Intersection multiplicities and Grothendieck spaces

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Contents

Preface

This text constitutes my dissertation for the PhD degree in mathematics at the University of Copenhagen. It is the result of work done from 2004 to 2007 under the supervision of Hans-Bjørn Foxby.

I would like to express my gratitude to my adviser, Hans-Bjørn Foxby, for his interest and support throughout the years. I am also pleased to thank Sankar P. Dutta and Paul C. Roberts for letting me visit them and their departments for long periods of time during my studies. Further, I thank Anders J. Frankild, who has co-authored one of the articles in this dissertation, for many pleasant and interesting discussions. Finally, I thank David Jeffrey Breuer for copy-editing the introduction to this dissertation.

Esben Bistrup Halvorsen
Copenhagen, May 2007

Structure of the dissertation

The dissertation consists of three articles and an introduction. The articles are, in principle, the essential part of the dissertation, and the purpose of the introduction is to give an overview of the articles. Nevertheless, readers should be able to get a fairly good understanding of the content of the articles just by reading the introduction and using the articles only to look up proofs or additional information. The notation used in the introduction often differs from the notation used in the articles.

The three articles are:

I. Esben Bistrup Halvorsen, *Diagonalizing the Frobenius*, submitted.
II. Anders J. Frankild and Esben Bistrup Halvorsen, *Dualities and intersection multiplicities*, manuscript.
III. Hans-Bjørn Foxby and Esben Bistrup Halvorsen, *Grothendieck groups for categories of complexes*, submitted.

Page numbering is continuous throughout the entire dissertation but all four parts of the dissertation (the introduction and the three articles) have their own independent numbering of theorems, definitions etc. and of the references listed at the end of each part.

The order of the articles does not reflect the order in which they were produced or submitted, and it will probably not reflect the order in which they are published. Each article may change significantly before final publication; in particular, Article II may change dramatically even before being submitted.

Introduction

1. Overview

This introduction gives an overview of Articles I–III. The main focus is on *Grothendieck spaces*, which are introduced in Article I and further developed in Article II. The construction of Grothendieck spaces is quite similar to that of Grothendieck groups but targeted primarily at the study of intersection multiplicities. In this introduction, Grothendieck spaces are constructed from derived categories of complexes as done in Article II rather than from usual categories of complexes as done in Article I. The two constructions reveal the same space, so the reader should not worry about this.

Section 3 gives a classical introduction to Serre's intersection multiplicity, the Euler form and three variants of Dutta multiplicity. Sections 4–6 then present a set-up with derived categories before Sections 7–12 finally discuss Grothendieck spaces and their properties. The behavior of elements of Grothendieck spaces resembles that of local Chern characters, and Section 13 includes a comparison. Finally, Section 14 gives a few examples.

Article III focuses on Grothendieck groups of categories of complexes, the natural generalization of Grothendieck groups of categories of modules. Section 15 presents some of the results of Article III and discusses a consequence for Grothendieck spaces.

The construction of Grothendieck groups as well as of Grothendieck spaces is presented without paying much attention to set theoretical problems such as whether it is possible to represent the isomorphism classes in a category by a set. For a discussion of such problems, see Magurn [**9**, Section 3A].

2. Notation

Throughout this introduction, R denotes a fixed commutative ring. Unless otherwise stated, all ideals, modules and complexes are assumed to be ideals of R, R-modules and R-complexes, respectively. Modules are considered to be complexes concentrated in degree zero.

The *spectrum* of R, denoted $\operatorname{Spec} R$, is the set of prime ideals of R. It is equipped with the Zariski topology, in which the closed subsets are the subsets of the form

$$V(I) = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \supseteq I\}$$

for ideals I of R. A subset $\mathfrak{X}$ of $\operatorname{Spec} R$ is *specialization-closed* if it has the property

$$\mathfrak{p} \in \mathfrak{X} \implies V(\mathfrak{p}) \subseteq \mathfrak{X}$$

for prime ideals $\mathfrak{p}$. A closed subset is, in particular, specialization-closed.

The *support* of a complex X is the set

$$\operatorname{Supp} X = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathrm{H}(X_{\mathfrak{p}}) \neq 0\}.$$

A *finite* complex is a complex with bounded homology and finitely generated homology modules; the support of such a complex is a closed subset of $\operatorname{Spec} R$.

For a specialization-closed subset $\mathfrak{X}$ of $\operatorname{Spec} R$, the *dimension* of $\mathfrak{X}$, denoted $\dim \mathfrak{X}$, is the usual Krull dimension of $\mathfrak{X}$. When $\dim R$ is finite, the *co-dimension* of $\mathfrak{X}$, denoted $\operatorname{codim} \mathfrak{X}$, is the number $\dim R - \dim \mathfrak{X}$. For a finitely generated module M, the dimension and co-dimension of M, denoted $\dim M$ and $\operatorname{codim} M$, are the dimension and co-dimension of the support of M.

3. Intersection multiplicities

Assumption. Throughout this section, R is assumed to be Noetherian and local.

Let M and N be finitely generated modules with $\operatorname{length}(M \otimes_R N) < \infty$, and assume that either $\operatorname{pd} M < \infty$ or $\operatorname{pd} N < \infty$. The *intersection multiplicity* defined by Serre [**17**] of M and N is given by

$$\chi(M, N) = \sum_i (-1)^i \operatorname{length} \operatorname{Tor}_i^R(M, N).$$

The *intersection conjectures*, also proposed by Serre, state that

(*i*) $\dim M + \dim N \leq \dim R$;
(*ii*) $\chi(M, N) = 0$ whenever $\dim M + \dim N < \dim R$; and
(*iii*) $\chi(M, N) > 0$ whenever $\dim M + \dim N = \dim R$.

Conjecture (*ii*) is known as the *vanishing conjecture*, and conjecture (*iii*) is known as the *positivity conjecture*. In Serre's original formulation, R is assumed to be regular, and under this assumption, only the positivity conjecture remains open, and only in the case where R is ramified and of mixed characteristic. In the general setting presented above, however, neither the vanishing nor the positivity conjecture holds. This was first realized by Dutta, Hochster and McLaughlin [**4**], who presented a complete intersection ring of dimension 3 together with finitely generated modules M and N with $\dim M = 0$, $\dim N = 2$, $\operatorname{pd} M = 3$ and $\chi(M, N) = -1$. On the positive side, however, Foxby [**5**] proved that all three conjectures hold if the module that is *not* necessarily of finite projective dimension has dimension less than or equal to one.

A common weakening of the general intersection conjectures is to require both modules to have finite projective dimension. We shall denote the conjectures under this assumption the *weak intersection conjectures*; in particular, we shall refer to the *weak vanishing conjecture*. The weak intersection conjectures are still open. Roberts [**14**] and, independently, Gillet and Soulé [**6**] proved that the weak vanishing conjecture holds when R is a complete intersection ring, and Dutta [**3**] proved that it holds when R is a Gorenstein ring of dimension less than or equal to 5.

Assuming instead that either $\operatorname{pd} M < \infty$ or $\operatorname{id} N < \infty$, a natural analog of Serre's intersection multiplicity is the *Euler form*, introduced originally by Mori and Smith [**12**] and given by

$$\xi(M, N) = \sum_i (-1)^i \operatorname{length} \operatorname{Ext}_R^i(M, N).$$

Chan [**1**] suggested that the vanishing conjecture is connected with the general validity of the formula

$$\chi(M,N) = (-1)^{\operatorname{codim} M}\xi(M,N);$$

she proved, among other things, that the formula holds when R is a complete intersection, $\dim M + \dim N \leq \dim R$ and both M and N have finite projective dimension. Shortly after, Mori [**11**] proved other interesting connections between the above formula and the vanishing conjecture. Proposition 10 of Section 11 contributes to this discussion in the setting of complexes rather than modules.

Assumption. The remainder of this section assumes, in addition, that R is complete of prime characteristic p and has perfect residue field.

The *Frobenius endomorphism* $f\colon R \to R$ is given by $f(x) = x^p$. We denote ${}^fR$ the R-algebra which, as a ring, is R and which, as an R-module, has structure through f: that is, ${}^fR$ has R-module structure given by

$$r \cdot x = r^p x \quad \text{for } r \in R \text{ and } x \in {}^fR.$$

Note that ${}^fR$ under the current assumptions is a finitely generated R-module (see Roberts [**15**, Section 7.3]). The *Frobenius endofunctor* on the category of finitely generated modules is defined for a finitely generated module M by

$$F(M) = M \otimes_R {}^fR,$$

where the tensor product is a finitely generated module with R-structure obtained from the ring ${}^fR = R$: that is, with R-structure given by

$$r \cdot (m \otimes x) = m \otimes rx$$

for $r \in R$, $m \in M$ and $x \in {}^fR$. Note that here we also have

$$(rm) \otimes x = m \otimes (r \cdot x) = m \otimes r^p x.$$

Peskine and Szpiro [**13**, Théorème (1.7)] have proven that, if M has projective resolution X, then $F(M)$ has projective resolution $F(X)$; in particular if M has finite projective dimension, then so does $F(M)$.

Herzog [**7**] defined an analog of the Frobenius endofunctor. It is also an endofunctor on the category of finitely generated modules and is given for a finitely generated module N by

$$G(N) = \operatorname{Hom}_R({}^fR, N),$$

which is a finitely generated module with R-structure obtained from the ring ${}^fR = R$: that is, with R-structure given by

$$(r \cdot \varphi)(x) = \varphi(rx)$$

for $r \in R$, $x \in {}^fR$ and $\varphi \in \operatorname{Hom}_R({}^fR, N)$. Note that here we also have

$$r\varphi(x) = \varphi(r \cdot x) = \varphi(r^p x).$$

Herzog [**7**, Satz 5.2] proved that, if Y is an injective resolution of N, then $G(Y)$ is an injective resolution of $G(N)$; in particular, if N has finite injective dimension, then so does $G(N)$. We refer to the functor G as the *analogous Frobenius functor*. For a non-negative integer e, the e-fold composition of F and G is denoted F^e and G^e, respectively.

Dutta [**2**] introduced another intersection multiplicity that is now referred to as *Dutta multiplicity*. Let M and N be finitely generated modules such that length $M \otimes_R N < \infty$ and $\dim M + \dim \leq \dim R$. The Dutta multiplicity of M and N is defined by

$$\chi_\infty(M, N) = \lim_{e\to\infty} \frac{1}{p^{e \operatorname{codim} M}} \chi(F^e(M), N) \quad \text{whenever } \operatorname{pd} M < \infty.$$

This intersection multiplicity does, in fact, satisfy the vanishing conjecture, and Dutta showed that the general validity of the formula

$$\chi(M, N) = \chi_\infty(M, N)$$

is equivalent to the vanishing conjecture for the usual intersection multiplicity.

Article II defines two natural analogs of the Dutta multiplicity using the Euler form instead of Serre's intersection multiplicity. We let

$$\xi_\infty(M, N) = \lim_{e\to\infty} \frac{1}{p^{e \operatorname{codim} N}} \xi(M, G^e(N)) \quad \text{whenever } \operatorname{id} N < \infty; \quad \text{and}$$

$$\xi^\infty(M, N) = \lim_{e\to\infty} \frac{1}{p^{e \operatorname{codim} M}} \xi(F^e(M), N) \quad \text{whenever } \operatorname{pd} M < \infty.$$

As Section 6 shows, the two analogs of Dutta multiplicity can be expressed in terms of the original Dutta multiplicity.

4. Derived categories

Derived categories form a natural language to describe intersection multiplicities. This section and the following two sections explain how without delving too much into the theory of derived categories; see Weibel [**18**] for more details on this subject.

Assumption. Throughout this section, R is assumed to be Noetherian and local with maximal ideal $\mathfrak{m}$.

The derived category $\mathsf{D}(R)$ is obtained from the category of R-complexes by first equating homotopy-equivalent morphisms of complexes and then inverting quasi-isomorphisms by formally adjoining inverses (see Weibel [**18**, Chapter 10]). The objects of $\mathsf{D}(R)$ continue to be the usual R-complexes. We use the symbol $\simeq$ to denote isomorphisms in $\mathsf{D}(R)$ and the symbol $\sim$ to denote isomorphisms up to a shift.

For a specialization-closed subset $\mathfrak{X}$ of $\operatorname{Spec} R$, we define the following full subcategories of $\mathsf{D}(R)$.

$\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})$	$=$	the full subcategory of $\mathsf{D}(R)$ comprising the finite complexes with support contained in $\mathfrak{X}$.
$\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$	$=$	the full subcategory of $\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})$ comprising the complexes that are isomorphic to a bounded complex of projective modules.
$\mathsf{I}^{\mathsf{f}}(\mathfrak{X})$	$=$	the full subcategory of $\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})$ comprising the complexes that are isomorphic to a bounded complex of injective modules.

When $\mathfrak{X} = \operatorname{Spec} R$, we simply write $\mathsf{D}^{\mathsf{f}}_{\square}(R)$, $\mathsf{P}^{\mathsf{f}}(R)$ and $\mathsf{I}^{\mathsf{f}}(R)$, and when $\mathfrak{X} = \{\mathfrak{m}\}$, we write $\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{m})$, $\mathsf{P}^{\mathsf{f}}(\mathfrak{m})$ and $\mathsf{I}^{\mathsf{f}}(\mathfrak{m})$. The objects in $\mathsf{D}^{\mathsf{f}}_{\square}(R)$ are precisely the finite complexes.

The left-derived tensor product $- \otimes^{\mathbf{L}}_R -$ and the right-derived homomorphism functor $\mathbf{R}\mathrm{Hom}_R(-,-)$ are compositions on $\mathsf{D}(R)$ such that, when $\mathfrak{X}$ and $\mathfrak{Y}$ are specialization-closed subsets of $\operatorname{Spec} R$ and $X \in \mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$ and $Y \in \mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{Y})$ are complexes, then

$$
\begin{aligned}
X \otimes^{\mathbf{L}}_R Y &\in \mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X} \cap \mathfrak{Y}) && \text{if } X \in \mathsf{P}^{\mathrm{f}}(\mathfrak{X}) \text{ or } Y \in \mathsf{P}^{\mathrm{f}}(\mathfrak{Y});\\
X \otimes^{\mathbf{L}}_R Y &\in \mathsf{P}^{\mathrm{f}}(\mathfrak{X} \cap \mathfrak{Y}) && \text{if } X \in \mathsf{P}^{\mathrm{f}}(\mathfrak{X}) \text{ and } Y \in \mathsf{P}^{\mathrm{f}}(\mathfrak{Y});\\
X \otimes^{\mathbf{L}}_R Y &\in \mathsf{I}^{\mathrm{f}}(\mathfrak{X} \cap \mathfrak{Y}) && \text{if } X \in \mathsf{P}^{\mathrm{f}}(\mathfrak{X}) \text{ and } Y \in \mathsf{I}^{\mathrm{f}}(\mathfrak{Y});\\
X \otimes^{\mathbf{L}}_R Y &\in \mathsf{I}^{\mathrm{f}}(\mathfrak{X} \cap \mathfrak{Y}) && \text{if } X \in \mathsf{I}^{\mathrm{f}}(\mathfrak{X}) \text{ and } Y \in \mathsf{P}^{\mathrm{f}}(\mathfrak{Y});\\
\mathbf{R}\mathrm{Hom}_R(X,Y) &\in \mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X} \cap \mathfrak{Y}) && \text{if } X \in \mathsf{P}^{\mathrm{f}}(\mathfrak{X}) \text{ or } Y \in \mathsf{I}^{\mathrm{f}}(\mathfrak{Y});\\
\mathbf{R}\mathrm{Hom}_R(X,Y) &\in \mathsf{I}^{\mathrm{f}}(\mathfrak{X} \cap \mathfrak{Y}) && \text{if } X \in \mathsf{P}^{\mathrm{f}}(\mathfrak{X}) \text{ and } Y \in \mathsf{I}^{\mathrm{f}}(\mathfrak{Y});\\
\mathbf{R}\mathrm{Hom}_R(X,Y) &\in \mathsf{P}^{\mathrm{f}}(\mathfrak{X} \cap \mathfrak{Y}) && \text{if } X \in \mathsf{P}^{\mathrm{f}}(\mathfrak{X}) \text{ and } Y \in \mathsf{P}^{\mathrm{f}}(\mathfrak{Y}); \text{ and}\\
\mathbf{R}\mathrm{Hom}_R(X,Y) &\in \mathsf{P}^{\mathrm{f}}(\mathfrak{X} \cap \mathfrak{Y}) && \text{if } X \in \mathsf{I}^{\mathrm{f}}(\mathfrak{X}) \text{ and } Y \in \mathsf{I}^{\mathrm{f}}(\mathfrak{Y}).
\end{aligned}
\tag{1}
$$

A *dualizing complex* for R is a complex $D \in \mathsf{I}^{\mathrm{f}}(R)$ such that the homothety

$$R \to \mathbf{R}\mathrm{Hom}_R(D,D)$$

is an isomorphism. If D and D' are dualizing complexes, then $D \sim D'$. A complex $D \in \mathsf{D}^{\mathrm{f}}_{\square}(R)$ is dualizing if and only if $k \sim \mathbf{R}\mathrm{Hom}_R(k,D)$, and we say that D is *normalized* if $k \simeq \mathbf{R}\mathrm{Hom}_R(k,D)$. A normalized dualizing complex is unique up to isomorphism; in fact, such a complex is isomorphic to a complex in the form

$$0 \to D_{\dim R} \to \cdots \to D_1 \to D_0 \to 0,$$

where, for each i,

$$D_i = \bigoplus_{\substack{\mathfrak{p} \in \operatorname{Spec} R \\ \dim R/\mathfrak{p} = i}} E(R/\mathfrak{p}),$$

in which $E(R/\mathfrak{p})$ is the injective envelope of $R/\mathfrak{p}$.

If R is Cohen–Macaulay and D is a normalized dualizing complex, then $\mathrm{H}(D)$ is concentrated in degree $\dim R$ and $\mathrm{H}_{\dim R}(D)$ is the *canonical module* of R. If R is Gorenstein, $\mathsf{P}^{\mathrm{f}}(R) = \mathsf{I}^{\mathrm{f}}(R)$ and $\Sigma^{\dim R} R$ is a normalized dualizing complex. A ring admits a dualizing complex if it is a homomorphic image of a Gorenstein local ring; in particular, any complete local ring admits a dualizing complex.

5. Dagger, star, Foxby and Frobenius functors

Assumption. Throughout this section, R is assumed to be Noetherian and local. Further, it is assumed that R admits a normalized dualizing complex D.

Let $\mathfrak{X}$ be a specialization-closed subset of $\operatorname{Spec} R$. The *dagger duality functor* is the contravariant functor

$$(-)^{\dagger} = \mathbf{R}\mathrm{Hom}_R(-,D),$$

which gives a duality on $\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$ in the sense that, for all complexes $X \in \mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$, there is a natural isomorphism

$$X^{\dagger\dagger} = \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(X,D),D) \xleftarrow{\simeq} X.$$

According to (1), since $D \in \mathsf{I}^{\mathrm{f}}(R)$, dagger duality restricts to a duality between $\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ and $\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$.

The *Foxby equivalence functors* are the covariant functors

$$D \otimes^{\mathbf{L}}_R - \quad \text{and} \quad \mathbf{R}\mathrm{Hom}_R(D, -).$$

These give an equivalence between $\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ and $\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ in the sense that, for complexes $X \in \mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ and $Y \in \mathsf{I}^{\mathrm{f}}(\mathfrak{X})$, there are natural isomorphisms

$$\mathbf{R}\mathrm{Hom}_R(D, D \otimes^{\mathbf{L}}_R X) \xleftarrow{\simeq} X \quad \text{and} \quad D \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(D, Y) \xrightarrow{\simeq} Y.$$

The *star duality functor* is the contravariant functor

$$(-)^* = \mathbf{R}\mathrm{Hom}_R(-, R),$$

which gives a duality on $\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ in the sense that, for all complexes $X \in \mathsf{P}^{\mathrm{f}}(\mathfrak{X})$, there is a natural isomorphism

$$X^{**} = \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(X, R), R) \xleftarrow{\simeq} X.$$

The star duality functor can also be described as the composition of a dagger and a Foxby functor. In fact, there are the following isomorphisms of contravariant endofunctors on $\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$.

$$(-)^* \simeq (D \otimes^{\mathbf{L}}_R -)^\dagger \simeq \mathbf{R}\mathrm{Hom}_R(D, -^\dagger).$$

A natural dual of this picture is to start in the category $\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ and then compose a dagger and a Foxby functor. And indeed, this is the same as conjugating the star functor with either the dagger functors or with the Foxby functors, so that there are the following isomorphisms of contravariant endofunctors on $\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$.

$$(-)^{\dagger * \dagger} \simeq D \otimes^{\mathbf{L}}_R (\mathbf{R}\mathrm{Hom}_R(D, -)^*) \simeq D \otimes^{\mathbf{L}}_R (-^\dagger) \simeq \mathbf{R}\mathrm{Hom}_R(D, -)^\dagger.$$

We also refer to this as a *star duality functor* and we denote it $(-)^\star$. With these terms, the dagger, Foxby and star functors on $\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ and $\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ always commute pairwise, and the composition of two of these three kinds of functors always yields the third kind of functor: for example, the star and dagger functors commute, and their composition is a Foxby functor, since

$$(-)^{*\dagger} \simeq (-)^{\dagger\star} \simeq D \otimes^{\mathbf{L}}_R - \quad \text{and} \quad (-)^{\star\dagger} \simeq (-)^{\dagger *} \simeq \mathbf{R}\mathrm{Hom}_R(D, -).$$

Here, the first pair of isomorphisms are isomorphisms of covariant functors $\mathsf{P}^{\mathrm{f}}(\mathfrak{X}) \to \mathsf{I}^{\mathrm{f}}(\mathfrak{X})$, and the second pair of isomorphisms are isomorphisms of covariant functors $\mathsf{I}^{\mathrm{f}}(\mathfrak{X}) \to \mathsf{P}^{\mathrm{f}}(\mathfrak{X})$.

If R is Gorenstein, $\mathsf{P}^{\mathrm{f}}(\mathfrak{X}) = \mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ and $D \simeq \Sigma^{\dim R} R$, and hence the Foxby functors are just shifts of $\pm \dim R$ degrees, whereas $(-)^\dagger = \Sigma^{\dim R}(-)^* = \Sigma^{\dim R}(-)^\star$.

All the functors described above fit into the following diagram.

$$\begin{tikzcd}
\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}) \arrow[rr, shift left, "(-)^\dagger"] & & \mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}) \arrow[ll, shift left, "(-)^\dagger"] \\
\mathsf{P}^{\mathrm{f}}(\mathfrak{X}) \arrow[u] \arrow[loop left, "(-)^*"] \arrow[rr, shift left, "(-)^\dagger"] \arrow[rr, bend left, "D \otimes^{\mathbf{L}}_R -"] & & \mathsf{I}^{\mathrm{f}}(\mathfrak{X}) \arrow[u] \arrow[loop right, "(-)^\star"] \arrow[ll, shift left, "(-)^\dagger"] \arrow[ll, bend left, "\mathbf{R}\mathrm{Hom}_R(D,-)"]
\end{tikzcd} \tag{2}$$

Here, the corresponding straight horizontal arrows are mutually inverse and form a duality of categories; the curly horizontal arrows are mutually inverse and form an equivalence of categories; the circular arrows are self-inverse and form dualities of categories; and the vertical arrows are full embeddings of categories.

Assumption. The remainder of this section assumes, in addition, that R is complete of prime characteristic p and has perfect residue field.

Denote by $\mathbf{L}F$ and $\mathbf{R}G$ the left- and right-derived functors of F and G, respectively. We can think of these functors as

$$\mathbf{L}F(-) = - \otimes^{\mathbf{L}}_R {}^f\!R \quad \text{and} \quad \mathbf{R}G(-) = \mathbf{R}\mathrm{Hom}_R({}^f\!R, -),$$

where we require the derivation to take place in the blank variable, so that the resulting R-structure can be obtained from the ring ${}^f\!R = R$. For any specialization-closed subset $\mathfrak{X}$ of $\operatorname{Spec} R$, the functors $\mathbf{L}F$ and $\mathbf{R}G$ are covariant endofunctors on $\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ and $\mathsf{I}^{\mathsf{f}}(\mathfrak{X})$, respectively. For a non-negative integer e, the e-fold compositions of $\mathbf{L}F$ and $\mathbf{R}G$ are denoted $\mathbf{L}F^e$ and $\mathbf{R}G^e$, respectively; these are the same as the left- and right-derived functors of F^e and G^e.

The endofunctor $\mathbf{L}F$ on $\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ and its analog $\mathbf{R}G$ on $\mathsf{I}^{\mathsf{f}}(\mathfrak{X})$ commute with the dagger, star and Foxby functors in the sense that the following isomorphisms of functors exist (see Article II, Lemma 2.11).

$$\begin{aligned}
&\mathbf{L}F(-^*) \simeq \mathbf{L}F(-)^*; && \mathbf{R}G(-^\star) \simeq \mathbf{R}G(-)^\star;\\
&\mathbf{R}G(-^\dagger) \simeq \mathbf{L}F(-)^\dagger; && \mathbf{R}G(D \otimes^{\mathbf{L}}_R -) \simeq D \otimes^{\mathbf{L}}_R \mathbf{L}F(-);\\
&\mathbf{L}F(-^\dagger) \simeq \mathbf{R}G(-)^\dagger; \quad \text{and} && \mathbf{L}F(\mathbf{R}\mathrm{Hom}_R(D,-)) \simeq \mathbf{R}\mathrm{Hom}_R(D, \mathbf{R}G(-)).
\end{aligned} \tag{3}$$

Here, the isomorphisms in the first row are isomorphisms of contravariant endofunctors on $\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ and $\mathsf{I}^{\mathsf{f}}(\mathfrak{X})$, respectively; the isomorphisms in the second row are isomorphisms of contravariant and covariant functors $\mathsf{P}^{\mathsf{f}}(\mathfrak{X}) \to \mathsf{I}^{\mathsf{f}}(\mathfrak{X})$, respectively; and the isomorphisms in the third row are isomorphisms of contravariant and covariant functors $\mathsf{I}^{\mathsf{f}}(\mathfrak{X}) \to \mathsf{P}^{\mathsf{f}}(\mathfrak{X})$, respectively. When R is Gorenstein, the formulas show that $\mathbf{L}F = \mathbf{R}G$.

6. Intersection multiplicities in derived categories

Assumption. Throughout this section, R is assumed to be Noetherian and local with maximal ideal $\mathfrak{m}$. Further, it is assumed that R admits a dualizing complex.

The *Euler characteristic* of a complex $Z \in \mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{m})$ is the integer

$$\chi(Z) = \sum_i (-1)^i \operatorname{length} \mathrm{H}_i(Z).$$

Let X and Y be complexes in $\mathsf{D}^{\mathsf{f}}_{\square}(R)$ and assume that $\operatorname{Supp} X \cap \operatorname{Supp} Y = \{\mathfrak{m}\}$. The *intersection multiplicity* of X and Y is defined by

$$\chi(X, Y) = \chi(X \otimes^{\mathbf{L}}_R Y) \quad \text{whenever } X \in \mathsf{P}^{\mathsf{f}}(R) \text{ or } Y \in \mathsf{P}^{\mathsf{f}}(R),$$

and the *Euler form* of X and Y is defined by

$$\xi(X, Y) = \chi(\mathbf{R}\mathrm{Hom}_R(X, Y)) \quad \text{whenever } X \in \mathsf{P}^{\mathsf{f}}(R) \text{ or } Y \in \mathsf{I}^{\mathsf{f}}(R).$$

The following identities hold (see Mori [**11**, Lemma 4.3(1)–(2)]).

$$\begin{aligned} \chi(X,Y) &= \xi(X,Y^\dagger) \quad \text{whenever } X \in \mathsf{P}^{\mathrm{f}}(R) \text{ or } Y \in \mathsf{P}^{\mathrm{f}}(R); \\ \chi(X^*,Y) &= \chi(X,Y^\dagger) \quad \text{whenever } X \in \mathsf{P}^{\mathrm{f}}(R); \text{ and} \\ \xi(X,Y^\star) &= \xi(X^\dagger,Y) \quad \text{whenever } Y \in \mathsf{I}^{\mathrm{f}}(R). \end{aligned} \tag{4}$$

In the special case where X and Y are modules, the intersection multiplicity and Euler form defined here are the same as the ones defined in Section 3.

We generalize Serre's vanishing conjecture to the setting of complexes by saying that R *satisfies vanishing* if, in the case where $X \in \mathsf{P}^{\mathrm{f}}(R)$ or $Y \in \mathsf{P}^{\mathrm{f}}(R)$,

$$\chi(X,Y) = 0 \quad \text{whenever } \dim(\operatorname{Supp} X) + \dim(\operatorname{Supp} Y) < \dim R.$$

Likewise, we say that R *satisfies weak vanishing* if the above holds under the restriction that both $X \in \mathsf{P}^{\mathrm{f}}(R)$ *and* $Y \in \mathsf{P}^{\mathrm{f}}(R)$.

According to the first formula in (4) and since the dagger duality functor does not change the support of a complex, R satisfies vanishing if and only if, in the case where $X \in \mathsf{P}^{\mathrm{f}}(R)$ or $Y \in \mathsf{I}^{\mathrm{f}}(R)$,

$$\xi(X,Y) = 0 \quad \text{whenever } \dim(\operatorname{Supp} X) + \dim(\operatorname{Supp} Y) < \dim R,$$

and R satisfies weak vanishing if and only if this holds under the restriction that both $X \in \mathsf{P}^{\mathrm{f}}(R)$ *and* $Y \in \mathsf{I}^{\mathrm{f}}(R)$.

Assumption. The remainder of this section assumes, in addition, that R is complete of prime characteristic p and has perfect residue field.

The Dutta multiplicity and its two analogs from Section 3 can also be generalized. Let X and Y be complexes in $\mathsf{D}^{\mathrm{f}}_{\square}(R)$, and assume that $\operatorname{Supp} X \cap \operatorname{Supp} Y = \{\mathfrak{m}\}$ and $\dim(\operatorname{Supp} X) + \dim(\operatorname{Supp} Y) \leq \dim R$. We define generalizations of the Dutta multiplicity and its analogs by setting

$$\begin{aligned} \chi_\infty(X,Y) &= \lim_{e\to\infty} \frac{1}{p^{e \operatorname{codim} \mathfrak{X}}} \chi(\mathbf{L}F^e(X),Y) \quad \text{whenever } X \in \mathsf{P}^{\mathrm{f}}(R); \\ \xi_\infty(X,Y) &= \lim_{e\to\infty} \frac{1}{p^{e \operatorname{codim} \mathfrak{Y}}} \xi(X,\mathbf{R}G^e(Y)) \quad \text{whenever } Y \in \mathsf{I}^{\mathrm{f}}(R); \text{ and} \\ \xi^\infty(X,Y) &= \lim_{e\to\infty} \frac{1}{p^{e \operatorname{codim} \mathfrak{X}}} \xi(\mathbf{L}F^e(X),Y) \quad \text{whenever } X \in \mathsf{P}^{\mathrm{f}}(R). \end{aligned}$$

The formulas in (3) and (4) show that

$$\xi_\infty(X,Y) = \chi_\infty(Y^\dagger,X) \quad \text{and} \quad \xi^\infty(X,Y) = \chi_\infty(X^*,Y),$$

whenever defined. Corollary 5 of Section 10 presents some interesting formulas to calculate these multiplicities.

7. Grothendieck spaces

This section introduces Grothendieck spaces as topological $\mathbb{Q}$-vector spaces modelled on the derived categories $\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$, $\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ and $\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$. This is the construction carried out in Article II. The construction in Article I is modelled on non-derived categories, but the resulting spaces are the same. The spaces denoted $\mathbb{G}\mathsf{P}(\mathfrak{X})$ and $\mathbb{G}\mathsf{C}(\mathfrak{X})$ in Article I are denoted $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ and $\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$, respectively, in this introduction.

Assumption. Throughout this section, R is assumed to be Noetherian and local with maximal ideal $\mathfrak{m}$.

For any given specialization-closed subset $\mathfrak{X}$ of $\operatorname{Spec} R$, a new specialization-closed subset is defined by

$$\mathfrak{X}^c = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{X} \cap V(\mathfrak{p}) = \{\mathfrak{m}\} \text{ and } \dim V(\mathfrak{p}) \leq \operatorname{codim} \mathfrak{X}\}.$$

This set is maximal under inclusion among specialization-closed subsets of $\operatorname{Spec} R$ with respect to the properties

$$\mathfrak{X} \cap \mathfrak{X}^c = \{\mathfrak{m}\} \quad \text{and} \quad \dim \mathfrak{X} + \dim \mathfrak{X}^c \leq \dim R.$$

In fact, when $\mathfrak{X}$ is closed,

$$\dim \mathfrak{X} + \dim \mathfrak{X}^c = \dim R.$$

Note also that $\mathfrak{X} \subseteq \mathfrak{X}^{cc}$. (In fact, we always have $\mathfrak{X}^c = \mathfrak{X}^{ccc}$, so the association $\mathfrak{X} \mapsto \mathfrak{X}^{cc}$ is actually a closure operator on $\operatorname{Spec} R$.)

If X is a complex in $\mathsf{P}^{\mathrm{f}}(R)$ with support equal to $\mathfrak{X}$, then $\mathfrak{X}^c$ is the maximum allowed support for a complex Y in $\mathsf{D}^{\mathrm{f}}_{\square}(R)$ in the definition of the Dutta multiplicity $\chi_\infty(X, Y)$. When dealing with the vanishing conjecture, it suffices to consider complexes X and Y with supports $\mathfrak{X}$ and $\mathfrak{Y}$, respectively, such that $\mathfrak{Y} \subseteq \mathfrak{X}^c$ or, equivalently, $\mathfrak{X} \subseteq \mathfrak{Y}^c$. Note that Serre's conjecture (i) from Section 3 states that these inclusions always hold in the special case where X and Y are modules.

Let $\mathfrak{X}$ be a specialization-closed subset of $\operatorname{Spec} R$. The *Grothendieck space* of $\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ is the $\mathbb{Q}$-vector space $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ presented by elements $[X]$, one for each isomorphism class of a complex X in $\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$, and relations

$$[X] = [X'] \quad \text{whenever } \chi(X, -) = \chi(X', -)$$

as metafunctions (that is, "functions" from a category to a set) $\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}^c) \to \mathbb{Q}$. Likewise, the *Grothendieck space* of $\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ is the $\mathbb{Q}$-vector space $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ presented by elements $[Y]$, one for each isomorphism class of a complex Y in $\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$, and relations

$$[Y] = [Y'] \quad \text{whenever } \xi(-, Y) = \xi(-, Y')$$

as metafunctions $\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}^c) \to \mathbb{Q}$. Finally, the *Grothendieck space* of $\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$ is the $\mathbb{Q}$-vector space $\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$ presented by elements $[Z]$, one for each isomorphism class of a complex Z in $\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$, and relations

$$[Z] = [Z'] \quad \text{whenever } \chi(-, Z) = \chi(-, Z')$$

as metafunctions $\mathsf{P}^{\mathrm{f}}(\mathfrak{X}^c) \to \mathbb{Q}$; when R admits a dualizing complex, the identities in (4) from Section 6 show that these relations are equivalent to the relations

$$[Z] = [Z'] \quad \text{whenever } \xi(Z, -) = \xi(Z', -)$$

as metafunctions $\mathsf{I}^{\mathrm{f}}(\mathfrak{X}^c) \to \mathbb{Q}$.

Since the intersection multiplicity and the Euler form in either variable are additive on short exact sequences and zero on exact complexes, the Grothendieck spaces $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$, $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ and $\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$ are tensor products of $\mathbb{Q}$ and a homomorphic image of the Grothendieck groups of certain non-derived versions of $\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$, $\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ and $\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$. This means, for instance, that $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ is the tensor product of $\mathbb{Q}$ and a homomorphic image of $K_0(\mathsf{C}(\mathfrak{X}))$, where $K_0(\mathsf{C}(\mathfrak{X}))$ is the Grothendieck group of the (non-derived) category $\mathsf{C}(\mathfrak{X})$ of bounded complexes supported at $\mathfrak{X}$ and consisting of finitely generated projective modules. Section 15 discusses Grothendieck groups of categories of complexes.

We usually use lowercase Greek letters α, β, γ etc. to denote elements of Grothendieck spaces. An element of a Grothendieck space can always be written in the form $r[X]$ for a rational number r and a complex X in the corresponding category (see Article II, Proposition 4.3).

The Grothendieck spaces are equipped with the initial topology of the $\mathbb{Q}$-linear maps induced by the metafunctions that define the relations in the presentation of the space: the topology on $\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$, for instance, is the coarsest topology such that, for each $Y \in \mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X}^c)$, the induced $\mathbb{Q}$-linear map

$$\chi(-,Y)\colon \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X}) \to \mathbb{Q} \quad \text{given by} \quad \chi([X],Y) = \chi(X,Y)$$

for $X \in \mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ is continuous. (The fact that the map is well defined follows from the definition of the Grothendieck space.) In this way, $\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ becomes a topological $\mathbb{Q}$-vector space, and the same is true for $\mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X})$ and $\mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})$. We always think of Grothendieck spaces as topological $\mathbb{Q}$-vector spaces, so that such words as "homomorphism" and "isomorphism" denote maps that preserve topological as well as $\mathbb{Q}$-vector space structure. Thus, for example, a homomorphism from one Grothendieck space to another is a continuous, $\mathbb{Q}$-linear map, and an isomorphism between two Grothendieck spaces is a $\mathbb{Q}$-linear homeomorphism.

The Grothendieck spaces have been constructed as topological $\mathbb{Q}$-vector spaces through which the intersection multiplicity and the Euler form, respectively, factor as homomorphisms of topological $\mathbb{Q}$-vector spaces. For example, the Grothendieck space $\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ is a topological $\mathbb{Q}$-vector space with the property that the metafunction

$$\chi(-,Y)\colon \mathsf{P}^{\mathsf{f}}(\mathfrak{X}) \to \mathbb{Q}$$

for all $Y \in \mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X}^c)$ factors through $\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ as a topological $\mathbb{Q}$-vector space homomorphism, which we, by abuse of notation, denote likewise:

$$\begin{tikzcd}
\mathsf{P}^{\mathsf{f}}(\mathfrak{X}) \arrow[rr, "\chi(-,Y)"] \arrow[dr, "X\mapsto[X]"'] & & \mathbb{Q} \\
& \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X}) \arrow[ur, "\chi(-,Y)"'] &
\end{tikzcd}$$

In fact, by definition, $\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ is the $\mathbb{Q}$-vector space with the universal property that there is a metafunction $\mathsf{P}^{\mathsf{f}}(\mathfrak{X}) \to \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ that identifies complexes X and X' whenever the intersection multiplicities $\chi(X,-)$ and $\chi(X',-)$ are identical as metafunctions $\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X}^c) \to \mathbb{Q}$. Similar remarks hold for the Grothendieck spaces $\mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X})$ and $\mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})$.

If R is Gorenstein, $\mathsf{P}^{\mathsf{f}}(\mathfrak{X}) = \mathsf{I}^{\mathsf{f}}(\mathfrak{X})$, and hence, a priori, two different Grothendieck spaces are associated with the same category. However, in this situation $(-)^{\dagger} = \Sigma^{\dim R}(-)^{*}$, so for all $X \in \mathsf{P}^{\mathsf{f}}(\mathfrak{X}) = \mathsf{I}^{\mathsf{f}}(\mathfrak{X})$ and all $Y \in \mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X}^c)$, the formulas in (4) yield that

$$\chi(X,Y) = \chi(X^{*},Y^{\dagger}) = \chi(Y^{\dagger},X^{*}) = \xi(Y^{\dagger},X^{*\dagger}) = (-1)^{\dim R}\xi(Y^{\dagger},X),$$

and hence the metafunctions defining the Grothendieck spaces $\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ and $\mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X})$ are the same, which implies that the spaces are identical. If R is regular then, in addition, $\mathsf{P}^{\mathsf{f}}(\mathfrak{X}) = \mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})$, but in this case $\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X}^c) = \mathsf{P}^{\mathsf{f}}(\mathfrak{X}^c)$, which clearly implies that the metafunctions defining the Grothendieck spaces $\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ and $\mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})$ are the same.

8. Induced maps of Grothendieck spaces

Assumption. Throughout this section, R is assumed to be Noetherian and local with maximal ideal $\mathfrak{m}$. Further, it is assumed that R admits a normalized dualizing complex D.

The Euler characteristic $\chi\colon \mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{m}) \to \mathbb{Q}$ induces an isomorphism (that is, a $\mathbb{Q}$-linear homeomorphism)

$$\chi\colon \mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{m}) \to \mathbb{Q} \quad \text{given by} \quad \chi([Z]) = \chi(Z),$$

for $Z \in \mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{m})$; see Article I, Proposition 2(vi), for more details.

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be specialization-closed subsets of $\operatorname{Spec} R$ with $\mathfrak{X} \cap \mathfrak{Y} = \{\mathfrak{m}\}$ and $\dim \mathfrak{X} + \dim \mathfrak{Y} \leq \dim R$. The derived tensor product $- \otimes^{\mathbf{L}}_R -$ and the derived homomorphism functor $\mathbf{R}\mathrm{Hom}_R(-,-)$ induce a collection of Grothendieck space bi-homomorphisms (maps that are continuous and $\mathbb{Q}$-linear in each variable)

$$\begin{array}{ccc}
\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X}) \times \mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{Y}) \to \mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{m}); & & \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X}) \times \mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{Y}) \to \mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{m}); \\
\mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X}) \times \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{Y}) \to \mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{m}); & & \mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X}) \times \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{Y}) \to \mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{m}); \\
\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X}) \times \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{Y}) \to \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{m}); & & \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X}) \times \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{Y}) \to \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{m}); \\
\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X}) \times \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{Y}) \to \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{m}); & & \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X}) \times \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{Y}) \to \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{m}); \\
\mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X}) \times \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{Y}) \to \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{m}); & \text{and} & \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X}) \times \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{Y}) \to \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{m}).
\end{array}$$

The bi-homomorphisms in the first column are induced by the derived tensor product; they are all denoted $- \otimes -$ and given by

$$([X],[Y]) \mapsto [X] \otimes [Y] \overset{\text{def}}{=} [X \otimes^{\mathbf{L}}_R Y]$$

for complexes X and Y in the appropriate categories. The bi-homomorphisms in the second column are induced by the derived homomorphism functor; they are all denoted $\mathrm{Hom}(-,-)$ and given by

$$([X],[Y]) \mapsto \mathrm{Hom}([X],[Y]) \overset{\text{def}}{=} [\mathbf{R}\mathrm{Hom}_R(X,Y)]$$

for complexes X and Y in the appropriate categories. See Article II, Proposition 4.7 for further details.

Let $\mathfrak{X} \subseteq \mathfrak{X}'$ be specialization-closed subsets of $\operatorname{Spec} R$ and consider the inclusion functors pictured in the diagram below.

$$\begin{tikzcd}
\mathsf{P}^{\mathsf{f}}(\mathfrak{X}) \arrow[r] \arrow[d] & \mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X}) \arrow[d] & \mathsf{I}^{\mathsf{f}}(\mathfrak{X}) \arrow[l] \arrow[d] \\
\mathsf{P}^{\mathsf{f}}(\mathfrak{X}') \arrow[r] & \mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X}') & \mathsf{I}^{\mathsf{f}}(\mathfrak{X}') \arrow[l]
\end{tikzcd}$$

These functors induce natural Grothendieck space homomorphisms, all given by $[X] \mapsto [X]$, where X is a complex in the appropriate (domain) category and the first $[X]$ is an element in the domain of the homomorphism, whereas the second is an element in the range of the homomorphism. The result is the following commutative diagram of homomorphisms of Grothendieck spaces.

$$\begin{tikzcd}
\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X}) \arrow[r] \arrow[d] & \mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X}) \arrow[d] & \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X}) \arrow[l] \arrow[d] \\
\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X}') \arrow[r] & \mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X}') & \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X}') \arrow[l]
\end{tikzcd}$$

These maps are referred to as *inclusion homomorphisms*, and they are often denoted by the symbol $\overline{(-)}$ so that, for example, if α is an element of $\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$, then $\overline{\alpha}$ denotes the image of α in $\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X}')$ under the inclusion homomorphism $\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X}) \to \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X}')$. If the range of the inclusion homomorphism needs to be emphasized, we write $\overline{\alpha} \in \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X}')$ instead of just $\overline{\alpha}$. If a property of an element $\alpha \in \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ holds after an application of the inclusion homomorphism $\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X}) \to \mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})$, we often say that the property holds *in* $\mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})$ or that it holds *numerically*. Similar remarks hold for elements of $\mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X})$.

The duality and equivalence functors in (2) of Section 5 naturally induce isomorphisms of Grothendieck spaces. By abuse of notation, the maps induced from the dagger and star duality functors are denoted by the same symbol; the maps induced by the Foxby equivalence functors are denoted $D \otimes -$ and $\operatorname{Hom}(D, -)$, respectively. In this way, for example,

$$[X]^{\dagger} \overset{\text{def}}{=} [X^{\dagger}], \quad [X]^{*} \overset{\text{def}}{=} [X^{*}] \quad \text{and} \quad D \otimes [X] \overset{\text{def}}{=} [D \otimes^{\mathbf{L}}_{R} X]$$

for a complex $X \in \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$. As with the underlying functors (see Section 5), these three kinds of isomorphisms commute pairwise, and the composition of two of the three kinds of isomorphisms gives an isomorphism of the third kind. The situation looks as follows.

$$\begin{tikzcd}
\mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X}) \arrow[r, shift left, "(-)^{\dagger}"] & \mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X}) \arrow[l, shift left, "(-)^{\dagger}"] \\
\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X}) \arrow[u] \arrow["(-)^{*}"', loop left] \arrow[r, bend left, "D\otimes -"] \arrow[r, shift left, "(-)^{\dagger}"] & \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X}) \arrow[u] \arrow["(-)^{*}"', loop right] \arrow[l, shift left, "(-)^{\dagger}"] \arrow[l, bend left, "\operatorname{Hom}(D,-)"]
\end{tikzcd}$$

Here, the corresponding horizontal arrows are mutually inverse isomorphisms; the circular arrows are self-inverse automorphisms; and the vertical arrows are inclusion homomorphisms as discussed above. See Article II, Proposition 4.8 for further details.

Assumption. The remainder of this section assumes, in addition, that R is complete of prime characteristic p and has perfect residue field.

Let $\mathfrak{X}$ be a specialization-closed subset of $\operatorname{Spec} R$. The left-derived Frobenius endofunctor $\mathbf{L}F$ on $\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ induces an endomorphism on $\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ denoted $F_{\mathfrak{X}}$, and the right-derived endofunctor $\mathbf{R}G$ on $\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ induces an endomorphism on $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ denoted $G_{\mathfrak{X}}$. These endomorphisms are given for complexes $X \in \mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ and $Y \in \mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ by

$$F_{\mathfrak{X}}([X]) = [\mathbf{L}F(X)] \quad \text{and} \quad G_{\mathfrak{X}}([Y]) = [\mathbf{R}G(Y)].$$

See Article II, Section 4.10 for further details.

Theorems 3 and 4 in Section 10 show that $F_{\mathfrak{X}}$ and $G_{\mathfrak{X}}$ are automorphisms. We define normalized versions of $F_{\mathfrak{X}}$ and $G_{\mathfrak{X}}$ by

$$\Phi_{\mathfrak{X}} = p^{-\operatorname{codim}\mathfrak{X}} F_{\mathfrak{X}} \quad \text{and} \quad \Psi_{\mathfrak{X}} = p^{-\operatorname{codim}\mathfrak{X}} G_{\mathfrak{X}}.$$

These play an important role in Section 10 when we discuss decompositions of elements in Grothendieck spaces. A consequence of Theorems 3 and 4 in Section 10 is that the e-fold compositions $\Phi_{\mathfrak{X}}^e$ and $\Psi_{\mathfrak{X}}^e$ converge to endomorphisms $\lim_{e\to\infty}\Phi_{\mathfrak{X}}^e$ and $\lim_{e\to\infty}\Psi_{\mathfrak{X}}^e$ on $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ and $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$, respectively. When R is Gorenstein and $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ is identical to $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$, $F_{\mathfrak{X}} = G_{\mathfrak{X}}$ and $\Phi_{\mathfrak{X}} = \Psi_{\mathfrak{X}}$.

9. Intersection multiplicities in Grothendieck spaces

Assumption. Throughout this section, R is assumed to be Noetherian and local with maximal ideal $\mathfrak{m}$. Further, it is assumed that R admits a normalized dualizing complex D.

The fact that the Euler characteristic induces an isomorphism $\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{m}) \to \mathbb{Q}$ means that the intersection multiplicity and the Euler form can represented by elements in $\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{m})$ of the form

$$\delta \otimes \varepsilon \quad \text{and} \quad \operatorname{Hom}(\delta, \varepsilon).$$

For example, if X and Y are finite complexes with $\mathfrak{X} = \operatorname{Supp} X$ and $\mathfrak{Y} = \operatorname{Supp} Y$ such that $\mathfrak{X} \cap \mathfrak{Y} = \{\mathfrak{m}\}$ and $\dim \mathfrak{X} + \dim \mathfrak{Y} \leq \dim R$, then, when $X \in \mathsf{P}^{\mathrm{f}}(R)$ or $Y \in \mathsf{P}^{\mathrm{f}}(R)$, the intersection multiplicity $\chi(X,Y) = \chi(X \otimes^{\mathbf{L}}_R Y)$ is the image in $\mathbb{Q}$ under the isomorphism $\chi\colon \mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{m}) \to \mathbb{Q}$ of the element $[X \otimes^{\mathbf{L}}_R Y] = [X] \otimes [Y]$ in $\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{m})$, where $[X] \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ and $[Y] \in \mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{Y})$, or $[X] \in \mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$ and $[Y] \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{Y})$.

Being able to discuss the vanishing conjecture, however, requires introducing the concept of dimension for elements of Grothendieck spaces. Let $\mathfrak{X}$ be a specialization-closed subset of R and let $\alpha \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$, $\beta \in \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ and $\gamma \in \mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$. The *dimensions* of α, β and γ are defined as

$$\dim \alpha = \inf\{\dim(\operatorname{Supp} X) \mid \alpha = r[X] \text{ for some } r \in \mathbb{Q} \text{ and } X \in \mathsf{P}^{\mathrm{f}}(\mathfrak{X})\};$$

$$\dim \beta = \inf\{\dim(\operatorname{Supp} Y) \mid \beta = s[Y] \text{ for some } s \in \mathbb{Q} \text{ and } Y \in \mathsf{I}^{\mathrm{f}}(\mathfrak{X})\}; \text{ and}$$

$$\dim \gamma = \inf\{\dim(\operatorname{Supp} Z) \mid \gamma = t[Z] \text{ for some } t \in \mathbb{Q} \text{ and } Z \in \mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})\}.$$

Thus, the dimension of an element in a Grothendieck space is the minimum dimension of the support of a complex representing that element.

We say that α *satisfies vanishing* if

$$\alpha \otimes \delta = 0 \quad \text{in } \mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{m}) \text{ for all } \delta \in \mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}^c) \text{ with } \dim \delta < \operatorname{codim}\mathfrak{X}.$$

Further, we say that α *satisfies weak vanishing* if

$$\overline{\alpha \otimes \varepsilon} = 0 \quad \text{in } \mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{m}) \text{ for all } \varepsilon \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X}^c) \text{ with } \dim \varepsilon < \operatorname{codim}\mathfrak{X}.$$

Similarly, we say that β *satisfies vanishing* if

$$\operatorname{Hom}(\delta, \beta) = 0 \quad \text{in } \mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{m}) \text{ for all } \delta \in \mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}^c) \text{ with } \dim \delta < \operatorname{codim}\mathfrak{X},$$

and we say that β *satisfies weak vanishing* if

$$\overline{\operatorname{Hom}(\varepsilon,\beta)} = 0 \quad \text{in } \mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{m}) \text{ for all } \varepsilon \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X}^c) \text{ with } \dim \varepsilon < \operatorname{codim} \mathfrak{X}.$$

If $\alpha = [X]$ for a complex $X \in \mathsf{P}^{\mathrm{f}}(R)$ with $\operatorname{Supp} X = \mathfrak{X}$, then α satisfies vanishing exactly when

$$\chi(X,Y) = 0 \quad \text{for all } Y \in \mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}^c) \text{ with } \dim(\operatorname{Supp} X) + \dim(\operatorname{Supp} Y) < \dim R,$$

and α satisfies weak vanishing exactly when this is true under the extra assumption that $Y \in \mathsf{P}^{\mathrm{f}}(\mathfrak{X}^c)$. This corresponds to the generalization of Serre's vanishing and weak vanishing conjectures to the setting of complexes rather than modules as described in Section 6. In particular, R satisfies vanishing or weak vanishing, respectively, exactly when α satisfies vanishing or weak vanishing, respectively, for all specialization-closed subsets $\mathfrak{X}$ of $\operatorname{Spec} R$ and all $\alpha \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$.

Likewise, if $\beta = [Y]$ for a complex $Y \in \mathsf{I}^{\mathrm{f}}(R)$ with $\operatorname{Supp} Y = \mathfrak{X}$, then β satisfies vanishing exactly when

$$\xi(X,Y) = 0 \quad \text{for all } X \in \mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}^c) \text{ with } \dim(\operatorname{Supp} X) + \dim(\operatorname{Supp} Y) < \dim R,$$

and β satisfies weak vanishing exactly when this is true under the extra assumption that $X \in \mathsf{P}^{\mathrm{f}}(\mathfrak{X}^c)$. This corresponds to an analog of Serre's vanishing and weak vanishing conjectures for complexes, using the Euler form instead of the intersection multiplicity. In particular, the discussion in Section 6 shows that R satisfies vanishing or weak vanishing, respectively, exactly when β satisfies vanishing or weak vanishing, respectively, for all specialization-closed subsets $\mathfrak{X}$ of $\operatorname{Spec} R$ and all $\beta \in \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$.

The *vanishing dimensions* of $\alpha \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ and $\beta \in \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ are defined as

$$\operatorname{vdim} \alpha = \inf \left\{ u \in \mathbb{Z} \,\middle|\, \begin{matrix} \alpha \otimes \delta = 0 \text{ for all } \delta \in \mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}^c) \\ \text{with } \dim \delta < \operatorname{codim} \mathfrak{X} - u \end{matrix} \right\}; \text{ and}$$
$$\operatorname{vdim} \beta = \inf \left\{ v \in \mathbb{Z} \,\middle|\, \begin{matrix} \operatorname{Hom}(\delta,\beta) = 0 \text{ for all } \delta \in \mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}^c) \\ \text{with } \dim \delta < \operatorname{codim} \mathfrak{X} - v \end{matrix} \right\}.$$

In particular, $\operatorname{vdim} \alpha = -\infty$ if and only if $\alpha = 0$, and $\operatorname{vdim} \beta = -\infty$ if and only if $\beta = 0$. Further, $\operatorname{vdim} \alpha \leq 0$ if and only if α satisfies vanishing, and $\operatorname{vdim} \beta \leq 0$ if and only if β satisfies vanishing.

If $\alpha = [X]$ for a complex $X \in \mathsf{P}^{\mathrm{f}}(R)$ with $\operatorname{Supp} X = \mathfrak{X}$, then the vanishing dimension of α measures the extent to which vanishing fails to hold: $\operatorname{vdim} \alpha$ is the infimum among integers u such that

$$\chi(X,Y) = 0 \quad \text{for all } Y \in \mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}^c) \text{ with } \dim(\operatorname{Supp} X) + \dim(\operatorname{Supp} Y) < \dim R - u.$$

Similarly, if $\beta = [Y]$ for a complex $Y \in \mathsf{I}^{\mathrm{f}}(R)$ with $\operatorname{Supp} Y = \mathfrak{X}$, then the vanishing dimension of β measures the extent to which vanishing fails to hold: $\operatorname{vdim} \beta$ is the infimum among integers v such that

$$\xi(X,Y) = 0 \quad \text{for all } X \in \mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}^c) \text{ with } \dim(\operatorname{Supp} X) + \dim(\operatorname{Supp} Y) < \dim R - v.$$

A notable feature of dimension and vanishing dimension is that they are invariant under the dagger, star and Foxby isomorphisms from Section 8. Indeed, for a specialization-closed subset $\mathfrak{X}$ of $\operatorname{Spec} R$ and elements $\alpha \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$, $\beta \in \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ and

$\gamma \in \mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})$, $\dim \gamma = \dim \gamma^{\dagger}$ and

$$\begin{aligned} \dim \alpha &= \dim \alpha^{\dagger} = \dim \alpha^{*} = \dim(D \otimes \alpha);\\ \dim \beta &= \dim \beta^{\dagger} = \dim \beta^{\star} = \dim \operatorname{Hom}(D, \beta);\\ \operatorname{vdim} \alpha &= \operatorname{vdim} \alpha^{\dagger} = \operatorname{vdim} \alpha^{*} = \operatorname{vdim}(D \otimes \alpha);\ \text{and}\\ \operatorname{vdim} \beta &= \operatorname{vdim} \beta^{\dagger} = \operatorname{vdim} \beta^{\star} = \operatorname{vdim} \operatorname{Hom}(D, \beta). \end{aligned}$$

See Article II, Remark 5.3 for further details. A consequence of the formulas is that the definitions of dimension and vanishing dimension are not contradictory when R is Gorenstein and $\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ is identical to $\mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X})$.

The following two propositions from Article I, Proposition 24 and Article II, Proposition 5.6 shed additional light on what it means to have a certain vanishing dimension.

Proposition 1. *Let $\mathfrak{X}$ be a specialization-closed subset of* $\operatorname{Spec} R$*, let $\alpha \in \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ and let u be a non-negative integer. The following are equivalent.*

- (i) $\operatorname{vdim} \alpha \leq u$.
- (ii) $\alpha \otimes \delta = 0$ *for all* $\delta \in \mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X}^{c})$ *with* $\dim \delta < \operatorname{codim} \mathfrak{X} - u$.
- (iii) $\overline{\alpha} = 0$ *in* $\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X}')$ *for any specialization-closed subset* $\mathfrak{X}'$ *of* $\operatorname{Spec} R$ *with* $\mathfrak{X} \subseteq \mathfrak{X}'$ *and* $\operatorname{codim} \mathfrak{X}' < \operatorname{codim} \mathfrak{X} - u$.
- (iv) $\overline{\alpha} = 0$ *in* $\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X}')$ *for any specialization-closed subset* $\mathfrak{X}'$ *of* $\operatorname{Spec} R$ *with* $\mathfrak{X} \subseteq \mathfrak{X}'$ *and* $\operatorname{codim} \mathfrak{X}' = \operatorname{codim} \mathfrak{X} - u - 1$.

Proposition 2. *Let $\mathfrak{X}$ be a specialization-closed subset of* $\operatorname{Spec} R$*, let $\beta \in \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X})$ and let v be a non-negative integer. The following are equivalent.*

- (i) $\operatorname{vdim} \beta \leq v$.
- (ii) $\operatorname{Hom}(\delta, \beta) = 0$ *for all* $\delta \in \mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X}^{c})$ *with* $\dim \delta < \operatorname{codim} \mathfrak{X} - v$.
- (iii) $\overline{\beta} = 0$ *in* $\mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X}')$ *for any specialization-closed subset* $\mathfrak{X}'$ *of* $\operatorname{Spec} R$ *with* $\mathfrak{X} \subseteq \mathfrak{X}'$ *and* $\operatorname{codim} \mathfrak{X}' < \operatorname{codim} \mathfrak{X} - v$.
- (iv) $\overline{\beta} = 0$ *in* $\mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X}')$ *for any specialization-closed subset* $\mathfrak{X}'$ *of* $\operatorname{Spec} R$ *with* $\mathfrak{X} \subseteq \mathfrak{X}'$ *and* $\operatorname{codim} \mathfrak{X}' = \operatorname{codim} \mathfrak{X} - v - 1$.

Foxby [**5**] has proven that Serre's vanishing conjecture holds when the module that is *not* necessarily of finite projective dimension has dimension less than or equal to one. The proof easily extends to complexes, and hence it follows that all elements of $\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ satisfy vanishing as long as $\operatorname{codim} \mathfrak{X} \leq 2$. Using Propositions 1 and 2, this implies that

$$\operatorname{vdim} \alpha \leq \max(0, \operatorname{codim} \mathfrak{X} - 2) \quad \text{and} \quad \operatorname{vdim} \beta \leq \max(0, \operatorname{codim} \mathfrak{X} - 2) \tag{5}$$

for all specialization-closed subsets $\mathfrak{X}$ of $\operatorname{Spec} R$, all $\alpha \in \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ and all $\beta \in \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X})$.

Assumption. The remainder of this section assumes, in addition, that R is complete of prime characteristic p and has perfect residue field.

