On Mortality and Investment Risk in Life Insurance

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Ph.D. Thesis

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Preface

This thesis has been prepared in partial fulfillment of the requirements for the Ph.D. degree at the Laboratory at Actuarial Mathematics, Institute of Mathematical Sciences, University of Copenhagen, Denmark. The work has been carried out in the period from November 2002 to October 2005 under the supervision of Professor Thomas Mikosch, University of Copenhagen, Associate Professor Mogens Steffensen, University of Copenhagen, and Thomas Møller, PFA Pension (Assistant Professor at University of Copenhagen until February 2003).

In the thesis each chapter is self-contained and can be read independently of the rest of the thesis. This structure is chosen to ease the submission of parts of the thesis. The independence has resulted in some notational discrepancies among the different chapters.

The present version differs from the original version submitted for the Ph.D. degree in that a minor number of misprints have been corrected and some statements, in particular in Chapter 8, have been clarified.

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Summary

This thesis is concerned with analyzing the risks faced by a life insurance company. In general life insurance companies are exposed to a large number of financial and insurance risks. Usually these risks are well understood, and models have been developed and studied extensively in the literature. However, some of the risks have received less attention both in the literature and in practice. In this thesis we study the modelling of these risks in detail. An important discipline for life insurance companies is to valuate their liabilities. We apply methods from financial mathematics and in particular the principle of no arbitrage. This principle rests on the reasonable assumption that without any initial capital, it is impossible to obtain a riskfree gain. In complete financial markets this principle leads to unique prices for all possible contracts. However, since life insurance contracts are not traded in the financial market, we study an incomplete market, and in this case the no arbitrage principle is not sufficient to obtain unique arbitrage free prices. Hence, in addition to the no arbitrage principle, we consider the mean-variance indifference pricing principles developed in order to obtain unique prices in incomplete financial markets. In addition to valuating their liabilities, life insurance companies are concerned with possible methods to decrease their risk. In this thesis the main emphasis is on the possibility of hedging the life insurance contracts in the financial market. However, other possibilities are mentioned as well. We apply hedging principles used to determine optimal hedging strategies in incomplete financial markets. Here focus is on the criterion of risk-minimization and the optimal hedging strategies associated with the mean-variance indifference principles. Risk-minimizing strategies have the nice property that they decompose the risk associated with the contracts into a hedgeable and an unhedgeable part.

In the first part of the thesis we consider the problem of determining a fair distribution of assets between the equity capital and the portfolio of insured in the case, where the insurance contracts include a periodic interest rate guarantee. We study a distribution mechanism, where the equity capital is accumulated with a rate of return, which exceeds the riskfree rate, in periods where the combined development of the investment return and the insurance portfolio is favorable. This additional rate represents the price for the guarantee in the accumulation period. We consider an insurance company whose insurance portfolio consists of either capital insurances or pure endowments and a simple financial market given by the complete and arbitrage free Black–Scholes model. Given an investment strategy we apply the principle of no arbitrage to obtain an implicit equation for the fair additional rate of return to the equity capital in periods, when such an additional

rate of return is possible. In the case of a portfolio of pure endowments the equation depends on the market's attitude towards unsystematic mortality risk. The investment strategies considered are: A buy and hold strategy and a strategy with constant relative portfolio weights, both with and without stop-loss in case solvency is threatened.

In the second part we focus on the so-called systematic mortality risk, which is the uncertainty associated with the future mortality intensity. In order to describe this uncertainty we model the mortality intensity as a stochastic process. We note that the relative impact of systematic mortality risk cannot be reduced by increasing the size of the portfolio. Hence, we cannot use the well-established actuarial pricing principle of diversification to price life insurance contracts in the presence of systematic mortality risk. Instead we apply the no arbitrage principle to derive market reserves. Since the life insurance contracts are not traded in the financial market, we do not obtain a unique market reserve. In particular we have that the market reserves depend on the market's attitude towards the systematic mortality risk. In order to obtain a unique reserve we apply the mean-variance indifference pricing principles. We study different methods for the company to lower the exposure to the systematic mortality risk. One possibility is to trade in the financial market. Here, we consider the criteria of risk-minimization and the optimal strategies associated with the mean-variance indifference prices. Alternatively, the company can trade so-called mortality derivatives, i.e. contracts which depend on the development of the mortality intensity. As a last option we discuss the possibility of transferring the systematic mortality risk to the insured by issuing contracts, where the premiums and/or benefits are linked to the development of the mortality intensity.

In practice only bonds with a limited time to maturity are traded in the market. Hence, companies issuing long term contracts are exposed to an uncertainty associated with the initial price of a new bond issued in the market. In the literature this risk is usually ignored, since the bond market is assumed to include bonds with all times to maturity. The third part of thesis is devoted to the modelling of this so-called reinvestment risk. For financial contracts the reinvestment risk is usually non-existing due to the short term of the contracts. However, for life insurance companies this risk is of importance, since life insurance contracts usually are very long term contracts. We propose a discrete-time model for the reinvestment risk. At each trading time a bond matures and a new long term bond is introduced in the market. The entry price of the new bond depends on the prices of existing bonds and a stochastic term independent of the existing bond prices. Within this purely financial model we determine risk-minimizing strategies. Danish legislation force the life insurance companies to value their long term liabilities using a level long term yield curve. In a numerical example we compare this principle to the related principle of a level long term forward rate curve and the financial principle of super-replication. In addition to the discrete-time model, we also propose a continuous-time model with fixed times of issue. Here, the uncertainty of the initial prices of bonds issued in the market is modelled by letting the extension of the forward rate curve be stochastic. In this case we also derive risk-minimizing strategies.

In the fourth and last part, we consider a model including a large number of the risks faced by a life insurance company. In particular, this model includes the systematic mortality

risk and the reinvestment risk. Within this refined model we determine market reserves and mean-variance indifference prices for life insurance contracts. Furthermore the hedging aspect is addressed by the derivation of risk-minimizing strategies and the optimal hedging strategies associated with the mean-variance indifference principles. A numerical study of market reserves and the alternative principles of a level long term yield curve, a level long term forward rate curve and super-replication of reinvestment risk is carried out. This numerical study also includes the risk measures of Value at Risk and tail conditional expectation.

Resumé

I denne afhandling analyseres de forskellige risici som et livsforsikringsselskab er eksponeret for. Generelt er livsforsikringsselskaber eksponeret for et stort antal finansielle og forsikringsmæssige risici. Som regel er der et udbredt kendskab til og en indgående forståelse af disse risici, og der er udviklet modeller, som er studeret detaljeret i litteraturen. Enkelte risici har dog ikke fået samme opmærksomhed, hverken i litteraturen eller i praksis. I denne afhandling foretages et detaljeret studie af modelleringen af disse risici. En vigtig opgave for livsforsikringsselskaber er at værdiansætte deres forpligtigelser. Vi anvender metoder fra finansmatematikken og specielt princippet om fraværet af arbitragemuligheder. Dette princip bygger på den rimelige antagelse om, at man uden startkapital ikke kan opnå en risikofri gevinst. I fuldstændige finansielle markeder fører dette princip til entydige arbitragefri priser for alle kontakter. Da livsforsikringskontrakter ikke handles på det finansielle marked, betragter vi et ufuldstændigt marked, og i dette tilfælde er princippet om fravær af arbitrage ikke tilstrækkeligt til at sikre entydige arbitragefri priser. Vi betragter derfor også mean-variance indifferens prisfastsættelses principper udviklet med henblik på at opnå entydige priser i ufuldstændige finansielle markeder. Udover at værdiansætte deres forpligtigelser er livsforsikringsselskaber optaget af mulige metoder til at mindske deres risiko. I denne afhandling er hovedfokus på muligheden for at hedge (afdække) livsforsikringskontrakter i det finansielle marked, men andre muligheder vil også blive nævnt. Vi anvender afdækningsprincipper, som normalt anvendes til at bestemme optimale handelsstrategier i ufuldstændige finansielle markeder. Her er fokus på kriteriet risiko-minimering og på de optimale handelsstrategier forbundet med mean-variance indifferens principperne. Risiko-minimerende strategier har den pæne egenskab, at de dekomponerer risikoen forbundet med kontrakterne i en del som kan elimineres ved at handle på det finansielle marked, og en del som ikke kan elimineres.

I den første del af afhandlingen betragter vi problemet med at bestemme en fair fordeling af aktiverne mellem egenkapitalen og forsikringsporteføljen i det tilfælde, hvor forsikringskontakterne indeholder en rentegaranti. Vi betragter en fordelingsmekanisme, hvor egenkapitalen forrentes med en rente, der er højere end den risikofri rente i perioder, hvor den samlede udvikling af investeringerne og forsikringsporteføljen er favorabel. Denne ekstra forrentning af egenkapitalen repræsenterer prisen for garantien i perioden. Vi betragter et forsikringsselskab, hvis forsikringsportefølje udelukkende består af enten kapitalforsikringer eller rene overlevelsesforsikringer og et simpelt finansielt marked beskrevet ved den fuldstændige og arbitragefri Black–Scholes model. For en given investeringsstrategi

anvender vi princippet om fravær af arbitrage til at bestemme en implicit ligning for den ekstra forrentning af egenkapitalen i perioder, hvor en sådan ekstra rente er mulig. I tilfældet hvor forsikringsporteføljen består af rene oplevelsesforsikringer, afhænger ligningen af markedets attitude til usystematisk dødsrisiko. Vi betragter følgende investeringsstrategier: En buy and hold strategi og en strategi med konstante relative porteføljevægte. I begge tilfælde betragtes både tilfældet med og uden stop-loss, hvis selskabets solvens er truet.

I den anden del fokuserer vi på den såkaldte systematiske dødsrisiko, som er usikkerheden forbundet med den fremtidige dødelighed. For at kunne beskrive denne usikkerhed modellerer vi dødeligheden som en stokastisk proces. Vi bemærker at den relative effekt af den systematiske dødsrisiko ikke kan reduceres ved at øge størrelsen af forsikringsporteføljen. Vi kan derfor ikke benytte det veletablerede aktuar prisfastsættelsesprincip, diversifikation, til at prisfastsætte livsforsikringskontrakter i forbindelse med systematisk dødsrisiko. I stedet anvender vi princippet om fravær af arbitrage til at udlede markedsreserver. Da livsforsikringskontrakter ikke handles på det finansielle marked, giver dette ikke en entydig markedsreserve. Specielt gælder det, at markedsreserverne afhænger af markedets attitude til systematisk dødsrisiko. For at opnå en entydig reserve anvender vi mean-variance indifferens prisfastsættelses principperne. Vi betragter forskellige metoder for selskabet til at mindske eksponeringen til den systematiske dødsrisiko. En mulighed er at handle i det finansielle marked. Her betragter vi kriteriet risiko-minimering og de optimale handelsstrategier forbundet med mean-variance indifferens priserne. Alternativt kan selskabet handle med såkaldte dødelighedsderivater, som er kontrakter, der afhænger af udviklingen i dødeligheden. Som en sidste mulighed diskuterer vi muligheden for at overføre den systematiske dødsrisiko til de forsikrede ved at udstede kontrakter, hvor præmierne og/eller ydelserne er afhængige af udviklingen i dødeligheden.

I praksis handles kun obligationer med en begrænset løbetid på det finansielle marked. Selskaber, der udsteder kontrakter med lang løbetid, er derfor eksponeret til en usikkerhed forbundet med introduktionsprisen, når nye obligationer udstedes på det finansielle marked. I litteraturen ignoreres denne risiko typisk, da man antager at obligationer med alle løbetider handles i markedet. Den tredje del af afhandlingen behandler modelleringen af denne såkaldte geninvesteringsrisiko. For rent finansielle kontrakter er geninvesteringsrisikoen normalt ikke eksisterende, da de som regel har en kort tidshorisont. For livsforsikringskontrakter er denne risiko imidlertid af stor vigtighed, da livsforsikringskontrakter generelt har en meget lang tidshorisont. Vi opstiller en diskret-tids model for geninvesteringsrisikoen. På ethvert handelstidspunkt udløber en obligation og en ny obligation med lang løbetid udstedes i markedet. Udstedelsesprisen afhænger af prisen på de eksisterende obligationer og et stokastisk led uafhængigt heraf. I denne rent finansielle model bestemmer vi risiko-minimerende strategier. Ifølge dansk lovgivning skal livsforsikringsselskaber anvende en rentekurve, som er flad ved lange løbetider, til at værdiansætte deres forpligtigelser med lang tidshorisont. I et numerisk eksempel sammenligner vi dette princip med det relaterede princip om at anvende en forwardrentekurve, som er flad for lange løbetider, og det finansielle princip super-replikering. Udover diskret-tids modellen opstilles en model i kontinuert tid med faste udstedelsestidspunkter. Her modellerer vi usikkerheden forbundet med udstedelsesprisen på nye obligationer, ved at lade fortsættelsen af forwardrentekurven være stokastisk. I dette tilfælde udledes også risiko-minimerende strategier.

I den fjerde og sidste del betragter vi en model, der inkluderer et stort antal af de risici, som et livsforsikringsselskab er eksponeret for. Specielt indeholder modellen systematisk dødsrisiko og geninvesteringsrisko. I denne forfinede model bestemmes markedsreserver og mean-variance indifferens priser for livsforsikringskontrakter. Yderligere er afdækningsproblematikken belyst ved udledningen af risiko-minimerende strategier og de optimale handelsstrategier forbundet med mean-variance indifferens principperne. Der foretages et numerisk studie af markedsreserver og de alternative principper om en flad rentekurve for lange tidshorisonter, en flad forwardrentekurve for lange tidshorisonter og super-replikering af geninvesteringsrisiko. Dette numeriske studie inkluderer også risikomålene Value at Risk and tail conditional expectation.

Contents

Chapter 1

Introduction

In this thesis we focus on the risks to which an insurance company is exposed when selling life insurance contracts. Here, we use the term risk to describe a source of uncertainty, even though it may lead to a surplus as well as a loss. We are interested in identifying and modelling the sources of risk in order to measure and control the risk of the insurance company, and to value life insurance contracts. The exposition relies heavily on methods from financial mathematics. In particular we apply the no arbitrage principle and methods used for valuation and hedging in incomplete markets.

1.1 Risks in life insurance

A life insurance contract specifies a stream of payments between the insured and the insurance company contingent on some predetermined insurance events. Payments from the insured are called premiums, and payments to the insured are referred to as benefits. The premiums usually consist of a lump sum premium at initiation of the contract and continuous premiums paid until retirement as long as the insured is alive (and active). Standard textbook examples of benefits are: Pure endowment, term insurance and (temporary or whole) life annuity. For an explanation of these insurance contracts and an introduction to life insurance in general, we refer to Gerber (1997), in discrete time, and Norberg (2000), in continuous time.

When entering the contract the qualitative nature of the premiums and benefits is agreed upon by the insured and the insurance company. Furthermore, the insured specifies either the premiums or benefits quantitatively, and it is left to the insurance company to calculate the remaining quantity. Hence, both the quantitative and the qualitative nature of benefits and premiums are stated in the insurance contract. Typically the quantitative specifications serve as a guarantee to the insured leaving the company unable to lower benefits, or equivalently increase premiums, if it observes an adverse development of the financial market and/or the insurance portfolio. Thus, since the company is unable to alter the specifications in the contract in order to take an unfavorable development of the financial market and/or the insurance portfolio into account, it is of importance for the company to understand the risks associated with entering the insurance contract. Hence, the company should be able to identify and adequately model the (major) sources of risk, such that it is able to price the contract correctly. However, an adequate description of the risks is not only of importance when pricing the contract. It is important throughout the course of the contract, both for internal control purposes and for measuring the impact of different scenarios as described in the so-called "traffic light system" introduced by the regulatory authorities in Denmark. Furthermore it is believed that future solvency rules will require the company to constantly monitor and measure the risks of the company. Having measured the risks it is natural for the company to consider methods to reduce the risk, and thereby lower the effect of the different scenarios in the traffic light system (and in the future the solvency requirements). Here, some possibilities are trading in the financial market and purchasing reinsurance.

In this chapter we consider the case where the insured specify the benefits quantitatively, and the company has to calculate the premiums. This is no loss of generality, since the alternative case can be handled similarly. Throughout the chapter we restrict calculations to the case of a portfolio of pure endowments paid by single premiums, since these are the simplest life insurance contract involving a dependence on the death or survival of the insured. In particular this allows us to consider benefits at a fixed time only, such that we avoid considering payment processes. However, all qualitative statements in this chapter hold for payment processes as well.

1.1.1 Types of risk

We focus on two main types of risk for the insurance company: *Financial risk* and *mortality risk*. In the literature mortality risk is sometimes referred to as insurance risk. The company is naturally exposed to other types of risk as well. We mention operational risk and risk associated with future administration costs, such as wages, purchase of computer systems, rent and general maintenance of business operations. The Basel Committee's definition of operational risk is "the risk of losses resulting from inadequate or failed internal processes, people and systems or from external events". Hence, the operational risk covers all losses resulting from errors connected to running the business. This includes both human and system errors. For a detailed description of, and an approach to modelling, operational risk we refer to King (2001) and Cruz (2002).

Here, we further split the financial risk into *equity risk* and *interest rate risk*. Hence, in this exposition we disregard other types of risk, such as credit risk, which is the risk associated with the default of the counterparty in a financial transaction. For a detailed description of credit risk see e.g. Lando (2004). The equity risk covers the uncertainty associated with risky investments except bonds, and interest rate risk covers uncertainty associated with future interest rates and hence bond prices. Here, we further divide the interest rate risk into *standard interest rate risk*, which is uncertainty associated with the development of the currently traded bonds (the currently observable yield curve) and *reinvestment risk*, which measures the additional uncertainty associated with the entry prices, when new bonds are issued in the market. The reinvestment risk is naturally only of interest if bonds with sufficiently long time to maturity are not traded at the time of consideration. This use of the term reinvestment risk differs from the one of e.g. Luenberger (1998), who uses it to describe the risk associated with the unknown rate of return, when currently owned bonds mature in the future, and the capital is reinvested in the bond market. Hence, Luenberger (1998) does not distinguish between whether or not the bonds in which the capital is reinvested were traded at the time of purchase of the first bonds. In our terminology, the reinvestment risk only covers the case, where no bonds with sufficiently long time horizon are traded initially, whereas the risk associated with the future rate of return of bonds presently traded is covered by the standard interest rate risk.

The mortality risk consists of two fundamentally different sources of risk: *Systematic* and *unsystematic mortality risk*. Here, the unsystematic mortality risk refers to the risk associated with the random development of an insurance portfolio with known mortality intensity. From the strong law of large numbers we know that the relative impact of the unsystematic mortality risk is a decreasing function of the number of insured, and if the insurance portfolio is infinitely large, the unsystematic mortality risk is eliminated. Thus, the unsystematic mortality risk is diversifiable. The systematic mortality risk refers to the uncertainty associated with changes in the underlying mortality intensity. Since changes in the underlying mortality intensity affect all insured, the systematic mortality risk is an increasing function of the number of insured with similar contracts. Hence, in contrast to the unsystematic mortality risk the systematic mortality risk is non-diversifiable. However, a reduction (elimination) of the systematic mortality risk is possible, if the company both sells contracts, where the payoff is contingent on survival, and contracts, where the payoff is contingent on death. Note that similar considerations can be made for other transition intensities, e.g. disability, recovery etc. Hence, we can interpret the mortality risk as covering all biometric risks.

