t', the inner integral is merely the Mellin transform of u:

$$[\mathrm{op}^\gamma_M f]u(t) = \frac{1}{2\pi i}\int_{L_\gamma} t^{-z} f(t, z)\,\widetilde{u}(z)\, dz. \tag{5.2.42}$$

Now, if $\widetilde{g}(x_n, y_n, \xi')$ is a quasi-homogeneous (B-)singular Green symbol-kernel, such that

$$\widetilde{g}(t, t', \xi') = (t')^{-1}\, \widetilde{g}\big(\tfrac{t}{t'}, 1, t'\xi'\big), \tag{5.2.43}$$

then, using (5.2.39) and setting $p = r/t'$, we get

$$\begin{aligned}
\int_0^\infty \widetilde{g}(t, t', \xi')\, u(t')\, dt' &= \frac{1}{2\pi i} \int_{L_\gamma} t^{-z} \int_0^\infty r^{z-1} \int_0^\infty \widetilde{g}(r, t', \xi')\, u(t')\, dt'\, dr\, dz \\
&= \frac{1}{2\pi i} \int_{L_\gamma} t^{-z} \int_0^\infty \int_0^\infty r^{z-1} \widetilde{g}(r, t', \xi')\, dr\, u(t')\, dt'\, dz \qquad (5.2.44) \\
&= \frac{1}{2\pi i} \int_{L_\gamma} t^{-z} \int_0^\infty \int_0^\infty r^{z-1}\, (t')^{-1}\, \widetilde{g}\big(\tfrac{r}{t'}, 1, t'\xi'\big)\, dr\, u(t')\, dt'\, dz \\
&= \frac{1}{2\pi i} \int_{L_\gamma} t^{-z} \int_0^\infty \int_0^\infty p^{z-1}\, (t')^{z-2}\, \widetilde{g}(p, 1, t'\xi')\, t'\, dp\, u(t')\, dt'\, dz \\
&= \frac{1}{2\pi i} \int_{L_\gamma} \int_0^\infty (t/t')^{-z} \int_0^\infty p^{z-1}\, \widetilde{g}(p, 1, t'\xi')\, dp\; u(t')\, \frac{dt'}{t'}\; dz.
\end{aligned}$$

Defining

$$\mathsf{g}(t, t', \xi', z) = \int_0^\infty p^{z-1} \widetilde{g}(p, 1, t'\xi')\, dp = \mathcal{M}_{p\to z}\big(\, \widetilde{g}(p, 1, t'\xi')\,\big) \tag{5.2.45}$$

we arrive at

$$g(\xi', D_n) u(t) = \int_0^\infty \widetilde{g}(t, t', \xi')\, u(t')\, dt' = [\mathrm{op}_M^\gamma\, \mathsf{g}] u(t). \tag{5.2.46}$$

We conclude that the principal boundary operator $g(x', \xi', D_n)$ of a singular Green operator of order 0, with quasi-homogeneous symbol-kernel $\widetilde{g}(x', x_n, y_n, \xi')$, can be described as a Mellin operator with symbol $\mathsf{g}(x', t, t', \xi', z)$ given by (5.2.45).

Note that the symbol in (5.2.45) is independent of t, while most of the literature on Mellin operators deals with t'-independent symbols. However, there are formulas to express symbols depending on (t, t') as symbols depending only on t: we can find $\mathsf{g}_1(x', t, \xi', z)$ such that

$$g(x', \xi', D_n) = \mathrm{op}_M^\gamma\, \mathsf{g} \sim \mathrm{op}_M^\gamma\, \mathsf{g}_1, \tag{5.2.47}$$

where the $\sim$ essentially means "modulo smoothing operators" (but is more technical), cf. [SS95, Theorem 2.3.3].

In the case of our model operator we find the Mellin symbol to be

$$\mathsf{g}(t, t', \xi', z) = \int_0^\infty p^{z-1}\, \frac{e^{-t'|\xi'|(1+p)}}{1+p}\, dp = \Gamma(z)\, \Gamma\big(1-z, t'|\xi'|\big), \tag{5.2.48}$$

where $\Gamma(a,x)$ is the (upper) incomplete Gamma function

$$\Gamma(a,x) = \int_x^\infty t^{a-1} e^{-t} dt, \quad a \in \mathbb{C},\, x \in \mathbb{R}_+. \tag{5.2.49}$$

For $t'|\xi'| > 0$, $\mathsf{g}(t,t',\xi',z)$ is a meromorphic function of z, with simple poles in $z \in -\mathbb{N}_0$. When $t'|\xi'| = 0$, it simplifies to

$$\mathsf{g}(t,0,0,z) = \Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \tag{5.2.50}$$

as in (5.2.21).

More generally, one would investigate the Mellin symbol $\mathsf{g}_{\theta,\varphi}(t,t',z)$ from (5.2.45) where $\widetilde{g}_{\theta,\varphi}$ is as in Section 5.2.1. In particular, plugging in (5.2.6) gives us

$$\mathsf{g}_{\theta,\varphi}(t,t',z) = \frac{i}{2\pi} \int_0^\infty \int_{\Gamma_{\theta,\varphi}} p^{z-1}\, \widetilde{g}(p,1,t'\xi',\lambda)\, d\lambda\, dz. \tag{5.2.51}$$

However, it seems there are no immediate advantages to this, i.e., changing the order of integration does not seem to simplify.

We considered different pseudodifferential calculi incorporating Mellin operators; there are several to choose from, e.g. cone algebras, but the appearance of the $t'\xi'$ in (5.2.45) — which intuitively corresponds to $x_n\partial_{x'}$, a tangential differential operator which degenerates at the boundary — makes it natural to look at some form of an edge calculus:

Let $A = \sum_{|\alpha|\le m} a_\alpha(x) D_x^\alpha$ be a differential operator in $\mathbb{R}^n_+$ with coefficients a_α in $C^\infty(\overline{\mathbb{R}}^n_+)$. Then A can be rewritten

$$A = x_n^{-m} \sum_{k+|\beta|\le m} a_{k,\beta}(x',x_n)(-x_n\partial_n)^k (x_n D_{x'})^\beta, \quad a_{k,\beta} \in C^\infty(\overline{\mathbb{R}}^n_+). \tag{5.2.52}$$

Schulze and Seiler [SSe02] call a differential operator of the form (5.2.52) *edge-degenerate*, and we see the appearance of $x_n\partial_{x'}$ as mentioned above. It should be mentioned that the class of edge-degenerate operators is much larger than that induced by operators with smooth coefficients. (The notion of edge-degenerate operators is closely connected to Mazzeo's *zero calculus*, cf. e.g. Albin and Melrose [AM06] and the references there.)

One description of an edge algebra is given in [SSe02]; we spent quite some time studying this subject before it was ultimately decided to be too time-consuming to get to the bottom of edge algebras compared to the expected gains (it is not at all clear how one would define a residue on the edge algebra).

5.2.6 The Hilbert transform

Let us finish this discussion with a quick look at an example in one dimension; we originally looked at this to get an intuitive feel for the problem, but the example is interesting in its own right.

We look — once again — at Example 4.4.1, transferred to the 1d case by setting $\xi = (0, 0, 0, \xi_4)$. We denote ξ_4 by ξ for ease of notation; we get

$$\pi_{\theta,\varphi}(\xi) = \frac{1}{2|\xi|}\begin{pmatrix} |\xi| & 0 & i\xi & 0 \\ 0 & |\xi| & 0 & i\xi \\ -i\xi & 0 & |\xi| & 0 \\ 0 & -i\xi & 0 & |\xi| \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 & i\,\mathrm{sgn}(\xi) & 0 \\ 0 & 1 & 0 & i\,\mathrm{sgn}(\xi) \\ -i\,\mathrm{sgn}(\xi) & 0 & 1 & 0 \\ 0 & -i\,\mathrm{sgn}(\xi) & 0 & 1 \end{pmatrix}. \tag{5.2.53}$$

The symbol-kernel of the singular Green part becomes

$$\widetilde{g}_{\theta,\varphi}(x_n, y_n) = \frac{i}{2\pi}\int_{-\infty}^{\infty} \widetilde{g}(x_n, y_n, t)\, dt \tag{5.2.54}$$

where

$$\widetilde{g}(x_n, y_n, \lambda) = \frac{1}{2\sigma}\begin{pmatrix} i\sigma & 0 & -\lambda & 0 \\ 0 & i\sigma & 0 & -\lambda \\ -\lambda & 0 & -i\sigma & 0 \\ 0 & -\lambda & 0 & -i\sigma \end{pmatrix} e^{-\sigma(x_n,y_n)} \tag{5.2.55}$$

with $\sigma = \sqrt{\lambda^2}$. Plugging (5.2.55) into (5.2.54) we obtain

$$\widetilde{g}(x_n, y_n) = \frac{1}{2\pi}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \frac{1}{x_n + y_n}. \tag{5.2.56}$$

The Hilbert transform $\mathscr{H}$,

$$\mathscr{H} f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(y)}{x - y}\, dy, \tag{5.2.57}$$

is a classical ψdo of order 0 with symbol $-i\,\mathrm{sgn}(\xi)$; this is essentially the symbol of (5.2.53). Using the reflection operator $J : f(x) \mapsto f(-x)$ we define the operators H and G as

$$\begin{aligned} Hf(x) &= r^+ \mathscr{H} e^+ f(x) = \frac{1}{\pi}\int_0^{\infty} \frac{f(y)}{x - y}\, dy, \\ Gf(x) &= r^+ \mathscr{H} J e^+ f(x) = \frac{1}{\pi}\int_0^{\infty} \frac{f(y)}{x + y}\, dy, \end{aligned} \tag{5.2.58}$$

and find that

$$\Pi_{\theta,\varphi}(P)_+ = \frac{1}{2}\begin{pmatrix} I & 0 & -H & 0 \\ 0 & I & 0 & -H \\ H & 0 & I & 0 \\ 0 & H & 0 & I \end{pmatrix}, \; G_{\theta,\varphi} = \frac{1}{2}\begin{pmatrix} -G & 0 & 0 & 0 \\ 0 & -G & 0 & 0 \\ 0 & 0 & G & 0 \\ 0 & 0 & 0 & G \end{pmatrix}. \tag{5.2.59}$$

Since $\Pi_{\theta,\varphi} = \Pi_{\theta,\varphi}(P)_+ + G_{\theta,\varphi}$ is idempotent, we obtain the identities

$$G^2 - H^2 = I, \quad GH = HG, \tag{5.2.60}$$

which are probably well-known. The one on the left is connected to the fact that $\mathscr{H}^2 = -I$.

5.3 Logarithms

Finally, we have spent a little time working on the idea of an algebra containing $(\log P)_+$, the truncation of $\log P$ to $\mathbb{R}^n_+$, for a ψdo P on $\mathbb{R}^n$. As is well known, the symbol of $\log P$ is $m\log[\xi] + l(x,\xi)$, where m is the order of P, $[\xi]$ is a smooth positive function that equals $|\xi|$ for $|\xi| \geq 1$, and $l(x,\xi)$ is a classical ψdo symbol of order zero; it does not satisfy the transmission condition in general.

We briefly studied the boundary symbol operator $p(\xi', D_n)_+$ in $L_2(\mathbb{R}_+)$ for $p(\xi) = \log|\xi|$; what we proved is that one cannot draw any immediate parallels to Èskin's algebra.

5.3.1 Analogy to Èskin

Èskin gave the following description of a ψdo of order zero [Èsk73]: Let $a(\xi)$ be homogeneous of degree 0, and let $\widetilde{a}(\xi', x_n) = \mathcal{F}^{-1}_{\xi_n \to x_n} a(\xi', \xi_n)$, $a_\pm = a(0, \pm 1)$ and $u_+ = e^+ u$. Then

$$a(\xi', D_n)_+ u = \widetilde{a} * u_+ = a_+ \delta_- * u_+ + a_- \delta_+ * u_+ + b * u_+ \tag{5.3.1}$$

where $*$ is convolution, $\widetilde{b}$ is in $L_{2,x_n}(\mathbb{R})$, and

$$\delta_\pm = \frac{\delta}{2} \pm \frac{1}{2\pi i}\,\mathrm{PV}\,\frac{1}{x}. \tag{5.3.2}$$

This description was central for Èskin in showing that the operators in $\mathfrak{E}$ in fact form an algebra. It is an essential detail that the operator $\Pi^\pm : u \mapsto \delta_\pm * u_+$ can be described as a Mellin multiplier with symbol

$$\widetilde{\mu}^{+}(s) = \frac{1}{1-e^{2\pi i s}}, \qquad \widetilde{\mu}^{-}(s) = -\frac{e^{2\pi i s}}{1-e^{2\pi i s}}. \tag{5.3.3}$$

While $\widetilde{\mu}^{\pm}$ is not in $\mathcal{M}(\mathscr{M}^{\infty})$, it is analytic in the strip $\{0 < \operatorname{Re} s < 1\}$ and bounded in closed substrips; hence, for each $m \in \mathscr{M}^{\infty}$, the product $\widetilde{\mu}^{\pm}(s)\widetilde{m}(s)$ is in $\mathcal{M}(\mathscr{M}^{\infty})$. Below, we look into the possibility of a similar scenario for $\log|\xi|$.

The idea behind (5.3.1) is to control the behavior of $a(\xi)$ for $\xi_n \to \pm\infty$: let H denote the Heaviside function, and split up a as

$$a(\xi', \xi_n) = a_+ H(\xi_n) + a_- H(-\xi_n) + \big\{a(\xi', \xi_n) - a_+ H(\xi_n) - a_- H(-\xi_n)\big\}. \tag{5.3.4}$$

The expression enclosed in $\{\ \}$ is in L_2 wrt. ξ_n and $\widetilde{b}$ is then the inverse Fourier transform of this, while the $\delta_{\mp}$ arises as the inverse Fourier transform of $H(\pm\xi_n)$.

We seek to emulate this for the symbol $p(\xi) = \log|\xi| = \frac{1}{2}\log(\xi^2)$. Here

$$b(\xi) = p(\xi) - \log|\xi_n| = \tfrac{1}{2}\log(|\xi'|^2 + \xi_n^2) - \log|\xi_n| = \tfrac{1}{2}\log\Big(1 + \frac{|\xi'|^2}{\xi_n^2}\Big) \tag{5.3.5}$$

is in L_2 wrt. ξ_n, so if we let

$$\ell = \mathcal{F}^{-1}_{\xi_n \to x_n} \log|\xi_n| \quad \text{and} \quad \widetilde{b}(\xi', x_n) = \mathcal{F}^{-1}_{\xi_n \to x_n} b(\xi), \tag{5.3.6}$$

then the action of the boundary symbol operator is given by

$$p(\xi', D_n)_+ u = \ell * u_+ + \widetilde{b} * u_+. \tag{5.3.7}$$

Here ℓ is a temperate distribution and $\widetilde{b}(\xi', x_n)$ is in $L_{2,x_n}(\mathbb{R})$ for fixed ξ'; in fact, the dependence of the L_2 norm on ξ' is such that $b \in S^{\frac{1}{2}}(\mathbb{R}^{n-1}; L_2(\mathbb{R}))$.

We find ℓ to be

$$\ell = -\frac{1}{2}\operatorname{Pf}\frac{1}{|x|} - \gamma\,\delta, \tag{5.3.8}$$

where γ is Euler's gamma constant and the "pseudofunction" is defined by

$$\operatorname{Pf}\frac{1}{|x|} = \lim_{\varepsilon\to 0^+}\Big[\frac{H(x-\varepsilon) - H(-x-\varepsilon)}{x} + 2\log\varepsilon\ \delta\Big], \tag{5.3.9}$$

such that convolution with a function $u_+ = e^+ u$ becomes

$$\Big(\operatorname{Pf}\frac{1}{|x|} * u_+\Big)(x) = \lim_{\varepsilon\to 0^+}\Big[\int_0^{x-\varepsilon}\frac{u(y)}{x-y}\,dy - \int_{x+\varepsilon}^{\infty}\frac{u(y)}{x-y}\,dy + 2\log\varepsilon\ u(x)\Big]. \tag{5.3.10}$$

However, ℓ is not a homogeneous distribution, and this prevents convolution with it from being a Mellin multiplier. The easiest way to show this is with a concrete example: let $a > 0$ and consider the function

$$u(y) = e^{-ay}, \qquad \widetilde{u}(s) = \int_0^\infty y^{s-1}\, u(y)\, dy = a^{-s}\,\Gamma(s). \tag{5.3.11}$$

The convolution of $\operatorname{Pf}\frac{1}{|x|}$ with u_+ is given by the above formula, and we get

$$\begin{aligned}\Big(\operatorname{Pf}\frac{1}{|x|} * u_+\Big)(x) &= \lim_{\varepsilon\to 0^+}\Big[e^{-ax}\big(\operatorname{Ei}(ax) - \operatorname{Ei}(a\,\varepsilon) - \Gamma(0, a\,\varepsilon) + 2\log\varepsilon\big)\Big] \\ &= e^{-ax}\big(\operatorname{Ei}(ax) - 2\gamma - 2\log a\big),\end{aligned} \tag{5.3.12}$$

where Ei is the exponential integral function

$$\operatorname{Ei}(z) = -\operatorname{PV}\int_{-z}^\infty \frac{e^{-t}}{t}\, dt, \tag{5.3.13}$$

which is smooth on $\mathbb{R}_+$ (and square integrable on $[0, 1]$).

Convolution with δ is the identity, such that

$$\big(\ell * u_+\big)(x) = -\tfrac{1}{2}\, e^{-ax}\big(\operatorname{Ei}(ax) - 2\gamma - 2\log a\big) - \gamma\, e^{-ax} = e^{-ax}\big(\log a - \tfrac{1}{2}\operatorname{Ei}(ax)\big). \tag{5.3.14}$$

The Mellin transform of this is

$$\begin{aligned}\mathcal{M}\big(\ell * u_+\big)(s) &= \int_0^\infty x^{s-1}e^{-ax}\big(\log a - \tfrac{1}{2}\operatorname{Ei}(ax)\big)dx \\ &= a^{-s}\,\Gamma(s)\big(\log a - \tfrac{\pi}{2}\cot(\pi s)\big) \\ &= \big(\log a - \tfrac{\pi}{2}\cot(\pi s)\big)\widetilde{u}(s).\end{aligned} \tag{5.3.15}$$

The $\log a$ term ensures that we cannot find a Mellin symbol $h(s)$ such that

$$\mathcal{M}\big(\ell * v\big)(s) = h(s)\widetilde{v}(s) \tag{5.3.16}$$

for arbitrary v, and we have shown that convolution with ℓ is not a Mellin multiplier.

Remark 5.3.1. The operator $p(\xi', D_n)$ above can also be described as convolution with $\widetilde{p}(\xi', x_n)$, the inverse Fourier transform of $p(\xi) = \log|\xi|$. It is given by

$$\widetilde{p}(\xi, x_n) = \mathcal{F}^{-1}_{\xi_n\to x_n} p(\xi) = -\frac{1}{2}\, e^{-|\xi'|\,|x_n|}\operatorname{Pf}\frac{1}{|x_n|} - \gamma\,\delta. \tag{5.3.17}$$

For $x_n > 0$, this equals $-\frac{e^{-|\xi'|x_n}}{2x_n}$, which clearly resembles the symbol-kernel of our model operator quite a lot.

The symbol $p(\xi)$ corresponds to the operator $P = \frac{1}{2}\log(-\Delta)$ on $\mathbb{R}^n$; let $J : f(x', x_n) \mapsto f(x', -x_n)$ be the reflection map; then the generalized singular Green operators, cf. [GG08, Example 2.8],

$$G^+(P) = r^+ P e^- J \quad \text{and} \quad G^-(P) = J r^- P e^+ \tag{5.3.18}$$

have symbol-kernels

$$\begin{aligned} \widetilde{g}^+(x_n, y_n, \xi') &= \left(r^+_{z_n} \mathcal{F}^{-1}_{\xi_n \to z_n} p(\xi)\right)\big|_{z_n = x_n + y_n} = -\frac{e^{-|\xi'|(x_n+y_n)}}{2\,(x_n + y_n)}, \\ \widetilde{g}^-(x_n, y_n, \xi') &= \left(r^-_{z_n} \mathcal{F}^{-1}_{\xi_n \to z_n} p(\xi)\right)\big|_{z_n = -x_n - y_n} = -\frac{e^{-|\xi'|(x_n+y_n)}}{2\,(x_n + y_n)}. \end{aligned} \tag{5.3.19}$$

We see that the generalized singular Green operator G, with the model kernel (5.2.9) as its symbol-kernel, arises as $G = G^\pm(-2\,P) = -G^\pm\big(\log(-\Delta)\big)$.