The Dutta multiplicity and its two analogs from Section 6 correspond to limit points in $\mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{m})$ of the form

$$\lim_{e \to \infty} (\Phi^{e}_{\mathfrak{X}}(\alpha) \otimes \gamma), \quad \lim_{e \to \infty} \operatorname{Hom}(\gamma, \Psi^{e}_{\mathfrak{X}}(\beta)) \quad \text{and} \quad \lim_{e \to \infty} \operatorname{Hom}(\Phi^{e}_{\mathfrak{X}}(\alpha), \gamma)$$

for a specialization-closed subset $\mathfrak{X}$ of $\operatorname{Spec} R$ and elements $\alpha \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$, $\beta \in \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ and $\gamma \in \mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}^c)$. For example, if X and Y are finite complexes with $\mathfrak{X} = \operatorname{Supp} X$ and $\mathfrak{Y} = \operatorname{Supp} Y$ such that $\mathfrak{X} \cap \mathfrak{Y} = \{\mathfrak{m}\}$ and $\dim \mathfrak{X} + \dim \mathfrak{Y} \leq \dim R$, then, when $X \in \mathsf{P}^{\mathrm{f}}(R)$, the Dutta multiplicity

$$\chi_\infty(X, Y) = \lim_{e\to\infty} \frac{1}{p^{e \operatorname{codim} \mathfrak{X}}} \chi(\mathbf{L}F^e(X), Y) = \lim_{e\to\infty} \frac{1}{p^{e \operatorname{codim} \mathfrak{X}}} \chi(\mathbf{L}F^e(X) \otimes^{\mathbf{L}}_R Y)$$

is the image in $\mathbb{Q}$ under the isomorphism $\chi\colon \mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{m}) \to \mathbb{Q}$ of the element

$$\lim_{e\to\infty} \frac{1}{p^{e \operatorname{codim} \mathfrak{X}}} [\mathbf{L}F^e(X) \otimes^{\mathbf{L}}_R Y] = \lim_{e\to\infty} (\Phi^e_{\mathfrak{X}}([X]) \otimes [Y])$$

in $\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{m})$, where $[X] \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ and $[Y] \in \mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{Y})$. These limits converge in $\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{m})$ since the corresponding sequences in $\mathbb{Q}$ converge. In fact, Theorems 3 and 4 of Section 10 show that we can "move the limits inside" and instead write

$$(\lim_{e\to\infty} \Phi^e_{\mathfrak{X}}(\alpha)) \otimes \gamma, \quad \operatorname{Hom}(\gamma, \lim_{e\to\infty} \Psi^e_{\mathfrak{X}}(\beta)) \quad \text{and} \quad \operatorname{Hom}(\lim_{e\to\infty} \Phi^e_{\mathfrak{X}}(\alpha), \gamma).$$

10. Decompositions in Grothendieck spaces

Assumption. Throughout this section, R is assumed to be complete, Noetherian and local of prime characteristic p with maximal ideal $\mathfrak{m}$ and perfect residue field. Further, D denotes a normalized dualizing complex.

The following is the diagonalization theorem for the Frobenius from Article I, Theorem 19.

Theorem 3. *Let $\mathfrak{X}$ be a specialization-closed subset of $\operatorname{Spec} R$, let $\alpha \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ and suppose that u is a non-negative integer with $u \geq \operatorname{vdim} \alpha$. Then*

$$(p^u \Phi_{\mathfrak{X}} - \mathrm{id}) \circ \cdots \circ (p\Phi_{\mathfrak{X}} - \mathrm{id}) \circ (\Phi_{\mathfrak{X}} - \mathrm{id})(\alpha) = 0.$$

Further, there is a unique decomposition

$$\alpha = \alpha^{(0)} + \cdots + \alpha^{(u)}$$

in which each $\alpha^{(i)}$ is either zero or an eigenvector for $\Phi_{\mathfrak{X}}$ with eigenvalue p^{-i}. The components $\alpha^{(0)}, \ldots, \alpha^{(u)}$ are recursively defined by

$$\alpha^{(0)} = \lim_{e\to\infty} \Phi^e_{\mathfrak{X}}(\alpha) \quad \textit{and} \quad \alpha^{(i)} = \lim_{e\to\infty} p^{ie} \Phi^e_{\mathfrak{X}}(\alpha - (\alpha^{(0)} + \cdots + \alpha^{(i-1)})),$$

and there is a formula

$$\begin{pmatrix} \alpha^{(0)} \\ \vdots \\ \alpha^{(u)} \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & p^{-1} & \cdots & p^{-u} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & p^{-u} & \cdots & p^{-u^2} \end{pmatrix}^{-1} \begin{pmatrix} \alpha \\ \Phi_{\mathfrak{X}}(\alpha) \\ \vdots \\ \Phi^u_{\mathfrak{X}}(\alpha) \end{pmatrix}.$$

The following theorem from Article II, Theorem 6.2 presents a diagonalization theorem for the analogous Frobenius functor defined by Herzog.

Theorem 4. *Let $\mathfrak{X}$ be a specialization-closed subset of $\operatorname{Spec} R$, let $\beta \in \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ and suppose that v is a non-negative integer with $v \geq \operatorname{vdim} \beta$. Then*

$$(p^v \Psi_{\mathfrak{X}} - \mathrm{id}) \circ \cdots \circ (p\Psi_{\mathfrak{X}} - \mathrm{id}) \circ (\Psi_{\mathfrak{X}} - \mathrm{id})(\beta) = 0.$$

Further, there is a unique decomposition

$$\beta = \beta^{(0)} + \cdots + \beta^{(v)}$$

in which each $\beta^{(i)}$ is either zero or an eigenvector for $\Psi_{\mathfrak{X}}$ with eigenvalue p^{-i}. The components $\beta^{(0)}, \dots, \beta^{(v)}$ are recursively defined by

$$\beta^{(0)} = \lim_{e\to\infty} \Psi^e_{\mathfrak{X}}(\beta) \quad \textit{and} \quad \beta^{(i)} = \lim_{e\to\infty} p^{ie}\Psi^e_{\mathfrak{X}}(\beta - (\beta^{(0)} + \cdots + \beta^{(i-1)})),$$

and there is a formula

$$\begin{pmatrix} \beta^{(0)} \\ \vdots \\ \beta^{(v)} \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & p^{-1} & \cdots & p^{-v} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & p^{-v} & \cdots & p^{-v^2} \end{pmatrix}^{-1} \begin{pmatrix} \beta \\ \Psi_{\mathfrak{X}}(\beta) \\ \vdots \\ \Psi^v_{\mathfrak{X}}(\beta) \end{pmatrix}.$$

The i'th component operation $(-)^{(i)}$ on $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ and $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$, respectively, is an endomorphism of Grothendieck spaces. Let $\mathfrak{X} \subseteq \mathfrak{X}'$ be specialization-closed subsets of $\operatorname{Spec} R$ and set $s = \operatorname{codim}\mathfrak{X} - \operatorname{codim}\mathfrak{X}'$. For $\alpha \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$, $\beta \in \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ and all integers i (see Article I, Remark 20 and Article II, Remark 6.6),

$$\overline{\alpha^{(i)}} = \overline{\alpha}^{(i-s)} \text{ in } \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X}') \quad \text{and} \quad \overline{\beta^{(i)}} = \overline{\beta}^{(i-s)} \text{ in } \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X}'),$$

where we apply the convention that components of negative degree are zero.

The dagger, star and Foxby isomorphisms between $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ and $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ commute with the i'th component endomorphisms in that (see Article II, Propositions 6.3 and 7.3)

$$(\alpha^{\dagger})^{(i)} = \alpha^{(i)\dagger}; \qquad (\beta^{\dagger})^{(i)} = \beta^{(i)\dagger};$$
$$(\alpha^{*})^{(i)} = \alpha^{(i)*}; \qquad (\beta^{\star})^{(i)} = \beta^{(i)\star};$$
$$(D\otimes\alpha)^{(i)} = D\otimes\alpha^{(i)}; \quad \text{and} \quad \operatorname{Hom}(D,\beta)^{(i)} = \operatorname{Hom}(D,\beta^{(i)}).$$

The star automorphisms on $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ and $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ can be explicitly described in terms of i'th components; see Theorem 12 in Section 11.

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be specialization-closed subsets of $\operatorname{Spec} R$ such that $\mathfrak{X}\cap\mathfrak{Y} = \{\mathfrak{m}\}$ and $\dim\mathfrak{X} + \dim\mathfrak{Y} \leq \dim R$. Set $s = \dim R - (\dim\mathfrak{X} + \dim\mathfrak{Y})$. If (δ,ε) is a pair of elements from

$$\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X}) \times \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{Y}), \quad \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X}) \times \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{Y}) \quad \text{or} \quad \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X}) \times \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{Y}),$$

then $\delta\otimes\varepsilon$ is a well-defined element of $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{m})$ or $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{m})$, and

$$(\delta\otimes\varepsilon)^{(i)} = \sum_{m+n=i+s} \delta^{(m)}\otimes\varepsilon^{(n)}.$$

If instead (δ,ε) is a pair of elements from

$$\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X}) \times \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{Y}), \quad \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X}) \times \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{Y}) \quad \text{or} \quad \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X}) \times \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{Y}),$$

then $\operatorname{Hom}(\delta,\varepsilon)$ is a well-defined element of $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{m})$ or $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{m})$, and

$$\operatorname{Hom}(\delta,\varepsilon)^{(i)} = \sum_{m+n=i+s} \operatorname{Hom}(\delta^{(m)},\varepsilon^{(n)}).$$

See Article II, Proposition 6.9 for further details.

Let $\mathfrak{X}$ be a specialization-closed subset of $\operatorname{Spec} R$ and let $\alpha \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$, $\beta \in \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ and $\gamma \in \mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}^c)$. Using continuity of the maps $-\otimes\gamma$, $\operatorname{Hom}(\gamma,-)$ and

$\operatorname{Hom}(-,\gamma)$, Theorems 3 and 4 together with previous remarks show that the Dutta multiplicity and its two analogs can be described by elements in $\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{m})$ of the form

$$\lim_{e\to\infty}(\Phi^e_{\mathfrak{X}}(\alpha)\otimes\gamma) = (\lim_{e\to\infty}\Phi^e_{\mathfrak{X}}(\alpha))\otimes\gamma = \alpha^{(0)}\otimes\gamma;$$
$$\lim_{e\to\infty}\operatorname{Hom}(\gamma,\Psi^e_{\mathfrak{X}}(\beta)) = \operatorname{Hom}(\gamma,\lim_{e\to\infty}\Psi^e_{\mathfrak{X}}(\beta)) = \operatorname{Hom}(\gamma,\beta^{(0)});\ \text{and}$$
$$\lim_{e\to\infty}\operatorname{Hom}(\Phi^e_{\mathfrak{X}}(\alpha),\gamma) = \operatorname{Hom}(\lim_{e\to\infty}\Phi^e_{\mathfrak{X}}(\alpha),\gamma) = \operatorname{Hom}(\alpha^{(0)},\gamma).$$

Translating this back to complexes and exploiting the formulas in Theorems 3 and 4 reveals the following proposition from Article I, Remark 21 and Article II, Corollary 6.5, showing that the Dutta multiplicity and its analogs can be described as $\mathbb{Q}$-linear combinations of ordinary intersection multiplicities or Euler forms.

Corollary 5. *Let X and Y be finite complexes. Set $\mathfrak{X} = \operatorname{Supp} X$ and $\mathfrak{Y} = \operatorname{Supp} Y$ and assume that $\mathfrak{X}\cap\mathfrak{Y} = \{\mathfrak{m}\}$ and $\dim\mathfrak{X}+\dim\mathfrak{Y}\le\dim R$. Further, set $t = \operatorname{codim}\mathfrak{X}$ and $s = \operatorname{codim}\mathfrak{Y}$. If $X\in\mathsf{P}^{\mathrm{f}}(R)$ and u is an integer with $u\ge\operatorname{vdim}[X]$ for $[X]\in\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$, then*

$$\chi_\infty(X,Y) = \begin{pmatrix}1 & 0 & \cdots & 0\end{pmatrix}\begin{pmatrix}1 & 1 & \cdots & 1\\ p^t & p^{t-1} & \cdots & p^{t-u}\\ \vdots & \vdots & \ddots & \vdots\\ p^{ut} & p^{u(t-1)} & \cdots & p^{u(t-u)}\end{pmatrix}^{-1}\begin{pmatrix}\chi(X,Y)\\ \chi(\mathbf{L}F(X),Y)\\ \vdots\\ \chi(\mathbf{L}F^u(X),Y)\end{pmatrix}.$$

Similarly, under the same assumptions,

$$\xi^\infty(X,Y) = \begin{pmatrix}1 & 0 & \cdots & 0\end{pmatrix}\begin{pmatrix}1 & 1 & \cdots & 1\\ p^t & p^{t-1} & \cdots & p^{t-u}\\ \vdots & \vdots & \ddots & \vdots\\ p^{ut} & p^{u(t-1)} & \cdots & p^{u(t-u)}\end{pmatrix}^{-1}\begin{pmatrix}\xi(X,Y)\\ \xi(\mathbf{L}F(X),Y)\\ \vdots\\ \xi(\mathbf{L}F^u(X),Y)\end{pmatrix}.$$

If instead $Y\in\mathsf{I}^{\mathrm{f}}(R)$ and v is an integer with $v\ge\operatorname{vdim}[Y]$ for $[Y]\in\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{Y})$, then

$$\xi_\infty(X,Y) = \begin{pmatrix}1 & 0 & \cdots & 0\end{pmatrix}\begin{pmatrix}1 & 1 & \cdots & 1\\ p^s & p^{s-1} & \cdots & p^{s-v}\\ \vdots & \vdots & \ddots & \vdots\\ p^{vs} & p^{v(s-1)} & \cdots & p^{v(s-v)}\end{pmatrix}^{-1}\begin{pmatrix}\xi(X,Y)\\ \xi(X,\mathbf{R}G(Y))\\ \vdots\\ \xi(X,\mathbf{R}G^v(Y))\end{pmatrix}.$$

With the assumption of prime characteristic sustained in this section, Propositions 1 and 2 of Section 9 can be extended by additional equivalent conditions. These are contained in Propositions 6 and 7 below from Article I, Propositions 23 and 24 and Article II, Propositions 6.7 and 6.8.

Proposition 6. *Let $\mathfrak{X}$ be a specialization-closed subset of $\operatorname{Spec} R$, let $\alpha\in\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ and let u be a non-negative integer. The following are equivalent.*

(*i*) $\operatorname{vdim}\alpha\le 0$ *(that is, α satisfies vanishing).*
(*ii*) $\alpha = \alpha^{(0)}$.
(*iii*) $\alpha = \Phi_{\mathfrak{X}}(\alpha)$.
(*iv*) $\alpha = \Phi^e_{\mathfrak{X}}(\alpha)$ *for some $e\in\mathbb{N}$.*
(*v*) $\alpha = \lim_{e\to\infty}\Phi^e_{\mathfrak{X}}(\alpha)$.

Further, the following are equivalent.

(*vi*) $\operatorname{vdim} \alpha \leq u$.
(*vii*) $\alpha = \alpha^{(0)} + \cdots + \alpha^{(u)}$.
(*viii*) $(p^u \Phi_{\mathfrak{X}} - \mathrm{id}) \circ \cdots \circ (p\Phi_{\mathfrak{X}} - \mathrm{id}) \circ (\Phi_{\mathfrak{X}} - \mathrm{id})(\alpha) = 0$.

Proposition 7. *Let $\mathfrak{X}$ be a specialization-closed subset of* $\operatorname{Spec} R$*, let $\beta \in \mathbb{GI}^{\mathsf{f}}(\mathfrak{X})$ and let v be a non-negative integer. The following are equivalent.*

(*i*) $\operatorname{vdim} \beta \leq 0$ *(that is, β satisfies vanishing).*
(*ii*) $\beta = \beta^{(0)}$.
(*iii*) $\beta = \Psi_{\mathfrak{X}}(\beta)$.
(*iv*) $\beta = \Psi_{\mathfrak{X}}^{e}(\beta)$ *for some* $e \in \mathbb{N}$.
(*v*) $\beta = \lim_{e\to\infty} \Psi_{\mathfrak{X}}^{e}(\beta)$.

Further, the following are equivalent.

(*vi*) $\operatorname{vdim} \beta \leq v$.
(*vii*) $\beta = \beta^{(0)} + \cdots + \beta^{(v)}$.
(*viii*) $(p^v \Psi_{\mathfrak{X}} - \mathrm{id}) \circ \cdots \circ (p\Psi_{\mathfrak{X}} - \mathrm{id}) \circ (\Psi_{\mathfrak{X}} - \mathrm{id})(\beta) = 0$.

Let $\mathfrak{X}$ be a specialization-closed subset of $\operatorname{Spec} R$ and let $\alpha \in \mathbb{GP}^{\mathsf{f}}(\mathfrak{X})$ and $\beta \in \mathbb{GI}^{\mathsf{f}}(\mathfrak{X})$. Propositions 6 and 7 state that α and β satisfy vanishing exactly when they are equal to their zeroth components; in particular, this implies that the Dutta multiplicity and its analogs satisfy vanishing.

We say that α *satisfies numerical vanishing* if $\overline{\alpha} = \overline{\alpha^{(0)}}$ in $\mathbb{GD}^{\mathsf{f}}_{\square}(\mathfrak{X})$, and we say that β *satisfies numerical vanishing* if $\overline{\beta} = \overline{\beta^{(0)}}$ in $\mathbb{GD}^{\mathsf{f}}_{\square}(\mathfrak{X})$. Further, we say that R *satisfies numerical vanishing* if all elements α in $\mathbb{GP}^{\mathsf{f}}(\mathfrak{X})$ for all specialization-closed subsets $\mathfrak{X}$ of $\operatorname{Spec} R$ satisfy numerical vanishing; this is equivalent to requiring all elements β in $\mathbb{GI}^{\mathsf{f}}(\mathfrak{X})$ for all specialization-closed subsets $\mathfrak{X}$ of $\operatorname{Spec} R$ to satisfy numerical vanishing. When R is Gorenstein and we identify $\mathbb{GP}^{\mathsf{f}}(\mathfrak{X})$ and $\mathbb{GI}^{\mathsf{f}}(\mathfrak{X})$, there are, as always, no contradictions in these definitions.

Proposition 8 below combines Article I, Proposition 31 and Article II, Proposition 6.12 and sheds a little more light on what it means for R to satisfy numerical vanishing.

Proposition 8. *The following are equivalent.*

(*i*) *R satisfies numerical vanishing.*
(*ii*) *All elements of $\mathbb{GP}^{\mathsf{f}}(\mathfrak{X})$ satisfy numerical vanishing for all specialization-closed subsets $\mathfrak{X}$ of* $\operatorname{Spec} R$.
(*iii*) *All elements of $\mathbb{GP}^{\mathsf{f}}(\mathfrak{m})$ satisfy numerical vanishing.*
(*iv*) *All elements of $\mathbb{GI}^{\mathsf{f}}(\mathfrak{X})$ satisfy numerical vanishing for all specialization-closed subsets $\mathfrak{X}$ of* $\operatorname{Spec} R$.
(*v*) *All elements of $\mathbb{GI}^{\mathsf{f}}(\mathfrak{m})$ satisfy numerical vanishing.*
(*vi*) $\chi(\mathbf{L}F(X)) = p^{\dim R}\chi(X)$ *for all complexes X in $\mathsf{P}^{\mathsf{f}}(\mathfrak{m})$.*
(*vii*) $\lim_{e\to\infty} p^{-e\dim R}\chi(\mathbf{L}F^{e}(X)) = \chi(X)$ *for all complexes X in $\mathsf{P}^{\mathsf{f}}(\mathfrak{m})$.*
(*viii*) $\chi(\mathbf{R}G(Y)) = p^{\dim R}\chi(Y)$ *for all complexes Y in $\mathsf{I}^{\mathsf{f}}(\mathfrak{m})$.*
(*ix*) $\lim_{e\to\infty} p^{-e\dim R}\chi(\mathbf{R}G^{e}(Y)) = \chi(Y)$ *for all complexes Y in $\mathsf{I}^{\mathsf{f}}(\mathfrak{m})$.*

When R is Cohen–Macaulay, $\mathbb{GP}^{\mathsf{f}}(\mathfrak{m})$ and $\mathbb{GI}^{\mathsf{f}}(\mathfrak{m})$ are generated by modules, and hence the above conditions are equivalent to each of the following conditions.

(*x*) $\operatorname{length} F(M) = p^{\dim R} \operatorname{length} M$ *for all modules* M *of finite length and finite projective dimension.*
(*xi*) $\lim_{e\to\infty} p^{-e\dim R} \operatorname{length} F^e(M) = \operatorname{length} M$ *for all modules* M *of finite length and finite projective dimension.*
(*xii*) $\operatorname{length} G(N) = p^{\dim R} \operatorname{length} N$ *for all modules* N *of finite length and finite injective dimension.*
(*xiii*) $\lim_{e\to\infty} p^{-e\dim R} \operatorname{length} G^e(N) = \operatorname{length} N$ *for all modules* N *of finite length and finite injective dimension.*

In particular, if R *is a complete intersection or* R *is Gorenstein of dimension less than or equal to three, then* R *satisfies numerical vanishing.*

Numerical vanishing is clearly a weaker condition than vanishing, and demonstrating that numerical vanishing is a stronger condition than weak vanishing is easy; see Proposition 13 of Section 12.

11. Self-duality

Assumption. Throughout this section, R is assumed to be Noetherian and local with maximal ideal $\mathfrak{m}$.

Let $\mathfrak{X}$ be a specialization-closed subset of $\operatorname{Spec} R$. If an element $\alpha \in \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ satisfies the formula
$$\alpha = (-1)^{\operatorname{codim}\mathfrak{X}} \alpha^*,$$
we say that α is *self-dual*. If the formula holds after the inclusion homomorphism $\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X}) \to \mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})$ is applied so that $\overline{\alpha} = (-1)^{\operatorname{codim}\mathfrak{X}}\overline{\alpha^*}$ in $\mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})$, we say that α is *numerically self-dual*. Similarly, if an element $\beta \in \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X})$ satisfies the formula
$$\beta = (-1)^{\operatorname{codim}\mathfrak{X}} \beta^\star,$$
we say that β is *self-dual*, and if the formula holds after the inclusion homomorphism $\mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X}) \to \mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})$ is applied so that $\overline{\beta} = (-1)^{\operatorname{codim}\mathfrak{X}}\overline{\beta^\star}$ in $\mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})$, we say that β is *numerically self-dual*.

If all elements $\alpha \in \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ for all specialization-closed subsets $\mathfrak{X}$ of $\operatorname{Spec} R$ are self-dual or numerically self-dual, respectively, we say that R *satisfies self-duality* or *numerical self-duality*, respectively; this is equivalent to requiring the same conditions for all elements $\beta \in \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X})$ for all specialization-closed subsets $\mathfrak{X}$ of $\operatorname{Spec} R$. Once again, these definitions are not contradictory when R is Gorenstein and $\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ is identified with $\mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X})$.

The following theorem from Article II, Proposition 7.4 shows that the conditions of satisfying vanishing, weak vanishing, self-duality and numerical self-duality are correlated; see also Theorem 13 from Section 12.

Theorem 9. *Let* $\mathfrak{X}$ *be a specialization-closed subset of* $\operatorname{Spec} R$ *and let* $\alpha \in \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ *and* $\beta \in \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X})$. *If* α *satisfies vanishing, then* α *is self-dual, and if* β *satisfies vanishing, then* β *is self-dual. Further,* R *satisfies vanishing if and only if* R *satisfies self-duality, and if* R *satisfies numerical self-duality, then* R *satisfies weak vanishing.*

Theorem 12 contains a consequence of this theorem in prime characteristic. The following proposition from Article II, Proposition 7.9 contributes to the discussion initiated by Chan [**1**] concerning the connection between the vanishing conjecture and an equation relating the intersection multiplicity and the Euler form.

Proposition 10. *The ring R satisfies vanishing if and only if*
$$\alpha \otimes \gamma = (-1)^{\operatorname{codim} \mathfrak{X}} \operatorname{Hom}(\alpha, \gamma)$$
in $\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{m})$ for all specialization-closed subsets $\mathfrak{X}$ of $\operatorname{Spec} R$, all $\alpha \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ and all $\gamma \in \mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}^c)$, and R satisfies numerical self-duality if and only if the equality holds when requiring $\gamma \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X}^c)$ instead. In other words, R satisfies vanishing if and only if
$$\chi(X, Y) = (-1)^{\operatorname{codim}(\operatorname{Supp} X)} \xi(X, Y)$$
for complexes $X \in \mathsf{P}^{\mathrm{f}}(R)$ and $Y \in \mathsf{D}^{\mathrm{f}}_{\square}(R)$ with $\operatorname{Supp} X \cap \operatorname{Supp} Y = \{\mathfrak{m}\}$ and $\dim(\operatorname{Supp} X) + \dim(\operatorname{Supp} Y) \leq \dim R$, and R satisfies numerical self-duality if and only if the equality holds when restricting to complexes $Y \in \mathsf{P}^{\mathrm{f}}(R)$.

Article II, Proposition 7.11 contains the following result.

Proposition 11. *Assume that R is Gorenstein. Then all elements of $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ and $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ are numerically self-dual for all specialization-closed subsets $\mathfrak{X}$ of R with $\dim \leq 2$; in particular, all elements of $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{m})$ and $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{m})$ are numerically self-dual. If $\dim R \leq 5$, then R satisfies numerical self-duality.*

Assumption. The remainder of this section assumes, in addition, that R is complete of prime characteristic p and has perfect residue field.

A consequence of Theorem 9 in prime characteristic is the following theorem from Article II, Theorem 7.5.

Theorem 12. *Let $\mathfrak{X}$ be a specialization-closed subset of $\operatorname{Spec} R$, let α be an element of $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ with $\operatorname{vdim} \alpha = u$ and let β be an element of $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ with $\operatorname{vdim} \beta = v$. Then*
$$(-1)^{\operatorname{codim} \mathfrak{X}} \alpha^* = \alpha^{(0)} - \alpha^{(1)} + \alpha^{(2)} - \cdots + (-1)^u \alpha^{(u)}$$
and
$$(-1)^{\operatorname{codim} \mathfrak{X}} \beta^\star = \beta^{(0)} - \beta^{(1)} + \beta^{(2)} - \cdots + (-1)^v \beta^{(v)}.$$
If α satisfies numerical vanishing, then α is numerically self-dual, and if β satisfies numerical vanishing, then β is numerically self-dual. In particular, if R satisfies numerical vanishing, then R satisfies numerical self-duality.

Theorem 12 shows that an element $\alpha \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ is self-dual exactly when α decomposes as
$$\alpha = \alpha^{(0)} + \alpha^{(2)} + \alpha^{(4)} + \cdots ,$$
and α is numerically self-dual when the equality above holds numerically: that is, when
$$\overline{\alpha} = \overline{\alpha^{(0)}} + \overline{\alpha^{(2)}} + \overline{\alpha^{(4)}} + \cdots$$
in $\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$. Similar remarks hold for elements $\beta \in \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$.

12. Implications

This section discusses the relationships between well-known ring properties and the concepts introduced here.

Assumption. Throughout this section, R is assumed to be Noetherian and local with maximal ideal $\mathfrak{m}$.

The following proposition compiles the results from Proposition 8, Theorem 9, Proposition 11 and Theorem 12.

Proposition 13. *Consider the following implications of properties for the ring R.*

$$
\begin{tikzcd}
 & \textit{self-duality} \arrow[d, Leftrightarrow] & \\
\textit{regular} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \textit{vanishing} \arrow[d, Rightarrow] & \dim \leq 2 \arrow[l, Rightarrow] \\
\textit{complete intersection} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \textit{numerical vanishing} \arrow[d, Rightarrow] & \begin{array}{l}\textit{Gorenstein}\\ \textit{of } \dim \leq 3\end{array} \arrow[l, Rightarrow] \\
\textit{Gorenstein} \arrow[d, Rightarrow] & \textit{numerical self-duality} \arrow[d, Rightarrow] & \begin{array}{l}\textit{Gorenstein}\\ \textit{of } \dim \leq 5\end{array} \arrow[l, Rightarrow] \\
\textit{Cohen–Macaulay} & \textit{weak vanishing} & \dim \leq 4 \arrow[l, Rightarrow]
\end{tikzcd}
$$

All implications hold when R is complete of prime characteristic p with perfect residue field. When this is not the case, the dashed part of the diagram no longer makes sense and can be removed, and the remaining implications still hold, including the implication from "vanishing" and "complete intersection" to "numerical self-duality". All implications except for the single bi-implication are strict.

None of the references preceding Proposition 13 actually contain the fact that the implication from "complete intersection" to "numerical self-duality" holds in arbitrary characteristic, but it is not hard to prove using local Chern characters; the proof for modules by Chan [**1**, Theorem 4] easily extends to complexes. Section 13 briefly discusses, among other things, how to make a decomposition (and hence how to define the concept of numerical vanishing) in arbitrary characteristic.

Looking at the diagram above, it is tempting to ask whether there is an implication from "Gorenstein" to "numerical self-duality". Proposition 11 shows that an implication from "Gorenstein" to "numerical self-duality" would follow if only a sufficient condition for numerical self-duality to hold globally is that it holds in $\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{m})$ or $\mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{m})$, just as numerical vanishing holds globally if only it holds in $\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{m})$ or $\mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{m})$, as seen in Proposition 8.

13. Comparison with local Chern characters

Many of the properties of Grothendieck spaces seem to resemble properties of local Chern characters. This section briefly describes this relationship without delving into the theory of local Chern characters; for more details on local Chern characters, see Roberts [**15**]. The reader should be warned that this section has no precise results—the statements here are a matter of opinion rather than based on proof.

Assumption. Throughout this section, R is assumed to be complete, Noetherian and local of prime characteristic p with maximal ideal $\mathfrak{m}$ and perfect residue field.

Let X be complex in $\mathsf{P}^{\mathrm{f}}(R)$ and set $\operatorname{Supp} X = \mathfrak{X}$ and $d = \dim R$. The *local Chern character* of X is an object $\operatorname{ch}(X)$ (whose definition is not discussed further here) with a decomposition

$$\operatorname{ch}(X) = \operatorname{ch}_d(X) + \cdots + \operatorname{ch}_0(X).$$

In the same way, we can think of the element $[X] \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ as an object with a decomposition

$$[X] = [X]^{(0)} + \cdots + [X]^{(d)}.$$

There seems to be some sort of connection

$$\operatorname{ch}_{\operatorname{codim}\mathfrak{X}-i}(X) \longleftrightarrow [X]^{(i)}.$$

For example, in the case where $\operatorname{codim}\mathfrak{X} = 0$, $\operatorname{ch}_0(X) \neq 0$ if and only if the alternating sum of Betti numbers of X is non-zero (see Roberts [**16**, pp. 6–7]), which happens if and only if $[X]^{(0)} \neq 0$. Likewise, in the case where $\operatorname{codim}\mathfrak{X} = d$, $\operatorname{ch}_d(X)$ can be used to describe the asymptotic Euler characteristic of X (see Roberts [**15**, Theorem 12.7.1]): that is, $\operatorname{ch}_d(X)$ can be used to describe the number $\chi_\infty(X) = \lim_{e\to\infty} p^{-ed}\chi(F^e(X))$, which is simply the usual (induced) Euler characteristic applied to $[X]^{(0)}$.

The similarities between local Chern characters and elements of Grothendieck spaces are listed below. Let X, X' and X'' be complexes in $\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ such that there is a short exact sequence (in the non-derived category)

$$0 \to X' \to X \to X'' \to 0.$$

Further, let Y be a complex in $\mathsf{P}^{\mathrm{f}}(\mathfrak{X}^c)$. We then have the following equations of local Chern characters and elements of Grothendieck spaces.

Local Chern characters	Elements of Grothendieck spaces
$\operatorname{ch}(X) = \operatorname{ch}(X') + \operatorname{ch}(X'')$	$[X] = [X'] + [X'']$
$\operatorname{ch}(X \otimes^{\mathbf{L}}_R Y) = \operatorname{ch}(X)\operatorname{ch}(Y)$	$[X \otimes^{\mathbf{L}}_R Y] = [X] \otimes [Y]$
$\operatorname{ch}_i(X)\operatorname{ch}_j(Y) = \operatorname{ch}_j(X)\operatorname{ch}_i(Y)$	$[X]^{(i)} \otimes [Y]^{(j)} = [Y]^{(j)} \otimes [X]^{(i)}$
$\operatorname{ch}_i(X^*) = (-1)^i \operatorname{ch}_i(X)$	$[X^*]^{(i)} = (-1)^{i+\operatorname{codim}\mathfrak{X}}[X]^{(i)}$

An element in the Grothendieck space $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ of the form $[X]$ for a complex $X \in \mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ is uniquely determined by the intersection multiplicities $\chi(X,Y)$ of X with complexes $Y \in \mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}^c)$. The image $\overline{[X]}$ in $\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$ of $[X]$ under the inclusion homomorphism $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X}) \to \mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$ is uniquely determined by the intersection multiplicities $\chi(X,Y)$ of X with complexes $Y \in \mathsf{P}^{\mathrm{f}}(\mathfrak{X}^c)$. Thus, numerical properties, which are properties that hold after the inclusion homomorphism $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X}) \to \mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$ is applied, are verified by investigating a restricted set of intersection multiplicities.

Numerical properties seem to be encoded in the *Todd class* $\tau(R)$ of R, which is an element on which local Chern characters can operate. In other words, we are investigating a restricted set of operations of local Chern characters: the operations of local Chern characters on $\tau(R)$. For example, if R is a complete intersection, then the Todd class $\tau(R)$ of R is equal to its component $\tau_d(R)$ in degree $d = \dim R$; this is reflected in the fact that complete intersection rings satisfy numerical vanishing, so that $[X]$ and $[X]^{(0)}$ are numerically the same. Further, if R is Gorenstein, then $\tau_{d-i}(R) = 0$ if i is odd; this is reflected in the fact that Gorenstein rings satisfy numerical self-duality for all elements of $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{m})$, so that $[X]^{(i)}$ is numerically zero if i is odd whenever $[X] \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{m})$.

Of course, Grothendieck spaces with their single built-in application to the study of intersection multiplicities do not even come close to local Chern characters, with their many applications to all kinds of questions in commutative algebra. Even in studying intersection multiplicities, local Chern characters have one clear advantage over Grothendieck spaces: local Chern characters have nice properties in any characteristic, whereas the pleasant structure of Grothendieck spaces only emerges in prime characteristic p. A natural question to ask is whether Grothendieck spaces have an equally pleasant structure in arbitrary characteristic; here we discuss a special case where such a structure can be obtained.

Let $\mathfrak{X}$ be a specialization-closed subset of $\operatorname{Spec} R$ and let $\alpha \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$. The aim is to find a decomposition of α that also works in arbitrary characteristic. We show how this can be done when $\operatorname{vdim} \alpha \leq 1$. In this case, there is a decomposition

$$\alpha = \alpha^{(0)} + \alpha^{(1)}.$$

From Theorem 12 follows that

$$\alpha^{(0)} = \frac{1}{2}(\alpha + (-1)^{\operatorname{codim} \mathfrak{X}}\alpha^*) \quad \text{and} \quad \alpha^{(1)} = \frac{1}{2}(\alpha - (-1)^{\operatorname{codim} \mathfrak{X}}\alpha^*). \tag{6}$$

The right sides of these formulas make sense in general characteristic too, and hence we can simply use the above to *define* a decomposition of α. To demonstrate that this decomposition is natural, even in general characteristic, let $\mathfrak{X}'$ be a specialization-closed subset of $\operatorname{Spec} R$ with $\mathfrak{X} \subseteq \mathfrak{X}'$ and $\operatorname{codim} \mathfrak{X}' = \operatorname{codim} \mathfrak{X} - 1$, and consider the image $\overline{\alpha}$ of α in $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X}')$. Since the vanishing dimension of α is less than or equal to one, $\overline{\alpha}$ satisfies vanishing; this is a consequence of Proposition 1. In particular, according to Theorem 9, $\overline{\alpha}$ is self-dual, and hence the image of $\alpha^{(0)}$ in $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X}')$ is

$$\overline{\alpha^{(0)}} = \frac{1}{2}(\overline{\alpha} + (-1)^{\operatorname{codim} \mathfrak{X}}\overline{\alpha}^*) = \frac{1}{2}(\overline{\alpha} - (-1)^{\operatorname{codim} \mathfrak{X}'}\overline{\alpha}^*) = 0.$$

According to Proposition 1, this shows that $\alpha^{(0)}$ satisfies vanishing, even in general characteristic, just as one would expect of a zeroth component.

Further investigation is required to determine whether a decomposition can be obtained when α has vanishing dimension higher than one.

14. Examples

This section discusses three examples of rings and modules of interest when discussing intersection multiplicities. These are the example by Dutta, Hochster and McLaughlin [**4**] and two examples by Miller and Singh [**10**]. These examples are constructed to allow the assumption that the ring in play is complete of prime characteristic p and has perfect residue field: we can simply *choose* the residue field to be perfect of prime characteristic and then consider the completion of the ring, which does not change intersection multiplicities.

Example 14 (the example by Dutta, Hochster and McLaughlin)**.** In this example, we have the Noetherian, local ring

$$R = k[u, v, x, y]_{\mathfrak{m}}/(ux - vy),$$

where k is a field and $\mathfrak{m}$ is the maximal ideal (u, v, x, y). This ring is a complete intersection of dimension 3. We also have a module M of length 15 with finite projective dimension and a module $N = R/\mathfrak{q}$ of dimension 2, where $\mathfrak{q}$ is the prime ideal generated by u and v. The computations performed by Dutta, Hochster and McLaughlin show that

$$\chi(M, N) = -1.$$

Since R has dimension 3, all elements of $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ satisfy vanishing except when $\operatorname{codim}\mathfrak{X} = 3$, which is when $\mathfrak{X} = \{\mathfrak{m}\}$. A result for Grothendieck groups by Levine [**8**, Theorem 4.2] implies that $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{m}) = \mathbb{Q} \oplus \mathbb{Q}$; see Article I, Example 35, for a few more details. When R is complete of prime characteristic p and has perfect residue field, $[M] \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{m})$ decomposes as

$$[M] = [M]^{(0)} + [M]^{(1)},$$

where, using the formula in Theorem 3,

$$\begin{aligned} [M]^{(0)} &= \frac{-1}{p-1}[M] + \frac{1}{p^2(p-1)}[F(M)]; \quad \text{and} \\ [M]^{(1)} &= \frac{p}{p-1}[M] - \frac{1}{p^2(p-1)}[F(M)]. \end{aligned}$$

Since R is a complete intersection, R satisfies numerical vanishing, so

$$\chi([M]^{(0)}) = \chi(M) = \operatorname{length} M = 15 \quad \text{and} \quad \chi([M]^{(1)}) = 0.$$

In particular, $[M]^{(0)}$ is non-zero. Since $[M]^{(0)}$ satisfies vanishing,

$$\chi([M]^{(0)}, N) = 0 \quad \text{and} \quad \chi([M]^{(1)}, N) = \chi(M, N) = -1,$$

so $[M]^{(1)}$ must also be non-zero. Thus, $[M]^{(0)}$ and $[M]^{(1)}$ are linearly independent and generate $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{m})$. These formulas imply that

$$\operatorname{length} F^e(M) = 15p^{3e} \quad \text{and} \quad \chi(F^e(M), N) = -p^{2e}$$

for all non-negative integers e. Since R is Cohen–Macaulay, the complex $\Sigma^3 M^*$ in $\mathsf{P}^{\mathrm{f}}(\mathfrak{m})$ is isomorphic to a module

$$M^\vee = \operatorname{Ext}^3_R(M, R),$$

and Theorem 12 yields that the element $[M^\vee] \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{m})$ decomposes as

$$[M^\vee] = [M]^{(0)} - [M]^{(1)}.$$

As discussed at the end of Section 13, we can also write

$$[M]^{(0)} = \frac{1}{2}([M] + [M^\vee]) \quad \text{and} \quad [M]^{(1)} = \frac{1}{2}([M] - [M^\vee]),$$

which makes sense in general characteristic too.

Example 15 (the first example by Miller and Singh)**.** In this example, we have the Noetherian, local ring

$$R = k[u, v, w, x, y, z]_{\mathfrak{m}}/(ux + vy + wz),$$

where k is a field and $\mathfrak{m}$ is the maximal ideal (u, v, w, x, y, z). This ring is a complete intersection of dimension 5. We also have a module M of length 55 with finite projective dimension and a module $N = R/\mathfrak{q}$ of dimension 3, where $\mathfrak{q}$ is the prime ideal generated by u, v and w. The computations performed by Miller and Singh show that

$$\chi(M,N) = -2.$$

Again, the result by Levine [**8**, Theorem 4.2] implies that $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{m}) = \mathbb{Q} \oplus \mathbb{Q}$. When R is complete of prime characteristic p and has perfect residue field, Article I, Example 35, shows that $[M]$ can be decomposed as

$$[M] = [M]^{(0)} + [M]^{(2)},$$

where, using the formula in Theorem 3,

$$[M]^{(0)} = \frac{1}{(p-1)^2(p+1)}[M] - \frac{1}{p^4(p-1)^2}[F(M)] + \frac{1}{p^7(p-1)^2(p+1)}[F^2(M)]; \quad \text{and}$$

$$[M]^{(2)} = \frac{p^3}{(p-1)^2(p+1)}[M] - \frac{1}{p^2(p-1)^2}[F(M)] + \frac{1}{p^6(p-1)^2(p+1)}[F^2(M)].$$

Using the fact that $[M]^{(1)} = 0$, it is even possible to obtain

$$[M]^{(0)} = \frac{-1}{p^2-1}[M] + \frac{1}{p^3(p^2-1)}[F(M)]; \quad \text{and}$$

$$[M]^{(2)} = \frac{p^2}{p^2-1}[M] - \frac{1}{p^3(p^2-1)}[F(M)].$$

Since R is a complete intersection, R satisfies numerical vanishing, so

$$\chi([M]^{(0)}) = \chi(M) = \operatorname{length} M = 55 \quad \text{and} \quad \chi([M]^{(2)}) = 0.$$

In particular, $[M]^{(0)}$ is non-zero. Since $[M]^{(0)}$ satisfies vanishing,

$$\chi([M]^{(0)}, N) = 0 \quad \text{and} \quad \chi([M]^{(2)}, N) = \chi(M,N) = -2,$$

so $[M]^{(2)}$ must also be non-zero. Thus, $[M]^{(0)}$ and $[M]^{(2)}$ are linearly independent and generate $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{m})$. These formulas imply that

$$\operatorname{length} F^e(M) = 55p^{5e} \quad \text{and} \quad \chi(F^e(M), N) = -2p^{3e}$$

for all non-negative integers e.

Example 16 (the second example by Miller and Singh)**.** Using the ring R from the previous example in the characteristic 2 case, Miller and Singh construct a module-finite extension of R by setting

$$S = R\big[\sqrt{uyz}, \sqrt{vxz}, \sqrt{wxy}, \sqrt{uvw}, \sqrt{uwyz}\big].$$

This ring is a Gorenstein normal domain of dimension 5. The S-module $K = M \otimes_R S$ has length 222 and finite projective dimension, and from Article I, Example 35, we still have the decomposition

$$[K] = [K]^{(0)} + [K]^{(2)}$$

in $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{n})$, where $\mathfrak{n}$ is the maximal ideal of S. The computations by Miller and Singh imply that

$$\chi([K]^{(0)}]) = 220 \quad \text{and} \quad \chi([K]^{(2)}]) = 2.$$

Miller and Singh also construct another S-module L of length 218 with finite projective dimension. This module also decomposes in $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{n})$ as

$$[L] = [L]^{(0)} + [L]^{(2)},$$

and the computations performed by Miller and Singh imply that

$$\chi([L]^{(0)}]) = 220 \quad \text{and} \quad \chi([L]^{(2)}]) = -2.$$

Unlike the two previous examples, this example includes a Dutta multiplicity that is neither zero nor equal to the usual intersection multiplicity:

$$\chi(K, S) = 222 \neq 220 = \chi_\infty(K, S).$$

Corollary 5 implies that the Dutta multiplicity is always rational, but in this case it is, apparently, even an integer. There are no known examples where the Dutta multiplicity is *not* an integer. In relation to this, it is notable that the decomposition from (6) in Section 13 shows that the Dutta multiplicity $\chi_\infty(X, Y)$ of complexes $X \in \mathsf{P}^{\mathrm{f}}(R)$ and $Y \in \mathsf{D}^{\mathrm{f}}_{\square}(R)$ with $\operatorname{Supp} X \cap \operatorname{Supp} Y = \{\mathfrak{m}\}$ and $\dim(\operatorname{Supp} X) + \dim(\operatorname{Supp} Y) \leq \dim R$ is a number in $\frac{1}{2}\mathbb{Z}$ whenever the element $[X] \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\operatorname{Supp} X)$ has vanishing dimension at most one.

15. Grothendieck groups

This section presents some of the results from Article III. Although the content of this article is not directly related to Grothendieck spaces, the section concludes with an application to Grothendieck spaces.

Let C be a full subcategory of the (non-derived) category of bounded complexes of finitely generated modules. The *Grothendieck group* of C is defined as the Abelian group $K_0(\mathsf{C})$ presented by generators $[X]$, one for each isomorphism class of a complex X in C, and relations

$$\begin{aligned} [X] &= 0 && \text{whenever } X \text{ is exact; and} \\ [X] &= [X'] + [X''] && \text{whenever } 0 \to X' \to X \to X'' \to 0 \end{aligned}$$

is a short exact sequence in C. If C contains only modules (complexes concentrated in degree zero), then the first requirement is contained in the second one.

Let $S = (S_1, \ldots, S_d)$ be a family of multiplicative systems of R: that is, multiplicatively closed subsets of R containing the unit element. A module M is said to be *S-torsion* if $S_i^{-1}M = 0$ for $i = 1, \ldots, d$, and a complex X is said to be *homologically S-torsion* if all its homology modules $\mathrm{H}_n(X)$ are S-torsion; in other words, if $S_i^{-1}X$ is exact for $i = 1, \ldots, d$. We define the following (non-derived) categories.

$\mathsf{C}(S\text{-tor})$	$=$	the category of bounded, homologically S-torsion complexes of finitely generated, projective modules.
$\mathsf{C}_d(S\text{-tor})$	$=$	the full subcategory of $\mathsf{C}(S\text{-tor})$ comprising the complexes concentrated in degrees $d, \ldots, 0$.
$\mathsf{M}(S\text{-tor})$	$=$	the category of finitely generated, S-torsion modules of finite projective dimension.

The inclusion $\mathsf{C}_d(S\text{-tor}) \to \mathsf{C}(S\text{-tor})$ of categories naturally induces a homomorphism

$$\mathcal{I}\colon K_0(\mathsf{C}_d(S\text{-tor})) \to K_0(\mathsf{C}(S\text{-tor})) \quad \text{given by} \quad \mathcal{I}([X]) = [X],$$

where X is a complex in $\mathsf{C}_d(S\text{-tor})$ and the first $[X]$ is an element of $K_0(\mathsf{C}_d(S\text{-tor}))$, whereas the second is an element of $K_0(\mathsf{C}(S\text{-tor}))$. The following is the Main Theorem of Article III.

Theorem 17. *Suppose that d is a non-negative integer and that $S = (S_1, \dots, S_d)$ is a family of multiplicative systems of R. Then the homomorphism*

$$\mathcal{I}\colon K_0(\mathsf{C}_d(S\text{-tor})) \to K_0(\mathsf{C}(S\text{-tor}))$$

is an isomorphism.

This theorem has many interesting consequences, and a few are mentioned here. For an element $x \in R$, we define a multiplicative system

$$S(x) = \{x^n \mid n \in \mathbb{N}_0\},$$

and if $x = (x_1, \dots, x_d)$ is a d-tuple of elements of R, we let

$$S(x) = (S(x_1), \dots, S(x_d)).$$

In the case where R is Noetherian and local of dimension d and $x = (x_1, \dots, x_d)$ is a system of parameters for R, the category $\mathsf{C}(S(x)\text{-tor})$ is simply the category of bounded complexes of finitely generated projective modules with finite-length homology. Theorem 17 states that the Grothendieck group of this category is isomorphic to the Grothendieck group of the full subcategory of complexes concentrated in degrees $d, \dots, 0$ through the homomorphism induced by the inclusion of categories—an interesting result compared with the *new intersection theorem* (see Roberts [**15**, Theorem 13.4.1]), which states that the complexes concentrated in degrees $d, \dots, 0$ in some sense are "minimal" within this category.

Another interesting consequence of Theorem 17 is the interplay between the Grothendieck groups of the categories $\mathsf{C}_d(S\text{-tor})$, $\mathsf{C}(S\text{-tor})$ and $\mathsf{M}(S\text{-tor})$. When R is Noetherian, a module in $\mathsf{M}(S\text{-tor})$ has a projective resolution in $\mathsf{C}(S\text{-tor})$, and this association induces a homomorphism

$$\mathcal{R}\colon K_0(\mathsf{M}(S\text{-tor})) \to K_0(\mathsf{C}(S\text{-tor})) \quad \text{given by} \quad \mathcal{R}([M]) = [X],$$

where M is a module in $\mathsf{M}(S\text{-tor})$ and X is a complex in $\mathsf{C}(S\text{-tor})$, which is a projective resolution of M. Under certain additional assumptions, discussed below, the homology of complexes in $\mathsf{C}_d(S\text{-tor})$ is concentrated in degree zero, meaning that any complex in $\mathsf{C}_d(S\text{-tor})$ is a projective resolution of a module in $\mathsf{M}(S\text{-tor})$. When this is true, taking homology induces a homomorphism

$$\mathcal{H}\colon K_0(\mathsf{C}_d(S\text{-tor})) \to K_0(\mathsf{M}(S\text{-tor})) \quad \text{given by} \quad \mathcal{H}([X]) = [\mathrm{H}(X)],$$

where X is a complex in $\mathsf{C}_d(S\text{-tor})$ and $\mathrm{H}(X)$ is its homology complex, which is concentrated in degree zero and hence is a module.

The following is a list of sufficient conditions for $\mathcal{R}$ and $\mathcal{H}$ to be defined.

(*i*) R is Noetherian and local and $S = S(x)$ for a regular sequence $x = (x_1, \dots, x_d)$ where $d > 0$;
(*ii*) R is Noetherian and local and S is a single multiplicative system containing only non-zerodivisors; and
(*iii*) R is Noetherian and S is empty: that is, it is trivial to be S-torsion and $d = 0$.

The following proposition derives from Article III, Corollaries 8, 9 and 10.

Proposition 18. *Under each of the assumptions (i)–(iii) above, the homomorphisms $\mathcal{R}$, $\mathcal{H}$ and $\mathcal{I}$ are all isomorphisms and fit together in the following commutative diagram.*

$$\begin{tikzcd} K_0(\mathsf{C}_d(S\text{-tor})) \arrow[rr, "\mathcal{I}"] \arrow[dr, "\mathcal{H}"'] & & K_0(\mathsf{C}(S\text{-tor})) \\ & K_0(\mathsf{M}(S\text{-tor})) \arrow[ur, "\mathcal{R}"'] & \end{tikzcd}$$

Under each of the assumptions (*i*)–(*iii*), any module in $\mathsf{M}(S\text{-tor})$ must have grade at least equal to d, and hence the above diagram shows that the Grothendieck group of $\mathsf{C}(S\text{-tor})$ is generated by projective resolutions of perfect modules of projective dimension d. A consequence of this observation in case (*ii*) is that $K_0(\mathsf{M}(S\text{-tor}))$ is generated by elements of the form $[R/x]$ for a non-zerodivisor x; Foxby [**5**] used this to prove Serre's intersection conjectures when the module that is *not* necessarily of finite projective dimension has dimension less than or equal to one. This introduction concludes by using the observation in case (*i*) to deduce a result for Grothendieck spaces.

Assume from now on that R is Noetherian and local, and let $x = (x_1, \ldots, x_c)$ be a regular sequence in R. Then the Grothendieck group of the category $\mathsf{C}(S(x)\text{-tor})$ is generated by complexes concentrated in degrees $c, \ldots, 0$ that are minimal projective resolutions of finitely generated perfect modules. Being $S(x)$-torsion for a finite complex is the same as having support contained in $V(x)$, so the non-derived category $\mathsf{C}(S(x)\text{-tor})$ can be used to represent the derived category $\mathsf{P}^{\mathrm{f}}(V(x))$: any complex in $\mathsf{P}^{\mathrm{f}}(V(x))$ is isomorphic (within $\mathsf{P}^{\mathrm{f}}(V(x))$) to a complex from $\mathsf{C}(S(x)\text{-tor})$. Since the relations defining the Grothendieck space $\mathbb{G}\mathsf{P}^{\mathrm{f}}(V(x))$ include the relations defining the Grothendieck group $K_0(\mathsf{C}(S(x)\text{-tor})$, there must be a surjection

$$\mathbb{Q} \otimes_{\mathbb{Z}} K_0(\mathsf{C}(S(x)\text{-tor})) \to \mathbb{G}\mathsf{P}^{\mathrm{f}}(V(x)) \quad \text{given by } r \otimes [X] \mapsto r[X],$$

where $r \in \mathbb{Q}$, $X \in \mathsf{C}(S(x)\text{-tor})$ and the first $[X]$ is an element of $K_0(\mathsf{C}(S(x)\text{-tor}))$, whereas the second is an element of $\mathbb{G}\mathsf{P}^{\mathrm{f}}(V(x))$. Thus, $\mathbb{G}\mathsf{P}^{\mathrm{f}}(V(x))$ is the tensor product of $\mathbb{Q}$ and a homomorphic image of $K_0(\mathsf{C}(S(x)\text{-tor})$, and consequently $\mathbb{G}\mathsf{P}^{\mathrm{f}}(V(x))$ is generated by the same kind of elements as $K_0(\mathsf{C}(S(x)\text{-tor}))$; in fact, following the proof of Article III, Corollary 8, and recalling that all elements α in $\mathbb{G}\mathsf{P}^{\mathrm{f}}(V(x))$ can be written as $\alpha = r[X]$ for a rational number r and a complex X in $\mathsf{P}^{\mathrm{f}}(V(x))$ leads to the following.

Proposition 19. *Assume that R is Noetherian and local, and let $x = (x_1, \ldots, x_c)$ be a regular sequence in R. Then any element $\alpha \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(V(x))$ can be written in the form*

$$\alpha = r\big([M] - m[R/x]\big)$$

for a rational number r, a non-negative integer m and a finitely generated perfect module M with $\operatorname{pd} M = c$.

If R is Cohen–Macaulay, the regular sequence x can be extended by another regular sequence $y = (y_1, \ldots, y_d)$ so that $(x, y) = (x_1, \ldots, x_c, y_1, \ldots, y_d)$ is a regular sequence of maximal length $c + d = \dim R$. An element β in $\mathbb{G}\mathsf{P}^{\mathrm{f}}(V(y))$ can then be written in the form

$$\beta = s\big([N] - n[R/y]\big)$$

for a rational number s, a non-negative integer n and a finitely generated perfect module N with $\operatorname{pd} N = d$. Computing $\alpha \otimes \beta$ is now easy since both α and β are expressed in terms of perfect modules:

$$\begin{aligned}\alpha \otimes \beta &= rs\big([M] \otimes [N] - m[R/x] \otimes [N] - n[M] \otimes [R/y] + mn[R/x] \otimes [R/y]\big) \\ &= rs\big([M \otimes_R N] - m[N/xN] - n[M/yM] + mn[R/x, y]\big)\end{aligned}$$

If $\dim \beta < \operatorname{codim} V(x) = c$, then $[R/x] \otimes \beta = 0$, since $[R/x]$ satisfies vanishing, and then

$$\alpha \otimes \beta = rs\big([M \otimes_R N] - n[M/yM]\big).$$

Determining whether vanishing holds for α is now a question of determining whether

$$\operatorname{length}(M \otimes_R N) = n \operatorname{length}(M/yM).$$

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Article I

Diagonalizing the Frobenius

DIAGONALIZING THE FROBENIUS

ESBEN BISTRUP HALVORSEN

Abstract. This paper explores the interplay between the Frobenius functor and Serre's vanishing conjecture over a Noetherian local ring R of prime characteristic p. We show that the Frobenius functor induces a diagonalizable map on certain $\mathbb{Q}$-vector spaces, which are tensor products of $\mathbb{Q}$ with quotients of Grothendieck groups. This allows us to decompose an element (representing a bounded complex of finitely generated projective modules) into eigenvectors for the Frobenius: the component with eigenvalue 1 describes the Dutta multiplicity of the element, and the remaining components describe the extent to which Serre's vanishing conjecture fails to hold. As a consequence, we explicitly describe the Dutta multiplicity as a $\mathbb{Q}$-linear combination of finitely many terms in a sequence of intersection multiplicities; and we show that, over a Cohen–Macaulay ring, a sufficient condition for the weak version of Serre's vanishing conjecture (the one in which both modules are assumed to have finite projective dimension) to hold is that the Frobenius functor changes the length of modules of finite projective dimension by a factor $p^{\dim R}$.

1. Introduction

For finitely generated modules M and N over a commutative, Noetherian, local ring R with $\operatorname{pd} M < \infty$ and $\ell(M \otimes_R N) < \infty$, the *intersection multiplicity* defined by Serre [13] is given by

$$\chi^R(M, N) = \sum_i (-1)^i \, \ell(\operatorname{Tor}_i^R(M, N)).$$

The *intersection conjectures* state that

(i) $\dim M + \dim N \leq \dim R$;
(ii) $\chi^R(M, N) = 0$ whenever $\dim M + \dim N < \dim R$; and
(iii) $\chi^R(M, N) > 0$ whenever $\dim M + \dim N = \dim R$.

Serre's original conjectures require R to be regular, but the conjectures make sense in the more general setting presented above. Part (ii) is known as the *vanishing conjecture* and part (iii) is known as the *positivity conjecture*. Serre proved that, when R is regular, (i) holds and that, when R is regular and of equal characteristic or unramified of mixed characteristic, vanishing and positivity hold. The vanishing conjecture was later proven by Roberts [10] and, independently, by Gillet and Soulé [5] in the more general setting where the requirement that R be regular is weakened to the requirement that R be a complete intersection and both modules have finite projective dimension. Foxby [3] proved that all three conjectures hold when $\dim N \leq 1$.

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However, neither the vanishing nor the positivity conjecture hold in the full generality presented above. This was shown in the famous counterexample by Dutta, Hochster and McLaughlin [2]. Subsequently, other counterexamples have emerged, such as the one by Miller and Singh [8].

For rings with prime characteristic p, a different, and in some sense "better", intersection multiplicity was introduced by Dutta [1]. The *Dutta multiplicity* is given by

$$\chi_\infty^R(M,N) = \lim_{e\to\infty} \frac{1}{p^{e \operatorname{codim} M}} \chi(F_R^e(M), N),$$

where F_R denotes the Frobenius functor. The Dutta multiplicity satisfies the vanishing conjecture and is equal to the usual intersection multiplicity whenever this satisfies vanishing.

In this paper, we will study the interplay between the vanishing conjecture and the Frobenius functor and obtain a new way to describe the Dutta multiplicity. The main result is Theorem 19, which, in a certain sense, describes how to decompose a bounded complex of finitely generated projective modules into eigenvectors for the Frobenius functor. It should be noted that the diagonalizability of the Frobenius functor has been discussed by Kurano [6], but that the approach taken in this paper is new, at least to the knowledge of this author.

For each specialization-closed subset $\mathfrak{X} \subseteq \operatorname{Spec} R$ we let $\mathsf{P}(\mathfrak{X})$ denote the the category of bounded complexes with support in $\mathfrak{X}$ and consisting of finitely generated projective modules, and we let $\mathsf{C}(\mathfrak{X})$ denote the category of homologically bounded complexes with support in $\mathfrak{X}$ and with finitely generated homology modules. We shall introduce the *Grothendieck spaces* $\mathbb{G}\mathsf{P}(\mathfrak{X})$ and $\mathbb{G}\mathsf{C}(\mathfrak{X})$, which are tensor products of $\mathbb{Q}$ with quotients of the usual Grothendieck groups of the categories $\mathsf{P}(\mathfrak{X})$ and $\mathsf{C}(\mathfrak{X})$, respectively. The Grothendieck spaces $\mathbb{G}\mathsf{P}(\mathfrak{X})$ and $\mathbb{G}\mathsf{C}(\mathfrak{X})$ will be equipped with a topology, allowing us to discuss properties such as convergence and continuity, and we shall generalize Serre's intersection multiplicity to a map $\mathbb{G}\mathsf{P}(\mathfrak{X}) \otimes_{\mathbb{Q}} \mathbb{G}\mathsf{C}(\mathfrak{X}^c) \to \mathbb{G}\mathsf{C}(\{\mathfrak{m}\}$ induced by the tensor product of complexes, where $\mathfrak{X}^c$ is the maximal subset of $\operatorname{Spec} R$ such that $\mathfrak{X} \cap \mathfrak{X}^c = \{\mathfrak{m}\}$ and $\dim \mathfrak{X} + \dim \mathfrak{X}^c \leq \dim R$. Given modules M and N as above with $\dim M + \dim N \leq \dim R$, we can set $\mathfrak{X} = \operatorname{Supp} M$ and represent M and N by elements $\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{X})$ and $\beta \in \mathbb{G}\mathsf{C}(\mathfrak{X}^c)$, respectively. The intersection multiplicity $\chi^R(M,N)$ can then be represented by the element $\alpha \otimes \beta \in \mathbb{G}\mathsf{C}(\{\mathfrak{m}\})$.