1.1.2 A qualitative classification of the types of risk

In order to obtain a qualitative description of the types of risk we classify them according to the exposure of the company. First we concentrate on the contract and classify the types of risk according to whether the company is exposed to the risk as a consequence of entering the contract. Hence, the different types of risk are divided into the following two classes:

- *Contractual risks*: The types of risk to which the insurance company is exposed as a consequence of entering the contract.
- *Non-contractual risks*: The types of risk, which are not contractual risks.

Note that if entering the contract does not expose the company to a type of risk, it is a non-contractual risk. Within the class of contractual risks we further distinguish between whether the company is able to eliminate the type of risk by trading in the financial market. We say that a type of risk can be eliminated if all uncertainty associated with the type of risk can be eliminated by trading in the financial market. To determine whether this is the case, we consider the contingent model where the particular type of risk accounts for all uncertainty. Now all uncertainty can be eliminated if the company can invest a fixed initial amount and trade in the financial market, such that it always has exactly the desired amount. Hence, the class of contractual risks consists of the following sub-classes:

- *Hedgeable contractual risks*: The types of contractual risk for which the company, given a certain fixed initial investment, can eliminate all uncertainty by trading in the financial market.
- *Unhedgeable contractual risks*: The types of contractual risk for which the company, given a certain fixed initial investment, *cannot* eliminate all uncertainty by trading in the financial market.

The definitions above are closely related to the definition of hedging in financial theory, see Section 1.3 for more details. Here, it is important to note that in the contingent model it may be possible to eliminate the so-called short-fall risk, which is the risk of holding insufficient funds, related to an unhedgeable contractual risk by investing a sufficiently large amount at initiation of the contract. However, since the company in this case has a (large) positive probability of holding more than required to cover the benefits, the risk is not eliminated. Hence, one cannot turn an unhedgeable contractual risk into a hedgeable contractual risk by investing a sufficiently large amount. The idea of eliminating the shortfall risk is closely related to so-called super-replicating (super-hedging) strategies, see Section 1.3.1.

The classes and sub-classes above are connected to the contract only, so it can be interpreted as a classification of the risks on the liability side. However, it is important to note that the effect of the different types of risk on the balance sheet depends on both the considered insurance contract and the investment strategy. Hence, in order to correctly describe the exposure of the company to the different types of risk, one should involve the asset side as well. Here, the assets only refer to the assets associated with the liabilities, whereas the assets corresponding to the equity capital is disregarded. The importance of including the assets has also been observed by the life insurance companies, which in general devote a large amount of effort to ALM (asset liability modelling/management). The necessity to involve the asset allocation arises since the value of the assets and liabilities may increase or decrease at the same time. Hence, in some cases the company may be able to reduce a type of unhedgeable contractual risk by traded wisely in the financial market. On the other hand the company may decide not to eliminate the uncertainty associated with a hedgeable contractual risk. It may even expose the balance sheet to non-contractual risks. In order to describe the types of risk to which the insurance company is exposed, when taking the asset allocation into account, we introduce:

- *Business risks*: The types of risk to which the combined balance sheet of the company is exposed.

As noted above it holds that even for a company, which is aware of the contractual risks to which it is exposed, the mere possibility to eliminate or reduce a type of risk by trading in the financial market is not equivalent to the fact that the company actually decides to do so. Hence, in some cases the company exposes the balance sheet to risk(s) that could have been avoided. This behavior can be explained by the fact that the company follows an investment strategy which also focuses on the expected rate of return. In particular, the belief that the long term return is higher on stocks than on bonds encourages many insurance companies to invest in stocks even when the financial risk associated with the contract only consists of interest rate risk. For a specific contract (or portfolio of contracts) and a given investment strategy the business risks consist of the following three sub-classes:

- *Non-hedged hedgeable contractual risks*: The types of hedgeable contractual risk, which the company has not eliminated.

- *Unhedgeable contractual risks*: The types of contractual risk for which the company, given a certain fixed initial investment, *cannot* eliminate all uncertainty by trading in the financial market.

- *Gambling risks*: Non-contractual risks to which the company is exposed as a consequence of the investment strategy.

The method available to the company in order to eliminate/convert a certain type of business risk depends on the sub-class to which the risk belongs. A non-hedged hedgeable contractual risk can by definition be eliminated by trading in the financial market, whereas it is impossible to eliminate an unhedgeable contractual risk once the contract is signed. However, risks, which otherwise would be unhedgeable contractual risks may be transferred to the insured and thus they may be converted into non-contractual risks by designing the contract cleverly. The so-called mortality-linked contracts introduced in Chapter 3 is an example of a type of contracts designed to convert an unhedgeable contractual risk into a non-contractual risk. Here, the systematic mortality risk is transferred from the insurance company to the insured. The gambling risks can naturally be eliminated simply by altering the investment strategy, such that is does not include investments in the assets which expose the company to the non-contractual risk.

The classification of the risks is of importance when pricing and reserving for the contract, as well as for risk management. The contractual risks influence prices and reserves, whereas the business risks influence the sensitivity to the different scenarios in e.g. the traffic light system and in the future possibly the solvency requirements. It could be argued that the investment strategy, and hence the business risks, also should be of importance when pricing the contract, since a company which follows a risky investment strategy has a larger risk of default and hence exposes the policy-holder to a larger credit risk. However, we ignore this aspect, since the traffic light system and the solvency rules essentially should eliminate the credit risk of the policy-holders.

1.1.3 An illustration of risks in life insurance

In order to illustrate the ideas in Sections 1.1.1 and 1.1.2 we now identify and classify (qualitatively) the different types of risk in a simple example. Consider a portfolio of n insured of age x all purchasing a pure endowment of K paid by a single premium π at time 0. Hence, at time 0 the company receives the premium π from each of the insured, such that the total premiums received are $n\pi$, and at the time of maturity, T, the surviving policy-holders receive K. Let $N(T)$ denote the number of deaths in the portfolio until time T. Hence, the number of survivors is given by $n - N(T)$, such that the total benefits to the policy-holders are

$$H = (n - N(T))K.$$

Here, we first identify and classify the types of risk associated with the contractual payments described above. In order to identify possible financial risks, we assume the random course of the insured lives are known, such that the number of survivors at time T, and hence the benefits, are known at time 0. Since K is a fixed benefit, no specific dependence on stocks is stated in the contract, so the equity risk is a non-contractual risk. Hence, among the financial risks only the interest rate risks may be contractual risks. To classify the interest rate risks we distinguish between whether the time of maturity of the insurance contract lies before or after the time of maturity of the longest bond traded at time 0. In the first case the standard interest rate risk is a contractual risk and the reinvestment risk is a non-contractual risk, whereas both interest rate risks are contractual risks in the second case. The company is able to eliminate the contractual interest rate risk(s) if it can invest a fixed amount at time 0 and trade in the financial market, such that it is certain to hold $(n - N(T))K$ at time T. This is the case if there exists a so-called zero coupon bond (a bond, which always pays one at time of maturity) with the same time of maturity as the insurance contract, since purchasing $(n - N(T))K$ zero coupon bonds at time 0 leaves the company with exactly $(n - N(T))K$ at time T. This situation corresponds to the first case with sufficiently long bonds. Hence, in this case the standard interest rate risk is a hedgeable contractual risk and the reinvestment risk is a non-contractual risk. If, on the other hand, both the standard interest rate risk and the reinvestment risk are contractual risks, then the company is unable to pursue an investment strategy which guarantees exactly $(n-N(T))K$ at time T. Hence, in this case at least one of the interest rate risks is an unhedgeable contractual risk. Considering the two types of interest rate risk separately, we find that the standard interest rate risk still is hedgeable, since the risk associated with the movement of the bond prices between the times of issue of new bonds can be eliminated by trading in the bonds. On the contrary the reinvestment risk cannot be eliminated by trading in bonds already in the market, such that it is an unhedgeable contractual risk.

In order to identify and classify the mortality risks we consider the contingent model where the future stock and bond prices are known. As mentioned in Section 1.1.1, the uncertainty regarding the number of survivors at time T can be slit into unsystematic and systematic mortality risk. Here, we first turn our attention to the unsystematic mortality risk. Hence, we assume that the future mortality intensity is known and consider the uncertainty associated with the number of survivors. In this case, the survival probability of each individual is known, and we know from a diversification argument that in a large portfolio the number of survivors is approximately equal to the expected number of survivors given by the product of the survival probability and the number of insured. However, since the size of the portfolio is finite the number of survivors is not exactly equal to the expected number of survivors. Hence, in this case the unsystematic mortality risk accounts for the uncertainty associated with the number of survivors at time T given the underlying mortality intensity. In addition to the unsystematic mortality risk the company is exposed to a risk associated with the actual development of the mortality intensity, the so-called systematic mortality risk. Here, the company will experience a surplus (loss) if the mortality intensity increases (decreases) more than expected, such that the realized expected number of survivors is lower (higher) than the expected number of survivors calculated at time 0. Since we assume that the financial market only consists of bonds and stocks, the company is unable to eliminate the uncertainty associated with the number of survivors by trading in the financial market. Hence, the mortality risks are unhedgeable contractual risks.

Thus, when considering a portfolio of pure endowments with fixed benefits, the contractual risks include both types of mortality risk and standard interest rate risk, whereas the equity risk in a non-contractual risk. Whether the reinvestment risk is a contractual or non-contractual risk depends on the time to maturity of the bonds compared to the time to maturity of the contracts. The standard interest rate risk is a hedgeable contractual risk and the mortality risks are unhedgeable contractual risks. If the reinvestment risk is a contractual risk it is an unhedgeable contractual risk.

As noted in Section 1.1.2, the class of business risks depends on the investment strategy. In order to illustrate this dependence we consider two different investment strategies. First consider the case where the company eliminates the standard interest rate risk by investing in bonds. In this case the business risks consists of the unhedgeable contractual risks: The mortality risks and possibly the reinvestment risk. The standard interest rate risk, which is the only hedgeable contractual risk has been eliminated so there are no non-hedged contractual risks, and since the company does not invest in stocks, there are no gambling risks. As a second example we consider the case where the company invests in a mixture of bonds and stocks, such that is does not entirely eliminate the standard interest rate risk, and thus, since the classification is qualitative, the standard interest rate risk is a non-hedged contractual risk. The unhedgeable contractual risks are independent of the investment strategy, so the sub-class is unaltered. The investment in stocks introduces equity risk as a gambling risk. Hence, in this second example all types of risk are business risks.

1.2 Traditional approach to risk in life insurance

Traditional life insurance contracts include some guaranteed benefits. The two most common types of guarantees are maturity guarantees and periodic interest rate guarantees. A maturity guarantee states a minimal benefit, whereas a periodic interest rate guarantee states a minimum return in each accumulation period. In order to calculate the premiums, the insurance company typically applies the principle of equivalence using a constant interest rate, $\widetilde{r}$, and a deterministic mortality intensity, $\widetilde{\mu}$, which is independent of calender time (henceforth referred to as time-independent). The pair $(\widetilde{r}, \widetilde{\mu})$ is usually referred to as the technical basis or the first order basis, see e.g. Norberg (1999). The principle of equivalence states that the expected value of the discounted (guaranteed) benefits and premiums must be equal. Hence, in the case of deterministic guaranteed benefits the calculations depend on the specification of the future interest rate and mortality intensity. Since the technical basis is deterministic, the derivation of the premiums is particularly simple. For a contract with deterministic benefits the calculations necessary to determine a lump sum premium simply requires the company first to replace the uncertain course of the random life by the expected development using the technical mortality intensity and second to determine the present value of the resulting deterministic benefits using the technical interest rate.

Consider a portfolio consisting of n pure endowments with guaranteed benefits K paid by a single premium. In this case, the individual premium calculated by the principle of equivalence using the technical basis is given by

$$\pi = e^{-\int_0^T \widetilde{\mu}(x+u)du} e^{-\widetilde{r}T} K. \tag{1.2.1}$$

Here, $\widetilde{\mu}(x+t)$ is the technical mortality intensity at time t for a person aged x at time 0, where the contract was issued. Hence, $\exp(-\int_0^T \widetilde{\mu}(x+u)du)$ is the survival probability for a person of age x from time 0 to T using the technical basis.

The basic idea in traditional risk management in life insurance is to choose the technical basis to the safe side, as seen from the company's point of view, such that the future interest rate and portfolio-wide mortality intensity never behaves worse (again seen from the company's point of view) than the technical basis. In the case of a pure endowment this corresponds to applying a technical interest rate and mortality intensity, which are too low. Thus, at any time t the reserve

$$V(t) = e^{-\int_t^T \widetilde{\mu}(x+u)du} e^{-\widetilde{r}(T-t)} K,$$

is on average (more than) sufficient to cover the guaranteed benefit K at time T given survival until time t. However, the company receives the portfolio-wide premium $n\pi$ at time 0, which it invests in the financial market. If we let $\widehat{r}$ denote the rate of return obtained by the company, the portfolio-wide assets at time t are given by $n\pi \exp(\int_0^t \widehat{r}(u)du)$. If we further denote by $\widehat{\mu}(x, u)$ the observed mortality intensity in the portfolio at time u for an insured of age x at time 0, then the observed number of survivors at time t is given by $n \exp(-\int_0^t \widehat{\mu}(x, u)du)$. So the assets per survivor are $\pi \exp(\int_0^t \widehat{r}(u) + \widehat{\mu}(x, u)du)$. Now the choice of technical basis ensures that the company is able to choose an investment strategy, such that the individual assets are sufficient to cover the reserve calculated with the technical basis, i.e.

$$\pi e^{\int_0^t \widehat{r}(u)+\widehat{\mu}(x,u)du} \geq V(t).$$

One such strategy is to invest in very short term bonds. Here, the rate of return is r, which by assumption is larger than or equal to $\widetilde{r}$. Thus, the company is able to generate a systematic surplus by obtaining an investment return which always exceeds the technical interest rate and by observing a mortality intensity in the portfolio larger than the technical mortality intensity. Note that the company measures the surplus generated by the mortality by comparing the observed mortality in the portfolio with the technical mortality intensity, so no distinction is made between the systematic and unsystematic mortality risk. In Danish legislation the so-called contribution principle states that a systematic surplus must be returned to the group of insured, and the distribution mechanism should take the contribution of each individual to the surplus into account. This return of surplus to the insured is usually referred to as bonus, see Norberg (1999). Traditionally bonus has been used to purchase additional coverage calculated using the technical basis. Thus, the guaranteed benefits are increased during the course of the contract as bonus is allocated to the individual insured. However, legislation does not prescribe when the surplus must be returned as bonus, and since the distribution mechanism is not specified in the contract either, it is left to the company as a decision parameter within some legislative bounds. Hence, a company may follow a very aggressive (conservative) bonus strategy by returning bonus immediately (as late as possible). The surplus not distributed to the individual insured is kept by the company as a portfolio-wide buffer. In the literature this buffer is often referred to as the bonus reserve and in recent Danish legislation it is known as the collective bonus potential. This buffer serves two purposes. Firstly, it is used to cover the deficit in the case, where the company, in an attempt to maximize the investment return, invests in stocks as well as bonds, and observe an investment return below the technical interest rate. Secondly, the buffer serves to smooth the bonus to the insured over the course of the contract, such that the insured observe a steady development of the individual account. We note that the legislative bounds on the return mechanism allows for some redistribution among the different generations. Hence, the actual benefits depend on the investment return obtained by the company, the realized development of the insurance portfolio, the competition in the market and the capital of the company at initiation of the contract.

Summing up we conclude that the traditional method of risk management consists of collecting premiums, which in all circumstances are sufficient to cover the guaranteed benefits and to redistribute the observed surplus among the insured as bonus. Thus, the traditional risk management can be viewed as a static risk management, which works as long as the technical basis really is to the safe side. However, both the interest rate and the number of survivors are stochastic, so a natural question is whether the company really is able to determine a deterministic interest rate and mortality intensity, such that the observed quantities at all times are to the safe side. In the case of a pure endowment this could be obtained by using an interest rate and a mortality intensity of zero. However, the guarantee should be of interest to the insured, since they otherwise would seek alternative methods for saving to retirement, and this is not the case if the guarantee is calculated with interest rate and mortality intensity zero. Thus, competition forces the companies to calculate guarantees using strictly positive technical elements, and these are not to the safe side for all possible future scenarios. Moreover, the fact that the companies often allocate some of the surplus as bonus immediately after it is observed implies that even though the premium includes a large safety loading, the company may find itself unable to cover the guarantees later. This problem is further enlarged by the use of bonus to calculate additional guaranteed benefits on the technical basis. The latter problem seems to be outdated or at least reduced, since recent additional benefits typically either are unguaranteed or calculated using an interest rate lower than the technical interest rate. The magnitude of the risk of a company following a traditional risk management approach naturally depends on the technical basis, the aggressiveness of the bonus strategy and the investment strategy.

1.3 Financial theory

In this section we describe the approach of modern financial mathematics to risk. Some recent standard references are Musiela and Rutkowski (1997) and Björk (2004). Let T denote a fixed finite time horizon and consider a financial market consisting of $d+1$ traded assets: A savings account earning a (possibly) stochastic rate of interest and d risky assets (stocks, bonds, real estate etc.). The price processes, which are given by $B = (B(t))_{0\leq t\leq T}$ and $X = (X(t))_{0\leq t\leq T}$, respectively, are defined on a probability space $(\Omega, \mathcal{F}, P)$ with filtration $\mathbb{F} = (\mathcal{F}(t))_{0\leq t\leq T}$. Here, $\mathcal{F}(t)$ can be interpreted as the information available at time t. This covers information regarding the price processes, and may in general include other information as well.

A *trading strategy* is a process $\varphi = (\vartheta, \eta)$ satisfying certain integrability conditions. Here, ϑ is predictable, and η is adapted to the filtration $\mathbb{F}$. The pair $\varphi(t) = (\vartheta(t), \eta(t))$ is interpreted as the portfolio held at time t. Here, ϑ is a d-dimensional vector denoting the number of the d risky assets in the portfolio, whereas η is the discounted deposit in the savings account. The value process $\mathcal{V}(\varphi)$ associated with φ is defined by

$$\mathcal{V}(t, \varphi) = \vartheta(t)X(t) + \eta(t)B(t), \quad 0 \leq t \leq T.$$

A trading strategy is called *self-financing* if

$$\mathcal{V}(t, \varphi) = \mathcal{V}(0, \varphi) + \int_0^t \vartheta(u)dX(u) + \int_0^t \eta(u)dB(u). \tag{1.3.1}$$

Hence, the value of the portfolio at time t is the initial value $\mathcal{V}(0, \varphi)$ added trading gains, $\int_0^t \vartheta(u)dX(u)$, and interest on the savings account, $\int_0^t \eta(u)dB(u)$. Thus, no in- or outflow of capital to/from the portfolio has occurred in $(0, t]$. A self-financing strategy is a so-called *arbitrage* if $\mathcal{V}(0, \varphi) = 0$ and $\mathcal{V}(T, \varphi) \geq 0$ P-a.s. with $P(\mathcal{V}(T, \varphi) > 0) > 0$. Thus, an arbitrage is the possibility without an initial investment to obtain a riskfree gain. If the model allows for arbitrage possibilities an investor has a positive probability to become infinitely rich, without risking any money. Hence, in practice all investors would pursue such a strategy and thereby force prices to correct themselves, such that no arbitrage possibilities would exist. Thus, a reasonable model is arbitrage free.