Another interesting observation in this connection: the function $\widetilde{b}(\xi', x_n)$ from above is given, for $x_n > 0$, by

$$\widetilde{b}(\xi', x_n) = \mathcal{F}^{-1}_{\xi_n \to x_n} b(\xi) = -\frac{1}{2}\Big(\frac{e^{-|\xi'|x_n}}{x_n} - \frac{1}{x_n}\Big), \tag{5.3.20}$$

such that

$$-\,2\left(r^+_{z_n} \mathcal{F}^{-1}_{\xi_n \to z_n} b(\xi)\right)\big|_{z_n = x_n + y_n} = \frac{e^{-|\xi'|(x_n+y_n)}}{x_n + y_n} - \frac{1}{x_n + y_n}, \tag{5.3.21}$$

which equals $h(x_n, y_n, \xi')$ from (5.2.24).

Remark 5.3.2. In our preprints for [GG08], equation (2.9) there,

$$(\log P)_+ \;:\; H^t(X, E) \to H^{t-\varepsilon}(X, E) \qquad \text{for} \;\; |t| < \tfrac{1}{2}, \tag{5.3.22}$$

was stated as being valid for all $t > -\frac{1}{2}$. This was a formulational error, for in fact, it is not.

We can show this easily with the example $u(y) = e^{-y}$: while u is in $H^t(\mathbb{R}_+)$ for any t, the boundary symbol operator maps it to the function

$$v(x) = p(\xi', D_n)_+ u(x) = \tfrac{1}{2}\, e^{-x}\big(\Gamma(0, x|\xi'| - x) + \log(|\xi'|^2 - 1)\big) \tag{5.3.23}$$

which is in $L_2(\mathbb{R}_+)$, but not even in $H^1(\mathbb{R}_+)$:

$$v'(x) = -\frac{e^{-|\xi'|x}}{2x} - v(x) \notin L_2(\mathbb{R}_+). \tag{5.3.24}$$

Looking at the product $\varphi(x')u(x_n)$ for some $\varphi \in C_0^\infty(\mathbb{R}^{n-1})$ and carrying the situation over to the manifold X, it follows that the statement in (5.3.22) above is not valid for $t > 1$.

The error was corrected in the galley proofs.

Remark 5.3.3. Another reason we studied edge algebras was the following: It is clear from Section 4.3, that an algebra of logarithms automatically would give us an algebra containing the sectorial projections. Schulze and Seiler [SSe02, Theorem 4.3.4] describe an edge boundary operator D^μ with principal symbol $|\xi|^\mu$; from this, one could possibly obtain an operator with symbol $\log|\xi|$ as

$$\frac{d}{d\mu} D^\mu \Big|_{\mu=0}. \tag{5.3.25}$$

However, as mentioned earlier, the edge algebra work was ended when it seemed to take us too far astray.

5.4 APS operators

In our definition of the sectorial projection of a boundary problem $B = (P+G)_T$, we required that the boundary problem $\{P_+ + G - \lambda, T\}$ was parameter-elliptic — in the sense of Grubb — for λ in two sectors.

Since normality is a necessary condition for parameter-ellipticity to occur, cf. [Gru96, Lemma 1.5.7], this requirement excludes an important class of non-normal boundary value problems, namely the Atiyah-Patodi-Singer (APS) operators. An APS operator is the realization of a certain class of boundary problems for first order Dirac operators [APS75].

In other words, our Definition 4.2.3 of noncommutative residue does not apply to sectorial projections of APS operators, since the definition of the sectorial projection itself does not apply.

However, for a self-adjoint APS operator $\mathcal{D}$, the projection $\Pi_>$ onto the eigenspaces of the eigenvalues in $\mathbb{R}_+$ can obviously be interpreted as the sectorial projection $\Pi_{\downarrow\uparrow}(\mathcal{D})$, where $\downarrow$ ($\uparrow$) corresponds to any angle in the lower (upper) halfplane. Likewise for $\Pi_<$ and $\Pi_{\uparrow\downarrow}(\mathcal{D})$.

For such an operator $\mathcal{D}$, there are — to our knowledge — no results on trace expansions coefficients $C_{0,\downarrow}(\mathcal{D})$ and $C_{0,\uparrow}(\mathcal{D})$, but we can instead define the residue analogously to Proposition 4.2.5:

$$\operatorname{res}\Pi_{>} = \operatorname{res}\Pi_{\downarrow\uparrow}(\mathcal{D}) = \tfrac{1}{2}\operatorname{res}_{s=0}\eta(\mathcal{D},s). \tag{5.4.1}$$

The right hand side has been extensively examined — for more general APS-type problems too — in e.g. [GSe96, Corollary 2.4], [BL99, Theorem 3.4], Wojciechowski [Woj99, Theorem 0.2], Grubb [Gru03, Theorem 5.9], and vanishes in certain cases.

However, [GSe96, Corollary 2.4.(2)] also gives examples of APS-type operators $P_{\geq}$, for which the right hand side *may* be non-zero (see [GSe96] for notation):

$$\operatorname{res}\Pi_{>} = \tfrac{1}{2}\operatorname{res}_{s=0}\eta(P_{\geq},s) = \operatorname{res}_{s=0}\eta(P_{\geq},2s) = \tfrac{1}{4\pi}\operatorname{Tr}(\sigma\Pi_0(A)). \tag{5.4.2}$$

Likewise, [Gru03, p. 276–278] indicates that there might be cases where $R(P)$ is nonzero. In both cases, a closer analysis would be needed in order to verify this.

5.5 Other ideas

Finally, we sketch a few other ideas related to the question $\operatorname{res}\Pi_{\theta,\varphi}(B) = 0$. We have not spent any significant time on these approaches, though.

5.5.1 Brüning-Lesch

Brüning and Lesch [BL99, Lemma 2.7] showed that, on a closed manifold, the vanishing of $\operatorname{res}_{s=0}\eta(P,s)$ for self-adjoint ψdos P is equivalent to the vanishing of $\operatorname{res}\Pi$ for any idempotent ψdo Π.

If one was to examine whether a similar equivalence holds on manifolds with boundary, Proposition 4.2.5 is clearly relevant; however, $\operatorname{res}_{s=0}\eta(B,s)$ has not been studied thoroughly for general self-adjoint realizations B and not much is known, as mentioned in connection with the proposition.

5.5.2 Adjoining projections to $\mathfrak{A}$

Another approach might be to use entirely operator algebraic methods; for instance, one could consider the smallest C^*-algebra generated by $\mathscr{A}$ and all the sectorial projections $\Pi_{\theta,\varphi}(B)$, but a simpler path in this direction is to adjoin a single sectorial projection $\Pi = \Pi_{\theta,\varphi}(P_T)$ to $\mathscr{A}$ to obtain the sub-C^*-algebra $\mathfrak{C} = C^*(\mathscr{A},\Pi) \subseteq \mathcal{B}(\mathcal{H})$ generated by $\mathscr{A}$ and Π.

Hopefully, one could then use some kind of K-theoretic arguments on this new algebra $\mathfrak{C}$ to extend the definition of residue and show its (possible) vanishing.

The description of the set of commutators $[\Pi, \mathscr{A}]$ is probably important for this approach; for instance, if $[\Pi, \mathscr{A}] \subseteq \mathscr{A}\Pi + \mathscr{A}$, then $\langle \mathscr{A}, \Pi \rangle = \mathscr{A}\Pi + \mathscr{A}$, since Π is idempotent and any combination of Π with operators from $\mathscr{A}$ can then be commuted.

5.5.3 Deformation

Given a projection $\Pi_{\theta,\varphi}(B)$, one final approach could be to make a continuous deformation (through projections) to a projection which is essentially supported in the interior. The idea of continuous deformations go back to Boutet de Monvel himself, but also Albin and Melrose worked on this in their recent paper [AM06].

However, it does not immediately seem to be a promising idea: deformations usually go through elliptic elements, and the projections are not elliptic. One could imagine making a homotopy $B = P_T$ to $B' = P'_{T'}$, and then looking at the path from $\Pi_{\theta,\varphi}(B)$ to $\Pi_{\theta,\varphi}(B')$ but it seems highly unlikely that this would work, since continuous deformations are bad at “conserving the spectrum”, i.e., it is hard to imagine that all intermediate realizations in the path would satisfy the requirements on the spectrum such that the sectorial projection could even be defined.

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LOGARITHMS AND SECTORIAL PROJECTIONS FOR ELLIPTIC BOUNDARY PROBLEMS

ANDERS GAARDE and GERD GRUBB

Abstract

On a compact manifold with boundary, consider the realization B of an elliptic, possibly pseudodifferential, boundary value problem having a spectral cut (a ray free of eigenvalues), say R_-. In the first part of the paper we define and discuss in detail the operator $\log B$; its residue (generalizing the Wodzicki residue) is essentially proportional to the zeta function value at zero, $\zeta(B,0)$, and it enters in an important way in studies of composed zeta functions $\zeta(A,B,s) = \mathrm{Tr}(AB^{-s})$ (pursued elsewhere).

There is a similar definition of the operator $\log_\theta B$, when the spectral cut is at a general angle θ. When B has spectral cuts at two angles $\theta < \varphi$, one can define the sectorial projection $\Pi_{\theta,\varphi}(B)$ whose range contains the generalized eigenspaces for eigenvalues with argument in $]\theta,\varphi[$; this is studied in the last part of the paper. The operator $\Pi_{\theta,\varphi}(B)$ is shown to be proportional to the difference between $\log_\theta B$ and $\log_\varphi B$, having slightly better symbol properties than they have. We show by examples that it belongs to the Boutet de Monvel calculus in many special cases, but lies outside the calculus in general.

1. Introduction

The purpose of this paper is to set up logarithms and sectorial projections for elliptic boundary value problems, and to establish and analyze residue definitions associated with these operators. Let us first recall the situation for boundaryless manifolds:

For a classical elliptic pseudodifferential operator (ψdo) P of order $m > 0$, acting in a vector bundle $\widetilde{E}$ over a closed (i.e., compact boundaryless) n-dimensional manifold $\widetilde{X}$, certain functions of the operator have been studied with great interest for many years. Assuming that P has no eigenvalues on some ray, say R_-, one has from Seeley's work [16] that the complex powers P^{-s} can be defined as ψdo's by use of the resolvent $(P-\lambda)^{-1}$. Moreover, the zeta function $\zeta(P,s) = \mathrm{Tr}(P^{-s})$ has a meromorphic extension to $s \in \mathsf{C}$ with at most simple poles at the real numbers $\{(n-j)/m \mid j \in \mathsf{N}\}$ (we denote $\{0,1,2,\dots\} = \mathsf{N}$). There is no pole at $s = 0$ (for $j = n$), and the value $\zeta(P,0)$ plays an important role in index formulas. Let us define the *basic zeta value*

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$C_0(P)$ by

$$C_0(P) = \zeta(P, 0) + \nu_0, \tag{1.1}$$

where ν_0 is the algebraic multiplicity of the zero eigenvalue of P (if any). It is well-known how $C_0(P)$ can be calculated in local coordinates from finitely many homogeneous terms of the symbol of P.

Another interesting function of P is $\log P$, defined on smooth functions by

$$\log P = \lim_{s \searrow 0} \frac{i}{2\pi} \int_{\mathscr{C}} \lambda^{-s} \log \lambda \, (P - \lambda)^{-1} \, d\lambda; \tag{1.2}$$

here λ^{-s} and $\log \lambda$ are taken with branch cut $\mathbb{R}_-$, and $\mathscr{C}$ is a contour in $\mathbb{C} \setminus \overline{\mathbb{R}}_-$ going around the nonzero spectrum of P in the positive direction. By use of the fact that $\log P = -\frac{d}{ds} P^{-s}\big|_{s=0}$, Scott [15] showed that

$$C_0(P) = -\tfrac{1}{m} \operatorname{res}(\log P), \tag{1.3}$$

where $\operatorname{res}(\log P)$ is a slight generalization of Wodzicki's noncommutative residue ([20], Guillemin [11]).

In the case of a compact n-dimensional manifold X with boundary $\partial X = X'$ (smoothly imbedded in an n-dimensional manifold $\widetilde{X}$ without boundary), one can study the analogous operators and constants defined from a realization B of a pseudodifferential (or differential) elliptic boundary value problem. Here $B = (P + G)_T$, defined from a system $\{P_+ + G, T\}$ of order $m > 0$ ($m \in \mathbb{Z}$) in the Boutet de Monvel calculus [2], where P is a ψdo on $\widetilde{X}$ and P_+ is its truncation to X (acting in $E = \widetilde{E}|_X$), G is a singular Green operator (s.g.o.) and T is a system of trace operators. B is the operator acting like $P_+ + G$ with domain

$$D(B) = \{u \in H^m(X, E) \mid Tu = 0\}, \tag{1.4}$$

where $H^m(X, E)$ is the Sobolev space of order m. In the differential operator case, $G = 0$. Assuming that for λ on a ray, say $\mathbb{R}_-$, $\{P_+ + G - \lambda, T\}$ satisfies the hypotheses of parameter-ellipticity of Grubb [6, Sect. 3.3] (consistent with those of Seeley [17] in the differential operator case), one can define the complex powers by functional analysis and study the pole structure of $\zeta(B, s) = \operatorname{Tr}(B^{-s})$ [6, Sect. 4.4], and in particular discuss the basic zeta value $C_0(B)$ defined similarly to (1.1). However, in contrast with the closed manifold case, the powers B^{-s} do not lie in the calculus we are using (in particular their ψdo part does not satisfy the transmission condition of [2]). Then it is advantageous to build the analysis more directly on the resolvent, which does belong to the parameter-dependent calculus set up in [6]. In fact, for $N > n/m$

(such that $(B-\lambda)^{-N}$ is trace-class), there is a trace expansion for $\lambda \to \infty$ in a sector V around R_-:

$$\operatorname{Tr}(B-\lambda)^{-N} = \sum_{0\le j\le n} c_j^{(N)}(-\lambda)^{(n-j)/m-N} + O(\lambda^{-N-\varepsilon}) \tag{1.5}$$

($\varepsilon > 0$), and here

$$C_0(B) = c_n^{(N)}, \tag{1.6}$$

independently of N. It is shown in [8] that for a generalization of (1.3) to B,

$$C_0(B) = -\tfrac{1}{m}\operatorname{res}(\log B), \tag{1.7}$$

it is sufficient to be able to define $\log B$; the complex powers B^{-s} are not needed.

The present paper gives in Sections 2 and 3 a detailed study of $\log B$. For one thing, this allows a more precise interpretation of the formula (1.7), initiated in [8]. Another important purpose is to open up for the use of compositions of $\log B$ with other operators. These are needed for the consideration of composed zeta functions $\zeta(A, B, s) = \operatorname{Tr}(AB^{-s})$ with general A from the calculus of [2], or rather, trace expansion formulas for composed resolvents $A(B-\lambda)^{-N}$. Such a study is carried out in [9] using the results on $\log B$ obtained in the present paper. We show in Section 2 that

$$\log B = (\log P)_+ + G^{\log}, \tag{1.8}$$

where $G^{\log}$ is a generalized singular Green operator satisfying a specific part of the usual symbol estimates for s.g.o.s; its principal part has a singularity at the boundary. In Section 3 we study its residue.

If, more generally than R_-, the ray free of eigenvalues for B (the spectral cut) is $e^{i\theta}\mathsf{R}_+$ for some angle θ, the corresponding operator functions will be defined by formulas where λ^{-s} and $\log\lambda$ (as in (1.2)) are replaced by λ_θ^{-s} and $\log_\theta \lambda$ with branch cut $e^{i\theta}\mathsf{R}_+$, and the integration curve runs in $\mathsf{C}\setminus e^{i\theta}\overline{\mathsf{R}}_+$. The functions are then provided with an index θ;

$$\zeta_\theta(B,s) = \operatorname{Tr}(B_\theta^{-s}), \qquad \log_\theta B = (\log_\theta P)_+ + G^{\log_\theta}. \tag{1.9}$$

When B has spectral cuts at θ and φ for some $\theta < \varphi < \theta + 2\pi$, it is of interest to study the *sectorial projection* $\Pi_{\theta,\varphi}(B)$, a projection whose range contains the generalized eigenspace of B for the sector $\Lambda_{\theta,\varphi} = \{ re^{i\omega} \mid r > 0, \theta < \omega < \varphi \}$ and whose nullspace contains the generalized eigenspace of B for $\Lambda_{\varphi,\theta+2\pi}$; it was considered earlier by Burak [3], and in the boundaryless case by Wodzicki

[20], Ponge [14]. We show in Section 4 that it equals $\frac{i}{2\pi}(\log_\theta B - \log_\varphi B)$ and has the form

$$\Pi_{\theta,\varphi}(B) = (\Pi_{\theta,\varphi}(P))_+ + G_{\theta,\varphi}. \tag{1.10}$$

Here $\Pi_{\theta,\varphi}(P)$ is a zero-order classical ψdo, which satisfies the transmission condition when m is even, and $G_{\theta,\varphi}$ is a generalized s.g.o., bounded in L_2 in the differential operator case. There are natural types of examples where $G_{\theta,\varphi}$ is a standard s.g.o. as in [2], but in general it will be of a generalized type satisfying only part of the standard symbol estimates.

We expect to take up elsewhere the study of its residue, whose possible vanishing is important for the study of eta functions associated with B.

2. The singular Green part of the logarithm

Let X be a compact n-dimensional C^∞ manifold with boundary $\partial X = X'$, provided with a hermitian C^∞ vector bundle E. We can assume that X is smoothly imbedded in an n-dimensional manifold $\widetilde{X}$ without boundary and that E is the restriction to X of a bundle $\widetilde{E}$ over $\widetilde{X}$. Consider a system $\{P_+ + G, T\}$ of operators in the Boutet de Monvel calculus [2] (pseudodifferential boundary operators, ψdbo's). Here P is defined as a ψdo of order $m > 0$ on $\widetilde{X}$ acting on the sections of $\widetilde{E}$, and its truncation to X is

$$P_+ = r^+ P e^+, \qquad r^+ \text{ restricts from } \widetilde{X} \text{ to } X^\circ,\ e^+ \text{ extends by } 0. \tag{2.1}$$

To assure that P_+ maps $C^\infty(X, E)$ into itself, P is assumed to satisfy the transmission condition, which means that in local coordinate systems at the boundary, where the manifold is replaced by $\mathsf{R}^n_+ = \{x = (x_1, \dots, x_n) \mid x_n > 0\}$, with notation $x' = (x_1, \dots, x_{n-1})$,

$$\begin{aligned}\partial_x^\beta \partial_\xi^\alpha p_{m-j}(x', 0, 0, -\xi_n)& \\ &= (-1)^{m-j-|\alpha|} \partial_x^\beta \partial_\xi^\alpha p_{m-j}(x', 0, 0, \xi_n) \qquad \text{for} \quad |\xi_n| \ge 1,\end{aligned} \tag{2.2}$$

for all indices; m is integer. (A discussion of such conditions can be found in Grubb and Hörmander [10].) G is a singular Green operator in E of order and class m, and $T = \{T_0, \dots, T_{m-1}\}$ is a system of trace operators T_k of order and class k, going from E to bundles F_k over ∂X, defining an elliptic boundary value problem. In particular,

$$\sum_{0 \le k \le m-1} \dim F_k = \tfrac{1}{2} m \dim E. \tag{2.3}$$

Details on these operator types can be found in [2], [6].

We assume that the system $\{P_+ + G - \lambda, T\}$ satisfies the conditions of parameter-ellipticity in [6, Def. 3.3.1] for λ on the rays in a sector V around R_-. In particular, it can be a differential operator system; here P and T are differential, and G is omitted. A classical example is the Laplace operator on a domain in R^n, together with the Dirichlet trace operator $T = \gamma_0$.

It should be noted that the hypotheses imply that the trace operator is normal, as accounted for in [6, Section 1.5].

The system has a certain regularity number ν in the sense of [6]; it is an integer or half-integer in $\left[\frac{1}{2}, m\right]$ for pseudodifferential problems, $+\infty$ for purely differential problems.

From the system we define the realization $B = (P + G)_T$ as the operator acting like $P_+ + G$ with domain (1.4). By [6, Ch. 3], the resolvent $R_\lambda = (B - \lambda)^{-1}$ exists on each ray in V for sufficiently large $|\lambda|$, and is $O(\lambda^{-1})$ in L_2 operator norm there. It has the structure

$$R_\lambda = Q_{\lambda,+} + G_\lambda, \tag{2.4}$$

where $Q_\lambda = (P - \lambda)^{-1}$ on $\widetilde{X}$ (which can be assumed to be compact), and G_λ is the singular Green part. Since the spectrum of B is discrete, we can assume (after a small rotation if necessary) that R_- is free of eigenvalues of B, and likewise for P.