In prime characteristic p, the Frobenius functor F_R induces an endomorphism $F_{\mathfrak{X}}$ on $\mathbb{G}\mathsf{P}(\mathfrak{X})$, and we shall study the endomorphism $\Phi_{\mathfrak{X}}$ defined as $p^{-\operatorname{codim}\mathfrak{X}}$ times $F_{\mathfrak{X}}$. The endomorphism $\Phi_{\mathfrak{X}}$ is continuous, and it turns out that an element of $\mathbb{G}\mathsf{P}(\mathfrak{X})$ is a fixed point of $\Phi_{\mathfrak{X}}$ if and only if it satisfies vanishing: that is, its intersection multiplicity with elements in $\mathbb{G}\mathsf{C}(\mathfrak{X}^c)$ of dimension smaller than the codimension of $\mathfrak{X}$ vanishes (Proposition 17). A consequence of this is that $\Phi_{\mathfrak{X}}$ must be diagonalizable with eigenvalues $1, 1/p, 1/p^2, \ldots$, and hence we can decompose any element $\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{X})$ into eigenvectors for $\Phi_{\mathfrak{X}}$: that is, we can write $\alpha = \alpha^{(0)} + \cdots + \alpha^{(u)}$, where $\alpha^{(i)}$ is an eigenvector for $\Phi_{\mathfrak{X}}$ with eigenvalue $1/p^i$ (Theorem 19). The number u, which is the largest number such that the component $\alpha^{(u)}$ is nonzero, will be called the *vanishing dimension* of α. It measures, in a sense, how far an element in $\mathbb{G}\mathsf{P}(\mathfrak{X})$ is from satisfying vanishing (Proposition 24). The vanishing dimension turns out to be bounded by $\max(0, \operatorname{codim}\mathfrak{X} - 2)$.

The decomposition of α into eigenvectors allows for an easy formula to compute $\Phi^e_{\mathfrak{X}}(\alpha)$ as a $\mathbb{Q}$-linear combination of $\alpha^{(0)}, \dots, \alpha^{(u)}$. However, the formula can also be reversed, allowing us to describe each $\alpha^{(i)}$ as a $\mathbb{Q}$-linear combination of $\alpha, \Phi_{\mathfrak{X}}(\alpha), \dots, \Phi^u_{\mathfrak{X}}(\alpha)$. In fact, we have the following formula (Theorem 19):

$$\begin{pmatrix} \alpha^{(0)} \\ \vdots \\ \alpha^{(u)} \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1/p & \cdots & 1/p^u \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1/p^u & \cdots & 1/p^{u^2} \end{pmatrix}^{-1} \begin{pmatrix} \alpha \\ \Phi_{\mathfrak{X}}(\alpha) \\ \vdots \\ \Phi^u_{\mathfrak{X}}(\alpha) \end{pmatrix}.$$

We shall also give a different description of the element $\alpha^{(i)}$ as the limit of a certain sequence; in particular, we shall show that $\alpha^{(0)}$ is the limit of $\Phi^e_{\mathfrak{X}}(\alpha)$ as e tends to infinity. It then follows that the Dutta multiplicity of α and an element from $\mathbb{G}\mathsf{C}(\mathfrak{X}^c)$ can be calculated as the usual intersection multiplicity of $\alpha^{(0)}$ and that element. From the formula above describing $\alpha^{(0)}$ as a $\mathbb{Q}$-linear combination of $\alpha, \Phi_{\mathfrak{X}}(\alpha), \dots, \Phi^u_{\mathfrak{X}}(\alpha)$, we obtain, in particular, an explicit formula for calculating the Dutta multiplicity (Remark 21). In fact, for finitely generated modules M and N with $\operatorname{pd} M < \infty$, $\ell(M \otimes_R N) < \infty$ and $t = \operatorname{codim} M \geq \dim N$ such that the vanishing dimension of the element represented by a projective resolution of M is less than or equal to u, we have the general formula

$$\chi^R_\infty(M,N) = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ p^t & p^{t-1} & \cdots & p^{t-u} \\ \vdots & \vdots & \ddots & \vdots \\ p^{ut} & p^{u(t-1)} & \cdots & p^{u(t-u)} \end{pmatrix}^{-1} \begin{pmatrix} \chi^R(M,N) \\ \chi^R(F_R(M),N) \\ \vdots \\ \chi^R(F^u_R(M),N) \end{pmatrix}.$$

This can be useful information, for example when using a computer to calculate Dutta multiplicity.

Finally, we shall introduce the concept of *numerical vanishing*, a condition which holds if the vanishing conjecture holds, and which implies a weaker version of the vanishing conjecture, namely the one in which both modules are required to have finite projective dimension. A feature of numerical vanishing is that it holds globally (that is, for all elements of all Grothendieck space) if and only if it holds for all elements in the Grothendieck space $\mathbb{G}\mathsf{P}(\{\mathfrak{m}\})$. A consequence is that, over a Cohen–Macaulay ring, a sufficient condition for the weak version of the vanishing conjecture to hold is that $\ell(F_R(M)) = p^{\dim R}\,\ell(M)$ for all modules M of finite length and finite projective dimension (Example 33).

2. Notation

Throughout this paper, R will denote a commutative, Noetherian, local ring with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. Modules and complexes are, unless otherwise stated, assumed to be R-modules and R-complexes, respectively. Modules are considered to be complexes concentrated in degree zero.

The *spectrum* of R, denoted $\operatorname{Spec} R$, is the set of prime ideals of R. A subset $\mathfrak{X} \subseteq \operatorname{Spec} R$ is *specialization-closed* if, for any inclusion $\mathfrak{p} \subseteq \mathfrak{q}$ of prime ideals, $\mathfrak{p} \in \mathfrak{X}$ implies $\mathfrak{q} \in \mathfrak{X}$. A closed subset of $\operatorname{Spec} R$ is, in particular, specialization-closed. Throughout, whenever we deal with subsets of the spectrum of a ring, it is implicitly assumed that they are non-empty and specialization-closed.

For every $\mathfrak{X} \subseteq \operatorname{Spec} R$, the *dimension* of $\mathfrak{X}$, denoted $\dim \mathfrak{X}$, shall mean the usual Krull dimension of $\mathfrak{X}$, and the *co-dimension* of $\mathfrak{X}$, denoted $\operatorname{codim} \mathfrak{X}$, shall mean the number $\dim R - \dim \mathfrak{X}$. The dimension and co-dimension of a complex X (and hence also of a module) is the dimension and co-dimension of its support: that is, of the set $\operatorname{Supp}_R X = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathrm{H}(X_\mathfrak{p}) \neq 0\}$.

3. Grothendieck spaces and vanishing

For every (non-empty, specialization-closed) $\mathfrak{X} \subseteq \operatorname{Spec} R$, we consider the following categories.

$\mathsf{P}(\mathfrak{X})$	$=$	the category of bounded complexes with support in $\mathfrak{X}$ and consisting of finitely generated projective modules.
$\mathsf{C}(\mathfrak{X})$	$=$	the category of homologically bounded complexes with support in $\mathfrak{X}$ and with finitely generated homology modules.

If $\mathfrak{X} = \{\mathfrak{m}\}$, we shall simply write $\mathsf{P}(\mathfrak{m})$ and $\mathsf{C}(\mathfrak{m})$.

The *Euler characteristic* of a complex X in $\mathsf{C}(\mathfrak{m})$ is defined as

$$\chi^R(X) = \sum_i (-1)^i \, \ell(\mathrm{H}_i(X)).$$

If M and N are finitely generated modules with $\operatorname{pd} M < \infty$ and $\ell(M \otimes_R N) < \infty$, and X is a projective resolution of M, $X \otimes_R N$ is a complex in $\mathsf{C}(\mathfrak{m})$, and the intersection multiplicity $\chi^R(M,N)$ of M and N is the number $\chi^R(X \otimes_R N)$. There is no problem in letting N be a complex rather than just a module, so we can extend the definition of intersection multiplicity to an even more general setting: for subsets $\mathfrak{X}, \mathfrak{Y} \subseteq \operatorname{Spec} R$ with $\mathfrak{X} \cap \mathfrak{Y} = \{\mathfrak{m}\}$ and complexes $X \in \mathsf{P}(\mathfrak{X})$ and $Y \in \mathsf{C}(\mathfrak{Y})$, we define the intersection multiplicity of X and Y by

$$\chi^R(X,Y) = \chi^R(X \otimes_R Y) = \sum_i (-1)^i \, \ell(\mathrm{H}_i(X \otimes_R Y)).$$

We shall shortly construct the "Grothendieck spaces" in which we identify all complexes in $\mathsf{P}(\mathfrak{X})$ or $\mathsf{C}(\mathfrak{X})$ whose intersection multiplicity with other complexes are the same. To describe what "other complexes" we will look at, we define, for each $\mathfrak{X} \subseteq \operatorname{Spec} R$, a subset

$$\mathfrak{X}^c = \{\mathfrak{q} \in \operatorname{Spec} R \mid \mathfrak{X} \cap V(\mathfrak{q}) = \{\mathfrak{m}\} \text{ and } \dim V(\mathfrak{q}) \leq \operatorname{codim} \mathfrak{X}\}.$$

The set $\mathfrak{X}^c$ shall play the role of a sort of "complement" of $\mathfrak{X}$, and the idea is to consider only the intersection multiplicity of complexes from $\mathsf{P}(\mathfrak{X})$ with complexes from $\mathsf{C}(\mathfrak{X}^c)$ (and, conversely, of complexes from $\mathsf{C}(\mathfrak{X})$ with complexes from $\mathsf{P}(\mathfrak{X}^c)$). Note that $\mathfrak{X}^c$ is a specialization-closed subset of $\operatorname{Spec} R$ and that the set $\mathfrak{X}^c$ is the largest subset of $\operatorname{Spec} R$ such that $\mathfrak{X} \cap \mathfrak{X}^c = \{\mathfrak{m}\}$ and $\dim \mathfrak{X} + \dim \mathfrak{X}^c \leq \dim R$. (In fact it is not hard to see that, when $\mathfrak{X}$ is closed, $\dim \mathfrak{X} + \dim \mathfrak{X}^c = \dim R$.) The latter requirement corresponds to the assumption that $\dim M + \dim N \leq \dim R$, which is necessary in order to define the Dutta multiplicity.

Definition 1. The *Grothendieck space* of the category $\mathsf{P}(\mathfrak{X})$ is the $\mathbb{Q}$-vector space $\mathbb{G}\mathsf{P}(\mathfrak{X})$ presented by elements $[X]_{\mathsf{P}(\mathfrak{X})}$, one for each isomorphism class of a complex X in $\mathsf{P}(\mathfrak{X})$, and relations

$$[X]_{\mathsf{P}(\mathfrak{X})} = [\widetilde{X}]_{\mathsf{P}(\mathfrak{X})} \quad \text{whenever} \quad \chi^R(X \otimes_R -) = \chi^R(\widetilde{X} \otimes_R -)$$

as functions $\mathsf{C}(\mathfrak{X}^c) \to \mathbb{Q}$. Similarly, the *Grothendieck space* of the category $\mathsf{C}(\mathfrak{X})$ is the $\mathbb{Q}$-vector space $\mathbb{G}\mathsf{C}(\mathfrak{X})$ presented by elements $[Y]_{\mathsf{C}(\mathfrak{X})}$, one for each isomorphism class of a complex Y in $\mathsf{C}(\mathfrak{X})$, and relations

$$[Y]_{\mathsf{C}(\mathfrak{X})} = [\widetilde{Y}]_{\mathsf{C}(\mathfrak{X})} \quad \text{whenever} \quad \chi^R(- \otimes_R Y) = \chi^R(- \otimes_R \widetilde{Y})$$

as functions $\mathsf{P}(\mathfrak{X}^c) \to \mathbb{Q}$. If $\mathfrak{X} = \{\mathfrak{m}\}$, we shall simply write $\mathbb{G}\mathsf{P}(\mathfrak{m})$ and $\mathbb{G}\mathsf{C}(\mathfrak{m})$, and if we need to emphasize what the underlying ring is, we write $\mathbb{G}\mathsf{P}^R(\mathfrak{X})$, $\mathbb{G}\mathsf{C}^R(\mathfrak{X})$, $[X]^R_{\mathsf{P}(\mathfrak{X})}$ and $[Y]^R_{\mathsf{C}(\mathfrak{X})}$.

In the construction of Grothendieck spaces, we are, basically, identifying a complex $X \in \mathsf{P}(\mathfrak{X})$ with the map $\chi^R(X \otimes_R -)\colon \mathsf{C}(\mathfrak{X}^c) \to \mathbb{Q}$ and a complex $Y \in \mathsf{C}(\mathfrak{X})$ with the map $\chi^R(- \otimes_R Y)\colon \mathsf{P}(\mathfrak{X}^c) \to \mathbb{Q}$. Since intersection multiplicity is additive on short exact sequences and trivial on exact complexes, the Grothendieck spaces $\mathbb{G}\mathsf{P}(\mathfrak{X})$ and $\mathbb{G}\mathsf{C}(\mathfrak{X})$ can also be regarded as the tensor product of $\mathbb{Q}$ with quotients of the Grothendieck groups $K_0(\mathsf{P}(\mathfrak{X}))$ and $K_0(\mathsf{C}(\mathfrak{X}))$ of the categories $\mathsf{P}(\mathfrak{X})$ and $\mathsf{C}(\mathfrak{X})$. (For more information on Grothendieck groups of categories of complexes, see [4].) In particular, any equation that holds in one of these Grothendieck groups also holds in the corresponding Grothendieck space.

The choice of $\mathbb{Q}$ as underlying field is not inevitable: any extension of $\mathbb{Q}$, for example $\mathbb{R}$ or $\mathbb{C}$, could have been chosen, but $\mathbb{Q}$ will do for our purposes. Note that the spaces $\mathbb{G}\mathsf{P}(\mathfrak{X})$ and $\mathbb{G}\mathsf{C}(\mathfrak{X})$ in general should not be expected to be finite-dimensional. However, as we shall see in Proposition 2 below, $\mathbb{G}\mathsf{C}(\mathfrak{m})$ is always one-dimensional (and, in fact, so is $\mathbb{G}\mathsf{P}(\operatorname{Spec} R)$).

Proposition 2. *Suppose that $\mathfrak{X}, \mathfrak{Y} \subseteq \operatorname{Spec} R$.*

(*i*) *If $0 \to X \to Y \to Z \to 0$ is a short exact sequence of complexes in $\mathsf{P}(\mathfrak{X})$ (or $\mathsf{C}(\mathfrak{X})$, respectively), then $[Y]_{\mathsf{P}(\mathfrak{X})} = [X]_{\mathsf{P}(\mathfrak{X})} + [Z]_{\mathsf{P}(\mathfrak{X})}$ in $\mathbb{G}\mathsf{P}(\mathfrak{X})$ (or $[Y]_{\mathsf{C}(\mathfrak{X})} = [X]_{\mathsf{C}(\mathfrak{X})} + [Z]_{\mathsf{C}(\mathfrak{X})}$ in $\mathbb{G}\mathsf{C}(\mathfrak{X})$, respectively).*

(*ii*) *If $\varphi\colon X \to Y$ is a quasi-isomorphism of complexes in $\mathsf{P}(\mathfrak{X})$ (or $\mathsf{C}(\mathfrak{X})$, respectively), then $[X]_{\mathsf{P}(\mathfrak{X})} = [Y]_{\mathsf{P}(\mathfrak{X})}$ (or $[X]_{\mathsf{C}(\mathfrak{X})} = [Y]_{\mathsf{C}(\mathfrak{X})}$, respectively). In particular, if X is exact, then $[X]_{\mathsf{P}(\mathfrak{X})} = 0$ (or $[X]_{\mathsf{C}(\mathfrak{X})} = 0$, respectively).*

(*iii*) *If X is a complex in $\mathsf{P}(\mathfrak{X})$ (or $\mathsf{C}(\mathfrak{X})$, respectively), then $[\Sigma^n X]_{\mathsf{P}(\mathfrak{X})} = (-1)^n [X]_{\mathsf{P}(\mathfrak{X})}$ (or $[\Sigma^n X]_{\mathsf{C}(\mathfrak{X})} = (-1)^n [X]_{\mathsf{C}(\mathfrak{X})}$, respectively). (Here, $\Sigma^n(-)$ denotes the shift functor, taking a complex X to the complex $\Sigma^n X$ defined by $(\Sigma^n X)_i = X_{i-n}$ and $\partial_i^{\Sigma^n X} = (-1)^n \partial_{i-n}^X$.)*

(*iv*) *Any element in $\mathbb{G}\mathsf{P}(\mathfrak{X})$ (or $\mathbb{G}\mathsf{C}(\mathfrak{X})$, respectively) can be written in the form $r[X]_{\mathsf{P}(\mathfrak{X})}$ (or $r[X]_{\mathsf{C}(\mathfrak{X})}$, respectively) for a rational number $r \in \mathbb{Q}$ and a complex X in $\mathsf{P}(\mathfrak{X})$ (or $\mathsf{C}(\mathfrak{X})$, respectively).*

(*v*) *$\mathbb{G}\mathsf{C}(\mathfrak{X})$ is generated by the elements $[R/\mathfrak{q}]_{\mathsf{C}(\mathfrak{X})}$ for prime ideals $\mathfrak{p} \in \mathfrak{X}$.*

(*vi*) *The Euler characteristic $\mathsf{C}(\mathfrak{m}) \to \mathbb{Q}$ induces an isomorphism $\mathbb{G}\mathsf{C}(\mathfrak{m}) \xrightarrow{\cong} \mathbb{Q}$ given by $[X]_{\mathsf{C}(\mathfrak{m})} \mapsto \chi^R(X)$.*

(*vii*) *The inclusion $\mathsf{P}(\mathfrak{X}) \to \mathsf{C}(\mathfrak{X})$ of categories induces a $\mathbb{Q}$-linear map $\mathbb{G}\mathsf{P}(\mathfrak{X}) \to \mathbb{G}\mathsf{C}(\mathfrak{X})$ given by $[X]_{\mathsf{P}(\mathfrak{X})} \mapsto [X]_{\mathsf{C}(\mathfrak{X})}$.*

(*viii*) *If $\mathfrak{X} \subseteq \mathfrak{Y}$, the inclusions $\mathsf{P}(\mathfrak{X}) \to \mathsf{P}(\mathfrak{Y})$ and $\mathsf{C}(\mathfrak{X}) \to \mathsf{C}(\mathfrak{Y})$ of categories induce $\mathbb{Q}$-linear maps*

$$\mathbb{G}\mathsf{P}(\mathfrak{X}) \to \mathbb{G}\mathsf{P}(\mathfrak{Y}) \quad \textit{and} \quad \mathbb{G}\mathsf{C}(\mathfrak{X}) \to \mathbb{G}\mathsf{C}(\mathfrak{Y}),$$

respectively, given by $[X]_{\mathsf{P}(\mathfrak{X})} \mapsto [X]_{\mathsf{P}(\mathfrak{Y})}$ and $[Y]_{\mathsf{C}(\mathfrak{X})} \mapsto [Y]_{\mathsf{C}(\mathfrak{Y})}$, respectively.

(*ix*) *The tensor product of complexes induces* $\mathbb{Q}$*-linear maps*

$$\begin{aligned} \mathbb{G}\mathsf{P}(\mathfrak{X}) \otimes_{\mathbb{Q}} \mathbb{G}\mathsf{C}(\mathfrak{X}^c) &\to \mathbb{G}\mathsf{C}(\mathfrak{m}) \quad \textit{and} \\ \mathbb{G}\mathsf{P}(\mathfrak{X}) \otimes_{\mathbb{Q}} \mathbb{G}\mathsf{P}(\mathfrak{X}^c) &\to \mathbb{G}\mathsf{P}(\mathfrak{m}), \end{aligned}$$

respectively, given by

$$\begin{aligned} [X]_{\mathsf{P}(\mathfrak{X})} \otimes [Y]_{\mathsf{C}(\mathfrak{X}^c)} &\mapsto [X \otimes_R Y]_{\mathsf{C}(\mathfrak{m})} \quad \textit{and} \\ [X]_{\mathsf{P}(\mathfrak{X})} \otimes [Y]_{\mathsf{P}(\mathfrak{X}^c)} &\mapsto [X \otimes_R Y]_{\mathsf{P}(\mathfrak{m})}, \end{aligned}$$

respectively.

Proof. Properties (*i*), (*ii*) and (*iii*) hold since they hold for the corresponding Grothendieck groups (see, for example, [4]).

We show that (*iv*) holds for elements in $\mathbb{G}\mathsf{P}(\mathfrak{X})$; the argument for elements in $\mathbb{G}\mathsf{C}(\mathfrak{X})$ is identical. Note first that any element in $\mathbb{G}\mathsf{P}(\mathfrak{X})$ can be written as a sum $\sum_i r_i[X^i]_{\mathsf{P}(\mathfrak{X})}$ for various complexes X^i in $\mathsf{P}(\mathfrak{X})$. By using (*iii*), we can assume that all r_i are positive, and by choosing a greatest common divisor, we can write the element in the form $r\sum_i a_i[X^i]_{\mathsf{P}(\mathfrak{X})}$ for a rational number r and positive integers a_i. Because of (*i*), a sum of two elements represented by complexes is equal to the element represented by their direct sum, and hence the sum $\sum_i a_i[X^i]_{\mathsf{P}(\mathfrak{X})}$ can be replaced by a single element $[X]_{\mathsf{P}(\mathfrak{X})}$, where X is the direct sum over i of a_i copies of X^i.

Property (*v*) holds since it holds for the corresponding Grothendieck group. This is easily seen by using short exact sequences to transform a complex in $\mathsf{C}(\mathfrak{X})$ first into a bounded complex, then into the alternating sum of its homology modules, and finally, by taking filtrations, into a linear combination of modules in the form $R/\mathfrak{q}$ for prime ideals $\mathfrak{q} \in \mathfrak{X}$.

A consequence of (*v*) is that $\mathbb{G}\mathsf{C}(\mathfrak{m})$ must be generated by the element $[k]_{\mathsf{C}(\mathfrak{m})}$, so that $\mathbb{G}\mathsf{C}(\mathfrak{m})$ necessarily is isomorphic to $\mathbb{Q}$ or 0. Since the Euler characteristic naturally induces a non-trivial map $\mathbb{G}\mathsf{C}(\mathfrak{m}) \to \mathbb{Q}$, it follows that this map must be an isomorphism. This proves (*vi*).

To see (*vii*) and (*viii*), it suffices to note that, since $\mathsf{C}(\mathfrak{X}^c)$ contains $\mathsf{P}(\mathfrak{X}^c)$ as well as $\mathsf{C}(\mathfrak{Y}^c)$ whenever $\mathfrak{X} \subseteq \mathfrak{Y}$ (because then $\mathfrak{Y}^c \subseteq \mathfrak{X}^c$), any relation in $\mathbb{G}\mathsf{P}(\mathfrak{X})$ is also a relation in $\mathbb{G}\mathsf{C}(\mathfrak{X})$ and $\mathbb{G}\mathsf{P}(\mathfrak{Y})$.

Finally, to see (*ix*), note that the first tensor product map is well-defined (in both variables) because of (*vi*) and by definition of Grothendieck spaces. (For the second variable, use (*viii*) together with the fact that $(\mathfrak{X}^c)^c \supseteq \mathfrak{X}$.) To see that the second tensor product map is well-defined, we restrict attention to the first variable; by symmetry (and again using (*viii*) together with the fact that $(\mathfrak{X}^c)^c \supseteq \mathfrak{X}$), the argument for the second variable is identical. So fix $Y \in \mathsf{P}(\mathfrak{X}^c)$ and suppose that $X, \widetilde{X} \in \mathsf{P}(\mathfrak{X})$ are such that $[X]_{\mathsf{P}(\mathfrak{X})} = [\widetilde{X}]_{\mathsf{P}(\mathfrak{X})}$. Then $\chi^R(X \otimes_R Z) = \chi^R(\widetilde{X} \otimes_R Z)$ for all complexes $Z \in \mathbb{G}\mathsf{C}(\mathfrak{X}^c)$, and hence $\chi^R(X \otimes_R Y \otimes_R Z) = \chi^R(\widetilde{X} \otimes_R Y \otimes_R Z)$ for all $Z \in \mathbb{G}\mathsf{C}(\{\mathfrak{m}\}^c) = \mathbb{G}\mathsf{C}(\operatorname{Spec} R)$ since, in this case, $Y \otimes_R Z \in \mathbb{G}\mathsf{C}(\mathfrak{X}^c)$ Thus, $[X \otimes_R Y]_{\mathsf{P}(\mathfrak{m})} = [\widetilde{X} \otimes_R Y]_{\mathsf{P}(\mathfrak{m})}$. $\square$

Using (*v*) to write an element $\alpha \in \mathbb{G}\mathsf{C}(\mathfrak{X})$ as a linear combination of elements $[R/\mathfrak{q}]_{\mathsf{C}(\mathfrak{X})}$ will be called *taking a filtration* of α. The $\mathbb{Q}$-linear maps in (*vii*) and (*viii*) will be denoted *inclusion homomorphisms* although they in general are not injective. By abuse of notation, the image under an inclusion homomorphism of an element α is likewise denoted by α. It should be apparent from the context which Grothendieck space α is considered an element of.

By a slight abuse of notation, we denote the map induced by the Euler characteristic by χ^R. By more abuse of notation, we denote both of the tensor product maps by $-\otimes-$, so that, for example, if $\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{X})$ can be written as $\alpha = r[X]_{\mathsf{P}(\mathfrak{X})}$ for $X \in \mathsf{P}(\mathfrak{X})$ and $r \in \mathbb{Q}$, and $\beta \in \mathbb{G}\mathsf{C}(\mathfrak{X}^c)$ can be written as $\beta = s[Y]_{\mathsf{C}(\mathfrak{X}^c)}$ for $Y \in \mathsf{C}(\mathfrak{X}^c)$ and $s \in \mathbb{Q}$, then $\alpha\otimes\beta \in \mathbb{G}\mathsf{C}(\mathfrak{m})$ can be written as $\alpha\otimes\beta = rs[X\otimes_R Y]_{\mathsf{C}(\mathfrak{m})}$.

The tensor product also allows us to change rings:

Proposition 3. *Suppose that S is another commutative, Noetherian, local ring, and that $R \to S$ is a local ring homomorphism such that S is finitely generated as an R-module. Let $\mathfrak{X} \subseteq \operatorname{Spec} R$ and $\mathfrak{Y} \subseteq \operatorname{Spec} S$. Suppose first that the following conditions are satisfied.*

(*i*) *If $X \in \mathsf{P}^R(\mathfrak{X})$ then $X \otimes_R S \in \mathsf{P}^S(\mathfrak{Y})$.*
(*ii*) *If $Y \in \mathsf{C}^S(\mathfrak{Y}^c)$ then $Y \in \mathsf{C}^R(\mathfrak{X}^c)$.*

Then the extension of scalars tensor product $-\otimes_R S$ induces a $\mathbb{Q}$-linear map

$$\mathbb{G}\mathsf{P}^R(\mathfrak{X}) \to \mathbb{G}\mathsf{P}^S(\mathfrak{Y}) \quad \text{given by} \quad [X]^R_{\mathsf{P}(\mathfrak{X})} \mapsto [X \otimes_R S]^S_{\mathsf{P}(\mathfrak{Y})}.$$

Suppose instead that the following conditions are satisfied

(*iii*) *If $Y \in \mathsf{C}^S(\mathfrak{Y})$ then $Y \in \mathsf{C}^R(\mathfrak{X})$.*
(*iv*) *If $X \in \mathsf{P}^R(\mathfrak{X}^c)$ then $X \otimes_R S \in \mathsf{P}^S(\mathfrak{Y}^c)$.*

Then the restriction of scalars induces a $\mathbb{Q}$-linear map

$$\mathbb{G}\mathsf{C}^S(\mathfrak{Y}) \to \mathbb{G}\mathsf{C}^R(\mathfrak{X}) \quad \text{given by} \quad [Y]^S_{\mathsf{C}(\mathfrak{Y})} \mapsto [Y]^R_{\mathsf{C}(\mathfrak{X})}.$$

Proof. Since the homomorphism $R \to S$ is local and finite, the residue field of S is a k-vector space of some finite dimension d, and for the Euler characteristic, we have $\chi^R(-) = d\chi^S(-)$ under the restriction of scalars.

Suppose first that conditions (*i*) and (*ii*) hold. Condition (*i*) ensures that the extension of scalars functor maps into the right category. To see that the functor induces a well-defined $\mathbb{Q}$-linear map on Grothendieck spaces, assume that $X, \widetilde{X} \in \mathsf{P}^R(\mathfrak{X})$ are such that $[X]_{\mathsf{P}(\mathfrak{X})} = [\widetilde{X}]_{\mathsf{P}(\mathfrak{X})}$ in $\mathbb{G}\mathsf{P}^R(\mathfrak{X})$. Then $\chi^R(X \otimes_R Z) = \chi^R(\widetilde{X} \otimes_R Z)$ for all $Z \in \mathsf{C}^R(\mathfrak{X}^c)$, and hence

$$\chi^S((X \otimes_R S) \otimes_S Y) = d^{-1}\chi^R(X \otimes_R Y) = d^{-1}\chi^R(\widetilde{X} \otimes_R Y) = \chi^S((\widetilde{X} \otimes_R S) \otimes_S Y)$$

for all $Y \in \mathsf{C}^S(\mathfrak{Y}^c)$, since such a Y as an R-complex lies in $\mathsf{C}^R(\mathfrak{X}^c)$ by condition (*ii*). Thus, $[X\otimes_R S]_{\mathsf{P}(\mathfrak{Y})} = [\widetilde{X}\otimes_R S]_{\mathsf{P}(\mathfrak{Y})}$ in $\mathbb{G}\mathsf{P}^S(\mathfrak{Y})$, and the induced map is well-defined.

Suppose instead that conditions (*iii*) and (*iv*) hold. Condition (*iii*) ensures that the restriction of scalars functor maps into the right category. To see that the functor induces a well-defined $\mathbb{Q}$-linear map on Grothendieck spaces, assume that $Y, \widetilde{Y} \in \mathsf{C}^S(\mathfrak{Y})$ are such that $[Y]_{\mathsf{C}(\mathfrak{Y})} = [\widetilde{Y}]_{\mathsf{C}(\mathfrak{Y})}$ in $\mathbb{G}\mathsf{C}^S(\mathfrak{Y})$. Then $\chi^S(Z \otimes_S Y) = \chi^S(Z \otimes_S \widetilde{Y})$ for all $Z \in \mathsf{P}^S(\mathfrak{Y}^c)$, and hence

$$\chi^R(X \otimes_R Y) = d\chi^S((X \otimes_R S) \otimes_S Y) = d\chi^S((X \otimes_R S) \otimes_S \widetilde{Y}) = \chi^R(X \otimes_R \widetilde{Y})$$

for all $X \in \mathsf{P}^R(\mathfrak{X}^c)$, since, for such an X, $X \otimes_R S$ lies in $\mathsf{P}^S(\mathfrak{Y}^c)$ by condition (*iv*). Thus, $[Y]_{\mathsf{C}(\mathfrak{X})} = [\widetilde{Y}]_{\mathsf{C}(\mathfrak{X})}$ in $\mathbb{G}\mathsf{C}^R(\mathfrak{X})$, and the induced map is well-defined. ☐

The map induced by $-\otimes_R S$ shall, by abuse of notation, also be denoted by $-\otimes_R S$. An example of a change of rings where the conditions in Proposition 3 are satisfied is the quotient map $R \to R/I$ for an ideal I of R and the subsets $\mathfrak{X} = \{\mathfrak{m}\}$ and $\mathfrak{Y} = \{\mathfrak{m}/I\}$.

Suppose that M and N are finitely generated modules with $\operatorname{pd} M < \infty$ and $\ell(M \otimes_R N) < \infty$, such that the intersection multiplicity of M and N is defined. Suppose further that $\dim M + \dim N \leq \dim R$ and set $\mathfrak{X} = \operatorname{Supp} M$. Let $X \in \mathsf{P}(\mathfrak{X})$ be a projective resolution of M. Then we must have $\operatorname{Supp} N \subseteq \mathfrak{X}^c$ and thereby $N \in \mathsf{C}(\mathfrak{X}^c)$, and hence

$$\chi^R(M,N) = \chi^R(X \otimes_R N) = \chi^R([X \otimes_R N]_{\mathsf{C}(\mathfrak{m})}) = \chi^R([X]_{\mathsf{P}(\mathfrak{X})} \otimes [N]_{\mathsf{C}(\mathfrak{X}^c)}),$$

which is the image in $\mathbb{Q}$ of $[X]_{\mathsf{P}(\mathfrak{X})} \otimes [N]_{\mathsf{C}(\mathfrak{X}^c)}$ under the isomorphism $\mathbb{G}\mathsf{C}(\mathfrak{m}) \cong \mathbb{Q}$ induced by the Euler characteristic. Thus, the intersection multiplicity generalizes to the $\mathbb{Q}$-linear map $\mathbb{G}\mathsf{P}(\mathfrak{X}) \otimes \mathbb{G}\mathsf{C}(\mathfrak{X}^c) \to \mathbb{G}\mathsf{C}(\mathfrak{m})$ induced by the tensor product.

Definition 4. Given $\mathfrak{X} \subseteq \operatorname{Spec} R$ and an element α in $\mathbb{G}\mathsf{P}(\mathfrak{X})$ (or $\mathbb{G}\mathsf{C}(\mathfrak{X})$, respectively), we define the *dimension* of α to be the number $\dim \alpha$ defined as the infimum over n such that α can be written in $\mathbb{G}\mathsf{P}(\mathfrak{X})$ (or $\mathbb{G}\mathsf{C}(\mathfrak{X})$, respectively) as a $\mathbb{Q}$-linear combination of complexes of dimension less than or equal to n. In particular, $\dim \alpha = -\infty$ if and only if $\alpha = 0$.

So, for example, for an element $\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{X})$ of dimension t, we can write α as a sum $\alpha = \sum_i r_i [X^i]_{\mathsf{P}(\mathfrak{X})}$ for rational numbers r_i and complexes $X^i \in \mathsf{P}(\mathfrak{X})$ of dimension less than or equal to t, and t is the smallest number such that this is possible.

Definition 5. Suppose that $\mathfrak{X} \subseteq \operatorname{Spec} R$ and let $\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{X})$. We say that α *satisfies vanishing* if, for all β in $\mathbb{G}\mathsf{C}(\mathfrak{X}^c)$, $\alpha \otimes \beta = 0$ whenever $\dim \beta < \operatorname{codim} \mathfrak{X}$. We define the *vanishing co-dimension* of α, denoted by $\operatorname{vcodim} \alpha$, to be largest integer t such that α satisfies vanishing as an element of $\mathbb{G}\mathsf{P}(\mathfrak{Y})$ (that is, after applying the inclusion homomorphism $\mathbb{G}\mathsf{P}(\mathfrak{X}) \to \mathbb{G}\mathsf{P}(\mathfrak{Y})$) for all $\mathfrak{Y} \supseteq \mathfrak{X}$ with $\operatorname{codim} \mathfrak{Y} \leq t$. We define the *vanishing dimension* of α to be the integer $\operatorname{vdim} \alpha = \operatorname{codim} \mathfrak{X} - \operatorname{vcodim} \alpha$. If $\alpha = 0$ we set $\operatorname{vcodim} \alpha = \infty$ and $\operatorname{vdim} \alpha = -\infty$.

To "satisfy vanishing" for an element α generalizes the traditional way of satisfying vanishing for a module of finite projective dimension: if M is finitely generated and of finite projective dimension, and X is a projective resolution of M, then the element $[X]_{\mathsf{P}(\mathfrak{X})}$ in $\mathbb{G}\mathsf{P}(\mathfrak{X})$, where $\mathfrak{X} = \operatorname{Supp} M$, satisfies vanishing exactly when $\chi^R(M,N) = 0$ for all finitely generated modules N with $\ell(M \otimes_R N) < \infty$ and $\dim M + \dim N < \dim R$ (we need only consider modules because of Proposition 2(v)).

The vanishing co-dimension of an element measures the "co-dimension level" that we need to move that element to, using an inclusion homomorphism, in order to be sure that vanishing will hold. The vanishing dimension measures, relatively, how many co-dimension levels we have to move down from $\operatorname{codim} \mathfrak{X}$ in order for vanishing to hold. In this sense, the vanishing dimension measures how far an element is from satisfying vanishing. In particular, the vanishing dimension of a non-trivial element is zero if and only if the element satisfies vanishing.

Example 6. A result by Foxby [3] shows that vanishing holds for all $\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{X})$ whenever $\operatorname{codim} \mathfrak{X} \leq 2$. In particular, for all $\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{X})$,

$$\operatorname{vcodim} \alpha \geq \min(2, \operatorname{codim} \mathfrak{X}),$$

and hence $\operatorname{vdim} \alpha \leq \max(0, \operatorname{codim} \mathfrak{X} - 2)$.

Remark 7. It is not hard to see that an element $\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{X})$ satisfies vanishing if and only if, for all $\mathfrak{Y} \supseteq \mathfrak{X}$ with $\operatorname{codim} \mathfrak{Y} < \operatorname{codim} \mathfrak{X}$, the image in $\mathbb{G}\mathsf{P}(\mathfrak{Y})$ of α under the inclusion map $\mathbb{G}\mathsf{P}(\mathfrak{X}) \to \mathbb{G}\mathsf{P}(\mathfrak{Y})$ vanishes. For if $\beta \in \mathbb{G}\mathsf{C}(\mathfrak{X}^c)$ has $\dim \beta < \operatorname{codim} \mathfrak{X}$, β is generated by complexes with supports of dimension strictly less than $\operatorname{codim} \mathfrak{X}$. Letting $\mathfrak{Z}$ denote the union of these supports and setting $\mathfrak{Y} = \mathfrak{Z}^c$, we must have $\mathfrak{X} \subseteq \mathfrak{Y}$ and $\operatorname{codim} \mathfrak{Y} < \operatorname{codim} \mathfrak{X}$. Since $\mathfrak{Z} \subseteq (\mathfrak{Z}^c)^c = \mathfrak{Y}^c$, β is the image of an element in $\mathbb{G}\mathsf{C}(\mathfrak{Y}^c)$, and to see whether $\alpha \otimes \beta$ vanishes for $\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{X})$, it therefore suffices to replace α by its image in $\mathbb{G}\mathsf{P}(\mathfrak{Y})$ under the inclusion map.

Suppose that $\mathfrak{X} \subseteq \mathfrak{Y}$, let $\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{X})$ and denote by $\bar{\alpha}$ the image in $\mathbb{G}\mathsf{P}(\mathfrak{Y})$ of α under the inclusion map. Then $\operatorname{vcodim} \bar{\alpha} \geq \operatorname{vcodim} \alpha$, and hence

$$\operatorname{vdim} \bar{\alpha} \leq \operatorname{vdim} \alpha - (\operatorname{codim} \mathfrak{X} - \operatorname{codim} \mathfrak{Y}).$$

It is always possible to find a $\mathfrak{Y} \supseteq \mathfrak{X}$ with any given co-dimension larger than $\operatorname{vcodim} \alpha$ and smaller than $\operatorname{codim} \mathfrak{X}$ such that the above is an equality.

We now introduce a topology on the Grothendieck spaces. The topology will be induced by a family of semi-norms.

Definition 8. Suppose that $\mathfrak{X} \subseteq \operatorname{Spec} R$. For each $\beta \in \mathbb{G}\mathsf{C}(\mathfrak{X}^c)$, we define a map

$$\| - \|_\beta = |\chi^R(- \otimes \beta)| \colon \mathbb{G}\mathsf{P}(\mathfrak{X}) \to \mathbb{Q},$$

and for each $\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{X}^c)$, we define a map

$$\| - \|_\alpha = |\chi^R(\alpha \otimes -)| \colon \mathbb{G}\mathsf{C}(\mathfrak{X}) \to \mathbb{Q}.$$

The fact that the maps $\| - \|_\beta$ and $\| - \|_\alpha$ are well-defined follows from the definition of $\mathbb{G}\mathsf{P}(\mathfrak{X})$ and $\mathbb{G}\mathsf{C}(\mathfrak{X})$. The abusive notation should not cause any confusion: if $\|\gamma\|_\delta$ is defined, then it is equal to $\chi^R(\gamma \otimes \delta)$ no matter what spaces γ and δ lie in. If γ is represented by a complex X and δ is represented by a complex Y, we shall occasionally also denote $\|\gamma\|_\delta$ by $\|X\|_\delta$, $\|\gamma\|_Y$ or $\|X\|_Y$. Note that we always have

$$\|\gamma\|_\delta = \|\delta\|_\gamma = \|R\|_{\gamma \otimes \delta} = \|\gamma \otimes \delta\|_R.$$

Proposition 9. *Given $\mathfrak{X} \subseteq \operatorname{Spec} R$ and $\beta \in \mathbb{G}\mathsf{C}(\mathfrak{X}^c)$, the map $\| - \|_\beta$ satisfies, for all $\alpha, \alpha' \in \mathbb{G}\mathsf{P}(\mathfrak{X})$ and $r \in \mathbb{Q}$,*

(*i*) $\|\alpha\|_\beta \geq 0$;
(*ii*) $\|r\alpha\|_\beta = |r| \|\alpha\|_\beta$;
(*iii*) $\|\alpha + \alpha'\|_\beta \leq \|\alpha\|_\beta + \|\alpha'\|_\beta$; *and*
(*iv*) $\alpha = 0$ *in* $\mathbb{G}\mathsf{P}(\mathfrak{X})$ *if and only if* $\|\alpha\|_\beta = 0$ *for all* $\beta \in \mathbb{G}\mathsf{C}(\mathfrak{X}^c)$.

In particular, $\| - \|_\beta$ is a semi-norm on $\mathbb{G}\mathsf{P}(\mathfrak{X})$. Similarly, given $\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{X}^c)$, $\| - \|_\alpha$ satisfies, for all $\beta, \beta' \in \mathbb{G}\mathsf{C}(\mathfrak{X})$ and $r \in \mathbb{Q}$,

(*i'*) $\|\beta\|_\alpha \geq 0$;
(*ii'*) $\|r\beta\|_\alpha = |r| \|\beta\|_\alpha$;
(*iii'*) $\|\beta + \beta'\|_\alpha \leq \|\beta\|_\alpha + \|\beta'\|_\alpha$; *and*
(*iv'*) $\beta = 0$ *in* $\mathbb{G}\mathsf{C}(\mathfrak{X})$ *if and only if* $\|\beta\|_\alpha = 0$ *for all* $\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{X}^c)$.

In particular, $\| - \|_\alpha$ is a semi-norm on $\mathbb{G}\mathsf{C}(\mathfrak{X})$.

Proof. Properties (*i*)–(*iii*) and (*i'*)–(*iii'*) follow immediately from the corresponding properties for numerical value and from the linearity of the map induced by the Euler characteristic. Properties (*iv*) and (*iv'*) follow from the definition of $\mathbb{G}\mathsf{P}(\mathfrak{X})$ and $\mathbb{G}\mathsf{C}(\mathfrak{X})$. ∎

Note that, for elements γ and γ' in either $\mathbb{G}\mathsf{P}(\mathfrak{X})$ or $\mathbb{G}\mathsf{C}(\mathfrak{X})$ and $r \in \mathbb{Q}$, we also have

$$\| - \|_{r\gamma} = |r| \| - \|_{\gamma} \quad \text{and} \quad \| - \|_{\gamma+\gamma'} \leq \| - \|_{\gamma} + \| - \|_{\gamma'}.$$

We equip $\mathbb{G}\mathsf{P}(\mathfrak{X})$ (and $\mathbb{G}\mathsf{C}(\mathfrak{X})$, respectively) with the *initial topology* of the family $(\| - \|_{\beta})_{\beta \in \mathbb{G}\mathsf{C}(\mathfrak{X}^c)}$ (or $(\| - \|_{\alpha})_{\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{X}^c)}$, respectively) of semi-norms: that is, the coarsest topology such that each of the maps $\| - \|_{\beta}$ (or $\| - \|_{\alpha}$, respectively) is continuous. With the initial topology, continuity of maps to and from Grothendieck spaces can be verified by a comparison of semi-norms; in fact, any map from a topological space to a Grothendieck space is continuous exactly when its composition with each semi-norm is continuous. Using this fact, it is easy to verify continuity of addition and scalar multiplication on Grothendieck spaces. We can also readily verify continuity of the other maps that we have introduced so far:

Proposition 10. *Suppose that $\mathfrak{X} \subseteq \operatorname{Spec} R$. Then the tensor product homomorphisms $\mathbb{G}\mathsf{P}(\mathfrak{X}) \otimes_{\mathbb{Q}} \mathbb{G}\mathsf{C}(\mathfrak{X}^c) \to \mathbb{G}\mathsf{C}(\mathfrak{m})$ and $\mathbb{G}\mathsf{P}(\mathfrak{X}) \otimes_{\mathbb{Q}} \mathbb{G}\mathsf{P}(\mathfrak{X}^c) \to \mathbb{G}\mathsf{P}(\mathfrak{m})$ are continuous in each variable. Furthermore, if $\mathfrak{X} \subseteq \mathfrak{Y} \subseteq \operatorname{Spec} R$, then the inclusion homomorphisms $\mathbb{G}\mathsf{P}(\mathfrak{X}) \to \mathbb{G}\mathsf{C}(\mathfrak{X})$, $\mathbb{G}\mathsf{P}(\mathfrak{X}) \to \mathbb{G}\mathsf{P}(\mathfrak{Y})$ and $\mathbb{G}\mathsf{C}(\mathfrak{X}) \to \mathbb{G}\mathsf{C}(\mathfrak{Y})$ are continuous.*

Proof. Continuity of the tensor product homomorphisms follows since

$$\|\alpha \otimes \beta\|_{\gamma} = \|\alpha\|_{\beta \otimes \gamma} = \|\beta\|_{\alpha \otimes \gamma},$$

when $\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{X})$, $\beta \in \mathbb{G}\mathsf{C}(\mathfrak{X}^c)$ and $\gamma \in \mathbb{G}\mathsf{P}(\{\mathfrak{m}\}^c) = \mathbb{G}\mathsf{P}(\operatorname{Spec} R)$ as well as when $\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{X})$, $\beta \in \mathbb{G}\mathsf{P}(\mathfrak{X}^c)$ and $\gamma \in \mathbb{G}\mathsf{C}(\{\mathfrak{m}\}^c) = \mathbb{G}\mathsf{C}(\operatorname{Spec} R)$. Continuity of the inclusion homomorphisms is immediate, since they all preserve the semi-norms that are defined for the co-domain of the inclusion. $\square$

4. Frobenius and vanishing dimension

Notation. Throughout this section, R is assumed to be complete of prime characteristic p, and k is assumed to be a perfect field.

Note that, although the assumptions that R be complete and k be perfect may seem restrictive, they really are not when it comes to dealing with intersection multiplicities; for more details, see Dutta [1, p. 425].

The Frobenius ring homomorphism $f\colon R \to R$ is given by $f(r) = r^p$; the composition of e copies of f is the ring homomorphism $f^e\colon R \to R$ given by $f(r) = r^{p^e}$. We denote by ${}^{f^e}\!R$ the bi-R-algebra R having the structure of an R-algebra from the left by f^e and from the right by the identity map: that is, if $x \in {}^{f^e}\!R$ and $r, s \in R$, then $r \cdot x \cdot s = r^{p^e} xs$.

Definition 11. Two functors, ${}^{f^e}(-)$ and F_R^e, are defined on the category of R-modules by

$${}^{f^e}(-) = {}^{f^e}\!R \otimes_R - \quad \text{and} \quad F_R^e(-) = - \otimes_R {}^{f^e}\!R,$$

where, for a module M, ${}^{f^e}\!M$ is viewed through its *left* structure, whereas $F_R^e(M)$ is viewed through its *right* structure. The functor F_R is called the *Frobenius functor*.

We now note a few facts about the functors ${}^{f^e}(-)$ and F_R^e.

Proposition 12. *The following hold.*

(*i*) *For a module N, ${}^{f^e}\!N$ is the module N with R-module structure defined by restriction of scalars via f^e: that is, if $n \in {}^{f^e}\!N$ and $r \in R$, then $r \cdot n = r^{p^e} n$.*
(*ii*) *For a homomorphism $\varphi\colon N \to N'$, ${}^{f^e}\!\varphi = \varphi$.*
(*iii*) *The functor ${}^{f^e}(-)$ is exact.*
(*iv*) *${}^{f^e}(-)$ and F_R^e are the compositions of e copies of ${}^{f}(-)$ and F_R, respectively.*
(*v*) *If X is a complex of finitely generated projective modules, then $F_R^e(X)$ is a complex with the same modules as X and with matrices representing the differentials given by raising the entries in the corresponding matrices for X to the p^eth power.*
(*vi*) *If X and Y are bounded complexes of finitely generated projective modules, then $F_R^e(X \otimes_R Y) \cong F_R^e(X) \otimes_R F_R^e(Y)$.*
(*vii*) *The functor F_R^e preserves exactness of bounded complexes of finitely generated projective modules.*
(*viii*) *If S is another commutative, Noetherian, local ring of characteristic p, and $R \to S$ is a ring homomorphism, then, for any bounded complex X of finitely generated projective modules, $F_R^e(X) \otimes_R S \cong F_S^e(X \otimes_R S)$; in particular, for a prime ideal $\mathfrak{p}$, $F_R^e(X)_\mathfrak{p} \cong F_{R_\mathfrak{p}}^e(X_\mathfrak{p})$.*

Proof. All properties are readily verified. For more details, see, for example, Peskine and Szpiro [9] or Roberts [12]. □

The fact that k is perfect implies that ${}^{f^e}\!R$ is finitely generated as a left R-module, and hence ${}^{f^e}(-)$ defines a functor $\mathsf{C}(\mathfrak{X}) \to \mathsf{C}(\mathfrak{X})$ for every $\mathfrak{X} \subseteq \operatorname{Spec} R$. Since ${}^{f^e}(-)$ is exact, for any complex $Y \in \mathsf{C}(\mathfrak{m})$,

$$\chi^R({}^{f^e}Y) = \chi^R(Y)\,\ell({}^{f^e}k) = \chi^R(Y),$$

where the last equation follows since $k \cong {}^{f^e}k$. Now, suppose that $X \in \mathsf{P}(\mathfrak{X})$ and $Y \in \mathsf{C}(\mathfrak{X}^c)$. It is not hard to see that ${}^{f^e}(F_R^e(X) \otimes_R Y) \cong X \otimes_R {}^{f^e}Y$, and it follows that

$$\chi^R(F_R^e(X) \otimes_R Y) = \chi^R(X \otimes_R {}^{f^e}Y).$$

It is an immediate consequence of (*v*), (*vii*) and (*viii*) that F_R^e defines a functor $\mathsf{P}(\mathfrak{X}) \to \mathsf{P}(\mathfrak{X})$ for every $\mathfrak{X} \subseteq \operatorname{Spec} R$. If $[X]_{\mathsf{P}(\mathfrak{X})} = [\widetilde{X}]_{\mathsf{P}(\mathfrak{X})}$ in $\mathbb{G}\mathsf{P}(\mathfrak{X})$ for complexes $X, \widetilde{X} \in \mathsf{P}(\mathfrak{X})$, then $\chi^R(X \otimes_R Y) = \chi^R(\widetilde{X} \otimes_R Y)$ for all $Y \in \mathsf{C}(\mathfrak{X}^c)$, and hence

$$\chi^R(F_R^e(X) \otimes_R Y) = \chi^R(X \otimes_R {}^{f^e}Y) = \chi^R(\widetilde{X} \otimes_R {}^{f^e}Y) = \chi^R(F_R^e(\widetilde{X}) \otimes_R Y)$$

for all $Y \in \mathsf{C}(\mathfrak{X}^c)$, and we conclude that $[F_R(X)]_{\mathsf{P}(\mathfrak{X})} = [F_R(\widetilde{X})]_{\mathsf{P}(\mathfrak{X})}$. Thus, F_R^e induces an endomorphism on $\mathbb{G}\mathsf{P}(\mathfrak{X})$ given by $[X]_{\mathsf{P}(\mathfrak{X})} \mapsto [F_R(X)]_{\mathsf{P}(\mathfrak{X})}$.

Definition 13. Given $\mathfrak{X} \subseteq \operatorname{Spec} R$ and $e \in \mathbb{N}_0$, the endomorphism on $\mathbb{G}\mathsf{P}(\mathfrak{X})$ induced by F_R^e shall be denoted by $F_\mathfrak{X}^e$. Further, we define

$$\Phi_\mathfrak{X}^e = \frac{1}{p^{e \operatorname{codim} \mathfrak{X}}} F_\mathfrak{X}^e.$$

For $\mathfrak{X} = \{\mathfrak{m}\}$ we shall simply write $F_\mathfrak{m}^e$ and $\Phi_\mathfrak{m}^e$.

Lemma 14. *Suppose that $\mathfrak{X} \subseteq \operatorname{Spec} R$ and let $\beta \in \mathbb{G}\mathsf{C}(\mathfrak{X}^c)$. There exist elements $\gamma_1, \ldots, \gamma_t \in \mathbb{G}\mathsf{C}(\mathfrak{X}^c)$ with $\dim \gamma_i \leq \dim \beta$ for all i such that*

$$\frac{1}{p^{e \dim \beta}} \|F_\mathfrak{X}^e(-)\|_\beta \leq \sum_i \|-\|_{\gamma_i}$$

for all $e \in \mathbb{N}_0$*.*

Proof. The proof is by induction on $m = \dim \beta$. Let $\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{X})$. By taking a filtration of β with elements of dimension less than or equal to $\dim \beta$, we can assume that $\beta = [R/\mathfrak{q}]_{\mathsf{C}(\mathfrak{X}^c)}$ for a prime ideal $\mathfrak{q} \in \mathfrak{X}^c$ with $\dim R/\mathfrak{q} = \dim \beta = m$. Further, we can assume that $\alpha = [X]_{\mathsf{P}(\mathfrak{X})}$ for some minimal complex $X \in \mathsf{P}(\mathfrak{X})$. The case $m = 0$ is now easy since, in this case, $\mathfrak{q} = \mathfrak{m}$ so that $\beta = [k]_{\mathsf{C}(\mathfrak{m})}$, and minimality of X means that $F_R(X) \otimes_R k = F_k(X \otimes_R k) = X \otimes_R k$, so that $\|F^e_{\mathfrak{X}}(\alpha)\|_k = \|\alpha\|_k$ for all $e \in \mathbb{N}_0$.

Now, suppose that $m > 0$ and that the statement holds for smaller values of m. The ring $R/\mathfrak{q}$ is a complete domain of dimension m with perfect residue field, so the torsion-free rank of ${}^f(R/\mathfrak{q})$ over $R/\mathfrak{q}$ is p^m (see, for example, Dutta [1, p. 426]). Thus, we have short exact sequence

$$0 \to (R/\mathfrak{q})^{p^m} \to {}^f(R/\mathfrak{q}) \to Q \to 0,$$

where Q is a finitely generated module with $\dim Q < m$. By tensoring with $F^j_R(X)$, we get the short exact sequence

$$0 \to (F^j_R(X) \otimes_R R/\mathfrak{q})^{p^m} \to F^{j+1}_R(X) \otimes_R R/\mathfrak{q} \to F^j_R(X) \otimes_R Q \to 0.$$

It follows that

$$\|\frac{1}{p^{(j+1)m}} F^{j+1}_{\mathfrak{X}}(X) - \frac{1}{p^{jm}} F^j_{\mathfrak{X}}(X)\|_{R/\mathfrak{q}} = \|\frac{1}{p^{(j+1)m}} F^j_{\mathfrak{X}}(X)\|_Q.$$

Since $\dim Q < m$, the induction hypothesis implies that there exist elements $\gamma'_1, \ldots, \gamma'_{t-1} \in \mathbb{G}\mathsf{C}(\mathfrak{X}^c)$ with $\dim \gamma'_i \leq \dim Q < \dim R/\mathfrak{q}$ for all i, such that this is bounded by $p^{-j} \sum_i \|X\|_{\gamma'_i}$. Thus,

$$\begin{aligned}\|\frac{1}{p^{em}} F^e_{\mathfrak{X}}(X)\|_{R/\mathfrak{q}} &\leq \|X\|_{R/\mathfrak{q}} + \sum_{j=0}^{e-1} \|\frac{1}{p^{(j+1)m}} F^{j+1}_{\mathfrak{X}}(X) - \frac{1}{p^{jm}} F^j_{\mathfrak{X}}(X)\|_{R/\mathfrak{q}} \\ &\leq \|X\|_{R/\mathfrak{q}} + (1 + \frac{1}{p} + \cdots + \frac{1}{p^{e-1}}) \sum_i \|X\|_{\gamma'_i}\end{aligned}$$

and the result now follows by setting $\gamma_i = (1 - 1/p)^{-1} \gamma'_i$ for each $i = 1, \ldots, t-1$ and $\gamma_t = [R/\mathfrak{q}]_{\mathsf{C}(\mathfrak{X}^c)}$. ☐

The proof of Lemma 14 contains another result which will become useful later and which we therefore state separately as Lemma 15 below.

Lemma 15. *Suppose that* $\mathfrak{X} \subseteq \operatorname{Spec} R$ *and let* $\beta \in \mathbb{G}\mathsf{C}(\mathfrak{X}^c)$*. Then there exist elements* $\gamma_1, \ldots, \gamma_t \in \mathbb{G}\mathsf{C}(\mathfrak{X}^c)$ *with* $\dim \gamma_i < \dim \beta$ *for all* i *such that*

$$\|\frac{1}{p^{(e+1)\dim \beta}} F^{e+1}_{\mathfrak{X}}(-) - \frac{1}{p^{e \dim \beta}} F^e_{\mathfrak{X}}(-)\|_\beta \leq \frac{1}{p^e} \sum_i \| - \|_{\gamma_i}$$

for all $e \in \mathbb{N}_0$*.* ☐

Proposition 16. *For all* $\mathfrak{X} \subseteq \operatorname{Spec} R$*, the endomorphism* $\Phi_{\mathfrak{X}}$ *on* $\mathbb{G}\mathsf{P}(\mathfrak{X})$ *is continuous.*

Proof. This follows immediately from Lemma 14. ☐

Proposition 17. *Suppose that* $\mathfrak{X} \subseteq \operatorname{Spec} R$ *and let* $\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{X})$*. Then* α *satisfies vanishing if and only if* $\alpha = \Phi_{\mathfrak{X}}(\alpha)$*.*

Proof. If α satisfies vanishing, it follows immediately from Lemma 14 and 15 that, for $\beta \in \mathbb{G}\mathsf{C}(\mathfrak{X}^c)$, $\|\alpha\|_\beta = \|\Phi_{\mathfrak{X}}(\alpha)\|_\beta = 0$ whenever $\dim \beta < \operatorname{codim} \mathfrak{X}$ and $\|\alpha - \Phi_{\mathfrak{X}}(\alpha)\|_\beta = 0$ whenever $\dim \beta = \operatorname{codim} \mathfrak{X}$, so that, in any case, $\|\alpha - \Phi_{\mathfrak{X}}(\alpha)\|_\beta = 0$, and hence $\alpha = \Phi_{\mathfrak{X}}(\alpha)$. Conversely, if $\alpha = \Phi_{\mathfrak{X}}(\alpha)$, it follows immediately from Lemma 14 that, for all $\beta \in \mathbb{G}\mathsf{C}(\mathfrak{X}^c)$ with $\dim \beta < \operatorname{codim} \mathfrak{X}$, $\|\alpha\|_\beta$ is bounded by a sequence converging to 0, so that $\|\alpha\|_\beta = 0$ and α satisfies vanishing. □

Definition 18. Suppose that $\mathfrak{X} \subseteq \operatorname{Spec} R$ and let $\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{X})$. For $i \in \mathbb{N}_0$ we define

$$p^{i\infty}\Phi_{\mathfrak{X}}^{\infty}(\alpha) = \lim_{e \to \infty} p^{ie}\Phi_{\mathfrak{X}}^{e}(\alpha)$$

whenever this converges; in particular, for $i = 0$, we define $\Phi_{\mathfrak{X}}^{\infty}(\alpha) = \lim_{e\to\infty} \Phi^e(\alpha)$.