A *contingent claim* H with maturity T is an $\mathcal{F}(T)$-measurable random variable, i.e. the value of H is known at time T. If H only depends on the terminal value of the price processes it is called a *simple* contingent claim. The contingent claim H is called *attainable* if there exists a self-financing trading strategy φ^H such that $\mathcal{V}(T,\varphi^H) = H$ P-a.s. The strategy φ^H is called the perfect replicating (hedging) strategy for H. From (1.3.1) we see that H is attainable if and only if there exists a self-financing strategy φ^H such that

$$H = \mathcal{V}(0,\varphi^H) + \int_0^T \vartheta^H(u)dX(u) + \int_0^T \eta^H(u)dB(u). \tag{1.3.2}$$

If on the other hand no perfect replicating strategy exists, H is called *unattainable*. If all contingent claims are attainable the model is *complete*, and otherwise it is called *incomplete*. Here, we note that if the market is incomplete there are infinitely many unattainable claims. To observe this we assume that the claim H is unattainable. However, if H is unattainable, then it follows from (1.3.2) that cH is unattainable for $c \in \mathbb{R}\backslash\{0\}$. If this was not the case we could perfectly replicate H by the strategy $\varphi^H = \varphi^{cH}/c$, where φ^{cH} is the perfect replicating strategy for cH.

The fundamental pricing principle in financial mathematics is the *no arbitrage principle* due to Black and Scholes (1973) and Merton (1973). The no arbitrage principle states that the financial market still should be arbitrage free after the introduction of a new asset. Hence, for an attainable claim H with replicating strategy φ^H, the *unique arbitrage free price* is given by $\mathcal{V}(0,\varphi^H)$, since this is the only price which excludes arbitrage possibilities, see e.g. Møller (2002) for a simple argument. If on the other hand H is unattainable no perfect replicating strategy exists and thus no unique arbitrage free price exists. In fact is can be shown that there exists an interval of arbitrage free prices. We note that the no arbitrage principle leads to relative prices only, such that prices of new assets rest heavily on the prices of the original traded assets. Black and Scholes (1973) and Merton (1973) used the no arbitrage principle to derive partial differential equations for the prices of attainable claims. They observed that for simple contingent claims the partial differential equations differ by their boundary conditions only. This way Black and Scholes obtained the celebrated Black–Scholes formula for the price of a so-called European call option.

Additional insight in no arbitrage pricing was obtained by Harrison and Kreps (1979), in discrete time, and Harrison and Pliska (1981), in continuous time. They observed a connection between on one side the properties of completeness and absence of arbitrage and on the other side so-called *equivalent martingale measures*. Recall that Q is an equivalent martingale measure for the model $(B, X, \mathbb{F})$ if Q is a probability measure, Q and P are equivalent (for all $A \in \mathcal{F}$: $Q(A) = 0 \Leftrightarrow P(A) = 0$) and all discounted price processes associated with traded assets are martingales under Q. They observed that for a complete and arbitrage free model there exists a unique equivalent martingale measure and all prices are given by the expectation under this unique equivalent martingale measure of the discounted value of claim. Hence, in a complete and arbitrage free model the price of H is given by

$$F^H(0) = E^Q\left[B(T)^{-1}H\right]. \tag{1.3.3}$$

If the model is incomplete there exist infinitely many equivalent martingale measures. In this case the arbitrage free prices of an unattainable claim, H, are still given by (1.3.3). However, now Q is an equivalent martingale measure rather than the unique one. Recall that in a arbitrage free and incomplete market an attainable claim still has a unique price, since all equivalent martingale measures actually give the same price. The insight obtained by the link to equivalent martingale measures have further opened for the connection between the partial differential equations for prices and the pricing formula in (1.3.3) given by the Feynman-Kač stochastic representation formula, see e.g. Björk (2004).

In general a model is arbitrage free and complete if it in addition to a (locally) riskfree savings account includes the same number of risky assets as the number of fundamental stochastic processes (Wiener processes and counting processes) accounting for the uncertainty. A simple example of an incomplete market is if the contingent claims are allowed to depend on a complete financial market and independent insurance events.

In the following we shall decorate a discounted claim, price process or value process by an asterisk (*).

1.3.1 Valuation and hedging in incomplete markets

As noted above the principle of no arbitrage yields unique prices and perfect replicating strategies for attainable claims in both complete and incomplete markets. However, for unattainable claims the principle gives no unique arbitrage free price and replicating strategy. Hence, in order to determine a unique price and a hedging strategy for an unattainable claim more structure has to be added. The different criteria proposed in the literature have their primary focus either on the hedging or the pricing aspect and consider the other quantity as a secondary information. Here, we review several different principles proposed in the literature. The principles considered are naturally inspired by a possible use throughout the thesis. The review is somewhat similar to the one in Møller (2002).

Super-replication

The basic idea in *super-replication* (super-hedging) is to determine the lowest possible initial investment and the corresponding self-financing strategy, which eliminates the shortfall risk of the hedger. Mathematically this corresponds to

$$\min_{\varphi} \mathcal{V}(0, \varphi)$$

under the constraint that $P(\mathcal{V}(T, \varphi) \geq H) = 1$. Obviously this criterion is not suitable for determining prices, since it would introduce arbitrage possibilities. For instance the super-replicating price of a capital insurance and a pure endowment are identical even though the capital insurance always pays out the benefits at time of maturity, whereas the pure endowment only pays out the benefits in case of survival of the insured. The theory of super-replication has been applied in Chapter 5. For more details on super-replication we refer to El Karoui and Quenez (1995).

Quadratic hedging

The quadratic hedging approaches focus on hedging in incomplete markets, and as a secondary result the initial capital necessary to construct the optimal hedging strategy can be interpreted as a possible price. For a review of quadratic hedging approaches, see Schweizer (2001a).

Mean-variance hedging
The idea of *mean-variance hedging* was introduced in Bouleau and Lamberton (1989) and Duffie and Richardson (1991). With mean-variance hedging the aim is to determine the self-financing strategy, $\hat{\varphi}$, which minimizes

$$E^P\left[(H - \mathcal{V}(T,\varphi))^2\right].$$

Since we consider self-financing strategies only, the strategy is uniquely determined by the pair $(\mathcal{V}(0,\hat{\varphi}),\hat{\vartheta})$. Here, $\mathcal{V}(0,\hat{\varphi})$ and $\hat{\vartheta}$ are known as *the approximation price for* H and *the mean-variance optimal hedging strategy*, respectively.

Risk-minimization
The criterion of *risk-minimization* was originally proposed by Föllmer and Sondermann (1986) for contingent claims. They considered the special case, where the discounted price processes are P-martingales. The approach was extended to the general semi-martingale case by Schweizer (1991), who introduced the idea of *local risk-minimization.* Schweizer also observed that the local risk-minimizing strategy essentially corresponds to the risk-minimizing strategy under the so-called *minimal martingale measure*, see also Schweizer (2001a). Møller (2001c) extended the approach in a different direction by allowing for payment processes. Here, we consider a fixed but arbitrary equivalent martingale measure, Q, for the considered model, such that discounted price processes indeed are Q-martingales. The criterion of risk-minimization is closely related to the cost process $C(\varphi)$ defined by

$$C(t,\varphi) = \mathcal{V}^*(t,\varphi) - \int_0^t \vartheta(u)dX^*(u). \tag{1.3.4}$$

From (1.3.4) we observe that the accumulated costs $C(t,\varphi)$ at time t are the discounted value $\mathcal{V}^*(t,\varphi)$ of the portfolio reduced by discounted trading gains, $\int_0^t \xi(u)dX^*(u)$. A strategy is called risk-minimizing, if it minimizes

$$R(t,\varphi) = E^Q\left[(C(T,\varphi) - C(t,\varphi))^2 \middle| \mathcal{F}(t)\right]$$

for all t with respect to all (not necessarily self-financing) strategies and $\mathcal{V}(T,\varphi) = H$. The process $R(\varphi)$ is called the risk process. Föllmer and Sondermann (1986) realized that the risk-minimizing strategies are related to the so-called Galtchouk-Kunita-Watanabe decomposition of the Q-martingale

$$V^{*,Q}(t) = E^Q\left[H^* \middle| \mathcal{F}(t)\right].$$

The process $V^{*,Q}$ is usually called to as the intrinsic value process. Furthermore they observed that the cost process is a Q-martingale and that the discounted value process for the risk-minimizing strategy coincides with the intrinsic value process. In this thesis risk-minimization is applied in Chapters 4, 5, 6 and 7.

Utility approaches

Utility functions have traditionally been applied in micro-economics and non-life insurance to determine prices, and in recent years they have be applied to derive prices in incomplete financial markets. Here, we focus on *marginal utility indifference pricing* and a special case of *utility indifference pricing* called *mean-variance indifference pricing*.

Marginal utility indifference pricing
Davis (1997) proposed to value contingent claims in incomplete markets by a "marginal rate of substitution argument". Hence, p is a fair price for the claim H if the maximal achievable expected terminal utility is indifferent to whether an agent at time 0 invests a small amount of capital in the contingent claim. To express the idea mathematically, Davis (1997) introduced the function

$$L(\delta, c, p) = \sup_{\vartheta} E^P \left[u \left(c - \delta + \int_0^T \vartheta(u) dX(u) + \frac{\delta}{p} H \right) \right],$$

where u is a utility function, c is the initial capital of the agent and δ is the capital invested in H. Now provided that the partial derivative of L with respect to δ exists at $\delta = 0$ and there exists a unique solution, $\hat{p}$, to

$$\frac{\partial}{\partial \delta} L(\delta, c, p)|_{\delta=0} = 0,$$

then $\hat{p}$ is the price of H.

Mean-variance indifference pricing
Denote by A^* the discounted wealth of the insurer at time T and consider the mean-variance utility-functions

$$u_i(A^*) = E^P[A^*] - a_i \left(\mathrm{Var}^P[A^*] \right)^{\beta_i}, \tag{1.3.5}$$

$i = 1, 2$, where $a_i > 0$ are so-called risk-loading parameters, and where we take $\beta_1 = 1$ and $\beta_2 = 1/2$. It can be shown that calculating premiums using the equations $u_i(A^*) = u_i(0)$ indeed leads to the premiums assigned by the classical actuarial variance (i=1) and standard deviation premium principle (i=2), respectively, see e.g. Møller (2001b). These classical actuarial principles have traditionally been applied in non-life insurance.

Schweizer (2001b) proposed to apply the mean-variance utility functions (1.3.5) in an indifference argument which takes into consideration the possibility to trade in the financial market. Denote by c the insurer's initial capital at time 0. The u_i-indifference price v_i associated with the claim H is defined via

$$\sup_{\vartheta} u_i \left(c + v_i + \int_0^T \vartheta(u) dX^*(u) - H^* \right) = \sup_{\widetilde{\vartheta}} u_i \left(c + \int_0^T \widetilde{\vartheta}(u) dX^*(u) \right). \tag{1.3.6}$$

The strategy ϑ^* which maximizes the left hand side of (1.3.6) is called the optimal strategy for H. The optimal strategies associated with the mean-variance indifference prices are derived in Møller (2001b). Note that the price assigned to a claim using this criterion is not necessarily arbitrage free, see Møller (2002) for an example where the price lies outside the interval of arbitrage free prices. We determine mean-variance indifference prices in Chapters 4 and 7.

The idea of mean-variance indifference pricing is slightly different from the general set-up in utility indifference pricing, since the mean-variance utility functions taken on a random variable returns deterministic value. In general one considers standard utility functions, which still return a random variable. In this case the indifference price for a claim H is the price, which leaves the investor indifferent between purchasing the claim or not. Here, the indifference refers to the fact that the investor is able to obtain the same maximal expected utility of terminal wealth in the two cases. For an overview of utility indifference pricing we refer to Henderson and Hobson (2004). We note that the indifference price depends on the choice of utility function.

Quantile hedging and minimization of expected shortfall

A major disadvantage of the quadratic hedging approaches is that gains and losses are considered equally unattractive. In an attempt to eliminate this disadvantage, Föllmer and Leukert (1999) proposed the criterion of *quantile hedging*. Here, two related problems are solved. The first problem is for a given a fixed initial capital, say c, to determine the maximal obtainable probability of a successful hedge and the associated self-financing strategy. Hence, one has to solve

$$\max_{\varphi} P\left[\mathcal{V}(T,\varphi) \geq H\right]$$

under the constraint $\mathcal{V}(0,\varphi) \leq c$. The related problem is for a given minimal probability of a successful hedge, say $1-\varepsilon$, to determine the minimal necessary initial capital and the associated self-financing strategy. Here, the agent is interested in determining

$$\min_{\varphi} \mathcal{V}(0,\varphi)$$

under the constraint $P\left[\mathcal{V}(T,\varphi) \geq H\right] \geq 1-\varepsilon$. Here, one easily observes that the initial investment converges to the super-replicating price as ε converges to zero. A natural criticism of quantile hedging is that the agent in case of a shortfall is indifferent to the size of the shortfall. In order to meet this criticism Föllmer and Leukert (2000) and Cvitanić (2000) introduced the criterion of *minimizing the expected shortfall*. For a given initial capital they derived the self-financing strategy, which minimizes the expected losses from hedging the considered claim. As an alternative they fixed the maximum expected shortfall and derived the self-financing strategy, which requires the minimal initial capital. Föllmer and Leukert (2000) introduced a so-called *loss function*, which is an increasing convex function with $\ell(0) = 0$, such that they solved the problem

$$\min_{\varphi} E^{P}\left[\ell\left(H - \mathcal{V}(T,\varphi)\right)^{+}\right]$$

under the constraint $\mathcal{V}(0,\varphi) \leq c$. Hence, the method of Föllmer and Leukert (2000), which they refer to as *efficient hedging*, could be referred to as *minimizing the expected* *adjusted shortfall*. Here, the special case $\ell(x) = x$ corresponds to minimizing the expected shortfall.

Even though the criteria of quantile hedging and especially minimizing the expected shortfall are advantageous compared to the quadratic approaches they are not pursued in the thesis, since explicit results are extremely hard to obtain.

1.4 Applying financial methods in life insurance

Even though the fields of (life) insurance and finance originally were separate fields the interplay between the two fields have increased over the past decades, see e.g. Embrechts (2000) and Møller (2002). New insurance contracts linked directly to the financial market have been introduced and old ones have gained increased popularity. In life insurance, such contracts linked directly to the financial market are called *unit-linked* contracts. Similarly financial contracts linked to insurance events have been introduced. We mention catastrophe insurance (CAT) futures, catastrophe-linked bonds and mortality dependent bonds.

The increased interplay has in turn increased the need for comparable methods for pricing and reserving in the fields of financial and insurance, since prices and solvency requirements should be independent of whether the seller is a bank or an insurance company. Since financial methods are compatible with observed prices, legislation has forced life insurance companies to use these methods to calculate prices and reserves. Reserves calculated by the use of financial mathematics are called *market reserves*.

1.4.1 Traditional insurance contracts

As mentioned above, legislation has forced life insurance companies to apply methods from financial mathematics to value their assets and liabilities. Whereas the value of the assets easily is obtained from the prices quoted, the liabilities represent a greater problem, since they involve a mixture of financial and insurance elements.

Persson (1998) introduced market reserves in the case of standard interest rate risk and unsystematic mortality risk. Combining the insurance valuation principle of diversification (for the unsystematic mortality risk) and the financial valuation principle of no arbitrage (for the standard interest rate risk), he obtained unique market reserves. The valuation is carried out in two steps. First, the principle of diversification is applied to replace the uncertain course of the insured life by the expected development, and second the resulting purely financial contract is priced uniquely by the no arbitrage principle. For a pure endowment the approach corresponds to exchanging the claim $I(T)K$ by $\exp(-\int_0^T \mu(x,t)dt)K$, where $I(T)$ is an indicator function denoting whether the insured is alive at time T. Now, under the assumption of the existence of sufficiently long bonds the only remaining risk is the standard interest rate risk. Thus, the market reserve at time 0 of the guaranteed benefit, K, is uniquely given by

$$V^Q(0) = E^Q\left[e^{-\int_0^T \mu(x,t)dt} B(T)^{-1} K\right] = e^{-\int_0^T \mu(x,t)dt} P(0,T)K, \qquad (1.4.1)$$

where $P(0,T)$ is the price at time 0 of a zero coupon bond maturing at time T. Hence, the market reserve in (1.4.1) corresponds to the traditional reserve except now the discount factor is the price of a zero coupon bond. However, the market value in (1.4.1) is only one of the infinitely many arbitrage free prices for a pure endowment in the case where we only consider unsystematic mortality risk and standard interest rate risk. In particular, if we only apply the no arbitrage principle we can define a market reserve by

$$V^{Q,g}(0) = E^Q\left[I(T)B(T)^{-1}K\right] = e^{-\int_0^T (1+g(u))\mu(x,u)du} P(0,T)K \qquad (1.4.2)$$

for each choice of stochastic process $g > -1$ adapted to the filtration generated by the insurance events, see Steffensen (2000). Here, we essentially calculate the survival probability with a mortality intensity, $(1 + g(u))\mu(x,u)$, which may differ from the real one. Comparing (1.4.1) and (1.4.2) we observe that (1.4.1) is a special case of (1.4.2) obtained by letting $g = 0$. This choice of g corresponds to assuming risk-neutrality with respect to unsystematic mortality risk.

In this thesis the principle of no arbitrage is applied to determine market reserves in the case of a stochastic mortality intensity, see Chapter 3. Since we only apply the principle of no arbitrage, no unique market value is obtained. In this case appealing to diversification arguments is not sufficient to obtain a unique market reserve, since this only eliminates the pricing uncertainty regarding the unsystematic mortality risk, whereas the pricing uncertainty regarding the systematic mortality risk still remains.

Since the insured receive bonus if the investment return exceeds the technical interest rate and the technical interest rate serves as a guarantee, the contracts have imbedded an interest rate option. Traditionally these options have been ignored, since they at the time of issue have been considered highly unlikely to have any effect. For instance a major part of the Danish life insurance contracts with a technical rate of 4.5% was issued in the 1980's where the interest rate was 15-20%. However, the decreasing interest rates over the past years have implied that these options have become very valuable. This, in turn, has increased the necessity for the derivation of correct prices. However, this is in general not a simple task, since the benefit of the insured and thus the price of the options depend on the bonus strategy of the company. Numerous papers consider the simple case where the investment return immediately after realization is distributed between the individual accounts of the insured and the equity capital, see e.g. Briys and de Varenne (1997), Aase and Persson (1997), Miltersen and Persson (1999) and Bacinello (2001). The case including a bonus reserve is studied in Grosen and Jørgensen (2000), Hansen and Miltersen (2002) and Miltersen and Persson (2003). These papers differ by considering different distribution mechanisms. Another feature encountered in practice is the possibility for the insured to surrender. This has been studied in Grosen and Jørgensen (2000) and Steffensen (2002), where the similarities with American options from finance are exploited. These similarities also holds for the free-policy option of the insured, which has been considered in Steffensen (2002).