We shall define the operator $\log(B) = \log((P + G)_T)$, also written $\log B$, $\log(P + G)_T$, by

$$\log(P + G)_T = \lim_{s \searrow 0} \frac{i}{2\pi} \int_{\mathscr{C}} \lambda^{-s} \log \lambda \, R_\lambda \, d\lambda, \tag{2.5}$$

to be further explained below; here $\mathscr{C}$ is a Laurent loop

$$\mathscr{C} = \{re^{i\pi} \mid \infty > r > r_0\} \cup \{r_0 e^{i\omega} \mid \pi \geq \omega \geq -\pi\} \cup \{re^{-i\pi} \mid r_0 < r < \infty\} \tag{2.6}$$

going around the nonzero spectrum of $(P + G)_T$ in the positive direction.

Insertion of the decomposition (2.4) in the defining formula (2.5) shows that $Q_{\lambda,+}$ contributes with

$$\lim_{s \searrow 0} \frac{i}{2\pi} \int_{\mathscr{C}} \lambda^{-s} \log \lambda \, r^+ Q_\lambda e^+ \, d\lambda = r^+ (\log P) e^+ = (\log P)_+, \tag{2.7}$$

where $\log P$ is well-known from the closed manifold case, cf. (1.2). Its symbol in local coordinates is of the form

$$\operatorname{symb}(\log P) = m \log[\xi] + l(x, \xi), \tag{2.8}$$

where $l(x,\xi)$ is a classical ψdo symbol of order 0 (see also the lemma below), and $[\xi]$ is a smooth positive function that equals $|\xi|$ for $|\xi|\geq 1$. The operator is continuous from $H^t(\widetilde{X},\widetilde{E})$ to $H^{t-\varepsilon}(\widetilde{X},\widetilde{E})$ for any $\varepsilon>0$; hence

$$(\log P)_+\colon H^t(X,E)\to H^{t-\varepsilon}(X,E) \qquad \text{for} \quad |t|<\tfrac{1}{2}. \tag{2.9}$$

(The limit for $s\to 0$ in (2.7) can be taken in this operator norm.)

In even-order cases, the transmission condition satisfied by P carries over to $l(x,\xi)$:

LEMMA 2.1. *When m is even, $l(x,\xi)$ satisfies the transmission condition.*

PROOF. As shown e.g. in Okikiolu [13], the symbol of $\log P$ is calculated in local coordinates from the symbol $q(x,\xi,\lambda)$ of Q_λ by integration with $\log\lambda$ around the spectrum of the principal symbol p_m of P; here the quasi-homogeneous terms in the expansion $q(x,\xi,\lambda)\sim\sum_{j\in\mathsf{N}} q_{-m-j}(x,\xi,\lambda)$ (homogeneous of degree $-m-j$ in $(\xi,|\lambda|^{\frac{1}{m}})$ on each ray) contribute as follows:

$$\begin{aligned}
&\frac{i}{2\pi}\int_{\mathscr{C}(x,\xi)}\log\lambda\, q_{-m}(x,\xi,\lambda)\,d\lambda\\
&\qquad=\frac{i}{2\pi}\int_{\mathscr{C}(x,\xi)}\log\lambda(p_m(x,\xi)-\lambda)^{-1}\,d\lambda=\log p_m(x,\xi)\\
&\qquad=\log([\xi]^m)+\log([\xi]^{-m}p_m(x,\xi))=m\log[\xi]+l_0(x,\xi),\\
&\frac{i}{2\pi}\int_{\mathscr{C}(x,\xi)}\log\lambda\, q_{-m-j}(x,\xi,\lambda)\,d\lambda=l_{-j}(x,\xi)\qquad\text{for}\quad j>0,
\end{aligned}\tag{2.10}$$

where $\mathscr{C}(x,\xi)$ is a closed curve in $\mathsf{C}\setminus\overline{\mathsf{R}}_-$ around the spectrum of $p_m(x,\xi)$. Each l_{-j} is homogeneous in ξ of degree $-j$ for $|\xi|\geq 1$; for $j=0$ it follows since $[\xi]^{-m}p_m(x,\xi)$ is so, and for $j\geq 1$ it is seen e.g. as follows (where we set $\lambda=t^m\varrho$):

$$\begin{aligned}
l_{-j}(x,t\xi)&=\frac{i}{2\pi}\int_{\mathscr{C}(x,t\xi)}\log\lambda\, q_{-m-j}(x,t\xi,\lambda)\,d\lambda\\
&=\frac{i}{2\pi}\int_{t^{-m}\mathscr{C}(x,t\xi)}(\log\varrho+m\log t)t^{-m-j}q_{-m-j}(x,\xi,\varrho)t^m\,d\varrho\\
&=t^{-j}l_{-j}(x,\xi)+mt^{-j}\log t\,\frac{i}{2\pi}\int_{\mathscr{C}(x,\xi)}q_{-m-j}(x,\xi,\varrho)\,d\varrho,
\end{aligned}$$

where the last term is zero since q_{-m-j} is $O(|\varrho|^{-2})$ for $|\varrho|\to\infty$ when $j>0$.

When m is even, we see that the transmission condition (2.2) carries over through the calculations (2.10) to the corresponding property for $l(x,\xi)$, since the parity of $-j$ is the same as that of $-j-m$.

Now consider the contribution from G_λ. Here we shall use the following observations:

$$(2.11)\quad \begin{aligned} &Q_\lambda + \lambda^{-1} = Q_\lambda + \lambda^{-1}(P-\lambda)Q_\lambda = \lambda^{-1}PQ_\lambda \quad \text{on } \widetilde{X},\\ &R_\lambda + \lambda^{-1}\\ &\quad = R_\lambda + \lambda^{-1}(P_+ + G - \lambda)R_\lambda = \lambda^{-1}(P_+ + G)(Q_{\lambda,+} + G_\lambda)\\ &\quad = \lambda^{-1}[(PQ_\lambda)_+ - L(P,Q_\lambda) + GQ_{\lambda,+} + (P_+ + G)G_\lambda]\\ &\quad = Q_{\lambda,+} + \lambda^{-1} + \lambda^{-1}[-L(P,Q_\lambda) + GQ_{\lambda,+} + (P_+ + G)G_\lambda] \quad \text{on } X; \end{aligned}$$

they imply in view of (2.4) that G_λ may be written as

$$(2.12)\qquad G_\lambda = \lambda^{-1}[-L(P,Q_\lambda) + GQ_{\lambda,+} + (P_+ + G)G_\lambda].$$

Here $L(P,Q_\lambda) = G^+(P)G^-(Q_\lambda)$ in local coordinates. (The latter formula is accounted for in [6, (1.2.49–50) and Sect. 2.6]; we recall that $G^+(P) = r^+Pe^-J$ and $G^-(P) = Jr^-Pe^+$, where $e^\pm$ extends by zero from $\mathbb{R}^n_\pm$ to $\mathbb{R}^n$, $r^\pm$ restricts from $\mathbb{R}^n$ to $\mathbb{R}^n_\pm$, and J is the reflection map $J\colon u(x',x_n)\mapsto u(x',-x_n)$.) By [6, Th. 3.3.2], G_λ is of order $-m$ and regularity ν; moreover, (2.12) shows that it is λ^{-1} times an s.g.o. of order 0 and regularity ν (by the composition rules in [6, Th. 2.7.6–7]).

Since

$$\begin{aligned} &Q_\lambda\colon L_2(\widetilde{X},\widetilde{E}) \to H^{m-\varepsilon}(\widetilde{X},\widetilde{E}),\\ &G_\lambda\colon L_2(X,E) \to H^{m-\varepsilon}(X,E), \end{aligned}\quad \text{with norms}\quad O(\lambda^{-\varepsilon/m}),$$

for $\varepsilon\in[0,m]$ (a standard observation used also in [6, pp. 409–410]), each of the terms in [ ] in (2.12) maps $L_2(X,E)$ to $H^{-\varepsilon}(X,E)$ with norm $O(\lambda^{-\varepsilon/m})$. Then we can perform the integration in this operator norm (letting $s\to 0$), defining the s.g.o.-like part $G^{\log}$ of $\log(P+G)_T$ by

$$(2.13)\quad \begin{aligned} G^{\log} &= \frac{i}{2\pi}\int_{\mathscr{C}} \log\lambda\, G_\lambda\, d\lambda\\ &= \frac{i}{2\pi}\int_{\mathscr{C}} \lambda^{-1}\log\lambda[-L(P,Q_\lambda) + GQ_{\lambda,+} + (P_+ + G)G_\lambda]\, d\lambda, \end{aligned}$$

also written as

$$(2.14)\quad G^{\log} = -G^+(P)\frac{i}{2\pi}\int_{\mathscr{C}} \lambda^{-1}\log\lambda\, G^-(Q_\lambda)\,d\lambda$$
$$+ G\frac{i}{2\pi}\int_{\mathscr{C}} \lambda^{-1}\log\lambda\, Q_{\lambda,+}\,d\lambda + (P_+ + G)\frac{i}{2\pi}\int_{\mathscr{C}} \lambda^{-1}\log\lambda\, G_\lambda\,d\lambda,$$

when localized. It is a bounded operator from $L_2(X, E)$ to $H^{-\varepsilon}(X, E)$. Summing up, we have found:

THEOREM 2.2. *The logarithm of the realization $B = (P + G)_T$ satisfies*

$$(2.15)\qquad \log B = \log(P + G)_T = (\log P)_+ + G^{\log},$$

where $\log P$ *is the logarithm of P on $\widetilde{X}$, and $G^{\log}$ is defined by* (2.13), (2.14)*; the terms are bounded operators from $L_2(X, E)$ to $H^{-\varepsilon}(X, E)$ (any $\varepsilon > 0$).*

The operator $G^{\log}$ is a generalized singular Green operator, in the same spirit as the generalized s.g.o.s $G^{(-s)}$ studied in [6, Sect. 4.4] (the s.g.o.-like parts of the powers B^{-s}), and one can show as in [6, Th. 4.4.4] that there is a symbol-kernel satisfying part of the usual $L_{2,x_n,y_n}(\mathsf{R}^2_{++})$ estimates for s.g.o.s, allowing $D^\beta_{x'}$, $D^\alpha_{\xi'}$, $(x_n D_{x_n})^k$ and $(y_n D_{y_n})^l$ in arbitrarily high powers (with exceptions for the principal term), and allowing some applications of $x_n^k D^{k'}_{x_n}$ and $y_n^l D^{l'}_{y_n}$, limited by the regularity and other restrictions. We account for this in Theorem 2.6 below; let us first consider an example.

EXAMPLE 2.3. Let $P = 1 - \Delta$ on R^n_+. It is easy to see that the solution operator for the Dirichlet problem for $P-\lambda = 1-\Delta-\lambda$, $\lambda \in V = \mathsf{C}\setminus \mathsf{R}_+$, is $R_\lambda = Q_{\lambda,+} + G_\lambda$, where Q_λ is the ψdo $(1-\lambda-\Delta)^{-1}$ with symbol $(\langle\xi\rangle^2-\lambda)^{-1}$, and G_λ is the singular Green operator with symbol-kernel $\frac{-1}{2\kappa_1}e^{-\kappa_1(x_n+y_n)}$; $\kappa_1 = (\langle\xi'\rangle^2-\lambda)^{\frac12}$. (We here use the well-known notation $\langle x\rangle = (x_1^2+\cdots+x_n^2+1)^{\frac12}$.) It follows that

$$(2.16)\qquad \log P = \mathrm{OP}(2\log\langle\xi\rangle).$$

To find out how $G^{\log}$ acts on functions $\varphi \in C_0^\infty(\mathsf{R}^n_+)$, we write (using that $e^{-\kappa_1(x_n+y_n)}$ is rapidly decreasing in λ on the rays in V when y_n is in the support of φ):

$$G^{\log}\varphi = \frac{i}{2\pi}\int_{\mathscr{C}} \log\lambda\, G_\lambda\varphi\,d\lambda$$
$$= \frac{i}{2\pi}\int_{\mathscr{C}}\int_{\mathsf{R}^{n-1}}\int_0^\infty \log\lambda\, e^{ix'\cdot\xi'}\frac{-1}{2\kappa_1}e^{-\kappa_1(x_n+y_n)}\acute{\varphi}(\xi', y_n)\,dy_n\,đ\xi'\,d\lambda,$$

with $\acute{\varphi}$ denoting the partial Fourier transform $\acute{\varphi}(\xi', y_n) = \mathscr{F}_{y'\to\xi'}\varphi(y', y_n)$. Here we can calculate
(2.17)

$$\begin{aligned}\frac{i}{2\pi}\int_{\mathscr{C}} \log\lambda \frac{-1}{2\kappa_1} e^{-\kappa_1(x_n+y_n)}\, d\lambda &= \int_{-\infty}^{0} \frac{1}{2(\langle\xi'\rangle^2 - t)^{\frac12}} e^{-(\langle\xi'\rangle^2-t)^{\frac12}(x_n+y_n)}\, dt \\ &= \int_{0}^{\infty} \frac{1}{2(\langle\xi'\rangle^2 + s)^{\frac12}} e^{-(\langle\xi'\rangle^2+s)^{\frac12}(x_n+y_n)}\, ds \\ &= \int_{\langle\xi'\rangle}^{\infty} \frac{1}{2u} e^{-u(x_n+y_n)}\, 2u\, du \\ &= \frac{1}{x_n+y_n} e^{-\langle\xi'\rangle(x_n+y_n)},\end{aligned}$$

using that the $\log|\lambda|$ contributions cancel out (as in [8, Lemma 1.2]). Thus

$$G^{\log}\varphi = \int_{\mathsf{R}^{n-1}}\int_0^\infty e^{ix'\cdot\xi'} \frac{1}{x_n+y_n} e^{-\langle\xi'\rangle(x_n+y_n)} \acute{\varphi}(\xi', y_n)\, dy_n đ\xi'.$$

This shows that $G^{\log}$ is a generalized kind of s.g.o. with symbol-kernel

$$\tilde{g}^{\log}(x', x_n, y_n, \xi') = \frac{1}{x_n+y_n} e^{-\langle\xi'\rangle(x_n+y_n)}. \tag{2.18}$$

Since the operator with kernel $\frac{1}{x_n+y_n}$ is bounded in $L_2(\mathsf{R}_+)$ (as a truncation of the Hilbert transform), it follows that $G^{\log}$ is a bounded operator in $L_2(\mathsf{R}^n_+)$.

Note that $\partial_{\xi_1}\tilde{g}^{\log}$ is a standard s.g.o. symbol-kernel, and that $x_n\tilde{g}^{\log}$ is bounded.

The same calculations with $\langle\xi'\rangle$ replaced by $|\xi'|$ show that for $P = -\Delta$, $G^{\log}$ has symbol-kernel $\frac{1}{x_n+y_n}e^{-|\xi'|(x_n+y_n)}$ for $|\xi'| \geq 1$.

In the general *differential operator* case, $G^{\log}$ is qualitatively very much like in this example. Here one can directly use the symbol-kernel estimates and boundedness considerations worked out by Seeley in [17], [18]. Notationally, we follow [8]; in particular, the enumeration of quasi-homogeneous (resp. homogeneous) terms in the asymptotic expansions of singular Green symbol-kernels (resp. symbols) have been shifted by one step in comparison with [6], in order to have the same index on an s.g.o. symbol-kernel (resp. symbol) and its normal trace. For example, the principal part of a symbol-kernel $\tilde{g}$ of order $-m$ is denoted $\tilde{g}_{-m}$ (although the corresponding symbol g_{-m} has homogeneity degree $-m-1$). We shall use the notation $\dot{\leq}$ (resp. $\dot{\geq}$) to indicate "less than or equal (resp. greater than or equal) to a constant times", and $\doteq$ to indicate that both $\dot{\leq}$ and $\dot{\geq}$ hold.

THEOREM 2.4. *Consider the case where P is a differential operator, $G = 0$, and the trace operators $T_0, \dots, T_{m-1}$ are differential operators. In this case, the singular Green part G_λ of the resolvent is of regularity $+\infty$ and its symbol-kernel in local coordinates $\tilde g \sim \sum_{j\ge 0} \tilde g_{-m-j}$, expanded in quasi-homogeneous terms*

$$\tilde g_{-m-j}\Big(x', \frac{x_n}{t}, \frac{y_n}{t}, t\xi', t^m\lambda\Big) = t^{-m+1-j}\tilde g_{-m-j}(x', x_n, y_n, \xi', \lambda) \quad \text{for} \quad t \ge 1, |\xi'| \ge 1, \tag{2.19}$$

satisfies estimates on the rays in V, with $\kappa = |\xi'| + |\lambda|^{\frac{1}{m}}$:

$$\big|D^\beta_{x'} D^\alpha_{\xi'} x_n^k D^{k'}_{x_n} y_n^l D^{l'}_{y_n} D^p_\lambda \tilde g_{-m-j}\big| \dot\le \kappa^{1-m-|\alpha|-k+k'-l+l'-j-mp} e^{-c\kappa(x_n+y_n)} \tag{2.20}$$

for all indices, when $\kappa \ge \varepsilon$.

Then $G^{\log}$ is, in local coordinates near X', a generalized singular Green operator

$$\begin{aligned} G^{\log}u(x) &= \int_{\mathbb{R}^{n-1}} \int_0^\infty e^{ix'\cdot\xi'} \tilde g^{\log}(x', x_n, y_n, \xi') \acute u(\xi', y_n)\, dy_n\, đ\xi' \\ &= \operatorname{OPG}(\tilde g^{\log}(x', x_n, y_n, \xi'))u(x) \end{aligned} \tag{2.21}$$

with $\tilde g^{\log} \sim \sum_{j\in\mathbb{N}} \tilde g^{\log}_{-j}$; here the j'th term is quasihomogeneous:

$$\tilde g^{\log}_{-j}\Big(x', \frac{x_n}{t}, \frac{y_n}{t}, t\xi'\Big) = t^{1-j}\tilde g^{\log}_{-j}(x', x_n, y_n, \xi') \quad \text{for } t \ge 1 \text{ and } |\xi'| \ge 1, \tag{2.22}$$

and satisfies, when $|\xi'| \ge \varepsilon$,

$$\big|D^\beta_{x'} D^\alpha_{\xi'} x_n^k D^{k'}_{x_n} y_n^l D^{l'}_{y_n} \tilde g^{\log}_{-j}\big| \dot\le |\xi'|^{-|\alpha|-k+k'-l+l'-j} \tfrac{1}{x_n+y_n} e^{-c|\xi'|(x_n+y_n)} \tag{2.23}$$

for the indices satisfying

$$-k + k' - l + l' - |\alpha| - j \le 0. \tag{2.24}$$

It follows in particular that $G^{\log}$ is a bounded operator in $L_p(X, E)$ for $1 < p < \infty$.

PROOF. The estimates (2.20) were shown in [17, (29)], [18]. Because of the fall-off in λ, they allow us to define the j'th term in the symbol-kernel of $G^{\log}$

for $|\xi'| \geq \varepsilon$ by

$$\begin{aligned}\tilde{g}^{\log}_{-j}(x', x_n, y_n, \xi') &= \frac{i}{2\pi} \int_{\mathscr{C}} \log \lambda \tilde{g}_{-m-j}(x', x_n, y_n, \xi', \lambda)\, d\lambda \\ &= \int_0^\infty \tilde{g}_{-m-j}(x', x_n, y_n, \xi', -s)\, ds;\end{aligned} \tag{2.25}$$

here we rewrote the integral as in (2.17) (and [8, Lemma 1.2]). The homogeneity is seen from the last integral, using (2.19). The function is estimated as follows, for the indices satisfying (2.24), when we use that $|\xi'| + s^{\frac{1}{m}} \doteq (|\xi'|^m + s)^{\frac{1}{m}}$:

(2.26)

$$\begin{aligned}&\big|D^\beta_{x'} D^\alpha_{\xi'} x_n^k D^{k'}_{x_n} y_n^l D^{l'}_{y_n} \tilde{g}^{\log}_{-j}\big| \\ &= \left|\int_0^\infty D^\beta_{x'} D^\alpha_{\xi'} x_n^k D^{k'}_{x_n} y_n^l D^{l'}_{y_n} \tilde{g}_{-m-j}(x', x_n, y_n, \xi', -s)\, ds\right| \\ &\dot{\leq} |\xi'|^{-|\alpha|-k+k'-l+l'-j} \int_0^\infty \big((|\xi'|^m + s)^{\frac{1}{m}}\big)^{1-m}\, e^{-c(|\xi'|^m+s)^{\frac{1}{m}}(x_n+y_n)}\, ds \\ &= |\xi'|^{-|\alpha|-k+k'-l+l'-j} \int_{|\xi'|}^\infty u^{1-m} e^{-cu(x_n+y_n)} m u^{m-1}\, du \\ &= |\xi'|^{-|\alpha|-k+k'-l+l'-j} \tfrac{m}{c(x_n+y_n)} e^{-c|\xi'|(x_n+y_n)}.\end{aligned}$$

The operator $G^{\log}$ is defined from a finite number of these symbol terms multiplied with an excision function $\zeta(|\xi'|)$, where

(2.27)

$$\zeta(t) \in C^\infty(\mathsf{R}), \qquad \zeta(t) = 0 \text{ for } |t| \leq \delta_1, \quad \zeta(t) = 1 \text{ for } |t| \geq \delta_2,$$

plus an integral as in (2.13) of the remainder of G_λ, which can be taken with arbitrarily high smoothness of the kernel and decrease for $\lambda \to \infty$, cf. [18, (2.14)]. Applying the arguments of Theorem 1 of [18] (using Lemmas 1 and 2 there invoking Mihlin's theorem and the Hilbert transform) one finds that $G^{\log}$ is L_p-continuous as asserted.