We are now ready to prove the main theorem of this paper.

Theorem 19. *Suppose that $\mathfrak{X} \subseteq \operatorname{Spec} R$, let $\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{X})$ and suppose that u is a non-negative integer with $u \geq \operatorname{vdim} \alpha$. Then*

$$(p^u\Phi_{\mathfrak{X}} - \mathrm{id}) \circ \cdots \circ (p\Phi_{\mathfrak{X}} - \mathrm{id}) \circ (\Phi_{\mathfrak{X}} - \mathrm{id})(\alpha) = 0. \tag{1}$$

Further, there exists a decomposition $\alpha = \alpha^{(0)} + \cdots + \alpha^{(u)}$ in which each $\alpha^{(i)}$ is either zero or an eigenvector for $\Phi_{\mathfrak{X}}$ with eigenvalue $1/p^i$. The elements $\alpha^{(0)}, \ldots, \alpha^{(u)}$ can be recursively defined by

$$\alpha^{(0)} = \Phi_{\mathfrak{X}}^{\infty}(\alpha) \quad \textit{and} \quad \alpha^{(i)} = p^{i\infty}\Phi_{\mathfrak{X}}^{\infty}(\alpha - (\alpha^{(0)} + \cdots + \alpha^{(i-1)})),$$

and there is a formula

$$\begin{pmatrix} \alpha^{(0)} \\ \vdots \\ \alpha^{(u)} \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1/p & \cdots & 1/p^u \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1/p^u & \cdots & 1/p^{u^2} \end{pmatrix}^{-1} \begin{pmatrix} \alpha \\ \Phi_{\mathfrak{X}}(\alpha) \\ \vdots \\ \Phi_{\mathfrak{X}}^u(\alpha) \end{pmatrix}. \tag{2}$$

Proof. We prove (1) by induction on u. The case $u = 0$ is trivial since Proposition 17 in this situation yields that $(\Phi_{\mathfrak{X}} - \mathrm{id})(\alpha) = 0$. Now, suppose that $u > 0$ and that the formula holds for smaller values of u. By Proposition 17 and commutativity of the involved maps, equation (1) holds if and only if vanishing holds for the element

$$\beta = (p^u\Phi_{\mathfrak{X}} - \mathrm{id}) \circ \cdots \circ (p\Phi_{\mathfrak{X}} - \mathrm{id})(\alpha).$$

Now, satisfying vanishing is equivalent to being in the kernel of any inclusion homomorphism $\mathbb{G}\mathsf{P}(\mathfrak{X}) \to \mathbb{G}\mathsf{P}(\mathfrak{Y})$, where $\mathfrak{Y} \supseteq \mathfrak{X}$ has $\operatorname{codim} \mathfrak{Y} = \operatorname{codim} \mathfrak{X} - 1$. But since, in $\mathbb{G}\mathsf{P}(\mathfrak{Y})$ for such a $\mathfrak{Y}$, $\Phi_{\mathfrak{X}}(\alpha) = p^{-1}\Phi_{\mathfrak{Y}}(\bar{\alpha})$, where $\bar{\alpha}$ denotes the image of α in $\mathbb{G}\mathsf{P}(\mathfrak{Y})$, we get that, in $\mathbb{G}\mathsf{P}(\mathfrak{Y})$,

$$\beta = (p^{u-1}\Phi_{\mathfrak{Y}} - \mathrm{id}) \circ \cdots \circ (p\Phi_{\mathfrak{Y}} - \mathrm{id}) \circ (\Phi_{\mathfrak{Y}} - \mathrm{id})(\bar{\alpha}) = 0,$$

where the last equation follows by induction, since $\operatorname{vdim} \bar{\alpha} \leq u - 1$ by Remark 7. This proves (1).

By applying $\Phi_{\mathfrak{X}}^{e-u}$ to (1), we get a recursive formula to calculate $\Phi_{\mathfrak{X}}^{e+1}(\alpha)$ from $\Phi_{\mathfrak{X}}^{e}(\alpha), \ldots, \Phi_{\mathfrak{X}}^{e-u}(\alpha)$. The characteristic polynomial for the recursion is

$$(p^u x - 1) \cdots (px - 1)(x - 1),$$

which has $u + 1$ distinct roots, $1, 1/p, \ldots, 1/p^u$. Thus, there is a general formula

$$\Phi_{\mathfrak{X}}^{e}(\alpha) = \alpha^{(0)} + \frac{1}{p^e}\alpha^{(1)} + \cdots + \frac{1}{p^{ue}}\alpha^{(u)} \tag{3}$$

for suitable $\alpha^{(0)}, \dots, \alpha^{(u)} \in \mathbb{G}\mathsf{P}(\mathfrak{X})$, where each $\alpha^{(i)}$ satisfies

$$\Phi_{\mathfrak{X}}^{e}(\alpha^{(i)}) = \frac{1}{p^{ei}}\alpha^{(i)} \tag{4}$$

and hence is an eigenvector for $\Phi_{\mathfrak{X}}$ with eigenvalue $1/p^i$.

We obtain the recursive definition of $\alpha^{(i)}$ by induction on i. The case $i = 0$ follows immediately from (3) by letting e go to infinity. Suppose now that $i > 0$ and that the result holds for smaller values of i. From (4) we then get

$$\begin{aligned} p^{ie}\Phi^{e}(\alpha - (\alpha^{(0)} + \cdots + \alpha^{(i-1)})) &= p^{ie}\Phi^{e}(\alpha^{(i)} + \cdots + \alpha^{(u)}) \\ &= \alpha^{(i)} + \frac{1}{p^{e}}\alpha^{(i+1)} + \cdots + \frac{1}{p^{e(u-i)}}\alpha^{(u)}, \end{aligned}$$

and letting e go to infinity, we obtain the desired formula.

From (3) we know that $\alpha^{(0)}, \dots, \alpha^{(u)}$ solve the following system of equations with rational coefficients.

$$\begin{array}{ccccccccc} \alpha^{(0)} & + & \alpha^{(1)} & + & \cdots & + & \alpha^{(u)} & = & \alpha \\ \alpha^{(0)} & + & \frac{1}{p}\alpha^{(1)} & + & \cdots & + & \frac{1}{p^u}\alpha^{(u)} & = & \Phi_{\mathfrak{X}}(\alpha) \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ \alpha^{(0)} & + & \frac{1}{p^u}\alpha^{(1)} & + & \cdots & + & \frac{1}{p^{u^2}}\alpha^{(u)} & = & \Phi_{\mathfrak{X}}^{u}(\alpha) \end{array}$$

Formula (2) now follows. (The matrix is the Vandermonde matrix of the elements $1, 1/p, \dots, 1/p^u$ with determinant $\prod_{0\le i<j\le u}(1/p^j - 1/p^i) \neq 0$.) $\square$

Remark 20. Since $\mathbb{G}\mathsf{P}(\mathfrak{X})$ splits up into a direct sum of eigenspaces of $\Phi_{\mathfrak{X}}$, it is clear that the decomposition of an element into eigenvectors for $\Phi_{\mathfrak{X}}$ is unique. We also clearly have that

$$(r\alpha)^{(i)} = r\alpha^{(i)} \quad \text{and} \quad (\alpha + \beta)^{(i)} = \alpha^{(i)} + \beta^{(i)}$$

for all $i \in \mathbb{N}_0$, $r \in \mathbb{Q}$ and $\alpha, \beta \in \mathbb{G}\mathsf{P}(\mathfrak{X})$. We obviously have $(\Phi_{\mathfrak{X}}^{e}(\alpha))^{(i)} = p^{-ie}\alpha^{(i)}$.

It is also easy to see that, for $\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{X})$ and $\beta \in \mathbb{G}\mathsf{P}(\mathfrak{X}^c)$,

$$(\alpha \otimes \beta)^{(t)} = \sum_{i+j=t} \alpha^{(i)} \otimes \beta^{(j)}$$

in $\mathbb{G}\mathsf{P}(\mathfrak{m})$. In particular, $(\alpha \otimes \beta)^{(0)} = \alpha^{(0)} \otimes \beta^{(0)}$.

Suppose now that $\mathfrak{X} \subseteq \mathfrak{Y} \subseteq \operatorname{Spec} R$ and let $s = \operatorname{codim} \mathfrak{X} - \operatorname{codim} \mathfrak{Y}$. Let $\bar{\alpha}$ denote the image of α in $\mathbb{G}\mathsf{P}(\mathfrak{Y})$ under the inclusion homomorphism $\mathbb{G}\mathsf{P}(\mathfrak{X}) \to \mathbb{G}\mathsf{P}(\mathfrak{Y})$. Since the image of $\Phi_{\mathfrak{X}}(\alpha)$ in $\mathbb{G}\mathsf{P}(\mathfrak{Y})$ is equal to $p^{-s}\Phi_{\mathfrak{Y}}(\bar{\alpha})$, it follows from Theorem 19 that the image of $\alpha^{(i)}$ in $\mathbb{G}\mathsf{P}(\mathfrak{Y})$ is $\bar{\alpha}^{(i-s)}$ whenever $i \geq s$ and zero whenever $i < s$.

Remark 21. From Theorem 19 we see that the general formula for $\Phi_{\mathfrak{X}}^{e}(\alpha)$ is

$$\Phi_{\mathfrak{X}}^{e}(\alpha) = \begin{pmatrix} 1 & 1/p^e & \cdots & 1/p^{eu} \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1/p & \cdots & 1/p^u \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1/p^u & \cdots & 1/p^{u^2} \end{pmatrix}^{-1} \begin{pmatrix} \alpha \\ \Phi_{\mathfrak{X}}(\alpha) \\ \vdots \\ \Phi_{\mathfrak{X}}^{u}(\alpha) \end{pmatrix}$$

and, in particular, that

$$\Phi_{\mathfrak{X}}^{\infty}(\alpha) = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1/p & \cdots & 1/p^u \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1/p^u & \cdots & 1/p^{u^2} \end{pmatrix}^{-1} \begin{pmatrix} \alpha \\ \Phi_{\mathfrak{X}}(\alpha) \\ \vdots \\ \Phi_{\mathfrak{X}}^u(\alpha) \end{pmatrix}.$$

The Dutta multiplicity of an element α and arbitrary elements from $\mathbb{G}\mathsf{C}(\mathfrak{X}^c)$ is given by application of the function

$$\chi_\infty^R(\alpha, -) = \lim_{e\to\infty} \chi^R(\Phi_{\mathfrak{X}}^e(\alpha) \otimes -) = \chi^R(\Phi_{\mathfrak{X}}^\infty(\alpha) \otimes -).$$

Thus, the Dutta multiplicity is a rational number and we need not find a limit to calculate it. This is valuable knowledge, for instance, when using a computer to calculate Dutta multiplicity. Translating this into the usual setup with finitely generated modules M and N with $\operatorname{pd} M < \infty$, $\ell(M \otimes_R N) < \infty$ and $t = \operatorname{codim} M \geq \dim N$, we get the general formula

$$\begin{aligned} \chi_\infty^R(M,N) &= \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1/p & \cdots & 1/p^u \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1/p^u & \cdots & 1/p^{u^2} \end{pmatrix}^{-1} \begin{pmatrix} \chi^R(M,N) \\ p^{-t}\chi^R(F_R(M),N) \\ \vdots \\ p^{-ut}\chi^R(F_R^u(M),N) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ p^t & p^{t-1} & \cdots & p^{t-u} \\ \vdots & \vdots & \ddots & \vdots \\ p^{ut} & p^{u(t-1)} & \cdots & p^{u(t-u)} \end{pmatrix}^{-1} \begin{pmatrix} \chi^R(M,N) \\ \chi^R(F_R(M),N) \\ \vdots \\ \chi^R(F_R^u(M),N) \end{pmatrix}. \end{aligned}$$

For example, the case $\operatorname{vdim} \alpha \leq 1$ yields

$$\Phi_{\mathfrak{X}}^\infty(\alpha) = \frac{1}{p-1}\left(p\Phi_{\mathfrak{X}}(\alpha) - \alpha\right),$$

so that

$$\chi_\infty^R(\alpha,-) = \frac{1}{p-1}\left(p\chi^R(\Phi_{\mathfrak{X}}(\alpha)\otimes -) - \chi^R(\alpha\otimes -)\right).$$

In other words, for M and N as above,

$$\chi_\infty^R(M,N) = \frac{1}{p-1}\left(\frac{1}{p^{t-1}}\chi^R(F_R(M),N) - \chi^R(M,N)\right).$$

In the case $\operatorname{vdim}\alpha \leq 2$, we get

$$\Phi_{\mathfrak{X}}^\infty(\alpha) = \frac{1}{p^3-p^2-p+1}\left(p^3\Phi_{\mathfrak{X}}^2(\alpha) - p(p+1)\Phi_{\mathfrak{X}}(\alpha) + \alpha\right),$$

so that

$$\begin{aligned} \chi_\infty^R(\alpha,-) = \frac{1}{p^3-p^2-p+1}\big(&p^3\chi^R(\Phi_{\mathfrak{X}}^2(\alpha)\otimes -) \\ &- p(p+1)\chi^R(\Phi_{\mathfrak{X}}(\alpha)\otimes -) + \chi^R(\alpha\otimes -)\big). \end{aligned}$$

In other words, for M and N as above,

$$\chi^R_\infty(M,N) = \frac{1}{p^3-p^2-p+1}\Big(\frac{1}{p^{2t-3}}\chi^R(F_R^2(M),N) \\ - \frac{p+1}{p^{t-1}}\chi^R(F_R(M),N) + \chi^R(M,N)\Big).$$

The fact that Dutta multiplicity satisfies vanishing is also immediate:

Corollary 22. *Suppose that $\mathfrak{X} \subseteq \operatorname{Spec} R$ and let $\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{X})$. Then $\Phi^\infty_\mathfrak{X}(\alpha)$ satisfies vanishing.*

Proof. By continuity, $\Phi_\mathfrak{X}(\Phi^\infty_\mathfrak{X}(\alpha)) = \Phi^\infty_\mathfrak{X}(\alpha)$, and Proposition 17 yields that $\Phi^\infty_\mathfrak{X}(\alpha)$ satisfies vanishing. □

With Theorem 19 in hand, we can now in multiple ways describe what it means to have a certain vanishing dimension. We first look at the vanishing dimension zero case.

Proposition 23. *Suppose that $\mathfrak{X} \subseteq \operatorname{Spec} R$ and let $\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{X})$. The following are equivalent.*

(*i*) *α satisfies vanishing.*
(*ii*) *$\alpha \otimes \beta = 0$ for all $\beta \in \mathbb{G}\mathsf{C}(\mathfrak{X}^c)$ with $\dim \beta < \operatorname{codim} \mathfrak{X}$.*
(*iii*) *$\alpha = \alpha^{(0)}$.*
(*iv*) *$\alpha = \Phi_\mathfrak{X}(\alpha)$.*
(*v*) *$\alpha = \Phi^e_\mathfrak{X}(\alpha)$ for some $e \in \mathbb{N}$.*
(*vi*) *$\alpha = \Phi^\infty_\mathfrak{X}(\alpha)$.*
(*vii*) *$\alpha = 0$ as an element of $\mathbb{G}\mathsf{P}(\mathfrak{Y})$ for any $\mathfrak{Y} \supseteq \mathfrak{X}$ with $\operatorname{codim} \mathfrak{Y} < \operatorname{codim} \mathfrak{X}$.*
(*viii*) *$\alpha = 0$ as an element of $\mathbb{G}\mathsf{P}(\mathfrak{Y})$ for any $\mathfrak{Y} \supseteq \mathfrak{X}$ with $\operatorname{codim} \mathfrak{Y} = \operatorname{codim} \mathfrak{X} - 1$.*
(*ix*) *$\operatorname{vdim} \alpha \leq 0$.*

Proof. (*i*) is equivalent to (*ii*) by definition; (*i*) is equivalent to (*iv*) by Proposition 17; (*iv*) is equivalent to (*iii*) and (*vi*) by Theorem 19; (*iv*) implies (*v*) implies (*vi*), so these must all be equivalent; (*i*) is equivalent to (*vii*) and (*viii*) by Remark 7; and (*i*) is equivalent to (*ix*) by definition of vanishing dimension. □

We can also establish equivalent conditions in the general case for having a certain vanishing dimension:

Proposition 24. *Suppose that $\mathfrak{X} \subseteq \operatorname{Spec} R$, let $\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{X})$ and let $u \in \mathbb{N}_0$. The following are equivalent.*

(*i*) *$\alpha \otimes \beta = 0$ for all $\beta \in \mathbb{G}\mathsf{C}(\mathfrak{X}^c)$ with $\dim \beta < \operatorname{codim} \mathfrak{X} - u$.*
(*ii*) *$\alpha = \alpha^{(0)} + \cdots + \alpha^{(u)}$.*
(*iii*) *$(p^u \Phi_\mathfrak{X} - \mathrm{id}) \circ \cdots \circ (p\Phi_\mathfrak{X} - \mathrm{id}) \circ (\Phi_\mathfrak{X} - \mathrm{id})(\alpha) = 0$.*
(*iv*) *$\alpha = 0$ as an element of $\mathbb{G}\mathsf{P}(\mathfrak{Y})$ for any $\mathfrak{Y} \supseteq \mathfrak{X}$ with $\operatorname{codim} \mathfrak{Y} < \operatorname{codim} \mathfrak{X} - u$.*
(*v*) *$\alpha = 0$ as an element of $\mathbb{G}\mathsf{P}(\mathfrak{Y})$ for any $\mathfrak{Y} \supseteq \mathfrak{X}$ with $\operatorname{codim} \mathfrak{Y} = \operatorname{codim} \mathfrak{X} - u - 1$.*
(*vi*) *$\operatorname{vdim} \alpha \leq u$.*

Proof. The proof of Theorem 19 shows how (*vi*) implies (*iii*) which again implies (*ii*); (*vi*) is equivalent to (*i*) by definition of vanishing dimension; (*i*) is clearly equivalent to (*iv*) and (*v*); and (*ii*) implies (*v*) by Remark 20. □

Having vanishing dimension exactly equal to $u > 0$ of course means that the above conditions are satisfied and that the same conditions fail to hold if u is replaced by $u-1$. In particular, if $\operatorname{vdim}\alpha = u$, then $\alpha^{(u)} \neq 0$ and there exists a $\beta \in \mathbb{G}\mathsf{C}(\mathfrak{X}^c)$ with $\dim\beta = \operatorname{codim}\mathfrak{X} - u$ such that $\alpha\otimes\beta = \alpha^{(u)}\otimes\beta \neq 0$. Consequently, if the term $\alpha^{(i)}$ is non-zero, then it has vanishing dimension i and can be regarded as "the part of α that allows a counterexample to vanishing where the difference of co-dimension and dimension is equal to i".

We conclude this section by examining how the decomposition into eigenvectors behaves under extension of scalars.

Proposition 25. *Suppose that S is a another commutative, complete, Noetherian local ring of characteristic p and with perfect residue field and that $R \to S$ is a local ring homomorphism, such that S is finitely generated as an R-module. Let $\mathfrak{X} \subseteq \operatorname{Spec} R$ and $\mathfrak{Y} \subseteq \operatorname{Spec} S$ be such that conditions (i) and (ii) of Proposition 3 are satisfied, so that the extension of scalars map $-\otimes_R S\colon \mathbb{G}\mathsf{P}^R(\mathfrak{X}) \to \mathbb{G}\mathsf{P}^S(\mathfrak{Y})$ is defined. Let $\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{X})$ and set $t = \operatorname{codim}\mathfrak{X} - \operatorname{codim}\mathfrak{Y}$. Then $\alpha^{(i)} \otimes_R S = (\alpha \otimes_R S)^{(i-t)}$.*

Proof. Within $\mathbb{G}\mathsf{P}^S(\mathfrak{Y})$ we have that

$$\begin{aligned}\Phi_{\mathfrak{Y}}(\alpha^{(i)} \otimes_R S) &= \frac{1}{p^{\operatorname{codim}\mathfrak{Y}}} F_{\mathfrak{Y}}(\alpha^{(i)} \otimes_R S)\\ &= \frac{1}{p^{\operatorname{codim}\mathfrak{Y}}} F_{\mathfrak{X}}(\alpha^{(i)}) \otimes_R S\\ &= \frac{1}{p^{-t}} \Phi_{\mathfrak{X}}(\alpha^{(i)}) \otimes_R S\\ &= \frac{1}{p^{i-t}} \alpha^{(i)} \otimes_R S.\end{aligned}$$

This proves that $\alpha^{(i)} \otimes_R S$ in $\mathbb{G}\mathsf{P}^S(\mathfrak{Y})$ is an eigenvector for $\Phi_{\mathfrak{Y}}$ with eigenvalue $1/p^{i-t}$, and since $\alpha = \sum_i \alpha^{(i)}$, it follows that $\alpha \otimes_R S = \sum_i \alpha^{(i)} \otimes_R S$, and hence that $\alpha^{(i)} \otimes_R S = (\alpha \otimes_R S)^{(i-t)}$. $\square$

5. Numerical vanishing

Notation. Throughout this section, we continue to assume that R is complete of prime characteristic $p > 0$, and that k is a perfect field.

Although the relation $\alpha = \Phi_{\mathfrak{X}}^{\infty}(\alpha)$ might not hold in $\mathbb{G}\mathsf{P}(\mathfrak{X})$, there is still a chance that it holds after an application of the inclusion homomorphism $\mathbb{G}\mathsf{P}(\mathfrak{X}) \to \mathbb{G}\mathsf{C}(\mathfrak{X})$, and this situation has interesting consequences as well.

Definition 26. Suppose that $\mathfrak{X} \subseteq \operatorname{Spec} R$ and let $\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{X})$. We shall say that α satisfies *numerical vanishing* if $\alpha = \Phi_{\mathfrak{X}}^{\infty}(\alpha)$ in $\mathbb{G}\mathsf{C}(\mathfrak{X})$. We shall say that α satisfies *weak vanishing* if, for all $\beta \in \mathbb{G}\mathsf{P}(\mathfrak{X}^c)$, $\alpha \otimes \beta = 0$ in $\mathbb{G}\mathsf{C}(\mathfrak{m})$ whenever $\dim\beta < \operatorname{codim}\mathfrak{X}$.

The reason for the word "numerical" is that numerical vanishing can be verified "numerically" for elements of $\mathbb{G}\mathsf{P}(\mathfrak{m})$ (see Remark 28), and that numerical vanishing of all elements of all Grothendieck groups can be verified in this way (see Remark 32).

The concept of "weak vanishing" corresponds to a weaker version of Serre's vanishing conjecture in which both modules are assumed to have finite projective dimension—in fact, weak vanishing is a little stronger: if M is finitely generated and of finite projective dimension, and X is a projective resolution of M, then the element $[X]_{\mathsf{P}(\mathfrak{X})}$ in $\mathbb{G}\mathsf{P}(\mathfrak{X})$, where $\mathfrak{X} = \operatorname{Supp} M$, satisfies weak vanishing *only if* $\chi^R(M, N) = 0$ for all finitely generated modules N of finite projective dimension with $\ell(M \otimes_R N) < \infty$ and $\dim M + \dim N < \dim R$. (To get an "if and only if" statement, we would have to replace N by an arbitrary bounded complex of finitely generated projective modules.)

Proposition 27. *Suppose that* $\mathfrak{X} \subseteq \operatorname{Spec} R$ *and let* $\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{X})$. *For the following conditions, each condition implies the next.*

(*i*) α *satisfies vanishing.*
(*ii*) α *satisfies numerical vanishing*
(*iii*) α *satisfies weak vanishing*

Proof. It is clear that vanishing implies numerical vanishing. Suppose that α satisfies numerical vanishing and let $\beta \in \mathbb{G}\mathsf{P}(\mathfrak{X}^c)$ be such that $\dim \beta < \operatorname{codim} \mathfrak{X}$. When calculating $\alpha \otimes \beta$ in $\mathbb{G}\mathsf{C}(\mathfrak{m})$, we are allowed to consider α an element of $\mathbb{G}\mathsf{C}(\mathfrak{X})$. But then $\alpha \otimes \beta = \Phi^\infty_{\mathfrak{X}}(\alpha) \otimes \beta = 0$, since $\Phi^\infty_{\mathfrak{X}}(\alpha)$ satisfies vanishing, and we conclude that α satisfies weak vanishing. □

As Example 33 will show, the implications in Proposition 27 are strict.

Remark 28. If X is a complex in $\mathsf{P}(\mathfrak{m})$, then $[X]_{\mathsf{P}(\mathfrak{m})}$ satisfies numerical vanishing if and only if

$$\lim_{e\to\infty} \frac{1}{p^{e \dim R}} \chi^R(F^e_R(X)) = \chi^R(X).$$

In particular, if X is a projective resolution of a module M, then $[X]_{\mathsf{P}(\mathfrak{m})}$ satisfies numerical vanishing if and only if

$$\lim_{e\to\infty} \frac{1}{p^{e \dim R}} \ell(F^e_R(M)) = \ell(M); \tag{5}$$

this follows easily from the fact that the Euler characteristic is an isomorphism on $\mathbb{G}\mathsf{C}(\mathfrak{m})$ together with the result by Peskine and Szpiro [9, Theorem 1.7] that $F^e_R(X)$ is a projective resolution of $F^e_R(M)$ for all $e \in \mathbb{N}_0$. As we shall see in Proposition 29 below, for (5) to hold, it suffices (but need not be necessary) to verify that the equation

$$\ell(F^e_R(M)) = p^{e \dim R}\, \ell(M)$$

holds for $\operatorname{vdim}([X]_{\mathsf{P}(\mathfrak{m})})$ distinct values of $e > 0$.

Proposition 29. *Suppose that* $\mathfrak{X} \subseteq \operatorname{Spec} R$ *and let* $\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{X})$. *A sufficient condition for* α *to satisfy numerical vanishing is that* $\alpha = \Phi^e_{\mathfrak{X}}(\alpha)$ *holds in* $\mathbb{G}\mathsf{C}(\mathfrak{X})$ *for* $\operatorname{vdim}(\alpha)$ *distinct values of* $e > 0$.

Proof. Let $u = \operatorname{vdim}(\alpha)$. The difference $\alpha - \Phi^e_{\mathfrak{X}}(\alpha)$ in $\mathbb{G}\mathsf{C}(\mathfrak{X})$ is obtained by inserting $x = 1/p^e$ in the polynomial

$$(\alpha^{(0)} - \alpha) + x\alpha^{(1)} + \cdots + x^u \alpha^{(u)}.$$

The polynomial always has the root $x = 1$. If there are u additional roots, it must be the zero-polynomial, so that $\alpha = \Phi^e_{\mathfrak{X}}(\alpha)$ for all $e \in \mathbb{N}_0$, and hence $\alpha = \Phi^\infty_{\mathfrak{X}}(\alpha)$. □

Definition 30. We shall say that R satisfies *vanishing* (or *numerical vanishing* or *weak vanishing*, respectively) if all elements of all Grothendieck spaces over R satisfy vanishing (or numerical vanishing or weak vanishing, respectively).

A nice property of numerical vanishing, and one of the reasons that we have even bothered to introduce the concept, is that, in order to verify that numerical vanishing holds for all elements of all Grothendieck spaces, it suffices to restrict attention to the elements in $\mathbb{G}\mathsf{P}(\mathfrak{m})$:

Proposition 31. *The following are equivalent.*

(i) *R satisfies numerical vanishing.*
(ii) *$\alpha = \Phi_{\mathfrak{X}}(\alpha)$ in $\mathbb{G}\mathsf{C}(\mathfrak{X})$ for all $\mathfrak{X} \subseteq \operatorname{Spec} R$ and $\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{X})$.*
(iii) *$\alpha = \Phi_{\mathfrak{m}}(\alpha)$ in $\mathbb{G}\mathsf{C}(\mathfrak{m})$ for all $\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{m})$.*
(iv) *$\alpha = \Phi^{\infty}_{\mathfrak{X}}(\alpha)$ in $\mathbb{G}\mathsf{C}(\mathfrak{X})$ for all $\mathfrak{X} \subseteq \operatorname{Spec} R$ and $\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{X})$.*
(v) *$\alpha = \Phi^{\infty}_{\mathfrak{m}}(\alpha)$ in $\mathbb{G}\mathsf{C}(\mathfrak{m})$ for all $\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{m})$.*

Proof. By definition, (i) is equivalent to (iv). It is clear that (ii) implies (iii) and that (iv) implies (v). It is also clear that (ii) implies (iv) and that (iii) implies (v). Thus, it only remains to prove that (v) implies (ii). So assume (v) and let $\mathfrak{X} \subseteq \operatorname{Spec} R$ and $\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{X})$. To conclude that $\alpha - \Phi_{\mathfrak{X}}(\alpha)$ vanishes in $\mathbb{G}\mathsf{C}(\mathfrak{X})$, we simply note that, for arbitrary $\beta \in \mathbb{G}\mathsf{P}(\mathfrak{X}^c)$,

$$\begin{aligned}
\|\alpha - \Phi_{\mathfrak{X}}(\alpha)\|_{\beta} &= \|(\alpha - \Phi_{\mathfrak{X}}(\alpha)) \otimes \beta\|_R \\
&= \|\big((\alpha - \Phi_{\mathfrak{X}}(\alpha)) \otimes \beta\big)^{(0)}\|_R && \text{(by the assumption)} \\
&= \|(\alpha - \Phi_{\mathfrak{X}}(\alpha))^{(0)} \otimes \beta^{(0)}\|_R && \text{(by Remark 20)} \\
&= \|(\alpha^{(0)} - \alpha^{(0)}) \otimes \beta^{(0)}\|_R && \text{(also by Remark 20)} \\
&= 0.
\end{aligned}$$

□

Remark 32. Comparing Remark 28 with Proposition 31, we see that, if $\mathbb{G}\mathsf{P}(\mathfrak{m})$ is generated by acyclic complexes, a necessary and sufficient condition for R to satisfy numerical vanishing is that

$$\ell(F_R(M) = p^{\dim R}\,\ell(M) \tag{6}$$

for all modules M of finite length and finite projective dimension.

Example 33. If R is Cohen–Macaulay, $\mathbb{G}\mathsf{P}(\mathfrak{m})$ is generated by acyclic complexes: $\mathbb{G}\mathsf{P}(\mathfrak{m})$ is the tensor product of $\mathbb{Q}$ with a quotient of the Grothendieck group of the category $\mathsf{P}(\mathfrak{m})$, and this Grothendieck group is generated by acyclic complexes (see [4]). So if R is Cohen–Macaulay, numerical vanishing holds if and only if condition (6) holds, and condition (6) implies the weak version of Serre's vanishing conjecture.

Dutta [1] has proven that condition (6) holds when R is Gorenstein of dimension (at most) 3 or a complete intersection (of any dimension). Since the ring in these situations is Cohen–Macaulay, numerical (and hence weak) vanishing must hold. The rings in the counterexamples by Dutta, Hochster and McLaughlin [2] and Miller and Singh [8] are complete intersections (which can easily be transformed into complete rings of characteristic p with perfect residue fields), and hence they satisfy numerical vanishing without satisfying vanishing.

Any ring of dimension at most 4 will satisfy weak vanishing; this follows from Example 6. Roberts [11] has shown the existence of a Cohen–Macaulay ring of dimension 3 (which can also be transformed into a complete ring of characteristic p with perfect residue field) such that condition (6) does not hold. Thus, this ring satisfies weak vanishing without satisfying numerical vanishing.

Proposition 34. *Suppose that $\mathfrak{X} \subseteq \operatorname{Spec} R$ and let $\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{X})$. Suppose that I is an ideal such that R/I has support in $\mathfrak{X}^c$ and as a ring satisfies numerical vanishing. Set $t = \operatorname{codim} \mathfrak{X} - \dim R/I$. Then $\alpha \otimes [R/I]_{\mathsf{C}(\mathfrak{X}^c)} = \alpha^{(t)} \otimes [R/I]_{\mathsf{C}(\mathfrak{X}^c)}$.*

Proof. Consider the local ring homomorphism $R \to R/I$ and note that the subsets $\mathfrak{X} \subseteq \operatorname{Spec} R$ and $\{\mathfrak{m}/I\} \subseteq \operatorname{Spec} R/I$ satisfy conditions (i) and (ii) of Proposition 3 and that the subsets $\{\mathfrak{m}/I\} \subseteq \operatorname{Spec} R/I$ and $\{\mathfrak{m}\} \subseteq \operatorname{Spec} R$ satisfy conditions (iii) and (iv) of Proposition 3. The element $\alpha \otimes [R/I]_{\mathsf{C}(\mathfrak{X}^c)}$ in $\mathbb{G}\mathsf{C}^R(\mathfrak{m})$ is the image of α under the composition

$$\mathbb{G}\mathsf{P}^R(\mathfrak{X}) \xrightarrow{-\otimes_R R/I} \mathbb{G}\mathsf{P}^{R/I}(\mathfrak{m}/I) \longrightarrow \mathbb{G}\mathsf{C}^{R/I}(\mathfrak{m}/I) \longrightarrow \mathbb{G}\mathsf{C}^R(\mathfrak{m}),$$

in which the first homomorphism is the extension of scalars, which is well-defined by Proposition 3, the second homomorphism is the inclusion homomorphism, and the third homomorphism is the restriction of scalars, which is well-defined also by Proposition 3. Since R/I satisfies numerical vanishing, the inclusion homomorphism maps $\alpha \otimes_R R/I$ to $(\alpha \otimes_R R/I)^{(0)}$, and according to Proposition 25, this is equal to $\alpha^{(t)} \otimes_R R/I$. □

Example 35. We will now investigate the counterexamples by Dutta, Hochster and McLaughlin [2] and Miller and Singh [8]. We start by recalling the setup in these examples.

In the counterexample by Dutta, Hochster and McLaughlin, we have the following: the ring $R = k[u, v, x, y]_{\mathfrak{m}}/(ux - vy)$ of dimension three, where $\mathfrak{m}$ is the maximal ideal (u, v, x, y); a module M of finite length and finite projective dimension; and the module $R/\mathfrak{q}$ of dimension two, where $\mathfrak{q}$ is the prime ideal generated by u and v. We have that $\chi^R(M, R/\mathfrak{q}) = -1$.

In the counterexample by Miller and Singh, we have the following: the ring $R = k[u, v, w, x, y, z]_{\mathfrak{m}}/(ux + vy + wz)$ of dimension five, where $\mathfrak{m}$ is the maximal ideal (u, v, w, x, y, z); a module M of finite length and finite projective dimension; and the module $R/\mathfrak{q}$ of dimension three, where $\mathfrak{q}$ is the prime ideal generated by u, v and w. We have that $\chi^R(M, R/\mathfrak{q}) = -2$.

A result by Levine [7] shows that, in both of these examples, the Grothendieck space $\mathbb{G}\mathsf{P}(\mathfrak{m})$ is generated by Koszul complexes and one additional element. Thus, in both examples, the quotient of $\mathbb{G}\mathsf{P}(\mathfrak{m})$ with the subspace of fixed points for $\Phi_{\mathfrak{m}}$ has dimension one. This leaves room for one more eigenspace of dimension one; the corresponding eigenvalue is $1/p^i$ for some $i > 0$. We will now find i in the two examples.

In the counterexample by Dutta, Hochster and McLaughlin, the dimension of the ring is three, and hence, by Example 6, $\operatorname{vdim}(\alpha) \leq 1$ for all $\alpha \in \mathbb{G}\mathsf{P}(\mathfrak{m})$, so we must have $i = 1$ in this case.

In the counterexample by Miller and Singh, we first note that $R/\mathfrak{q} \cong k[x, y, z]$ is a regular ring and hence satisfies numerical vanishing. Letting X denote a projective resolution of the module M, Proposition 34 now yields that

$$0 \neq [X]_{\mathsf{P}(\mathfrak{m})} \otimes [R/\mathfrak{q}]_{\mathsf{C}(\operatorname{Spec} R)} = [X]^{(2)}_{\mathsf{P}(\mathfrak{m})} \otimes [R/\mathfrak{q}]_{\mathsf{C}(\operatorname{Spec} R)},$$

and it follows that $[X]^{(2)}_{\mathsf{P}(\mathfrak{m})} \neq 0$ and hence that $i = 2$.

In the characteristic 2 case, Miller and Singh also present a Gorenstein normal domain $\bar{R}$ of dimension 5, which is a module-finite extension of R, such that the element $[\bar{X}]_{\mathsf{P}(\bar{\mathfrak{m}})} \in \mathbb{G}\mathsf{P}(\bar{\mathfrak{m}})$, where $\bar{X} = X \otimes_R \bar{R}$ is a projective resolution of the

$\bar{R}$-module $\bar{M} = M \otimes_R \bar{R}$, does not satisfy numerical vanishing. In fact, their calculations show that $\chi^{\bar{R}}([\bar{X}]^{(0)}_{\mathsf{P}(\bar{\mathfrak{m}})}) = 220$ and $\chi^{\bar{R}}([\bar{X}]^{(2)}_{\mathsf{P}(\bar{\mathfrak{m}})}) = 2$. Miller and Singh also construct a module N of finite length and finite projective dimension such that, if Y is a projective resolution of N, then the element $[\bar{Y}]_{\mathsf{P}(\bar{\mathfrak{m}})} \in \mathbb{G}\mathsf{P}(\bar{\mathfrak{m}})$, where $\bar{Y} = Y \otimes_R \bar{R}$ is a projective resolution of the $\bar{R}$-module $\bar{N} = N \otimes_R \bar{R}$, satisfies that $\chi^{\bar{R}}([\bar{Y}]^{(0)}_{\mathsf{P}(\bar{\mathfrak{m}})}) = 220$ and $\chi^{\bar{R}}([\bar{Y}]^{(2)}_{\mathsf{P}(\bar{\mathfrak{m}})}) = -2$. Of course, the fact that $[\bar{X}]_{\mathsf{P}(\bar{\mathfrak{m}})}$ and $[\bar{Y}]_{\mathsf{P}(\bar{\mathfrak{m}})}$ do not satisfy numerical vanishing implies, according to Proposition 27, that they do not satisfy vanishing either.

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Article II

Dualities and intersection multiplicities

DUALITIES AND INTERSECTION MULTIPLICITIES

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ABSTRACT. Let R be a commutative, noetherian, local ring. Topological $\mathbb{Q}$–vector spaces modelled on full subcategories of the derived category of R are constructed in order to study intersection multiplicities.

1. INTRODUCTION

Let R be a commutative, noetherian, local ring and let X and Y be homologically bounded complexes over R with finitely generated homology and supports intersecting at the maximal ideal. When the projective dimension of X or Y is finite, their intersection multiplicity is defined as

$$\chi(X, Y) = \chi(X \otimes^{\mathbf{L}}_R Y),$$

where $\chi(-)$ denotes the *Euler characteristic* defined as the alternating sum of the lengths of the homology modules. When X and Y are modules, this definition agrees with the intersection multiplicity defined by Serre [22].

The ring R is said to *satisfy vanishing* when

$$\chi(X, Y) = 0 \quad \text{provided } \dim(\operatorname{Supp} X) + \dim(\operatorname{Supp} Y) < \dim R.$$

If the above holds under the restriction that both complexes have finite projective dimension, R is said to *satisfy weak vanishing*.

Assume, in addition, that $\dim(\operatorname{Supp} X) + \dim(\operatorname{Supp} Y) \leqslant \dim R$ and that R has prime characteristic p. The *Dutta multiplicity* of X and Y is defined when X has finite projective dimension as the limit

$$\chi_\infty(X, Y) = \lim_{e\to\infty} \frac{1}{p^{e \operatorname{codim}(\operatorname{Supp} X)}} \chi(\mathbf{L}F^e(X), Y),$$

where $\mathbf{L}F^e$ denotes the e-fold composition of the left-derived Frobenius functor; the Frobenius functor F was systematically used in the classical work by Peskine and Szpiro [18]. When X and Y are modules, $\chi_\infty(X, Y)$ is the usual Dutta multiplicity; see Dutta [6].

Let $\mathfrak{X}$ be a specialization-closed subset of $\operatorname{Spec} R$ and let $\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$ denote the full subcategory of the derived category of R comprising the homologically bounded complexes with finitely generated homology and support contained in $\mathfrak{X}$. The symbols $\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ and $\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ denote the full subcategories of $\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$ comprising the complexes that are isomorphic to a complex of projective or injective modules, respectively. The *Grothendieck spaces* $\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$, $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ and $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ are topological $\mathbb{Q}$–vector spaces modelled on these categories. The first two of these spaces were introduced in [11] but were there modelled on ordinary non-derived categories of complexes. The construction of Grothendieck spaces is similar to that of Grothendieck groups but targeted at the study of intersection multiplicities.

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The main result of [11] is a diagonalization theorem in prime characteristic p for the automorphism on $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ induced by the Frobenius functor. A consequence of this theorem is that every element $\alpha \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ can be decomposed as

$$\alpha = \alpha^{(0)} + \alpha^{(1)} + \cdots + \alpha^{(u)},$$

where the component of degree zero describes the Dutta multiplicity, whereas the components of higher degree describe the extent to which vanishing fails to hold for the the intersection multiplicity. This paper presents (see Theorem 6.2) a similar diagonalization theorem for a functor that is analogous to the Frobenius functor and has been studied by Herzog [13]. A consequence is that every element $\beta \in \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ can be decomposed as

$$\beta = \beta^{(0)} + \beta^{(1)} + \cdots + \beta^{(v)},$$

where the component of degree zero describes an analog of the Dutta multiplicity, whereas the components of higher degree describe the extent to which vanishing fails to hold for the Euler form, introduced by Mori and Smith [16]. Another consequence (see Theorem 6.12) is that R satisfies weak vanishing if only the Euler characteristic of homologically bounded complexes with finite-length homology changes by a factor $p^{\dim R}$ when the analogous Frobenius functor is applied.

The star duality endofunctor $(-)^* = \mathbf{R}\mathrm{Hom}_R(-,R)$ on $\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ induces an automorphism on $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ which in prime characteristic p is given by (see Theorem 7.5)

$$(-1)^{\operatorname{codim}\mathfrak{X}}\alpha^* = \alpha^{(0)} - \alpha^{(1)} + \cdots + (-1)^u\alpha^{(u)}.$$

Even in arbitrary characteristic, R satisfies vanishing if and only if all elements $\alpha \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ are *self-dual* in the sense that $\alpha = (-1)^{\operatorname{codim}\mathfrak{X}}\alpha^*$; and R satisfies weak vanishing if all elements $\alpha \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ are *numerically self-dual*, meaning that $\alpha - (-1)^{\operatorname{codim}\mathfrak{X}}\alpha^*$ is in the kernel of the homomorphism $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X}) \to \mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$ induced by the inclusion of the underlying categories (see Theorem 7.4). Rings for which all elements of the Grothendieck spaces $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ are numerically self-dual include Gorenstein rings of dimension less than or equal to five (see Proposition 7.11) and complete intersections (see Proposition 7.7 together with [11, Example 33]).

Notation

Throughout, R denotes a commutative, noetherian, local ring with unique maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. Unless otherwise stated, modules and complexes are assumed to be R–modules and R–complexes, respectively.

2. Derived categories and functors

In this section we review notation and results from the theory of derived categories, and we introduce a new star duality and derived versions of the Frobenius functor and its natural analog. For details on the derived category and derived functors, consult [9, 12, 23].

2.1. **Derived categories.** A complex X is a sequence $(X_i)_{i\in\mathbb{Z}}$ of modules equipped with a differential $(\partial_i^X)_{i\in\mathbb{Z}}$ lowering the homological degree by one. The homology complex $\mathrm{H}(X)$ of X is the complex whose modules are

$$\mathrm{H}(X)_i = \mathrm{H}_i(X) = \operatorname{Ker}\partial_i^X / \operatorname{Im}\partial_{i+1}$$

and whose differentials are trivial.

A morphism of complexes $\sigma\colon X \to Y$ is a family $(\sigma_i)_{i\in\mathbb{Z}}$ of homomorphisms commuting with the differentials in X and Y. The morphism of complexes σ is a *quasi-isomorphism* if the induced map on homology $\mathrm{H}_i(\sigma)\colon \mathrm{H}_i(X) \to \mathrm{H}_i(Y)$ is an isomorphism in every degree. Two morphisms of complexes $\sigma,\rho\colon X \to Y$ are *homotopic* if there exists a family $(s_i)_{i\in\mathbb{Z}}$ of maps $s_i\colon X_i \to Y_{i+1}$ such that

$$\sigma_i - \rho_i = \partial^Y_{i+1}s_i + s_{i-1}\partial^X_i .$$

Homotopy yields an equivalence relation in the group $\mathrm{Hom}_R(X,Y)$ of morphisms of complexes, and the *homotopy category* $\mathsf{K}(R)$ is obtained from the category of complexes $\mathsf{C}(R)$ by declaring

$$\mathrm{Hom}_{\mathsf{K}(R)}(X,Y) = \mathrm{Hom}_{\mathsf{C}(R)}(X,Y)/\,\text{homotopy}.$$

The collection S of quasi-isomorphisms in the triangulated category $\mathsf{K}(R)$ form a multiplicative system of morphisms. The *derived category* $\mathsf{D}(R)$ is obtained by (categorically) localizing $\mathsf{K}(R)$ with respect to S. Thus, quasi-isomorphisms become isomorphisms in $\mathsf{D}(R)$; in the sequel, they are denoted $\simeq$.

Let n be an integer. The symbol $\Sigma^n X$ denotes the complex X shifted (or translated or suspended) n degrees to the left; that is, against the direction of the differential. The modules in $\Sigma^n X$ are given by $(\Sigma^n X)_i = X_{i-n}$, and the differentials are $\partial_i^{\Sigma^n X} = (-1)^n \partial^X_{i-n}$. The symbol $\sim$ denotes isomorphisms up to a shift in the derived category.

The full subcategory of $\mathsf{D}(R)$ consisting of complexes with bounded, finitely generated homology is denoted $\mathsf{D}^{\mathrm{f}}_{\square}(R)$. Complexes from $\mathsf{D}^{\mathrm{f}}_{\square}(R)$ are called *finite* complexes. The symbols $\mathsf{P}^{\mathrm{f}}(R)$ and $\mathsf{I}^{\mathrm{f}}(R)$ denote the full subcategories of $\mathsf{D}^{\mathrm{f}}_{\square}(R)$ consisting of complexes that are isomorphic in the derived category to a bounded complex of projective modules and isomorphic to a bounded complex of injective modules, respectively. Note that $\mathsf{P}^{\mathrm{f}}(R)$ coincides with the full subcategory $\mathsf{F}^{\mathrm{f}}(R)$ of $\mathsf{D}^{\mathrm{f}}_{\square}(R)$ consisting of complexes isomorphic to a complex of flat modules.

2.2. **Support.** The *spectrum* of R, denoted $\operatorname{Spec} R$, is the set of prime ideals of R. A subset $\mathfrak{X}$ of $\operatorname{Spec} R$ is *specialization-closed* if it has the property

$$\mathfrak{p} \in \mathfrak{X} \text{ and } \mathfrak{p} \subseteq \mathfrak{q} \implies \mathfrak{q} \in \mathfrak{X}$$

for all prime ideals $\mathfrak{p}$ and $\mathfrak{q}$. A subset that is closed in the Zariski topology is, in particular, specialization-closed.

The *support* of a complex X is the set

$$\operatorname{Supp} X = \left\{ \mathfrak{p} \in \operatorname{Spec} R \;\middle|\; \mathrm{H}(X_{\mathfrak{p}}) \neq 0 \right\}.$$

A *finite* complex is a complex with bounded homology and finitely generated homology modules; the support of such a complex is a closed and hence specialization-closed subset of $\operatorname{Spec} R$.

For a specialization-closed subset $\mathfrak{X}$ of $\operatorname{Spec} R$, the *dimension* of $\mathfrak{X}$, denoted $\dim \mathfrak{X}$, is the usual Krull dimension of $\mathfrak{X}$. When $\dim R$ is finite, the *co-dimension* of $\mathfrak{X}$, denoted $\operatorname{codim} \mathfrak{X}$, is the number $\dim R - \dim \mathfrak{X}$. For a finitely generated module M, the dimension and co-dimension of M, denoted $\dim M$ and $\operatorname{codim} M$, are the dimension and co-dimension of the support of M.

For a specialization-closed subset $\mathfrak{X}$ of $\operatorname{Spec} R$, the symbols $\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$, $\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$, and $\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ denote the full subcategories of $\mathsf{D}^{\mathrm{f}}_{\square}(R)$, $\mathsf{P}^{\mathrm{f}}(R)$, and $\mathsf{I}^{\mathrm{f}}(R)$, respectively, consisting of complexes whose support is contained in $\mathfrak{X}$. In the case where $\mathfrak{X}$ equals $\{\mathfrak{m}\}$, we simply write $\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{m})$, $\mathsf{P}^{\mathrm{f}}(\mathfrak{m})$ and $\mathsf{I}^{\mathrm{f}}(\mathfrak{m})$, respectively.

2.3. **Derived functors.** A complex P is said to be semi-projective if the functor $\operatorname{Hom}_R(P,-)$ sends surjective quasi-isomorphisms to surjective quasi-isomorphisms. If a complex is bounded to the right and consists of projective modules, it is semi-projective. A semi-projective resolution of M is a quasi-isomorphism $\pi\colon P \to X$ where P is semi-projective.

Dually, a complex I is said to be semi-injective if the functor $\operatorname{Hom}_R(-,I)$ sends injective quasi-isomorphisms to surjective quasi-isomorphisms. If a complex is bounded to the left and consists of injective modules, it is semi-injective. A semi-injective resolution of Y is a quasi-isomorphism $\iota\colon Y \to I$ where I is semi-injective. For existence of semi-projective and semi-injective resolutions see [2].

Let X and Y be complexes. The left-derived tensor product $X \otimes^{\mathbf{L}}_R Y$ in $\mathsf{D}(R)$ of X and Y is defined by

$$P \otimes_R Y \simeq X \otimes^{\mathbf{L}}_R Y \simeq X \otimes_R Q,$$

where $P \xrightarrow{\simeq} X$ is a semi-projective resolution of X and $Q \xrightarrow{\simeq} Y$ is a semi-projective resolution of Y. The right-derived homomorphism complex $\mathbf{R}\operatorname{Hom}_R(X,Y)$ in $\mathsf{D}(R)$ of X and Y is defined by

$$\operatorname{Hom}_R(P,Y) \simeq \mathbf{R}\operatorname{Hom}_R(X,Y) \simeq \operatorname{Hom}_R(X,I),$$

where $P \xrightarrow{\simeq} X$ is a semi-projective resolution of X and $Y \xrightarrow{\simeq} I$ is a semi-injective resolution of Y. When M and N are modules,

$$\mathrm{H}_n(M \otimes^{\mathbf{L}}_R N) \cong \operatorname{Tor}^R_n(M,N) \quad \text{and} \quad \mathrm{H}_{-n}(\mathbf{R}\operatorname{Hom}_R(M,N)) \cong \operatorname{Ext}^n_R(M,N)$$

for all integers n.

2.4. **Stability.** Let $\mathfrak{X}$ and $\mathfrak{Y}$ be specialization-closed subsets of $\operatorname{Spec} R$ and let X be a complex in $\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$ and Y be a complex in $\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{Y})$. Then

$$(2.4.1)\qquad \begin{aligned} X \otimes^{\mathbf{L}}_R Y &\in \mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}\cap\mathfrak{Y}) && \text{if } X \in \mathsf{P}^{\mathrm{f}}(\mathfrak{X}) \text{ or } Y \in \mathsf{P}^{\mathrm{f}}(\mathfrak{Y}),\\ X \otimes^{\mathbf{L}}_R Y &\in \mathsf{P}^{\mathrm{f}}(\mathfrak{X}\cap\mathfrak{Y}) && \text{if } X \in \mathsf{P}^{\mathrm{f}}(\mathfrak{X}) \text{ and } Y \in \mathsf{P}^{\mathrm{f}}(\mathfrak{Y}),\\ X \otimes^{\mathbf{L}}_R Y &\in \mathsf{I}^{\mathrm{f}}(\mathfrak{X}\cap\mathfrak{Y}) && \text{if } X \in \mathsf{P}^{\mathrm{f}}(\mathfrak{X}) \text{ and } Y \in \mathsf{I}^{\mathrm{f}}(\mathfrak{Y}),\\ X \otimes^{\mathbf{L}}_R Y &\in \mathsf{I}^{\mathrm{f}}(\mathfrak{X}\cap\mathfrak{Y}) && \text{if } X \in \mathsf{I}^{\mathrm{f}}(\mathfrak{X}) \text{ and } Y \in \mathsf{P}^{\mathrm{f}}(\mathfrak{Y}),\\ \mathbf{R}\operatorname{Hom}_R(X,Y) &\in \mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}\cap\mathfrak{Y}) && \text{if } X \in \mathsf{P}^{\mathrm{f}}(\mathfrak{X}) \text{ or } Y \in \mathsf{I}^{\mathrm{f}}(\mathfrak{Y}),\\ \mathbf{R}\operatorname{Hom}_R(X,Y) &\in \mathsf{P}^{\mathrm{f}}(\mathfrak{X}\cap\mathfrak{Y}) && \text{if } X \in \mathsf{P}^{\mathrm{f}}(\mathfrak{X}) \text{ and } Y \in \mathsf{P}^{\mathrm{f}}(\mathfrak{Y}),\\ \mathbf{R}\operatorname{Hom}_R(X,Y) &\in \mathsf{I}^{\mathrm{f}}(\mathfrak{X}\cap\mathfrak{Y}) && \text{if } X \in \mathsf{P}^{\mathrm{f}}(\mathfrak{X}) \text{ and } Y \in \mathsf{I}^{\mathrm{f}}(\mathfrak{Y}) \text{ and}\\ \mathbf{R}\operatorname{Hom}_R(X,Y) &\in \mathsf{P}^{\mathrm{f}}(\mathfrak{X}\cap\mathfrak{Y}) && \text{if } X \in \mathsf{I}^{\mathrm{f}}(\mathfrak{X}) \text{ and } Y \in \mathsf{I}^{\mathrm{f}}(\mathfrak{Y}). \end{aligned}$$

2.5. **Functorial isomorphisms.** Throughout, we will make use of the functorial isomorphisms stated below. As we will not need them in the most general setting, the reader should bear in mind that not all the boundedness conditions imposed on the complexes are strictly necessary. For details the reader is referred e.g., to [5, A.4] and the references therein.

Let S be another commutative, noetherian, local ring. Let $K, L, M \in \mathsf{D}(R)$, let $P \in \mathsf{D}(S)$ and let $N \in \mathsf{D}(R,S)$, the derived category of R–S–bi-modules. There are the next functorial isomorphisms in $\mathsf{D}(R,S)$.

(Comm) $$M \otimes^{\mathbf{L}}_R N \xrightarrow{\simeq} N \otimes^{\mathbf{L}}_R M.$$

(Assoc) $$(M \otimes^{\mathbf{L}}_R N) \otimes^{\mathbf{L}}_S P \xrightarrow{\simeq} M \otimes^{\mathbf{L}}_R (N \otimes^{\mathbf{L}}_S P).$$

(Adjoint) $$\mathbf{R}\mathrm{Hom}_S(M \otimes^{\mathbf{L}}_R N, P) \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_R(M, \mathbf{R}\mathrm{Hom}_S(N,P)).$$

(Swap) $$\mathbf{R}\mathrm{Hom}_R(M, \mathbf{R}\mathrm{Hom}_S(P,N)) \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_S(P, \mathbf{R}\mathrm{Hom}_R(M,N)).$$

Moreover, there are the following evaluation morphisms.

(Tensor-eval) $$\sigma_{KLP}\colon\ \mathbf{R}\mathrm{Hom}_R(K,L) \otimes^{\mathbf{L}}_S P \to \mathbf{R}\mathrm{Hom}_R(K, L \otimes^{\mathbf{L}}_S P).$$

(Hom-eval) $$\rho_{PLM}\colon\ P \otimes^{\mathbf{L}}_S \mathbf{R}\mathrm{Hom}_R(L,M) \to \mathbf{R}\mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_R(P,L), M).$$

In addition,

- the morphism σ_{KLP} is invertible if K is finite, $\mathrm{H}(L)$ is bounded, and either $P \in \mathsf{P}(S)$ or $K \in \mathsf{P}(R)$; and
- the morphism ρ_{PLM} is invertible if P is finite, $\mathrm{H}(L)$ is bounded, and either $P \in \mathsf{P}(R)$ or $M \in \mathsf{I}(R)$.

2.6. **Dualizing complexes.** A finite complex D is a *dualizing complex* for R if

$$D \in \mathsf{I}^{\mathrm{f}}(R) \quad \text{and} \quad R \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_R(D,D).$$

Dualizing complexes are essentially unique: if D and D' are dualizing complexes for R, then $D \sim D'$. To check whether a finite complex D is dualizing is equivalent to checking whether

$$k \sim \mathbf{R}\mathrm{Hom}_R(k,D).$$

A dualizing complex D is said to be *normalized* when $k \simeq \mathbf{R}\mathrm{Hom}_R(k,D)$. If R is a Cohen–Macaulay ring of dimension d and D is a normalized dualizing complex, then $\mathrm{H}(D)$ is concentrated in degree d, and the module $\mathrm{H}_d(D)$ is the (so-called) *canonical module*; see [3, Chapter 3]. Observe that $\operatorname{Supp} D = \operatorname{Spec} R$.

If D is a normalized dualizing complex for R, then it is isomorphic to a complex

$$0 \to D_{\dim R} \to D_{\dim R-1} \to \cdots \to D_1 \to D_0 \to 0$$

consisting of injective modules, where

$$D_i = \bigoplus_{\dim R/\mathfrak{p}=i} E_R(R/\mathfrak{p})$$

and $E_R(R/\mathfrak{p})$ is the injective hull (or envelope) of $R/\mathfrak{p}$ for a prime ideal $\mathfrak{p}$; in particular, it follows that $D_0 = E_R(k)$.

When R is a homomorphic image of a local Gorenstein ring Q, then the R–complex $\mathsf{\Sigma}^n\, \mathbf{R}\mathrm{Hom}_Q(R,Q)$, where $n = \dim Q - \dim R$, is a normalized dualizing complex over R. In particular, it follows from Cohen's structure theorem for complete local rings that any complete ring admits a dualizing complex. Conversely, if a local ring admits a dualizing complex, then it must be a homomorphic image of a Gorenstein ring; this follows from Kawasaki's proof of Sharp's conjecture; see [14].

2.7. **Dagger duality.** Assume that R admits a normalized dualizing complex D and consider the duality morphism of functors

$$\mathrm{id}_{\mathsf{D}(R)} \to \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(-,D),D).$$

It follows essentially from (Hom-eval) that the contravariant functor

$$(-)^\dagger = \mathbf{R}\mathrm{Hom}_R(-,D)$$

provides a duality on the category $\mathsf{D}^{\mathsf{f}}_{\Box}(R)$ which restricts to a duality between $\mathsf{P}^{\mathsf{f}}(R)$ and $\mathsf{I}^{\mathsf{f}}(R)$. This duality is sometimes referred to as *dagger duality.* According to (2.4.1), if $\mathfrak{X}$ is a specialization-closed subset of $\operatorname{Spec} R$, then dagger duality gives a duality on $\mathsf{D}^{\mathsf{f}}_{\Box}(\mathfrak{X})$ which restricts to a duality between $\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ and $\mathsf{I}^{\mathsf{f}}(\mathfrak{X})$ as described by the following commutative diagram.

$$\begin{tikzcd}
\mathsf{D}^{\mathsf{f}}_{\Box}(\mathfrak{X}) \arrow[r, shift left, "(-)^\dagger"] & \mathsf{D}^{\mathsf{f}}_{\Box}(\mathfrak{X}) \arrow[l, shift left, "(-)^\dagger"] \\
\mathsf{P}^{\mathsf{f}}(\mathfrak{X}) \arrow[u] \arrow[r, shift left, "(-)^\dagger"] & \mathsf{I}^{\mathsf{f}}(\mathfrak{X}). \arrow[u] \arrow[l, shift left, "(-)^\dagger"]
\end{tikzcd}$$

Here the vertical arrows are full embeddings of categories. For more details on dagger duality, see [12].