1.4.2 Unit-linked life insurance

Unit-linked (equity-linked) life insurance contracts were introduced as an alternative to the traditional life insurance contracts in the United States, Netherlands and United Kingdom in the 1950's, see Turner (1971). In the United States the unit-linked contracts are known as variable life insurance contracts. In a unit-linked contract the benefits are linked directly to the development of a specific reference portfolio. The reference portfolio can be chosen by the insured or the company depending on the desire of the insured, and it may change during the insurance period. In recent years life insurance companies (at least in Denmark) have experienced an increasing demand for unit-linked contracts. This increase was catalyzed by the explosive development of the stock prices in the late 1990's, since there is a possibility to invest more in stocks in unit-linked contracts than in traditional life insurance contracts. Furthermore, from the insured's point of view, unit-linked contracts have the advantages that the investment profile can be adapted to the desire of the individual and that the insured easily are able to identify their individual savings, such that no distribution among the different generations can take place. For the company, unit-linked contracts are advantageous, since the financial risk is easily eliminated using the hedging approach of modern financial mathematics. However, they may also prove disadvantageous for the company, since it cannot use surplus generated by the financial risks to cover a possible deficit resulting from the mortality risks and vice versa. Hence, there is an increased need for the insurance company to correctly understand, model and control the individual risks. Assume that the vector of risky assets X includes a stock with price process $S = (S(t))_{0 \leq t \leq T}$ and consider a portfolio of unit-linked pure endowments linked to the development of the stock. In this case the total liability of the company is given by

$$H = (n - N(T))f(S),$$

where f is a function of the entire path of the stock price, S. Traditionally the literature distinguish between *pure unit-linked life insurance contracts*, where $f(S) = S(T)$ and *unit-linked life insurance contracts with guarantee*, where $f(S) = \max(S(T), G(T,S))$. Some possible guarantees include: A maturity guarantee of a fixed amount

$$G(T,S) = K,$$

a periodic guarantee on the return of the stock

$$G(T,S) = K \prod_{i=1}^{T} \max\left(1 + \frac{S_i - S_{i-1}}{S_{i-1}}, 1 + \delta_i\right),$$

where δ_i is the guarantee in period i and a quantile guarantee

$$G(T,S) = \alpha \sup_{0 \leq t \leq T} S(t),$$

where $\alpha \in [0,1]$. The fixed and periodic guarantees are common in practice, whereas the quantile guarantee is quite rare. However, we mention that a product including a discrete version of the quantile guarantee is sold by the Danish life and pension company "Danica Pension".

In the early years the unit-linked contracts were usually without a guarantee, in which case no advanced financial mathematics is necessary to eliminate the financial risk of the company. However, contracts could include a minimum guarantee, and prior to the introduction of the no arbitrage principle, the early approach in order value the guarantees was to involve statistical methods for the development of the assets and use simulation studies to determine an adequate reserve, see e.g. Turner (1969), Kahn (1971) and Wilkie (1978). Inspired by the introduction of modern financial mathematics by Black and Scholes (1973) and Merton (1973), Brennan and Schwartz (1976, 1979a, 1979b) considered a large portfolio of unit-linked insurance contracts with deterministic mortality intensity and a Black–Scholes model for the financial market, such that they considered a complete financial market with a constant interest rate. Using a diversification argument they exchanged the random course of the insured lives by the expected development. Upon this replacement they obtained a purely financial contract, which could be priced and hedged using the no arbitrage principle. Hence, their approach essentially corresponds to considering the claim

$$\widetilde{H} = ne^{-\int_0^T \mu(x,u)du} f(S).$$

They considered the case $f(S) = \max(S(T), K)$ and as in Black and Scholes (1973) and Merton (1973), they obtained prices by solving a partial differential equation. Delbaen (1986) was the first to apply the martingale methods of Harrison and Kreps (1979) and Harrison and Pliska (1981) in order to valuate unit-linked contracts. Since then numerous papers have used the martingale approach to valuate unit-linked contracts with guarantees. Bacinello and Ortu (1993a) consider the case of a constant interest rate and endogenous guarantees. The case of a stochastic interest rate has been considered by Bacinello and Ortu (1993b), who derive a closed form solution in the case of a single premium pure endowment and a so-called Vasiček model for the interest rate. Nielsen and Sandmann (1995) consider a stochastic interest rate in association with periodic premiums and periodic guarantees. They observe that the guarantee introduces a discretely sampled Asian option. In order to obtain results they use numerical methods. Aase and Persson (1994) are the first to consider instantaneous death probabilities. However, even though they consider only one insured, they also quickly insert the expected values regarding survival.

So far the mentioned papers have applied the diversification principle at an early point, such that the pricing problem reduces to pricing contingent claims in a complete financial market. Furthermore, the papers considering the hedging aspect all consider the purely financial contract resulting from the use of the diversification principle. Brennan and Schwartz (1976) refer to such a strategy as "riskless", even though it only eliminates the financial risk, since the company of course still is exposed to unsystematic mortality risk. Hence, a more appropriate name would be a "financially riskless strategy".

In contrast to the papers above Møller (1998) does not apply the diversification principle at an early point in the pricing and hedging problem. Hence, he considers an incomplete financial market consisting of a complete financial market and a counting process counting the number of deaths in the portfolio. When assuming risk-neutrality with respect to mortality, he obtains prices identical to those in the papers above. However, similarly to traditional insurance contracts, there exists infinitely many arbitrage free prices and the one mentioned above is just one particular arbitrage free price. The incomplete market setting becomes particularly important when discussing hedging strategies. Here, Møller (1998) considers the criterion of risk-minimization and derives risk-minimizing strategies for unit-linked life insurance contracts payable at a fixed time. Hence, whereas the previous papers main purpose is to derive prices for unit-linked life insurance contracts the main purpose of Møller (1998) is to determine a hedging strategy. The hedging result obtained essentially corresponds to the "riskless" hedging strategy of Brennan and Schwartz (1976). However, now the strategy is adjusted continuously to the expected number of survivors given the current development of the insurance portfolio. The work is extended to cover payment processes in Møller (2001c) such that a more realistic insurance contracts can be considered. Møller (2001a) essentially considers a discrete time version of Møller (1998). The use of indifference pricing to value insurance contracts in an incomplete market have been studied in Becherer (2003), who worked with exponential utility functions, and Møller (2001b, 2003a, 2003b), who considered mean-variance indifference utility functions.

All of the above papers disregard the systematic mortality risk. In this thesis prices and hedging strategies are pursued in the presence of systematic mortality risk. We determine prices using the no arbitrage principle only. Similarly to the case of traditional life insurance contracts we refer to these prices as market values. Furthermore, we determine mean-variance indifference prices. For the hedging aspect emphasis is on the criterion risk-minimization and the optimal hedging strategies associated with the mean-variance indifference prices.

1.5 Quantifying the types of risk

The qualitative description of the types of risk in Section 1.1.2 provides valuable knowledge regarding if and how the insurance company is exposed to the different types of risk. However, the main interest for the company is a quantitative description. In order to obtain a quantitative description the company has to specify a model for the sources of risk and the criterion measuring the risk. Hence, the criterion is of great importance when defining the strategy which minimizes the business risk and in order to compare the business risk for different strategies. However, the hedging criterion used to determine the optimal strategy is not necessarily the criterion used to quantify the business risk. Here, the company often use a different criterion which is more easily interpretable (and perhaps demanded by the regulators). In order to obtain an adequate model for the sources of risk it should involve all the relevant (main) sources of risk, and the description of the types of risk should mirror real life as closely as possible. Methods used to quantify the risk can be found in Section 1.3.1. In this section we give an overview of the development of the modelling of the different sources of risk.

1.5.1 Equity risk

The modelling of stock prices in continuous time was initiated by Bachelier (1900), who proposed to model the stock price by

$$dS(t) = \alpha dt + \sigma dW(t),$$

where α and σ are constants, and $W = (W(t))_{0\leq t\leq T}$ is a Wiener process. Hence, the change in the stock price in a short interval of length Δt is independent of earlier changes and follows a normal distribution with mean $\alpha\Delta t$ and variance $\sigma^2\Delta t$. This model however has the undesirable property that the stock price may become negative. This flaw was eliminated in Samuelson (1965), who proposed to model the stock price by a so-called geometric Brownian motion, where the dynamics are given by

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t). \tag{1.5.1}$$

It was within this framework that Black and Scholes obtained their famous result. Today this model serves a standard example and reference in financial mathematics. A natural extension of (1.5.1) is to allow for time-dependent functions α and σ. All results obtained with constant parameters are easily extended to this case. Also the extension to a multi-dimensional Wiener process is straightforward. Heston (1993) extended the model to include stochastic volatility, i.e. he modelled the diffusion parameter σ as a stochastic process. Despite empirical evidence that an adequate model should allow for jumps most of the literature on stock prices consider diffusion models, i.e. models where the stock prices are driven by Wiener processes. However, a model including jumps was already considered in Merton (1976), who introduced the jumps in a particular nice fashion, such that option prices are infinite sums of Black–Scholes option prices. Recently Levy processes have attracted attention, see e.g. Chan (1999), Eberlein (2001) and Cont and Tankov (2004). The advantage of Levy processes is twofold: They constitute a flexible class of models, such that they can provide an adequate description of the stock prices and at the same time are they mathematically "nice". An interesting simple alternative can be found in Norberg (2003), who considers a market driven by a finite state Markov chain.

1.5.2 Interest rate risk

Since it is inconvenient to model bond prices directly, the standard approach in the literature is to model interest rates instead. Here, the first approach was to model the dynamics of the short rate, r, as a diffusion process, i.e. by

$$dr(t) = \alpha(t, r(t))dt + \sigma(t, r(t))dW(t).$$

Within the class of short rate diffusion models especially those given by

$$\begin{aligned}\alpha(t, r(t)) &= \gamma^{\alpha}(t) + \delta^{\alpha}(t)r(t),\\ \sigma(t, r(t)) &= \sqrt{\gamma^{\sigma}(t) + \delta^{\sigma}(t)r(t)}.\end{aligned}$$

have received a lot of attention, since models of this form give an *affine term structure*, which is particularly nice from a pricing perspective, see e.g. Björk (2004). The class of affine short rate models includes the famous Vasiček and Cox–Ingersoll-Ross (CIR) models, see Vasiček (1977) and Cox, Ingersoll and Ross (1985). Extensions of both models to the time-inhomogeneous case can be found in Hull and White (1990).

A different approach was proposed in Heath, Jarrow and Morton (1992) who modelled the entire forward rate curve. For a fixed time of maturity τ they modelled forward rate dynamics by

$$df(t,\tau) = \alpha(t,\tau)dt + \sigma(t,\tau)dW(t),$$

for some adapted processes α and σ. A related work can be found in Musiela (1993), where the forward rates are parameterized with time to maturity instead of time of maturity. For an overview of interest rate models without jumps, we refer to Brigo and Mercurio (2001). Shirakawa (1991) extended the approach in Heath et al. (1992) to the case, where the forward rates are driven by a Wiener process and Poisson driven jumps of a fixed magnitude. A general description of a bond market including jumps can be found in Björk, Di Masi, Kabanov and Runggaldier (1997) and Björk, Kabanov and Runggaldier (1997).

All of the above papers ignore the reinvestment risk by assuming the existence of sufficiently long bonds. This assumption is usually justified, when considering purely financial products, since these, as opposed to life insurance contracts, traditionally are short term contracts. The first attempt to include the reinvestment risk is Sommer (1997). In this thesis models for the reinvestment risk are proposed in Chapters 5 and 6.

1.5.3 Unsystematic mortality risk

In general the development of a single contract or a portfolio of similar contracts can be described by a finite state Markov chain, see Hoem (1969). In the single insurance case the Markov chain describes the different states of health, which are of importance for the insurance contract. In the portfolio case the insured are assumed to be a homogeneous group of lives, which are mutually independent given the underlying mortality intensity. Hence, the Markov chain counts the number of insured in each state of health. We note that the insured lives not in general are independent, since the underlying mortality intensity affects all the insured. The unsystematic mortality risk is now the uncertainty related to the development of the Markov chain with known transition intensities. The case of a portfolio of identical pure endowments is particularly simple, since the development of the insurance portfolio in this case can be described by a counting process, $N = (N(t))_{0\leq t\leq T}$, counting the number of deaths in the portfolio.

Note that while the assumption of conditional independence of the insured lives in general reasonable when considering the development of a portfolio over a long time horizon, it may not be applicable, when considering the short term probability of a large number of deaths, since catastrophes, such as terrorism, hurricanes and earthquakes may affect a specific group of persons.

1.5.4 Systematic mortality risk

In the literature the standard approach has been to consider a deterministic (and time-independent) mortality intensity, such that the model excludes systematic mortality risk. This is in correspondence with practice, where the life insurance companies, in at least Denmark, so far have worked with a mortality intensity, which depends on the age of the insured, only. The companies have been aware that the mortality intensity has decreased over the years and to incorporate this they have adjusted their mortality intensities yearly. However, regardless of the frequency of these mortality investigations, the estimation of a time-homogeneous mortality intensity still only measures the current level. In order to obtain a more accurate prediction of the future mortality intensity the company has to capture trends in the mortality intensity, such that a time and age dependent mortality intensity is necessary. If company also wants to capture the stochastic nature of the future mortality, the mortality intensity should be modelled as a stochastic process. In recent years several papers have considered modelling the future mortality and pricing mortality derivatives. One of the best known models for the future mortality is the Lee-Carter model, see Lee and Carter (1992) and Lee (2000). Here, the yearly death rates are modelled by three factors: Two age-dependent and one time-dependent. By modelling the time-dependent factor as a time-series the model can be used for forecasting. The first paper to introduce a stochastic mortality intensity is Milevsky and Promislow (2001), who consider a so-called mean-reverting Gompertz model under the equivalent martingale measure. Milevsky and Promislow (2001) are also the first to consider a model including both standard interest rate and systematic mortality risk. Dahl (2004b), see Chapter 3, considers a general diffusion model and discusses the change of measure with respect to the mortality. A general affine jump diffusion model for the mortality intensity can be found in Biffis and Millossovich (2004). These models for the mortality intensity are inspired by the short rate models used to describe standard interest rate risk, see Section 1.5.2. An overview over interest rate approaches applicable to describe the systematic mortality risk can be found in Cairns, Blake and Dowd (2004). An entirely different approach is taken in Olivieri and Pitacco (2002), where Baysian methods are considered.

1.6 Overview and contributions of the thesis

The aim of the thesis is to analyze risks in life insurance. We propose models for the different types of risk and use methods from financial mathematics to value and hedge life insurance contracts. The thesis consists of four main parts. The first part, Chapter 2, considers the problem of determining a fair distribution of assets between the equity capital and the portfolio of insured in the case where the insurance contract includes a periodic interest rate guarantee. The second part, Chapters 3 and 4, consider the impact of modelling the mortality intensity as a stochastic process. Here, Chapter 3 focus on determining market reserve and on possible ways to transfer the systematic mortality risk to the insured or agents in the financial market. Chapter 4 includes a derivation of hedging strategies for life-insurance contracts and a numerical comparison life expectancies using a time-independent, a time-dependent and a stochastic mortality intensity. In part three we propose models for the reinvestment risk, and derive optimal hedging strategies. Here, Chapter 5 contains a discrete-time model, whereas a continuous-time model is introduced in Chapter 6. In the fourth and last part, we essentially combine Chapters 4 and 6, such that we obtain a model including a large number of the risks faced by a life insurance company. Here, Chapter 7 covers the theoretical derivation of hedging strategies, whereas Chapter 8 includes a numerical comparison of different reservation principles. Now we give a more detailed description of the individual chapters.

Fair Distribution of Assets in Life Insurance
When issuing life insurance contracts with a periodic interest rate guarantee, the equity capital of the company is exposed to the risk of low or even negative payoffs at the end of an accumulation period. In the worst case scenario, where the guarantee can not be covered, all equity capital is lost and the company is declared bankrupt. To compensate the owners for the risk of low returns on equity capital imposed by the guarantee, the equity capital should be accumulated by a rate, which exceeds the riskfree rate in periods, where the investment return and development of the insurance portfolio allows for such a high return on equity capital. In Chapter 2, based on Dahl (2004a), we consider an insurance company whose insurance portfolio consists of either capital insurances or pure endowments with a periodic interest rate guarantee. Since the financial market is given by the complete and arbitrage free Black–Scholes model, we can for a given investment strategy apply the principle of no arbitrage to obtain an equation for the fair additional payoff to the equity capital in periods, when such an additional payoff is possible. The investment strategies considered are: A buy and hold strategy and a strategy with constant relative portfolio weights, both with and without stop-loss in case solvency is threatened. In order to study the magnitude of the fair additional rate of interest and the dependence on parameter values, initial distribution of capital and investment strategy, we supply numerical results.

Stochastic Mortality in Life Insurance: Market Reserves and Mortality-Linked Insurance Contracts
In life insurance, actuaries have traditionally calculated premiums and reserves using a deterministic mortality intensity, which is a function of the age of the insured only. However, the future mortality intensity is unknown, so it should be modelled as a stochastic process. In Chapter 3, based on Dahl (2004b), we model the mortality intensity as a diffusion process. This allows us to consider unit-linked contracts in a model including equity and standard interest rate risk, as well as both types of mortality risks. Within this model we derive market reserves and study possible ways of transferring the systematic mortality risk to other parties. One possibility is to introduce mortality-linked insurance contracts. Here, the premiums and/or benefits are linked to the development of the mortality intensity, thereby transferring the systematic mortality risk to the insured. Alternatively the insurance company can transfer some or all of the systematic mortality risk to agents in the financial market by trading derivatives depending on the mortality intensity. We derive a general partial differential equation for mortality derivatives and show an example of how a mortality derivative can be used to eliminate or reduce the systematic mortality risk of the company.

Valuation and Hedging of Life Insurance Liabilities with Systematic Mortality Risk
Chapter 4 considers the problem of valuating and hedging a portfolio of life insurance contracts that are subject to systematic mortality risk as well as the usual sources of risk, namely standard interest rate risk and unsystematic mortality risk. Since the mortality risks are unhedgeable they cannot be eliminated by trading in the financial market. Furthermore, since the systematic mortality risk is a non-diversifiable risk it cannot be reduced by increasing the size of the portfolio and appealing to the law of large numbers. Hence, we propose to apply techniques from incomplete markets in order to hedge and valuate these contracts. We derive market reserves and mean-variance indifference prices. The hedging aspect is addressed by determining risk-minimizing strategies and optimal hedging strategies associated with the mean-variance indifference prices. The chapter includes empirical evidence supporting the modelling of the mortality intensity as a stochastic process, and a numerical example comparing the life expectancies using a time-independent, a time-dependent and a stochastic mortality intensity. This chapter is based on Dahl and Møller (2005).