REMARK 2.5. The lower order terms in $\tilde{g}^{\log}$ and the derivatives are not as singular for $x_n + y_n \to 0$ as (2.23) indicates. In fact, the symbol-kernels one step down can be estimated as follows:

$$\text{When } -k + k' - l + l' - |\alpha| - j \leq -1, \tag{2.28}$$

$$\begin{aligned}\big|D^{\beta}_{x'}D^{\alpha}_{\xi'}x_n^k D^{k'}_{x_n}y_n^l D^{l'}_{y_n}\tilde g^{\log}_{-j}\big| &\dot\le |\xi'|^{-|\alpha|-k+k'-l+l'-j+1}\int_{|\xi'|}^{\infty} u^{-1-\varepsilon}u^{\varepsilon}e^{-cu(x_n+y_n)}\,du\\ &\dot\le |\xi'|^{-|\alpha|-k+k'-l+l'-j+1+\varepsilon}\sup_{u\in\mathsf{R}_+}\big|u^{\varepsilon}e^{-cu(x_n+y_n)}\big|\\ &\dot\le |\xi'|^{-|\alpha|-k+k'-l+l'-j+1+\varepsilon}(x_n+y_n)^{-\varepsilon},\end{aligned}$$

for $\varepsilon > 0$. The symbol-kernels two steps down are bounded for $x_n + y_n \to 0$:

$$\text{When } -k+k'-l+l'-|\alpha|-j\le -2, \tag{2.29}$$

$$\begin{aligned}\big|D^{\beta}_{x'}D^{\alpha}_{\xi'}x_n^k D^{k'}_{x_n}y_n^l D^{l'}_{y_n}\tilde g^{\log}_{-j}\big| &\dot\le |\xi'|^{-|\alpha|-k+k'-l+l'-j+2}\int_0^{\infty}\big(|\xi'|+s^{\frac1m}\big)^{-m-1}\,ds\\ &\dot\le |\xi'|^{-|\alpha|-k+k'-l+l'-j+1},\end{aligned}$$

and the smoothness at 0 increases with increasing $|\alpha|$ and j.

Now let us turn to the pseudodifferential case and the methods of [6, Sect. 4.4].

THEOREM 2.6. *Let $\{P_+ + G, T\}$ have regularity $\nu \in \big[\frac12, \infty\big[$, and define $G^{\log}$ by* (2.13)*. Then $G^{\log}$ is, in local coordinates near X', a generalized singular Green operator as in* (2.21) *with $\tilde g^{\log} \sim \sum_{j\in\mathsf{N}} \tilde g^{\log}_{-j}$; here the j'th term is quasihomogeneous as in* (2.22) *when $j > 0$, and the series approximates $\tilde g^{\log}$ asymptotically in the sense that*

$$\Big\|D^{\beta}_{x'}D^{\alpha}_{\xi'}x_n^k D^{k'}_{x_n}y_n^l D^{l'}_{y_n}\Big[\tilde g^{\log}-\sum_{j<J}\tilde g^{\log}_{-j}\Big]\Big\|_{L_{2,x_n,y_n}} \dot\le \langle\xi'\rangle^{-|\alpha|-k+k'-l+l'-J} \tag{2.30}$$

holds for the indices satisfying

$$\begin{gathered}-k+k'-l+l'-|\alpha|-J<0,\\ [k-k']_- + [l-l']_- < \nu.\end{gathered} \tag{2.31}$$

Moreover,

$$\big\|D^{\beta}_{x'}D^{\alpha}_{\xi'}x_n^k D^{k'}_{x_n}y_n^l D^{l'}_{y_n}\tilde g^{\log}_{-J}\big\|_{L_{2,x_n,y_n}} \dot\le \langle\xi'\rangle^{-|\alpha|-k+k'-l+l'-J} \tag{2.32}$$

holds for these indices.

With $\zeta(t)$ defined as in (2.27)*, the above symbol-kernels multiplied with $\zeta(x_n)\zeta(y_n)$ satisfy estimates for all $\alpha, \beta, J, k, k', l, l'$ with $\langle\xi'\rangle^{-M}$, any M, in the right-hand side.*

PROOF. This is modeled after the proof of [6, Th. 4.4.4] and the remarks preceding it.

We recall from [6, Th. 3.3.9] that the symbol-kernel $\tilde g(x', x_n, y_n, \xi', \lambda)$ of G_λ (in a local coordinate system) has an expansion in quasi-homogeneous terms $\tilde g \sim \sum_{j\geq 0} \tilde g_{-m-j}$ satisfying (2.19) in V, and that one has for all indices, denoting $\lambda = -\mu^m e^{i\omega}$ ($\mu > 0$), $(|\xi'|^2 + \mu^2 + 1)^{\frac12} = \langle \xi', \mu\rangle$:

$$
\begin{aligned}
&\Big\| D^\beta_{x',\omega} D^\alpha_{\xi'} x_n^k D^{k'}_{x_n} y_n^l D^{l'}_{y_n}\Big[\tilde g - \sum_{j<J} \tilde g_{-m-j}\Big]\Big\|_{L_{2,x_n,y_n}} \\
&\quad \dot\leq \big(\langle \xi'\rangle^{\nu-M'} + \langle \xi', \mu\rangle^{\nu-M'}\big)\langle \xi',\mu\rangle^{-m-\nu+M''} \\
&\quad \dot\leq \begin{cases} \langle \xi',\mu\rangle^{-m-M'+M''}, & \text{when } M' \leq \nu, \\ \langle \xi'\rangle^{\nu-M'}\langle \xi',\mu\rangle^{-m-\nu+M''}, & \text{when } M' \geq \nu, \end{cases}
\end{aligned}
\tag{2.33}
$$

with

$$
\begin{aligned}
M' &= [k-k']_+ + [l-l']_+ + |\alpha| + J, \\
M'' &= [k-k']_- + [l-l']_-; \qquad \text{so} \\
-M' + M'' &= -k + k' - l + l' - |\alpha| - J.
\end{aligned}
\tag{2.34}
$$

The notation $N_\pm = \max\{\pm N, 0\}$ is used, and we have (as recalled earlier) changed the indexation from [6] by one step as in [8].

Let us first observe that the "error terms" and remainders in the resolvent construction, that are negligible in the class of operators of order $-m$ and regularity ν, give rise to generalized s.g.o. error terms G' here, satisfying estimates of the type (as in [6, Lemma 2.3.11])

$$
\begin{aligned}
&\Big\| D^\beta_{x'} D^\alpha_{\xi'} x_n^k D^{k'}_{x_n} y_n^l D^{l'}_{y_n} \tilde g'\Big\|_{L_{2,x_n,y_n}} \\
&\quad \dot\leq \langle \xi'\rangle^{-M}\Big|\int_{\mathscr C} \log\lambda\,\langle\lambda\rangle^{-1-(\nu-[k-k']_- -[l-l']_-)/m}\,d\lambda\Big| \\
&\quad \dot\leq \langle \xi'\rangle^{-M}, \qquad \text{for any } M, \text{ when} \quad [k-k']_- + [l-l']_- < \nu.
\end{aligned}
\tag{2.35}
$$

It follows that the corresponding kernels $\mathscr K_{G'}(x,y)$ satisfy, for these indices:

$$
\sup_{x',y'} \big\| D^\gamma_{x',y'} x_n^k D^{k'}_{x_n} y_n^l D^{l'}_{y_n} \mathscr K_{G'}\big\|_{L_{2,x_n,y_n}} < \infty. \tag{2.36}
$$

For $j > 0$ the L_{2,x_n,y_n}-norm of $\tilde g_{-m-j}$ is $O(\lambda^{-1-1/2m})$ since $\nu \geq \frac12$, so the corresponding term $\tilde g^{\log}_{-j}$ can be defined directly for $|\xi'| \geq 1$ by Cauchy integrals as in (2.25), convergent in the L_{2,x_n,y_n}-norm. The quasi-homogeneity of $\tilde g^{\log}_{-j}$ is seen as in (2.25) by using [8, Lemma 1.2] in L_{2,x_n,y_n}-norm.

We use the estimates (2.33) to see that for $\tilde g^{\log} - \sum_{j<J} \tilde g^{\log}_{-j}$ with $J > 0$ (so that the first term is excluded), the integrand in the corresponding Cauchy integral is $O(\lambda^{-1-\varepsilon})$ in L_{2,x_n,y_n}-norm (some $\varepsilon > 0$), when

$$(2.37)\quad -k+k'-l+l'-|\alpha|-J < 0,\ \text{if } [k-k']_+ + [l-l']_+ + |\alpha| + J \le \nu,$$

and when

$$(2.38)\quad [k-k']_- + [l-l']_- < \nu,\ \text{if } [k-k']_+ + [l-l']_+ + |\alpha| + J \ge \nu.$$

Then the integral converges and defines a symbol-kernel satisfying the asserted estimate. Since

$$-k+k'-l+l'-|\alpha|-J = [k-k']_- + [l-l']_- - ([k-k']_+ + [l-l']_+ + |\alpha| + J),$$

we see that the conditions "if ..." can be left out in (2.37)–(2.38), leading to the formulation (2.31).

We still have to consider the first term $\tilde g^{\log}_0$ in $\tilde g^{\log}$, defined from the principal part $\tilde g_{-m}$ of $\tilde g$. Here we use that $\tilde g_{-m}$ can be found by performing the resolvent construction on the principal boundary symbol level for the corresponding operators on $L_2(\mathsf{R}_+)$, and that they obey a one-dimensional version of the identities in (2.11). So we can replace $\tilde g_{-m}$ by the symbol-kernel of the principal boundary symbol version of (2.12), which gives a convergent Cauchy integral, when the λ-independent factors are pulled outside of the integration. In a formal sense, we can ascribe it a symbol-kernel $\tilde g^{\log}_0(x', x_n, y_n, \xi')$. The resulting boundary symbol operator is continuous from $L_2(\mathsf{R}_+)$ to $H^{-\varepsilon}(\mathsf{R}_+)$ for $\varepsilon > 0$, at each (x', ξ'). If we define the functions derived from $\tilde g^{\log}_0$ "weakly" by

$$\begin{aligned} &D^\beta_{x'} D^\alpha_{\xi'} x_n^k D^{k'}_{x_n} y_n^l D^{l'}_{y_n} \tilde g^{\log}_0(x', x_n, y_n, \xi') \\ &\qquad = \frac{i}{2\pi} \int_{\mathscr{C}} \log\lambda\, D^\beta_{x'} D^\alpha_{\xi'} x_n^k D^{k'}_{x_n} y_n^l D^{l'}_{y_n} \tilde g_{-m}(x', x_n, y_n, \xi', \lambda)\, d\lambda, \end{aligned}$$

we can use that the integral converges in L_{2,x_n,y_n}-norm when the indices satisfy (2.31). In this sense, the estimates (2.30) hold also when $J = 0$ in (2.31).

The estimates (2.32) of the individual terms follow from (2.30) since $\tilde g^{\log}_{-J} = \big(\tilde g^{\log} - \sum_{j<J} \tilde g^{\log}_{-j}\big) - \big(\tilde g^{\log} - \sum_{j<J+1} \tilde g^{\log}_{-j}\big)$.

Finally, for the statements on the symbol-kernels multiplied with $\zeta(x_n)\zeta(y_n)$, note that $\zeta(t)$ can for any $k \in \mathsf{N}$ be written as $t^k \zeta_k(t)$ with a bounded smooth function ζ_k, so from the already shown estimates we can infer arbitrarily rapid fall-off in ξ' by rewriting with arbitrarily high powers of x_n and y_n.

If R_λ has infinite regularity, ν can be arbitrarily large in the second line of (2.31), so the line can be left out. Note that even then there is a limitation on the indices for which we get standard s.g.o. estimates.

While $G^{\log}$ is the primary s.g.o.-type operator to consider in this connection, it is also of interest to study some other s.g.o.-type operators here, namely, in local coordinates, $G^+(\log P) = r^+(\log P)e^- J$ and $G^-(\log P) = Jr^-(\log P)e^+$, with notation as in the text after (2.12). The operators $G^\pm(\log P)$ have properties very similar to those of $G^{\log}$:

THEOREM 2.7. *The operators $G^\pm(\log P)$ are defined in local coordinates by*

$$\begin{aligned} G^+(\log P) &= r^+ \log P e^- J = r^+ \frac{i}{2\pi} \int_{\mathscr{C}} \log\lambda\, Q_\lambda\, d\lambda\, e^- J \\ &= \frac{i}{2\pi} \int_{\mathscr{C}} \lambda^{-1} \log\lambda\, G^+(PQ_\lambda)\, d\lambda, \\ G^-(\log P) &= Jr^- \log P e^+ = Jr^- \frac{i}{2\pi} \int_{\mathscr{C}} \log\lambda\, Q_\lambda\, d\lambda\, e^+ \\ &= \frac{i}{2\pi} \int_{\mathscr{C}} \lambda^{-1} \log\lambda\, G^-(PQ_\lambda)\, d\lambda. \end{aligned} \tag{2.39}$$

Their symbol-kernels $\tilde{g}^\pm(\log p)$ have properties like those of $\tilde{g}^{\log}$ in Theorem 2.6, with $\nu = m$.

In particular, when P is a differential operator, the s.g.o.s $G^\pm(Q_\lambda)$ satisfy Seeley's estimates (2.20)*, and hence the operators $G^\pm(\log P)$ have symbol estimates and boundedness properties like those of $G^{\log}$ in Theorem 2.4, Remark 2.5.*

PROOF. The defining integrals are established by use of the first formula in (2.11), noting that $G^\pm(\lambda^{-1}) = 0$. By [6, Th. 2.7.4], $G^\pm(Q_\lambda)$ is a parameter-dependent polyhomogeneous family of s.g.o.s of order $-m$ and regularity $m-\varepsilon$ (any $\varepsilon > 0$), since Q_λ is of order $-m$ and regularity m. The symbol-kernel then satisfies estimates like those for $\tilde{g}$ in Theorem 2.6, with $\nu = m - \varepsilon$. The method of Theorem 2.6 leads to the conclusion that the resulting symbol-kernel $\tilde{g}^\pm(\log p)$ has properties like those stated for $\tilde{g}^{\log}$, with $\nu = m - \varepsilon$; here ε can be removed since the second inequality in (2.31) is sharp.

For the second statement, we must show that the Seeley estimates (2.20) are valid for the homogeneous terms in the symbol-kernel of $G^\pm(Q_\lambda)$. But this is easy. Consider e.g. $G^+(Q_\lambda)$. Using the Taylor expansion of the symbol of Q_λ at $x_n = 0$:

$$q(x', x_n, \xi, \lambda) \sim \sum_{l \in \mathbb{N}} \tfrac{1}{l!} x_n^l \partial_{x_n}^l q(x', 0, \xi, \lambda)$$

we have from [6, Th. 2.7.4] that

$$g^+(q)(x', \xi, \eta_n, \lambda) \sim \sum_{l \in \mathbb{N}} \tfrac{1}{l!} \overline{D}^l_{\xi_n} g^+[\partial^l_{x_n} q(x', 0, \xi, \lambda)],$$

where $g^+[f](\xi_n, \eta_n)$ is the s.g.o. symbol corresponding to the symbol-kernel $\tilde{g}^+[f](x_n, y_n)$ defined by:

$$\tilde{g}^+[f](x_n, y_n) = \big(r^+_{z_n}[\mathscr{F}^{-1}_{\xi_n \to z_n} f]\big)|_{z_n = x_n + y_n}.$$

The homogeneous terms in the symbols $\partial^l_{x_n} q(x', 0, \xi, \lambda)$ are rational functions of ξ_n with $\frac{1}{2} m \dim E$ poles in $\mathbb{C}_\pm = \{z \in \mathbb{C} \mid \operatorname{Im} z \gtrless 0\}$, lying inside a circle of radius $C\kappa$ and having a distance $\geq c\kappa$ from the real axis, for suitable positive constants $C > c$. (A more detailed description is given e.g. in [6, Remark 3.3.7].) For simplicity of notation, consider the j'th term q_{-m-j} itself. The inverse Fourier transform evaluated at $z_n > 0$ can be written as an integral of $e^{iz_n \xi_n} q_{-m-j}(x', 0, \xi', \xi_n)$ over the curve bounding the intersection of the circle $\{|\xi_n| = C\kappa\}$ with the halfplane $\{\operatorname{Im} \xi_n \geq c\kappa\}$ (lying in $\mathbb{C}_+$). We get the factor $e^{-c\kappa z_n}$ since $|e^{iz_n \xi_n}| \leq e^{-c\kappa z_n}$ on the curve. (Similarly, the inverse Fourier transform evaluated at $z_n < 0$ can be written as an integral over a closed curve in $\mathbb{C}_-$ with $\operatorname{Im} \xi_n \leq -c\kappa$.) For the resulting symbol-kernel, this gives the factor $e^{-c\kappa(x_n + y_n)}$; the power of κ in front is seen from the degree of the rational function.

Once the estimates (2.20) are established, the rest of the proof goes as in Theorem 2.4.

EXAMPLE 2.8. For $P = 1 - \Delta$ as in Example 2.3, one finds by direct calculation of the inverse Fourier transform w.r.t. ξ_n that $G^\pm(Q_\lambda)$ both have the symbol-kernel

$$\tilde{g}^+ = \tilde{g}^- = \frac{1}{2\kappa_1} e^{-\kappa_1 (x_n + y_n)}, \tag{2.40}$$

with $\kappa_1 = (\langle \xi' \rangle^2 - \lambda)^{\frac{1}{2}}$. Then the calculations of Example 2.3 can be used again, to see that

(2.41)

$$\tilde{g}^+(\log p)(x', x_n, y_n, \xi') = \tilde{g}^-(\log p)(x', x_n, y_n, \xi') = \frac{-1}{x_n + y_n} e^{-\langle \xi' \rangle (x_n + y_n)}.$$

For $P = -\Delta$, the calculations give that the symbol-kernel of $G^\pm(\log P)$ is $\frac{-1}{x_n + y_n} e^{-|\xi'|(x_n + y_n)}$ for $|\xi'| \geq 1$; the same holds for $P = \mathrm{OP}([\xi]^2)$.

When the order m is even, there is a remarkable simplification in view of Lemma 2.1:

PROPOSITION 2.9. *When $m = 2k$, k integer > 0, then in local coordinates, the symbol-kernel of $G^{\pm}(\log P)$ satisfies for $|\xi'| \geq 1$:*

$$\tilde{g}^{\pm}(\log p)(x', x_n, y_n, \xi') = \frac{-k}{x_n + y_n} e^{-|\xi'|(x_n+y_n)} + \tilde{g}^{\pm,0}(x', x_n, y_n, \xi'), \tag{2.42}$$

where $\tilde{g}^{\pm,0}(x', x_n, y_n, \xi')$ is a standard singular Green symbol of order and class 0.

PROOF. We here have in view of Lemma 2.1 that the symbol of $\log P$ is the sum of $k \log[\xi]^2$ and a symbol $l(x, \xi)$ of order 0 satisfying the transmission condition. Then we can apply Example 2.8 to the first term and the standard $G^{\pm}$ construction (of [6]) to the second term.

Thus in the even-order case, the terms in $G^{\pm}(\log P)$ of order < 0 satisfy *all* the standard s.g.o. estimates.