2.8. **Foxby equivalence.** Assume that R admits a normalized dualizing complex D and consider the two contravariant adjoint functors

$$D \otimes^{\mathbf{L}}_R - \quad \text{and} \quad \mathbf{R}\mathrm{Hom}_R(D,-),$$

which come naturally equipped the unit and co-unit morphisms

$$\eta\colon \mathrm{id}_{\mathsf{D}(R)} \to \mathbf{R}\mathrm{Hom}_R(D, D\otimes^{\mathbf{L}}_R -) \quad \text{and} \quad \varepsilon\colon D\otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(D,-) \to \mathrm{id}_{\mathsf{D}(R)}\,.$$

It follows essentially from an application of (Tensor-eval) and (Hom-eval) that the categories $\mathsf{P}(R)$ and $\mathsf{I}(R)$ are naturally equivalent via the above two functors. This equivalence is usually known as *Foxby equivalence* and was introduced in [1], to which the reader is referred for further details.

According to (2.4.1), for a specialization-closed subset $\mathfrak{X}$ of $\operatorname{Spec} R$, Foxby equivalence restricts to an equivalence between $\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ and $\mathsf{I}^{\mathsf{f}}(\mathfrak{X})$ as described by the following diagram.

$$\begin{tikzcd}
\mathsf{P}^{\mathsf{f}}(\mathfrak{X}) \arrow[r, shift left, "D\otimes^{\mathbf{L}}_R -"] & \mathsf{I}^{\mathsf{f}}(\mathfrak{X}). \arrow[l, shift left, "\mathbf{R}\mathrm{Hom}_R(D,-)"]
\end{tikzcd}$$

2.9. **Star duality.** Consider the duality morphism of functors

$$\mathrm{id}_{\mathsf{D}(R)} \to \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(-,R),R).$$

From an application of (Hom-eval) it is readily seen that the functor

$$(-)^* = \mathbf{R}\mathrm{Hom}_R(-,R)$$

provides a duality on the category $\mathsf{P}^{\mathrm{f}}(R)$. According to (2.4.1), for a specialization-closed subset $\mathfrak{X}$ of $\operatorname{Spec} R$, star duality restricts to a duality on $\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ as described by following diagram.

$$\begin{tikzcd} \mathsf{P}^{\mathrm{f}}(\mathfrak{X}) \arrow[r, shift left, "(-)^*"] & \mathsf{P}^{\mathrm{f}}(\mathfrak{X}). \arrow[l, shift left, "(-)^*"] \end{tikzcd}$$

When R admits a dualizing complex D, the star functor can also be described in terms of the dagger and Foxby functors. Indeed, it is straightforward to show that the following three contravariant endofunctors on $\mathsf{P}^{\mathrm{f}}(R)$ are isomorphic.

$$(-)^*, \quad \mathbf{R}\mathrm{Hom}_R(D, -^{\dagger}), \quad \text{and} \quad (D \otimes^{\mathbf{L}}_R -)^{\dagger}.$$

It is equally straightforward to show that the following four contravariant endofunctors on $\mathsf{I}^{\mathrm{f}}(R)$ are isomorphic.

$$(-)^{\dagger\,*\,\dagger}, \quad \mathbf{R}\mathrm{Hom}_R(D, -)^{\dagger}, \quad D \otimes^{\mathbf{L}}_R (\mathbf{R}\mathrm{Hom}_R(D, -)^*) \quad \text{and} \quad D \otimes^{\mathbf{L}}_R (-)^{\dagger}.$$

They provide a duality on $\mathsf{I}^{\mathrm{f}}(R)$. In the sequel, the four isomorphic functors are denoted $(-)^{\star}$. According to (2.4.1), for a specialization-closed subset $\mathfrak{X}$ of $\operatorname{Spec} R$, this new kind of star duality restricts to a duality on $\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ as described by the following diagram.

$$\begin{tikzcd} \mathsf{I}^{\mathrm{f}}(\mathfrak{X}) \arrow[r, shift left, "(-)^{\star}"] & \mathsf{I}^{\mathrm{f}}(\mathfrak{X}). \arrow[l, shift left, "(-)^{\star}"] \end{tikzcd}$$

The dagger duality, Foxby equivalence and star duality functors fit together in the following diagram.

(2.9.1)
$$\begin{tikzcd}[column sep=huge] \mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}) \arrow[r, shift left, "(-)^{\dagger}"] & \mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}) \arrow[l, shift left, "(-)^{\dagger}"] \\ \mathsf{P}^{\mathrm{f}}(\mathfrak{X}) \arrow[u] \arrow[loop left, "(-)^*"] \arrow[r, bend left, "D\otimes^{\mathbf{L}}_R -"] \arrow[r, shift left, "(-)^{\dagger}"] & \mathsf{I}^{\mathrm{f}}(\mathfrak{X}) \arrow[u] \arrow[loop right, "(-)^{\star}"] \arrow[l, shift left, "(-)^{\dagger}"] \arrow[l, bend left, "\mathbf{R}\mathrm{Hom}_R(D,-)"] \end{tikzcd}$$

In the lower part of the diagram, the three types of functors, dagger, Foxby and star, always commute pairwise, and the composition of two of the three types yields a functor of the third type. For example, star duality and dagger duality always commute and compose to give Foxby equivalence, since we have

$$(-)^{*\dagger} \simeq (-)^{\dagger\star} \simeq D \otimes^{\mathbf{L}}_R - \quad \text{and} \quad (-)^{\star\dagger} \simeq (-)^{\dagger *} \simeq \mathbf{R}\mathrm{Hom}_R(D, -).$$

2.10. **Frobenius endofunctors.** Assume that R is complete of prime characteristic p and with perfect residue field k. The endomorphism

$$f \colon R \to R \quad \text{defined by} \quad f(r) = r^p$$

for $r \in R$ is called the *Frobenius endomorphism* on R. The n-fold composition of f, denoted f^n, operates on a generic element $r \in R$ by $f^n(r) = r^{p^n}$. We let ${}^{f^n}\!R$ denote the R–algebra which, as a ring, is identical to R but, as a module, is viewed through f^n. Thus, the R–module structure on ${}^{f^n}\!R$ is given by

$$r \cdot x = r^{p^n} x \quad \text{for } r \in R \text{ and } x \in {}^{f^n}\!R.$$

Under the present assumptions on R, the R–module ${}^{f^n}R$ is finitely generated (see, for example, Roberts [21, Section 7.3]).

We define two functors from the category of R–modules to the category of ${}^{f^n}R$–modules by

$$F^n(-) = - \otimes_R {}^{f^n}R \quad \text{and} \quad G^n(-) = \operatorname{Hom}_R({}^{f^n}R, -),$$

where the resulting modules are finitely generated modules with R–structure obtained from the ring ${}^{f^n}R = R$. The functor F^n is called the *Frobenius functor* and has been studied by Peskine and Szpiro [18]. The functor G^n has been studied by Herzog [13] and is analogous to F^n in a sense that will be described below. We call this the *analogous Frobenius functor*. The R–structure on $F^n(M)$ is given by

$$r \cdot (m \otimes x) = m \otimes rx$$

for $r \in R$, $m \in M$ and $x \in {}^{f^n}R$, and the R–structure on $G^n(N)$ is given by

$$(r \cdot \varphi)(x) = \varphi(rx)$$

for $r \in R$, $\varphi \in \operatorname{Hom}_R({}^{f^n}R, N)$ and $x \in {}^{f^n}R$. Note that here we also have

$$(rm) \otimes x = m \otimes (r \cdot x) = m \otimes r^p x \quad \text{and} \quad r\varphi(x) = \varphi(r \cdot x) = \varphi(r^p x).$$

Peskine and Szpiro [18, Théorème (1.7)] have proven that, if M has finite projective dimension, then so does $F(M)$, and Herzog [13, Satz 5.2] has proven that, if N has finite injective dimension, then so does $G(N)$.

It follows by definition that the functor F^n is right-exact while the functor G^n is left-exact. We denote by $\mathbf{L}F^n(-)$ the left-derived of $F^n(-)$ and by $\mathbf{R}G^n(-)$ the right-derived of $G^n(-)$. When X and Y are R–complexes with semi-projective and semi-injective resolutions

$$P \xrightarrow{\simeq} X \quad \text{and} \quad Y \xrightarrow{\simeq} I,$$

respectively, these derived functors are obtained as

$$\mathbf{L}F^n(X) = P \otimes_R {}^{f^n}R \quad \text{and} \quad \mathbf{R}G^n(Y) = \operatorname{Hom}_R({}^{f^n}R, I),$$

where the resulting complexes are viewed through their ${}^{f^n}R$–structure, which makes them R–complexes since ${}^{f^n}R$ as a ring is just R. Observe that we may identify these functors with

$$\mathbf{L}F^n(X) = X \otimes^{\mathbf{L}}_R {}^{f^n}R \quad \text{and} \quad \mathbf{R}G^n(Y) = \mathbf{R}\operatorname{Hom}_R({}^{f^n}R, Y).$$

2.11. **Lemma.** *Let R be a complete ring of prime characteristic and with perfect residue field, and let $\mathfrak{X}$ be a specialization-closed subset of* $\operatorname{Spec} R$. *Then the Frobenius functors commute with dagger and star duality in the sense that*

$$\begin{aligned} &\mathbf{L}F^n(-)^\dagger \simeq \mathbf{R}G^n(-^\dagger), && \mathbf{R}G^n(-)^\dagger \simeq \mathbf{L}F^n(-^\dagger), \\ &\mathbf{L}F^n(-)^* \simeq \mathbf{L}F^n(-^*) \quad \textit{and} && \mathbf{R}G^n(-)^\star \simeq \mathbf{R}G^n(-^\star). \end{aligned}$$

Here the first row contains isomorphisms of functors between $\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ *and* $\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$, *while the second row contains isomorphisms of endofunctors on* $\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ *and* $\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$, *respectively. Finally, the Frobenius functors commute with Foxby equivalence in the sense that*

$$\begin{gathered} D \otimes^{\mathbf{L}}_R \mathbf{L}F^n(-) \simeq \mathbf{R}G^n(D \otimes^{\mathbf{L}}_R -) \quad \textit{and} \\ \mathbf{R}\operatorname{Hom}_R(D, \mathbf{R}G^n(-)) \simeq \mathbf{L}F^n(\mathbf{R}\operatorname{Hom}_R(D, -)) \end{gathered}$$

as functors from $\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ *to* $\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ *and from* $\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ *to* $\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$, *respectively.*

Proof. Let $\varphi\colon R \to S$ be a local homomorphism making S into a finitely generated R–module, and let D^R denote a normalized dualizing complex for R. Then $D^S = \mathbf{R}\mathrm{Hom}_R(S, D^R)$ is a normalized dualizing complex for S. Pick an R–complex X and consider the next string of natural isomorphisms.

$$\begin{aligned}
\mathbf{R}\mathrm{Hom}_S(X \otimes^{\mathbf{L}}_R S, D^S) &= \mathbf{R}\mathrm{Hom}_S(X \otimes^{\mathbf{L}}_R S, \mathbf{R}\mathrm{Hom}_R(S, D^R)) \\
&\xleftarrow{\simeq} \mathbf{R}\mathrm{Hom}_R(X \otimes^{\mathbf{L}}_R S, D^R) \\
&\xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_R(S, \mathbf{R}\mathrm{Hom}_R(X, D^R)).
\end{aligned}$$

Here, the two isomorphism follow from (Adjoint). The computation shows that

$$(- \otimes^{\mathbf{L}}_R S)^{\dagger_S} \simeq \mathbf{R}\mathrm{Hom}_R(S, -^{\dagger_R})$$

in $\mathsf{D}(S)$. A similar computation using the natural isomorphisms (Adjoint) and (Hom-eval) shows that

$$(-)^{\dagger_R} \otimes^{\mathbf{L}}_R S \simeq \mathbf{R}\mathrm{Hom}_R(S, -)^{\dagger_S}.$$

Under the present assumptions, the n-fold composition of the Frobenius endomorphism $f^n\colon R \to R$ is module-finite map. Therefore, the above isomorphisms of functors yield

$$\mathbf{L}F^n(-)^\dagger \simeq \mathbf{R}G^n(-^\dagger) \quad \text{and} \quad \mathbf{L}F^n(-^\dagger) \simeq \mathbf{R}G^n(-)^\dagger.$$

Similar considerations establish the remaining isomorphisms of functors. □

2.12. **Corollary.** *Let R be a complete ring of prime characteristic and with perfect residue field, and let $\mathfrak{X}$ be a specialization-closed subset of* $\operatorname{Spec} R$*. Then the Frobenius functor $\mathbf{R}G^n$ is an endofunctor on $\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$.*

Proof. From the above lemma, we learn that

$$\mathbf{R}G^n(-) \simeq (-)^\dagger \circ \mathbf{L}F^n \circ (-)^\dagger$$

and since $\mathbf{L}F^n$ is an endofunctor on $\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ the conclusion is immediate. □

2.13. **Lemma.** *Let R be a complete ring of prime characteristic and with perfect residue field. For complexes $X, X' \in \mathsf{P}^{\mathrm{f}}(R)$ and $Y, Y' \in \mathsf{I}^{\mathrm{f}}(R)$ there are isomorphisms*

$$\begin{aligned}
\mathbf{L}F^n(X \otimes^{\mathbf{L}}_R X') &\simeq \mathbf{L}F^n(X) \otimes^{\mathbf{L}}_R \mathbf{L}F^n(X'), \\
\mathbf{R}G^n(X \otimes^{\mathbf{L}}_R Y) &\simeq \mathbf{L}F^n(X) \otimes^{\mathbf{L}}_R \mathbf{R}G^n(Y), \\
\mathbf{R}G^n(\mathbf{R}\mathrm{Hom}_R(X, Y)) &\simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{L}F^n(X), \mathbf{R}G^n(Y)) \\
\mathbf{L}F^n(\mathbf{R}\mathrm{Hom}_R(X, X')) &\simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{L}F^n(X), \mathbf{L}F^n(X')) \quad \textit{and} \\
\mathbf{L}F^n(\mathbf{R}\mathrm{Hom}_R(Y, Y')) &\simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}G^n(Y), \mathbf{R}G^n(Y')).
\end{aligned}$$

Proof. We prove the first and the third isomorphism. The rest are obtained in a similar manner using Lemma 2.11 and the functorial isomorphisms.

Let $F \xrightarrow{\simeq} X$ and $F' \xrightarrow{\simeq} X'$ be finite free resolutions. Then it follows

$$\begin{aligned}
\mathbf{L}F^n(X \otimes^{\mathbf{L}}_R X') &\simeq F^n(F \otimes_R F') \\
&\simeq F^n(F) \otimes_R F^n(F') \\
&\simeq \mathbf{L}F^n(X) \otimes^{\mathbf{L}}_R \mathbf{L}F^n(X').
\end{aligned}$$

Here the first isomorphism follows as $F \otimes_R F'$ is isomorphic to $X \otimes^{\mathbf{L}}_R X'$; second isomorphism follows from e.g., [11, Proposition 12(vi)].

From Corollary 2.12 we learn that

$$\mathbf{R}G^n(Y) \simeq (\mathbf{L}F^n(Y^\dagger))^\dagger,$$

and therefore we may compute as follows.

$$\begin{aligned}
\mathbf{R}\mathrm{Hom}_R(\mathbf{L}F^n(X), \mathbf{R}G^n(Y)) &\simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{L}F^n(X), (\mathbf{L}F^n(Y^\dagger))^\dagger) \\
&\simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{L}F^n(X) \otimes^{\mathbf{L}}_R \mathbf{L}F^n(Y^\dagger), D) \\
&\simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{L}F^n(X \otimes^{\mathbf{L}}_R Y^\dagger), D) \\
&\simeq \mathbf{L}F^n(X \otimes^{\mathbf{L}}_R Y^\dagger)^\dagger \\
&\simeq (\mathbf{L}F^n(\mathbf{R}\mathrm{Hom}_R(X, Y)^\dagger)^\dagger \\
&\simeq \mathbf{R}G^n(\mathbf{R}\mathrm{Hom}_R(X, Y)).
\end{aligned}$$

Here the second isomorphism follows by (Adjoint); the third from the first statement in the Lemma; the fourth from definition; the fifth isomorphism follows from (Hom-eval); and the last isomorphism follows from Corollary 2.12. $\square$

2.14. **Remark.** Any complex in $\mathsf{P}^{\mathrm{f}}(R)$ is isomorphic to a bounded complex of finitely generated, free modules, and it is well-known that the Frobenius functor acts on such a complex by simply raising the entries in the matrices representing the differentials to the p^n'th power. To be precise, if X is a complex in the form

$$X = \cdots \longrightarrow R^m \xrightarrow{(a_{ij})} R^n \longrightarrow \cdots \longrightarrow 0,$$

then $\mathbf{L}F^n(X) = F^n(X)$ is a complex in the form

$$\mathbf{L}F^n(X) = \cdots \longrightarrow R^m \xrightarrow{(a_{ij}^{p^n})} R^n \longrightarrow \cdots \longrightarrow 0.$$

If R is Cohen–Macaulay with canonical module ω, then it follows from dagger duality that any complex in $\mathsf{I}^{\mathrm{f}}(R)$ is isomorphic to a complex Y in the form

$$Y = 0 \longrightarrow \cdots \longrightarrow \omega^n \xrightarrow{(a_{ji})} \omega^m \longrightarrow \cdots,$$

and $\mathbf{R}G^n$ acts on Y by raising the entries in the matrices representing the differentials to the p^n'th power, so that $\mathbf{R}G^n(Y) = G^n(Y)$ is a complex in the form

$$\mathbf{R}G^n(Y) = 0 \longrightarrow \cdots \longrightarrow \omega^n \xrightarrow{(a_{ji}^{p^n})} \omega^m \longrightarrow \cdots.$$

3. Intersection multiplicities

3.1. **Serre's intersection multiplicity.** If Z is a complex in $\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{m})$, then its finitely many homology modules all have finite length, and the *Euler characteristic* of Z is defined by

$$\chi(Z) = \sum_i (-1)^i \operatorname{length} \mathrm{H}_i(Z).$$

Let X and Y be finite complexes with $\operatorname{Supp} X \cap \operatorname{Supp} Y = \{\mathfrak{m}\}$. The *intersection multiplicity* of X and Y is defined by

$$\chi(X, Y) = \chi(X \otimes^{\mathbf{L}}_R Y) \quad \text{when either } X \in \mathsf{P}^{\mathrm{f}}(R) \text{ or } Y \in \mathsf{P}^{\mathrm{f}}(R).$$

In the case where X and Y are finitely generated modules, $\chi(X, Y)$ coincides with Serre's intersection multiplicity; see [22].

Serre's vanishing conjecture can be generalized to the statement that

$$\chi(X, Y) = 0 \quad \text{if } \dim(\operatorname{Supp} X) + \dim(\operatorname{Supp} Y) < \dim R \tag{3.1.1}$$

when either $X \in \mathsf{P}^{\mathrm{f}}(R)$ or $Y \in \mathsf{P}^{\mathrm{f}}(R)$. We will say that R *satisfies vanishing* when the above holds; note that this, in general, is a stronger condition than Serre's vanishing conjecture for modules. It is known that R satisfies vanishing in certain cases, for example when R is regular. However, it does not hold in general, as demonstrated by Dutta, Hochster and McLaughlin [8].

If we require that both $X \in \mathsf{P}^{\mathrm{f}}(R)$ *and* $Y \in \mathsf{P}^{\mathrm{f}}(R)$, condition (3.1.1) becomes weaker. When this weaker condition is satisfies, we say that R *satisfies weak vanishing*. It is known that R satisfies weak vanishing in many cases, for example if R is a complete intersection; see Roberts [19] or Gillet and Soulé [10]. There are, so far, no counterexamples preventing it from holding in full generality.

3.2. **Euler form.** Let X and Y be finite complexes with $\operatorname{Supp} X \cap \operatorname{Supp} Y = \{\mathfrak{m}\}$. The *Euler form* of X and Y is defined by

$$\xi(X, Y) = \chi(\mathbf{R}\mathrm{Hom}_R(X, Y)) \quad \text{when either } X \in \mathsf{P}^{\mathrm{f}}(R) \text{ or } Y \in \mathsf{I}^{\mathrm{f}}(R).$$

In the case where X and Y are finitely generated modules, $\chi(X, Y)$ coincides with the Euler form introduced by Mori and Smith [16].

If R admits a dualizing complex, then from Mori [17, Lemma 4.3(1) and (2)] and the definition of $(-)^{\star}$, we obtain

$$\begin{aligned} \xi(X, Y) &= \chi(X, Y^{\dagger}) \quad \text{whenever } X \in \mathsf{P}^{\mathrm{f}}(R) \text{ or } Y \in \mathsf{I}^{\mathrm{f}}(R), \\ \chi(X^{*}, Y) &= \chi(X, Y^{\dagger}) \quad \text{whenever } X \in \mathsf{P}^{\mathrm{f}}(R), \quad \text{and} \\ \xi(X, Y^{\star}) &= \xi(X^{\dagger}, Y) \quad \text{whenever } Y \in \mathsf{I}^{\mathrm{f}}(R). \end{aligned} \tag{3.2.1}$$

Since the dagger functor does not change supports of complexes, the first formula in (3.2.1) shows that R satisfies vanishing exactly when

$$\xi(X, Y) = 0 \quad \text{if } \dim(\operatorname{Supp} X) + \dim(\operatorname{Supp} Y) < \dim R \tag{3.2.2}$$

when either $X \in \mathsf{P}^{\mathrm{f}}(R)$ or $Y \in \mathsf{I}^{\mathrm{f}}(R)$, and that R satisfies weak vanishing exactly when (3.2.2) holds when we require both $X \in \mathsf{P}^{\mathrm{f}}(R)$ *and* $Y \in \mathsf{I}^{\mathrm{f}}(R)$.

3.3. **Dutta multiplicity.** Assume that R is complete of prime characteristic p and with perfect residue field. Let X and Y be finite complexes with

$$\operatorname{Supp} X \cap \operatorname{Supp} Y = \{\mathfrak{m}\} \quad \text{and} \quad \dim(\operatorname{Supp} X) + \dim(\operatorname{Supp} Y) \leqslant \dim R.$$

The *Dutta multiplicity* of X and Y is defined by

$$\chi_{\infty}(X, Y) = \lim_{e \to \infty} \frac{1}{p^{e \operatorname{codim}(\operatorname{Supp} X)}} \chi(\mathbf{L}F^{e}(X), Y) \quad \text{when } X \in \mathsf{P}^{\mathrm{f}}(R).$$

When X and Y are finitely generated modules, $\chi_{\infty}(X, Y)$ coincides with the Dutta multiplicity defined in [6].

The Euler form prompts to two natural analogs of the Dutta multiplicity. We define

$$\xi_\infty(X,Y) = \lim_{e\to\infty} \frac{1}{p^{e\,\mathrm{codim}(\mathrm{Supp}\,Y)}} \xi(X, \mathbf{R}G^e(Y)) \quad \text{when } Y \in \mathsf{I}^{\mathsf{f}}(R), \text{ and}$$

$$\xi^\infty(X,Y) = \lim_{e\to\infty} \frac{1}{p^{e\,\mathrm{codim}(\mathrm{Supp}\,X)}} \xi(\mathbf{L}F^e(X), Y) \quad \text{when } X \in \mathsf{P}^{\mathsf{f}}(R).$$

We immediately note, using (3.2.1) together with Lemma 2.11, that

$$\xi_\infty(X,Y) = \chi_\infty(Y^\dagger, X) \quad \text{whenever } Y \in \mathsf{I}^{\mathsf{f}}(\mathfrak{Y}), \text{ and}$$
$$\xi^\infty(X,Y) = \chi_\infty(X^*, Y) \quad \text{whenever } X \in \mathsf{P}^{\mathsf{f}}(\mathfrak{X}).$$

4. Grothendieck spaces

In this section we present the definition and basic properties of Grothendieck spaces. We will introduce three types of Grothendieck spaces, two of which were introduced in [11]. The constructions in *loc. cit.* are different from the ones here but yield the same spaces.

4.1. **Complement.** For any specialization-closed subset $\mathfrak{X}$ of $\mathrm{Spec}\,R$, a new subset is defined by

$$\mathfrak{X}^c = \left\{ \mathfrak{p} \in \mathrm{Spec}\,R \,\middle|\, \mathfrak{X} \cap V(\mathfrak{p}) = \{\mathfrak{m}\} \text{ and } \dim V(\mathfrak{p}) \leqslant \mathrm{codim}\,\mathfrak{X} \right\}.$$

This set is engineered to be the largest subset of $\mathrm{Spec}\,R$ such that

$$\mathfrak{X} \cap \mathfrak{X}^c = \{\mathfrak{m}\} \qquad \text{and} \qquad \dim \mathfrak{X} + \dim \mathfrak{X}^c \leqslant \dim R.$$

In fact, when $\mathfrak{X}$ is closed,

$$\dim \mathfrak{X} + \dim \mathfrak{X}^c = \dim R.$$

Note that $\mathfrak{X}^c$ is specialization-closed and that $\mathfrak{X} \subseteq \mathfrak{X}^{cc}$.

4.2. **Grothendieck space.** Let $\mathfrak{X}$ be a specialization-closed subset of $\mathrm{Spec}\,R$. The *Grothendieck space* of the category $\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ is the $\mathbb{Q}$–vector space $\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ presented by elements $[X]_{\mathsf{P}^{\mathsf{f}}(\mathfrak{X})}$, one for each isomorphism class of a complex $X \in \mathsf{P}^{\mathsf{f}}(\mathfrak{X})$, and relations

$$[X]_{\mathsf{P}^{\mathsf{f}}(\mathfrak{X})} = [\widetilde{X}]_{\mathsf{P}^{\mathsf{f}}(\mathfrak{X})} \quad \text{whenever} \quad \chi(X,-) = \chi(\widetilde{X},-)$$

as metafunctions ("functions" from a category to a set) $\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X}^c) \to \mathbb{Q}$.

Similarly, the Grothendieck space of the category $\mathsf{I}^{\mathsf{f}}(\mathfrak{X})$ is the $\mathbb{Q}$–vector space $\mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X})$ presented by elements $[Y]_{\mathsf{I}^{\mathsf{f}}(\mathfrak{X})}$, one for each isomorphism class of a complex $Y \in \mathsf{I}^{\mathsf{f}}(\mathfrak{X})$, and relations

$$[Y]_{\mathsf{I}^{\mathsf{f}}(\mathfrak{X})} = [\widetilde{Y}]_{\mathsf{I}^{\mathsf{f}}(\mathfrak{X})} \quad \text{whenever} \quad \xi(-,Y) = \xi(-,\widetilde{Y})$$

as metafunctions $\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X}^c) \to \mathbb{Q}$.

Finally, the Grothendieck space of the category $\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})$ is the $\mathbb{Q}$–vector space $\mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})$ presented by elements $[Z]_{\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})}$, one for each isomorphism class of a complex $Z \in \mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})$, and relations

$$[Z]_{\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})} = [\widetilde{Z}]_{\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})} \quad \text{whenever} \quad \chi(-,Z) = \chi(-,\widetilde{Z})$$

as metafunctions $\mathsf{P}^{\mathrm{f}}(\mathfrak{X}^c) \to \mathbb{Q}$. Because of (3.2.1), these relations are exactly the same as the relations

$$[Z]_{\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})} = [\widetilde{Z}]_{\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})} \quad \text{whenever} \quad \xi(Z,-) = \xi(\widetilde{Z},-)$$

as metafunctions $\mathsf{I}^{\mathrm{f}}(\mathfrak{X}^c) \to \mathbb{Q}$.

By definition of the Grothendieck space $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ there is, for each complex Z in $\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}^c)$, a well-defined $\mathbb{Q}$–linear map

$$\chi(-,Z)\colon \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X}) \to \mathbb{Q} \quad \text{given by} \quad [X]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})} \mapsto \chi(X,Z).$$

We equip $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ with the *initial topology* induced by the family of maps in the above form. This topology is the coarsest topology on $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ making the above map continuous for all Z in $\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}^c)$. Likewise, for each complex Z in $\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}^c)$, there is a well-defined $\mathbb{Q}$–linear map

$$\xi(Z,-)\colon \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X}) \to \mathbb{Q} \quad \text{given by} \quad [Y]_{\mathsf{I}^{\mathrm{f}}(\mathfrak{X})} \mapsto \xi(Z,Y),$$

and we equip $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ with the initial topology induced by the family of maps in the above form. Finally, for each complex X in $\mathsf{P}^{\mathrm{f}}(\mathfrak{X}^c)$, there is a well-defined $\mathbb{Q}$–linear map

$$\chi(X,-)\colon \mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}) \to \mathbb{Q} \quad \text{given by} \quad [Z]_{\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})} \mapsto \chi(X,Z),$$

and we equip $\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$ with the initial topology induced by the family of maps in the above form. By (3.2.1), this topology is the same as the initial topology induced by the family of (well-defined, $\mathbb{Q}$–linear) maps in the form

$$\xi(-,Y)\colon \mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}) \to \mathbb{Q} \quad \text{given by} \quad [Z]_{\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})} \mapsto \xi(Z,Y),$$

for complexes Y in $\mathsf{I}^{\mathrm{f}}(\mathfrak{X}^c)$.

It is straightforward to see that addition and scalar multiplication are continuous operations on Grothendieck spaces, making $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$, $\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$ and $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ topological $\mathbb{Q}$–vector spaces. We shall always consider Grothendieck spaces as topological $\mathbb{Q}$–vector spaces, so that, for example, a "homomorphism" between Grothendieck spaces means a homomorphism of topological $\mathbb{Q}$–vector spaces: that is, a continuous, $\mathbb{Q}$–linear map.

The following proposition is an improved version of [11, Proposition 2(iv) and (v)].

4.3. **Proposition.** *Let $\mathfrak{X}$ be a specialization-closed subset of* $\operatorname{Spec} R$.

(*i*) *Any element in $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ can be written in the form $r[X]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})}$ for some $r \in \mathbb{Q}$ and some $X \in \mathsf{P}^{\mathrm{f}}(\mathfrak{X})$, any element in $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ can be written in the form $s[Y]_{\mathsf{I}^{\mathrm{f}}(\mathfrak{X})}$ for some $s \in \mathbb{Q}$ and some $Y \in \mathsf{I}^{\mathrm{f}}(\mathfrak{X})$, and any element in $\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$ can be written in the form $t[Z]_{\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})}$ for some $t \in \mathbb{Q}$ and some $Z \in \mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$. Moreover, X, Y and Z may be chosen so that*

$$\operatorname{codim}(\operatorname{Supp} X) = \operatorname{codim}(\operatorname{Supp} Y) = \operatorname{codim}(\operatorname{Supp} Z) = \operatorname{codim} \mathfrak{X}.$$

(*ii*) *For any complex $Z \in \mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$, we have the identity*

$$[Z]_{\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})} = [\mathrm{H}(Z)]_{\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})}.$$

In particular, the $\mathbb{Q}$–vector space $\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$ is generated by elements in the form $[R/\mathfrak{p}]_{\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})}$ for prime ideals $\mathfrak{p}$ in $\mathfrak{X}$.

Proof. (i) By construction, any element α in $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ is a $\mathbb{Q}$–linear combination

$$\alpha = r_1[X^1]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})} + \cdots + r_n[X^n]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})}$$

where $r_i \in \mathbb{Q}$ and $X^i \in \mathsf{P}^{\mathrm{f}}(\mathfrak{X})$. Since a shift of a complex changes the sign of the corresponding element in the Grothendieck space, we can assume that $r_i > 0$ for all i. Choosing a greatest common denominator for the r_i's, we can find $r \in \mathbb{Q}$ such that

$$\alpha = r(m_1[X^1]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})} + \cdots + m_n[X^n]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})}) = r[X]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})},$$

where the m_i's are natural numbers and X is the direct sum over i of m_i copies of X^i.

In order to prove the last statement of (i), choose a prime ideal $\mathfrak{p} = (a_1, \ldots, a_t)$ in $\mathfrak{X}$ which is first in a chain $\mathfrak{p} = \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_t = \mathfrak{m}$ of prime ideals in $\mathfrak{X}$ of maximal length $t = \operatorname{codim} \mathfrak{X}$. Note that $\mathfrak{X} \supseteq V(\mathfrak{p})$ and that the Koszul complex $K = K(a_1, \ldots, a_t)$ has support exactly equal to $V(\mathfrak{p})$. It follows that

$$\alpha = \alpha + 0 = r[X]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})} + r[K]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})} - r[K]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})} = r[X \oplus K \oplus \Sigma K]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})},$$

where $\operatorname{codim}(\operatorname{Supp}(X \oplus K \oplus \Sigma K)) = \operatorname{codim} \mathfrak{X}$. The same argument applies to elements of $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ and $\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$.

(ii) Any complex in $\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$ is isomorphic to a bounded complex. After an appropriate shift, we may assume that Z is a complex in $\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$ in the form

$$0 \to Z_n \to \cdots \to Z_1 \to Z_0 \to 0$$

for some natural number n. Since $\mathrm{H}_n(Z)$ is the kernel of the map $Z_n \to Z_{n-1}$, we can construct a short exact sequence of complexes

$$0 \to \Sigma^n \mathrm{H}_n(Z) \to Z \to Z' \to 0,$$

where Z' is a complex in $\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$ concentrated in the same degrees as Z. The complex Z' is exact in degree n, and $\mathrm{H}_i(Z') = \mathrm{H}_i(Z)$ for $i = n-1, \ldots, 0$. In the Grothendieck space $\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$, we then have

$$[Z]_{\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})} = [\Sigma^n \mathrm{H}_n(Z)]_{\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})} + [Z']_{\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})}.$$

Again, Z' is isomorphic to a complex concentrated in degree $n-1, \cdots, 0$, so we can repeat the process a finite number of times and achieve that

$$\begin{aligned} [Z]_{\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})} &= [\Sigma^n \mathrm{H}_n(Z)]_{\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})} + \cdots + [\Sigma\, \mathrm{H}_1(Z)]_{\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})} + [\mathrm{H}_0(Z)]_{\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})} \\ &= [\Sigma^n \mathrm{H}_n(Z) \oplus \cdots \oplus \Sigma\, \mathrm{H}_1(Z) \oplus \mathrm{H}_0(Z)]_{\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})} \\ &= [\mathrm{H}(Z)]_{\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})}. \end{aligned}$$

The above analysis shows that any element of $\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$ can be written in the form

$$r[Z]_{\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})} = r \sum_i (-1)^i [\mathrm{H}_i(Z)]_{\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})},$$

which means that $\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$ is generated by modules. Taking a filtration of a module establishes that $\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$ must be generated by elements of the form $[R/\mathfrak{p}]_{\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})}$ for prime ideals $\mathfrak{p}$ in $\mathfrak{X}$. $\square$

4.4. **Induced Euler characteristic.** The Euler characteristic $\chi\colon \mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{m}) \to \mathbb{Q}$ induces an isomorphism[1]

$$\mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{m}) \xrightarrow{\cong} \mathbb{Q} \quad \text{given by} \quad [Z]_{\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{m})} \mapsto \chi(Z). \tag{4.4.1}$$

See [11] for more details. We also denote this isomorphism by χ. The isomorphism means that we can identify the intersection multiplicity $\chi(X,Y)$ and the Euler form $\xi(X,Y)$ of complexes X and Y with elements in $\mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{m})$ of the form

$$[X \otimes^{\mathbf{L}}_R Y]_{\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{m})} \quad \text{and} \quad [\mathbf{R}\mathrm{Hom}_R(X,Y)]_{\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{m})},$$

respectively.

4.5. **Induced inclusion.** Let $\mathfrak{X}$ be a specialization-closed subset of $\operatorname{Spec} R$. It is straightforward to verify that the full embeddings of $\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ and $\mathsf{I}^{\mathsf{f}}(\mathfrak{X})$ into $\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})$ induce homomorphisms[2]

$$\begin{aligned} \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X}) &\to \mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X}) \quad \text{given by} \quad [X]_{\mathsf{P}^{\mathsf{f}}(\mathfrak{X})} \mapsto [X]_{\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})}, \text{ and} \\ \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X}) &\to \mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X}) \quad \text{given by} \quad [Y]_{\mathsf{I}^{\mathsf{f}}(\mathfrak{X})} \mapsto [Y]_{\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})}. \end{aligned}$$

If $\mathfrak{X}$ and $\mathfrak{X}'$ are specialization-closed subsets of $\operatorname{Spec} R$ such that that $\mathfrak{X} \subseteq \mathfrak{X}'$, then it is straightforward to verify that the full embeddings of $\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ into $\mathsf{P}^{\mathsf{f}}(\mathfrak{X}')$, $\mathsf{I}^{\mathsf{f}}(\mathfrak{X})$ into $\mathsf{I}^{\mathsf{f}}(\mathfrak{X}')$ and $\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})$ into $\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X}')$ induce homomorphisms

$$\begin{aligned} \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X}) &\to \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X}') \quad \text{given by} \quad [X]_{\mathsf{P}^{\mathsf{f}}(\mathfrak{X})} \mapsto [X]_{\mathsf{P}^{\mathsf{f}}(\mathfrak{X}')}, \\ \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X}) &\to \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X}') \quad \text{given by} \quad [Y]_{\mathsf{I}^{\mathsf{f}}(\mathfrak{X})} \mapsto [Y]_{\mathsf{I}^{\mathsf{f}}(\mathfrak{X}')}, \text{ and} \\ \mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X}) &\to \mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X}') \quad \text{given by} \quad [Z]_{\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})} \mapsto [Z]_{\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X}')}. \end{aligned}$$

The maps obtained in this way are called *inclusion homomorphisms*, and we shall often denote them by an overline: if σ is an element in a Grothendieck space, then $\overline{\sigma}$ denotes the image of σ after an application of an inclusion homomorphisms.

4.6. **Induced tensor product and Hom.** Proposition 4.7 below shows that the left-derived tensor product functor and the right-derived Hom-functor induce bi-homomorphisms[3] on Grothendieck spaces. To clarify the contents of the proposition, let $\mathfrak{X}$ and $\mathfrak{Y}$ be specialization-closed subsets of $\operatorname{Spec} R$ such that $\mathfrak{X} \cap \mathfrak{Y} = \{\mathfrak{m}\}$ and $\dim \mathfrak{X} + \dim \mathfrak{Y} \leqslant \dim R$. Proposition 4.7 states, for example, that the right-derived Hom-functor induces a bi-homomorphism

$$\mathrm{Hom}\colon \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X}) \times \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{Y}) \to \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{m}).$$

Given elements $\sigma \in \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ and $\tau \in \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{Y})$, we can, by Proposition 4.3, write

$$\sigma = r[X]_{\mathsf{P}^{\mathsf{f}}(\mathfrak{X})} \quad \text{and} \quad \tau = s[Y]_{\mathsf{I}^{\mathsf{f}}(\mathfrak{Y})},$$

where r and s are rational numbers, X is a complex in $\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ and Y is a complex in $\mathsf{I}^{\mathsf{f}}(\mathfrak{Y})$. The bi-homomorphism above is then given by

$$(\sigma,\tau) \mapsto \mathrm{Hom}(\sigma,\tau) = rs[\mathbf{R}\mathrm{Hom}_R(X,Y)]_{\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{m})}. \tag{4.6.1}$$

We shall use the symbol "$\otimes$" to denote any bi-homomorphism on Grothendieck spaces induced by the left-derived tensor product and the symbol "Hom" to denote any bi-homomorphism induced by right-derived Hom-functor. Together with the

[1] That is, a $\mathbb{Q}$–linear homeomorphism.

[2] That is, continuous, $\mathbb{Q}$–linear maps.

[3] That is, maps that are continuous and $\mathbb{Q}$–linear in each variable.

isomorphism in (4.4.1) it follows that the intersection multiplicity $\chi(X,Y)$ and Euler form $\xi(X,Y)$ can be identified with elements in $\mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{m})$ of the form

$$[X]_{\mathsf{P}^{\mathsf{f}}(\mathfrak{X})} \otimes [Y]_{\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{Y})}, \qquad [X]_{\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})} \otimes [Y]_{\mathsf{P}^{\mathsf{f}}(\mathfrak{Y})},$$
$$\operatorname{Hom}([X]_{\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X})}, [Y]_{\mathsf{I}^{\mathsf{f}}(\mathfrak{Y})}) \quad \text{and} \quad \operatorname{Hom}([X]_{\mathsf{P}^{\mathsf{f}}(\mathfrak{X})}, [Y]_{\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{Y})}).$$

4.7. **Proposition.** *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be specialization-closed subsets of* Spec R *such that* $\mathfrak{X} \cap \mathfrak{Y} = \{\mathfrak{m}\}$ *and* $\dim \mathfrak{X} + \dim \mathfrak{Y} \leqslant \dim R$. *The left-derived tensor product induces bi-homomorphisms as in the first column below, and the right-derived Hom-functor induces bi-homomorphisms as in the second column below.*

$$\begin{array}{cc}
\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X}) \times \mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{Y}) \to \mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{m}), & \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X}) \times \mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{Y}) \to \mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{m}), \\
\mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X}) \times \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{Y}) \to \mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{m}), & \mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X}) \times \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{Y}) \to \mathbb{G}\mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{m}), \\
\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X}) \times \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{Y}) \to \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{m}), & \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X}) \times \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{Y}) \to \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{m}), \\
\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X}) \times \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{Y}) \to \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{m}), & \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X}) \times \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{Y}) \to \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{m}), \\
\mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X}) \times \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{Y}) \to \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{m}) \quad \textit{and} & \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X}) \times \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{Y}) \to \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{m}).
\end{array}$$

Proof. We verify that the map

$$\operatorname{Hom}\colon \mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X}) \times \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{Y}) \to \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{m})$$

given as in (4.6.1) is a well-defined bi-homomorphism, leaving the same verifications for the remaining maps as an easy exercise for the reader.

Therefore, assume that X and $\widetilde{X}$ are complexes from $\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ and that Y and $\widetilde{Y}$ are complexes from $\mathsf{I}^{\mathsf{f}}(\mathfrak{Y})$ such that

$$\sigma = [X]_{\mathsf{P}^{\mathsf{f}}(\mathfrak{X})} = [\widetilde{X}]_{\mathsf{P}^{\mathsf{f}}(\mathfrak{X})} \quad \text{and} \quad \tau = [Y]_{\mathsf{I}^{\mathsf{f}}(\mathfrak{Y})} = [\widetilde{Y}]_{\mathsf{I}^{\mathsf{f}}(\mathfrak{Y})}.$$

In order to show that the map is a well-defined $\mathbb{Q}$–bi-linear map, we are required to demonstrate that

$$[\mathbf{R}\mathrm{Hom}_R(X,Y)]_{\mathsf{I}^{\mathsf{f}}(\mathfrak{m})} = [\mathbf{R}\mathrm{Hom}_R(\widetilde{X},\widetilde{Y})]_{\mathsf{I}^{\mathsf{f}}(\mathfrak{m})}.$$

To this end, let Z be an arbitrary complex in $\mathsf{D}^{\mathsf{f}}_{\square}(\{\mathfrak{m}\}^c) = \mathsf{D}^{\mathsf{f}}_{\square}(R)$. We want to show that

$$\xi(Z, \mathbf{R}\mathrm{Hom}_R(X,Y)) = \xi(Z, \mathbf{R}\mathrm{Hom}_R(\widetilde{X},\widetilde{Y})).$$

Without loss of generality, we may assume that R is complete; in particular, we may assume that R admits a normalized dualizing complex. Observe that

$$Z \otimes_R X \in \mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X}) \subseteq \mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{Y}^c) \quad \text{and} \quad Z \otimes^{\mathbf{L}}_R Y^{\dagger} \in \mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{Y}) \subseteq \mathsf{D}^{\mathsf{f}}_{\square}(\mathfrak{X}^c).$$

Applying (3.2.1), (Hom-eval) and (Assoc), we learn that

$$\begin{aligned}
\xi(Z, \mathbf{R}\mathrm{Hom}_R(X,Y)) &= \chi(Z, \mathbf{R}\mathrm{Hom}_R(X,Y)^{\dagger}) \\
&= \chi(Z, X \otimes^{\mathbf{L}}_R Y^{\dagger}) \\
&= \chi(X, Z \otimes^{\mathbf{L}}_R Y^{\dagger}).
\end{aligned} \tag{4.7.1}$$

A similar computation shows that $\xi(Z, \mathbf{R}\mathrm{Hom}_R(\widetilde{X},Y)) = \chi(\widetilde{X}, Z \otimes^{\mathbf{L}}_R Y^{\dagger})$, and since $[X]_{\mathsf{P}^{\mathsf{f}}(\mathfrak{X})} = [\widetilde{X}]_{\mathsf{P}^{\mathsf{f}}(\mathfrak{X})}$, we conclude that

$$\xi(Z, \mathbf{R}\mathrm{Hom}_R(X,Y)) = \xi(Z, \mathbf{R}\mathrm{Hom}_R(\widetilde{X},Y)).$$

An application of (Adjoint) yields that

$$\xi(Z, \mathbf{R}\mathrm{Hom}_R(\widetilde{X},Y)) = \xi(Z \otimes^{\mathbf{L}}_R \widetilde{X}, Y),$$

and similarly $\xi(Z, \mathbf{R}\mathrm{Hom}_R(\widetilde{X}, \widetilde{Y})) = \xi(Z \otimes^{\mathbf{L}}_R \widetilde{X}, \widetilde{Y})$. Since $[Y]_{\mathsf{I}^{\mathrm{f}}(\mathfrak{Y})} = [\widetilde{Y}]_{\mathsf{I}^{\mathrm{f}}(\mathfrak{Y})}$, we conclude that

$$\xi(Z, \mathbf{R}\mathrm{Hom}_R(\widetilde{X}, Y)) = \xi(Z, \mathbf{R}\mathrm{Hom}_R(\widetilde{X}, \widetilde{Y})).$$

Thus, we have that

$$\xi(Z, \mathbf{R}\mathrm{Hom}_R(X, Y)) = \xi(Z, \mathbf{R}\mathrm{Hom}_R(\widetilde{X}, \widetilde{Y})),$$

which establishes well-definedness.

By definition, the induced Hom-map is $\mathbb{Q}$–linear. To establish that it is continuous in, say, the first variable it suffices for fixed $\tau \in \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{Y})$ to show that, to every $\varepsilon > 0$ and every complex $Z \in \mathsf{D}^{\mathrm{f}}_{\square}(\{\mathfrak{m}\}^c) = \mathsf{D}^{\mathrm{f}}_{\square}(R)$, there exists a $\delta > 0$ and a complex $Z' \in \mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}^c)$ such that

$$|\chi(\sigma, Z')| < \delta \implies |\xi(Z, \mathrm{Hom}(\sigma, \tau))| < \varepsilon.$$

We can write $\tau = r[Y]_{\mathsf{I}^{\mathrm{f}}(\mathfrak{Y})}$ for an $Y \in \mathsf{I}^{\mathrm{f}}(\mathfrak{Y})$ and a rational number $r > 0$. According to (4.7.1), the implication above is then achieved with $Z' = Z \otimes^{\mathbf{L}}_R Y^\dagger$ and $\delta = \varepsilon/r$. Continuity in the second variable is shown by similar arguments. $\square$

In Proposition 4.8 below, we will show that the dagger, Foxby and star functors from diagram (2.9.1) induce isomorphisms of Grothendieck spaces. We shall denote the isomorphisms induced by the star and dagger duality functors by the same symbol as the original functor, whereas the isomorphisms induced by the Foxby functors will be denoted according to Proposition 4.7 by $D \otimes -$ and $\mathrm{Hom}(D, -)$. In this way, for example,

$$[X]^\dagger_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})} = [X^\dagger]_{\mathsf{I}^{\mathrm{f}}(\mathfrak{X})}, \quad [X]^*_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})} = [X^*]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})} \quad \text{and} \quad D \otimes [X]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})} = [D \otimes^{\mathbf{L}}_R X]_{\mathsf{I}^{\mathrm{f}}(\mathfrak{X})}.$$

4.8. **Proposition.** *Let $\mathfrak{X}$ be a specialization-closed subset of* $\operatorname{Spec} R$*, and assume that R admits a dualizing complex. The functors from diagram* (2.9.1) *induce isomorphisms of Grothendieck spaces as described by the horizontal and circular arrows in the following commutative diagram.*

$$\begin{tikzcd}[column sep=huge]
\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}) \arrow[r, shift left, "(-)^\dagger"] & \mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}) \arrow[l, shift left, "(-)^\dagger"] \\
\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X}) \arrow[u] \arrow[loop left, "(-)^*"] \arrow[r, bend left, "D\otimes^{\mathbf{L}}_R-"] \arrow[r, shift left, "(-)^\dagger"] & \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X}) \arrow[u] \arrow[loop right, "(-)^*"] \arrow[l, shift left, "(-)^\dagger"] \arrow[l, bend left, "\mathbf{R}\mathrm{Hom}_R(D,-)"]
\end{tikzcd}$$

Proof. The fact that the dagger, star and Foxby functors induce homomorphisms on Grothendieck spaces follows immediately from Proposition 4.7. The fact that the induced homomorphisms are isomorphisms follows immediately from 2.7, 2.8 and 2.9, since the underlying functors define dualities or equivalences of categories. $\square$

4.9. **Proposition.** *Let $\mathfrak{X}$ be a specialization-closed subset of* $\operatorname{Spec} R$ *and consider the following elements of Grothendieck spaces.*

$$\alpha \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X}), \quad \beta \in \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X}), \quad \gamma \in \mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}^c) \quad \textit{and} \quad \sigma \in \mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{m}).$$

Then $\sigma^\dagger = \sigma$ holds in $\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{m})$, and so do the following identities.

$$\begin{gathered}\alpha \otimes \gamma = \operatorname{Hom}(\gamma, \alpha^\dagger) = \operatorname{Hom}(\alpha, \gamma^\dagger) = \operatorname{Hom}(\alpha^*, \gamma)\\ \operatorname{Hom}(\alpha, \gamma) = \alpha \otimes \gamma^\dagger = \operatorname{Hom}(\gamma, D \otimes \alpha) = \alpha^* \otimes \gamma\\ \operatorname{Hom}(\gamma, \beta) = \beta^\dagger \otimes \gamma = \operatorname{Hom}(\operatorname{Hom}(D, \beta), \gamma)\\ \operatorname{Hom}(\beta^\dagger, \gamma) = \operatorname{Hom}(\gamma^\dagger, \beta) = \operatorname{Hom}(D, \beta) \otimes \gamma = \operatorname{Hom}(\gamma, \beta^\star)\end{gathered}$$

Proof. Recall from 2.9 that the Foxby functors can be written as the composition of a star and a dagger functor. All identities follow from the formulas in (3.2.1). The formula for σ is a consequence of the first formula in (3.2.1) in the case $X = R$. □

4.10. **Frobenius endomorphism.** Assume that R is complete of prime characteristic p and with perfect residue field. Let $\mathfrak{X}$ be a specialization-closed subset of $\operatorname{Spec} R$, and let n be a non-negative integer. The derived Frobenius endofunctor $\mathbf{L}F^n$ on $\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ induces an endomorphism[4] on $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$, which will be denoted $F^n_{\mathfrak{X}}$; see [11] for further details. It is given for a complex $X \in \mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ by

$$F^n_{\mathfrak{X}}([X]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})}) = [\mathbf{L}F^n(X)]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})}.$$

Let

$$\Phi^n_{\mathfrak{X}} = \frac{1}{p^{n \operatorname{codim} \mathfrak{X}}} F^n_{\mathfrak{X}} \colon \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X}) \to \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X}).$$

According to [11, Theorem 19], the endomorphism $\Phi^n_{\mathfrak{X}}$ is diagonalizable.

In Lemma 2.11, we established that the functor $\mathbf{R}G^n$ is an endofunctor on $\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ which can be written as

$$\mathbf{R}G^n(-) = (-)^\dagger \circ \mathbf{L}F^n \circ (-)^\dagger.$$

Thus, $\mathbf{R}G^n$ is composed of functors that induce homomorphisms on Grothendieck spaces, and hence it too induces a homomorphism $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X}) \to \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$. We denote this endomorphism on $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ by $G^n_{\mathfrak{X}}$. It is given for a complex $Y \in \mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ by

$$G^n_{\mathfrak{X}}([Y]_{\mathsf{I}^{\mathrm{f}}(\mathfrak{X})}) = [\mathbf{R}G^n(Y)]_{\mathsf{I}^{\mathrm{f}}(\mathfrak{X})}.$$

Let

$$\Psi^n_{\mathfrak{X}} = \frac{1}{p^{n \operatorname{codim} \mathfrak{X}}} G^n_{\mathfrak{X}} \colon \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X}) \to \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X}).$$

Theorem 6.2 shows that $\Psi^n_{\mathfrak{X}}$ also is a diagonalizable automorphism.

For complexes $X \in \mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ and $Y \in \mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ we shall write $\Phi^n_{\mathfrak{X}}(X)$ and $\Psi^n_{\mathfrak{X}}(Y)$ instead of $\Phi^n_{\mathfrak{X}}([X]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})})$ and $\Psi^n_{\mathfrak{X}}([Y]_{\mathsf{I}^{\mathrm{f}}(\mathfrak{X})})$, respectively. The isomorphism in (4.4.1) together with Proposition 4.7 shows that the Dutta multiplicity $\chi_\infty(X, Y)$ and its two analogs $\xi_\infty(X, Y)$ and $\xi^\infty(X, Y)$ from Section 3.3 can be identified with elements in $\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{m})$ of the form

$$\lim_{e\to\infty} (\Phi^e_{\mathfrak{X}}(X) \otimes [Y]_{\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{Y})}), \quad \lim_{e\to\infty} \operatorname{Hom}([X]_{\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})}, \Psi^e_{\mathfrak{Y}}(Y)) \text{ and } \lim_{e\to\infty} \operatorname{Hom}(\Phi^e_{\mathfrak{X}}(X), [Y]_{\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{Y})}).$$

[4]That is, a continuous, $\mathbb{Q}$–linear operator.

5. Vanishing

5.1. **Vanishing.** Let $\mathfrak{X}$ be a specialization-closed subset of $\operatorname{Spec} R$ and consider an element α in $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$, an element β in $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ and an element γ in $\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$. The *dimensions* of α, β and γ are defined as

$$\dim\alpha = \inf\Big\{\dim(\operatorname{Supp} X)\,\Big|\,\alpha = r[X]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})} \text{ for some } r\in\mathbb{Q} \text{ and } X\in\mathsf{P}^{\mathrm{f}}(\mathfrak{X})\Big\},$$

$$\dim\beta = \inf\Big\{\dim(\operatorname{Supp} Y)\,\Big|\,\alpha = s[Y]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})} \text{ for some } s\in\mathbb{Q} \text{ and } Y\in\mathsf{I}^{\mathrm{f}}(\mathfrak{X})\Big\} \text{ and}$$

$$\dim\gamma = \inf\Big\{\dim(\operatorname{Supp} Z)\,\Big|\,\gamma = t[Z]_{\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})} \text{ for some } t\in\mathbb{Q} \text{ and } Z\in\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})\Big\}.$$

In particular, the dimension of an element in a Grothendieck space is $-\infty$ if and only if the element is trivial. We say that α *satisfies vanishing* if

$$\alpha\otimes\sigma = 0 \quad \text{in } \mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{m}) \text{ for all } \sigma\in\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}^c) \text{ with } \dim\sigma < \operatorname{codim}\mathfrak{X},$$

and that α *satisfies weak vanishing* if

$$\overline{\alpha\otimes\tau} = 0 \quad \text{in } \mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{m}) \text{ for all } \tau\in\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X}^c) \text{ with } \dim\tau < \operatorname{codim}\mathfrak{X}.$$

Similarly, we say that β *satisfies vanishing* if

$$\operatorname{Hom}(\sigma,\beta) = 0 \quad \text{in } \mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{m}) \text{ for all } \sigma\in\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}^c) \text{ with } \dim\sigma < \operatorname{codim}\mathfrak{X},$$

and that β *satisfies weak vanishing* if

$$\overline{\operatorname{Hom}(\tau,\beta)} = 0 \quad \text{in } \mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{m}) \text{ for all } \tau\in\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X}^c) \text{ with } \dim\tau < \operatorname{codim}\mathfrak{X}.$$

The *vanishing dimension* of α and β is defined as the numbers

$$\operatorname{vdim}\alpha = \inf\left\{u\in\mathbb{Z}\,\middle|\,\begin{array}{l}\alpha\otimes\sigma = 0 \text{ for all } \sigma\in\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}^c)\\ \text{with } \dim\sigma < \operatorname{codim}\mathfrak{X} - u\end{array}\right\} \text{ and}$$

$$\operatorname{vdim}\beta = \inf\left\{v\in\mathbb{Z}\,\middle|\,\begin{array}{l}\operatorname{Hom}(\sigma,\beta) = 0 \text{ for all } \sigma\in\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}^c)\\ \text{with } \dim\sigma < \operatorname{codim}\mathfrak{X} - v\end{array}\right\}.$$

In particular, the vanishing dimension of an element in a Grothendieck space is $-\infty$ if and only if the element is trivial, and the vanishing dimension is less than or equal to 0 if and only if the element satisfies vanishing.

5.2. **Remark.** If X is a complex in $\mathsf{P}^{\mathrm{f}}(R)$ with $\mathfrak{X} = \operatorname{Supp} X$, then the element $\alpha = [X]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})}$ in $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ satisfies vanishing exactly when

$$\chi(X,Y) = 0 \quad \text{for all complexes } Y\in\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}^c) \text{ with } \dim(\operatorname{Supp} Y) < \operatorname{codim}\mathfrak{X},$$

and α satisfies weak vanishing exactly when

$$\chi(X,Y) = 0 \quad \text{for all complexes } Y\in\mathsf{P}^{\mathrm{f}}(\mathfrak{X}^c) \text{ with } \dim(\operatorname{Supp} Y) < \operatorname{codim}\mathfrak{X}.$$

The vanishing dimension of α measures the extent to which vanishing fails to hold: the vanishing dimension of α is the infimum of integers u such that

$$\chi(X,Y) = 0 \quad \text{for all complexes } Y\in\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}^c) \text{ with } \dim(\operatorname{Supp} Y) < \operatorname{codim}\mathfrak{X} - u.$$

It follows that the ring R satisfies vanishing (or weak vanishing, respectively) as defined in 3.1, if and only if all elements of $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ for all specialization-closed subsets $\mathfrak{X}$ of $\operatorname{Spec} R$ satisfy vanishing (or weak vanishing, respectively).

If Y is a complex in $\mathsf{I}^{\mathrm{f}}(R)$ with $\mathfrak{X} = \operatorname{Supp} Y$, then the element $\beta = [Y]_{\mathsf{I}^{\mathrm{f}}(\mathfrak{X})}$ in $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ satisfies vanishing exactly when

$$\xi(X,Y) = 0 \quad \text{for all complexes } X\in\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}^c) \text{ with } \dim(\operatorname{Supp} X) < \operatorname{codim}\mathfrak{X}.$$

and β satisfies weak vanishing exactly when

$$\xi(X,Y)=0 \quad \text{for all complexes } X\in \mathsf{P}^{\mathrm{f}}(\mathfrak{X}^c) \text{ with } \dim(\operatorname{Supp} X)<\operatorname{codim}\mathfrak{X}.$$

The vanishing dimension of β measures the extent to which vanishing of the Euler form fails to hold: the vanishing dimension of β is the infimum of integers v such that

$$\xi(X,Y)=0 \quad \text{for all complexes } X\in \mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}^c) \text{ with } \dim(\operatorname{Supp} X)<\operatorname{codim}\mathfrak{X}-v.$$

Because of the formulas in (3.2.1), it follows that the ring R satisfies vanishing (or weak vanishing, respectively) if and only all elements of $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ for all specialization-closed subsets $\mathfrak{X}$ of $\operatorname{Spec} R$ satisfy vanishing (or weak vanishing, respectively).

5.3. **Remark.** For a specialization closed subset $\mathfrak{X}$ of $\operatorname{Spec} R$ and elements $\alpha\in\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$, $\beta\in\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ and $\gamma\in\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$, we have the following formulas for dimension.

$$\dim\gamma=\dim\gamma^{\dagger},$$
$$\dim\alpha=\dim\alpha^{\dagger}=\dim\alpha^{*}=\dim(D\otimes\alpha) \text{ and}$$
$$\dim\beta=\dim\beta^{\dagger}=\dim\beta^{\star}=\dim\operatorname{Hom}(D,\beta).$$

These follow immediately from the fact that the dagger, star and Foxby functors do not change supports of complexes. Further, we have the following formulas for vanishing dimension.

$$\operatorname{vdim}\alpha=\operatorname{vdim}\alpha^{\dagger}=\operatorname{vdim}\alpha^{*}=\operatorname{vdim}(D\otimes\alpha) \text{ and}$$
$$\operatorname{vdim}\beta=\operatorname{vdim}\beta^{\dagger}=\operatorname{vdim}\beta^{\star}=\operatorname{vdim}\operatorname{Hom}(D,\beta).$$

These follow immediately from the above together with (3.2.1).