A Discrete-Time Model for Reinvestment Risk in Bond Markets
In the literature bond markets usually include bonds with all times *to* maturity. However, in practice the liquid bonds traded have a fixed maximum time to maturity. Hence, a life insurance company selling long term contracts is exposed to an unhedgeable reinvestment risk, associated with the entry prices of newly issued bonds. In Chapter 5, which is based on Dahl (2005b), we propose a discrete-time model for a bond market, where the reinvestment risk is present. The analysis is carried out in discrete time in order to explain the ideas in a framework, where the technical details are kept to a minimum. At each trading time a bond matures and a new bond is introduced in the market. The entry price of the new bond depends on the prices of existing bonds and a stochastic term independent of the existing bond prices. In order to determine optimal hedging strategies we consider the criteria of super-replication and risk-minimization. Furthermore, a link between super-replication and the maximal guarantees for which the short fall interest rate risk can be eliminated is observed. Finally, we consider a numerical example, where we compare our stochastic analysis with the deterministic pricing principle of a level long term yield curve. In this example we also introduce the alternative deterministic pricing principle of a level long term forward rate curve.

A Continuous-Time Model for Reinvestment Risk in Bond Markets
In Chapter 6, based on Dahl (2005a), we propose a continuous bond market model including reinvestment risk. We consider a model, where only bonds with a limited time *to* maturity are traded in the market. At fixed times new bonds with stochastic initial prices are introduced in the market. Here, the new price is allowed to depend on the existing bond prices and all past information, such that we obtain a flexible model. To quantify and control the reinvestment risk we apply the criterion of risk-minimization.

Valuation and Hedging of Unit-Linked Life Insurance Contracts Subject to Reinvestment and Mortality Risks
In Chapter 7, based on Dahl (2005d), we consider a model covering a large number of the risks faced by a company issuing unit-linked life insurance contracts. Here, the financial

market consists of a bond market including reinvestment risk and a stock, whereas the insurance part involves a stochastic mortality intensity. Hence, we consider a model, which combine the systematic mortality risk considered in Chapters 3 and 4 with the reinvestment risk from Chapters 5 and 6. The valuation and hedging results in Chapter 4 are then extended to this more refined model.

A Numerical Study of Reserves and Risk Measures in Life Insurance
Reserving and risk management are of great importance for life insurance companies. In Chapter 8, based onDahl (2005c), we provide a numerical investigation of different reservation principles and risk measures. We consider market reserves calculated by the no arbitrage principle, only. Furthermore, we consider the following alternative approaches to pricing the dependence on the reinvestment risk: Super-replication and the principles of a level long term yield/forward rate curve. Combined with the no arbitrage principle for the remaining risks, these principles give reserves, which can be compared to the market reserves. The risk measures considered are Value at Risk and tail conditional expectation.

Chapter 2

Fair Distribution of Assets in Life Insurance

(This chapter is an adapted version of Dahl (2004a))

When a life insurance company distributes assets between the equity capital and the portfolio of insured, possible periodic guarantees to the insured must be covered whenever possible. Hence, depending on the development of the financial market and the portfolio of insured, the equity capital may experience periods with low or even negative payoffs. In the worst case scenario, where the guarantee can not be covered, the company is declared bankrupt, and the entire equity capital is lost. To compensate the owners for the risk of low returns on equity capital, the equity capital should be accumulated by a rate, which exceeds the riskfree rate in periods, where the investment return and development of the insurance portfolio allows for such a high return on equity capital. We consider an insurance company with a very simple insurance portfolio: It consists of either capital insurances or pure endowments. The financial market is described by a Black–Scholes model. Given an investment strategy for the company, the principle of no arbitrage gives an equation for the fair additional payoff to the equity capital in periods, when such an additional payoff is possible. The investment strategies considered are: A buy and hold strategy and a strategy with constant relative portfolio weights, both with and without stop-loss in case solvency is threatened. To investigate the magnitude of the fair additional rate of interest and the dependence on parameter values, initial distribution of capital and investment strategy, we supply numerical results.

2.1 Introduction

When issuing life insurance contracts with a guarantee, the insurance companies are exposed to a risk, since the guarantee must be covered whenever possible. The two most common types of guarantees are: A maturity guarantee, where the company guarantees a minimal total accumulation for the entire duration of the contract, and guaranteed periodic accumulation factors (guaranteed periodic interest rates), where the company guarantees a minimal accumulation factor for each period. Even though the most common type of guarantee in Denmark is a maturity guarantee, we consider the case of guaranteed periodic accumulation factors, since it allows us to consider each accumulation period independently. When guaranteeing periodic accumulation factors, the equity capital of the company might experience low or even negative payoffs in periods with low returns on investments and/or an adverse development of the insurance portfolio. In the extreme case, where the guarantee can not be covered, the interest of the insured take precedence over the interests of the company, and all assets are paid to the insured.

Guaranteed periodic accumulation factors implicitly introduce a string of European call options on the investment gain in the insurance contract. Historically, the guarantees have in practice been chosen far out of the money, and therefore they have been ignored when pricing the insurance contracts. However, the decreasing interest rates in recent years has caused the guarantees to become an important element of some old contracts. This, in turn, has increased the importance for correct pricing of the options imbedded in the insurance contracts, see e.g. Briys and de Varenne (1997), Aase and Persson (1997), Miltersen and Persson (1999) and Bacinello (2001). In practice, insurance companies use a bonus account for undistributed surplus in order to smooth the accumulation factors over time. When including a bonus account, the price of an insurance contract depends on the bonus mechanism. For some different possible bonus mechanisms and their impact on prices, see Grosen and Jørgensen (2000), Hansen and Miltersen (2002) and Miltersen and Persson (2003). Another feature encountered in practice is the possibility for the insured to surrender, which is included e.g. in Grosen and Jørgensen (2000). The bankruptcy of major life insurance companies in England and Japan have also underlined the importance of including the risk of the company defaulting. This is done inBriys and de Varenne (1997).

The main purpose of the above mentioned papers is essentially to obtain the arbitrage free price of an insurance contract by considering the development of the insurance contract until termination. The aim of the present chapter is slightly different from that of pricing individual contracts. Here, the goal is to determine a fair distribution of assets between the owners of the insurance company and the portfolio of insured at the end of each accumulation period. Thus, the model considered is essentially a 1-period model with one accumulation period. In the model the accumulation factor, announced by the company prior to the accumulation period, is viewed as an exogenous parameter. Hence, we avoid the modelling of the announced accumulation factor, which is quite difficult since competition seems to play a major role in the decision process. In contrast to many companies, we do not view the announced accumulation factor as binding. Thus, the actual and announced accumulation factors may differ when experiencing poor investment returns and/or an adverse development of the insurance portfolio. To determine the distribution of the assets between the deposit, the bonus reserve and the equity capital at the end of the accumulation period, we define a distribution scheme. Within this scheme, the only unknown parameter is the interest rate used, in addition to the riskfree interest rate, to accumulate the equity capital in periods when possible. We assume that the company is allowed to invest in a financial market described by a Black-Scholes model. This market is known to be complete and arbitrage free. A distribution scheme is considered as fair, if it does not introduce arbitrage possibilities for the owners or the insurance portfolio. When considering a portfolio of capital insurances, the distribution scheme depends entirely on the development of the financial market, and since the financial market is complete and arbitrage free, we can derive a simple equation, which has to be fulfilled by a distribution scheme in order not to introduce arbitrage possibilities. Thus, we are able to find an equation for the unique fair additional interest rate. For a portfolio of pure endowments the distribution scheme depends on both the financial market and the development of the insurance portfolio. Hence, we are in an incomplete market. Thus, infinitely many equivalent martingale measures exist, such that the principle of no arbitrage yields infinitely many possible equations from which to derive a fair distribution. However, for a fixed equivalent martingale measure, we again have a unique equation for the fair additional interest rate. Since the equations derived for the fair additional interest rate are implicit equations, we have to use numerical techniques to derive the result. Hence, in contrast to other papers including bonus accounts, no simulation is necessary.

We point out that the results in this chapter for the fair additional interest rate are based on a simple financial model with constant interest rate and a deterministic mortality intensity. Hence, we only take the financial risk associated with investments in stocks and the unsystematic mortality risk into account. The fair additional interest rate would be larger if we were to add interest rate risk and/or systematic mortality risk to the model. Note that we distinguish between systematic mortality risk, referring to the future development of the underlying mortality intensity, and unsystematic mortality risk, referring to a possible adverse development of the insured portfolio with known mortality intensity, see Chapter 3. Furthermore expenses and the associated risk have been disregarded in the study. In addition to the measurable risks mentioned above one could consider operational risk as well. Thus, the fair additional interest rate determined in this chapter serves as a lower bound for the fair additional interest rate in practice.

The chapter is organized as follows: In Section 2.2, a simplified balance sheet and a short description of the accounts are given. The financial model and the relevant financial terminology is introduced in Section 2.3. In Section 2.4, a company with an insurance portfolio of capital insurances is considered. Given different investment strategies, we decompose the terminal equity capital into payoffs from standard options, such that each investment strategy leads to an equation for the fair additional interest rate. Section 2.5 studies the case of a portfolio of pure endowments. In this case, the value for the fair additional interest rate depends on the chosen equivalent martingale measure. Since the equations obtained in Sections 2.4 and 2.5 for the the fair additional interest rate are implicit equations only, we supply numerical results in Section 2.6. In Section 2.7 we discuss some possible changes to the distribution mechanism and their impact on the results. A discussion on the realism and versatility of the model is given in Section 2.8, whereas Section 2.9 contains a conclusion. Proofs and calculations of some technical results can be found in Section 2.10

2.2 The balance sheet

To describe the assets and liabilities of the insurance company we use the following simplified balance sheet.

Assets	Liabilities
A	V
	U
	E
A	A

The asset side consists of the account A only, while the liability side is comprised of three accounts: V, U and E. The bottom line of the balance sheet just states that the assets and liabilities must balance, i.e. $V + U + E = A$. We now give a detailed description of the individual accounts.

Account V (the deposit) is the total deposit of the insurance portfolio. The deposit is allocated to the insured on an individual basis. In case of a capital insurance or a pure endowment, the individual deposit at time of termination is the sum paid to the insured. Whenever an insurance contract states a guaranteed periodic accumulation factor, the guarantee applies to the deposit. Capital allocated to the deposit belongs to the individual owning the actual account, and cannot be transferred to the deposit of another insured or other accounts on the liability side.

Account U (the bonus reserve) is the undistributed surplus allocated to the insurance portfolio as a whole. It is used by the company to smooth deposit accumulation factors over time. Capital allocated to the bonus reserve cannot freely be transferred to the equity capital. Such a transfer may only take place as a payment to the equity capital for the risk associated with the insurance contracts.

Account E (the equity capital) is the capital belonging to the owners of the company.

Account A (the assets) describes the value of the assets of the insurance company. We assume that the insurance company invests in the financial market described in Section 2.3. In order to consider the risk associated with the insurance contracts only, we assume that the company invests the amount E_0 in the savings account, and the amount $V_0 + U_0$ in an admissible strategy $\varphi = (\vartheta, \eta)$ with value process $\mathcal{V}(\varphi)$. Thus, at time t, $t \in [0, T]$, we have

$$A_t = e^{rt} E_0 + \frac{\mathcal{V}_t(\varphi)}{\mathcal{V}_0(\varphi)} (V_0 + U_0).$$

It now follows from the following argument that we without loss of generality may assume that

$$V_0 + U_0 = \mathcal{V}_0(\varphi) \tag{2.2.1}$$

such that $A_t = e^{rt}E_0 + \mathcal{V}_t(\varphi)$. Assume that (2.2.1) does not hold. Then the self-financing strategy given by

$$\widetilde{\varphi} = \frac{V_0 + U_0}{\mathcal{V}_0(\varphi)}\varphi = \left(\frac{V_0 + U_0}{\mathcal{V}_0(\varphi)}\vartheta, \frac{V_0 + U_0}{\mathcal{V}_0(\varphi)}\eta\right)$$

fulfills

$$e^{rt}E_0 + \mathcal{V}_t(\widetilde{\varphi}) = e^{rt}E_0 + \frac{V_0 + U_0}{\mathcal{V}_0(\varphi)}\mathcal{V}_t(\varphi) = A_t, \quad t \in [0, T].$$

A similar simplified balance sheet is used in Grosen and Jørgensen (2000), Hansen and Miltersen (2002) and Miltersen and Persson (2003). However, the number of accounts on the liability side of the balance sheet, and their interpretation varies.

2.3 The financial model

We consider a financial market described by the standard Black–Scholes model. Here, the market consists of two traded assets: A risky asset with price process S and a riskfree asset with price process B. The risky asset is usually referred to as a stock and the riskfree asset as a savings account. The price processes are defined on a probability space $(\Omega, \mathcal{F}, P)$, and the P-dynamics of the price processes are given by

$$\begin{aligned} dS_t &= \alpha S_t dt + \sigma S_t d\widetilde{W}_t, \quad S_0 > 0, \\ dB_t &= rB_t dt, \quad B_0 = 1, \end{aligned} \tag{2.3.1}$$

where $(\widetilde{W}_t)_{0 \le t \le T}$ is a Wiener process on the interval $[0, T]$ under P, with T being a fixed finite time horizon. The coefficient σ is a strictly positive constant, while α and r are non-negative constants. The filtration $\mathbb{G} = (\mathcal{G}_t)_{0 \le t \le T}$ is the P-augmentation of the natural filtration generated by (B, S), i.e. $\mathcal{G}_t = \mathcal{G}_t^+ \vee \mathcal{N}$, where $\mathcal{N}$ is the σ-algebra generated by all P-null sets and

$$\mathcal{G}_t^+ = \sigma\{(B_u, S_u), u \le t\} = \sigma\{S_u, u \le t\} = \sigma\{\widetilde{W}_u, u \le t\}.$$

Here, we have used the strict positivity of σ in the last equality. We interpret α as the mean rate of return of the stock, σ as the standard deviation of the rate of return and r as the short rate of interest. The constant ν defined by $\nu = \frac{\alpha - r}{\sigma}$ is known as the market price of risk associated with S. It is well-known, see e.g. Musiela and Rutkowski (1997), that in the Black–Scholes model, the probability measure Q^0 defined by

$$\frac{dQ^0}{dP} \equiv O_T = e^{-\nu \widetilde{W}_T - \frac{1}{2}\nu^2 T}$$

is the unique equivalent martingale measure. Hence, Q^0 is a probability measure equivalent to P under which all discounted price processes on the financial market are (local) martingales. The Q^0-dynamics of the price processes are

$$\begin{aligned} dS_t &= rS_t dt + \sigma S_t dW_t, \quad S_0 > 0, \\ dB_t &= rB_t dt, \quad B_0 = 1, \end{aligned} \tag{2.3.2}$$

where $(W_t)_{0\leq t\leq T}$ is a Wiener process on the interval $[0,T]$ under Q^0.

A trading strategy is an adapted process $\varphi = (\vartheta, \eta)$ satisfying certain integrability conditions. The pair $\varphi_t = (\vartheta_t, \eta_t)$ is interpreted as the portfolio held at time t. Here, ϑ_t and η_t, respectively, denote the number of stocks and the discounted deposit in the savings account in the portfolio at time t. The value process $\mathcal{V}(\varphi)$ associated with φ is given by

$$\mathcal{V}_t(\varphi) = \vartheta_t S_t + \eta_t B_t.$$

A strategy φ is called self-financing if

$$\mathcal{V}_t(\varphi) = \mathcal{V}_0(\varphi) + \int_0^t \vartheta_u dS_u + \int_0^t \eta_u dB_u.$$

Thus, the value at any time t of a self-financing strategy is the initial value added trading gains from investing in stocks and interest earned on the deposit in the savings account; withdrawals and additional deposits are not allowed during $(0,T)$. A self-financing strategy $\varphi = (\vartheta, \eta)$ is called admissible if $(\vartheta, \eta) \geq 0$, which guarantees that $\mathcal{V}_t(\varphi) \geq 0$ P-a.s. for all $t \in [0,T]$. We restrict the investment strategies of the insurance company to admissible strategies. A self-financing strategy is a so-called arbitrage if $\mathcal{V}_0(\varphi) = 0$ and $\mathcal{V}_T(\varphi) \geq 0$ P-a.s. with $P(\mathcal{V}_T(\varphi) > 0) > 0$. A contingent claim (or a derivative) in the model $(B, S, \mathbb{G})$ with maturity T is a $\mathcal{G}_T$-measurable, Q^0-square integrable random variable X. A contingent claim is called attainable if there exists a self-financing strategy such that $\mathcal{V}_T(\varphi) = X$, P-a.s. An attainable claim can thus be replicated perfectly by investing $\mathcal{V}_0(\varphi)$ at time 0 and investing during the interval $[0,T]$ according to the self-financing strategy φ. Hence, at any time t, there is no difference between holding the claim X and the portfolio φ_t. In this sense, the claim X is redundant in the market, and from the assumption of no arbitrage it follows that the price of X at time t must be $\mathcal{V}_t(\varphi)$. Thus, the initial investment $\mathcal{V}_0(\varphi)$ is the unique arbitrage free price of X. If all contingent claims are attainable, the model is called complete and otherwise it is called incomplete. It is well-known from the financial literature, see e.g. Björk (2004), that the Black–Scholes model is complete and arbitrage free, and that the discounted value process associated with any self-financing trading strategy is a Q^0-martingale. Throughout the chapter, we denote by S^* the discounted stock price and by $\mathcal{V}^*(\varphi)$ the discounted value process. Furthermore we use the asterisk to denote that a constant or function has been multiplied by e^{-rT}, i.e. discounted from time T to time 0.

2.4 Capital insurances

Consider a life insurance company whose insurance portfolio constitutes capital insurances exclusively. A capital insurance pays out a sum insured at a specified time, whether the insured is alive or not. For simplicity we assume that no payments between the company and the insured take place during $(0,T)$. In this case, we can disregard the individual contracts and focus on the total insurance portfolio.

The aim of this section is, while respecting the general terms of the contracts, to determine an arbitrage free distribution of the assets at time T among the accounts on the liability side. We shall refer to such a distribution as fair, see Section 2.4.2 for more details. We assume that all insured are promised the same accumulation factor G_T on the deposit in the period $(0, T)$. In practice, we often have $G_T \geq 1$. The consequence of the guarantee is that the total deposit should be at least $G_T V_0$ at time T whenever possible. If the company is unable to cover the guarantee, all assets are allocated to the deposit and paid to the insured in cash, while the company is declared bankrupt.

Remark 2.4.1 Two possible choices for the guaranteed accumulation factor are $1 + gT$ and e^{gT}, depending on whether g is expressed in terms of a periodical or a continuously compounding rate. If $T = 1$ and time is measured in years, then $G_1 = 1 + g$ corresponds to a guaranteed annual interest rate of g.

$\square$

Remark 2.4.2 For the company to survive in the long run, we should have $G_T \leq e^{rT}$. However, since we are interested in short term conditions only, also the reverse situation is relevant.

$\square$

To be consistent with common practice, the company at time 0 announces a deposit accumulation factor K_T, $K_T \geq G_T$, by which they intend to accumulate the deposit at time T. In contrast to G_T, we do not consider K_T as legally binding. Hence, at time T the company is allowed to use an accumulation factor different from K_T for the actual accumulation. However, using an accumulation factor different from K_T affects the credibility of the company, and thus, it is not done frequently in practice. In order to model this reluctancy in a simple way without removing the possibility of using an accumulation factor different from K_T, we assume that the company uses K_T unless the value of the risky investments at time T, $\mathcal{V}_T(\varphi)$, is less than $K_T V_0(1+\gamma)$. Here, the factor γ, $\gamma \geq 0$, is the proportion of the deposit which is the target for the minimal bonus reserve, as decided by the management of the insurance company.