3. Trace formulas

The normal trace $\operatorname{tr}_n G$ of a singular Green operator G with symbol-kernel $\tilde{g}(x', x_n, y_n, \xi')$ in a local coordinate system is the ψdo $S = \operatorname{tr}_n G$ with symbol

$$s(x', \xi') = (\operatorname{tr}_n \tilde{g})(x', \xi') = \int_0^\infty \tilde{g}(x', x_n, x_n, \xi')\, dx_n. \tag{3.1}$$

In the differential operator case, we see from the estimates (2.23), (2.28), (2.29) that $\operatorname{tr}_n \tilde{g}^{\log}_{-j}$ is well-defined for $j \geq 1$. (Example 2.3 shows that this will generally not hold for the principal part.) In view of the homogeneity (2.22), $\operatorname{tr}_n \tilde{g}^{\log}_{-j}$ is homogeneous of degree $-j$ in ξ' for $|\xi'| \geq 1$, hence a classical ψdo symbol of degree $-j$. In the pseudodifferential case, we have when $\nu > 1$ and $j \geq 1$ that the L_{2,x_n,y_n}-estimates of $\tilde{g}^{\log}_{-j}$, $y_n \tilde{g}^{\log}_{-j}$, $\partial_{y_n} \tilde{g}^{\log}_{-j}$ and $y_n \partial_{y_n} \tilde{g}^{\log}_{-j}$ imply as in [6, pf. of Th. 3.3.9] that there is a well-defined normal trace, again a homogeneous classical symbol of order $-j$. This estimation applies also to remainders $\tilde{g}^{\log} - \sum_{j<J} \tilde{g}^{\log}_{-j}$ for $J \geq 1$.

For $\nu = \frac{1}{2}$ or 1, the estimates in Theorem 2.6 do not provide the estimates of $\partial_{y_n} \tilde{g}^{\log}_{-j}$ needed for this argument. However, it is still posssible to take the normal trace of G_λ, subtract the principal part, and integrate the remaining operator with $\log \lambda$ to get a classical ψdo of order -1.

THEOREM 3.1. *In a local coordinate system, let $S_\lambda = \operatorname{tr}_n G_\lambda$ with symbol $s(x', \xi', \lambda) = (\operatorname{tr}_n \tilde{g})(x', \xi', \lambda)$, expanded in terms $s_{-m-j}(x', \xi', \lambda) = (\operatorname{tr}_n \tilde{g}_{-m-j})(x', \xi', \lambda)$. Define the parts of G_λ and S_λ of order $-m-1$ by*

$$\begin{aligned} G_{\lambda,\mathrm{sub}} &= G_\lambda - \mathrm{OPG}(\tilde{g}_{-m}(x', x_n, y_n, \xi', \lambda)), \\ S_{\lambda,\mathrm{sub}} &= \operatorname{tr}_n G_{\lambda,\mathrm{sub}} = S_\lambda - \mathrm{OP}'(s_{-m}(x', \xi', \lambda)) \end{aligned} \tag{3.2}$$

(the remainders after subtracting principal parts), and let

$$G^{\log}_{\mathrm{sub}} = \frac{i}{2\pi}\int_{\mathscr{C}} \log\lambda\, G_{\lambda,\mathrm{sub}}\, d\lambda, \tag{3.3}$$

with symbol-kernel $\tilde g^{\log}_{\mathrm{sub}} = \tilde g^{\log} - \tilde g^{\log}_0$*. The formula*

$$S^{\log}_{\mathrm{sub}} = \frac{i}{2\pi}\int_{\mathscr{C}} \log\lambda\, S_{\lambda,\mathrm{sub}}\, d\lambda \tag{3.4}$$

defines a classical ψ*do of order* -1*, with symbol* $s^{\log}_{\mathrm{sub}}(x',\xi')$ *expanded in terms*

$$s^{\log}_{\mathrm{sub},-j}(x',\xi') = \frac{i}{2\pi}\int_{\mathscr{C}} \log\lambda\, s_{-m-j}(x',\xi',\lambda)\, d\lambda, \qquad j\ge 1. \tag{3.5}$$

When $\nu > 1$*,* $S^{\log}_{\mathrm{sub}}$ *is the normal trace of* $G^{\log}_{\mathrm{sub}}$*.*

Proof. Since G_λ and $G_{\lambda,\mathrm{sub}}$ are of regularity $\nu \ge \frac12$, S_λ and $S_{\lambda,\mathrm{sub}}$ are of regularity $\nu - \frac14 \ge \frac14$, cf. [8, Section 3]. In particular, the symbols in $S_{\lambda,\mathrm{sub}}$ are $O(\lambda^{-1-1/4m})$ on the rays in V so that the integrals in (3.4) and (3.5) make sense.

As accounted for in the text before the theorem, there are estimates in the cases $\nu > 1$ that allow interchange of the λ-integral with the x_n-integral involved in taking tr_n.

For the operator in Example 2.3, we note that $S_\lambda = \mathrm{tr}_n\, G_\lambda$ is the ψdo with symbol $-(2\kappa_1)^{-2} = -\frac14(\langle\xi'\rangle^2-\lambda)^{-1}$, so its log-integral gives $-\frac14\log(1-\Delta_{x'})$. This demonstrates that the "log-transform" of the principal part of S_λ will not in general be a classical ψdo.

Finally, we shall connect this with the study of the expansion coefficient $C_0(I,(P+G)_T)$ in the last section of [8]; we here write it simply as $C_0((P+G)_T)$ (or $C_0(B)$). It is known from [6, Sect. 3.3] that when $m > n$, the trace of the resolvent has an expansion in powers of $-\lambda$,

$$\mathrm{Tr}\, R_\lambda = \sum_{0\le l\le n} c_l(-\lambda)^{\frac{n-l}{m}-1} + O\big(\lambda^{-1-\frac{1}{4m}}\big), \tag{3.6}$$

and a similar proof shows that for general $m > 0$, the expansion holds for a sufficiently high iterate:

$$\mathrm{Tr}\, R^N_\lambda = \mathrm{Tr}\,\frac{\partial_\lambda^{N-1}}{(N-1)!}R_\lambda = \sum_{0\le l\le n} c^{(N)}_l(-\lambda)^{\frac{n-l}{m}-N} + O\big(\lambda^{-N-\frac{1}{4m}}\big). \tag{3.7}$$

Define *the basic zeta value* as the coefficient of $(-\lambda)^{-N}$:

$$C_0(B) = c_n^{(N)}, \tag{3.8}$$

it is independent of N. If B is invertible, $C_0(B)$ equals the value of the zeta function $\zeta(B, s)$ — the meromorphic extension of $\operatorname{Tr}(B^{-s})$ — at $s = 0$. If B has a nontrivial nullspace, the constants are connected by

$$C_0(B) = \zeta(B, 0) + \nu_0, \tag{3.9}$$

where ν_0 is the dimension of the generalized eigenspace of the zero eigenvalue.

There are similar expansions as in (3.7) of the traces of the ψdo iterates Q_λ^N on $\widetilde{X}$, truncated to X, that follow from integration over X of the diagonal kernel expansions, as established in [6, Sect. 3.3] (with remarks); it is the s.g.o. contribution that presents the greater challenge in [6]. In view of the identifications in [8, Sect. 1], the coefficient of $(-\lambda)^{-N}$ here equals $-\frac{1}{m}\operatorname{res}_+(\log P)$, where the plus-index indicates that the pointwise contribution to $-\frac{1}{m}\operatorname{res}(\log P)$ is integrated over X only. It can also be regarded as $-\frac{1}{m}\operatorname{res}((\log P)_+)$, extending the notation of [4].

The constant $C_0(B)$ was analyzed in [8, Sect. 5] in relation to residue formulas, and we can now improve the result with further information.

THEOREM 3.2. *One has that*

$$C_0(B) = -\tfrac{1}{m}\operatorname{res}_+(\log P) - \tfrac{1}{m}\operatorname{res}_{X'}(S^{\log}_{\mathrm{sub}}), \tag{3.10}$$

where the terms are calculated as sums of contributions from local coordinate patches of the form

$$\begin{aligned} &\int_{\mathbb{R}^n_+}\int_{|\xi|=1} \operatorname{tr} l_{-n}(x,\xi)\, đS(\xi)\, dx, \qquad \textit{resp.} \\ &\int_{\mathbb{R}^{n-1}}\int_{|\xi'|=1} \operatorname{tr} s^{\log}_{\mathrm{sub},1-n}(x',\xi')\, đS(\xi')\, dx'. \end{aligned} \tag{3.11}$$

The term $-\frac{1}{m}\operatorname{res}_+(\log P)$ *has an invariant meaning as the coefficient of* $(-\lambda)^{-N}$ *in the expansion similar to* (3.7) *of* $\operatorname{Tr}(((P-\lambda)^{-N})_+)$*, and hence the last term in the right-hand side of* (3.10) *likewise has an invariant meaning.*

When the problem is differential, or when the problem is pseudodifferential with regularity $\nu > 1$*, then* $\operatorname{res}_{X'}(S^{\log}_{\mathrm{sub}})$ *is, in local coordinates, the residue of the normal trace of* $G^{\log}_{\mathrm{sub}}$.

PROOF. It was shown in [8, Sect. 5] how $C_0(B)$ is found from integrals of the strictly homogeneous symbol terms of order $-m-n$ in $(P-\lambda)^{-1}$ resp. of order $-m-n+1$ in G_λ; the proof given for the case $m > n$ extends to general m when the iterates are used, cf. [8, Remark 3.12]. It was shown moreover that these integrals by use of [8, Lemmas 1.2, 1.3] could be turned into log-integrals as in (3.5). In those proofs, the log-integration is applied after the tr_n-integration, so the boundary term is really $\operatorname{res}(S^{\log}_{\text{sub}})$, as defined in Theorem 3.1.

When $\nu > 1$, in particular when the problem is differential so that $\nu = \infty$, Theorem 3.1 shows that $S^{\log}_{\text{sub}}$ is the normal trace of $G^{\log}_{\text{sub}}$, so the assertion for the residues follows.

What we gain here in comparison with [8, Sect. 5] is a little more insight into how the boundary term stems from the s.g.o.-like part of $\log B$, plus the inclusion of all orders $m > 0$. At any rate, since $C_0(B)$ is an invariant, we can propose it to be the residue of $-\frac{1}{m}\log B$:

DEFINITION 3.3. When $\{P_+ + G - \lambda, T\}$ satisfies the hypotheses of parameter-ellipticity given above, the residue of $\log(P+G)_T$ is defined to be the constant

$$\operatorname{res}(\log(P+G)_T) = -mC_0((P+G)_T) = \operatorname{res}_+(\log P) + \operatorname{res}_{X'}(S^{\log}_{\text{sub}}), \tag{3.12}$$

as calculated in Theorem 3.2.

This is consistent with the definition of [4]. We note that certain steps in an explicit calculation of this constant depend very much on localizations, e.g. in the steps of discarding the principal symbol and taking tr_n. A number of similar or more general residue definitions are made in [9] for compositions of ψdbo's with components of $\log P_T$ (when P_T is defined from an even-order differential problem). These residues do have a certain amount of traciality: $\operatorname{res}([A, \log P_T]) = 0$ holds for operators A of order and class zero (cf. Theorem 6.5 there).

It should be noted that Definition 3.3 does not cover the case of first-order differential operators with spectral boundary conditions, since such boundary conditions are not *normal*. But for such boundary problems (Atiyah-Patodi-Singer problems [1]) there exists a wealth of other treatments, adapted to the specific situation. The results there often depend on additional symmetry properties. (See e.g. [7] and its references.)

4. Sectorial projections

Now we turn our attention to a certain spectral projection connected to the realization $(P+G)_T$; namely a projection whose range contains the closure of the direct sum of the generalized eigenspaces for the eigenvalues in a sector of the complex plane. Such projections have been studied earlier by Burak [3], Wodzicki [20], and Ponge [14]; the latter gives a detailed deduction of the basic properties in the case of classical ψdo's on closed manifolds. We recall the properties below, supplying them with some additional information.

In order to apply the techniques to different types of operators, we first consider an abstract situation where A denotes an unbounded, densely defined, closed operator in a Hilbert space H. It is assumed to have the following properties:

A has a resolvent set containing two sectors V_θ and V_φ around $e^{i\theta}\mathsf{R}_+$ and $e^{i\varphi}\mathsf{R}_+$, respectively, for some $\theta < \varphi < \theta + 2\pi$, the resolvent $(A-\lambda)^{-1}$ is compact, and $\|(A-\lambda)^{-1}\|$ is $O(\lambda^{-1})$ for λ going to infinity on each ray of these sectors. (We refer to Kato [12] for general background theory.)

For $x \in D(A)$ and λ on a ray in either sector, we have

$$\|\lambda^{-1}A\,(A-\lambda)^{-1}x\| \le \|\lambda^{-1}(A-\lambda)^{-1}\| \cdot \|Ax\| = O(\lambda^{-2}), \tag{4.1}$$

so that $\lambda^{-1}A(A-\lambda)^{-1}x$ is integrable for $|\lambda| \to \infty$.

Then define the operator $\Pi_{\theta,\varphi}(A)$, the *sectorial projection*, with domain $D(A)$ to begin with, by

$$\Pi_{\theta,\varphi}(A)x = \frac{i}{2\pi}\int_{\Gamma_{\theta,\varphi}} \lambda^{-1}A\,(A-\lambda)^{-1}x\,d\lambda, \qquad x \in D(A), \tag{4.2}$$

where the integration goes along the sectorial contour
(4.3)

$$\Gamma_{\theta,\varphi} = \{re^{i\varphi} \mid \infty > r > r_0\} \cup \{r_0 e^{i\omega} \mid \varphi \ge \omega \ge \theta\} \cup \{re^{i\theta} \mid r_0 < r < \infty\},$$

with r_0 taken so small that 0 is the only possible eigenvalue in $\{|\lambda| < r_0\}$. If the operator is bounded in H-norm, we extend it to H. This operator is a spectral projection in the following sense:

For each $\lambda \in \sigma(A)$, denote the generalized eigenspace by E_λ,

$$E_\lambda = \bigcup_{k\in\mathsf{N}} \ker(A-\lambda)^k$$

(it equals $\ker(A-\lambda)^{k_0}$ for a sufficiently large k_0). For $\alpha < \beta$, set

$$\Lambda_{\alpha,\beta} = \{\, re^{i\omega} \mid r > 0,\ \alpha < \omega < \beta,\ \}, \qquad E_{\alpha,\beta} = \dot{+}_{\lambda\in\sigma(A)\cap\Lambda_{\alpha,\beta}} E_\lambda.$$

PROPOSITION 4.1. *$\Pi_{\theta,\varphi}(A)^2 = \Pi_{\theta,\varphi}(A)$, i.e. $\Pi_{\theta,\varphi}(A)$ is a (possibly unbounded) projection in H. Its range contains $E_{\theta,\varphi}$ and its kernel contains $E_0 \dotplus E_{\varphi,\theta+2\pi}$.*

(a) *If A has a complete system of root vectors, i.e. $\dotplus_{\lambda\in\sigma(A)} E_\lambda$ is dense in H, then $\Pi_{\theta,\varphi}(A)$ is the bounded projection onto $\overline{E_{\theta,\varphi}}$ along $E_0 \dotplus \overline{E_{\varphi,\theta+2\pi}}$.*

(b) *If A is normal, i.e. $A^*A = AA^*$, then $\Pi_{\theta,\varphi}(A)$ is the bounded orthogonal projection onto $\oplus_{\lambda\in\sigma(A)\cap\Lambda_{\theta,\varphi}} \ker(A-\lambda)$ along $\oplus_{\lambda\in\sigma(A)\setminus\Lambda_{\theta,\varphi}} \ker(A-\lambda)$.*

PROOF. Except for a few elementary considerations regarding the domain and closedness, the proofs of [14, Propositions 3.2, A.4, and A.5] carry over almost word for word to the present setting (it should be noted that some contours in [14] have the opposite orientation).

In (a) and (b), the boundedness of $\Pi_{\theta,\varphi}(A)$ follows from the fact that the kernel and range are closed.

In certain important cases, $\Pi_{\theta,\varphi}(A)$ can be seen to be bounded regardless of whether the hypotheses of (a) or (b) can be verified; as shown in [14, Proposition 3.1] this holds when A is a ψdo of order $m > 0$ on a closed manifold. We shall see below in Theorem 4.6 that it also holds for the realization of a differential elliptic boundary value problem.

As shown below, the sectorial projection has a direct connection with the choice of spectral cut in our definition of the logarithm of an operator. Using arguments as in Section 2, we can define the logarithm of A with a branch cut at the angle θ as

$$\log_\theta A = \lim_{s\searrow 0} \frac{i}{2\pi} \int_{\mathscr{C}_\theta} \lambda_\theta^{-s} \log_\theta \lambda (A-\lambda)^{-1}\, d\lambda \tag{4.4}$$

where the subscript θ indicates that $\lambda^{-s}\log\lambda$ is chosen to have a branch cut along $e^{i\theta}\mathsf{R}_+$, and the contour is the Laurent loop

$$\begin{aligned}\mathscr{C}_\theta = \{re^{i\theta} \mid \infty > r > r_0\} \\ \cup \{r_0 e^{i\omega} \mid \theta \ge \omega \ge \theta - 2\pi\} \cup \{re^{i(\theta-2\pi)} \mid r_0 < r < \infty\}.\end{aligned} \tag{4.5}$$

The following proposition eliminates the limiting procedure of (4.4) and gives a useful alternative description of $\Pi_{\theta,\varphi}(A)$. A proof can be found in the Appendix.

PROPOSITION 4.2. *For $x \in D(A)$ we have the identities*

$$\log_\theta Ax = \frac{i}{2\pi} \int_{\mathscr{C}_\theta} \lambda^{-1} \log_\theta \lambda A (A-\lambda)^{-1} x\, d\lambda \qquad \textit{and} \tag{4.6}$$

$$\Pi_{\theta,\varphi}(A)x = \frac{i}{2\pi} \int_{\Gamma_{\theta,\varphi}} (A-\lambda)^{-1} x\, d\lambda + \frac{\varphi-\theta}{2\pi} x, \tag{4.7}$$

where the integral in the right-hand side of (4.7) *is an improper integral.*

Next, we include a lemma which will be useful for our considerations regarding expressions involving different branches of the logarithm. Again, a proof is available in the Appendix.

LEMMA 4.3. *Let* $f(\lambda)$ *be a continuous (possibly vector-valued) function on the "punctuated double keyhole region"*

$$V_{r_0,\delta} = \{\lambda \in \mathbf{C} \mid |\lambda| < 2r_0 \text{ or } |\arg\lambda - \theta| < \delta \text{ or } |\arg\lambda - \varphi| < \delta\} \setminus \{0\}, \tag{4.8}$$

such that $f(\lambda)$ *is* $O(\lambda^{-1-\varepsilon})$ *for* $|\lambda| \to \infty$ *in* $V_{r_0,\delta}$*. Then*

$$\int_{\mathscr{C}_\theta} \log_\theta \lambda f(\lambda)\, d\lambda - \int_{\mathscr{C}_\varphi} \log_\varphi \lambda f(\lambda)\, d\lambda = -2\pi i \int_{\Gamma_{\theta,\varphi}} f(\lambda)\, d\lambda. \tag{4.9}$$

We can use this lemma to describe the relation between $\Pi_{\theta,\varphi}(A)$ and logarithms of A as follows:

PROPOSITION 4.4. *For* $x \in D(A)$*,*

$$\log_\theta Ax - \log_\varphi Ax = \int_{\Gamma_{\theta,\varphi}} \lambda^{-1} A(A-\lambda)^{-1} x\, d\lambda = -2\pi i \Pi_{\theta,\varphi}(A)x. \tag{4.10}$$

When $\Pi_{\theta,\varphi}(A)$ *is bounded, so is* $\log_\theta A - \log_\varphi A$*, and*

$$\Pi_{\theta,\varphi}(A) = \frac{i}{2\pi}(\log_\theta A - \log_\varphi A). \tag{4.11}$$

PROOF. For $x \in D(A)$, the expression $f(\lambda) = \lambda^{-1}A(A-\lambda)^{-1}x$ is holomorphic in $V_{r_0,\delta}$ for some $r_0, \delta > 0$, and $f(\lambda)$ is $O(\lambda^{-2})$ for $|\lambda| \to \infty$ in $V_{r_0,\delta}$ by (4.1).