5.4. **Proposition.** *Let $\mathfrak{X}$ be a specialization-closed subset of* $\operatorname{Spec} R$*, let $\alpha\in\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ and let $\beta\in\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$. Then the following hold.*

(*i*) *If* $\operatorname{codim}\mathfrak{X}\leqslant 2$ *then vanishing holds for all elements in $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ and $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$. In particular, we always have*

$$\operatorname{vdim}\alpha,\operatorname{vdim}\beta\leqslant\max(0,\operatorname{codim}\mathfrak{X}-2).$$

(*ii*) *Let $\mathfrak{X}'$ be a specialization-closed subset of* $\operatorname{Spec} R$ *with $\mathfrak{X}\subseteq\mathfrak{X}'$. Then*

$$\operatorname{vdim}\overline{\alpha}\leqslant\operatorname{vdim}\alpha-(\operatorname{codim}\mathfrak{X}-\operatorname{codim}\mathfrak{X}') \textit{ and}$$

for $\overline{\alpha}\in\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X}')$. For any given s in the range $0\leqslant s\leqslant\operatorname{vdim}\alpha$, we can always find an $\mathfrak{X}'$ with $s=\operatorname{codim}\mathfrak{X}-\operatorname{codim}\mathfrak{X}'$ such that the above inequality becomes an equality. Likewise,

$$\operatorname{vdim}\overline{\beta}\leqslant\operatorname{vdim}\beta-(\operatorname{codim}\mathfrak{X}-\operatorname{codim}\mathfrak{X}')$$

for $\overline{\beta}\in\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X}')$, and for any given s in the range $0\leqslant s\leqslant\operatorname{vdim}\beta$, we can always find an $\mathfrak{X}'$ with $s=\operatorname{codim}\mathfrak{X}-\operatorname{codim}\mathfrak{X}'$ such that the above inequality becomes an equality.

(*iii*) *The element α satisfies weak vanishing if and only if, for all specialization-closed subsets $\mathfrak{X}'$ with $\mathfrak{X}\subseteq\mathfrak{X}'$ and* $\operatorname{codim}\mathfrak{X}'=\operatorname{codim}\mathfrak{X}-1$*,*

$$\overline{\alpha}=0 \quad \textit{as an element of} \quad \mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}').$$

Similarly, the element β satisfies weak vanishing if and only if, for all specialization-closed subsets $\mathfrak{X}'$ with $\mathfrak{X}\subseteq\mathfrak{X}'$ and $\operatorname{codim}\mathfrak{X}'=\operatorname{codim}\mathfrak{X}-1$*,*

$$\overline{\beta}=0 \quad \textit{as an element of} \quad \mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}').$$

Proof. Because of Proposition 4.9 and the formulas in Remark 5.3, it suffices to consider the statements for α and $\mathbb{GP}^{\mathrm{f}}(\mathfrak{X})$. But the in this case, (i) and (ii) are already contained in [11, Example 6 and Remark 7], and (iii) follows by considerations similar to those proving (ii) in [11, Remark 7]. □

The following two propositions present conditions that are equivalent to having a certain vanishing dimension for elements of the Grothendieck space $\mathbb{GI}^{\mathrm{f}}(\mathfrak{X})$. There are similar results for elements of the Grothendieck space $\mathbb{GP}^{\mathrm{f}}(\mathfrak{X})$; see [11, Proposition 23 and 24].

5.5. **Proposition.** *Let $\mathfrak{X}$ be a specialization-closed subset of* $\operatorname{Spec} R$*, and let $\beta \in \mathbb{GI}^{\mathrm{f}}(\mathfrak{X})$. Then the following conditions are equivalent.*

(i) $\operatorname{vdim}\beta \leqslant 0$.
(ii) $\operatorname{Hom}(\gamma,\beta) = 0$ *for all* $\gamma \in \mathbb{GD}^{\mathrm{f}}_{\square}(\mathfrak{X}^c)$ *with* $\dim\gamma < \operatorname{codim}\mathfrak{X}$.
(iii) $\overline{\beta} = 0$ *in* $\mathbb{GI}^{\mathrm{f}}(\mathfrak{X}')$ *for any specialization-closed subset* $\mathfrak{X}'$ *of* $\operatorname{Spec} R$ *with* $\mathfrak{X} \subseteq \mathfrak{X}'$ *and* $\operatorname{codim}\mathfrak{X}' < \operatorname{codim}\mathfrak{X}$.
(iv) $\overline{\beta} = 0$ *in* $\mathbb{GI}^{\mathrm{f}}(\mathfrak{X}')$ *for any specialization-closed subset* $\mathfrak{X}'$ *of* $\operatorname{Spec} R$ *with* $\mathfrak{X} \subseteq \mathfrak{X}'$ *and* $\operatorname{codim}\mathfrak{X}' = \operatorname{codim}\mathfrak{X} - 1$.

Proof. By definition (i) is equivalent to (ii), and Proposition 4.3 in conjunction with Remark 5.2 shows that (i) implies (iii). Clearly (iii) is stronger than (iv), and (iv) in conjunction with Proposition 5.4 implies (ii). □

5.6. **Proposition.** *Let $\mathfrak{X}$ be a specialization-closed subset of* $\operatorname{Spec} R$*, let $\beta \in \mathbb{GI}^{\mathrm{f}}(\mathfrak{X})$, and let u be a non-negative integer. Then the following conditions are equivalent.*

(i) $\operatorname{vdim}\beta \leqslant v$.
(ii) $\operatorname{Hom}(\gamma,\beta) = 0$ *for all* $\gamma \in \mathbb{GD}^{\mathrm{f}}_{\square}(\mathfrak{X}^c)$ *with* $\dim\gamma < \operatorname{codim}\mathfrak{X} - v$.
(iii) $\overline{\beta} = 0$ *in of* $\mathbb{GI}^{\mathrm{f}}(\mathfrak{X}')$ *for any specialization-closed subset* $\mathfrak{X}'$ *of* $\operatorname{Spec} R$ *with* $\mathfrak{X} \subseteq \mathfrak{X}'$ *and* $\operatorname{codim}\mathfrak{X}' < \operatorname{codim}\mathfrak{X} - u$.
(iv) $\overline{\beta} = 0$ *in* $\mathbb{GI}^{\mathrm{f}}(\mathfrak{X}')$ *for any specialization-closed subset* $\mathfrak{X}'$ *of* $\operatorname{Spec} R$ *with* $\mathfrak{X} \subseteq \mathfrak{X}'$ *and* $\operatorname{codim}\mathfrak{X}' = \operatorname{codim}\mathfrak{X} - v - 1$.

Proof. The structure of the proof is similar to that of Proposition (5.5). □

6. Grothendieck spaces in prime characteristic

According to [11, Theorem 19] the endomorphism $\Phi_{\mathfrak{X}}$ on $\mathbb{GP}^{\mathrm{f}}(\mathfrak{X})$ is diagonalizable; the precise statement is recalled in the next theorem. This section establishes that the endomorphism $\Psi_{\mathfrak{X}}$ on $\mathbb{GI}^{\mathrm{f}}(\mathfrak{X})$ is also diagonalizable; the precise statement is Theorem 6.2 below.

6.1. **Theorem.** *Assume that R is complete of prime characteristic p and with perfect residue field, and let $\mathfrak{X}$ be a specialization-closed subset of* $\operatorname{Spec} R$*. If α is an element in $\mathbb{GP}^{\mathrm{f}}(\mathfrak{X})$ and u is a non-negative integer with $u \geqslant \operatorname{vdim}\alpha$, then*

$$(p^u\Phi_{\mathfrak{X}} - \mathrm{id}) \circ \cdots \circ (p\Phi_{\mathfrak{X}} - \mathrm{id}) \circ (\Phi_{\mathfrak{X}} - \mathrm{id})(\alpha) = 0,$$

and there exists a unique decomposition

$$\alpha = \alpha^{(0)} + \cdots + \alpha^{(u)}$$

in which each $\alpha^{(i)}$ *is either zero or an eigenvector for* $\Phi_{\mathfrak{X}}$ *with eigenvalue* p^{-i}*. The elements* $\alpha^{(i)}$ *can be computed according to the formula*

$$\begin{pmatrix}\alpha^{(0)}\\ \vdots\\ \alpha^{(u)}\end{pmatrix} = \begin{pmatrix}1 & 1 & \cdots & 1\\ 1 & p^{-1} & \cdots & p^{-u}\\ \vdots & \vdots & \ddots & \vdots\\ 1 & p^{-u} & \cdots & p^{-u^2}\end{pmatrix}^{-1}\begin{pmatrix}\alpha\\ \Phi_{\mathfrak{X}}(\alpha)\\ \vdots\\ \Phi_{\mathfrak{X}}^{u}(\alpha)\end{pmatrix},$$

and may also be recursively obtained as

$$\alpha^{(0)} = \lim_{e\to\infty}\Phi_{\mathfrak{X}}^{e}(\alpha) \quad \textit{and} \quad \alpha^{(i)} = \lim_{e\to\infty} p^{ie}\Phi_{\mathfrak{X}}^{e}(\alpha - (\alpha^{(0)} + \cdots + \alpha^{(i-1)})).$$

6.2. **Theorem.** *Assume that* R *is complete of prime characteristic* p *and with perfect residue field, and let* $\mathfrak{X}$ *be a specialization-closed subset of* $\operatorname{Spec} R$*. If* β *is an element in* $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ *and* v *is a non-negative integer with* $v \geqslant \operatorname{vdim}\beta$*, then*

$$(p^v\Psi_{\mathfrak{X}} - \mathrm{id})\circ\cdots\circ(p\Psi_{\mathfrak{X}} - \mathrm{id})\circ(\Psi_{\mathfrak{X}} - \mathrm{id})(\beta) = 0,$$

and there exists a unique decomposition

$$\beta = \beta^{(0)} + \cdots + \beta^{(v)},$$

in which each $\beta^{(i)}$ *is either zero or an eigenvector for* $\Psi_{\mathfrak{X}}$ *with eigenvalue* p^{-i}*. The elements* $\beta^{(i)}$ *can be computed according to the formula*

$$(6.2.1)\qquad \begin{pmatrix}\beta^{(0)}\\ \vdots\\ \beta^{(u)}\end{pmatrix} = \begin{pmatrix}1 & 1 & \cdots & 1\\ 1 & p^{-1} & \cdots & p^{-v}\\ \vdots & \vdots & \ddots & \vdots\\ 1 & p^{-v} & \cdots & p^{-v^2}\end{pmatrix}^{-1}\begin{pmatrix}\beta\\ \Psi_{\mathfrak{X}}(\beta)\\ \vdots\\ \Psi_{\mathfrak{X}}^{v}(\beta)\end{pmatrix},$$

and may also be recursively obtained as

$$\beta^{(0)} = \lim_{e\to\infty}\Psi_{\mathfrak{X}}^{e}(\beta) \quad \textit{and} \quad \beta^{(i)} = \lim_{e\to\infty} p^{ie}\Psi_{\mathfrak{X}}^{e}(\beta - (\beta^{(0)} + \cdots + \beta^{(i-1)})).$$

Proof. On the injective Grothendieck space $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$, the identities described in Lemma 2.11 imply that we have the following commutative diagram.

$$\begin{tikzcd}
\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X}) \arrow[r, "\Phi_{\mathfrak{X}}^{n}", "\cong"'] \arrow[d, "(-)^{\dagger}"', "\cong"] & \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X}) \\
\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X}) \arrow[r, "\Psi_{\mathfrak{X}}^{n}"] & \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X}) \arrow[u, "\cong", "(-)^{\dagger}"']
\end{tikzcd}$$

In particular,

$$\Psi_{\mathfrak{X}}(-) = (-)^{\dagger}\circ\Phi_{\mathfrak{X}}\circ(-)^{\dagger}.$$

By Remark 5.3, we have $v \geqslant \operatorname{vdim}\beta = \operatorname{vdim}\beta^{\dagger}$, so Theorem 6.1 and the above identity yields that

$$(6.2.2)\qquad (p^v\Psi_{\mathfrak{X}} - \mathrm{id})\circ\cdots\circ(p\Psi_{\mathfrak{X}} - \mathrm{id})\circ(\Psi_{\mathfrak{X}} - \mathrm{id})(\beta) = 0.$$

Applying $\Psi_{\mathfrak{X}}^{e-v}$ to (6.2.2) results in a recursive formula to compute $\Psi_{\mathfrak{X}}^{e+1}(\beta)$ from $\Psi_{\mathfrak{X}}^{e}(\beta),\ldots,\Psi_{\mathfrak{X}}^{e-v}(\beta)$. The characteristic polynomial for the recursion is

$$(p^vx - 1)\cdots(px - 1)(x - 1),$$

which has $v+1$ distinct roots $1, p^{-1}, \dots, p^{-v}$. Consequently, there exist elements $\beta^{(0)}, \dots, \beta^{(v)}$ such that

$$\Psi^e_{\mathfrak{X}}(\beta) = \beta^{(0)} + p^{-e}\beta^{(1)} + \cdots + p^{-ve}\beta^{(v)},$$

where each $\beta^{(i)}$ is an eigenvector for $\Psi_{\mathfrak{X}}$ with eigenvalue p^{-i}. Setting $e = 0$ obtains the decomposition $\beta = \beta^{(0)} + \cdots + \beta^{(v)}$, and solving the system of linear equations obtained by setting $e = 0, \dots, v$ shows (6.2.1); observe that the matrix is the Vandermonde matrix on $1, p^{-1}, \dots, p^{-v}$, which is invertible. The formula also immediately shows that $\lim_{e\to\infty} \Psi^e(\beta) = \beta^{(0)}$ and that

$$\begin{aligned}
\lim_{e\to\infty} p^{ie}\Psi^e_{\mathfrak{X}}(\beta - (\beta^{(0)} + \cdots + \beta^{(i-1)})) &= \lim_{e\to\infty} p^{ie}\Psi^e_{\mathfrak{X}}(\beta^{(i)} + \cdots + \beta^{(v)}) \\
&= \lim_{e\to\infty} (\beta^{(i)} + \cdots + p^{-(v-i)e}\beta^{(v)}) \\
&= \beta^{(i)}.
\end{aligned}$$

This concludes the argument. ∎

6.3. **Proposition.** *Assume that R is a complete ring of prime characteristic p and with perfect residue field, and let $\mathfrak{X}$ be a specialization-closed subset of* $\operatorname{Spec} R$. *Consider the following diagram.*

$$\begin{tikzcd}[column sep=huge]
\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X}) \arrow[loop left, "\Phi_{\mathfrak{X}}"] \arrow[r, bend left, "D\otimes -"] \arrow[r, shift left, "(-)^{\dagger}"] & \mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X}) \arrow[loop right, "\Psi_{\mathfrak{X}}"] \arrow[l, shift left, "(-)^{\dagger}"] \arrow[l, bend left, "\operatorname{Hom}(D,-)"]
\end{tikzcd}$$

For the Grothendieck space $\mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X})$, we have the following identities.

$$\Psi_{\mathfrak{X}}(-) = \Phi_{\mathfrak{X}}(-^{\dagger})^{\dagger} = D \otimes \Phi_{\mathfrak{X}}(\operatorname{Hom}(D,-)).$$
$$(-)^{(i)} = (-)^{\dagger(i)\dagger} = D \otimes (\operatorname{Hom}(D,-)^{(i)}).$$

Proof. The formulas in the first line are an immediate consequence of Lemma 2.11. Let β be an element in $\mathbb{G}\mathsf{I}^{\mathsf{f}}(\mathfrak{X})$. Using the decomposition in $\mathbb{G}\mathsf{P}^{\mathsf{f}}(\mathfrak{X})$ from Theorem 6.1, we can write

$$\beta = \beta^{\dagger\dagger} = \beta^{\dagger(0)\dagger} + \cdots + \beta^{\dagger(v)\dagger},$$

and since

$$\Psi_{\mathfrak{X}}(\beta^{\dagger(i)\dagger}) = \Phi_{\mathfrak{X}}(\beta^{\dagger(i)})^{\dagger} = p^{-i}\beta^{\dagger(i)\dagger},$$

we learn from the uniqueness of the decomposition that $\beta^{(i)} = \beta^{\dagger(i)\dagger}$. This proves the first equality in the second line. The last equality follows by similar considerations. ∎

6.4. **Remark.** In [11, Remark 21] it is established that the Dutta multiplicity is computable. Employing Theorems 6.1 and 6.2 together with the fact from Proposition 4.7 that the induced Hom-homomorphism on Grothendieck spaces is continuous in both variables, it follows, as will be shown below, that the two analogs of Dutta multiplicity are also computable.

Let X and Y be finite complexes. Set $\mathfrak{X} = \operatorname{Supp} X$ and $\mathfrak{Y} = \operatorname{Supp} Y$, and assume that $\mathfrak{X} \cap \mathfrak{Y} = \{\mathfrak{m}\}$ and $\dim \mathfrak{X} + \dim \mathfrak{Y} \leqslant \dim R$. Then, in the case where Y is in

$\mathsf{I}^{\mathrm{f}}(R)$, the multiplicity $\xi_\infty(X,Y)$ can be identified via (4.4.1) with the element

$$\begin{aligned}\lim_{e\to\infty}\operatorname{Hom}([X]_{\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{Y}^c)},\Psi^e_{\mathfrak{Y}}(Y)) &= \operatorname{Hom}([X]_{\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{Y}^c)},\lim_{e\to\infty}\Psi^e_{\mathfrak{Y}}(Y))\\ &= \operatorname{Hom}([X]_{\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{Y}^c)},[Y]^{(0)}_{\mathsf{I}^{\mathrm{f}}(\mathfrak{Y})}),\end{aligned}$$

whereas, in the case where X is in $\mathsf{P}^{\mathrm{f}}(R)$, the multiplicity $\xi^\infty(X,Y)$ can be identified via (4.4.1) with the element

$$\begin{aligned}\lim_{e\to\infty}\operatorname{Hom}(\Phi^e_{\mathfrak{X}}(X),[Y]_{\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}^c)}) &= \operatorname{Hom}(\lim_{e\to\infty}\Phi^e_{\mathfrak{X}}(X),[Y]_{\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}^c)})\\ &= \operatorname{Hom}([X]^{(0)}_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})},[Y]_{\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}^c)}).\end{aligned}$$

The formulas in Theorems 6.1 and 6.2 now yield formulas for $\xi_\infty(X,Y)$ and $\xi^\infty(X,Y)$ as presented in the corollary below.

6.5. **Corollary.** *Assume that R is a complete ring of prime characteristic p and with perfect residue field. Let X and Y be finite complexes with*

$$\operatorname{Supp}X\cap\operatorname{Supp}Y=\{\mathfrak{m}\}\quad\text{and}\quad\dim(\operatorname{Supp}X)+\dim(\operatorname{Supp}Y)\leqslant\dim R.$$

When $Y\in\mathsf{I}^{\mathrm{f}}(R)$, letting v denote the vanishing dimension of $[Y]_{\mathsf{I}^{\mathrm{f}}(\operatorname{Supp}Y)}$ and setting $t=\operatorname{codim}(\operatorname{Supp}Y)$, we have

$$\xi_\infty(X,Y)=\begin{pmatrix}1&0&\cdots&0\end{pmatrix}\begin{pmatrix}1&1&\cdots&1\\ p^t&p^{t-1}&\cdots&p^{t-v}\\ \vdots&\vdots&\ddots&\vdots\\ p^{vt}&p^{v(t-1)}&\cdots&p^{v(t-v)}\end{pmatrix}^{-1}\begin{pmatrix}\xi(X,Y)\\ \xi(X,\mathbf{R}G(Y))\\ \vdots\\ \xi(X,\mathbf{R}G^v(Y))\end{pmatrix},$$

and when $X\in\mathsf{P}^{\mathrm{f}}(R)$, letting u denote the vanishing dimension of $[X]_{\mathsf{P}^{\mathrm{f}}(\operatorname{Supp}X)}$ and setting $s=\operatorname{codim}(\operatorname{Supp}X)$, we have

$$\xi^\infty(X,Y)=\begin{pmatrix}1&0&\cdots&0\end{pmatrix}\begin{pmatrix}1&1&\cdots&1\\ p^s&p^{s-1}&\cdots&p^{s-u}\\ \vdots&\vdots&\ddots&\vdots\\ p^{us}&p^{u(s-1)}&\cdots&p^{u(s-u)}\end{pmatrix}^{-1}\begin{pmatrix}\xi(X,Y)\\ \xi(\mathbf{L}F(X),Y)\\ \vdots\\ \xi(\mathbf{L}F^u(X),Y)\end{pmatrix}.$$

Thus, it is possible to calculate $\xi_\infty(X,Y)$ and $\xi^\infty(X,Y)$ as $\mathbb{Q}$–linear combinations of ordinary Euler forms; in particular, they are rational numbers.

Note that the above corollary also can be obtained directly from [11, Remark 21] by employing Lemma 2.11 and the formulas in (3.2.1).

6.6. **Remark.** Let $\mathfrak{X}$ and $\mathfrak{X}'$ be specialization-closed subsets of $\operatorname{Spec}R$ such that $\mathfrak{X}\subseteq\mathfrak{X}'$. Set $s=\operatorname{codim}\mathfrak{X}-\operatorname{codim}\mathfrak{X}'$ and consider the inclusion homomorphism

$$\overline{(-)}\colon\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})\to\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X}').$$

Pick an element $\beta\in\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$, and apply the convention that $\beta^{(t)}=0$ for all negative integers t. It follows immediately that

$$\Psi_{\mathfrak{X}'}(\overline{\beta})=p^s\overline{\Psi_{\mathfrak{X}}(\beta)},$$

and employing Theorem 6.2 we obtain the identity $\overline{\beta^{(i)}} = \overline{\beta}^{(i-s)}$. The situation may be visualized as follows

$$\begin{tikzcd}[column sep=tiny]
\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X}) \ni & \beta \arrow[d] & = & \beta^{(0)} & + & \cdots & + & \beta^{(s)} \arrow[dllll] & + & \beta^{(s+1)} \arrow[dllll] & + & \cdots & + & \beta^{(v)} \arrow[dllll] \\
\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X}') \ni & \overline{\beta} & = & \overline{\beta}^{(0)} & + & \overline{\beta}^{(1)} & + & \cdots & + & \overline{\beta}^{(v-s)}. & & & &
\end{tikzcd}$$

There are similar results for elements $\alpha \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$; see [11, Remark 20].

The following two propositions characterize vanishing dimension for elements of the Grothendieck space $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$. They should be read in parallel with Propositions 5.5 and 5.6. There are similar results for the Grothendieck space $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$; see [11, Proposition 23 and 24].

6.7. **Proposition.** *Assume that R is complete of prime characteristic p and with perfect residue field. Let $\mathfrak{X}$ be a specialization-closed subset of* $\operatorname{Spec} R$ *and let $\beta \in \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$. The following are equivalent.*

(*i*) *β satisfies vanishing.*
(*ii*) $\operatorname{vdim} \beta \leqslant 0$.
(*iii*) $\beta = \beta^{(0)}$.
(*iv*) $\beta = \Psi_{\mathfrak{X}}(\beta)$.
(*v*) $\beta = \Psi_{\mathfrak{X}}^{e}(\beta)$ *for some $e \in \mathbb{N}$.*
(*vi*) $\beta = \lim_{e\to\infty} \Psi_{\mathfrak{X}}^{e}(\beta)$.

Proof. By definition (*i*) and (*ii*) are equivalent, and from Theorem 6.2 it follows that (*ii*) implies (*iii*). Moreover, Theorem 6.2 shows that the four conditions (*iii*)–(*vi*) are equivalent. Finally, condition (*iii*) implies condition (*i*) through a reference to Remark 6.6 and Proposition 5.5. □

6.8. **Proposition.** *Assume that R is complete of prime characteristic p and with perfect residue field. Let $\mathfrak{X}$ be a specialization-closed subset of* $\operatorname{Spec} R$*, let $\beta \in \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ and let v be a non-negative integer. The following are equivalent.*

(*i*) $\operatorname{vdim} \beta \leqslant v$.
(*ii*) $\beta = \beta^{(0)} + \cdots + \beta^{(v)}$.
(*iii*) $(p^{v}\Psi_{\mathfrak{X}} - \mathrm{id}) \circ \cdots \circ (p\Psi_{\mathfrak{X}} - \mathrm{id}) \circ (\Psi_{\mathfrak{X}} - \mathrm{id})(\beta) = 0$.

Proof. From Theorem 6.2 it follows that (*i*) implies (*ii*) which is equivalent to (*iii*). Since $\beta^{(i)} \neq 0$ implies $\operatorname{vdim} \beta^{(i)} = i$ by Remark 6.6 and Proposition 5.6, it follows that (*ii*) implies (*i*). □

6.9. **Proposition.** *Assume that R is complete of prime characteristic p and with perfect residue field. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be specialization-closed subsets of* $\operatorname{Spec} R$ *such that $\mathfrak{X} \cap \mathfrak{Y} = \{\mathfrak{m}\}$ and $\dim \mathfrak{X} + \dim \mathfrak{Y} \leqslant \dim R$, and set $e = \dim R - (\dim \mathfrak{X} + \dim \mathfrak{Y})$. If (σ, τ) is a pair of elements from*

$$\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X}) \times \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{Y}), \quad \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X}) \times \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{Y}) \quad \textit{or} \quad \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X}) \times \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{Y}),$$

so that $\sigma \otimes \tau$ is a well-defined element of $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{m})$ or $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{m})$, then

$$(\sigma \otimes \tau)^{(i)} = \sum_{m+n=i+e} \sigma^{(m)} \otimes \tau^{(n)}. \tag{6.9.1}$$

If instead (σ, τ) *is a pair of elements from*

$$\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X}) \times \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{Y}), \quad \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X}) \times \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{Y}) \quad \textit{or} \quad \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X}) \times \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{Y}),$$

so that $\mathrm{Hom}(\sigma, \tau)$ *is a well-defined element of* $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{m})$ *or* $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{m})$*, then*

$$\mathrm{Hom}(\sigma, \tau)^{(i)} = \sum_{m+n=i+e} \mathrm{Hom}(\sigma^{(m)}, \tau^{(n)}).$$

Proof. We will verify that (6.9.1) holds in the case where (σ, τ) is pair of elements from $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X}) \times \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{Y})$. The verification of the remaining statements is similar.

It suffices to argue that the element

$$\alpha = \sum_{m+n=i+e} \sigma^{(m)} \otimes \tau^{(n)} \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{m})$$

is an eigenvector for $\Phi_{\mathfrak{m}} = \Phi_{\mathfrak{X}\cap\mathfrak{Y}}$ with eigenvalue p^{-i}. We compute

$$\begin{aligned}
\Phi_{\mathfrak{m}}(\alpha) &= \sum_{m+n=i+e} p^{-\dim R} F_{\mathfrak{m}}(\sigma^{(m)} \otimes \tau^{(n)}) \\
&= p^{-\dim R} \sum_{m+n=i+e} F_{\mathfrak{X}}(\sigma^{(m)}) \otimes F_{\mathfrak{Y}}(\tau^{(n)}) \\
&= p^{-\dim R} \sum_{m+n=i+e} p^{\operatorname{codim} \mathfrak{X}} \Phi_{\mathfrak{X}}(\sigma^{(m)}) \otimes p^{\operatorname{codim} \mathfrak{Y}} \Phi_{\mathfrak{Y}}(\tau^{(n)}) \\
&= p^{-i} \sum_{m+n=i+e} \sigma^{(m)} \otimes \tau^{(n)} = p^{-i}\alpha.
\end{aligned}$$

Here, all equalities but the second are propelled only by definitions. The second equality follows from Proposition 2.13. □

In [11], the concept of "numerical vanishing" is introduced for elements α of the Grothendieck space $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$. We here repeat the definition and extend it to elements β in the Grothendieck space $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$.

6.10. **Definition.** Assume that R is complete of prime characteristic p and with perfect residue field, and let $\mathfrak{X}$ be a specialization-closed subset of $\operatorname{Spec} R$. An element $\alpha \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ is said to *satisfy numerical vanishing* if the images in $\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{m})$ of α and $\alpha^{(0)}$ coincide. An element $\beta \in \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ is said to *satisfy numerical vanishing* if the images in $\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{m})$ of β and $\beta^{(0)}$ coincide. The ring R is said to *satisfy numerical vanishing* if all elements of the Grothendieck space $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ satisfy numerical vanishing for all specialization-closed subsets $\mathfrak{X}$ of $\operatorname{Spec} R$.

6.11. **Remark.** R satisfies numerical vanishing precisely when all elements of the Grothendieck space $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ satisfy numerical vanishing for all specialization-closed subsets $\mathfrak{X}$ of $\operatorname{Spec} R$. To see this, simply note that, by Proposition 6.3, the element β in $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ satisfies numerical vanishing if and only if the corresponding element $\beta^{\dagger}$ in $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ does. This observation allows us in the following proposition to present an injective version of [11, Remark 28].

6.12. **Proposition.** *Assume that R is complete of prime characteristic p and with perfect residue field. A necessary and sufficient condition for R to satisfy numerical vanishing is that all element of* $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{m})$ *satisfy numerical vanishing: that is, that*

$$\chi(\mathbf{R}G(Y)) = p^{\dim R}\chi(Y)$$

for all complexes $Y \in \mathsf{I}^{\mathrm{f}}(\mathfrak{m})$. *If* R *is Cohen–Macaulay,* $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{m})$ *is generated by modules, and hence a necessary and sufficient condition for* R *to satisfy numerical vanishing is that*

$$\operatorname{length} G(N) = p^{\dim R} \operatorname{length} N$$

for all modules N *with finite length and finite injective dimension.*

Proof. The proposition follows immediately from [11, Remark 28] by applying the dagger duality isomorphism between $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{m})$ and $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{m})$ and by noting that $(-)^{\dagger}$ takes a module in $\mathsf{P}^{\mathrm{f}}(\mathfrak{m})$ to a module in $\mathsf{I}^{\mathrm{f}}(\mathfrak{m})$ and vice versa. □

7. Self-duality

Let $\mathfrak{X}$ be a specialization-closed subset of $\operatorname{Spec} R$, and let K be a Koszul complex in $\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ on $\operatorname{codim} \mathfrak{X}$ elements. It is a well-know fact that Koszul complexes are "self-dual" in the sense that $K \simeq \Sigma^{\operatorname{codim} \mathfrak{X}} K^*$. In particular, for the element $\alpha = [K]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})}$ in $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$, we have

$$\alpha = [\Sigma^{\operatorname{codim} \mathfrak{X}} K^*]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})} = (-1)^{\operatorname{codim} \mathfrak{X}} [K^*]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})} = (-1)^{\operatorname{codim} \mathfrak{X}} \alpha^*.$$

Proposition 7.4 below shows that this feature is displayed for all elements that satisfy vanishing.

7.1. **Definition.** Let $\mathfrak{X}$ be a specialization-closed subset of $\operatorname{Spec} R$ and consider an element $\alpha \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ and an element $\beta \in \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$. If

$$\alpha = (-1)^{\operatorname{codim} \mathfrak{X}} \alpha^*,$$

we say that α is *self-dual*, and if all elements in $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ for all specialization-closed subsets $\mathfrak{X}$ of $\operatorname{Spec} R$ are self-dual, we say that R *satisfies self-duality*. Moreover, if the above equality holds after an application of the inclusion homomorphism $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X}) \to \mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$ so that

$$\overline{\alpha} = (-1)^{\operatorname{codim} \mathfrak{X}} \overline{\alpha^*}$$

in $\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$, we say that α is *numerically self-dual*, and if all elements in $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ for all specialization-closed subsets $\mathfrak{X}$ of $\operatorname{Spec} R$ are numerically self-dual, we say that R *satisfies numerical self-duality.*

Similarly, if

$$\beta = (-1)^{\operatorname{codim} \mathfrak{X}} \beta^{\star},$$

we say that β is *self-dual*, and if the above equality holds after an application of the inclusion homomorphism $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X}) \to \mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$ so that

$$\overline{\beta} = (-1)^{\operatorname{codim} \mathfrak{X}} \overline{\beta^{\star}}$$

in $\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$, we say that β is *numerically self-dual.*

7.2. **Remark.** The commutativity of the star and dagger functors shows that an element $\beta \in \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ is self-dual if and only if the corresponding element $\beta^{\dagger} \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ is self-dual. Thus, R satisfies self-duality if and only if all elements in $\mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ for all specialization-closed subsets $\mathfrak{X}$ of $\operatorname{Spec} R$ are self-dual. A similar remark holds for numerical self-duality.

7.3. **Proposition.** *Let* $\mathfrak{X}$ *be a specialization-closed subset of* $\operatorname{Spec} R$*, let* $\alpha \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ *and let* $\beta \in \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$*. Then,*

$$\operatorname{vdim} \alpha^* = \operatorname{vdim} \alpha \quad \textit{and} \quad \operatorname{vdim} \beta^{\star} = \operatorname{vdim} \beta.$$

If, in addition, R is complete of prime characteristic p and with perfect residue field, we have

$$\Phi_{\mathfrak{X}}(\alpha^*) = \Phi_{\mathfrak{X}}(\alpha)^* \quad \textit{and} \quad \Psi_{\mathfrak{X}}(\beta^\star) = \Psi_{\mathfrak{X}}(\beta)^\star.$$

In particular, for all integers i, we have

$$(\alpha^*)^{(i)} = (\alpha^{(i)})^* \quad \textit{and} \quad (\beta^\star)^{(i)} = (\beta^{(i)})^\star.$$

Proof. The formulas for vanishing dimension follow from (3.2.1), since the dagger functor does not change the dimension of a complex. The second pair of formulas follow immediately from the commutativity of the star and Frobenius functors; see 2.10. Thus, it follows that

$$\Phi_{\mathfrak{X}}(\alpha^{(i)*}) = \Phi_{\mathfrak{X}}(\alpha^{(i)})^* = p^{-i}\alpha^{(i)*}.$$

That is to say, $\alpha^{(i)*}$ is an eigenvector for $\Phi_{\mathfrak{X}}$ with eigenvalue p^{-i}. Setting $u = \operatorname{vdim}\alpha$, the decomposition

$$\alpha^* = (\alpha^{(0)} + \cdots + \alpha^{(u)})^* = \alpha^{(0)*} + \cdots + \alpha^{(u)*}$$

now shows that $\alpha^{*(i)} = \alpha^{(i)*}$. A similar argument applies for β. □

7.4. **Proposition.** *Let $\mathfrak{X}$ be a specialization-closed subset of* $\operatorname{Spec} R$. *If an element $\alpha \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ satisfies vanishing, then α is self-dual, and if an element $\beta \in \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$ satisfies vanishing, then β is self-dual. Moreover, R satisfies vanishing if and only if R satisfies self-duality, and if R satisfies numerical self-duality, then R satisfies weak vanishing.*

Proof. We shall prove that, if α satisfies vanishing, then α is self-dual. The corresponding statement for β follows from dagger duality, since β is self-dual exactly when $\beta^\dagger$ is and satisfies vanishing exactly when $\beta^\dagger$ does.

By Proposition 4.3, it suffices to assume that $\alpha = [X]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})}$ for a complex X from $\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$. We are required to establish the identity

$$\chi(X^* \otimes^{\mathbf{L}}_R -) = (-1)^{\operatorname{codim}\mathfrak{X}}\chi(X \otimes^{\mathbf{L}}_R -) \tag{7.4.1}$$

viewed as metafunctions on $\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}^c)$. First, we translate this question into showing that, if R is a domain and $\mathfrak{X}$ equals $\mathfrak{m}$, then

$$\chi(X^*) = (-1)^{\dim R}\chi(X)$$

for all complexes X in $\mathsf{P}^{\mathrm{f}}(\mathfrak{m})$ such that $[X]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{m})}$ satisfies vanishing.

1° By assumption, α satisfies vanishing, and Proposition 7.3 implies that so does α^*. From Proposition 4.3 we see that, in order to show (7.4.1), it suffices to test with modules of the form $R/\mathfrak{p}$ for prime ideals $\mathfrak{p}$ from $\mathfrak{X}^c$ with $\dim R/\mathfrak{p} = \operatorname{codim}\mathfrak{X}$. Consider the following computation.

$$\begin{aligned} X^* \otimes^{\mathbf{L}}_R R/\mathfrak{p} &= \mathbf{R}\mathrm{Hom}_R(X,R) \otimes^{\mathbf{L}}_R R/\mathfrak{p} \\ &\simeq \mathbf{R}\mathrm{Hom}_R(X, R/\mathfrak{p}) \\ &\simeq \mathbf{R}\mathrm{Hom}_R(X, \mathbf{R}\mathrm{Hom}_{R/\mathfrak{p}}(R/\mathfrak{p}, R/\mathfrak{p})) \\ &\simeq \mathbf{R}\mathrm{Hom}_{R/\mathfrak{p}}(X \otimes^{\mathbf{L}}_R R/\mathfrak{p}, R/\mathfrak{p}). \end{aligned}$$

Here, the first isomorphism follows from (Tensor-eval); the second is trivial; and the third is due to (Adjoint). To keep notation simple, let

$$(-)^{*_{R/\mathfrak{p}}} = \mathbf{R}\mathrm{Hom}_{R/\mathfrak{p}}(-, R/\mathfrak{p}).$$

We are required to demonstrate that

$$\chi(X^* \otimes^{\mathbf{L}}_R R/\mathfrak{p}) = (-1)^{\dim R/\mathfrak{p}} \chi(X \otimes^{\mathbf{L}}_R R/\mathfrak{p}),$$

and since the Euler characteristics χ^R and $\chi^{R/\mathfrak{p}}$ are identical on all finite $R/\mathfrak{p}$–complexes with finite length homology, the computations above imply that we need to demonstrate that

$$\chi^{R/\mathfrak{p}}((X \otimes^{\mathbf{L}}_R R/\mathfrak{p})^{*_{R/\mathfrak{p}}}) = (-1)^{\dim R/\mathfrak{p}} \chi^{R/\mathfrak{p}}(X \otimes^{\mathbf{L}}_R R/\mathfrak{p}).$$

Having changed rings from R to the domain $R/\mathfrak{p}$, we need to verify that the element $[X \otimes^{\mathbf{L}}_R R/\mathfrak{p}]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{m}/\mathfrak{p})}$ in the Grothendieck space $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{m}/\mathfrak{p})$ over $R/\mathfrak{p}$ satisfies vanishing. But this follows from the fact that $\alpha = [X]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})}$ satisfies vanishing, since

$$\chi^{R/\mathfrak{p}}((X \otimes^{\mathbf{L}}_R R/\mathfrak{p}) \otimes^{\mathbf{L}}_{R/\mathfrak{p}} R/\mathfrak{a}) = \chi^R(X \otimes^{\mathbf{L}}_R R/\mathfrak{a}) = 0.$$

for all ideals $\mathfrak{a} \in V(\mathfrak{p})$ with $\dim R/\mathfrak{a} < \dim R/\mathfrak{p} = \operatorname{codim} \mathfrak{X}$. Thus, it suffices to show that

$$\chi(X^*) = (-1)^{\dim R} \chi(X)$$

when R is a domain, $\mathfrak{X}$ equals $\{\mathfrak{m}\}$ and $[X]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{m})}$ satisfies vanishing.

2° Without loss of generality, we may assume that R is complete; in particular, we may assume that R admits a normalized dualizing complex D. Letting $Y = R$ in (3.2.1) and applying Proposition 4.3, it follows that

$$\chi(X^*) = \chi(X^*, R) = \chi(X, D) = \chi(X, \mathrm{H}(D)).$$

According to 2.6 we may assume that the modules in the dualizing complex D have the form

$$D_i = \bigoplus_{\dim R/\mathfrak{p} = i} E_R(R/\mathfrak{p}). \tag{7.4.2}$$

Let $d = \dim R$ and observe that, since $[X]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{m})}$ satisfies vanishing and

$$\dim \mathrm{H}_i(D) \leqslant \dim D_i < d \quad \text{for all} \quad i < d,$$

it follows that

$$\chi^R(X^*) = \chi^R(X, \Sigma^d \mathrm{H}_d(D)) = (-1)^d \chi^R(X, \mathrm{H}_d(D)).$$

Since $\mathrm{H}_d(D)$ is a submodule of D_d, there is a short exact sequence

$$0 \to \mathrm{H}_d(D) \to D_d \to Q \to 0, \tag{7.4.3}$$

where Q is a submodule of D_{d-1}, so that $\dim Q \leqslant \dim D_{d-1} \leqslant d - 1$, where the last inequality follows from (7.4.2). Since R is assumed to be a domain,

$$D_d = E(R) = R_{(0)},$$

so localizing the short exact sequence (7.4.3) at the prime ideal (0), we obtain an isomorphism

$$\mathrm{H}_d(D)_{(0)} \xrightarrow{\cong} R_{(0)}.$$

This lifts to an R–homomorphism, producing an exact sequence of finitely generated R–modules

$$0 \to K \to \mathrm{H}_d(D) \to R \to C \to 0,$$

where K and C are not supported at the prime ideal (0). Thus, $\dim K$ and $\dim C$ are strictly smaller than $\dim R$. Consequently, since $[X]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})}$ satisfies vanishing and the intersection multiplicity is additive on short exact sequences,

$$\begin{aligned}\chi(X^*) &= (-1)^d\chi(X, \mathrm{H}_d(D)) \\ &= (-1)^d(\chi(X,K) + \chi(X,R) - \chi(X,C)) \\ &= (-1)^d\chi(X),\end{aligned}$$

which concludes the argument.

3° We have now shown that, if R satisfies vanishing, then R satisfies self-duality. To see the other implication, assume that R satisfies self-duality and let α be an element of $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ for some specialization-closed subset $\mathfrak{X}$ of $\operatorname{Spec} R$. For any specialization-closed subset $\mathfrak{X}'$ of $\operatorname{Spec} R$ with $\mathfrak{X} \subseteq \mathfrak{X}'$ and $\operatorname{codim} \mathfrak{X}' = \operatorname{codim} \mathfrak{X} - 1$, we now have, for the image $\overline{\alpha}$ of α in $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X}')$, that

$$(-1)^{\operatorname{codim} \mathfrak{X}'}\overline{\alpha}^* = \overline{\alpha} = \overline{(-1)^{\operatorname{codim} \mathfrak{X}}\alpha^*} = (-1)^{\operatorname{codim} \mathfrak{X}}\overline{\alpha}^*,$$

which means that $\overline{\alpha} = 0$. Thus, by Proposition 5.5, α satisfies vanishing, and since α was arbitrary, R must satisfy vanishing. Considering instead the image $\overline{\alpha}$ of α in $\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}')$ and applying Proposition 5.4, the same argument shows that, if α is numerically self-dual, then α satisfies weak vanishing. Thus, if R satisfies numerical self-duality, then R satisfies weak vanishing. $\square$

7.5. **Theorem.** *Assume that R is complete of prime characteristic p and with perfect residue field. Let $\mathfrak{X}$ be a specialization-closed subset of* $\operatorname{Spec} R$*, let $\alpha \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ and let $\beta \in \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$. Then, for all non-negative integers i*

(7.5.1) $$(\alpha^*)^{(i)} = (-1)^{i+\operatorname{codim} \mathfrak{X}}\alpha^{(i)} \quad \textit{and} \quad (\beta^\star)^{(i)} = (-1)^{i+\operatorname{codim} \mathfrak{X}}\beta^{(i)}.$$

Consequently, if u is the vanishing dimension of α, then

$$(-1)^{\operatorname{codim} \mathfrak{X}}\alpha^* = \alpha^{(0)} - \alpha^{(1)} + \alpha^{(2)} - \cdots + (-1)^u\alpha^{(u)},$$

and if v is the vanishing dimension of β, then

$$(-1)^{\operatorname{codim} \mathfrak{X}}\beta^\star = \beta^{(0)} - \beta^{(1)} + \beta^{(2)} - \cdots + (-1)^v\beta^{(v)}.$$

Proof. The last two statements of the proposition are immediate consequences of (7.5.1). We shall prove the formula for α in (7.5.1); the proof of the formula for β is similar.

The proof is by induction on i. For $i = 0$, since $\alpha^{(0)}$ satisfies vanishing, the statement follows from Propositions 7.3 and 7.4, since

$$(\alpha^*)^{(0)} = (\alpha^{(0)})^* = (-1)^{\operatorname{codim} \mathfrak{X}}\alpha^{(0)}.$$

Next, assume that $i > 0$ and that the statement holds for smaller values of i. Choose an arbitrary specialization-closed subset $\mathfrak{X}'$ of $\operatorname{Spec} R$ such that $\mathfrak{X} \subseteq \mathfrak{X}'$ and $\operatorname{codim} \mathfrak{X}' = \operatorname{codim} \mathfrak{X} - 1$, and consider the element

$$\sigma = (\alpha^*)^{(i)} - (-1)^{\operatorname{codim} \mathfrak{X}+i}\alpha^{(i)}.$$

We want to show that $\sigma = 0$. Applying the automorphism $\Phi_{\mathfrak{X}}$, we get by Proposition 7.3 that

$$\begin{aligned}\Phi_{\mathfrak{X}}(\sigma) &= \Phi_{\mathfrak{X}}((\alpha^*)^{(i)}) - (-1)^{\operatorname{codim} \mathfrak{X}+i}\Phi_{\mathfrak{X}}(\alpha^{(i)}) \\ &= p^{-i}((\alpha^*)^{(i)} - (-1)^{\operatorname{codim} \mathfrak{X}+i}\alpha^{(i)}) = p^{-i}\sigma,\end{aligned}$$

showing that σ is an eigenvector for $\Phi_{\mathfrak{X}}$ with eigenvalue p^{-i}; in particular, we have $\sigma = \sigma^{(i)}$. Denote by $\bar{\sigma}$ the image of σ in $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X}')$. Then, by [11, Remark 20] (which corresponds to Remark 6.6 but for elements of $\mathbb{G}(\mathfrak{X})$) and the induction hypothesis we obtain

$$\begin{aligned}\overline{\sigma} &= \overline{(\alpha^*)^{(i)}} - (-1)^{\operatorname{codim}\mathfrak{X}+i}\overline{\alpha^{(i)}} \\ &= \overline{\alpha}^{*(i-1)} - (-1)^{\operatorname{codim}\mathfrak{X}'+(i-1)}\overline{\alpha}^{(i-1)} = 0.\end{aligned}$$

Consequently, by [11, Proposition 23] (which corresponds to Proposition 5.5 but for elements of $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$), σ must satisfy vanishing: that is, $\sigma = \sigma^{(0)}$. But then $\sigma^{(i)} = \sigma = \sigma^{(0)}$ forcing $\sigma = 0$. □

7.6. **Remark.** Assume that R is complete of prime characteristic p and with perfect residue field. Let $\mathfrak{X}$ be a specialization-closed subset of $\operatorname{Spec} R$ and consider an element $\alpha \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$. In view of Theorem 6.1 we may decompose α into eigenvectors

$$\alpha = \alpha^{(0)} + \alpha^{(1)} + \alpha^{(2)} + \cdots + \alpha^{(u)}$$

where u is the vanishing dimension of α. Comparing it with the decomposition of α^* from Theorem 7.5

$$(-1)^{\operatorname{codim}\mathfrak{X}}\alpha^* = \alpha^{(0)} - \alpha^{(1)} + \alpha^{(2)} - \cdots + (-1)^u\alpha^{(u)}$$

shows that α is self-dual if and only if $\alpha^{(i)} = 0$ in $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ for all odd i: that is, if and only if

$$\alpha = \alpha^{(0)} + \alpha^{(2)} + \cdots$$

in $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$. Similarly, α is numerically self-dual if and only if $\overline{\alpha^{(i)}} = 0$ in $\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$ for all odd i: that is, if and only if

$$\overline{\alpha} = \overline{\alpha^{(0)}} + \overline{\alpha^{(2)}} + \cdots$$

in $\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X})$. Similar considerations apply for elements $\beta \in \mathbb{G}\mathsf{I}^{\mathrm{f}}(\mathfrak{X})$.

In Proposition 7.4, we proved that vanishing and self-duality are equivalent for R and that numerical self-duality implies weak vanishing. The following proposition shows that, in characteristic p, numerical vanishing logically lies between self-duality and numerical self-duality.

7.7. **Proposition.** *Assume that R is complete of prime characteristic p and with perfect residue field. For the following conditions, each condition implies the next. In fact,* (*i*) *and* (*ii*) *are equivalent.*

(*i*) *R satisfies vanishing.*
(*ii*) *R satisfies self-duality.*
(*iii*) *R satisfies numerical vanishing.*
(*iv*) *R satisfies numerical self-duality.*
(*v*) *R satisfies weak vanishing.*

Proof. The equivalence of (*i*) and (*ii*) and the fact that (*iv*) implies (*v*) is contained in Proposition 7.4. The fact that (*i*) implies (*iii*) is contained in [11, Proposition 27], and Remark 7.6 makes it clear that (*iii*) implies (*iv*). □

7.8. **Remark.** The constructions by Miller and Singh [15] shows that there can exist elements satisfying self-duality but not vanishing as well as elements satisfying numerical self-duality but not numerical vanishing; see [11, Example 35] for further details on this example. Roberts [20] has shown the existence of a ring satisfying weak vanishing but not numerical self-duality; see [11, Example 32] for further details. Thus, all the implications except the equivalence in the preceding proposition are strict.

7.9. **Proposition.** *R satisfies vanishing precisely when*

$$\alpha \otimes \gamma = (-1)^{\operatorname{codim} \mathfrak{X}} \operatorname{Hom}(\alpha, \gamma) \tag{7.9.1}$$

in $\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{m})$ for all specialization-closed subsets $\mathfrak{X}$ of $\operatorname{Spec} R$, all $\alpha \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ and all $\gamma \in \mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}^c)$, and R satisfies numerical self-duality precisely when (7.9.1) holds in $\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{m})$ when requiring $\gamma \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X}^c)$ instead. In other words, R satisfies vanishing precisely when the intersection multiplicity and the Euler form satisfy the identity

$$\chi(X, Y) = (-1)^{\operatorname{codim}(\operatorname{Supp} X)} \xi(X, Y) \tag{7.9.2}$$

for all complexes $X \in \mathsf{P}^{\mathrm{f}}(R)$ and $Y \in \mathsf{D}^{\mathrm{f}}_{\square}(R)$ with

$$\operatorname{Supp} X \cap \operatorname{Supp} Y = \{\mathfrak{m}\} \quad \text{and} \quad \dim(\operatorname{Supp} X) + \dim(\operatorname{Supp} Y) \leqslant \dim R,$$

and R satisfies numerical self-duality precisely when (7.9.2) holds when restricting to complexes $Y \in \mathsf{P}^{\mathrm{f}}(R)$.

Proof. Employing Proposition 4.9 it is readily verified that (7.9.1) is equivalent to

$$\alpha \otimes \gamma = (-1)^{\operatorname{codim} \mathfrak{X}} \alpha^* \otimes \gamma.$$

However, this identity is satisfied for all $\gamma \in \mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}^c)$ precisely when α is self-dual. From Proposition 7.4 it follows that R satisfies vanishing if and only if (7.9.1) holds for all specialization-closed subsets $\mathfrak{X}$ of $\operatorname{Spec} R$, all $\alpha \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ and all $\gamma \in \mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}^c)$.

On the other hand, applying the above argument to the case where $\gamma \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X}^c)$ shows that R satisfies numerical self-duality presily when (7.9.1) is satisfied for all specialization-closed subsets $\mathfrak{X}$ of $\operatorname{Spec} R$, all $\alpha \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ and all $\gamma \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X}^c)$.

Assume next that (7.9.1) holds in $\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{m})$ for all specialization-closed subsets $\mathfrak{X}$ of $\operatorname{Spec} R$, all $\alpha \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ and all $\gamma \in \mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}^c)$. If $X \in \mathsf{P}^{\mathrm{f}}(R)$ and $Y \in \mathsf{D}^{\mathrm{f}}_{\square}(R)$ are complexes such that

$$\operatorname{Supp} X \cap \operatorname{Supp} Y = \{\mathfrak{m}\} \quad \text{and} \quad \dim(\operatorname{Supp} X) + \dim(\operatorname{Supp} Y) \leqslant \dim R, \tag{7.9.3}$$

the identity (7.9.2) follows by setting

$$\mathfrak{X} = \operatorname{Supp} X, \quad \alpha = [X]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})} \quad \text{and} \quad \gamma = [Y]_{\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}^c)}$$

in (7.9.1). Conversely, if (7.9.2) holds for all complexes $X \in \mathsf{P}^{\mathrm{f}}(R)$ and $Y \in \mathsf{D}^{\mathrm{f}}_{\square}(R)$ such that (7.9.3) is satisfied, then (7.9.1) follows for all specialization-closed subsets $\mathfrak{X}$ of $\operatorname{Spec} R$, all $\alpha \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ and all $\gamma \in \mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{X}^c)$, since we by Proposition 4.3, $\alpha = r[X]_{\mathsf{P}^{\mathrm{f}}(\mathfrak{X})}$ for an $r \in \mathbb{Q}$ and a complex $X \in \mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ with $\operatorname{codim}(\operatorname{Supp} X) = \operatorname{codim} \mathfrak{X}$. Applying the same argument to elements $\gamma \in \mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X}^c)$ and complexes $Y \in \mathsf{P}^{\mathrm{f}}(R)$ proves the last part of the proposition. $\square$

7.10. **Remark.** Proposition 7.9 confirms Chan's supposition in [4], in the setting of complexes rather than modules, that the formula in (7.9.2) is equivalent to the vanishing conjecture. Note that, when restricting attention to complexes Y in $\mathsf{P}^{\mathrm{f}}(R)$, formula (7.9.2) is equivalent to numerical self-duality, which implies the weak vanishing conjecture but need not be equivalent to it. This negatively answers the question of whether the restriction of the formula in (7.9.2) to complexes Y in $\mathsf{P}^{\mathrm{f}}(R)$ is equivalent to the weak vanishing conjecture.

We already know that, if R is regular, then R satisfies vanishing, whereas, if R is a complete intersection (which is complete of prime characteristic p and with perfect residue field), then R satisfies numerical vanishing; see [11, Example 33]. The authors believe that this line of implications can be continued, at least in the characteristic p case, with the claim that, if R is Gorenstein, R satisfies numerical self-duality, so that we have the following implications of properties of R in the case where R is complete of prime characteristic p and with perfect residue field.

$$\begin{tikzcd}
\text{regular} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{vanishing} \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] & \text{self-duality} \\
\text{complete intersection} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{numerical vanishing} \arrow[d, Rightarrow] & \\
\text{Gorenstein} \arrow[r, Rightarrow, dashed] & \text{numerical self-duality} &
\end{tikzcd}$$

This supposition complies with the following proposition.

7.11. **Proposition.** *Assume that R is Gorenstein and let $\mathfrak{X}$ be a specialization-closed subset of* $\operatorname{Spec} R$. *If* $\dim \mathfrak{X} \leqslant 2$, *then all elements of* $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ *are numerically self-dual. In particular, if* $\dim R \leqslant 5$, *then R satisfies numerical self-duality.*

Proof. Let $\mathfrak{X}$ be a specialization-closed subset of $\operatorname{Spec} R$ with $\dim \mathfrak{X} \leqslant 2$ and consider elements α in $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ and β in $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X}^c)$. Then $\operatorname{codim} \mathfrak{X}^c \leqslant 2$, and therefore β satisfies vanishing by Proposition 5.4; in particular,

$$\beta^* = (-1)^{\operatorname{codim} \mathfrak{X}^c} \beta = (-1)^{\dim R - \operatorname{codim} \mathfrak{X}} \beta.$$

When R is Gorenstein, the complex $D = \Sigma^{\dim R} R$ is a normalized dualizing complex for R forcing $(-)^\dagger = \Sigma^{\dim R}(-)^*$. Thus, applying Proposition 4.9 the identity

$$\alpha^* \otimes \beta = \alpha \otimes \beta^\dagger = (-1)^{\dim R} \alpha \otimes \beta^* = (-1)^{\operatorname{codim} \mathfrak{X}} \alpha \otimes \beta$$

holds in $\mathbb{G}\mathsf{D}^{\mathrm{f}}_{\square}(\mathfrak{m})$. This proves that $\alpha^* = (-1)^{\operatorname{codim} \mathfrak{X}} \alpha$ so that α is numerically self-dual.

If $\dim R \leqslant 5$ then any specialization-closed subset $\mathfrak{X}$ of $\operatorname{Spec} R$ must either satisfy $\operatorname{codim} \mathfrak{X} \leqslant 2$, in which case vanishing holds in $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ by Proposition 5.4, or $\dim \mathfrak{X} \leqslant 2$. In either case, all elements of $\mathbb{G}\mathsf{P}^{\mathrm{f}}(\mathfrak{X})$ are numerically self-dual. $\square$

Since numerical self-duality implies weak vanishing, the preceding proposition shows that weak vanishing holds for any Gorenstein ring of dimension at most 5. Dutta [7] has already proven this fact.

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Article III

Grothendieck groups for categories of complexes

GROTHENDIECK GROUPS FOR CATEGORIES OF COMPLEXES

HANS-BJØRN FOXBY AND ESBEN BISTRUP HALVORSEN

Abstract. The new intersection theorem states that, over a Noetherian local ring R, for any non-exact complex concentrated in degrees $n, \dots, 0$ in the category $\mathsf{P}(\text{length})$ of bounded complexes of finitely generated projective modules with finite-length homology, we must have $n \geq d = \dim R$.

One of the results in this paper is that the Grothendieck group of $\mathsf{P}(\text{length})$ in fact is generated by complexes concentrated in the minimal number of degrees: if $\mathsf{P}_d(\text{length})$ denotes the full subcategory of $\mathsf{P}(\text{length})$ consisting of complexes concentrated in degrees $d, \dots, 0$, the inclusion $\mathsf{P}_d(\text{length}) \to \mathsf{P}(\text{length})$ induces an isomorphism of Grothendieck groups. When R is Cohen–Macaulay, the Grothendieck groups of $\mathsf{P}_d(\text{length})$ and $\mathsf{P}(\text{length})$ are naturally isomorphic to the Grothendieck group of the category $\mathsf{M}(\text{length})$ of finitely generated modules of finite length and finite projective dimension. This and a family of similar results are established in this paper.

1. Introduction

In this paper, we will prove the existence of isomorphisms between Grothendieck groups of various related categories of complexes. The paper presents a family of results that can all be formulated in a similar way. This introduction discusses only one of the results (as did the abstract); the remaining results can be obtained by replacing the property of "having finite length" with other properties of modules—see the next section for further details.

Let R be a commutative, Noetherian, local ring of dimension d. Let $\mathsf{P}(\text{length})$ denote the category of bounded complexes of finitely generated projective R-modules and with finite-length homology, and let $\mathsf{P}_d(\text{length})$ denote the full subcategory of complexes concentrated in degrees $d, \dots, 0$. We shall denote the Grothendieck groups of these two categories by $K_0\mathsf{P}(\text{length})$ and $K_0\mathsf{P}_d(\text{length})$, respectively. The inclusion of categories $\mathsf{P}_d(\text{length}) \to \mathsf{P}(\text{length})$ naturally induces a homomorphism

$$\mathcal{I}_d \colon K_0\mathsf{P}_d(\text{length}) \to K_0\mathsf{P}(\text{length}),$$

given by $\mathcal{I}_d([X]) = [X]$ for a complex $X \in \mathsf{P}_d(\text{length})$; here, the two $[X]$'s are different, since one is an element of $K_0\mathsf{P}_d(\text{length})$ and the other is an element of $K_0\mathsf{P}(\text{length})$. One of the results of this paper (Corollary 6) is that the above is an isomorphism. This is particularly interesting when comparing with the *new intersection theorem* (cf. [6, Theorem 13.4.1]), which states that, if a complex in $\mathsf{P}(\text{length})$ is non-exact and concentrated in degrees $n, \dots, 0$, then $n \geq d$. Thus, the Grothendieck group $K_0\mathsf{P}(\text{length})$ is generated by complexes concentrated in the smallest possible number of degrees.

Next let $\mathsf{M}(\text{length})$ denote the category of R-modules of finite length and finite projective dimension. We denote the Grothendieck group of $\mathsf{M}(\text{length})$ by $K_0\mathsf{M}(\text{length})$. Any module in $\mathsf{M}(\text{length})$ has a projective resolution in $\mathsf{P}(\text{length})$, and there is a natural homomorphism

$$\mathcal{R} \colon K_0\mathsf{M}(\text{length}) \to K_0\mathsf{P}(\text{length}),$$

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given by $\mathcal{R}([M]) = [X]$ for a module $M \in \mathsf{M}(\text{length})$ with projective resolution $X \in \mathsf{P}(\text{length})$.

Now, suppose further that R is Cohen-Macaulay. The acyclicity lemma by Peskine and Szpiro (see [6, Theorem 4.3.2]) implies that the complexes in $\mathsf{P}_d(\text{length})$ are acyclic: that is, they are projective resolutions of their zeroth homology module. Taking the homology of a complex induces a natural homomorphism

$$\mathcal{H}_d \colon K_0\mathsf{P}_d(\text{length}) \to K_0\mathsf{M}(\text{length}),$$

given by $\mathcal{H}_d([X]) = [\mathrm{H}_0(X)]$ for a complex $X \in \mathsf{P}_d(\text{length})$.