To compensate the equity capital for the exposure to the financial risk inherent in capital insurances, we introduce the parameter ρ, which represents the interest rate credited to the equity capital in addition to the riskfree interest rate, whenever such an additional return is possible.

2.4.1 Distribution scheme

The distribution scheme used by the company to distribute assets at time T between the three accounts on the liability side depends on the development of the assets of the company and hence on the financial market. We distinguish between the following three situations for the development of the assets:

1. $A_T < G_T V_0$: In this case, the company does not have sufficient assets to cover the guarantee. Since the interest of the insured takes priority over the interest of the

owners of the company, all capital is allocated to the deposit, and the equity is set to 0, that is

$$\begin{aligned} V_T &= A_T, \\ E_T &= 0, \\ U_T &= 0. \end{aligned}$$

2. $G_T V_0 \leq A_T < K_T V_0(1+\gamma) + e^{rT} E_0$: Here, the assets are sufficient to meet the guaranteed accumulation of the deposit. However, using the announced deposit accumulation factor would leave the company with a bonus reserve less than the target for the minimal bonus reserve, γV_T. Hence, the company chooses to accumulate the deposit by the guaranteed accumulation factor G_T. This way, the company obtains the maximal possible bonus reserve, which in some cases exceeds γV_T. The equity capital at time T is given by the equity capital at time 0 accumulated with the interest rate $r + \rho$ or the total assets deducted the deposit at time T, whichever is smallest. The bonus account is calculated residually as the assets subtracted the deposit and the equity capital. This leads to the following distribution:

$$\begin{aligned} V_T &= G_T V_0, \\ E_T &= \min\left(e^{(r+\rho)T} E_0, A_T - V_T\right), \\ U_T &= A_T - V_T - E_T. \end{aligned}$$

3. $e^{rT} E_0 + K_T V_0(1+\gamma) \leq A_T$: This outcome leaves the company with a bonus reserve larger than γV_T after accumulating the deposit with K_T. The distribution is given by an expression similar to the one in case 2 with G_T substituted by K_T. Thus

$$\begin{aligned} V_T &= K_T V_0, \\ E_T &= \min\left(e^{(r+\rho)T} E_0, A_T - V_T\right), \\ U_T &= A_T - V_T - E_T. \end{aligned}$$

Note that in the distribution scheme we first use the bonus reserve to cover the accumulation of the deposit, and if this is insufficient, we then use the equity capital.

In the distribution scheme, the only unknown parameter is ρ. Hence, determining the fair distribution scheme reduces to determining the fair value of ρ. Since $E_T \leq e^{(r+\rho)T} E_0$, a necessary requirement for a distribution scheme to be arbitrage free is $\rho \geq 0$. Hence, the referral to ρ as the *additional* rate of return. Furthermore, we immediately observe from the distribution scheme that E_T is non-decreasing in ρ for all ω. If further A_T is stochastic, i.e. if the company invests some capital in the risky asset, then $P(A_T - V_T \geq e^{(r+\rho)T} E_0) > 0$ for all finite ρ. Hence the set of ω's for which E_T is strictly increasing in ρ has a positive probability. We thus have that the fair value of ρ, if it exists, is unique.

2.4.2 Fair distribution

A distribution scheme is said to be fair if it does not introduce arbitrage opportunities for the insurance company or the insurance portfolio. Since the size of the accounts on the liability side of the balance sheet depends on the development of the financial market only, we can view E_T and $V_T + U_T$ as contingent claims in the complete and arbitrage free market $(B, S, \mathbb{G})$. Hence, the claims E_T and $V_T + U_T$ have unique prices. Thus, the distribution scheme is fair, if

$$E_0 = e^{-rT} \mathrm{E}^{Q^0} \left[E_T \right], \tag{2.4.1}$$

and

$$V_0 + U_0 = e^{-rT} \mathrm{E}^{Q^0} \left[V_T + U_T \right]. \tag{2.4.2}$$

Note that since we are interested in the distribution of assets between the insurance portfolio as a whole and the company, and not between the insured individuals, we do not distinguish between the deposit and the bonus reserve in (2.4.2). Depending on the bonus strategy of the company, the individual contracts may or may not be fair, but for the insured portfolio as a whole the contracts are fair if (2.4.2) is fulfilled. Since the assets are invested in a self-financing portfolio we have

$$\mathrm{E}^{Q^0}[e^{-rT} A_T] = A_0,$$

such that (2.4.1) is satisfied if and only if (2.4.2) is satisfied. Hence, determining the fair value of ρ, if it exists, amounts to solve (2.4.1) with respect to ρ.

2.4.3 Buy and hold strategy

Consider a buy and hold strategy, which is the simplest example of a self-financing strategy. In the buy and hold strategy the company invests ϑS_0 and η, respectively, in the risky and the riskfree asset at time 0 and no trading takes place until time T. Hence, the value at time T of the risky portfolio is

$$\mathcal{V}_T(\varphi) = \vartheta S_T + \eta e^{rT}.$$

Assume the company follows a buy and hold strategy with $\vartheta > 0$, i.e. with some investments in the risky asset. We now derive an implicit expression for the fair value of ρ by decomposing the value of the equity capital at time T into payoffs from standard European options on the stock.

Define the quantities s_1 and s_2 as the values of S_T which solve the two equations

$$G_T V_0 = e^{rT} E_0 + \vartheta S_T + \eta e^{rT}, \tag{2.4.3}$$

and

$$K_T V_0 (1 + \gamma) = \vartheta S_T + \eta e^{rT}, \tag{2.4.4}$$

respectively. Hence, s_1 is the lowest stock price at time T, which does not lead to bankruptcy of the insurance company, while s_2 is the lowest stock price for which, the company uses K_T as accumulation factor. Solving (2.4.3) and (2.4.4) for S_T we get

$$s_1 = \frac{G_T V_0 - \eta e^{rT} - e^{rT} E_0}{\vartheta}, \tag{2.4.5}$$

and

$$s_2 = \frac{K_T V_0(1+\gamma) - \eta e^{rT}}{\vartheta}. \tag{2.4.6}$$

Note that even though the stock price is positive, s_1 and s_2 might be negative. If s_1 is negative, the capital invested in the savings account is sufficient to ensure that the company is not bankrupted, whereas a negative value for s_2 corresponds to the case, where the capital invested in the savings account is sufficient to ensure that the company always uses K_T to accumulate the deposit. Using s_1 and s_2, we can rewrite the value of the equity capital at time T as

$$E_T^B = 1_{(S_T < s_1)} E_T^B + 1_{(s_1 \le S_T < s_2)} E_T^B + 1_{(s_2 \le S_T)} E_T^B \equiv E_T^{B1} + E_T^{B2} + E_T^{B3}.$$

Here, the superscript B indicates that we are working with a buy and hold strategy. The expressions for the equity capital in the different situations can be found in Section 2.4.1. Since E_T^{B1} is the equity capital in case of bankruptcy, it is equal to 0.

In order to decompose E_T^{B2}, we first recall that

$$E_T^{B2} = 1_{(s_1 \le S_T < s_2)} \min\left(e^{(r+\rho)T} E_0, \mathcal{V}_T(\varphi) + e^{rT} E_0 - G_T V_0\right). \tag{2.4.7}$$

In order to calculate (2.4.7), we determine s_3 which is the maximum value of S_T for which

$$\mathcal{V}_T(\varphi) + e^{rT} E_0 - G_T V_0 \le e^{(r+\rho)T} E_0. \tag{2.4.8}$$

Hence s_3 is the largest value for the stock price at time T for which the assets are insufficient to accumulate the equity capital with interest rate $r+\rho$, if we accumulate the deposit with G_T. Solving (2.4.8) we get

$$s_3 = \frac{(e^{\rho T} - 1) e^{rT} E_0 + G_T V_0 - \eta e^{rT}}{\vartheta}. \tag{2.4.9}$$

Rewriting s_3 as

$$s_3 = s_1 + \frac{e^{(r+\rho)T} E_0}{\vartheta},$$

and using that $\min(r, \rho) > -\infty$ and $\vartheta > 0$ we observe that $s_3 > s_1$, such that inserting in (2.4.7) gives

$$\begin{aligned} E_T^{B2} &= 1_{(s_1 \le S_T < \min(s_2,s_3))} \left(e^{rT} E_0 + \mathcal{V}_T(\varphi) - G_T V_0\right) + 1_{(\min(s_2,s_3) \le S_T < s_2)} e^{(r+\rho)T} E_0 \\ &= 1_{(s_1 \le S_T < \min(s_2,s_3))} \vartheta \left(S_T - s_1\right) + 1_{(\min(s_2,s_3) \le S_T < s_2)} e^{(r+\rho)T} E_0 \\ &= \vartheta \Bigg((S_T - s_1)^+ - (S_T - \min(s_2,s_3))^+ - (\min(s_2,s_3) - s_1)\, 1_{(\min(s_2,s_3) < S_T)} \Bigg) \\ &\quad + e^{(r+\rho)T} E_0 \left(1_{(\min(s_2,s_3) \le S_T)} - 1_{(s_2 \le S_T)}\right). \end{aligned} \tag{2.4.10}$$

Thus, E_T^{B2} can be decomposed into two terms. The first term is the number of stocks multiplied by the difference between the payoff from two European call options with strikes s_1 and $\min(s_2, s_3)$ subtracted the payoff from a so-called binary cash call option with strike $\min(s_2, s_3)$ and cash $\min(s_2, s_3) - s_1$. The second term is the equity capital accumulated with interest rate $r + \rho$ multiplied by the difference between the payoff from two binary cash call options with strikes $\min(s_2, s_3)$ and s_2. For a description of the these and other options mentioned in this chapter see Musiela and Rutkowski (1997).

In order to decompose E_T^{B3} we first determine s_4, which is the largest value of S_T for which

$$e^{rT}E_0 + \mathcal{V}_T(\varphi) - K_T V_0 \leq e^{(r+\rho)T}E_0.$$

Solving for S_T we get

$$s_4 = \frac{(e^{\rho T} - 1)e^{rT}E_0 + K_T V_0 - \eta e^{rT}}{\vartheta}. \tag{2.4.11}$$

The interpretation of s_4 is similar to that of s_3, however here the deposit is accumulated with K_T. Calculations similar to those for E_T^{B2} give

$$\begin{aligned}
E_T^{B3} &= 1_{(s_2 \leq S_T)} \min\left(e^{(r+\rho)T}E_0, e^{rT}E_0 + \mathcal{V}_T(\varphi) - K_T V_0\right) \\
&= 1_{(s_2 \leq S_T < \max(s_2, s_4))} \vartheta\,(S_T - s_5) + 1_{(\max(s_2,s_4) \leq S_T)} e^{(r+\rho)T}E_0 \\
&= \vartheta\Bigg((S_T - s_2)^+ - (S_T - \max(s_2, s_4))^+ + 1_{(s_2 \leq S_T)}(s_2 - s_5) \\
&\quad - 1_{(\max(s_2,s_4) \leq S_T)}\,(\max(s_2, s_4) - s_5)\Bigg) + 1_{(\max(s_2,s_4) \leq S_T)} e^{(r+\rho)T}E_0,
\end{aligned} \tag{2.4.12}$$

where we have used the notation

$$s_5 = \frac{K_T V_0 - \eta e^{rT} - e^{rT}E_0}{\vartheta}. \tag{2.4.13}$$

Hence, E_T^{B3} can be decomposed into two terms as well. The first term is the number of stocks multiplied by the payoff from known European options, and the second term is the equity capital accumulated with interest rate $r + \rho$ multiplied by the payoff from a binary cash call option. Denote by BCC and C, respectively, the price of a *binary cash call* and a *call option*. It is well-known that BCC and C are given by

$$BCC(x, S_0, \sigma) = E^{Q^0}\left[e^{-rT}1_{(x \leq S_T)}\right] = \begin{cases} e^{-rT}\Phi\left(\frac{\log\left(\frac{S_0}{x}\right) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right), & x > 0, \\ e^{-rT}, & x \leq 0, \end{cases}$$

and

$$\begin{aligned}
C(x, S_0, \sigma) &= E^{Q^0}\left[e^{-rT}(S_T - x)^+\right] \\
&= \begin{cases} S_0\Phi\left(\frac{\log\left(\frac{S_0}{x}\right) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) - e^{-rT}x\Phi\left(\frac{\log\left(\frac{S_0}{x}\right) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right), & x > 0, \\ S_0 - e^{-rT}x, & x \leq 0, \end{cases}
\end{aligned}$$

where Φ denotes the distribution function for the standard normal distribution. To simplify notation, we use the short hand notation $BCC(x)$ and $C(x)$ in expressions involving many option prices with the same initial value and volatility. Applying criterion (2.4.1) we obtain the following proposition

Proposition 2.4.3
If a company invests according to a buy and hold strategy the fair value of ρ satisfies

$$\begin{aligned}
E_0 = e^{(r+\rho)T} E_0 \Bigg(& BCC(\min(s_2, s_3)) - BCC(s_2) + BCC(\max(s_2, s_4))\Bigg) \\
& + \vartheta \Bigg(C(s_1) - C(\min(s_2, s_3)) + C(s_2) - C(\max(s_2, s_4)) \\
& - (\min(s_2, s_3) - s_1) BCC(\min(s_2, s_3)) \\
& + (s_2 - s_5) BCC(s_2) - (\max(s_2, s_4) - s_5) BCC(\max(s_2, s_4))\Bigg),
\end{aligned}$$

where $s_1 - s_5$ are given by (2.4.5), (2.4.6), (2.4.9), (2.4.11) and (2.4.13) and all option prices are calculated using initial value S_0 and volatility σ.

If $\vartheta = 0$, all assets are invested in the savings account. Hence, the value at time T of the assets is deterministic and equal to $A_T = e^{rT} A_0$. In this case we obtain the following result for the fair value of ρ.

Proposition 2.4.4
If a company invests in the savings account only, a fair value of ρ must satisfy

1. *If $e^{rT} A_0 < G_T V_0$ then no values of ρ exist for which the distribution scheme fair.*
2. *If $G_T V_0 \leq e^{rT} A_0 < K_T V_0 (1 + \gamma) + e^{rT} E_0$, then the distribution scheme is fair, if either of the following apply*
   (a) *$e^{rT} E_0 < e^{rT} A_0 - G_T V_0$ and $\rho = 0$.*
   (b) *$G_T = e^{rT} \frac{V_0 + U_0}{V_0}$ and $\rho \geq 0$.*
3. *If $K_T V_0 (1 + \gamma) + e^{rT} E_0 < e^{rT} A_0$, then the distribution scheme is fair, if either of the following apply*
   (a) *$e^{rT} E_0 < e^{rT} A_0 - K_T V_0$ and $\rho = 0$.*
   (b) *$K_T = e^{rT} \frac{V_0 + U_0}{V_0}$ and $\rho \geq 0$.*

Proof of Proposition 2.4.4: See Section 2.10.1.

□

Proposition 2.4.4 has the following interpretation: If the assets and hence the accounts on the liability side are deterministic at time T the distribution scheme is fair if only if $E_T = e^{rT}E_0$. Since this is intuitively clear, the proposition is not particularly interesting and stated for completeness, only.

We end this section with a result for the probability of ruin of the company at time T.

Proposition 2.4.5
The probability, $p_{ruin}(\varphi)$, that a company, using the buy and hold strategy φ, is ruined at time T is given by

$$p_{ruin}(\varphi) = \Phi\left(\frac{\log\left(\frac{s_1}{S_0}\right) - \left(\alpha - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right).$$

Proof of Proposition 2.4.5: The company is ruined at time T if $A_T < G_T V_T$. Hence,

$$p_{ruin}(\varphi) = P\left[A_T < G_T V_0\right] = P\left[S_T < \frac{G_T V_T - \eta e^{rT} - e^{rT}E_0}{\vartheta}\right] = P\left[S_T < s_1\right].$$

Here, we have used the definition of s_1 from (2.4.5). The result now follows by inserting the solution, $S_T = S_0 e^{(\alpha-\sigma^2/2)T+\sigma\widetilde{W}_T}$, to the stochastic differential equation for the dynamics of S under P given in (2.3.1).

□

2.4.4 Constant relative portfolio weights

Now consider the case where the company continuously adjusts the investment portfolio, such that at all times, $t \in [0,T]$, the proportion $\delta \in [0,1]$ of the portfolio value is invested in stocks. Hence,

$$\vartheta_t S_t = \delta\mathcal{V}_t(\varphi) \text{ and } \eta_t B_t = (1-\delta)\mathcal{V}_t(\varphi).$$

In this case the dynamics under Q^0 of the value process of the self-financing strategy are

$$\begin{aligned} d\mathcal{V}_t(\varphi) &= \vartheta_t dS_t + \eta_t dB_t \\ &= \vartheta_t(rS_t dt + \sigma S_t dW_t) + \eta_t r B_t dt \\ &= r\mathcal{V}_t(\varphi)dt + \delta\sigma\mathcal{V}_t(\varphi)dW_t. \end{aligned}$$

We note that the dynamics of the value process are of the same form as the dynamics of the stock price. For $\delta > 0$ calculations similar to those for a buy and hold strategy give

Proposition 2.4.6
When investing in a portfolio with constant relative portfolio weights the fair value of ρ solves the following equation

$$\begin{aligned} E_0 = e^{(r+\rho)T}E_0 &\left(BCC(\min(v_2,v_3)) - BCC(v_2) + BCC(\max(v_2,v_4))\right) \\ &+ C(v_1) - C(\min(v_2,v_3)) + C(v_2) - C(\max(v_2,v_4)) \\ &- (\min(v_2,v_3) - v_1)BCC(\min(v_2,v_3)) \\ &+ (v_2 - v_5)BCC(v_2) - (\max(v_2,v_4) - v_5)BCC(\max(v_2,v_4)), \end{aligned}$$

where

$$\begin{aligned}
v_1 &= G_T V_0 - e^{rT} E_0, \\
v_2 &= K_T V_0 (1+\gamma), \\
v_3 &= (e^{\rho T} - 1) e^{rT} E_0 + G_T V_0, \\
v_4 &= (e^{\rho T} - 1) e^{rT} E_0 + K_T V_0, \\
v_5 &= K_T V_0 - e^{rT} E_0,
\end{aligned}$$

and all option prices are calculated with initial value $V_0 + U_0$ and volatility $\delta\sigma$.

If $\delta = 0$ we are in exactly the same situation as in the buy and hold strategy with $\vartheta = 0$, so Proposition 2.4.4 applies.

Note that under P the dynamics of the value process for a self-financing strategy with constant relative portfolio weights are

$$\begin{aligned}
d\mathcal{V}_t(\varphi) &= \vartheta_t dS_t + \eta_t dB_t \\
&= \vartheta_t(\alpha S_t dt + \sigma S_t d\widetilde{W}_t) + \eta_t r B_t dt \\
&= (r + \delta(\alpha - r))\mathcal{V}_t(\varphi) dt + \delta\sigma \mathcal{V}_t(\varphi) d\widetilde{W}_t.
\end{aligned}$$

This leads to the following proposition for the probability of ruin at time T.