Hence we can apply Lemma 4.3, and insertion of the expression for $f(\lambda)$ into (4.9) gives

$$\begin{aligned}\int_{\mathscr{C}_\theta} \log_\theta \lambda\lambda^{-1} A(A-\lambda)^{-1} x\, d\lambda - \int_{\mathscr{C}_\varphi} \log_\varphi \lambda\lambda^{-1} A(A-\lambda)^{-1} x\, d\lambda \\ = -2\pi i \int_{\Gamma_{\theta,\varphi}} \lambda^{-1} A(A-\lambda)^{-1} x\, d\lambda.\end{aligned} \tag{4.12}$$

Then (4.10) follows from (4.2) and (4.6).

If $\Pi_{\theta,\varphi}(A)$ is bounded, (4.10) extends to all $x \in H$ since $D(A)$ is dense in H, and (4.11) follows.

With the results above at hand we return to the realization $(P + G)_T$. Modifying the assumption of Section 2 a little, we now assume $\{P_+ + G - \lambda, T\}$ to satisfy the conditions of parameter-ellipticity in [6, Def. 3.3.1] for λ on the rays of *two* sectors around $e^{i\theta}\mathsf{R}_+$ and $e^{i\varphi}\mathsf{R}_+$, respectively. Then the realization $B = (P + G)_T$ satisfies the requirements for A above (4.1), and we can define the sectorial projection accordingly:

$$\Pi_{\theta,\varphi}(B) = \frac{i}{2\pi}\int_{\Gamma_{\theta,\varphi}} \lambda^{-1} B R_\lambda \, d\lambda. \tag{4.13}$$

Here, and below, the integrals are understood to be in the strong sense, to simplify notation. Like in the case of the logarithm, we decompose it into the contributions from the pseudodifferential and singular Green parts.

For the ψdo P on the closed manifold $\widetilde{X}$, we can use Proposition 4.2 to see that

$$\frac{i}{2\pi}\int_{\Gamma_{\theta,\varphi}} Q_\lambda u \, d\lambda + \frac{\varphi - \theta}{2\pi} u = \Pi_{\theta,\varphi}(P)u, \qquad u \in D(P); \tag{4.14}$$

it is known from [20], [14], that $\Pi_{\theta,\varphi}(P)$ is a ψdo of order ≤ 0 on $\widetilde{X}$.

Using Proposition 4.2, (2.4), and the fact that $r^+ e^+ = I$, we can rewrite (4.13) as

$$\begin{aligned} \Pi_{\theta,\varphi}(B) &= \frac{i}{2\pi}\int_{\Gamma_{\theta,\varphi}} R_\lambda \, d\lambda + \frac{\varphi - \theta}{2\pi} \\ &= \frac{i}{2\pi}\int_{\Gamma_{\theta,\varphi}} [Q_{\lambda,+} + G_\lambda] \, d\lambda + \frac{\varphi - \theta}{2\pi} \\ &= r^+\Big(\frac{i}{2\pi}\int_{\Gamma_{\theta,\varphi}} Q_\lambda \, d\lambda + \frac{\varphi - \theta}{2\pi}\Big) e^+ + \frac{i}{2\pi}\int_{\Gamma_{\theta,\varphi}} G_\lambda \, d\lambda \\ &= \Pi_{\theta,\varphi}(P)_+ + \frac{i}{2\pi}\int_{\Gamma_{\theta,\varphi}} G_\lambda \, d\lambda; \end{aligned} \tag{4.15}$$

in the last line we moreover used (4.14). Now an application of Proposition 4.4 to P and B gives:

$$\begin{aligned} \Pi_{\theta,\varphi}(P)_+ &= \frac{i}{2\pi}\big((\log_\theta P)_+ - (\log_\varphi P)_+\big), \\ \Pi_{\theta,\varphi}(B) &= \frac{i}{2\pi}\big(\log_\theta B - \log_\varphi B\big). \end{aligned} \tag{4.16}$$

Using the contour $\mathscr{C}_\theta$ from (4.5) we can define an operator as in (2.13),

$$G^{\log_\theta} = \frac{i}{2\pi}\int_{\mathscr{C}_\theta} \log_\theta \lambda G_\lambda \, d\lambda, \tag{4.17}$$

and similarly define $G^{\log_\varphi}$ where θ is replaced by φ. By rotation it is obvious that $G^{\log_\theta}$ and $G^{\log_\varphi}$ have properties similar to those of $G^{\log}$ described in Section 2. Now (4.16) and (2.15) show that if we define $G_{\theta,\varphi}$ by

$$G_{\theta,\varphi} = \frac{i}{2\pi}\int_{\Gamma_{\theta,\varphi}} G_\lambda \, d\lambda, \tag{4.18}$$

then

$$G_{\theta,\varphi} = \frac{i}{2\pi}\big(G^{\log_\theta} - G^{\log_\varphi}\big). \tag{4.19}$$

In view of (4.15), we have then obtained:

THEOREM 4.5. *The sectorial projection for* $B = (P + G)_T$ *satisfies*

$$\Pi_{\theta,\varphi}(B) = \Pi_{\theta,\varphi}(P)_+ + G_{\theta,\varphi}, \tag{4.20}$$

where each term on the right hand side is known: $\Pi_{\theta,\varphi}(P)_+$ *is the truncation of a* ψ*do on* $\widetilde{X}$ *of order at most zero, in particular it is bounded on* $L_2(X, E)$*;* $G_{\theta,\varphi}$ *is a difference* (4.19) *of two terms of the log-type described in Section 2 and hence is a generalized singular Green operator, bounded from* $L_2(X, E)$ *to* $H^{-\varepsilon}(X, E)$.

Like $G^{\log}$, $G_{\theta,\varphi}$ acts as in (2.21). It has a symbol-kernel $\tilde{g}_{\theta,\varphi} \sim \sum_{j\in\mathbb{N}} \tilde{g}_{\theta,\varphi,-j}$, with terms given by

$$\begin{aligned} \tilde{g}_{\theta,\varphi,-j} &= \frac{i}{2\pi}\big(\tilde{g}^{\log_\theta}_{-j} - \tilde{g}^{\log_\varphi}_{-j}\big) \\ &= \frac{-1}{4\pi^2}\Big(\int_{\mathscr{C}_\theta} \log_\theta \lambda \, \tilde{g}_{-m-j} d\lambda - \int_{\mathscr{C}_\varphi} \log_\varphi \lambda \, \tilde{g}_{-m-j} \, d\lambda\Big). \end{aligned} \tag{4.21}$$

By Lemma 4.3 this is simplified to

$$\tilde{g}_{\theta,\varphi,-j}(x', x_n, y_n, \xi') = \frac{i}{2\pi}\int_{\Gamma_{\theta,\varphi}} \tilde{g}_{-m-j}(x', x_n, y_n, \xi', \lambda) \, d\lambda. \tag{4.22}$$

In view of (4.19) and (4.21), the results on $G^{\log}$ resp. $\tilde{g}^{\log}$ in Section 2 carry over immediately to $G_{\theta,\varphi}$ resp. $\tilde{g}_{\theta,\varphi}$. We shall not reproduce all the statements explicitly, but will just present the following important result obtained from Theorem 2.4.

THEOREM 4.6. *Assume that P is a differential operator, $G = 0$, and the trace operators $T_0, \dots, T_{m-1}$ are differential operators; hereby $B = P_T$.*

Then $G_{\theta,\varphi}$ is, in local coordinates near X', a generalized singular Green operator

$$G_{\theta,\varphi} = \mathrm{OPG}(\tilde{g}_{\theta,\varphi}) \tag{4.23}$$

with $\tilde{g}_{\theta,\varphi} \sim \sum_{j\in\mathbb{N}} \tilde{g}_{\theta,\varphi,-j}$; the j'th term is quasihomogeneous as in (2.22) *and satisfies estimates as in* (2.23).

$G_{\theta,\varphi}$ and $\Pi_{\theta,\varphi}(P_T)$ are bounded operators in $L_p(X, E)$ for $1 < p < \infty$. In particular, $\Pi_{\theta,\varphi}(P_T)$ is a bounded projection in $L_2(X, E)$.

PROOF. The claims regarding $\tilde{g}_{\theta,\varphi}$ follow immediately from Theorem 2.4 and (4.21).

The boundedness properties of $G_{\theta,\varphi}$ are obvious from Theorem 2.4 and (4.19). Since $\Pi_{\theta,\varphi}(P)_+$ is the truncation of a ψdo of order at most zero, this is also bounded in $L_p(X, E)$; then in view of (4.20) so is $\Pi_{\theta,\varphi}(P_T)$.

An interesting question is whether one can give criteria on P, G, and T assuring that the operator $\Pi_{\theta,\varphi}((P + G)_T)$ belongs to the Boutet de Monvel calculus.

Concerning the ψdo part $\Pi_{\theta,\varphi}(P)$, with symbol $\pi_{\theta,\varphi}(x, \xi)$ in local coordinates, we have easily by use of Lemma 2.1:

LEMMA 4.7. *When m is even, $\pi_{\theta,\varphi}(x, \xi)$ satisfies the transmission condition. Hence $\Pi_{\theta,\varphi}(P)_+$ is in the Boutet de Monvel calculus for even m.*

PROOF. We have that in view of (2.10) that

$$\mathrm{symb}(\log_\theta P) = m \log[\xi] + l_\theta(x, \xi), \qquad l_\theta(x, \xi) \sim \sum_{j\in\mathbb{N}} l_{\theta,-j}(x, \xi), \tag{4.24}$$

where $m \log[\xi] + l_{\theta,0}(x, \xi) = \log_\theta(p_m(x, \xi))$, with similar formulas for $\log_\varphi P$, so the symbols of $\log_\theta P$ and $\log_\varphi P$ have the same log-term $m \log[\xi]$. Then it is seen from the first line in (4.16) that

$$\pi_{\theta,\varphi}(x, \xi) = \frac{i}{2\pi}(l_\theta(x, \xi) - l_\varphi(x, \xi)), \tag{4.25}$$

which satisfies the transmission condition when m is even in view of Lemma 2.1.

This could also be based more directly on the fact, worked out in detail in [14], that $\pi_{\theta,\varphi}(x,\xi) \sim \sum_{j\in\mathbb{N}} \pi_{\theta,\varphi,-j}(x,\xi)$, where the terms are given by

$$\pi_{\theta,\varphi,-j}(x,\xi) = \frac{i}{2\pi}\int_{\mathscr{C}_{\theta,\varphi}(x,\xi)} q_{-m-j}(x,\xi,\lambda)\,d\lambda; \tag{4.26}$$

here $\mathscr{C}_{\theta,\varphi}(x,\xi)$ is a closed curve in the sector $\Lambda_{\theta,\varphi}$ going in the positive direction around the part of the spectrum of $p_m(x,\xi)$ lying in that sector.

When m is odd, one cannot expect $\Pi_{\theta,\varphi}(P)$ to satisfy the transmission condition. For example, for a first-order selfadjoint invertible elliptic differential operator A on $\widetilde{X}$ (e.g., a Dirac operator), $\Pi_{-\frac{\pi}{2},\frac{\pi}{2}}(A)$ equals $\Pi_>(A)$, the positive eigenprojection $\frac{1}{2}(I + A|A^2|^{-1/2})$, where $A|A^2|^{-1/2}$ *does not* satisfy the transmission condition (its even-order symbol terms are odd in ξ).

Next, let us consider the s.g.o. part $G_{\theta,\varphi}$. Example 4.8 below shows a differential operator realization where $G_{\theta,\varphi}$ is not a standard singular Green operator, already in a constant-coefficient principal symbol case. Example 4.9 on the other hand defines a general class of differential operator realizations where $G_{\theta,\varphi}$ is a standard s.g.o., and $\Pi_{\theta,\varphi}(B)$ belongs to the standard calculus. Here one finds however, that lower order perturbations can ruin the standard s.g.o.-properties.

EXAMPLE 4.8. Consider the differential operators A and P on $\mathbb{R}^4_+$ given by

$$A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} D_1 + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} D_2 + \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} D_3 + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} D_4, \tag{4.27}$$

and

$$P = \begin{pmatrix} 0 & -A^* \\ A & 0 \end{pmatrix}, \tag{4.28}$$

where A^* denotes the formal adjoint of A. (A and P are Dirac-type operators, with $A^*A = -\Delta I_2$, $(iP)^2 = -\Delta I_4$.)

Regarding this as a localization of a manifold situation, we seek the projection onto the (generalized) eigenspaces for the eigenvalues λ in the upper halfplane $\mathbb{C}_+$ for a certain realisation P_T of P, where the boundary condition is $B\gamma_0 u = 0$, with

$$B = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \tag{4.29}$$

i.e., $\gamma_0 u_1 + \gamma_0 u_3 = \gamma_0 u_2 + \gamma_0 u_4 = 0$, u_i being the i'th component of u.

Thus, in this localized situation we shall construct $\Pi_{\theta,\varphi}(P_T)$ with $\theta = 0$ and $\varphi = \pi$. In this case the contour $\Gamma_{\theta,\varphi}$ is a contour from $-\infty$ to ∞ passing above the origin.

P has symbol

$$p(\xi) = \begin{pmatrix} 0 & -{}^t\overline{a(\xi)} \\ a(\xi) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & i\xi_1 - \xi_4 & \xi_2 + i\xi_3 \\ 0 & 0 & -\xi_2 + i\xi_3 & -i\xi_1 - \xi_4 \\ i\xi_1 + \xi_4 & \xi_2 + i\xi_3 & 0 & 0 \\ -\xi_2 + i\xi_3 & -i\xi_1 + \xi_4 & 0 & 0 \end{pmatrix},$$

the eigenvalues of which are $\pm i|\xi|$. Hence $P - \lambda$ is parameter-elliptic for λ on all rays in $\mathbb{C} \setminus i\mathbb{R}$, with parametrix-symbol

$$q(\xi,\lambda) = (p(\xi) - \lambda)^{-1} = \frac{1}{|\xi|^2 + \lambda^2} \begin{pmatrix} -\lambda & 0 & -i\xi_1 + \xi_4 & -\xi_2 - i\xi_3 \\ 0 & -\lambda & \xi_2 - i\xi_3 & i\xi_1 + \xi_4 \\ -i\xi_1 - \xi_4 & -\xi_2 - i\xi_3 & -\lambda & 0 \\ \xi_2 - i\xi_3 & i\xi_1 - \xi_4 & 0 & -\lambda \end{pmatrix}.$$

We first find the ψdo part of $\Pi_{0,\pi}(P_T)$: According to (4.26) the symbol $\pi(\xi)$ of $\Pi_{0,\pi}(P)$ is obtained by integrating $q(\xi,\lambda)$ along a small closed curve, $\mathscr{C}_\xi$, enclosing the pole $i|\xi|$ in $\mathbb{C}_+$:

$$\pi(\xi) = \frac{i}{2\pi} \int_{\mathscr{C}_\xi} q(\xi,\lambda)\,d\lambda = -\operatorname*{Res}_{\lambda = i|\xi|}\big(q(\xi,\lambda)\big)$$

$$= \frac{1}{2|\xi|} \begin{pmatrix} |\xi| & 0 & \xi_1 + i\xi_4 & -i\xi_2 + \xi_3 \\ 0 & |\xi| & i\xi_2 + \xi_3 & -\xi_1 + i\xi_4 \\ \xi_1 - i\xi_4 & -i\xi_2 + \xi_3 & |\xi| & 0 \\ i\xi_2 + \xi_3 & -\xi_1 - i\xi_4 & 0 & |\xi| \end{pmatrix}. \tag{4.30}$$

The singular Green part G_λ of the resolvent $R_\lambda = (P_T - \lambda)^{-1}$ has symbol-kernel

$\tilde{g}(x_n, y_n, \xi', \lambda)$

$$= \frac{1}{2\sigma} \begin{pmatrix} -i\xi_1 + i\sigma & -\xi_2 - i\xi_3 & -\lambda & 0 \\ \xi_2 - i\xi_3 & i\xi_1 + i\sigma & 0 & -\lambda \\ -\lambda & 0 & -i\xi_1 - i\sigma & -\xi_2 - i\xi_3 \\ 0 & -\lambda & \xi_2 - i\xi_3 & i\xi_1 - i\sigma \end{pmatrix} e^{-\sigma(x_n+y_n)},$$

where $\sigma = \sqrt{|\xi'|^2 + \lambda^2}$. Note that σ is holomorphic (and $\operatorname{Re}\sigma > 0$) for $\lambda \in \mathsf{C} \setminus \pm i(|\xi'|, \infty)$; in particular $\{P - \lambda, B\gamma_0\}$ is parameter-elliptic for λ on any ray in $\mathsf{C} \setminus i\mathsf{R}$.

The integration contour $\Gamma_{0,\pi}$ is homotopic in $\{re^{i\omega} \mid \omega \neq \pm\frac{\pi}{2} \text{ or } r < |\xi'|\}$ to the real line; thus, due to the exponential falloff of $e^{-(|\xi'|^2+\lambda^2)^{\frac{1}{2}}(x_n+y_n)}$ we get

$$\begin{aligned} \tilde{g}_{\theta,\varphi}(x_n, y_n, \xi') &= \frac{i}{2\pi} \int_{\Gamma_{\theta,\varphi}} \tilde{g}(x_n, y_n, \xi', \lambda)\, d\lambda \\ &= \frac{i}{2\pi} \int_{-\infty}^{\infty} \tilde{g}(x_n, y_n, \xi', t)\, dt. \end{aligned} \tag{4.31}$$

We can now verify that $\tilde{g}_{\theta,\varphi}$ is not a singular Green symbol-kernel: The 12-matrix entry of $\tilde{g}_{\theta,\varphi}$ becomes

$$\frac{-i\xi_2 + \xi_3}{4\pi} \int_{-\infty}^{\infty} \bigl(|\xi'|^2 + t^2\bigr)^{-\frac{1}{2}} e^{-(|\xi'|^2+t^2)^{\frac{1}{2}}(x_n+y_n)}\, dt, \tag{4.32}$$

which, for fixed ξ', is unbounded as $x_n + y_n$ goes to zero; hence, $\tilde{g}_{\theta,\varphi}$ is not in $\mathscr{S}_{++}$.

To see this note that, for fixed $a > 0$,

$$\begin{aligned} f(r) &= \frac{1}{2} \int_{-\infty}^{\infty} (a^2 + t^2)^{-\frac{1}{2}} e^{-r(a^2+t^2)^{\frac{1}{2}}}\, dt = \int_0^{\infty} (a^2 + t^2)^{-\frac{1}{2}} e^{-r(a^2+t^2)^{\frac{1}{2}}}\, dt \\ &\geq \int_0^{\infty} \frac{e^{-(a+t)r}}{a+t}\, dt = \int_{ar}^{\infty} \frac{e^{-u}}{u}\, du \end{aligned}$$

which diverges to $+\infty$ as $r \to 0^+$.

Example 4.9. Let X_0' be a closed $(n-1)$-dimensional manifold provided with an elliptic second-order differential operator S which is selfadjoint positive in $L_2(X_0')$. Let $X = X_0' \times [0, a]$ with points $x = (x', x_n)$, $x' \in X_0'$ and $x_n \in [0, a]$, and let B be the Dirichlet realization of $D_{x_n}^2 + S$ on X; it is selfadjoint positive in $L_2(X)$, with $D(B) = H^2(X) \cap H_0^1(X)$. Let A be the Dirichlet realization of

$$P = \begin{pmatrix} D_{x_n}^2 + S & S \\ S & -D_{x_n}^2 - S \end{pmatrix}. \tag{4.33}$$

on X, then in fact,

$$A = \begin{pmatrix} B & S \\ S & -B \end{pmatrix} \tag{4.34}$$

with domain $D(B) \times D(B)$. The resolvent is

$$(A-\lambda)^{-1} = \begin{pmatrix} -B-\lambda & -S \\ -S & B-\lambda \end{pmatrix} (\lambda^2 - B^2 - S^2)^{-1}, \tag{4.35}$$

where we used that S and B commute. Define $B_1 = (B^2+S^2)^{\frac{1}{2}}$. Here B^2+S^2 is the realization of the fourth-order elliptic differential operator $(D_{x_n}^2 + S)^2 + S^2$ determined by the boundary condition $\gamma_0 u = 0$, $\gamma_0 Bu = 0$. This is one of the particular cases where the square root of the interior operator does satisfy the transmission condition, cf. [6, (4.4.9)]. Moreover, the square root of the realization $B^2 + S^2$ represents a boundary condition consisting of exactly the part of the boundary condition for $B^2 + S^2$ that makes sense on $H^2(X)$, cf. [6, Cor. 4.4.3] (based on a result of Grisvard); so in fact B_1 is the realization of $((D_{x_n}^2 + S)^2 + S^2)^{\frac{1}{2}}$ determined by the Dirichlet condition $\gamma_0 u = 0$. This belongs to the standard calculus and enters nicely in the theory of [6], cf. Section 1.7 there. Note that $D(B_1) = D(B)$.