The three homomorphisms that we have introduced so far fit together in a commutative diagram:

$$\begin{tikzcd}
K_0\mathsf{P}_e(\text{length}) \arrow[rr, "\mathcal{I}_d"] \arrow[dr, dotted, "\mathcal{H}_d"'] & & K_0\mathsf{P}(\text{length}) \\
& K_0\mathsf{M}(\text{length}) \arrow[ur, "\mathcal{R}"'] &
\end{tikzcd}$$

Here, $\mathcal{H}_d$ is dotted to emphasize the fact that it required an extra assumption to be defined. The fact that $\mathcal{I}_d$ is an isomorphism yields that so are $\mathcal{R}$ and $\mathcal{H}_d$, whenever defined (Corollary 11).

When replacing the property of "having finite length" with other module properties, the same picture will emerge. The next section presents all the results of this paper in a general way—including the results mentioned in this introduction.

Historical note: This paper builds on the first author's incomplete preprint [2] whose results have been generalized and completely proven by the second author. The paper will become part of the second author's Ph.D. thesis. The results are generalizations of a result by Roberts and Srinivas [7, Proposition 2]

2. Grothendieck groups for categories of complexes

Notation. Throughout this paper, R denotes a non-trivial, unitary, commutative ring. All modules are, unless otherwise stated, assumed to be R-modules, and all complexes are, unless otherwise stated, assumed to be complexes of R-modules. Modules are considered to be complexes concentrated in degree zero.

Let d be a non-negative integer and let $S = (S_1, \dots, S_d)$ be a family of multiplicative systems of R. A module M is said to be S_i-*torsion* if $S_i^{-1}M = 0$, and M is said to be S-*torsion* if it is S_i-torsion for $i = 1, \dots, d$. The *grade* of M is the number

$$\operatorname{grade}_R M = \inf\{n \in \mathbb{N}_0 \mid \operatorname{Ext}_R^n(M, R) \neq 0\}.$$

If $M = 0$, we set $\operatorname{grade}_R M = \infty$. When R is Noetherian and M is non-trivial and finitely generated, $\operatorname{grade}_R M$ is the maximal length of a regular sequence in $\operatorname{Ann}_R M$. M is said to be d-*perfect* if $M = 0$ or $\operatorname{grade}_R M = d = \operatorname{pd}_R M$.

We shall use the following abbreviations for properties of modules.

S-tor:	being S-torsion;
length:	having finite length;
$\mathrm{gr} \geq d$:	having grade larger than or equal to d; and
d-perf:	being d-perfect.

Let e be a non-negative integer, and let the symbol # denote any of the module properties above. We define the following categories.

- M: the category of finitely generated modules of finite projective dimension;
- P: the category of bounded complexes of finitely generated projective modules;
- P_e: the full subcategory of P consisting of complexes concentrated in degrees $e, \ldots, 0$;
- $\mathsf{M}(\#)$: the full subcategory of M consisting of modules satisfying $\#$;
- $\mathsf{P}(\#)$: the full subcategory of P consisting of complexes whose homology modules satisfy $\#$;
- $\mathsf{P}_e(\#)$: the intersection of P_e and $\mathsf{P}(\#)$.

So, for example, $\mathsf{P}_e(S\text{-tor})$ denotes the category of complexes concentrated in degrees $e, \ldots, 0$ with finitely generated projective modules and S-torsion homology modules. We will allow the symbol $\#$ to be "empty" so that $\mathsf{M}(\#)$, $\mathsf{P}(\#)$ and $\mathsf{P}_e(\#)$ also can denote M, P and P_e, respectively. Similarly, we shall occasionally write $\mathsf{P}_\star(\#)$, where the symbol $\star$ either denotes a non-negative integer e or is "empty", in which case we are back with the category $\mathsf{P}(\#)$.

The isomorphism classes of any of the categories $\mathsf{M}(\#)$ and $\mathsf{P}_\star(\#)$ form a set. We shall occasionally abuse notation and use $\mathsf{M}(\#)$ and $\mathsf{P}_\star(\#)$ to denote the sets of isomorphism classes of the corresponding categories.

Definition 1. The *Grothendieck group* of a category $\mathsf{M}(\#)$ is the Abelian group $K_0\mathsf{M}(\#)$ presented by generators $[M]$, one for each isomorphism class in $\mathsf{M}(\#)$, and relations

$$[M] = [L] + [N] \quad \text{whenever} \quad 0 \to L \to M \to N \to 0$$

is a short exact sequence in $\mathsf{M}(\#)$.

The *Grothendieck group* of a category $\mathsf{P}_\star(\#)$ is the Abelian group $K_0\mathsf{P}_\star(\#)$ presented by generators $[X]$, one for each isomorphism class in $\mathsf{P}_\star(\#)$, and relations

$$[X] = 0 \quad \text{whenever } X \text{ is exact},$$

and

$$[Y] = [X] + [Z] \quad \text{whenever} \quad 0 \to X \to Y \to Z \to 0$$

is a short exact sequence in $\mathsf{P}_\star(\#)$.

So, for example, $K_0\mathsf{P}_e(S\text{-tor})$ denotes the Grothendieck group of the category $\mathsf{P}_e(S\text{-tor})$, whereas the usual *zeroth algebraic K-group of R* is the group $K_0(R) = K_0\mathsf{P}_0$: that is, the Grothendieck group of the category of P_0.

In the following three propositions, we list some useful properties of Grothendieck groups, which will be used throughout this paper. The properties can easily be verified and are stated without proof; for more details, the reader is referred to [4, pp. 8–10].

Proposition 2. *Any element in $K_0\mathsf{M}(\#)$ can be written in the form $[M]-[M']$ for modules $M, M' \in \mathsf{M}(\#)$, and any element in $K_0\mathsf{P}_\star(\#)$ can be written in the form $[X] - [X']$ for complexes $X, X' \in \mathsf{P}_\star(\#)$.* $\square$

If X is a complex, it can be *shifted n degrees to the left*, thereby yielding the complex $\Sigma^n X$ with modules $(\Sigma^n X)_\ell = X_{\ell-n}$ and differentials $\partial_\ell^{\Sigma^n X} = (-1)^n \partial_{\ell-n}^X$. In the case that $n = 1$, the operator $\Sigma^1(-)$ is simply denoted by $\Sigma(-)$.

Proposition 3. *Suppose that X is a complex in $\mathsf{P}_\star(\#)$ such that $\Sigma^n X$ is in $\mathsf{P}_\star(\#)$. Then $[\Sigma^n X] = (-1)^n[X]$ in $K_0\mathsf{P}_\star(\#)$.* $\square$

Proposition 4. *Suppose that $\phi\colon X \to Y$ is a quasi-isomorphism in $\mathsf{P}_\star(\#)$ such that ΣX is in $\mathsf{P}_\star(\#)$. Then $[X] = [Y]$ in $K_0\mathsf{P}_\star(\#)$.* $\square$

Note that, since quasi-isomorphisms become identities in the Grothendieck group, we might as well have modelled the Grothendieck groups on derived categories rather than usual categories.

Since $\mathsf{P}_e(\#)$ is a subcategory of $\mathsf{P}(\#)$, the inclusion of categories induces a natural group homomorphism

$$\mathcal{I}_e\colon K_0\mathsf{P}_e(\#) \to K_0\mathsf{P}(\#)$$

given by $\mathcal{I}_e([X]) = [X]$ where $X \in \mathsf{P}_e(\#)$. Note that the two $[X]$'s here are different: one is an element of $K_0\mathsf{P}_e(\#)$, whereas the other is an element of $K_0\mathsf{P}(\#)$. Note also that the fact that $\mathcal{I}_e$ is induced by an inclusion of the underlying categories does not mean that $\mathcal{I}_e$ is injective—it only ensures that $\mathcal{I}_e$ is well defined.

When R is Noetherian, a module in $\mathsf{M}(\#)$ always has a projective resolution in $\mathsf{P}(\#)$. It follows from Proposition 4 that different projective resolutions of the same module always represent the same element in the Grothendieck group $K_0\mathsf{P}(\#)$. Thus, we can, to each module $M \in \mathsf{M}(\#)$ with projective resolution $X \in \mathsf{P}(\#)$, associate the element $[X]$ in $K_0\mathsf{P}(\#)$. Since the modules in a short exact sequences have projective resolutions that fit together in a short exact sequence, this association induces a group homomorphism

$$\mathcal{R}\colon K_0\mathsf{M}(\#) \to K_0\mathsf{P}(\#)$$

given by $\mathcal{R}([M]) = [X]$ where $X \in \mathsf{P}(\#)$ is a projective resolution of $M \in \mathsf{M}(\#)$.

As we shall see in the next section, certain additional assumptions on the ring together with a sufficiently small choice of e can force the homology of complexes in $\mathsf{P}_e(\#)$ to be concentrated in degree zero and hence be modules in $\mathsf{M}(\#)$. Thus, in this case, we can, to every complex $X \in \mathsf{P}_e(\#)$, associate the element $[\mathrm{H}(X)]$ in $K_0\mathsf{M}(\#)$, where H denotes the homology functor. Since this association clearly preserves the relations in $K_0\mathsf{P}_e(\#)$, it induces a group homomorphism

$$\mathcal{H}_e\colon K_0\mathsf{P}_e(\#) \to K_0\mathsf{M}(\#)$$

given by $\mathcal{H}_e([X]) = [\mathrm{H}(X)]$ where $X \in \mathsf{P}_e(\#)$.

The homomorphisms $\mathcal{I}_e$, $\mathcal{H}_e$ and $\mathcal{R}$ are connected in a commutative diagram as shown below.

$$\begin{tikzcd}
K_0\mathsf{P}_e(\#) \arrow[rr, "\mathcal{I}_e"] \arrow[dr, dotted, "\mathcal{H}_e"'] & & K_0\mathsf{P}(\#) \\
& K_0\mathsf{M}(\#) \arrow[ur, "\mathcal{R}"'] &
\end{tikzcd}$$

$\mathcal{H}_e$ is here dotted to underline the fact that it required an extra assumption to be defined. The homomorphism $\mathcal{R}$ always requires R to be Noetherian in order to be defined.

Let $x = (x_1, \dots, x_d)$ denote a regular sequence, and let $S(x)$ denote the family $(S(x_1), \dots, S(x_d))$ of multiplicative systems $S(x_i) = \{x_i^n \mid n \in \mathbb{N}_0\}$. Further, let T denote a (single) multiplicative system such that $T \cap \mathrm{Zd}\, R = \emptyset$. In the next section we shall prove that the homomorphisms $\mathcal{H}_e$ and $\mathcal{R}$ are defined under the assumptions on e and R described in the table below.

$\#$	e	assumption on R
$S(x)$-tor	d	Noetherian, local
T-tor	1	Noetherian, local
$-$	0	Noetherian
length	$\dim R$	Noetherian, local, Cohen–Macaulay
$\mathrm{gr} \geq d$	d	Noetherian, local
d-perf	d	Noetherian, local

In this paper we will show that $\mathcal{I}_e$, $\mathcal{H}_e$ and $\mathcal{R}$ in all but the last of the above situations are isomorphisms and that, in the last situation, $\mathcal{I}_e$ and $\mathcal{R}$ are monomorphism and $\mathcal{H}_e$ is an isomorphism. These results will be derived as corollaries to the theorem below, which shall henceforth be referred to as the "Main Theorem". As the proof of the Main Theorem will show, the Grothendieck group $K_0\mathsf{P}_e(\#)$, where $\#$ is any but the last of the properties in the table above, is, in fact, isomorphic to $K_0\mathsf{P}(\#)$ whenever e is larger than or equal to the corresponding number in the table and trivial otherwise.

Main Theorem. *Suppose that d is a non-negative integer and that $S = (S_1, \dots, S_d)$ is a d-tuple of multiplicative systems of R. Then the homomorphism*

$$\mathcal{I}_d\colon K_0\mathsf{P}_d(S\text{-tor}) \to K_0\mathsf{P}(S\text{-tor})$$

given by $\mathcal{I}_d([X]) = [X]$ is an isomorphism.

Note that, in the setting of the Main Theorem, there are no additional requirements on R, and the homomorphisms $\mathcal{H}_d$ and $\mathcal{R}$ is not necessarily defined. However, when $\mathcal{H}_d$ and $\mathcal{R}$ are defined, we can immediately infer that $\mathcal{H}_d$ is injective and that $\mathcal{R}$ is surjective, and as it is not hard to see that $\mathcal{H}_d$ is surjective, it follows that all three homomorphisms are isomorphisms.

The Main Theorem says that any element of $K_0\mathsf{P}(S\text{-tor})$ can be represented by a linear combination of complexes concentrated in degrees $d, \dots, 0$. As we shall see, the inverse map $\mathcal{I}_d^{-1}\colon K_0\mathsf{P}(S\text{-tor}) \to K_0\mathsf{P}_d(S\text{-tor})$ is basically constructed from a procedure describing how to "make complexes smaller". When $\mathcal{H}_d$ is defined, the complexes become so small that they are forced to be resolutions of modules with projective dimension at most d.

When R is Noetherian and local, $d = 1$ and the multiplicative set T contains no zero-divisors, $\mathcal{H}_1\colon K_0\mathsf{P}_1(T\text{-tor}) \to K_0\mathsf{M}(T\text{-tor})$ is, as we shall see, defined and all of $\mathcal{I}_1$, $\mathcal{H}_1$ and $\mathcal{R}$ are isomorphisms. So in this case, the elements of $K_0\mathsf{M}(T\text{-tor})$ can be represented by elements in the form $[R^n/AR]$, where A is an injective $n \times n$-matrix. Using the localization sequence

$$K_1(R) \to K_1(T^{-1}R) \to K_0\mathsf{M}(T\text{-tor}) \to K_0(R) \to K_0(T^{-1}R)$$

of algebraic K-groups, it is not hard to see that $[R^n/AR^n] = [R/(\det A)R]$ in $K_0\mathsf{M}(T\text{-tor})$. Thus, $K_0\mathsf{P}_1(T\text{-tor})$ (and hence $K_0\mathsf{P}(T\text{-tor})$) is in fact generated by Koszul complexes. This property was fundamental in the proof in [3] of Serre's intersection conjectures in the case where the module that is *not* necessarily of finite projective dimension has dimension less than or equal to one.

The rather tedious proof of the Main Theorem is postponed until Section 4. For now, we will assume that it has been established and use it to derive all the other results.

3. Isomorphisms between Grothendieck groups

Definition 5. If x is an element of R, $S(x)$ denotes the multiplicative system $\{x^n \,|\, n \in \mathbb{N}_0\}$, and if $x = (x_1, \dots, x_d)$ is a d-tuple of elements from R, $S(x)$ denotes the d-tuple $(S(x_1), \dots, S(x_d))$ of multiplicative systems.

We begin our collection of corollaries to the Main Theorem with the result discussed in the abstract and the introduction.

Corollary 6. *If R is Noetherian and local with* $\dim R = d$*, then the group homomorphism* $\mathcal{I}_d\colon K_0\mathsf{P}_d(\text{length}) \to K_0\mathsf{P}(\text{length})$ *given by* $\mathcal{I}_d([X]) = [X]$ *is an isomorphism.*

Proof. Let $x = (x_1, \dots, x_d)$ be a system of parameters, and notice that a finitely generated module has finite length if and only if it is $S(x)$-torsion. Consequently, $K_0\mathsf{P}(\text{length}) = K_0\mathsf{P}(S(x)\text{-tor})$ and $K_0\mathsf{P}_d(\text{length}) = K_0\mathsf{P}_d(S(x)\text{-tor})$, and the result follows from the Main Theorem □

Lemma 7. *Suppose that R is Noetherian and let $x = (x_1, \dots, x_d)$ be a regular sequence of length $d > 0$. Then any complex X in $\mathsf{P}_d(S(x)\text{-tor})$ satisfies the condition that its homology complex $\mathrm{H}(X)$ is concentrated in degree 0: that is, $\mathrm{H}(X)$ is a module in $\mathsf{M}(S(x)\text{-tor})$.*

Proof. Let X be a non-exact complex in $\mathsf{P}_d(S(x)\text{-tor})$ and let t denote the largest integer such that $\mathrm{H}_t(X) \neq 0$; this exists since $\mathrm{H}(X) \neq 0$ and X is bounded. We already know that $t \geq 0$, so let us assume that $t > 0$ and try to reach a contradiction.

Let $\mathfrak{p}$ be an associated prime of $\mathrm{H}_t(X)$. Since $\mathrm{H}(X)$ is $S(x)$-torsion, we can find $N_1, \dots, N_d \in \mathbb{N}$ such that $x_1^{N_1}, \dots, x_d^{N_d} \in \operatorname{Ann}_R \mathrm{H}_t(X) \subseteq \mathfrak{p}$. Consequently, $(x_1/1, \dots, x_d/1)$ is an $R_\mathfrak{p}$-sequence in $\mathfrak{p}_\mathfrak{p}$, so $\operatorname{depth} R_\mathfrak{p} \geq d \geq 1$.

Now, the projective resolution

$$0 \to (X_d)_\mathfrak{p} \to \cdots \to (X_{t+1})_\mathfrak{p} \to (\operatorname{im} \partial_{t+1}^X)_\mathfrak{p} \to 0$$

of $(\operatorname{im} \partial_{t+1}^X)_\mathfrak{p}$ as an $R_\mathfrak{p}$-module shows that $\operatorname{pd}_{R_\mathfrak{p}} (\operatorname{im} \partial_{t+1}^X)_\mathfrak{p} \leq d - (t+1)$. From the Auslander–Buchsbaum formula (see, for example, [1, Theorem 1.3.3]), it now follows that

$$\operatorname{depth}_{R_\mathfrak{p}} (\operatorname{im} \partial_{t+1}^X)_\mathfrak{p} = \operatorname{depth} R_\mathfrak{p} - \operatorname{pd}_{R_\mathfrak{p}} (\operatorname{im} \partial_{t+1}^X)_\mathfrak{p} \geq t + 1 \geq 2.$$

Since $(\ker \partial_t^X)_\mathfrak{p}$ is a submodule of the non-trivial free $R_\mathfrak{p}$-module $(X_t)_\mathfrak{p}$ which has $\operatorname{depth}_{R_\mathfrak{p}} (X_t)_\mathfrak{p} = \operatorname{depth} R_\mathfrak{p} \geq d \geq 1$, we must also have $\operatorname{depth}_{R_\mathfrak{p}} (\ker \partial_t^X)_\mathfrak{p} \geq 1$. From the short exact sequence

$$0 \to (\operatorname{im} \partial_{t+1}^X)_\mathfrak{p} \to (\ker \partial_t^X)_\mathfrak{p} \to (\mathrm{H}_t(X))_\mathfrak{p} \to 0,$$

it now follows that $\operatorname{depth}_{R_\mathfrak{p}} (\mathrm{H}_t(X))_\mathfrak{p} \geq 1$ (see, for example, [1, Proposition 1.2.9]). This is a contradiction, however, because $\operatorname{depth}_{R_\mathfrak{p}} (\mathrm{H}_t(X))_\mathfrak{p} = 0$, since $\mathfrak{p}$ is associated to $\mathrm{H}_t(X)$. Thus, $t = 0$ as desired. ∎

Corollary 8. *If R is Noetherian and local, and $x = (x_1, \dots, x_d)$ is a regular sequence of length $d > 0$, then there is a commutative diagram*

$$\begin{tikzcd}
K_0\mathsf{P}_d(S(x)\text{-tor}) \arrow[rr, "\mathcal{I}_d"] \arrow[dr, "\mathcal{H}_d"'] & & K_0\mathsf{P}(S(x)\text{-tor}) \\
& K_0\mathsf{M}(S(x)\text{-tor}) \arrow[ur, "\mathcal{R}"'] &
\end{tikzcd}$$

in which $\mathcal{I}_d$, $\mathcal{H}_d$ and $\mathcal{R}$ are isomorphisms.

Proof. Lemma 7 shows that $\mathcal{I}_d$, $\mathcal{H}_d$ and $\mathcal{R}$ are well-defined homomorphisms, and the Main Theorem states that $\mathcal{I}_d$ is an isomorphism. Thus, we already know that $\mathcal{H}_d$ is injective and $\mathcal{R}$ is surjective.

Now, let M be a module in $\mathsf{M}(S(x)\text{-tor})$, and let us show by induction on $p = \operatorname{pd}_R M$ that $[M] \in \operatorname{im} \mathcal{H}_d$. If $p \leq d$, it is clear that $[M] \in \operatorname{im} \mathcal{H}$, since M in this case has a projective resolution in $\mathsf{P}_d(S(x)\text{-tor})$. So assume that $p > d$, and choose a finitely generated free module F and a surjective homomorphism $f \colon F \to M$. Next, using the fact that M is $S(x)$-torsion, choose $N_1, \dots, N_d \in \mathbb{N}$ so that $x_1^{N_1}, \dots, x_d^{N_d} \in \operatorname{Ann}_R M$, and let $\overline{F} = F/(x_1^{N_1}, \dots, x_d^{N_d})F$. The surjection f induces a surjection $\overline{f} \colon \overline{F} \to M$. Letting K denote the kernel of $\overline{f}$, we then have an exact sequence

$$0 \to K \to \overline{F} \to M \to 0,$$

and since $\operatorname{pd}_R \overline{F} = d < p = \operatorname{pd}_R M$, it follows that $\operatorname{pd}_R K = d - 1$. By construction, $\overline{F}$ and K are $S(x)$-torsion, so $\overline{F}$ and K are modules in $\mathsf{M}(S(x)\text{-tor})$, and the induction hypothesis yields $[M] = [\overline{F}] - [K] \in \operatorname{im} \mathcal{H}_d$. Consequently $\mathcal{H}_d$ is surjective, and it follows that $\mathcal{H}_d$ as well as $\mathcal{R}$ are isomorphisms. ∎

Corollary 8 also holds in the case $d = 0$, where the requirement of being $S(x)$-torsion drops out, even without the assumption that R is local. We state this as a separate corollary and leave the straightforward proof to the reader.

Corollary 9. *If R is Noetherian, then there is a commutative diagram*

$$\begin{tikzcd} K_0(R) = K_0\mathsf{P}_0 \arrow[rr, "\mathcal{I}_0"] \arrow[dr, "\mathcal{H}_0"'] & & K_0\mathsf{P} \\ & K_0\mathsf{M} \arrow[ur, "\mathcal{R}"'] & \end{tikzcd}$$

in which $\mathcal{I}_0$, $\mathcal{H}_0$ and $\mathcal{R}$ are isomorphisms. □

When R in addition is local, the Grothendieck groups in Corollary 9 are all isomorphic to $\mathbb{Z}$ through the rank on $K_0\mathsf{P}_0$. As the proof of the Main Theorem (Theorem 42) will show, the isomorphism $K_0\mathsf{P} \to \mathbb{Z}$ is given by taking an element $[X] \in K_0\mathsf{P}$ to the integer $\sum_{\ell\in\mathbb{Z}}(-1)^\ell \operatorname{rank}_R X_\ell$, whereas the isomorphism $K_0\mathsf{M} \to \mathbb{Z}$ is given by taking and element $[M] \in K_0\mathsf{M}$ to the Euler characteristic $\chi^R(M)$, defined as the alternating sum of the ranks in a finite free resolution of M.

The proofs of Lemma 7 and Corollary 8 in the case $d = 1$ clearly show that the multiplicative system $S(x) = S(x_1) = \{x_1^n \mid n \in \mathbb{N}_0\}$ can be replaced by *any* multiplicative system S containing only non-zerodivisors. This is because any element of such a multiplicative system in itself constitutes a regular sequence of length 1. We state this as a separate corollary.

Corollary 10. *If R is Noetherian and T is a multiplicative system with $T \cap \operatorname{Zd} R = \emptyset$, then there is a commutative diagram*

$$\begin{tikzcd} K_0\mathsf{P}_1(T\text{-tor}) \arrow[rr, "\mathcal{I}_1"] \arrow[dr, "\mathcal{H}_1"'] & & K_0\mathsf{P}(T\text{-tor}) \\ & K_0\mathsf{M}(T\text{-tor}) \arrow[ur, "\mathcal{R}"'] & \end{tikzcd}$$

in which $\mathcal{I}_1$, $\mathcal{H}_1$ and $\mathcal{R}$ are isomorphisms. □

Another special case of Corollary 8 that we would like to point out is the case $d = \dim R$, which is only possible when R is Cohen–Macaulay. In this case, the property of being S-torsion is identical to the property of having finite length. The result in this case was discussed in the abstract and the introduction, and we also state it as a separate corollary.

Corollary 11. *If R is a Noetherian, local Cohen-Macaulay ring of dimension d, then there is a commutative diagram*

$$\begin{tikzcd} K_0\mathsf{P}_d(\text{length}) \arrow[rr, "\mathcal{I}_d"] \arrow[dr, "\mathcal{H}_d"'] & & K_0\mathsf{P}(\text{length}) \\ & K_0\mathsf{M}(\text{length}) \arrow[ur, "\mathcal{R}"'] & \end{tikzcd}$$

in which $\mathcal{I}_d$, $\mathcal{H}_d$ and $\mathcal{R}$ are isomorphisms. □

As we shall see in Corollary 12 below, Corollary 8 can also be used to derive results concerning the property of having grader larger than or equal to d.

Corollary 12. *If R is Noetherian and local, and d is a positive integer, then there is a commutative diagram*

$$\begin{tikzcd} K_0\mathsf{P}_d(\mathrm{gr} \geq d) \arrow[rr, "\mathcal{I}_d"] \arrow[dr, "\mathcal{H}_d"'] & & K_0\mathsf{P}(\mathrm{gr} \geq d) \\ & K_0\mathsf{M}(\mathrm{gr} \geq d) \arrow[ur, "\mathcal{R}"'] & \end{tikzcd}$$

in which $\mathcal{I}_d$, $\mathcal{H}_d$ and $\mathcal{R}$ are isomorphisms.

Proof. If d is so large that there are no regular sequences in R of length d, then the involved Grothendieck groups are all trivial and the theorem holds. We can therefore assume that regular sequences of length d do exist.

If X is a complex in $\mathsf{P}_d(\mathrm{gr} \geq d)$, we can find a regular sequence $x = (x_1, \dots, x_d)$ of length d contained in the annihilator of *all* the homology modules of X. Then X will be homologically $S(x)$-torsion, and it follows from Lemma 7 that the homology of X is concentrated in degree 0. Consequently, $\mathcal{I}_d$, $\mathcal{H}_d$ and $\mathcal{R}$ are well-defined homomorphisms.

We define an equivalence relation on the set of regular sequences, letting a regular sequence $x = (x_1, \dots, x_d)$ be equivalent to a regular sequence $x' = (x'_1, \dots, x'_d)$ whenever

$$\mathrm{Rad}_R(x_1, \dots, x_d) = \mathrm{Rad}_R(x'_1, \dots, x'_d),$$

where the radical $\mathrm{Rad}_R\, I$ of an ideal I is the intersection of all prime ideals containing I. It is clear that this, indeed, is an equivalence relation. Denote the set of equivalence classes by E, and partially order E by reversed inclusion of radical ideals: that is,

$$x \preccurlyeq x' \overset{\text{def}}{\iff} \mathrm{Rad}_R(x_1, \dots, x_d) \supseteq \mathrm{Rad}_R(x'_1, \dots, x'_d)$$

for $x, x' \in E$. (It is of course the *equivalence classes* of x and x' that belong to E, but this unimportant technicality will be ignored here.) $E = (E, \preccurlyeq)$ is a directed set, for if x and x' are regular sequences of length d, then we can find a regular sequence x'' of length d contained in $(x) \cap (x')$ and hence satisfying the condition that $x, x' \preccurlyeq x''$.

Now, the category $\mathsf{M}(S(x)\text{-tor})$ is uniquely determined by the equivalence class of x in E, since, for any finitely generated module M,

$$\begin{aligned} M \text{ is } S(x)\text{-torsion} &\iff \forall \nu \in \{1, \dots, d\} \exists N_\nu \in \mathbb{N}_0 \colon x_\nu^{N_\nu} \in \mathrm{Ann}_R\, M \\ &\iff (x_1, \dots, x_d) \subseteq \mathrm{Rad}_R(\mathrm{Ann}_R\, M) \\ &\iff \mathrm{Rad}_R(x_1, \dots, x_d) \subseteq \mathrm{Rad}_R(\mathrm{Ann}_R\, M). \end{aligned}$$

Thus, we can consider the family of Grothendieck groups $K_0\mathsf{M}(S(x)\text{-tor})$ indexed by the equivalence classes in E. Given $x, x' \in E$ with $x \preccurlyeq x'$, there is a homomorphism

$$\mathcal{I}_{x,x'} \colon K_0\mathsf{M}(S(x)\text{-tor}) \to K_0\mathsf{M}(S(x')\text{-tor})$$

given by $\mathcal{I}_{x,x'}([M]) = [M]$ where $M \in \mathsf{M}(S(x)\text{-tor})$; this is well defined, since it is induced by an inclusion of categories as seen from the bi-implications above. Consequently, $(K_0\mathsf{M}(S(x)\text{-tor}), \mathcal{I}_{x,x'})_{x \preccurlyeq x'}$ is a direct system, and it is straightforward to see that the Grothendieck group $K_0\mathsf{M}(\mathrm{gr} \geq d)$ together with the natural homomorphisms $\tau_x \colon K_0\mathsf{M}(S(x)\text{-tor}) \to K_0\mathsf{M}(\mathrm{gr} \geq d)$ induced by the inclusion of the underlying categories and given by $\tau_x([M]) = [M]$, $x \in E$, satisfy the universal property required by a direct limit of this system.

We have now shown that $K_0\mathsf{M}(\mathrm{gr} \geq d)$ is the direct limit of the direct system $(K_0\mathsf{M}(S(x)\text{-tor}), \mathcal{I}_{x,x'})_{x \preccurlyeq x'}$. By the same methods one can show that $K_0\mathsf{P}_d(\mathrm{gr} \geq d)$ and $K_0\mathsf{P}(\mathrm{gr} \geq d)$ are the direct limits of the direct systems

$$(K_0\mathsf{P}_d(S(x)\text{-tor}), \mathcal{I}_{x,x'})_{x \preccurlyeq x'} \quad \text{and} \quad (K_0\mathsf{P}(S(x)\text{-tor}), \mathcal{I}_{x,x'})_{x \preccurlyeq x'},$$

respectively, where the homomorphisms $\mathcal{I}_{x,x'}$ now are given by $\mathcal{I}_{x,x'}([X]) = [X]$ for complexes X in $\mathsf{P}_d(S(x)\text{-tor})$ and $\mathsf{P}(S(x)\text{-tor})$, respectively. Now, we already know from Corollary 8 that there is a commutative diagram of isomorphisms

$$\begin{tikzcd}
K_0\mathsf{P}_d(S(x)\text{-tor}) \arrow[rr, "\mathcal{I}_d"] \arrow[dr, "\mathcal{H}_d"'] & & K_0\mathsf{P}(S(x)\text{-tor}) \\
& K_0\mathsf{M}(S(x)\text{-tor}) \arrow[ur, "\mathcal{R}"'] &
\end{tikzcd}$$

for all $x \in E$, and hence there must also be a commutative diagram of isomorphisms

$$\begin{tikzcd}
K_0\mathsf{P}_d(\mathrm{gr} \geq d) \arrow[rr, "\mathcal{I}_d"] \arrow[dr, "\mathcal{H}_d"'] & & K_0\mathsf{P}(\mathrm{gr} \geq d) \\
& K_0\mathsf{M}(\mathrm{gr} \geq d) \arrow[ur, "\mathcal{R}"'] &
\end{tikzcd}$$

involving the direct limits. □

Because of Lemma 7, the homology of any complex in $\mathsf{P}_d(\mathrm{gr} \geq d)$ must be a d-perfect module. Thus, $\mathsf{P}_d(\mathrm{gr} \geq d) = \mathsf{P}_d(d\text{-perf})$, and hence $K_0\mathsf{P}_d(\mathrm{gr} \geq d) = K_0\mathsf{P}_d(d\text{-perf})$. It follows that the isomorphisms $\mathcal{H}_d\colon K_0\mathsf{P}_d(\mathrm{gr} \geq d) \to K_0\mathsf{M}(\mathrm{gr} \geq d)$ and $\mathcal{I}_d\colon K_0\mathsf{P}_d(\mathrm{gr} \geq d) \to K_0\mathsf{P}(\mathrm{gr} \geq d)$ from Corollary 12 must factor through $K_0\mathsf{M}(d\text{-perf})$ and $K_0\mathsf{P}(d\text{-perf})$, respectively. This is discussed in Corollary 13 below, which extends Corollary 12, and where we let $\tau\colon K_0\mathsf{M}(d\text{-perf}) \to K_0\mathsf{M}(\mathrm{gr} \geq d)$ and $\overline{\tau}\colon K_0\mathsf{P}(d\text{-perf}) \to K_0\mathsf{P}(\mathrm{gr} \geq d)$ denote the natural homomorphisms induced by the inclusion of the underlying categories and given by $\tau([M]) = [M]$ for $M \in \mathsf{M}(d\text{-perf})$ and $\overline{\tau}([X]) = [X]$ for $X \in \mathsf{P}(d\text{-perf})$.

Corollary 13. *If R is Noetherian and local and d is a positive integer, then there is a commutative diagram*

$$\begin{tikzcd}
& K_0\mathsf{M}(d\text{-perf}) \arrow[ddd, "\tau"] \arrow[dr, "\mathcal{R}'"] & \\
K_0\mathsf{P}_d(d\text{-perf}) \arrow[ur, "\mathcal{H}'_d"] \arrow[rr, "\mathcal{I}'_d"] \arrow[d, equal] & & K_0\mathsf{P}(d\text{-perf}) \arrow[d, "\overline{\tau}"] \\
K_0\mathsf{P}_d(\mathrm{gr} \geq d) \arrow[rr, "\mathcal{I}_d"] \arrow[dr, "\mathcal{H}_d"'] & & K_0\mathsf{P}(\mathrm{gr} \geq d) \\
& K_0\mathsf{M}(\mathrm{gr} \geq d) \arrow[ur, "\mathcal{R}"'] &
\end{tikzcd}$$

in which $\mathcal{I}_d$, $\mathcal{H}_d$, $\mathcal{R}$, $\mathcal{H}'_d$ and τ are isomorphisms, $\mathcal{I}'_d$ and $\mathcal{R}'$ are monomorphisms and $\overline{\tau}$ is an epimorphism.

Proof. Commutativity of the diagram is clear, and we have already seen in Corollary 12 that $\mathcal{I}_d$, $\mathcal{H}_d$ and $\mathcal{R}$ are isomorphisms. From this it follows that $\mathcal{I}'_d$ and $\mathcal{H}'_d$ are injective, and that $\overline{\tau}$ and τ are surjective. However, $\mathcal{H}'_d$ is clearly also surjective, since any finitely generated d-perfect module has a resolution in $\mathsf{P}_d(d\text{-perf})$, and hence $\mathcal{H}'_d$ and τ are isomorphisms. □

Note that Corollary 13 (and hence Corollary 12) actually holds when $d = 0$, but that including this case is unnecessary, as it is already stated in Corollary 9.

4. Proving the Main Theorem

Establishing the Main Theorem is a cumbersome task. We will construct an inverse to $\mathcal{I}_d\colon K_0\mathsf{P}_d(S\text{-tor}) \to K_0\mathsf{P}(S\text{-tor})$ as follows. Given a complex $Y \in \mathsf{P}(S\text{-tor})$, we choose $n \in \mathbb{Z}$ so that the shifted complex $\Sigma^n Y$ is in $\mathsf{P}_e(S\text{-tor})$ for some $e > d$. To this complex we associate an element $w_e(\Sigma^n Y) \in K_0\mathsf{P}_{e-1}(S\text{-tor})$; this is the crucial step, in which we "make a complex smaller", starting with the complex $\Sigma^n Y$ of amplitude (at most) e and ending up with the element $w_e(\Sigma^n Y)$, which, as we shall see, is represented by the difference of two complexes of amplitude (at most) $e-1$. Repeating this process a finite number of times, we end up with an element $w_{d+1}\cdots w_e(\Sigma^n Y)$ in $K_0\mathsf{P}_d(S\text{-tor})$. This is the image of $[Y]$ under the inverse of $\mathcal{I}_d$.

4.1. **Contractions.**

Notation. Throughout Section 4.1, d denotes a non-negative integer and $S = (S_1,\dots,S_d)$ denotes a d-tuple of multiplicative systems of R.

Definition 14. Let X be a complex. A d-tuple $\alpha = (\alpha^1,\dots,\alpha^d)$ of families $\alpha^\nu = (\alpha^\nu_\ell)_{\ell\in\mathbb{Z}}$ of homomorphisms $\alpha^\nu_\ell\colon X_\ell \to X_{\ell+1}$ is an *S-contraction of X with weight* $s = (s_1,\dots,s_d) \in S_1\times\cdots\times S_d$ if

$$\partial^X_{\ell+1}\alpha^\nu_\ell + \alpha^\nu_{\ell-1}\partial^X_\ell = s_\nu \operatorname{id}_{X_\ell}$$

for all $\ell \in \mathbb{Z}$ and $\nu = 1,\dots,d$.

In the case that $d = 0$, the concept of S-contractions is meaningless, and the property of having an S-contraction is trivially satisfied. In any case, the existence of an S-contraction of X with weight $s = (s_1,\dots,s_d)$ is equivalent to the condition that the morphisms $s_\nu \operatorname{id}_X\colon X \to X$ for $\nu = 1,\dots,d$ are null-homotopic.

Proposition 15. *Each complex $X \in \mathsf{P}(S\text{-tor})$ has an S-contraction.*

Proof. For each ν the $S_\nu^{-1}R$-complex $S_\nu^{-1}X$ is exact, bounded and consists of finitely generated projective $S_\nu^{-1}R$-modules, so the identity morphism $\operatorname{id}_{S_\nu^{-1}X}$ on $S_\nu^{-1}X$ is null-homotopic (see, for example, [5, Theorem IV.4.1]). Thus, we can find $S_\nu^{-1}R$-homomorphisms $b^\nu_\ell\colon S_\nu^{-1}X_\ell \to S_\nu^{-1}X_{\ell+1}$ such that

$$\partial^{S_\nu^{-1}X}_{\ell+1} b^\nu_\ell + b^\nu_{\ell-1}\partial^{S_\nu^{-1}X}_\ell = \operatorname{id}_{S_\nu^{-1}X_\ell}$$

for all $\ell \in \mathbb{Z}$. Writing each b^ν_ℓ in the form β^ν_ℓ/t_ν for an R-homomorphism $\beta^\nu_\ell\colon X_\ell \to X_{\ell+1}$ and some common denominator $t_\nu \in S_\nu$, we now have in $S_\nu^{-1}X_\ell$ that, for any $x \in X_\ell$,

$$(\partial^X_{\ell+1}\beta^\nu_\ell + \beta^\nu_{\ell-1}\partial^X_\ell)(x)/t_\nu = x/1.$$

Consequently, we can find $u_{\nu,x} \in S_\nu$ depending on x so that in X_ℓ,

$$u_{\nu,x}(\partial^X_{\ell+1}\beta^\nu_\ell + \beta^\nu_{\ell-1}\partial^X_\ell)(x) = u_{\nu,x}t_\nu x.$$

Since X is bounded and consists of finitely generated modules, by multiplying a finite number of $u_{\nu,x}$'s, we can obtain an element $u_\nu \in S_\nu$, independent of x and of ℓ, such that $u_\nu(\partial^X_{\ell+1}\beta^\nu_\ell + \beta^\nu_{\ell-1}\partial^X_\ell)(x) = u_\nu t_\nu x$ for all $\ell \in \mathbb{Z}$ and all $x \in X_\ell$. Setting $\alpha^\nu_\ell = u_\nu\beta^\nu_\ell$ and $s_\nu = u_\nu t_\nu$, we see that $\alpha = (\alpha^1,\dots,\alpha^d)$, where $\alpha^\nu = (\alpha^\nu_\ell)_{\ell\in\mathbb{Z}}$, is an S-contraction of X with weight $s = (s_1,\dots,s_d)$. □

Definition 16. Let X and Y be complexes in P with S-contractions α and β, respectively, and let $\phi\colon X \to Y$ be a morphism of complexes. Then α and β are said to be *compatible with ϕ* if they have the same weight and $\phi_{\ell+1}\alpha^\nu_\ell = \beta^\nu_\ell\phi_\ell$ for all $\ell \in \mathbb{Z}$ and $\nu = 1,\dots,d$.

Theorem 17 below provides an example of a situation where an S-contraction of a complex induces an S-contraction of another complex. Although the hypotheses of the theorem are very specific, the theorem turns out to be applicable in several situations.

Theorem 17. *Let X be a complex in P_e, where $e > 1$, and suppose that α is an S-contraction of X with weight s. Let $\widetilde{X}$ be another complex in P_e, and suppose that the complex $\widetilde{X}$ is identical to X except for the modules and differentials in degrees e and $e-1$. Suppose further that $\widetilde{X}_e = 0$ and that a morphism $\phi\colon X \to \widetilde{X}$ exists such that $\phi_\ell = \mathrm{id}_{X_\ell}$ for $\ell = 0, \ldots, e-2$ and such that ϕ_{e-1} is surjective. Then the S-contraction α on X induces an S-contraction $\widetilde{\alpha}$ on $\widetilde{X}$ with weight s such that α and $\widetilde{\alpha}$ are compatible with the morphism ϕ; for $\nu = 1, \ldots, d$, $\widetilde{\alpha}^\nu$ is defined by setting $\widetilde{\alpha}^\nu_{e-2} = \phi_{e-1}\alpha^\nu_{e-2}$ and $\widetilde{\alpha}^\nu_\ell = \alpha^\nu_\ell$ for $\ell = 0, \ldots, e-3$.*

$$
\begin{tikzcd}
0 \arrow[r] & X_e \arrow[r, shift left, "\partial^X_e"] \arrow[d, "0"] & X_{e-1} \arrow[l, shift left, "\alpha^\nu_{e-1}"] \arrow[r, shift left, "\partial^X_{e-1}"] \arrow[d, "\phi_{e-1}"] & X_{e-2} \arrow[l, shift left, "\alpha^\nu_{e-2}"] \arrow[r, shift left, "\partial^X_{e-2}"] \arrow[d, "\mathrm{id}_{X_{e-2}}"] & \cdots \arrow[l, shift left, "\alpha^\nu_{e-3}"] \arrow[r, shift left] & X_1 \arrow[l, shift left] \arrow[r, shift left, "\partial^X_1"] \arrow[d, "\mathrm{id}_{X_1}"] & X_0 \arrow[l, shift left, "\alpha^X_0"] \arrow[r] \arrow[d, "\mathrm{id}_{X_0}"] & 0 \\
0 \arrow[r] & 0 \arrow[r, shift left, "0"] & \widetilde{X}_{e-1} \arrow[l, shift left, "0"] \arrow[r, shift left, "\partial^{\widetilde{X}}_{e-1}"] & X_{e-2} \arrow[l, shift left, "\phi_{e-1}\alpha^\nu_{e-2}"] \arrow[r, shift left, "\partial^X_{e-2}"] & \cdots \arrow[l, shift left, "\alpha^\nu_{e-3}"] \arrow[r, shift left] & X_1 \arrow[l, shift left] \arrow[r, shift left, "\partial^X_1"] & X_0 \arrow[l, shift left, "\alpha^\nu_0"] \arrow[r] & 0
\end{tikzcd}
$$

Proof. By inspection. □

Given an S-contraction α of X with weight $s = (s_1, \ldots, s_d)$ and a d-tuple $t = (t_1, \ldots, t_d) \in S_1 \times \cdots \times S_d$, we can construct an S-contraction $t\alpha$ of X with weight $st = (s_1t_1, \ldots, s_dt_d)$ by setting $t\alpha = (t_1\alpha^1, \ldots, t_d\alpha^d)$ where $t_\nu\alpha^\nu = (t_\nu\alpha^\nu_\ell)_{\ell\in\mathbb{Z}}$. We can also shift α n degrees to the left for some $n \in \mathbb{Z}$ to form an S-contraction $\Sigma^n\alpha$ of Σ^nX with weight s by setting $\Sigma^n\alpha = (\Sigma^n\alpha^1, \ldots, \Sigma^n\alpha^d)$, where $\Sigma^n\alpha^\nu = ((\Sigma^n\alpha^\nu)_\ell)_{\ell\in\mathbb{Z}} = ((-1)^n\alpha^\nu_{\ell-n})_{\ell\in\mathbb{Z}}$.

The following theorem shows how to construct a natural S-contraction of the mapping cone of a morphism between two complexes that both have S-contractions. Recall that the mapping cone of a morphism $\phi\colon X \to Y$ is the complex $C(\phi)$ defined by $C(\phi)_\ell = Y_\ell \oplus X_{\ell-1} = (Y \oplus \Sigma X)_\ell$ and

$$
\partial^{C(\phi)}_\ell = \begin{pmatrix} \partial^Y_\ell & \phi_{\ell-1} \\ 0 & -\partial^X_{\ell-1} \end{pmatrix} : \begin{matrix} Y_\ell \\ \oplus \\ X_{\ell-1} \end{matrix} \to \begin{matrix} Y_{\ell-1} \\ \oplus \\ X_{\ell-2} \end{matrix}
$$

for all $\ell \in \mathbb{Z}$. The (degreewise) inclusion $Y \hookrightarrow C\phi$ and the (degreewise) projection $C(\phi) \twoheadrightarrow \Sigma X$ are both morphisms of complexes, and together they form the canonical short exact sequence

$$0 \to Y \to C(\phi) \to \Sigma X \to 0.$$

Theorem 18. *Let $\phi\colon X \to Y$ be a morphism of complexes and let α and β be S-contractions of X and Y, respectively, with weights s and t, respectively. Define for $\nu = 1, \ldots, d$ and $\ell \in \mathbb{Z}$ the homomorphism*

$$
(\beta * \alpha)^\nu_\ell = \begin{pmatrix} s_\nu\beta^\nu_\ell & \beta^\nu_\ell\phi_\ell\alpha^\nu_{\ell-1} \\ 0 & -t_\nu\alpha^\nu_{\ell-1} \end{pmatrix} : C(\phi)_\ell = \begin{matrix} Y_\ell \\ \oplus \\ X_{\ell-1} \end{matrix} \to \begin{matrix} Y_{\ell+1} \\ \oplus \\ X_\ell \end{matrix} = C(\phi)_{\ell+1}.
$$

Then $(\beta * \alpha) = ((\beta * \alpha)^1, \ldots, (\beta * \alpha)^d)$, where $(\beta * \alpha)^\nu = ((\beta * \alpha)^\nu_\ell)_{\ell\in\mathbb{Z}}$, is an S-contraction of the mapping cone $C(\phi)$ of ϕ with weight $st = (s_1t_1, \ldots, s_dt_d)$, and the S-contractions $s\beta$, $(\beta * \alpha)$ and $\Sigma t\alpha$ are compatible with the morphisms in the canonical exact sequence

$$0 \to Y \to C(\phi) \to \Sigma X \to 0.$$

Proof. By inspection. □

4.2. **The idea behind the proof of the Main Theorem.**

Notation. Throughout Section 4.2, d denotes a non-negative integer and $S = (S_1, \dots, S_d)$ denotes a d-tuple of multiplicative systems of R. Furthermore, X denotes a fixed complex in $\mathsf{P}_e(S\text{-tor})$ for some integer $e > d$, and α denotes an S-contraction of X with weight $s = (s_1, \dots, s_d) \in S_1 \times \cdots \times S_d$.

Proving the Main Theorem involves the introduction of a complex $\Delta_e(X, s)$. More specifically, $\Delta_e(X, s)$ is the complex $\Sigma^{e-d} K(s, X_e)$: that is, the Koszul complex of the sequence $s = (s_1, \dots, s_d)$ with coefficients in X_e and shifted $e-d$ degrees to the left. For convenience we will now present an explicit description of $\Delta_e(X, s)$.

For any $\ell \in \mathbb{Z}$, let $\Upsilon(\ell)$ denote the set of ℓ-element subsets of $\{1, \dots, d\}$: that is, the set of subsets in the form $i = \{i_1, \dots, i_\ell\}$ where $1 \le i_1 < \cdots < i_\ell \le d$. In particular, $\Upsilon(0) = \{\emptyset\}$, $\Upsilon(d) = \{\{1, \dots, d\}\}$ and $\Upsilon(\ell) = \emptyset$ for all $\ell \notin \{0, \dots, d\}$. Thus, in any case, $\Upsilon(\ell)$ contains $\binom{d}{\ell}$ elements. An object $i \in \Upsilon(\ell)$ is called a *multi-index* and its elements are always denoted by $i_1, \dots, i_\ell$ in increasing order, so that $i = \{i_1, \dots, i_\ell\}$, where $1 \le i_1 < \cdots < i_\ell \le d$.

Definition 19. $\Delta_e(X, s)$ denotes the complex whose ℓ'th module is given by

$$\Delta_e(X, s)_\ell = \coprod_{i \in \Upsilon(e-\ell)} \Delta_e(X, s)^i_\ell, \quad \text{where} \quad \Delta_e(X, s)^i_\ell = X_e,$$

and whose ℓ'th differential $\partial_\ell^{\Delta_e(X,s)} \colon \Delta_e(X, s)_\ell \to \Delta_e(X, s)_{\ell-1}$ is given by the fact that its (j, i)-entry $(\partial_\ell^{\Delta_e(X,s)})_{j,i} \colon \Delta_e(X, s)^i_\ell \to \Delta_e(X, s)^j_{\ell-1}$ for $i \in \Upsilon(e-\ell)$ and $j \in \Upsilon(e - \ell + 1)$ is

$$(\partial_\ell^{\Delta_e(X,s)})_{j,i} = \begin{cases} (-1)^{u+1} s_{j_u} \operatorname{id}_{X_e}, & \text{if } j \backslash i = \{j_u\} \\ 0, & \text{if } j \not\supseteq i \end{cases}$$

So $\Delta_e(X, s)$ is a complex whose ℓ'th module $\Delta_e(X, s)_\ell$ consists of $\binom{d}{e-\ell}$ copies of X_e and whose ℓ'th differential as a map from the i'th copy of X_e in $\Delta_e(X, s)_\ell$ to the j'th copy of X_e in $\Delta_e(X, s)_{\ell+1}$ is non-zero only when $i \subseteq j$, in which case it is multiplication by $(-1)^{u+1} s_{j_u}$ for the unique j_u which is in j and not in i. In particular, if $d = 0$ the sequence s is empty and $\Delta_e(X, s)$ is the complex concentrated in degree e with $\Delta_e(X, s)_e = X_e$.

Proposition 20. *The complex $\Delta_e(X, s)$ is in $\mathsf{P}(S\text{-tor})$ and is concentrated in degrees $e, \dots, e-d$.*

Proof. The definition clearly implies that $\Delta_e(X, s)$ is concentrated in degrees $e, \dots, e-d$ and consists of finitely generated projective modules. Since $\Delta_e(X, s)$ is the Koszul complex of the sequence $s_1, \dots, s_d$, the homology modules of $\Delta_e(X, s)$ are annihilated by the ideal $(s_1, \dots, s_d)$ (see, for example, [1, Proposition 1.6.5]); in particular, the homology modules must be S_ν-torsion for $\nu = 1, \dots, d$. □

The complex $\Delta_e(X, s)$ comes naturally equipped with an S-contraction.

Theorem 21. *For each $\ell \in \mathbb{Z}$ and each $\nu = 1, \dots, d$, let the homomorphism $\delta_e(X, s)^\nu_\ell \colon \Delta_e(X, s)_\ell \to \Delta_e(X, s)_{\ell+1}$ be given by the fact that its (j, i)-entry for $i \in \Upsilon(e-\ell)$ and $j \in \Upsilon(e-\ell-1)$ is*

$$(\delta_e(X, s)^\nu_\ell)_{j,i} = \begin{cases} (-1)^{w+1} \operatorname{id}_{X_e}, & \text{if } i \backslash j = \{i_w\} = \{\nu\}, \\ 0, & \text{if } i \not\supseteq j. \end{cases}$$

Then $\delta_e(X, s) = (\delta_e(X, s)^1, \dots, \delta_e(X, s)^d)$, where $\delta_e(X, s)^\nu = (\delta_e(X, s)^\nu_\ell)_{\ell \in \mathbb{Z}}$, is an S-contraction of $\Delta_e(X, s)$ with weight s

Proof. This is a matter of verification. For each $\nu \in \{1, \dots, d\}$, $\ell \in \mathbb{Z}$ and $i, i' \in \Upsilon(d-\ell)$, the (i', i)-entry of $\partial_{\ell+1}^{\Delta_e(X,s)} \delta_e(X,s)_\ell^\nu$ is

$$\begin{array}{cl} s_\nu \operatorname{id}_{X_e}, & \text{if } i = i' \text{ and } \nu \in i, \\ (-1)^{u+w} s_{i'_u} \operatorname{id}_{X_e}, & \text{if } i \backslash i' = \{i_w\} = \{\nu\} \text{ and } i' \backslash i = \{i'_u\}, \text{ and} \\ 0, & \text{otherwise,} \end{array}$$

whereas the (i', i)-entry of $\delta_e(X,s)_{\ell-1}^\nu \partial_\ell^{\Delta_e(X,s)}$ is

$$\begin{array}{cl} s_\nu \operatorname{id}_{X_e}, & \text{if } i = i' \text{ and } \nu \notin i, \\ (-1)^{u+w+1} s_{i'_u} \operatorname{id}_{X_e}, & \text{if } i \backslash i' = \{i_w\} = \{\nu\} \text{ and } i' \backslash i = \{i'_u\}, \text{ and} \\ 0, & \text{otherwise.} \end{array}$$

Overall, we see that the (i', i)-entry of $\partial_{\ell+1}^{\Delta_e(X,s)} \delta_e(X,s)_\ell^\nu + \delta_e(X,s)_{\ell-1}^\nu \partial_\ell^{\Delta_e(X,s)}$ is $s_\nu \operatorname{id}_{X_e}$ if $i = i'$ and 0 otherwise. This is what we wanted to show. □

Definition 22. Let $\phi_e(X,\alpha)$ denote the family $(\phi_e(X,\alpha)_\ell)_{\ell\in\mathbb{Z}}$ of homomorphisms $\phi_e(X,\alpha)_\ell \colon X_\ell \to \Delta_e(X,s)_\ell = \coprod_{i\in\Upsilon(e-\ell)} X_e$ given by the fact that their i'th entries for $i \in \Upsilon(e-\ell)$ are

$$\phi_e(X,\alpha)_\ell^i = \alpha_{e-1}^{i_{e-\ell}} \alpha_{e-2}^{i_{e-\ell-1}} \cdots \alpha_\ell^{i_1}.$$

For $\ell = e$, this means that $\phi_e(X,\alpha)_e = \operatorname{id}_{X_e}$, and for $\ell \notin \{e, \dots, e-d\}$, it means that $\phi_e(X,\alpha)_\ell = 0$.

Proposition 23. *$\phi_e(X,\alpha) \colon X \to \Delta_e(X,s)$ is a morphism of complexes.*

Proof. Let $\Delta \overset{\text{def}}{=} \Delta_e(X,s)$ and $\phi \overset{\text{def}}{=} \phi_e(X,\alpha)$. To prove that ϕ is a morphism, we need to show that $\phi_{\ell-1}\partial_\ell^X = \partial_\ell^\Delta \phi_\ell$ for all $\ell \in \mathbb{Z}$: that is, we need to verify that the j'th entry, $\alpha_{e-1}^{j_{e-\ell+1}} \cdots \alpha_{\ell-1}^{j_1} \partial_\ell^X$, of the left side equals the j'th entry of $\partial_\ell^\Delta \phi_\ell$ for each $j \in \Upsilon(e-\ell+1)$. Since the (j,i)-entry of ∂_ℓ^Δ is $(-1)^{u+1} s_{j_u} \operatorname{id}_{X_e}$ whenever i is a subset of j with $j \backslash i = \{j_u\}$, that is, whenever $i = \{j_1, \dots, j_{u-1}, j_{u+1}, \dots, j_{e-\ell+1}\}$ for some $u \in \{1, \dots, e-\ell+1\}$, we see that the j'th coordinate of $\partial_\ell^\Delta \phi_\ell$ must be

$$\sum_{u=1}^{e-\ell+1} (-1)^{u+1} s_{j_u} \alpha_{e-1}^{j_{e-\ell+1}} \cdots \alpha_{\ell+u-1}^{j_{u+1}} \alpha_{\ell+u-2}^{j_{u-1}} \cdots \alpha_\ell^{j_1}.$$

So overall, we need to show that

$$\sum_{u=1}^{e-\ell+1} (-1)^{u+1} s_{j_u} \alpha_{e-1}^{j_{e-\ell+1}} \cdots \alpha_{\ell+u-1}^{j_{u+1}} \alpha_{\ell+u-2}^{j_{u-1}} \cdots \alpha_\ell^{j_1} = \alpha_{e-1}^{j_{e-\ell+1}} \cdots \alpha_{\ell-1}^{j_1} \partial_\ell^X \tag{1}$$

for all $j \in \Upsilon(e-\ell+1)$. We do this by descending induction on ℓ.

When $\ell > e$, the equation clearly holds since both sides are trivial, and in the case that $\ell = e$, (1) states that $s_{j_1} \operatorname{id}_{X_e} = \alpha_{e-1}^{j_1} \partial_e^X$, which is satisfied since α is an S-contraction of X with weight s. Suppose now that $\ell < e$ is arbitrarily chosen and that (1) holds for larger values of ℓ. We then have

$$\begin{aligned} \alpha_{e-1}^{j_{e-\ell+1}} \cdots \alpha_{\ell-1}^{j_1} \partial_\ell^X &= \alpha_{e-1}^{j_{e-\ell+1}} \cdots \alpha_\ell^{j_2} (s_{j_1} \operatorname{id}_{X_\ell} - \partial_{\ell+1}^X \alpha_\ell^{j_1}) \\ &= s_{j_1} \alpha_{e-1}^{j_{e-\ell+1}} \cdots \alpha_\ell^{j_2} \\ &\quad - \Big(\sum_{u=2}^{e-\ell+1} (-1)^u s_{j_u} \alpha_{e-1}^{j_{e-\ell+1}} \cdots \alpha_{\ell+u-1}^{j_{u+1}} \alpha_{\ell+u-2}^{j_{u-1}} \cdots \alpha_{\ell+1}^{j_2} \Big) \alpha_\ell^{j_1} \\ &= \sum_{u=1}^{e-\ell+1} (-1)^{u+1} s_{j_u} \alpha_{e-1}^{j_{e-\ell+1}} \cdots \alpha_{\ell+u-1}^{j_{u+1}} \alpha_{\ell+u-2}^{j_{u-1}} \cdots \alpha_\ell^{j_1}. \end{aligned}$$

Here the second equality follows from the induction hypothesis. This proves (1) by induction, so ϕ is a morphism of complexes. □

Definition 24. The mapping cone of $\phi_e(X,\alpha)$ is denoted by $C_e(X,\alpha)$.

Letting $\Delta \overset{\text{def}}{=} \Delta_e(X,s)$ and $\phi \overset{\text{def}}{=} \phi_e(X,\alpha)$, $C_e(X,\alpha)$ is the complex

$$0 \longrightarrow X_e \xrightarrow{\left(\begin{smallmatrix}\phi_e \\ -\partial_e^X\end{smallmatrix}\right)} \begin{matrix}\Delta_e \\ \oplus \\ X_{e-1}\end{matrix} \xrightarrow{\left(\begin{smallmatrix}\partial_e^\Delta & \phi_{e-1} \\ 0 & -\partial_{e-1}^X\end{smallmatrix}\right)} \begin{matrix}\Delta_{e-1} \\ \oplus \\ X_{e-2}\end{matrix} \longrightarrow \cdots$$

$$\cdots \longrightarrow \begin{matrix}\Delta_{e-d} \\ \oplus \\ X_{e-d-1}\end{matrix} \xrightarrow{\left(\begin{smallmatrix}0 & -\partial_{e-d-1}^X\end{smallmatrix}\right)} X_{e-d-2} \xrightarrow{-\partial_{e-d-2}^X} \cdots \longrightarrow X_0 \longrightarrow 0$$

concentrated in degrees $e+1,\dots,1$.

Since X and $\Delta_e(X,s)$ are equipped with S-contractions, Theorem 18 provides an S-contraction of $C_e(X,\alpha)$.

Definition 25. The S-contraction $\delta_e(X,s) * \alpha$ of $C_e(X,\alpha)$ is denoted by $\mu_e(X,\alpha)$.