Proposition 2.4.7
The probability of ruin, $p_{ruin}(\varphi)$, is given by

$$p_{ruin}(\varphi) = \Phi\left(\frac{\log\left(\frac{v_1}{V_0+U_0}\right) - \left(r + \delta(\alpha - r) - \frac{1}{2}(\delta\sigma)^2\right) T}{\delta\sigma\sqrt{T}}\right).$$

2.4.5 Buy and hold with stop-loss if solvency is threatened

Consider the case where the regulatory institutions set a solvency requirement for the insurance company. As in practice, the requirement considered here is a requirement on the equity capital. After accumulating the deposit at time T, the equity capital should be at least a proportion β of the deposit, i.e. $E_T \geq \beta V_T$. Since the solvency requirement must be satisfied at the end of each accumulation period we know that $E_0 \geq \beta V_0$. Otherwise the company would have been declared insolvent already. If further $e^{rT} E_0 \geq \beta K_T V_0$ and $A_0 \geq e^{-rT} G_T V_0 (1+\beta)$ the company may avoid insolvency by rebalancing the risky portfolio to include investments in the savings account only, if the value of the assets reaches the lower boundary

$$A_t = e^{-r(T-t)} G_T V_0 (1+\beta). \tag{2.4.14}$$

Now assume the company invests in a buy and hold strategy as introduced in Section 2.4.3, until a possible intervention. In this case we can write (2.4.14) in terms of the discounted stock price

$$S_t^* = \frac{e^{-rT} G_T V_0(1+\beta) - \eta - E_0}{\vartheta} \equiv H.$$

Remark 2.4.8 The stop-loss criterion in (2.4.14) is just one of many possible criterions. If $V_0 + U_0 \geq e^{-rT} K_T V_0(1+\gamma)$ the alternative criterion $\mathcal{V}_t(\varphi) = e^{-r(T-t)} K_T V_0(1+\gamma)$ in addition to solvency also guarantees that K_T is used as accumulation factor.

□

Decomposing the equity capital we first distinguish between whether the company has intervened or not

$$E_T^{BS} = 1_{(\inf_{0\leq t\leq T} S_t^* \leq H)} E_T^{BS} + 1_{(\inf_{0\leq t\leq T} S_t^* > H)} E_T^{BS} \equiv E_T^{BS1} + E_T^{BS2}.$$

Here, the superscript BS indicates that the company uses a buy and hold strategy with stop-loss. When $\inf_{0\leq t\leq T} S_t^* \leq H$ the asset value is deterministic and equal to $G_T V_0(1+\beta)$, such that

$$E_T^{BS1} = 1_{(\inf_{0\leq t\leq T} S_t^* \leq H)} \min\left(e^{(r+\rho)T} E_0, \beta G_T V_0\right) = 1_{(\inf_{0\leq t\leq T} S_t^* \leq H)} \beta G_T V_0.$$

Here, we have used that $e^{rT} E_0 \geq \beta K_T V_0 \geq \beta G_T V_0$ in both equalities and $\rho \geq 0$ in the last equality. We recognize this as the payoff from a down-and-in barrier option on the discounted stock price with the deterministic payoff $\beta G_T V_0$ when knocked in. When $\inf_{0\leq t\leq T} S_t^* > H$ it holds in particular that

$$S_T > \frac{G_T V_0(1+\beta) - \eta e^{rT} - e^{rT} E_0}{\vartheta} \equiv s_1^{\beta}.$$

The assumptions on the equity capital and the fact that $\rho \geq 0$ gives that $s_3 \geq s_1^{\beta}$. Hence, calculations similar to those leading to (2.4.10) and (2.4.12) gives

$$\begin{aligned}
E_T^{BS2} = 1_{(\inf_{0\leq t\leq T} S_t^* > H)} \Bigg(& e^{(r+\rho)T} E_0 \left(1_{(\min(s_2^*, s_3^*) \leq S_T^*)} - 1_{(s_2^* \leq S_T^*)} + 1_{(\max(s_2^*, s_4^*) \leq S_T^*)} \right) \\
& + \vartheta \Bigg(e^{rT} \Bigg(\left(S_T^* - s_1^{\beta,*}\right)^+ - (S_T^* - \min(s_2^*, s_3^*))^+ + (S_T^* - s_2^*)^+ - (S_T^* - \max(s_2^*, s_4^*))^+ \Bigg) \\
& + \left(s_1^{\beta} - s_1\right) 1_{(s_1^{\beta,*} < S_T^*)} - (\min(s_2, s_3) - s_1)\, 1_{(\min(s_2^*, s_3^*) < S_T^*)} \\
& + (s_2 - s_5) 1_{(s_2^* \leq S_T^*)} - 1_{(\max(s_2^*, s_4^*) \leq S_T^*)} (\max(s_2, s_4) - s_5) \Bigg) \Bigg).
\end{aligned}$$

Thus, the equity capital can be written in terms of payoffs from barrier options on the discounted stock price. To indicate that an option is written on the discounted stock price, we equip the option price by an asterisk (*). When working with barrier options we equip the notation for the corresponding European option, or 1 in case of a deterministic value, with two letters as a sub- or superscript depending on whether we are dealing with a *down* or an *up* barrier option. The first letter is the barrier and the second describe whether we are dealing with an *out*, denoted O, or an *in*, denoted I, option. Using Björk (2004, Theorem 18.8) we are able to write prices of the relevant barrier options on the discounted stock price in terms of prices of European options. For $S_0 > H$ we obtain the following option prices: A down-and-out option with payoff 1

$$\begin{aligned} 1^*_{HO}(S_0,\sigma) &= E^{Q^0}\left[e^{-rT}1_{(\inf_{0\le t\le T} S^*_t > H)}\right] \\ &= \begin{cases} e^{-rT}\Phi\left(\frac{\log\left(\frac{S_0}{H}\right)-\frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) - \left(\frac{S_0}{H}\right)e^{-rT}\Phi\left(\frac{\log\left(\frac{H}{S_0}\right)-\frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right), & H > 0, \\ e^{-rT}, & H \le 0, \end{cases} \end{aligned}$$

a down-and-out binary cash call option

$$\begin{aligned} BCC^*_{HO}(x,S_0,\sigma) &= E^{Q^0}\left[e^{-rT}1_{(\inf_{0\le t\le T} S^*_t > H)}1_{(x\le S^*_T)}\right] \\ &= \begin{cases} BCC^*(x,S_0,\sigma) - \frac{S_0}{H}BCC^*\left(x,\frac{H^2}{S_0},\sigma\right), & 0 < H \le x, \\ 1^*_{H0}(S_0,\sigma), & \max(0,x) \le H, \\ BCC^*(x,S_0,\sigma), & H \le 0 < x, \\ e^{-rT}, & \max(x,H) \le 0, \end{cases} \end{aligned}$$

and a down-and-out call option

$$\begin{aligned} &C^*_{HO}(x,S_0,\sigma) = E^{Q^0}\left[e^{-rT}1_{(\inf_{0\le t\le T} S^*_t > H)}(S^*_T - x)^+\right] \\ &= \begin{cases} C^*(x,S_0,\sigma) - \frac{S_0}{H}C^*\left(x,\frac{H^2}{S_0},\sigma\right), & 0 < H \le x, \\ C^*(x,S_0,\sigma), & H \le 0 < x, \\ e^{-rT}(S_0 - x), & \max(H,x) \le 0, \\ C^*(H,S_0,\sigma) - \frac{S_0}{H}C^*\left(H,\frac{H^2}{S_0},\sigma\right) + (H-x)1^*_{HO}(S_0,\sigma), & \max(0,x) \le H. \end{cases} \end{aligned}$$

Here the prices BCC^* and C^* can be calculated from the formulas for BBC and C using

$$\begin{aligned} BCC^*(x,S_0,\sigma) &= BCC(e^{rT}x,S_0,\sigma), \\ C^*(x,S_0,\sigma) &= e^{-rT}C(e^{rT}x,S_0,\sigma). \end{aligned}$$

For $S_0 \le H$ all down-and-out options have a price equal to 0. For a down-and-in option with payoff 1 we have for all S_0

$$1^*_{HI}(S_0,\sigma) = e^{-rT} - 1^*_{HO}(S_0,\sigma).$$

The following proposition now follows from applying criterion (2.4.1).

Proposition 2.4.9
If a company follows a buy and hold strategy with stop-loss in case solvency is threatened

the fair value of ρ must satisfy

$$\begin{aligned}
E_0 &= 1^*_{HI}\beta G_T V_0 \\
&\quad + e^{(r+\rho)T}E_0\Bigg(BCC^*_{HO}\left(\min\left(s_2^*,s_3^*\right)\right) - BCC^*_{HO}\left(s_2^*\right) + BCC^*_{HO}\left(\max(s_2^*,s_4^*)\right)\Bigg) \\
&\quad + \vartheta\Bigg(e^{rT}\left(C^*_{HO}\left(s_1^{\beta,*}\right) - C^*_{HO}\left(\min\left(s_2^*,s_3^*\right)\right) + C^*_{HO}\left(s_2^*\right) - C^*_{HO}\left(\max(s_2^*,s_4^*)\right)\right) \\
&\quad + \left(s_1^\beta - s_1\right)BCC^*_{HO}\left(s_1^{\beta,*}\right) - \left(\min\left(s_2,s_3\right)-s_1\right)BCC^*_{HO}\left(\min\left(s_2^*,s_3^*\right)\right) \\
&\quad + (s_2 - s_5)BCC^*_{HO}\left(s_2^*\right) - \left(\max(s_2,s_4)-s_5\right)BCC^*_{HO}\left(\max(s_2^*,s_4^*)\right)\Bigg),
\end{aligned}$$

where all option prices are calculated with initial value S_0 and volatility σ.

2.4.6 Constant relative amount δ in stocks until solvency is threatened

Now assume that a company, whose assets at time 0 fulfill $A_0 \geq e^{-rT}G_T V_0(1+\beta)$, initially invests in a portfolio with constant relative portfolio weights as described in Section 2.4.4. As in Section 2.4.5 the company rebalances the investment portfolio to include the riskfree asset only, the first time (2.4.14) holds. Written in terms of the discounted value process of the investment portfolio the company rebalances the portfolio if

$$\mathcal{V}_t^*(\varphi) = e^{-rT}G_T V_0(1+\beta) - E_0 \equiv \widetilde{H}.$$

As in Section 2.4.5 we know that $E_0 \geq \beta V_0$ and further assume that $e^{rT}E_0 \geq \beta K_T V_0$. The following proposition now follows from Proposition 2.4.9 in the same way as Proposition 2.4.6 followed from Proposition 2.4.3

Proposition 2.4.10
For a company investing in a portfolio with constant relative portfolio weights until solvency is threatened the fair value of ρ must satisfy

$$\begin{aligned}
E_0 &= 1^*_{\widetilde{H}I}\beta G_T V_0 \\
&\quad + e^{(r+\rho)T}E_0\left(BCC^*_{\widetilde{H}O}\left(\min\left(v_2^*,v_3^*\right)\right) - BCC^*_{\widetilde{H}O}\left(v_2^*\right) + BCC^*_{\widetilde{H}O}\left(\max(v_2^*,v_4^*)\right)\right) \\
&\quad + e^{rT}\left(C^*_{\widetilde{H}O}\left(v_1^{\beta,*}\right) - C^*_{\widetilde{H}O}\left(\min\left(v_2^*,v_3^*\right)\right) + C^*_{\widetilde{H}O}(v_2^*) - C^*_{\widetilde{H}O}(\max(v_2^*,v_4^*))\right) \\
&\quad + \left(v_1^\beta - v_1\right)BCC^*_{\widetilde{H}O}\left(v_1^{\beta,*}\right) - \left(\min\left(v_2,v_3\right)-v_1\right)BCC^*_{\widetilde{H}O}\left(\min\left(v_2^*,v_3^*\right)\right) \\
&\quad + (v_2 - v_5)BCC^*_{\widetilde{H}O}(v_2^*) - \left(\max(v_2,v_4)-v_5\right)BCC^*_{\widetilde{H}O}\left(\max(v_2^*,v_4^*)\right),
\end{aligned}$$

where all option prices are calculated with initial value $V_0 + U_0$ and volatility $\delta\sigma$ and

$$v_1^\beta = G_T V_0(1+\beta) - e^{rT}E_0.$$

2.5 Pure endowments

We now consider a company whose insurance portfolio consists of pure endowments. To carry out the study we first extend the probabilistic model to include the development of a portfolio of insured lives. This is done following the approach in Møller (1998).

2.5.1 The model for the insurance portfolio

Consider an insurance portfolio consisting at time 0 of Y_0 insured lives of the same age, say x. We assume that the individual remaining lifetimes at time 0 of the insured are described by a sequence $T_1, \ldots, T_{Y_0}$ of i.i.d. non-negative random variables defined on $(\Omega, \mathcal{F}, P)$. We further make the natural assumption that the distribution of T_i is absolutely continuous and $\mathrm{P}(T_i > T) > 0$, such that the mortality intensity μ_{x+t} is well-defined on $[0, T]$. The survival probability from time 0 to t, $t \in [0, T]$ for one individual in the insurance portfolio is given by

$$ {}_tp_x \equiv \mathrm{P}(T_i > t) = e^{-\int_0^t \mu_{x+u} du}. $$

Denote by ${}_tq_x$ the probability of death from time 0 to t, i.e. ${}_tq_x = 1 - {}_tp_x$. Now define the processes $Y = (Y_t)_{0 \le t \le T}$ and $N = (N_t)_{0 \le t \le T}$ by

$$ Y_t = \sum_{i=1}^{Y_0} 1_{(T_i > t)} \text{ and } N_t = \sum_{i=1}^{Y_0} 1_{(T_i \le t)}. $$

Then Y_t and N_t, respectively, denote the number of survivors and the number of deaths in the insurance portfolio at time t. The filtration $\mathbb{H} = (\mathcal{H}_t)_{0 \le t \le T}$ is the P-augmentation of the natural filtration generated by N, i.e. $\mathcal{H}_t = \mathcal{H}_t^+ \vee \mathcal{N}$, where

$$ \mathcal{H}_t^+ = \sigma\{N_u, u \le t\}. $$

Since the probability of two individuals dying at the same time is 0, then N is a 1-dimensional counting process. The i.i.d. assumption on the remaining lifetimes further gives that N is an $\mathbb{H}$-Markov process. The stochastic intensity process $\lambda = (\lambda_t)_{0 \le t \le T}$ of N under P can now be informally defined by

$$ \lambda_t dt \equiv \mathrm{E}^P \left[dN_t \middle| \mathcal{H}_{t-} \right] = (Y_0 - N_{t-}) \mu_{x+t} dt. $$

Thus, the probability of experiencing a death in the portfolio in the next short interval is the number of survivors multiplied by the probability of one person dying. It is well-known that the process M defined by

$$ M_t = N_t - \int_0^t \lambda_u du = N_t - \int_0^t (Y_0 - N_{u-}) \mu_{x+u} du, \quad 0 \le t \le T, $$

is an $\mathbb{H}$-martingale under P.

2.5.2 The combined model

Now introduce the filtration $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$ for the combined model of the economy and the insurance portfolio by

$$\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t.$$

Assume that the economy is stochastically independent of the development of the insurance portfolio, i.e. $\mathcal{G}_t$ and $\mathcal{H}_t$ are independent. This ensures that the properties of M and W are inherited in the larger filtration $\mathbb{F}$.

We now address the choice of equivalent martingale measure in the combined model. For any $\mathbb{F}$-predictable function h, $h > -1$, we can define a likelihood process $L = (L_t)_{0\le t\le T}$ by

$$\begin{aligned} dL_t &= L_{t-} h_t dM_t, \\ L_0 &= 1, \end{aligned}$$

and construct a new measure equivalent to P by

$$\frac{dQ^h}{dP} = O_T L_T. \tag{2.5.1}$$

We note that $h = 0$ corresponds to Q^0 defined in Section 2.3. The measure Q^h defined by (2.5.1) is a probability measure if $\mathrm{E}^{Q^0}[L_T] = 1$, or equivalently, if $\mathrm{E}^P[O_T L_T] = 1$. To preserve the independence between $\mathcal{G}_t$ and $\mathcal{H}_t$ under Q^h we restrict ourselves to functions h which are $\mathbb{H}$-predictable. Under this additional assumption, all measures Q^h defined by (2.5.1) are equivalent martingale measures if $\mathrm{E}^P[L_T] = 1$, see Møller (1998) for the necessary calculations. Girsanov's theorem for point processes, see e.g. Andersen, Borgan, Gill and Keiding (1993), gives that the stochastic intensity process $\lambda^h = (\lambda_t^h)_{0\le t\le T}$ for N under Q^h is given by

$$\lambda_t^h = (1 + h_t)\lambda_t = (Y_0 - N_{t-})(1 + h_t)\mu_{x+t}.$$

Hence, changing measure from Q^0 to Q^h can be interpreted as changing the mortality intensity from μ_{x+t} to $(1 + h_t)\mu_{x+t}$. With this interpretation the survival probability under Q^h is given by

$${}_tp_x^h = Q^h(T_i > t) = e^{-\int_0^t (1+h_u)\mu_{x+u}du}.$$

The probability of death under Q^h is given by ${}_tq_x^h = 1 - {}_tp_x^h$. We note that if h is on the form $h(t, N_{t-})$ then N is a Markov process under Q^h as well as under P. Since no unique equivalent martingale measure exists for the combined model, not all contingent claims in $(B, S, \mathbb{F})$ have unique prices. However, since $(B, S, \mathbb{G})$ is complete, all contingent claims depending only on the financial market still have unique prices. To find unique prices for contingent claims depending on the development of the insurance portfolio, we henceforth consider a fixed, but arbitrary, equivalent martingale measure Q^h. Motivated by the strong law of large numbers, the measure Q^0, corresponding to risk neutrality with respect to unsystematic mortality risk, is frequently used in the literature to price insurance contracts with financial risk, see e.g. Aase and Persson (1994) and Møller (1998). Møller (1998) also recognizes Q^0 as the minimal martingale measure for the considered model.