We can then calculate

$$\begin{aligned} &(\lambda^2 - (B^2+S^2))^{-1} \\ &\quad = (\lambda^2 - B_1^2)^{-1} = (B_1-\lambda)^{-1}(-B_1-\lambda)^{-1} \\ &\quad = (B_1-\lambda)^{-1}(2B_1)^{-1}(B_1+\lambda+B_1-\lambda)(-B_1-\lambda)^{-1}, \\ &\quad = -\frac{1}{2}B_1^{-1}\big((B_1-\lambda)^{-1} - (-B_1-\lambda)^{-1}\big) \end{aligned} \tag{4.36}$$

which leads to the formula:

$$\begin{aligned} &(A-\lambda)^{-1} \\ &\quad = \begin{pmatrix} -B+B_1-B_1-\lambda & -S \\ -S & B-B_1+B_1-\lambda \end{pmatrix} (B_1-\lambda)^{-1}(-B_1-\lambda)^{-1} \\ &\quad = \begin{pmatrix} (B_1-\lambda)^{-1} & 0 \\ 0 & (-B_1-\lambda)^{-1} \end{pmatrix} \\ &\qquad - \begin{pmatrix} B_1-B & -S \\ -S & B-B_1 \end{pmatrix} \frac{1}{2}B_1^{-1}\big((B_1-\lambda)^{-1} - (-B_1-\lambda)^{-1}\big), \end{aligned} \tag{4.37}$$

valid for λ outside the spectra of B_1 and $-B_1$. To determine the spectral projection $\Pi_{\theta,\varphi}(A)$ with $\theta = -\frac{\pi}{2}$, $\varphi = \frac{\pi}{2}$, we use the abstract machinery. It is seen from either of the formulas (4.2) or (4.7) that

$$\Pi_{-\frac{\pi}{2},\frac{\pi}{2}}(A) = \begin{pmatrix} \Pi_{-\frac{\pi}{2},\frac{\pi}{2}}(B_1) & 0 \\ 0 & \Pi_{-\frac{\pi}{2},\frac{\pi}{2}}(-B_1) \end{pmatrix} - \begin{pmatrix} B_1 - B & -S \\ -S & B - B_1 \end{pmatrix} \frac{1}{2} B_1^{-1}\big(\Pi_{-\frac{\pi}{2},\frac{\pi}{2}}(B_1) - \Pi_{-\frac{\pi}{2},\frac{\pi}{2}}(-B_1)\big). \tag{4.38}$$

Here

$$\Pi_{-\frac{\pi}{2},\frac{\pi}{2}}(B_1) = I, \qquad \Pi_{-\frac{\pi}{2},\frac{\pi}{2}}(-B_1) = 0, \tag{4.39}$$

in view of Proposition 4.1 and the fact that B_1 is selfadjoint positive. It follows that

$$\Pi_{-\frac{\pi}{2},\frac{\pi}{2}}(A) = \begin{pmatrix} \frac{1}{2} + \frac{1}{2} B B_1^{-1} & \frac{1}{2} S B_1^{-1} \\ \frac{1}{2} S B_1^{-1} & \frac{1}{2} - \frac{1}{2} B B_1^{-1} \end{pmatrix}. \tag{4.40}$$

The operator is in the Boutet de Monvel calculus. Note that the sum of the diagonal terms is I, so the residue of the operator is zero.

Inherent in this example are some symbol calculations where the poles of the resolvent symbol appear isolated in such a way that integrals over $\Gamma_{\theta,\varphi}$ can be turned into integrals over closed curves, reducing to simple residue calculations. Perturbations can easily introduce more complicated calculations where integrals as in (4.32) appear, leading to non-standard s.g.o.-symbols (we shall not reproduce examples here).

In view of Definition 3.3 and the formulas (4.16), the sectorial projection $\Pi_{\theta,\varphi}(B)$ has a well-defined residue. In the differential operator case where the order m is even, one can moreover define residues of the compositions of $\Pi_{\theta,\varphi}(B)$ with operators A in the Boutet de Monvel calculus; this is taken up in [9]. It is found there that if in addition, A is of order and class 0, the residue vanishes on the commutator of $\Pi_{\theta,\varphi}(B)$ and A.

It is still an open question whether the residue is zero on sectorial projections for boundary value problems, as it is in the closed manifold case; we expect to return to this question in a forthcoming work.

Appendix A. Proofs of auxiliary results in functional analysis

PROOF OF PROPOSITION 4.2. First we prove (4.6): Let, for $N \in \mathbb{N}$,

$$\mathscr{C}_\theta^N = \{re^{i\theta} \mid N \geq r \geq r_0\} \cup \{r_0 e^{i\omega} \mid \theta \geq \omega \geq \theta - 2\pi\} \cup \{re^{i(\theta-2\pi)} \mid r_0 \leq r \leq N\}. \tag{A.1}$$

Then, for $s > 0$

$$\text{(A.2)} \quad \int_{\mathscr{C}_\theta^N} \lambda_\theta^{-s-1} \log_\theta \lambda \, d\lambda = \Big[-\frac{1}{s^2} \lambda_\theta^{-s} (1 + s \log_\theta \lambda) \Big]_{Ne^{i(\theta-2\pi)}}^{Ne^{i\theta}} \longrightarrow 0 \quad \text{for } N \to \infty,$$

since N^{-s} and $N^{-s} \log N$ go to 0 for $N \to \infty$. It follows that

$$\text{(A.3)} \qquad \lim_{s \searrow 0} \lim_{N \to \infty} \int_{\mathscr{C}_\theta^N} \lambda_\theta^{-s-1} \log_\theta \lambda \, d\lambda = 0.$$

Observe that the order of the limits is important.

Using the resolvent identity $A(A-\lambda)^{-1} = 1 + \lambda(A-\lambda)^{-1}$ we now get for $x \in D(A)$:

(A.4)

$$\begin{aligned} \lim_{s \searrow 0} \int_{\mathscr{C}_\theta} & \lambda_\theta^{-s} \log_\theta \lambda (A-\lambda)^{-1} x \, d\lambda \\ &= \lim_{s \searrow 0} \lim_{N \to \infty} \int_{\mathscr{C}_\theta^N} \lambda_\theta^{-s} \log_\theta \lambda (A-\lambda)^{-1} x \, d\lambda \\ &= \lim_{s \searrow 0} \lim_{N \to \infty} \Big[\int_{\mathscr{C}_\theta^N} \lambda_\theta^{-s-1} \log_\theta \lambda x \, d\lambda + \int_{\mathscr{C}_\theta^N} \lambda_\theta^{-s} \log_\theta \lambda (A-\lambda)^{-1} x \, d\lambda \Big] \\ &= \lim_{s \searrow 0} \lim_{N \to \infty} \int_{\mathscr{C}_\theta^N} \lambda_\theta^{-s-1} \log_\theta \lambda \big[1 + \lambda (A-\lambda)^{-1} \big] x \, d\lambda \\ &= \lim_{s \searrow 0} \lim_{N \to \infty} \int_{\mathscr{C}_\theta^N} \lambda_\theta^{-s-1} \log_\theta \lambda A (A-\lambda)^{-1} x \, d\lambda, \end{aligned}$$

where we used (A.3) in the second line (adding zero). Then, since $\|(A-\lambda)^{-1}\| \dot{\le} |\lambda|^{-1}$,

$$\text{(A.5)} \qquad \big\| \lambda_\theta^{-s-1} \log_\theta \lambda A (A-\lambda)^{-1} x \big\| \dot{\le} |\log \lambda| |\lambda|^{-s-2} \|Ax\|,$$

so that the integrand in the last expression of (A.4) is integrable along $\mathscr{C}_\theta$ uniformly in $s > 0$, and

$$\text{(A.6)} \quad \lim_{s \searrow 0} \lim_{N \to \infty} \int_{\mathscr{C}_\theta^N} \lambda_\theta^{-s-1} \log_\theta \lambda \, A (A-\lambda)^{-1} x \, d\lambda = \int_{\mathscr{C}_\theta} \lambda^{-1} \log_\theta \lambda A (A-\lambda)^{-1} x \, d\lambda.$$

Combining (A.4) and (A.6) (and multiplying with $\frac{i}{2\pi}$) we obtain the desired result (4.6).

The identity (4.7) stems from [3] (we have corrected a sign here). For this, consider the integration contour

$$\text{(A.7)}\quad \Gamma^N_{\theta,\varphi} = \{re^{i\varphi} \mid N > r > r_0\} \cup \{r_0 e^{i\omega} \mid \varphi \geq \omega \geq \theta\} \cup \{re^{i\theta} \mid r_0 < r < N\}.$$

Using again $A(A-\lambda)^{-1} = 1 + \lambda(A-\lambda)^{-1}$ we obtain

$$\text{(A.8)}\quad \int_{\Gamma^N_{\theta,\varphi}} \lambda^{-1} A(A-\lambda)^{-1} x\, d\lambda = \int_{\Gamma^N_{\theta,\varphi}} (A-\lambda)^{-1} x\, d\lambda + \int_{\Gamma^N_{\theta,\varphi}} \lambda^{-1} x\, d\lambda.$$

For the second term we have, using a logarithm with branch cut disjoint from $\Lambda_{\theta,\varphi}$,

$$\text{(A.9)}\quad \int_{\Gamma^N_{\theta,\varphi}} \lambda^{-1} d\lambda = \big[\log \lambda\big]^{Ne^{i\theta}}_{Ne^{i\varphi}} = i(\theta - \varphi).$$

Thus

$$\text{(A.10)}\quad \frac{i}{2\pi}\int_{\Gamma^N_{\theta,\varphi}} \lambda^{-1} A(A-\lambda)^{-1} x\, d\lambda = \frac{i}{2\pi}\int_{\Gamma^N_{\theta,\varphi}} (A-\lambda)^{-1} x\, d\lambda + \frac{\varphi-\theta}{2\pi} x.$$

For $x \in D(A)$ the limit for $N \to \infty$ is well-defined on the left-hand side, and the limit of the first term on the right-hand side then exists as an improper integral, as indicated.

Proof of Lemma 4.3. The integral along $\mathscr{C}_\theta$ is, in detail:

$$\text{(A.11)}\quad \begin{aligned}\int_{\mathscr{C}_\theta} \log_\theta \lambda\, f(\lambda)\, d\lambda = &\int_\infty^{r_0} (\log r + i\theta) f(re^{i\theta}) e^{i\theta}\, dr \\ &+ \int_\theta^{\theta-2\pi} (\log r_0 + i\omega) f(r_0 e^{i\omega}) i r_0 e^{i\omega}\, d\omega \\ &+ \int_{r_0}^\infty (\log r + i\theta - 2\pi i) f(re^{i\theta-2\pi i}) e^{i\theta - 2\pi i}\, dr.\end{aligned}$$

Since $f(re^{i\theta-2\pi i})e^{i\theta-2\pi i} = f(re^{i\theta})e^{i\theta}$, the two terms with $(\log r + i\theta)$ cancel each other. Thus

$$\text{(A.12)}\quad \begin{aligned}&\int_{\mathscr{C}_\theta} \log_\theta \lambda f(\lambda)\, d\lambda \\ &= -\int_{\theta-2\pi}^\theta (\log r_0 + i\omega) f(r_0 e^{i\omega}) i r_0 e^{i\omega}\, d\omega - 2\pi i \int_{r_0}^\infty f(re^{i\theta}) e^{i\theta}\, dr.\end{aligned}$$

Denote the integrand in the first integral $g(\omega) = (\log r_0 + i\omega) f(r_0 e^{i\omega}) i r_0 e^{i\omega}$.

There is of course an identity similar to (A.12) with θ replaced by φ, and then

$$\begin{aligned}
&\int_{\mathscr{C}_\theta} \log_\theta \lambda f(\lambda)\, d\lambda - \int_{\mathscr{C}_\varphi} \log_\varphi \lambda f(\lambda)\, d\lambda \\
&\quad = \Big(-\int_{\theta-2\pi}^{\theta} + \int_{\varphi-2\pi}^{\varphi} \Big) g(\omega)\, d\omega \\
&\qquad - 2\pi i \Big(\int_{r_0}^{\infty} f(re^{i\theta}) e^{i\theta}\, dr - \int_{r_0}^{\infty} f(re^{i\varphi}) e^{i\varphi}\, dr \Big) \\
&\quad = \Big(-\int_{\theta-2\pi}^{\theta} + \int_{\varphi-2\pi}^{\varphi} \Big) g(\omega)\, d\omega \\
&\qquad - 2\pi i \int_{\infty}^{r_0} f(re^{i\varphi}) e^{i\varphi}\, dr - 2\pi i \int_{r_0}^{\infty} f(re^{i\theta}) e^{i\theta}\, dr.
\end{aligned} \tag{A.13}$$

The last two terms are recognized as the contributions to $-2\pi i \int_{\Gamma_{\theta,\varphi}} f(\lambda)\, d\lambda$ from the rays $e^{i\varphi}[r_0, \infty[$ and $e^{i\theta}[r_0, \infty[$. The first term is seen to give the contribution from the arc $\mathscr{C}_{r_0,\theta,\varphi} = \{ r_0 e^{i\omega} \mid \varphi \geq \omega \geq \theta \}$ as follows:

$$\begin{aligned}
&\Big(-\int_{\theta-2\pi}^{\theta} + \int_{\varphi-2\pi}^{\varphi} \Big) g(\omega)\, d\omega \\
&\quad = \Big(-\int_{\varphi}^{\theta} + \int_{\varphi-2\pi}^{\theta-2\pi} \Big) g(\omega)\, d\omega = \int_{\varphi}^{\theta} [-g(\omega) + g(\omega - 2\pi)]\, d\omega \\
&\quad = \int_{\varphi}^{\theta} \big[-(\log r_0 + i\omega) f(r_0 e^{i\omega}) i r_0 e^{i\omega} \\
&\qquad\quad + (\log r_0 + i(\omega - 2\pi)) f(r_0 e^{i\omega}) i r_0 e^{i\omega} \big]\, d\omega \\
&\quad = -2\pi i \int_{\varphi}^{\theta} f(r_0 e^{i\omega}) i r_0 e^{i\omega}\, d\omega = -2\pi i \int_{\mathscr{C}_{r_0,\theta,\varphi}} f(\lambda)\, d\lambda.
\end{aligned}$$

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NONCOMMUTATIVE RESIDUE OF PROJECTIONS IN BOUTET DE MONVEL'S CALCULUS

ANDERS GAARDE

Abstract. Employing results by Melo, Nest, Schick and Schrohe on the K-theory of Boutet de Monvel's calculus of boundary value problems, we show that the noncommutative residue introduced by Fedosov, Golse, Leichtnam and Schrohe vanishes on projections in the calculus.

This partially answers a question raised in a recent collaboration with Grubb, namely whether the residue is zero on sectorial projections for boundary value problems: This is confirmed to be true when the sectorial projection is in the calculus.

1. Introduction

Boutet de Monvel [2] constructed a calculus, often called the Boutet de Monvel calculus (or algebra), of pseudodifferential boundary operators on a manifold with boundary. It includes the classical differential boundary value problems as well of the parametrices of the elliptic elements:

Let X be a compact n-dimensional manifold with boundary ∂X; we consider X as an embedded submanifold of a closed n-dimensional manifold $\widetilde{X}$. Denote by X° the interior of X. Let E and F be smooth complex vector bundles over X and ∂X, respectively, with E the restriction to X of a bundle $\widetilde{E}$ over $\widetilde{X}$.

An operator in Boutet de Monvel's calculus — a (polyhomogeneous) Green operator — is a map A acting on sections of E and F, given by a matrix

$$A = \begin{pmatrix} P_+ + G & K \\ T & S \end{pmatrix} : \begin{matrix} C^\infty(X,E) \\ \oplus \\ C^\infty(\partial X, F) \end{matrix} \to \begin{matrix} C^\infty(X,E) \\ \oplus \\ C^\infty(\partial X, F), \end{matrix} \tag{1.1}$$

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where P is a pseudodifferential operator (ψdo) on $\widetilde{X}$ with the transmission property and P_+ is its truncation to X:

$$P_+ = r^+ P e^+, \quad r^+ \text{ restricts from } \widetilde{X} \text{ to } X^\circ,\ e^+ \text{ extends by } 0. \tag{1.2}$$

G is a singular Green operator, T a trace operator, K a Poisson operator, and S a ψdo on the closed manifold ∂X. See [2], Grubb [6], or Schrohe [15] for details.

Fedosov, Golse, Leichtnam and Schrohe [4] extended the notion of noncommutative residue known from closed manifolds (cf. Wodzicki [17], [18] and Guillemin [9]) to the algebra of Green operators. The noncommutative residue of A from (1.1) is defined to be

$$\begin{aligned} \operatorname{res}_X(A) = &\int_X \int_{S_x^* X} \operatorname{tr}_E p_{-n}(x,\xi)\, đS(\xi)\, dx \\ &+ \int_{\partial X} \int_{S_{x'}^* \partial X} \left[\operatorname{tr}_E (\operatorname{tr}_n g)_{1-n}(x',\xi') + \operatorname{tr}_F s_{1-n}(x',\xi') \right] đS(\xi')\, dx'. \end{aligned} \tag{1.3}$$

Here tr_E and tr_F are traces in $\operatorname{Hom}(E)$ and $\operatorname{Hom}(F)$, respectively; $đS(\xi)$ (resp. $đS(\xi')$) denotes the surface measure on the unit sphere of the cotangent bundle, divided by $(2\pi)^n$ (resp. $(2\pi)^{n-1}$); $\operatorname{tr}_n g$ is the symbol of $\operatorname{tr}_n G$ (the normal trace of G), a ψdo on ∂X; and the subscripts $-n$ and $1-n$ indicate that we consider only the homogeneous terms of degree $-n$ resp. $1-n$. Also, a sign error in [4] has been corrected, cf. Grubb and Schrohe [8, (1.5)].

It is well-known [17] that on a closed manifold, the noncommutative residue of a classical ψdo projection vanishes. In the present paper we show that the same holds in the case of Green operators:

Theorem 1.1. *The noncommutative residue of a projection in the Boutet de Monvel calculus is zero.*

In the proof, we use K-theoretic arguments (in a C^*-algebra setting) to reduce the problem to the known case of closed manifolds. We rely on results on the K-theory of Boutet de Monvel's algebra by Melo, Nest and Schrohe [10] and Melo, Schick and Schrohe [11].

In our recent collaboration with Grubb [5] we studied certain spectral projections: For the realization $B = (P+G)_T$ of an elliptic boundary value problem $\{P_+ + G, T\}$ of order $m > 0$ with two spectral cuts at angles θ and φ, one can define the *sectorial projection* $\Pi_{\theta,\varphi}(B)$. It is a (not necessarily self-adjoint) projection whose range contains the generalized eigenspace of B for the sector $\Lambda_{\theta,\varphi} = \{re^{i\omega} \mid r > 0, \theta < \omega < \varphi\}$ and whose nullspace contains the generalized eigenspace for $\Lambda_{\varphi,\theta+2\pi}$. It was considered earlier by Burak [3], and in the boundaryless case by Wodzicki [17] and Ponge [13].

In general this operator is not in Boutet de Monvel's calculus, but we showed that it has a residue in a slightly more general sense. The question was posed whether this residue vanishes.

The question of the noncommutative residue of projections is particularly interesting in the context of zeta-invariants as discussed by Grubb [7] and in [5]: The *basic zeta value* $C_{0,\theta}(B)$ for the realization B of a boundary value problem is defined via a choice of spectral cut in the complex plane; the difference in the basic zeta value based on two spectral cut angles θ and φ is then given as the noncommutative residue of the corresponding sectorial projection:

$$C_{0,\theta}(B) - C_{0,\varphi}(B) = \frac{2\pi i}{m} \operatorname{res}_X(\Pi_{\theta,\varphi}(B)). \tag{1.4}$$

Our results here show that the dependence of $C_{0,\theta}(B)$ upon θ is trivial whenever the projection $\Pi_{\theta,\varphi}(B)$ lies in Boutet de Monvel's calculus.

It should be noted that the literature in functional analysis and PDE-theory often uses "projection" as a synonym for idempotent, while C^*-algebraists furthermore require that projections are self-adjoint. We choose here the former terminology; that is, in this text projection and idempotent are synonymous.