Letting $\Delta \overset{\text{def}}{=} \Delta_e(X,s)$, $\phi \overset{\text{def}}{=} \phi_e(X,\alpha)$ and $\delta \overset{\text{def}}{=} \delta_e(X,s)$, $\mu_e(X,\alpha)$ is given by

$$\mu_e(X,\alpha)_\ell^\nu = \begin{pmatrix} s_\nu\delta_\ell^\nu & \delta_\ell^\nu\phi_\ell\alpha_{\ell-1}^\nu \\ \\ 0 & -s_\nu\alpha_{\ell-1}^\nu \end{pmatrix} : \begin{matrix}\Delta_\ell \\ \oplus \\ X_{\ell-1}\end{matrix} \to \begin{matrix}\Delta_{\ell+1} \\ \oplus \\ X_\ell\end{matrix}$$

for each $\ell \in \mathbb{Z}$ and $\nu \in \{1,\dots,d\}$. The weight of $\mu_e(X,\alpha)$ is $s^2 = (s_1^2,\dots,s_d^2)$.

Proposition 26. *$C_e(X,\alpha)$ is an object of $\mathsf{P}_{e+1}(S\text{-tor})$ concentrated in degrees $e+1,\dots,1$.*

Proof. $C_e(X,\alpha)$ is clearly concentrated in degrees $e+1,\dots,1$ and composed of finitely generated projective modules. To see that $C_e(X,\alpha)$ is homologically S-torsion, recall that the canonical short exact sequence

$$0 \to \Delta_e(X,s) \to C_e(X,\alpha) \to \Sigma X \to 0 \tag{2}$$

induces the long exact sequence

$$\cdots \to \mathrm{H}_\ell(\Delta_e(X,s)) \to \mathrm{H}_\ell(C_e(X,\alpha)) \to \mathrm{H}_\ell(\Sigma X) \to \cdots$$

on homology. By localizing at S_ν for $\nu = 1,\dots,d$ it follows that, since $\Delta_e(X,s)$ as well as ΣX are homologically S-torsion, $C_e(X,\alpha)$ must be homologically S-torsion as well. $\square$

Definition 27. Let ∂_{e-1}^D denote the homomorphism

$$\partial_{e-1}^D = \begin{pmatrix} -\phi_e(X,\alpha)_{e-1} \\ \\ \partial_{e-1}^X \end{pmatrix} : X_{e-1} \longrightarrow \begin{matrix}\Delta_e(X,s)_{e-1} \\ \oplus \\ X_{e-2}\end{matrix} = C_e(X,\alpha)_{e-1},$$

and let $D_e(X,\alpha)$ denote the complex

$$0 \longrightarrow X_{e-1} \xrightarrow{\partial_{e-1}^D} C_e(X,\alpha)_{e-1} \xrightarrow{-\partial_{e-1}^{C_e(X,\alpha)}} C_e(X,\alpha)_{e-2} \xrightarrow{-\partial_{e-2}^{C_e(X,\alpha)}} \cdots \longrightarrow C_e(X,\alpha)_1 \longrightarrow 0$$

concentrated in degrees $e-1,\dots,0$.

(One verifies easily that $D_e(X,\alpha)$ indeed is a complex. It is identical to the shifted mapping cone $\Sigma^{-1}C_e(X,\alpha)$ except in degrees $e+1$ and e.)

Proposition 28. *$D_e(X,\alpha)$ is an object of $\mathsf{P}_{e-1}(S\text{-tor})$.*

Proof. $D_e(X,\alpha)$ is clearly composed of finitely generated projective modules. The fact that $D_e(X,\alpha)$ is homologically S-torsion is a consequence of Theorem 29 below, from which it follows that $D_e(X,\alpha)$ is quasi-isomorphic to $\Sigma^{-1}C_e(X,\alpha)$. $\square$

Theorem 29. *Let B denote the exact complex $0 \to X_e \xrightarrow{\mathrm{id}} X_e \to 0$ concentrated in degrees e and $e-1$. There is then an exact sequence*

$$0 \to B \to \Sigma^{-1} C_e(X,\alpha) \to D_e(X,\alpha) \to 0.$$

Proof. Let $\Delta \overset{\mathrm{def}}{=} \Delta_e(X,s)$ and $\phi \overset{\mathrm{def}}{=} \phi_e(X,\alpha)$, and recall that $\Delta_e = X_e$ and $\phi_e = \mathrm{id}_{X_e}$. The situation is as follows.

$$\begin{tikzcd}
 & 0 \arrow[r] & B \arrow[r] & \Sigma^{-1}C_e(X,\alpha) \arrow[r] & D_e(X,\alpha) \arrow[r] & 0 \\
 & & \| & \| & \| & \\
\textit{degree} & & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
e & 0 \arrow[r] & X_e \arrow[r, "-\mathrm{id}_{X_e}"] \arrow[d, "\mathrm{id}_{X_e}"'] & X_e \arrow[r] \arrow[d, "\begin{pmatrix}-\phi_e \\ \partial^X_e\end{pmatrix}"] & 0 \arrow[r] \arrow[d] & 0 \\
e-1 & 0 \arrow[r] & X_e \arrow[r, "\begin{pmatrix}\mathrm{id}_{X_e} \\ -\partial^X_e\end{pmatrix}"] \arrow[d] & \begin{matrix}\Delta_e \\ \oplus \\ X_{e-1}\end{matrix} \arrow[r, "\begin{pmatrix}\partial^X_e & \mathrm{id}_{X_{e-1}}\end{pmatrix}"] \arrow[d, "\begin{pmatrix}-\partial^\Delta_e & -\phi_{e-1} \\ 0 & \partial^X_{e-1}\end{pmatrix}"] & X_{e-1} \arrow[r] \arrow[d, "\begin{pmatrix}-\phi_{e-1} \\ \partial^X_{e-1}\end{pmatrix}"] & 0 \\
e-2 & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & \begin{matrix}\Delta_{e-1} \\ \oplus \\ X_{e-2}\end{matrix} \arrow[r, "\mathrm{id}"] \arrow[d] & \begin{matrix}\Delta_{e-1} \\ \oplus \\ X_{e-2}\end{matrix} \arrow[r] \arrow[d] & 0 \\
\vdots & & \vdots & \vdots & \vdots &
\end{tikzcd}$$

It is straightforward to verify that the diagram commutes and that all the rows are exact. □

The morphism $\Sigma^{-1}C_e(X,\alpha) \to D_e(X,\alpha)$ from Theorem 29 is clearly in the form described in Theorem 17, so we are able to induce an S-contraction of $D_e(X,\alpha)$ with weight s^2 from the S-contraction $\Sigma^{-1}\mu_e(X,\alpha)$ on $\Sigma^{-1}C_e(X,\alpha)$.

Definition 30. The S-contraction of $D_e(X,\alpha)$ induced in the sense of Theorem 17 from $\Sigma^{-1}\mu_e(X,\alpha)$ through the morphism $\Sigma^{-1}C_e(X,\alpha) \to D_e(X,\alpha)$ from Theorem 29 is denoted by $\eta_e(X,\alpha)$.

Letting $\Delta \overset{\mathrm{def}}{=} \Delta_e(X,s)$, $\phi \overset{\mathrm{def}}{=} \phi_e(X,\alpha)$ and $\delta \overset{\mathrm{def}}{=} \delta_e(X,s)$, $\eta_e(X,\alpha)$ from the above definition is given by

$$\eta_e(X,\alpha)^\nu_\ell = \begin{pmatrix} -s_\nu \delta^\nu_{\ell+1} & -\delta_{\ell+1}\phi_{\ell+1}\alpha^\nu_\ell \\ 0 & s_\nu \alpha^\nu_\ell \end{pmatrix} : \begin{matrix}\Delta_{\ell+1} \\ \oplus \\ X_\ell\end{matrix} \to \begin{matrix}\Delta_{\ell+2} \\ \oplus \\ X_{\ell+1}\end{matrix}$$

whenever $\ell = e-3, \dots, 0$, and, as verified by a small calculation, by

$$\eta_e(X,\alpha)^\nu_{e-2} = \begin{pmatrix} -s_\nu \partial^X_e \delta^\nu_{e-1} & \alpha^\nu_{e-2}\partial^X_{e-1}\alpha^\nu_{e-2} \end{pmatrix} : \begin{matrix}\Delta_{e-1} \\ \oplus \\ X_{e-2}\end{matrix} \to X_{e-1}$$

whenever $\ell = e-2$.

From Theorem 29 and (2) in Proposition 26 it follows that

$$[X] = [\Sigma^{-1} C_e(X,\alpha)] - [\Sigma^{-1}\Delta_e(X,s)] = [D_e(X,\alpha)] - [\Sigma^{-1}\Delta_e(X,s)]$$

in $K_0\mathsf{P}_e(S\text{-tor})$. The complexes involved in the right end are both concentrated in degrees $e-1,\dots,0$. This gives us the idea of how to construct the inverse of the homomorphism $\mathcal{I}_d$ from the Main Theorem.

Definition 31. By $w_e(X,\alpha)$ we denote the element

$$w_e(X,\alpha) = [D_e(X,\alpha)] - [\Sigma^{-1}\Delta_e(X,s)]$$

in $K_0\mathsf{P}_{e-1}(S\text{-tor})$.

The next section is devoted to showing that $w_e(X,\alpha)$ is independent of the choice of α such that we can simply write $w_e(X)$; that the map $w_e\colon \mathsf{P}_e(S\text{-tor}) \to K_0\mathsf{P}_{e-1}(S\text{-tor})$ induces a homomorphism $\mathcal{W}_e\colon K_0\mathsf{P}_e(S\text{-tor}) \to K_0\mathsf{P}_{e-1}(S\text{-tor})$; and that the $\mathcal{W}_e$'s for different e's can be combined to form an inverse of $\mathcal{I}_d$.

4.3. Proving the Main Theorem.

Notation. Throughout Section 4.3, d denotes a non-negative integer and $S = (S_1,\dots,S_d)$ denotes a d-tuple of multiplicative systems of R. Furthermore, X denotes a fixed complex in $\mathsf{P}_e(S\text{-tor})$ for some integer $e > d$, and α denotes an S-contraction of X with weight $s = (s_1,\dots,s_d) \in S_1 \times \cdots \times S_d$.

We begin with a collection of useful lemmas.

Lemma 32. *If*

$$0 \longrightarrow \overline{Y} \xrightarrow{\overline{\psi}} Y \xrightarrow{\psi} \widetilde{Y} \longrightarrow 0$$

is an exact sequence in $\mathsf{P}_e(S\text{-tor})$, and if $\overline{\beta}$, β and $\widetilde{\beta}$ are S-contractions of $\overline{Y}$, Y and $\widetilde{Y}$, respectively, compatible with the morphisms in the above exact sequence (and thereby all having the same weight t), then there are exact sequences

$$0 \to \Delta_e(\overline{Y},t) \to \Delta_e(Y,t) \to \Delta_e(\widetilde{Y},t) \to 0, \tag{3}$$

$$0 \to C_e(\overline{Y},\overline{\beta}) \to C_e(Y,\beta) \to C_e(\widetilde{Y},\widetilde{\beta}) \to 0 \quad \textit{and} \tag{4}$$

$$0 \to D_e(\overline{Y},\overline{\beta}) \to D_e(Y,\beta) \to D_e(\widetilde{Y},\widetilde{\beta}) \to 0, \tag{5}$$

proving that $w_e(Y,\beta) = w_e(\overline{Y},\overline{\beta}) + w_e(\widetilde{Y},\widetilde{\beta})$ in $K_0\mathsf{P}_{e-1}(S\text{-tor})$. Furthermore, the S-contractions $\delta_e(\overline{Y},t)$, $\delta_e(Y,t)$ and $\delta_e(\widetilde{Y},t)$ are compatible with the morphisms in (3)*; the S-contractions $\mu_e(\overline{Y},\overline{\beta})$, $\mu_e(Y,\beta)$ and $\mu_e(\widetilde{Y},\widetilde{\beta})$ are compatible with the morphisms in* (4)*; and the S-contractions $\eta_e(\overline{Y},\overline{\beta})$, $\eta_e(Y,\beta)$ and $\eta_e(\widetilde{Y},\widetilde{\beta})$ are compatible with the morphisms in* (5).

Proof. According to the assumption, there is an exact sequence of modules

$$0 \longrightarrow \overline{Y}_e \xrightarrow{\overline{\psi}_e} Y_e \xrightarrow{\psi_e} \widetilde{Y}_e \longrightarrow 0,$$

which immediately induces the exact sequence in (3), because $\overline{\psi}_e$ and ψ_e clearly commute with each entry of the differentials in $\Delta_e(\overline{Y},t)$, $\Delta_e(Y,t)$ and $\Delta_e(\widetilde{Y},t)$. Since $\overline{\psi}_e$ and ψ_e also commute with each entry of the the S-contractions $\delta_e(\overline{Y},t)$, $\delta_e(Y,t)$ and $\delta_e(\widetilde{Y},t)$, these must be compatible with the morphisms in the sequence. In addition, the compatibility of the S-contractions $\overline{\beta}$, β and $\widetilde{\beta}$ with the morphisms $\overline{\psi}$ and ψ means that $\overline{\psi}_e\phi_e(\overline{Y},\overline{\beta})^i_\ell = \phi_e(Y,\beta)^i_\ell\overline{\psi}_\ell$ and $\psi_e\phi_e(Y,\beta)^i_\ell = \phi_e(\widetilde{Y},\widetilde{\beta})^i_\ell\psi_\ell$ for each $\ell \in \mathbb{Z}$ and $i \in \Upsilon(e-\ell)$, and hence that there is a commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & \overline{Y} \arrow[r] \arrow[d, "\phi_e(\overline{Y},\overline{\beta})"] & Y \arrow[r] \arrow[d, "\phi_e(Y,\beta)"] & \widetilde{Y} \arrow[r] \arrow[d, "\phi_e(\widetilde{Y},\widetilde{\beta})"] & 0 \\ 0 \arrow[r] & \Delta_e(\overline{Y},s) \arrow[r] & \Delta_e(Y,s) \arrow[r] & \Delta_e(\widetilde{Y},s) \arrow[r] & 0 \end{tikzcd} \tag{6}$$

From this we induce the exact sequence of the mapping cones in (4). Straightforward calculation easily verifies that the compatibility of the S-contractions $\overline{\beta}$, β and $\widetilde{\beta}$ with the morphisms $\overline{\psi}$ and ψ, the compatibility of the S-contractions $\delta_e(\overline{Y},t)$, $\delta_e(Y,t)$ and $\delta_e(\widetilde{Y},t)$ with the morphisms in (3) and the commutativity of diagram (6) imply that the S-contractions $\mu_e(\overline{Y},\overline{\beta})$, $\mu_e(Y,\beta)$ and $\mu_e(\widetilde{Y},\widetilde{\beta})$ are compatible with the morphisms in (4).

We now claim that the exact sequence in (4) induces the exact sequence in (5). To see this, let $\overline{B}$, B and $\widetilde{B}$ denote the exact complexes $0 \to \overline{Y}_e \to \overline{Y}_e \to 0$, $0 \to Y_e \to Y_e \to 0$ and $0 \to \widetilde{Y}_e \to \widetilde{Y}_e \to 0$ from Theorem 29, concentrated in degrees e and $e-1$. These three complexes come together in a short exact sequence $0 \to \overline{B} \to B \to \widetilde{B} \to 0$, induced by the short exact sequence $0 \to \overline{Y}_e \to Y_e \to \widetilde{Y}_e \to 0$. We claim that there is a commutative diagram

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & \overline{B} \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & \widetilde{B} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \Sigma^{-1}C_e(\overline{Y},\overline{\beta}) \arrow[r] \arrow[d] & \Sigma^{-1}C_e(Y,\beta) \arrow[r] \arrow[d] & \Sigma^{-1}C_e(\widetilde{Y},\widetilde{\beta}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & D_e(\overline{Y},\overline{\beta}) \arrow[r] \arrow[d] & D_e(Y,\beta) \arrow[r] \arrow[d] & D_e(\widetilde{Y},\widetilde{\beta}) \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

The columns are exact according to Theorem 29 and the top rectangles are readily verified to be commutative. A little diagram chase now shows that we can use the morphisms in the middle row to induce the morphisms in the bottom row, making the entire diagram commutative by construction. As we have seen, the two top rows are exact, so the exactness of the bottom row follows from the 9-lemma applied in each degree. This establishes the exact sequence in (5). Once again, straightforward calculation demonstrates that the S-contractions $\eta_e(\overline{Y},\overline{\beta})$, $\eta_e(Y,\beta)$ and $\eta_e(\widetilde{Y},\widetilde{\beta})$ are compatible with the morphisms in (5).

From (3) and (5), we now obtain that

$$\begin{aligned} w_e(Y,\beta) &= [D_e(Y,\beta)] - [\Sigma^{-1}\Delta_e(Y,t)] \\ &= [D_e(\overline{Y},\overline{\beta})] + [D_e(\widetilde{Y},\widetilde{\beta})] - [\Sigma^{-1}\Delta_e(\overline{Y},t)] - [\Sigma^{-1}\Delta_e(\widetilde{Y},t)] \\ &= w_e(\overline{Y},\overline{\beta}) + w_e(\widetilde{Y},\widetilde{\beta}), \end{aligned}$$

and the proof is complete. □

Lemma 33. *If X is exact, then $w_e(X,\alpha) = 0$ in $K_0\mathsf{P}_{e-1}(S\text{-tor})$.*

Proof. Let $\widetilde{\partial}_{e-1}$ denote the inclusion map $\operatorname{im}\partial^X_{e-1} \hookrightarrow X_{e-2}$, and let $\widetilde{X}$ denote the complex

$$0 \longrightarrow \operatorname{im}\partial^X_{e-1} \xrightarrow{\widetilde{\partial}_{e-1}} X_{e-2} \xrightarrow{\partial^X_{e-2}} X_{e-3} \longrightarrow \cdots \longrightarrow X_1 \xrightarrow{\partial^X_1} X_0 \longrightarrow 0$$

concentrated in degrees $e-1, \dots 0$. Since X is exact, $\widetilde{X}$ is exact, and it follows that $\operatorname{im}\partial^X_{e-1}$ is projective, and hence that $\widetilde{X}$ is a complex in $\mathsf{P}_{e-1}(S\text{-tor})$.

Letting B denote the exact complex $0 \to X_e \xrightarrow{\mathrm{id}} X_e \to 0$ from Theorem 29, there is an exact sequence

$$\begin{tikzcd}
 & 0 \arrow[r] & B \arrow[r] & X \arrow[r] & \widetilde{X} \arrow[r] & 0 \\
 & & \| & \| & \| & \\
\textit{degree} & & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
e & 0 \arrow[r] & X_e \arrow[r, "\mathrm{id}_{X_e}"] \arrow[d, "\mathrm{id}_{X_e}"] & X_e \arrow[r] \arrow[d, "\partial^X_e"] & 0 \arrow[r] \arrow[d] & 0 \\
e-1 & 0 \arrow[r] & X_e \arrow[r, "\partial^X_{e-1}"] \arrow[d] & X_{e-1} \arrow[r, "\partial^X_{e-1}"] \arrow[d, "\partial^X_{e-1}"] & \operatorname{im}\partial^X_{e-1} \arrow[r] \arrow[d, "\widetilde{\partial}_{e-1}"] & 0 \\
e-2 & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & X_{e-2} \arrow[r, "\mathrm{id}_{X_{e-2}}"] \arrow[d] & X_{e-2} \arrow[r] \arrow[d] & 0 \\
\vdots & & \vdots & \vdots & \vdots &
\end{tikzcd}$$

and we claim that there is a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Sigma^{-1}\Delta_e(X,s) \arrow[r] \arrow[d] & \Sigma^{-1}C_e(X,\alpha) \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Sigma^{-1}\Delta_e(X,s) \arrow[r] \arrow[d] & D_e(X,\alpha) \arrow[r] \arrow[d] & \widetilde{X} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The columns are exact (the middle one according to Theorem 29), and the top rectangles are readily verified to be commutative. A little diagram chase shows that we can use the morphisms in the middle row to induce the morphisms in the bottom row, so that the entire diagram is commutative by construction. Now, the two top rows are exact, so the exactness of the bottom row follows from the 9-lemma applied in each degree. Thus, we have constructed an exact sequence

$$0 \to \Sigma^{-1}\Delta_e(X,s) \to D_e(X,\alpha) \to \widetilde{X} \to 0 \tag{7}$$

of complexes in $\mathsf{P}_{e-1}(S\text{-tor})$. Since $\widetilde{X}$ is exact, it follows that

$$w_e(X,\alpha) = [D_e(X,\alpha)] - [\Sigma^{-1}\Delta_e(X,s)] = [\widetilde{X}] = 0$$

in $K_0\mathsf{P}_{e-1}(S\text{-tor})$ as desired. □

In the next lemma and the theorem that follows, we shall work with a number of similar Koszul complexes. We therefore introduce some convenient notation.

Definition 34. For $r \in S_1$, let $\Delta(r) \stackrel{\text{def}}{=} \Delta_e(X,(r,s_2,\dots,s_d))$; hence, in particular, $\Delta(s_1) = \Delta_e(X,s)$.

Lemma 35. *Suppose $r, r' \in S_1$, and define homomorphisms*

$$\pi(r,r')_\ell \colon \Delta(rr')_\ell \to \Delta(r)_\ell \quad \text{and} \quad \xi(r,r')\colon \Delta(r)_\ell \to \Delta(rr')_\ell$$

for each $\ell \in \mathbb{Z}$ by the fact that their (i',i)-entries for $i, i' \in \Upsilon(e-\ell)$ are

$$(\pi(r,r')_\ell)_{i',i} = \begin{cases} 0, & \text{if } i \neq i', \\ \mathrm{id}_{X_e}, & \text{if } i = i' \text{ and } 1 \in i, \\ r'\,\mathrm{id}_{X_e}, & \text{if } i = i' \text{ and } 1 \notin i, \end{cases}$$

and

$$(\xi(r,r')_\ell)_{i',i} = \begin{cases} 0, & \text{if } i \neq i', \\ r'\,\mathrm{id}_{X_e}, & \text{if } i = i' \text{ and } 1 \in i, \\ \mathrm{id}_{X_e}, & \text{if } i = i' \text{ and } 1 \notin i. \end{cases}$$

Then $\pi(r,r') = (\pi(r,r')_\ell)_{\ell\in\mathbb{Z}}$ is a morphism of complexes $\Delta(rr') \to \Delta(r)$ and $\xi(r,r') = (\xi(r,r')_\ell)_{\ell\in\mathbb{Z}}$ is a morphism of complexes $\Delta(r) \to \Delta(rr')$.

Proof. Assume that $i \in \Upsilon(e-\ell)$ and $j \in \Upsilon(e-\ell+1)$. A direct calculation then shows that the (j,i)-entries of $\partial_\ell^{\Delta(r)}\pi(r,r')_\ell$ and $\pi(r,r')_{\ell-1}\partial_\ell^{\Delta(rr')}$ are both given by

$$\begin{array}{cl} 0, & \text{if } j \not\supseteq i, \\ (-1)^{u+1}s_{j_u}\,\mathrm{id}_{X_e}, & \text{if } j\backslash i = \{j_u\} \text{ and } 1 \in i, \\ (-1)^{u+1}s_{j_u}r'\,\mathrm{id}_{X_e}, & \text{if } j\backslash i = \{j_u\} \neq \{1\} \text{ and } 1 \notin i, \text{ and} \\ rr'\,\mathrm{id}_{X_e}, & \text{if } j\backslash i = \{j_u\} = \{1\} \text{ and } 1 \notin i. \end{array}$$

This proves that $\pi(r,r')$ is a morphism of complexes.

Similarly, a direct calculation shows that the (j,i)-entries of $\partial_\ell^{\Delta(rr')}\xi(r,r')_\ell$ and $\xi(r,r')_{\ell-1}\partial_\ell^{\Delta(r)}$ are both given by

$$\begin{array}{cl} 0, & \text{if } j \not\supseteq i, \\ (-1)^{u+1}s_{j_u}r'\,\mathrm{id}_{X_e}, & \text{if } j\backslash i = \{j_u\} \text{ and } 1 \in i, \\ (-1)^{u+1}s_{j_u}\,\mathrm{id}_{X_e}, & \text{if } j\backslash i = \{j_u\} \neq \{1\} \text{ and } 1 \notin i, \text{ and} \\ rr'\,\mathrm{id}_{X_e}, & \text{if } j\backslash i = \{j_u\} = \{1\} \text{ and } 1 \notin i. \end{array}$$

This proves that $\xi(r,r')$ is a morphism of complexes. □

We are now ready to take the first step in proving that $w_e(X,\alpha)$ is independent of the S-contraction α.

Theorem 36. *Suppose that $t = (t_1,\dots,t_d) \in S_1 \times \cdots \times S_d$ and consider the S-contraction $t\alpha = (t_1\alpha^1,\dots,t_d\alpha^d)$ of X with weight $st = (s_1t_1,\dots,s_dt_d)$. Then $w_e(X,t\alpha) = w_e(X,\alpha)$ in $K_0\mathsf{P}_{e-1}(S\text{-tor})$.*

Proof. If only we can show the equation in the case where $t_\nu = 1$ for all but one of the ν's, then the equation follows since

$$t\alpha = (t_1,\dots,t_d)\alpha = (t_1,1\dots,1)\cdots(1,\dots,1,t_d)\alpha.$$

We will therefore assume that $t = (t_1,1,\dots,1)$; the other cases follow similarly (since we can permute the S_ν's).

To show the desired equation, it suffices to prove that the following equations hold in $K_0\mathsf{P}_{e-1}(S\text{-tor})$.

$$[\Sigma^{-1}\Delta(s_1t_1)] = [\Sigma^{-1}\Delta(s_1)] + [\Sigma^{-1}\Delta(t_1)]. \tag{8}$$

$$[D_e(X,t\alpha)] = [D_e(X,\alpha)] + [\Sigma^{-1}\Delta(t_1)]. \tag{9}$$

Since $\Delta(1)$ is exact (being the Koszul complex of a sequence involving a unit), the first equation follows if we can show that there is an exact sequence

$$0 \longrightarrow \Delta(s_1) \xrightarrow{\left(\begin{smallmatrix}\pi(1,s_1)\\ \xi(s_1,t_1)\end{smallmatrix}\right)} \begin{matrix}\Delta(1)\\ \oplus \\ \Delta(s_1t_1)\end{matrix} \xrightarrow{\left(\begin{smallmatrix}-\xi(1,t_1) & \pi(t_1,s_1)\end{smallmatrix}\right)} \Delta(t_1) \longrightarrow 0.$$

The two matrices clearly define morphisms of complexes, since $\pi(r,r')$ and $\xi(r,r')$ are morphisms of complexes for $r,r' \in S_1$ according to Lemma 35. Exactness at $\Delta(s_1)$ and $\Delta(t_1)$ is clear since there is always one identity map involved in either of $\pi(r,r')$ and $\xi(r,r')$ for $r,r' \in S_1$. Furthermore, $\xi(1,t_1)\pi(1,s_1)$ as well as $\pi(t_1,s_1)\xi(s_1,t_1)$ are defined in degree ℓ by the fact that their (i,i')-entries for $i,i' \in \Upsilon(e-\ell)$ are

$$\begin{array}{cl} 0, & \text{if } i \neq i', \\ t_1\,\mathrm{id}_{X_e}, & \text{if } i = i' \text{ and } 1 \in i, \text{ and} \\ s_1\,\mathrm{id}_{X_e}, & \text{if } i = i' \text{ and } 1 \notin i. \end{array}$$

To show the exactness of the sequence above, it therefore only remains to show that, for each $\ell \in \mathbb{Z}$, the kernel in degree ℓ of the second morphism is contained in the image in degree ℓ of the first. Since all (i,i')-entries of the maps involved are trivial except when $i = i'$, it suffices to consider an element (x,y) in the i-entry $\Delta(1)^i_\ell \oplus \Delta(s_1t_1)^i_\ell$ of the ℓ'th module of $\Delta(1) \oplus \Delta(s_1t_1)$. So suppose that such an element is in the kernel of the map in degree ℓ of the second morphism. If $1 \in i$, this means that $t_1x = y$, and in this case (x,y) is the image of x under the map in degree ℓ of the first morphism. If $1 \notin i$, it means that $x = s_1y$, and in this case (x,y) is the image of y under the map in degree ℓ of the first morphism. In either case, (x,y) is in the image of the map in degree ℓ of the first morphism, and hence the sequence is exact and equation (8) has been proven.

Moving on to equation (9), we first define for each $\ell \in \mathbb{Z}$ a homomorphism $\gamma_{\ell-1}\colon X_{\ell-1} \to \Delta(1)_\ell$ by letting its i'th entry for $i \in \Upsilon(e-\ell)$ be

$$\gamma^i_{\ell-1} = \begin{cases} 0, & \text{if } 1 \in i, \\ \alpha^{i_{e-\ell}}_{e-1}\cdots\alpha^{i_1}_{\ell}\alpha^{1}_{\ell-1}, & \text{if } 1 \notin i. \end{cases}$$

Another way of writing this is

$$\gamma_{\ell-1} = \coprod_{\substack{i\in\Upsilon(e-\ell)\\ 1\notin i}} \alpha^{i_{e-\ell}}_{e-1}\cdots\alpha^{i_1}_{\ell}\alpha^{1}_{\ell-1}.$$

We now claim that there are morphisms

$$\Phi\colon C_e(X,\alpha) \longrightarrow \begin{matrix}\Delta(1)\\ \oplus\\ C_e(X,t\alpha)\end{matrix} \quad \text{and} \quad \Psi\colon \begin{matrix}\Delta(1)\\ \oplus\\ C_e(X,t\alpha)\end{matrix} \longrightarrow \Delta(t_1)$$

given in degree ℓ by

$$\Phi_\ell = \begin{pmatrix} \pi(1,s_1)_\ell & \gamma_{\ell-1} \\ \xi(s_1,t_1)_\ell & 0 \\ 0 & \mathrm{id}_{X_{\ell-1}} \end{pmatrix} : \begin{matrix}\Delta(s_1)_\ell\\ \oplus\\ X_{\ell-1}\end{matrix} \longrightarrow \begin{matrix}\Delta(1)_\ell\\ \oplus\\ \Delta(s_1t_1)_\ell\\ \oplus\\ X_{\ell-1}\end{matrix}$$

and

$$\Psi_\ell = \begin{pmatrix} -\xi(1,t_1)_\ell & \pi(t_1,s_1)_\ell & \xi(1,t_1)_\ell\gamma_{\ell-1} \end{pmatrix} : \begin{matrix} \Delta(1)_\ell \\ \oplus \\ \Delta(s_1t_1)_\ell \\ \oplus \\ X_{\ell-1} \end{matrix} \longrightarrow \Delta(t_1)_\ell.$$

Proving that Φ and Ψ indeed are morphisms of complexes means proving that

$$\begin{pmatrix} \partial^{\Delta(1)}_{\ell+1} & 0 & 0 \\ 0 & \partial^{\Delta(s_1t_1)}_{\ell+1} & \phi_e(X,t\alpha)_\ell \\ 0 & 0 & -\partial^X_\ell \end{pmatrix} \Phi_{\ell+1} = \Phi_\ell \begin{pmatrix} \partial^{\Delta(s_1)}_{\ell+1} & \phi_e(X,\alpha)_\ell \\ 0 & -\partial^X_\ell \end{pmatrix}$$

and

$$\partial^{\Delta(t_1)}_{\ell+1}\Psi_{\ell+1} = \Psi_\ell \begin{pmatrix} \partial^{\Delta(1)}_{\ell+1} & 0 & 0 \\ 0 & \partial^{\Delta(s_1t_1)}_{\ell+1} & \phi_e(X,t\alpha)_\ell \\ 0 & 0 & -\partial^X_\ell \end{pmatrix}$$

for all $\ell \in \mathbb{Z}$. Since we already know from Lemma 35 that $\pi(r,u)$ and $\xi(r,u)$ are morphisms for $r,u \in S_1$, proving the above equations comes down to showing that the following hold for all $\ell \in \mathbb{Z}$:

$$\pi(1,s_1)_\ell\phi_e(X,\alpha)_\ell = \partial^{\Delta(1)}_{\ell+1}\gamma_\ell + \gamma_{\ell-1}\partial^X_\ell; \tag{10}$$

$$\phi_e(X,t\alpha)_\ell = \xi(s_1,t_1)_\ell\phi_e(X,\alpha)_\ell; \quad \text{and} \tag{11}$$

$$\pi(t_1,s_1)_\ell\phi_e(X,t\alpha)_\ell = \xi(1,t_1)_\ell\gamma_{\ell-1}\partial^X_\ell + \partial^{\Delta(t_1)}_{\ell+1}\xi(1,t_1)_{\ell+1}\gamma_\ell. \tag{12}$$

We verify (10) by brute force, calculating on the right hand side of the equation:

$$\begin{aligned}
\partial^{\Delta(1)}_{\ell+1}\gamma_\ell + \gamma_{\ell-1}\partial^X_\ell &= \partial^{\Delta(1)}_{\ell+1} \coprod_{\substack{j\in\Upsilon(e-\ell-1)\\ 1\notin j}} \alpha^{j_{e-\ell-1}}_{e-1}\cdots\alpha^{j_1}_{\ell+1}\alpha^1_\ell \\
&\quad + \coprod_{\substack{i\in\Upsilon(e-\ell)\\ 1\notin i}} \alpha^{i_{e-\ell}}_{e-1}\cdots\alpha^{i_1}_\ell\alpha^1_{\ell-1}\partial^X_\ell \\
&= \coprod_{\substack{i\in\Upsilon(e-\ell)\\ 1\in i}} \alpha^{i_{e-\ell}}_{e-1}\cdots\alpha^{i_2}_{\ell+1}\alpha^1_\ell \\
&\quad + \coprod_{\substack{i\in\Upsilon(e-\ell)\\ 1\notin i}} \Big(\sum_{u=1}^{e-\ell}(-1)^{u+1}s_{i_u}\alpha^{i_{e-\ell}}_{e-1}\cdots\alpha^{i_{u+1}}_{\ell+u}\alpha^{i_{u-1}}_{\ell+u-1}\cdots\alpha^{i_1}_{\ell+1}\alpha^1_\ell\Big) \\
&\quad + \coprod_{\substack{i\in\Upsilon(e-\ell)\\ 1\notin i}} \alpha^{i_{e-\ell}}_{e-1}\cdots\alpha^{i_1}_\ell\alpha^1_{\ell-1}\partial^X_\ell \\
&= \coprod_{\substack{i\in\Upsilon(e-\ell)\\ 1\in i}} \alpha^{i_{e-\ell}}_{e-1}\cdots\alpha^{i_1}_\ell \\
&\quad + \coprod_{\substack{i\in\Upsilon(e-\ell)\\ 1\notin i}} \alpha^{i_{e-\ell}}_{e-1}\cdots\alpha^{i_1}_\ell(\partial^X_{\ell+1}\alpha^1_\ell + \alpha^1_{\ell-1}\partial^X_\ell) \\
&= \coprod_{\substack{i\in\Upsilon(e-\ell)\\ 1\in i}} \alpha^{i_{e-\ell}}_{e-1}\cdots\alpha^{i_1}_\ell + \coprod_{\substack{i\in\Upsilon(e-\ell)\\ 1\notin i}} s_1\alpha^{i_{e-\ell}}_{e-1}\cdots\alpha^{i_1}_\ell \\
&= \pi(1,s_1)_\ell\phi_e(X,\alpha)_\ell.
\end{aligned}$$

Here, the third equality follows from (1) in Proposition 23. This proves the equation in (10). The equation in (11) is clear, since

$$\xi(s_1,t_1)_\ell \phi_e(X,\alpha)_\ell = \coprod_{\substack{i\in\Upsilon(e-\ell)\\ 1\in i}} t_1\phi_e(X,\alpha)^i_\ell + \coprod_{\substack{i\in\Upsilon(e-\ell)\\ 1\notin i}} \phi_e(X,\alpha)^i_\ell = \phi_e(X,t\alpha).$$

To prove that the equation in (12) holds, we apply (10) to the right side of (12):

$$\begin{aligned}\xi(1,t_1)_\ell\gamma_{\ell-1}\partial^X_\ell + \partial^{\Delta(t_1)}_{\ell+1}\xi(1,t_1)_{\ell+1}\gamma_\ell \\ &= \xi(1,t_1)_\ell(\gamma_{\ell-1}\partial^X_\ell + \partial^{\Delta(1)}_{\ell+1}\gamma_\ell) \\ &= \xi(1,t_1)_\ell\pi(1,s_1)_\ell\phi_e(X,\alpha)_\ell.\end{aligned}$$

In contrast, applying (11) to the left side of (12) yields

$$\pi(t_1,s_1)_\ell\phi_e(X,t\alpha)_\ell = \pi(t_1,s_1)_\ell\xi(s_1,t_1)_\ell\phi_e(X,\alpha)_\ell,$$

so proving equation (12) merely requires showing that

$$\xi(1,t_1)_\ell\pi(1,s_1)_\ell = \pi(t_1,s_1)_\ell\xi(s_1,t_1)_\ell. \tag{13}$$

This, however, follows since, for $i,i'\in\Upsilon(e-\ell)$, both sides of (13) have (i,i')-entries given by

$$\begin{matrix} 0, & \text{if } i\neq i', \\ s_1\,\mathrm{id}_{X_e}, & \text{if } i=i' \text{ and } 1\notin i, \text{ and} \\ t_1\,\mathrm{id}_{X_e}, & \text{if } i=i' \text{ and } 1\in i. \end{matrix}$$

Thus, we have verified equation (12), and we conclude that Φ and Ψ are morphisms of complexes.

We now claim that there is a short exact sequence

$$0 \longrightarrow C_e(X,\alpha) \xrightarrow{\Phi} \begin{matrix}\Delta(1)\\ \oplus \\ C_e(X,t\alpha)\end{matrix} \xrightarrow{\Psi} \Delta(t_1) \longrightarrow 0. \tag{14}$$

To see that the sequence is exact at $C_e(X,\alpha)$, suppose that, for some $\ell\in\mathbb{Z}$, the element $(x,y)\in\Delta(s_1)_\ell\oplus X_{\ell-1} = C_e(X,\alpha)_\ell$ maps to 0 under Φ_ℓ: that is,

$$0 = \begin{pmatrix}\pi(1,s_1)_\ell & \gamma_{\ell-1}\\ \xi(s_1,t_1)_\ell & 0 \\ 0 & \mathrm{id}_{X_{\ell-1}}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\pi(1,s_1)_\ell(x)+\gamma_{\ell-1}(y)\\ \xi(s_1,t_1)_\ell(x)\\ y\end{pmatrix}.$$

It immediately follows that $y=0$, and we are left with the equations $\pi(1,s_1)_\ell(x) = \xi(s_1,t_1)_\ell(x) = 0$ which imply that $x=0$. Thus, Φ_ℓ is injective and (14) is exact at $C_e(X,\alpha)$.

To see that the sequence is exact at $\Delta(t_1)$, suppose that $x\in\Delta(t_1)^i_\ell$ for some $\ell\in\mathbb{Z}$ and $i\in\Upsilon(e-\ell)$. Then, if $1\in i$,

$$\begin{pmatrix}-\xi(1,t_1)_\ell & \pi(t_1,s_1)_\ell & \xi(1,t_1)_\ell\gamma_{\ell-1}\end{pmatrix}\begin{pmatrix}0\\x\\0\end{pmatrix} = x,$$

and if $1\notin i$,

$$\begin{pmatrix}-\xi(1,t_1)_\ell & \pi(t_1,s_1)_\ell & \xi(1,t_1)_\ell\gamma_{\ell-1}\end{pmatrix}\begin{pmatrix}-x\\0\\0\end{pmatrix} = x.$$

In either case, x is in the image of Ψ_ℓ, and we conclude that Ψ_ℓ is surjective and that (14) is exact at $\Delta(t_1)$.

Equation (13) clearly shows that $\Psi\Phi=0$, so to show the exactness of (14), it only remains verify that the kernel of Ψ_ℓ is contained in the image of Φ_ℓ for all

$\ell \in \mathbb{Z}$. So suppose that $(x, y, z) \in \Delta(1)_\ell \oplus \Delta(s_1t_1)_\ell \oplus X_{\ell-1} = (\Delta(1) \oplus C_e(X, t\alpha))_\ell$ maps to 0 under Ψ_ℓ: that is,

$$-\xi(1, t_1)_\ell(x) + \pi(t_1, s_1)_\ell(y) + \xi(1, t_1)_\ell\gamma_{\ell-1}(z) = 0.$$

Here $x = (x_i)_{i\in\Upsilon(e-\ell)}$ and $y = (y_i)_{i\in\Upsilon(e-\ell)}$ are $\Upsilon(e-\ell)$-tuples, so the above equation states that, for $i \in \Upsilon(e - \ell)$,

$$\begin{aligned} -t_1x_i + y_i = 0, &\quad \text{when } 1 \in i, \text{ and} \\ -x_i + s_1y_i + \gamma^i_{\ell-1}(z) = 0, &\quad \text{when } 1 \notin i. \end{aligned}$$

Now let $w = (w_i)_{i\in\Upsilon(e-\ell)} \in \Delta(s_1)_\ell$ be defined by $w_i = x_i$ whenever $1 \in i$ and $w_i = y_i$ whenever $1 \notin i$. Then

$$\begin{aligned} \begin{pmatrix} \pi(1, s_1)_\ell & \gamma_{\ell-1} \\ \xi(s_1, t_1)_\ell & 0 \\ 0 & \mathrm{id}_{X_{\ell-1}} \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix} &= \begin{pmatrix} \pi(1, s_1)_\ell(w) + \gamma_{\ell-1}(z) \\ \xi(s_1, t_1)_\ell(w) \\ z \end{pmatrix} \\ &= \begin{pmatrix} \sum\limits_{\substack{i\in\Upsilon(e-\ell)\\ 1\in i}} x_i + \sum\limits_{\substack{i\in\Upsilon(e-\ell)\\ 1\notin i}} (s_1y_i + \gamma^i_{\ell-1}(z)) \\ \sum\limits_{\substack{i\in\Upsilon(e-\ell)\\ 1\in i}} t_1x_i + \sum\limits_{\substack{i\in\Upsilon(e-\ell)\\ 1\notin i}} y_i \\ z \end{pmatrix} \\ &= \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \end{aligned}$$

This proves that (x, y, z) is in the image of Φ_ℓ. We have now proved that (14) is exact.

Denoting by B the exact complex $0 \to X_e \xrightarrow{\mathrm{id}} X_e \to 0$, we now claim that there is a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & B \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Sigma^{-1}C_e(X,\alpha) \arrow[r] \arrow[d] & \begin{matrix}\Sigma^{-1}\Delta(1)\\ \oplus \\ \Sigma^{-1}C_e(X,t\alpha)\end{matrix} \arrow[r] \arrow[d] & \Sigma^{-1}\Delta(t_1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & D_e(X,\alpha) \arrow[r] \arrow[d] & \begin{matrix}\Sigma^{-1}\Delta(1)\\ \oplus \\ D_e(X,t\alpha)\end{matrix} \arrow[r] \arrow[d] & \Sigma^{-1}\Delta(t_1) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The columns are exact according to Theorem 29 and the top rectangles are readily verified to be commutative. A little diagram chase shows that we can use the morphisms in the middle row to induce the morphisms in the bottom row, so that the entire diagram is commutative by construction. Now, the top row is clearly exact, and we have just seen that the middle row is exact, so the exactness of the bottom row follows from the 9-lemma applied in each degree. Thus, we have constructed an exact sequence

$$0 \longrightarrow D_e(X,\alpha) \longrightarrow \begin{matrix} \Sigma^{-1}\Delta(1) \\ \oplus \\ D_e(X,t\alpha) \end{matrix} \longrightarrow \Sigma^{-1}\Delta(t_1) \longrightarrow 0$$

in $K_0\mathsf{P}_{e-1}(S\text{-tor})$, and since $\Delta(1)$ is exact, equation (9) follows. This proves the theorem. ☐

We are almost ready to take the final step in proving that $w_e(X,\alpha)$ is independent of the choice of α. But first a lemma.

Lemma 37. *If Y is an exact complex in $\mathsf{P}_{e+1}(S\text{-tor})$ and β is an S-contraction of Y with weight t, then $w_e(D_{e+1}(Y,\beta),\eta_{e+1}(Y,\beta)) \in K_0\mathsf{P}_{e-1}(S\text{-tor})$ does not depend on the choice of β (but still depends on the weight t).*

Proof. Let us consider the complex $\widetilde{Y} \in \mathsf{P}_e(S\text{-tor})$, constructed from Y in the way $\widetilde{X}$ was constructed from X in Lemma 33, and equip $\widetilde{Y}$ with the S-contraction $\widetilde{\beta}$ induced from β in the sense of Theorem 17:

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{im}\partial_e^Y \arrow[r, shift left, "\partial_e^{\widetilde{Y}}"] & Y_{e-1} \arrow[l, shift left, "\partial_e^Y\beta_{e-1}^\nu"] \arrow[r, shift left, "\partial_{e-1}^Y"] & \cdots \arrow[l, shift left, "\beta_{e-2}^\nu"] \arrow[r, shift left] & Y_1 \arrow[l, shift left] \arrow[r, shift left, "\partial_1^Y"] & Y_0 \arrow[l, shift left, "\beta_0^\nu"] \arrow[r] & 0. \end{tikzcd}$$

Recall that there is an S-contraction $\delta_{e+1}(Y,t)$ of $\Delta_{e+1}(Y,t)$ with weight t and an S-contraction $\mu_{e+1}(Y,\beta)$ of $C_{e+1}(Y,\beta)$ with weight t^2. According to Theorem 18, the S-contractions $\Sigma^{-1}t\delta_{e+1}(Y,t)$, $\Sigma^{-1}\mu_{e+1}(Y,\beta)$ and $t\beta$ are compatible with the morphisms in the short exact sequence

$$0 \to \Sigma^{-1}\Delta_{e+1}(Y,t) \to \Sigma^{-1}C_{e+1}(Y,\beta) \to Y \to 0. \tag{15}$$

Now, the S-contraction $\eta_{e+1}(Y,\beta)$ on $D_{e+1}(Y,\beta)$ is induced in the sense of Theorem 17 by the S-contraction $\Sigma^{-1}\mu_{e+1}(Y,\beta)$ on $\Sigma^{-1}C_{e+1}(Y,\beta)$ through the morphism $\Sigma^{-1}C_{e+1}(Y,\beta) \to D_{e+1}(Y,\beta)$. Similarly, as described above, the S-contraction $\widetilde{\beta}$ on $\widetilde{Y}$ is induced in the sense of Theorem 17 by the S-contraction β on Y through the morphism $Y \to \widetilde{Y}$. We claim that this implies that the S-contractions $\Sigma^{-1}t\delta_{e+1}(Y,t)$, $\eta_{e+1}(Y,\beta)$ and $t\widetilde{\beta}$ are compatible with the morphisms in the exact sequence

$$0 \to \Sigma^{-1}\Delta_{e+1}(Y,t) \to D_{e+1}(Y,\beta) \to \widetilde{Y} \to 0$$

from (7) in Lemma 33. This is easy: let $\Delta \stackrel{\text{def}}{=} \Delta_{e+1}(Y,t)$, $C \stackrel{\text{def}}{=} C_{e+1}(Y,\beta)$, $D \stackrel{\text{def}}{=} D_{e+1}(Y,\beta)$, $\delta \stackrel{\text{def}}{=} \delta_{e+1}(Y,t)$, $\mu \stackrel{\text{def}}{=} \mu_{e+1}(Y,\beta)$ and $\eta \stackrel{\text{def}}{=} \eta_{e+1}(Y,\beta)$. Proving, for example, that $\Sigma^{-1}t\delta$ and η are compatible with the morphism $\Sigma^{-1}\Delta \to D$ means proving the commutativity of the bottom rectangle of the following diagram for all $\ell \in \mathbb{Z}$ and $\nu = 1,\dots,d$.

$$\begin{tikzcd} & C_{\ell+1} \arrow[dd] & & C_\ell \arrow[ll, "-\mu_\ell"'] \arrow[dd] \\ \Delta_{\ell+1} \arrow[ur] \arrow[dd, "\mathrm{id}"'] & & \Delta_\ell \arrow[ll, "-t\delta_\ell"', crossing over] \arrow[ur] & \\ & D_\ell & & D_{\ell-1} \arrow[ll, "\eta_{\ell-1}"'] \\ \Delta_{\ell+1} \arrow[ur] & & \Delta_\ell \arrow[ll, "-t\delta_\ell"] \arrow[ur] \arrow[from=uu, crossing over, "\mathrm{id}"] & \end{tikzcd}$$

The top rectangle is commutative since $\Sigma^{-1}t\delta$ and $\Sigma^{-1}\mu$ are compatible with the first morphism in (15), and the back rectangle is commutative since η is induced from $\Sigma^{-1}\mu$ in the sense of Theorem 17. We have constructed the morphism $\Sigma^{-1}\Delta \to D$ by inducing it from $\Sigma^{-1}\Delta \to \Sigma^{-1}C$ via the morphism $\Sigma^{-1}C \to D$, so the rectangles on the left and right side must also be commutative. Thus all rectangles except possibly the bottom one are commutative. Since the vertical maps are all surjective, the bottom rectangle now lifts to the top rectangle, and it follows that the bottom rectangle must be commutative. A similar argument shows that η and $t\widetilde{\beta}$ are compatible with the morphism $D \to \widetilde{Y}$.

Recalling from Lemma 33 that the exactness of Y implies the exactness of $\widetilde{Y}$, we now get, using Lemmas 32 and 33, that

$$\begin{aligned} w_e(D,\eta) &= w_e(\Sigma^{-1}\Delta, \Sigma^{-1}t\delta) + w_e(\widetilde{Y}, t\widetilde{\beta}) \\ &= w_{e+1}(\Sigma^{-1}\Delta, \Sigma^{-1}t\delta), \end{aligned}$$

which does not depend on β (but apparently still depends on t). □

Theorem 38. *The element $w_e(X,\alpha) \in K_0\mathsf{P}_{e-1}(S\text{-tor})$ does not depend on the choice of α (nor on the weight s): that is, if β is an S-contraction of X with weight t, then $w_e(X,\alpha) = w_e(X,\beta)$.*

Proof. We can assume that the weight s of α equals the weight t of β: for if this is not the case, we consider instead the S-contractions $t\alpha$ and $s\beta$ whose weights are both st, and we know from Theorem 36 that $w_e(X,\alpha) = w_e(X,t\alpha)$ and $w_e(X,s\beta) = w_e(X,\beta)$.

Consider the mapping cone $C(\mathrm{id}_X)$ of the identity morphism $\mathrm{id}_X \colon X \to X$ and the canonical short exact sequence

$$0 \to X \to C(\mathrm{id}_X) \to \Sigma X \to 0.$$

According to Theorem 18, the S-contractions $s\beta$, $\beta * \alpha$ and $\Sigma s\alpha$ all have weight s^2 and are compatible with the morphisms in the above sequence.

Now, the above sequence, which is a sequence in $\mathsf{P}_{e+1}(S\text{-tor})$, induces by (5) from Lemma 32 the following exact sequence in $\mathsf{P}_e(S\text{-tor})$:

$$0 \to D_{e+1}(X, s\beta) \to D_{e+1}(C(\mathrm{id}_X), \beta*\alpha) \to D_{e+1}(\Sigma X, \Sigma s\alpha) \to 0.$$

According to the same lemma, the S-contractions $\eta_{e+1}(X,s\beta)$, $\eta_{e+1}(C(\mathrm{id}_X),\beta*\alpha)$ and $\eta_{e+1}(\Sigma X, \Sigma s\alpha)$, which all have weight s^4, are compatible with the morphisms in the above sequence.

In the construction of $D_{e+1}(X,s\beta)$ we have considered X as a complex concentrated in degrees $e+1,\dots,0$. Since X_{e+1} is the zero module, $\Delta_{e+1}(X,s^2)$ is the zero complex and $D_{e+1}(X,s\beta) = X$. Furthermore, it is straightforward to see that $\eta_{e+1}(X,s\beta)$ is the same as $s^3\beta$ considered as an S-contraction of X. It now follows from Theorem 36 and Lemma 32 that

$$\begin{aligned} w_e(X,\beta) &= w_e(X, s^3\beta) \\ &= w_e(D_{e+1}(X,s\beta), \eta_{e+1}(X,s\beta)) \\ &= w_e(D_{e+1}(C(\mathrm{id}_X),\beta*\alpha), \eta_{e+1}(C(\mathrm{id}_X),\beta*\alpha)) \\ &\quad - w_e(D_{e+1}(\Sigma X,\Sigma s\alpha), \eta_{e+1}(\Sigma X, \Sigma s\alpha)). \end{aligned}$$

Since $C(\mathrm{id}_X)$ is exact, Lemma 37 implies that the first term in the above difference does not depend on $\beta*\alpha$ and thereby not on β. The second term does not depend on β either, so it follows that the difference depends only on α. Replacing β by α, we therefore find that $w_e(X,\alpha)$ is equal to the same difference, and hence $w_e(X,\alpha) = w_e(X,\beta)$ as desired. □

Definition 39. In the light of Theorem 38, we shall write $w_e(X)$ to mean $w_e(X,\alpha)$ for any choice of S-contraction α of X.

We have now accomplished the first and hardest task in constructing an inverse to the homomorphism $\mathcal{I}_d$ from the Main Theorem. Our second task is achieved in the theorem below.

Theorem 40. *The map* $w_e\colon \mathsf{P}_e(S\text{-tor}) \to K_0\mathsf{P}_{e-1}(S\text{-tor})$ *induces a group homomorphism* $\mathcal{W}_e\colon K_0\mathsf{P}_e(S\text{-tor}) \to K_0\mathsf{P}_{e-1}(S\text{-tor})$ *defined by* $\mathcal{W}_e([X]) = w_e(X)$ *for* $X \in \mathsf{P}_e(S\text{-tor})$.

Proof. The only thing we need to show is that the relations in $K_0\mathsf{P}_e(S\text{-tor})$ are preserved under the map w_e.

If X is exact, we already know from Lemma 33 that $w_e(X) = 0$. Thus, it only remains to show that, if

$$0 \longrightarrow \overline{X} \xrightarrow{\overline{\psi}} X \xrightarrow{\psi} \widetilde{X} \to 0 \tag{16}$$

is an exact sequence in $\mathsf{P}_e(S\text{-tor})$, then $w_e(X) = w_e(\overline{X}) + w_e(\widetilde{X})$. In this case there exists a morphism $\rho\colon \Sigma^{-1}\widetilde{X} \to \overline{X}$ with the property that its mapping cone $C(\rho)$ is isomorphic to X. Now choose S-contractions $\overline{\alpha}$ and $\widetilde{\alpha}$ for $\overline{X}$ and $\widetilde{X}$, respectively, and let $\overline{s}$ and $\widetilde{s}$ denote the weights of $\overline{\alpha}$ and $\widetilde{\alpha}$, respectively. Recall that $\overline{\alpha} * \Sigma^{-1}\widetilde{\alpha}$ is an S-contraction of $C(\rho)$ with weight $\overline{s}\widetilde{s}$. We now have

$$\begin{aligned} w_e(X) &= w_e(C(\rho)) \\ &= w_e(C(\rho), \overline{\alpha} * \Sigma^{-1}\widetilde{\alpha}) \\ &= w_e(\overline{X}, \widetilde{s}\overline{\alpha}) + w_e(\widetilde{X}, \overline{s}\widetilde{\alpha}) \\ &= w_e(\overline{X}) + w_e(\widetilde{X}), \end{aligned}$$

where the third equality follows from Theorem 18 and Lemma 32. This proves the theorem. □

We are immediately able to show that our homomorphism $\mathcal{W}_e$ in fact is an isomorphism.

Theorem 41. *The group homomorphism*

$$\mathcal{I}'_{e-1}\colon K_0\mathsf{P}_{e-1}(S\text{-tor}) \to K_0\mathsf{P}_e(S\text{-tor})$$

given by $\mathcal{I}'_{e-1}([X]) = [X]$ *is an isomorphism; in fact, the inverse of* $\mathcal{I}'_{e-1}$ *is* $\mathcal{W}_e$.

Proof. If we shift the canonical exact sequence of the mapping cone $C_e(X,\alpha)$ one degree to the right, we get the exact sequence

$$0 \to \Sigma^{-1}\Delta_e(X,s) \to \Sigma^{-1}C_e(X,\alpha) \to X \to 0 \tag{17}$$

in $\mathsf{P}_e(S\text{-tor})$. Theorem 29 showed that there is an exact sequence

$$0 \to B \to \Sigma^{-1}C_e(X,\alpha) \to D_e(X,\alpha) \to 0 \tag{18}$$

in $\mathsf{P}_e(S\text{-tor})$, where B is the exact complex $0 \to X_e \xrightarrow{\mathrm{id}} X_e \to 0$ concentrated in degrees e and $e-1$. From the exact sequences in (17) and (18) it now follows that the following holds in $K_0\mathsf{P}_e(S\text{-tor})$.

$$\begin{aligned} [X] &= [\Sigma^{-1}C_e(X,\alpha)] - [\Sigma^{-1}\Delta_e(X,s)] \\ &= [D_e(X,\alpha)] - [\Sigma^{-1}\Delta_e(X,s)] \\ &= \mathcal{I}'_{e-1}\,\mathcal{W}_e([X]). \end{aligned}$$

Conversely, suppose that Y is a complex in $\mathsf{P}_{e-1}(S\text{-tor})$ and that β is an S-contraction of Y with weight t. Then, considering Y as a complex in $\mathsf{P}_e(S\text{-tor})$, $\Delta_e(Y,t) = 0$ and $D_e(Y,\beta) = Y$, and therefore

$$\begin{aligned}[][Y] &= [D_e(Y,\beta)] - [\Sigma^{-1}\Delta_e(Y,t)] \\ &= \mathcal{W}_e\,\mathcal{I}'_{e-1}([Y]).\end{aligned}$$

Thus, $\mathcal{I}'_{e-1}$ and $\mathcal{W}_e$ are mutually inverse, and the theorem is proved. □

Theorem 42 (Main Theorem)**.** *The group homomorphism*

$$\mathcal{I}_d\colon K_0\mathsf{P}_d(S\text{-tor}) \to K_0\mathsf{P}(S\text{-tor})$$

given by $\mathcal{I}_d([X]) = [X]$ *is an isomorphism.*

Proof. The sequence

$$K_0\mathsf{P}_d(S\text{-tor}) \xrightarrow{\mathcal{I}'_d} K_0\mathsf{P}_{d+1}(S\text{-tor}) \xrightarrow{\mathcal{I}'_{d+1}} \cdots$$

of Abelian groups $K_0\mathsf{P}_f(S\text{-tor})$ and homomorphisms $\mathcal{I}'_f$ for $f \geq d$ is a direct system, and it is straightforward to see that the Grothendieck group $K_0\mathsf{P}(S\text{-tor})$ together with the maps

$$\mathcal{I}_f\colon K_0\mathsf{P}_f(S\text{-tor}) \to K_0\mathsf{P}(S\text{-tor}) \quad \text{for } f \geq d$$

satisfies the universal property required by a direct limit of this system (since $K_0\mathsf{P}(S\text{-tor})$ is generated by complexes concentrated in non-negative degrees). In contrast, since all the homomorphisms $\mathcal{I}'_f$ are isomorphisms according to Theorem 41, the direct limit must be isomorphic to each of the groups $K_0\mathsf{P}_f(S\text{-tor})$ and $\mathcal{I}_f$ must be an isomorphism for each $f \geq d$. □

Exploiting the property of a direct limit, we see that the inverse of $\mathcal{I}_d$ must be the homomorphism $\mathcal{I}_d^{-1}$ making the following diagram commutative.

$$\begin{tikzcd}
K_0\mathsf{P}_f(S\text{-tor}) \arrow[d, "\mathcal{I}'_f"'] \arrow[dr, "\mathcal{I}_f"] \arrow[drr, bend left, "\mathcal{W}_{d+1}\cdots\mathcal{W}_f"] & & \\
 & K_0\mathsf{P}(S\text{-tor}) \arrow[r, dashed, "\mathcal{I}_d^{-1}"] & K_0\mathsf{P}_d(S\text{-tor}) \\
K_0\mathsf{P}_{f+1}(S\text{-tor}) \arrow[ur, "\mathcal{I}_{f+1}"'] \arrow[urr, bend right, "\mathcal{W}_{d+1}\cdots\mathcal{W}_{f+1}"'] & &
\end{tikzcd}$$

It follows that $\mathcal{I}_d^{-1}$ is given for $Y \in \mathsf{P}(S\text{-tor})$ by

$$\mathcal{I}_d^{-1}([Y]) = (-1)^n\,\mathcal{W}_{d+1}\cdots\mathcal{W}_f([\Sigma^n Y])$$

for n and f chosen sufficiently large that $\Sigma^n Y \in \mathsf{P}_f(S\text{-tor})$.

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