2. Preliminaries and notation

We employ Blackadar's [1] approach to K-theory: A pre-C^*-algebra B is called local if it, as a subalgebra of its C^*-completion $\overline{B}$, is closed under holomorphic function calculus. (Blackadar also required that all matrix algebras $\mathcal{M}_n(B)$ are closed under holomorphic function calculus, but this follows automatically, cf. Schweitzer [16].) Let $\mathcal{M}_\infty(B)$ denote the direct limit of the matrix algebras $\mathcal{M}_m(B)$, $m \in \mathbb{N}$. Define $\mathcal{IP}_\infty(B) = \operatorname{Idem}(\mathcal{M}_\infty(B))$ to be the set of all idempotent matrices with entries from B. Likewise, $\mathcal{IP}_m(B) = \operatorname{Idem}(\mathcal{M}_m(B))$ is the set of all $m \times m$ idempotents. Define the relation $\sim$ on $\mathcal{IP}_\infty(B)$ by

(2.1) $\quad x \sim y$ if there exist $a, b \in \mathcal{M}_\infty(B)$ such that $x = ab$ and $y = ba$.

If B has a unit we define $K_0(B)$ to be the Grothendieck group of the semigroup $V(B) = \mathcal{IP}_\infty(B)/\sim$. If B has no unit, we consider the scalar map from the unitization — indicated with a tilde as in $\widetilde{B}$ or $B^\sim$ — of B to the complex numbers $s : \widetilde{B} \to \mathbb{C}$ defined by $s(b + \lambda 1_{\widetilde{B}}) = \lambda$, and then define $K_0(B)$ as the kernel of the induced map $s_* : K_0(\widetilde{B}) \to K_0(\mathbb{C})$.

A fact that we shall use several times is that if B is local, then [1, p. 28]

$$V(B) \cong V(\overline{B}), \text{ and hence } K_0(B) \cong K_0(\overline{B}). \tag{2.2}$$

Combined with *the standard picture of* K_0 this implies that

$$K_0(\overline{B}) = \{\, [x]_0 - [y]_0 \mid x, y \in \mathcal{IP}_m(B), m \in \mathbb{N} \,\} \tag{2.3}$$

in the case where B is unital, and

$$K_0(\overline{B}) = \{\, [x]_0 - [y]_0 \mid x, y \in \mathcal{IP}_m(\widetilde{B}) \text{ with } x \equiv y \bmod \mathcal{M}_m(B), m \in \mathbb{N} \,\} \tag{2.4}$$

in the non-unital case [1].

Let $\mathscr{A}$ denote the set of Green operators as in (1.1) of order and class zero; it is equipped with a Fréchet topology, which makes it a Fréchet $*$-algebra (Schrohe [14]). Moreover, $\mathscr{A}$ is a $*$-subalgebra of the bounded operators on the Hilbert space $\mathcal{H} = L_2(X, E) \oplus H^{-1/2}(\partial X, F)$; we will denote by $\mathfrak{A}$ its C^*-closure in $\mathcal{B}(\mathcal{H})$. $\mathscr{A}$ is local [14], so $K_0(\mathscr{A}) \cong K_0(\mathfrak{A})$. Note that the definitions of $K_0(\mathscr{A})$ are equivalent whether we consider $\mathscr{A}$ as a Fréchet algebra or as a $*$-subalgebra of $\mathfrak{A}$, cf. Phillips [12].

We follow here the definition of order and class from [6], as opposed to the convention used in [11] where the operators are bounded on the Hilbert space $\mathcal{H}' = L_2(X, E) \oplus L_2(\partial X, F)$. It is explained in [10, 1.1] how the two approaches are equivalent for our purposes.

Furthermore, the K-theory of $\mathscr{A}$ is independent of the specific bundles [10, 1.5], so for simplicity we assume in this paper the simple case $E = X \times \mathbb{C}$ and $F = \partial X \times \mathbb{C}$.

$\mathscr{K}$ denotes the subalgebra of smoothing operators, $\mathfrak{K}$ its C^*-closure (the ideal of compact operators). We let $\mathscr{I}$ denote the set of elements in $\mathscr{A}$ of the form

$$\begin{pmatrix} \varphi P \psi + G & K \\ T & S \end{pmatrix} \tag{2.5}$$

with $\varphi, \psi \in C_c^\infty(X^\circ)$, P a ψdo on $\widetilde{X}$ of order zero, and G, K, T and S of negative order and class zero. $\mathfrak{I}$ will be the C^*-closure of $\mathscr{I}$ in $\mathfrak{A}$.

The noncommutative residue defined in [4] is a trace — a linear functional that vanishes on commutators — $\mathrm{res} : \mathscr{A} \to \mathbb{C}$. It is continuous with respect to the Fréchet topology in $\mathscr{A}$, and induces a group homomorphism $\mathrm{res}_* : K_0(\mathscr{A}) \to \mathbb{C}$ such that

$$\mathrm{res}_*([A]_0) = \mathrm{res}_X(A) \tag{2.6}$$

for any idempotent $A \in \mathscr{A}$. Our goal is to prove the vanishing of res_*, which obviously implies that $\mathrm{res}_X(A) = 0$ for all idempotent A.

The quotient map $q : \mathfrak{A} \to \mathfrak{A}/\mathfrak{K}$ induces an isomorphism $q_* : K_0(\mathfrak{A}) \to K_0(\mathfrak{A}/\mathfrak{K})$ [10, Prop. 13]. The isomorphisms $K_0(\mathscr{A}) \cong K_0(\mathfrak{A}) \cong K_0(\mathfrak{A}/\mathfrak{K})$ allow us to extend the noncommutative residue: For each $[\mathcal{A}+\mathfrak{K}]_0$ in $K_0(\mathfrak{A}/\mathfrak{K})$ there is an $A \in \mathcal{IP}_\infty(\mathscr{A})$ such that $q_*[A]_0 = [\mathcal{A} + \mathfrak{K}]_0$, and we then define

$$\widetilde{\mathrm{res}}_*[\mathcal{A} + \mathfrak{K}]_0 = \mathrm{res}_*[A]_0 = \mathrm{res}_X(A). \tag{2.7}$$

So $\widetilde{\mathrm{res}}_*$ is just $\mathrm{res}_* q_*^{-1}$, and is a group homomorphism $K_0(\mathfrak{A}/\mathfrak{K}) \to \mathbb{C}$.

3. K-theory and the residue

We employ results from Melo, Schick and Schrohe [11]: Theorem 1 there proves an isomorphism

$$K_0(\mathfrak{A}/\mathfrak{K}) \cong K_0(C(X)) \oplus K_1(C_0(T^*X^\circ)). \tag{3.1}$$

The intuitive interpretation of this isomorphism is that each K_0-class in $\mathfrak{A}/\mathfrak{K}$ is the sum of (the K_0-class of) a continuous function and (the K_0-class of) something vanishing on the boundary ∂X.

More precisely, we will use their observation

$$K_0(\mathfrak{A}/\mathfrak{K}) = q_* m_* K_0(C(X)) + i_* K_0(\mathfrak{I}/\mathfrak{K}). \tag{3.2}$$

Here $m : C(X) \to \mathfrak{A}$ sends f to the multiplication operator $\begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}$ and i is the inclusion $\mathfrak{I}/\mathfrak{K} \to \mathfrak{A}/\mathfrak{K}$; m_* and i_* are then the corresponding induced maps in K_0. We will in general suppress i and i_* to simplify notation.

We will show that $\widetilde{\mathrm{res}}_*$ vanishes on both terms in the right hand side of (3.2). The following lemma treats the first of these terms:

Lemma 3.1. $\widetilde{\mathrm{res}}_*$ *vanishes on* $q_* m_* K_0(C(X))$.

Proof. Recall that multiplication with a smooth function is a Green operator of order zero, whose noncommutative residue is clearly zero since it has no homogeneous term of order $-n$.

Let $f \in \mathcal{IP}_m(C^\infty(X))$; $m(f)$ acts by multiplication with a smooth (matrix) function and therefore lies in $\mathcal{IP}_m(\mathscr{A})$. Then $q_* m_*[f]_0 = q_*[m(f)]_0 = [m(f) + \mathfrak{K}]_0$, and according to (2.7)

$$\widetilde{\operatorname{res}}_*(q_* m_*[f]_0) = \operatorname{res}_*[m(f)]_0 = \operatorname{res}_X(m(f)) = 0. \tag{3.3}$$

Since $C^\infty(X)$ is local in $C(X)$ [1, 3.1.1-2], any element of $K_0(C(X))$ can be written as $[f]_0 - [g]_0$ for some $f, g \in \mathcal{IP}_m(C^\infty(X))$, cf. (2.3). The lemma follows from this. □

We now turn to the second term of (3.2); our strategy is to show that the elements of $K_0(\mathfrak{I}/\mathfrak{K})$ correspond to ψdos with symbols supported in the interior of X. This allows us to construct certain projections for which the noncommutative residue is given as the residue of a projection on the closed manifold $\widetilde{X}$.

The principal symbol induces an isomorphism $\mathfrak{I}/\mathfrak{K} \cong C_0(S^*X^\circ)$ [10, Theorem 1]. We denote the induced isomorphism in K_0 by σ_*, i.e.,

$$\sigma_* : K_0(\mathfrak{I}/\mathfrak{K}) \xrightarrow{\cong} K_0(C_0(S^*X^\circ)). \tag{3.4}$$

Like in Lemma 3.1 we wish to consider smooth functions instead of merely continuous functions; the following shows that instead of $C_0(S^*X^\circ)$, it suffices to look at smooth functions (symbols) compactly supported in the interior:

The algebra $C_c^\infty(S^*X^\circ)$, equipped with the sup-norm, is a local C^*-algebra [1, 3.1.1-2] with completion $C_0(S^*X^\circ)$. It follows from (2.2) that the injection $C_c^\infty(S^*X^\circ) \to C_0(S^*X^\circ)$ induces an isomorphism

$$K_0(C_c^\infty(S^*X^\circ)) \cong K_0(C_0(S^*X^\circ)). \tag{3.5}$$

We now show that each compactly supported symbol in $K_0(C_c^\infty(S^*X^\circ))$ gives rise to a ψdo projection Π_+ on X which is in fact the truncation of a ψdo projection on $\widetilde{X}$. This will allow us to calculate the residue of Π_+ from the residue of a projection on the closed manifold $\widetilde{X}$.

Lemma 3.2. *Let $p(x,\xi) \in \mathcal{IP}_m(C_c^\infty(S^*X^\circ)^\sim)$. There is a zero-order ψdo projection Π acting on $C^\infty(\widetilde{X}, \mathbb{C}^m)$, such that its symbol is constant on a neighborhood of $\widetilde{X} \setminus X^\circ$, its truncation Π_+ is an idempotent in $\mathcal{M}_m(\mathscr{I}^\sim)$, and*

$$\sigma_* q_*([\Pi_+]_0) = [p]_0. \tag{3.6}$$

Proof. By definition of the unitization of $C_c^\infty(S^*X^\circ)$, we can write p as a sum

$$p(x,\xi) = \alpha(x,\xi) + \beta, \tag{3.7}$$

with $\alpha \in \mathcal{M}_m(C_c^\infty(S^*X^\circ))$ and $\beta \in \mathcal{M}_m(\mathbb{C})$. Note that β itself is idempotent, since $p = \beta$ outside the support of α.

We extend α by zero to obtain a smooth function $\widetilde{\alpha}(x,\xi)$ on the closed manifold $S^*\widetilde{X}$. We get a ψdo symbol (also denoted $\widetilde{\alpha}$) of order zero on $\widetilde{X}$ by requiring $\widetilde{\alpha}$ to be homogeneous of degree zero in ξ. Let $\widetilde{p}(x,\xi) = \widetilde{\alpha}(x,\xi) + \beta$.

We now have an idempotent ψdo-symbol $\widetilde{p}$ on $\widetilde{X}$; we then construct a ψdo projection on $\widetilde{X}$ that has $\widetilde{p}$ as its principal symbol.

In [7, Chapter 3], Grubb constructed an operator that, for a suitable choice of atlas on the manifold, carries over to the Euclidean Laplacian in each chart, modulo smoothing operators. Hence, choose that particular atlas on $\widetilde{X}$ and let D denote this particular operator, i.e., with scalar symbol $d(x,\xi) = |\xi|^2$. Define the auxiliary second order ψdo $C = \mathrm{OP}(c(x,\xi))$, with symbol $c(x,\xi)$ given in the local coordinates of the specified charts as

$$c(x,\xi) = (2\widetilde{p}(x,\xi) - I)d(x,\xi). \tag{3.8}$$

Since $\widetilde{p}$ is idempotent, the eigenvalues of $2\widetilde{p} - I$ are ± 1, cf. (A.2), so C is an elliptic second order operator and $c(x,\xi) - \lambda$ is parameter-elliptic for λ on each ray in $\mathbb{C} \setminus \mathbb{R}$.

Then we can define the sectorial projection, cf. [13], [5], $\Pi = \Pi_{\theta,\varphi}(C)$ with angles $\theta = -\frac{\pi}{2}$, $\varphi = \frac{\pi}{2}$,

$$\Pi = \frac{i}{2\pi} \int_{\Gamma_{\theta,\varphi}} \lambda^{-1} C (C-\lambda)^{-1} \, d\lambda. \tag{3.9}$$

Π is a ψdo projection [13] on $\widetilde{X}$ with symbol π given in local coordinates by

$$\pi(x,\xi) = \frac{i}{2\pi} \int_{\mathcal{C}(x,\xi)} q(x,\xi,\lambda) \, d\lambda, \tag{3.10}$$

where $q(x,\xi,\lambda)$ is the symbol with parameter for a parametrix of $c(x,\xi) - \lambda$, and $\mathcal{C}(x,\xi)$ is a closed curve encircling the eigenvalues of $c_2(x,\xi)$ — the principal symbol of C — in the $\{\operatorname{Re} z > 0\}$ half-plane.

The eigenvalues of $c_2(x,\xi) = (2\widetilde{p}(x,\xi) - I)|\xi|^2$ are $\pm|\xi|^2$, so we can choose $\mathcal{C}(x,\xi)$ as the boundary of a small ball $B(|\xi|^2, r)$ around $+|\xi|^2$.

Then, the principal symbol of $\pi(x,\xi)$ is

$$\begin{aligned}\pi_0(x,\xi) &= \frac{i}{2\pi}\int_{\mathcal{C}(x,\xi)} q_{-2}(x,\xi,\lambda)\,d\lambda \\ &= \frac{i}{2\pi}\int_{\partial B(|\xi|^2,r)} [(2\widetilde{p}(x,\xi)-I)|\xi|^2-\lambda]^{-1}\,d\lambda = \widetilde{p}(x,\xi),\end{aligned} \tag{3.11}$$

according to Lemma A.1. So Π is a ψdo projection with principal symbol $\widetilde{p}(x,\xi)$, as desired.

Observe that for x outside the support of $\widetilde{\alpha}$, we have $c(x,\xi)=(2\beta-I)|\xi|^2$ and $q(x,\xi,\lambda)=q_{-2}(x,\xi,\lambda)=((2\beta-I)|\xi|^2-\lambda)^{-1}$ so $\pi(x,\xi)=\pi_0(x,\xi)=\beta$ there. (We cannot be sure that the full symbol of π equals $\widetilde{p}$ inside the support, since coordinate-dependence will in general influence the lower order terms of the parametrix.) In particular, $\pi(x,\xi)$ is constant equal to β for x outside $\widetilde{\alpha}$'s support, i.e., in a neighborhood of $\widetilde{X}\setminus X^\circ$.

Now consider the truncation Π_+. We have

$$(\Pi_+)^2 = (\Pi^2)_+ - L(\Pi,\Pi) = \Pi_+ - L(\Pi,\Pi), \tag{3.12}$$

where the singular Green operator $L(P,Q)$ is defined as $(PQ)_+ - P_+Q_+$ for ψdos P and Q. Since $\pi(x,\xi)$ equals the constant matrix β in a neighborhood of the boundary ∂X it follows, cf. [6, Theorem 2.7.5], that $L(\Pi,\Pi)=0$, so $(\Pi_+)^2=\Pi_+$.

The symbol of $\Pi-\beta$ is compactly supported within X°, so we can write $\Pi_+=\varphi P\psi+\beta$ for some φ,ψ,P, as in (2.5); hence Π_+ is in $\mathcal{M}_m(\mathscr{I}^\sim)$. Technically, Π_+ lies in the algebra where the boundary bundle F is the zero-bundle, but inserting zeros into Π_+'s matrix form will clearly allow us to augment it to the present case with $F=\partial X\times\mathbb{C}$.

Finally we take a look at (3.6): Since Π_+ is an idempotent in $\mathcal{M}_m(\mathscr{I}^\sim)$ it defines a K_0-class $[\Pi_+]_0$ in $K_0(\mathscr{I}^\sim)$. Then $q_*[\Pi_+]_0$ defines a class in $K_0(\mathfrak{I}/\mathfrak{K}^\sim)$, a class defined by its principal symbol. Since the principal symbol is exactly the idempotent $p(x,\xi)$ we obtain (3.6) by definition. $\square$

We now have all the tools to prove our main theorem:

Proof of Theorem 1.1. An idempotent Green operator necessarily has order and class zero, and thus lies in $\mathscr{A}$. So we need to show that $\operatorname{res}_X(A)$ is zero for any idempotent $A\in\mathscr{A}$; by (2.6) it suffices to show that res_* vanishes on $K_0(\mathscr{A})$. In turn, according to equation (3.2) and Lemma 3.1, we only need to show that $\widetilde{\operatorname{res}}_*$ vanishes on $K_0(\mathfrak{I}/\mathfrak{K})$.

So let $\omega \in K_0(\mathfrak{I}/\mathfrak{K})$. Employing (2.4), (3.4), and (3.5) we can find p, p' in $\mathcal{IP}_m(C_c^\infty(S^*X^\circ)^\sim)$ such that

$$\sigma_*\omega = [p]_0 - [p']_0. \tag{3.13}$$

Now, for p, p' we use Lemma 3.2 to find corresponding ψdos Π, Π' with the specific properties mentioned there. By (3.6) and (3.13) we see that

$$q_*[\Pi_+]_0 - q_*[\Pi'_+]_0 = \sigma_*^{-1}\big([p]_0 - [p']_0\big) = \omega. \tag{3.14}$$

Using equation (2.7) then gives us

$$\widetilde{\operatorname{res}}_*\omega = \operatorname{res}_X(\Pi_+) - \operatorname{res}_X(\Pi'_+). \tag{3.15}$$

Here

$$\operatorname{res}_X(\Pi_+) = \int_X \int_{S_x^*X} \operatorname{tr}\pi_{-n}(x,\xi)\,\bar{d}S(\xi)dx. \tag{3.16}$$

By construction, $\pi(x,\xi)$ is constant equal to β outside X; in particular $\pi_{-n}(x,\xi)$ is zero for $x \in \widetilde{X} \setminus X$ and therefore

$$\int_X \int_{S_x^*X} \operatorname{tr}\pi_{-n}(x,\xi)\,\bar{d}S(\xi)dx = \int_{\widetilde{X}} \int_{S_x^*\widetilde{X}} \operatorname{tr}\pi_{-n}(x,\xi)\,\bar{d}S(\xi)dx. \tag{3.17}$$

In other words

$$\operatorname{res}_X(\Pi_+) = \operatorname{res}_{\widetilde{X}}(\Pi), \tag{3.18}$$

where the latter is the noncommutative residue of a ψdo projection on a closed manifold. It is well-known [17], [18] that this always vanishes, so $\operatorname{res}_X(\Pi_+) = 0$. Likewise we obtain $\operatorname{res}_X(\Pi'_+) = 0$ and finally

$$\widetilde{\operatorname{res}}_*\omega = 0 \tag{3.19}$$

as desired. □

In [5], it was an open question whether the residue is zero on the sectorial projection for a boundary value problem. This theorem answers that question in the positive for the cases where the projection lies in $\mathscr{A}$.

It is not, at this time, clear for which boundary value problems this is true; however, we showed in [5] that there certainly are boundary value problems where the sectorial projection is not in $\mathscr{A}$.

A. Appendix

Lemma A.1. *Let* $M \in \mathcal{IP}_m(\mathbb{C})$*. Let* $d > 0$ *and let* $\partial B(d, r)$ *denote the closed curve in the complex plane along the boundary of the ball with center* d *and radius* $0 < r < d$*. Then*

$$\frac{i}{2\pi}\int_{\partial B(d,r)}[(2M - I)d - \lambda]^{-1}d\lambda = M. \tag{A.1}$$

Proof. A direct computation shows that, for $\lambda \neq \pm d$,

$$[(2M - I)d - \lambda]^{-1} = \frac{M}{d - \lambda} - \frac{I - M}{d + \lambda}. \tag{A.2}$$

The result in (A.1) then follows from the residue theorem. □

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