2.5.3 The development of the deposit in a 1-period model

Now assume all insured in the portfolio introduced in Section 2.5.1 have purchased identical pure endowments with termination at time T or later. If premiums are paid before or at time 0 and the portfolio of insured lives develop exactly as expected, the portfolio-wide deposit at time T is given by

$$V_T^{det} = \mathcal{E}_T V_0^{det}.$$

Here, $\mathcal{E}_T \in \{G_T, K_T\}$ is the deposit accumulation factor for the interval $(0, T]$, and the superscript *det* refers to a deterministic development of the insured portfolio. Dividing by the number of expected survivors we obtain an expression for the development of the deposit of one insured surviving to time T

$$V_T^{ind} = \mathcal{E}_T V_0^{ind} \frac{1}{{}_T p_x}.$$

Thus, the portfolio-wide deposit at time T is given by

$$V_T = Y_T V_T^{ind} = Y_T \mathcal{E}_T V_0^{ind} \frac{1}{{}_T p_x}. \tag{2.5.2}$$

2.5.4 Distribution scheme

Using (2.5.2) we define a distribution scheme, similar to the distribution scheme from Section 2.4.1, which is used by the company in case of a portfolio of pure endowments:

1. $A_T < Y_T G_T V_0^{ind} \frac{1}{{}_T p_x}$: Here, the assets are insufficient to meet the guaranteed deposit at time T for all the survivors in the insured portfolio. Hence, the company is declared bankrupt and all capital is allocated to the deposit.

$$\begin{aligned} V_T &= A_T, \\ E_T &= 0, \\ U_T &= 0. \end{aligned}$$

2. $Y_T G_T V_0^{ind} \frac{1}{{}_T p_x} \leq A_T < Y_T K_T V_0^{ind} \frac{1}{{}_T p_x}(1+\gamma) + e^{rT} E_0$: In this case the assets are sufficient to meet the guarantee. However, accumulating with the announced accumulation factor leaves the company with a bonus reserve less than the minimal target, γV_T. Thus, as in the case of capital insurances the company uses G_T to accumulate. Similarly to Section 2.4.1 the capital is distributed as follows

$$\begin{aligned} V_T &= Y_T G_T V_0^{ind} \frac{1}{{}_T p_x}, \\ E_T &= \min\left(e^{(r+\rho)T} E_0, A_T - V_T\right), \\ U_T &= A_T - V_T - E_T. \end{aligned}$$

3. $e^{rT}E_0 + Y_T K_T V_0^{ind}\frac{1}{{}_Tp_x}(1+\gamma) \leq A_T$: Here, the investments and the development of the insurance portfolio allow the company to accumulate using the announced deposit rate and still have a bonus reserve above the minimal target. The distribution is similar to the one above with G_T replaced by K_T

$$\begin{aligned} V_T &= Y_T K_T V_0^{ind}\frac{1}{{}_Tp_x}, \\ E_T &= \min\left(e^{(r+\rho)T}E_0, A_T - V_T\right), \\ U_T &= A_T - V_T - E_T. \end{aligned}$$

Note that we by the above distribution scheme implicitly consider the mortality intensity as guaranteed, since it is used even if the portfolio of insured behaves worse than anticipated. Thus, in the present situation the additional interest rate ρ is a compensation for both financial and unsystematic mortality risk. As in the case of capital insurances, the equity capital is only used to cover the accumulation of the deposit if the payoff generated by the deposit and bonus reserve is insufficient.

2.5.5 Fair distribution

From Section 2.5.4 we note that E_T and $V_T + U_T$ can be viewed as contingent claims in the combined model $(B, S, \mathbb{F})$. As in the case of capital insurances, we define the distribution scheme as fair if it does not include an arbitrage possibility for either the company or the portfolio of insured, i.e. if

$$E_0 = e^{-rT}\mathrm{E}^{Q^h}\left[E_T\right], \tag{2.5.3}$$

and

$$V_0 + U_0 = e^{-rT}\mathrm{E}^{Q^h}\left[V_T + U_T\right]. \tag{2.5.4}$$

The relation

$$E^{Q^h}[e^{-rT}A_T] = A_0,$$

now ensures that (2.5.3) holds if and only if (2.5.4) holds, such that we may consider (2.5.3) only.

Using the law of iterated expectations we can write (2.5.3) as

$$\begin{aligned} E^{Q^h}\left[E_T\right] &= E^{Q^h}E^{Q^h}\left[E_T|\mathcal{H}_T\right] \\ &= \sum_{n=0}^{Y_0}\binom{Y_0}{n}({}_Tp_x^h)^n({}_Tq_x^h)^{Y_0-n}E^{Q^h}\left[1_{\left(A_T < nG_T V_0^{ind}\frac{1}{{}_Tp_x}\right)}0\right. \\ &\quad + 1_{(nG_T V_0^{ind}\frac{1}{{}_Tp_x} \leq A_T < nK_T V_0^{ind}\frac{1}{{}_Tp_x}(1+\gamma)+e^{rT}E_0)}\min\left(e^{(r+\rho)T}E_0, A_T - nG_T V_0^{ind}\frac{1}{{}_Tp_x}\right) \\ &\quad \left. + 1_{(nK_T V_0\frac{1}{{}_Tp_x}(1+\gamma) \leq \mathcal{V}_T(\varphi))}\min\left(e^{(r+\rho)T}E_0, A_T - nK_T V_0^{ind}\frac{1}{{}_Tp_x}\right)\right]. \end{aligned} \tag{2.5.5}$$

Recall that with respect to the financial market all measures Q^h are identical. Thus, the expectation can be viewed as a weighted average of Y_0+1 portfolios of capital insurances with initial deposit $nV_0^{ind}\frac{1}{{}_Tp_x}$, $n=0,1,\ldots,Y_0$, respectively. Hence, most calculations necessary to derive an implicit equation for ρ are identical to those already carried out in Section 2.4.

Remark 2.5.1 Note that since all insured have identical contracts, the individual contracts are fair if the bonus reserve at the time of purchase was 0 and a possible bonus reserve at time of termination is distributed among the survivors in the insurance portfolio. □

2.5.6 Buy and hold

When the company follows a buy and hold strategy the fair value of ρ is given by the following proposition

Proposition 2.5.2
If an insurance company, whose portfolio consists of Y_0 pure endowments, follows a buy and hold strategy, then the fair value of ρ satisfies

$$\begin{aligned}E_0 = \sum_{n=0}^{Y_0}\binom{Y_0}{n}({}_Tp_x^h)^n({}_Tq_x^h)^{Y_0-n}\Bigg(&e^{(r+\rho)T}E_0\Big(BCC(\max(s_2^n,s_4^n))\\&+BCC(\min(s_2^n,s_3^n))-BCC\left(s_2^n\right)\Big)\\&+\vartheta\Bigg(C(s_1^n)-C(\min(s_2^n,s_3^n))+C(s_2^n)-C(\max(s_2^n,s_4^n))\\&-(\min(s_2^n,s_3^n)-s_1^n)BCC(\min(s_2^n,s_3^n))\\&+(s_2^n-s_5^n)BCC(s_2^n)-(\max(s_2^n,s_4^n)-s_5^n)BCC\left(\max(s_2^n,s_4^n)\right)\Bigg)\Bigg),\end{aligned}$$

where

$$\begin{aligned}s_1^n &= \frac{nG_TV_0^{ind}\frac{1}{{}_Tp_x}-\eta e^{rT}-e^{rT}E_0}{\vartheta},\\ s_2^n &= \frac{nK_TV_0^{ind}\frac{1}{{}_Tp_x}(1+\gamma)-\eta e^{rT}}{\vartheta},\\ s_3^n &= \frac{\left(e^{\rho T}-1\right)e^{rT}E_0+nG_TV_0^{ind}\frac{1}{{}_Tp_x}-\eta e^{rT}}{\vartheta},\\ s_4^n &= \frac{\left(e^{\rho T}-1\right)e^{rT}E_0+nK_TV_0^{ind}\frac{1}{{}_Tp_x}-\eta e^{rT}}{\vartheta},\\ s_5^n &= \frac{nK_TV_0^{ind}\frac{1}{{}_Tp_x}-\eta e^{rT}-e^{rT}E_0}{\vartheta}.\end{aligned}$$

Here, all option prices are calculated using initial value S_0 and volatility σ.

Again we are interested in the probability that the company is ruined at time T.

Proposition 2.5.3
The probability of ruin, $p_{ruin}(\varphi)$, at time T for a company following a buy and hold strategy is

$$p_{ruin}(\varphi) = \sum_{n=0}^{Y_0} \binom{Y_0}{n} ({}_Tp_x)^n ({}_Tq_x)^{Y_0-n} \Phi\left(\frac{\log\left(\frac{s_1^n}{S_0}\right) - \left(\alpha - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right).$$

Proof of Proposition 2.5.3: Using iterated expectations we get

$$\begin{aligned}
p_{ruin}(\varphi) &= P\left[A_T < Y_T G_T V_0^{ind} \frac{1}{{}_Tp_x}\right] \\
&= E^P\left[P\left[A_T < Y_T G_T V_0^{ind} \frac{1}{{}_Tp_x}\middle| \mathcal{H}_T\right]\right] \\
&= \sum_{n=0}^{Y_0} \binom{Y_0}{n} ({}_Tp_x)^n ({}_Tq_x)^{Y_0-n} P\left[A_T < n G_T V_0^{ind} \frac{1}{{}_Tp_x}\right].
\end{aligned}$$

The result now follows immediately from Proposition 2.4.5 and the definition of s_1^n.

□

2.5.7 Constant relative portfolio

In the case of investments in a portfolio with constant relative portfolio weights we obtain the following proposition from (2.5.5).

Proposition 2.5.4
For a company investing in a portfolio with constant relative portfolio weights the fair value of ρ is the solution to the following equation

$$\begin{aligned}
E_0 = \sum_{n=0}^{Y_0} \binom{Y_0}{n} ({}_Tp_x^h)^n ({}_Tq_x^h)^{Y_0-n} \Bigg(& e^{(r+\rho)T} E_0 \Big(BCC(\max(v_2^n, v_4^n)) \\
& + BCC(\min(v_2^n, v_3^n)) - BCC(v_2^n) \Big) \\
& + C(v_1^n) - C(\min(v_2^n, v_3^n)) + C(v_2^n) - C(\max(v_2^n, v_4^n)) \\
& - (\min(v_2^n, v_3^n) - v_1^n) BCC(\min(v_2^n, v_3^n)) \\
& + (v_2^n - v_5^n) BCC(v_2^n) - (\max(v_2^n, v_4^n) - v_5^n) BCC(\max(v_2^n, v_4^n)) \Bigg),
\end{aligned}$$

where

$$v_1^n = nG_T V_0^{ind} \frac{1}{{}_T p_x} - e^{rT} E_0, \tag{2.5.6}$$

$$v_2^n = nK_T V_0^{ind} \frac{1}{{}_T p_x}(1+\gamma), \tag{2.5.7}$$

$$v_3^n = \left(e^{\rho T} - 1\right) e^{rT} E_0 + nG_T V_0^{ind} \frac{1}{{}_T p_x},$$

$$v_4^n = \left(e^{\rho T} - 1\right) e^{rT} E_0 + nK_T V_0^{ind} \frac{1}{{}_T p_x},$$

$$v_5^n = nK_T V_0^{ind} \frac{1}{{}_T p_x} - e^{rT} E_0. \tag{2.5.8}$$

All option prices above are calculated using initial value $V_0 + U_0$ and volatility $\delta\sigma$.

Calculations similar to the case of investments in a buy and hold strategy gives the following result for the ruin probability.

Proposition 2.5.5
If a company, whose insurance portfolio consists of pure endowments, invests in a portfolio with constant relative portfolio weights, then the probability of ruin at time T is given by

$$p_{ruin}(\varphi) = \sum_{n=0}^{Y_0} \binom{Y_0}{n} ({}_T p_x)^n ({}_T q_x)^{Y_0 - n} \Phi \left(\frac{\log\left(\frac{v_1^n}{V_0 + U_0}\right) - \left(r + \delta(\alpha - r) - \frac{1}{2}(\delta\sigma)^2\right) T}{\delta\sigma\sqrt{T}} \right).$$

2.5.8 Buy and hold with stop-loss if solvency is threatened

Assume the solvency requirement determined by the regulatory institutions is given by $E_T \geq \beta Y_T V_T^{ind}$. Hence, $E_0 \geq \beta Y_0 V_0^{ind}$, since the company otherwise would be insolvent already at time 0. Here, we further assume that the initial assets of the company fulfills

$$A_0 \geq e^{-rT} Y_0 G_T V_0^{ind} \frac{{}_T p_x^h}{{}_T p_x}(1+\beta).$$

To avoid accumulating with K_T in situations where this leads to insolvency, we require that $e^{rT} E_0 \geq \beta K_T Y_0 V_0^{ind} \frac{1}{{}_T p_x}$. Thus, the factor $\frac{1}{{}_T p_x}$ makes the assumption on the initial equity capital stronger than in the case of capital insurances. At time 0 the company invests in a buy and hold strategy. However, to decrease the probability of insolvency the company rebalances the investment portfolio to include investments in the savings account only, if the assets hit the lower boundary

$$A_t = \mathrm{E}^{Q^h}\left[e^{-r(T-t)} Y_T G_T V_0^{ind} \frac{1}{{}_T p_x}(1+\beta) \right] = e^{-r(T-t)} Y_0 G_T V_0^{ind} \frac{{}_T p_x^h}{{}_T p_x}(1+\beta). \tag{2.5.9}$$

Thus, disregarding the information at time t about the development of the insurance portfolio the company rebalances the portfolio if the value of the solvency requirement is equal to the assets. The advantage of (2.5.9) is that it can be written as

$$S_t^* = \frac{e^{-rT} Y_0 G_T V_0^{ind} \frac{{}_T p_x^h}{{}_T p_x}(1+\beta) - \eta - E_0}{\vartheta} \equiv Z.$$

Hence, as in Section 2.4 the requirement on the assets can be transformed into a barrier problem for the discounted stock price with a constant barrier.

Remark 2.5.6 A natural extension of (2.5.9) is to take the development of the insurance portfolio into account. This gives the criterion

$$A_t = \mathrm{E}^{Q^h}\left[e^{-r(T-t)} Y_T G_T V_0^{ind} \frac{1}{{}_T p_x}(1+\beta) \middle| \mathcal{F}_t \right] = e^{-r(T-t)} Y_t G_T V_0^{ind} \frac{{}_{T-t} p_{x+t}^h}{{}_T p_x}(1+\beta). \tag{2.5.10}$$

However, this criterion does not allow us to write the problem as a constant barrier problem. Both criterion (2.5.9) and (2.5.10) leave the company with a positive probability of insolvency. To avoid insolvency almost surely, we could assume that

$$A_0 \geq e^{-rT} Y_0 G_T V_0^{ind} \frac{1}{{}_T p_x}(1+\beta),$$

and use the intervention criterion

$$A_t = e^{-r(T-t)} Y_t G_T V_0^{ind} \frac{1}{{}_T p_x}(1+\beta),$$

which corresponds to assuming that all insured persons, which are alive at time t survive to time T.

$\square$

In order to use (2.5.5) we consider a fixed number of survivors, say n. Given the number of survivors the equity capital can be decomposed into a term $E_T^{n,BS1}$, which is different from 0 if the company has intervened and a term $E_T^{n,BS2}$, which is non-zero if the company has not intervened. For $E_T^{n,BS1}$ we obtain

$$\begin{aligned}
E_T = {} & 1_{(\inf_{0\le t\le T} S_t^* \le Z)} \Bigg(1_{\left(Y_0 G_T V_0^{ind} \frac{{}_Tp_x^h}{{}_Tp_x}(1+\beta) < nG_T V_0^{ind} \frac{1}{{}_Tp_x}\right)} 0 \\
& + 1_{\left(nG_T V_0^{ind} \frac{1}{{}_Tp_x} \le Y_0 G_T V_0^{ind} \frac{{}_Tp_x^h}{{}_Tp_x}(1+\beta) < nK_T V_0^{ind} \frac{1}{{}_Tp_x}(1+\gamma) + e^{rT} E_0\right)} \\
& \times \min\left(e^{(r+\rho)T} E_0, Y_0 G_T V_0^{ind} \frac{{}_Tp_x^h}{{}_Tp_x}(1+\beta) - nG_T V_0^{ind} \frac{1}{{}_Tp_x} \right) \\
& + 1_{\left(nK_T V_0^{ind} \frac{1}{{}_Tp_x}(1+\gamma) \le Y_0 G_T V_0^{ind} \frac{{}_Tp_x^h}{{}_Tp_x}(1+\beta) - e^{rT} E_0\right.} \\
& \times \min\left(e^{(r+\rho)T} E_0, Y_0 G_T V_0^{ind} \frac{{}_Tp_x^h}{{}_Tp_x}(1+\beta) - nK_T V_0^{ind} \frac{1}{{}_Tp_x} \right) \Bigg) \\
= {} & 1_{(\inf_{0\le t\le T} S_t^* \le Z)} \Bigg(1_{(v_6 < v_1^n)} 0 + 1_{(v_1^n \le v_6 < v_2^n)} \min\left(e^{(r+\rho)T} E_0, v_6 - v_1^n \right) \\
& + 1_{(v_2^n \le v_6)} \min\left(e^{(r+\rho)T} E_0, v_6 - v_5^n \right) \Bigg),
\end{aligned}$$

where v_1^n, v_2^n and v_5^n are given by (2.5.6), (2.5.7) and (2.5.8), respectively, and

$$v_6 = Y_0 G_T V_0^{ind} \frac{{}_Tp_x^h}{{}_Tp_x}(1+\beta) - e^{rT} E_0.$$

For $E_T^{n,BS2}$ the calculations in Section 2.4.5 applies. Thus, we get

Proposition 2.5.7
In the situation with stop-loss the fair value of ρ must satisfy

$$\begin{aligned}
E_0 = {} & \sum_{n=0}^{Y_0} \binom{Y_0}{n} ({}_Tp_x^h)^n ({}_Tq_x^h)^{Y_0-n} \Bigg(1_{ZI}^* \left(1_{(v_1^n \le v_6 < v_2^n)} \min\left(e^{(r+\rho)T} E_0, v_6 - v_1^n \right) \right. \\
& \left. + 1_{(v_2^n \le v_6)} \min\left(e^{(r+\rho)T} E_0, v_6 - v_5^n \right) \right) + e^{(r+\rho)T} E_0 \Big(BCC_{ZO}^* \left(\max(s_2^{n,*}, s_4^{n,*}) \right) \\
& + BCC_{ZO}^* \left(\min\left(s_2^{n,*}, s_3^{n,*} \right) \right) - BCC_{ZO}^* \left(s_2^{n,*} \right) \Big) \\
& + \vartheta \Bigg(e^{rT} \left(C_{ZO}^*(s_1^{\beta,n,*}) - C_{ZO}^* \left(\min\left(s_2^{n,*}, s_3^{n,*} \right) \right) + C_{ZO}^*(s_2^{n,*}) - C_{ZO}^*(\max(s_2^{n,*}, s_4^{n,*})) \right) \\
& + (s_1^{\beta,n} - s_1^n) BCC_{ZO}^*(s_1^{\beta,n,*}) - \left(\min\left(s_2^n, s_3^n \right) - s_1^n \right) BCC_{ZO}^* \left(\min\left(s_2^{n,*}, s_3^{n,*} \right) \right) \\
& + (s_2^n - s_5^n) BCC_{ZO}^*(s_2^{n,*}) - \left(\max(s_2^n, s_4^n) - s_5^n \right) BCC_{ZO}^* \left(\max(s_2^{n,*}, s_4^{n,*}) \right) \Bigg) \Bigg),
\end{aligned}$$

where

$$s_1^{\beta,n} = \frac{nG_T V_0^{ind} \frac{1}{{}_Tp_x}(1+\beta) - \eta e^{rT} - e^{rT} E_0}{\vartheta}.$$

Here, all option prices are calculated with initial value S_0 and volatility σ.

The probability of insolvency for a company following the investment strategy described above is given in the following proposition

Proposition 2.5.8
For a company following a buy and hold strategy with stop-loss the probability of insolvency is given by

$$p_{ins}(\varphi) = \sum_{n=0}^{Y_0} \binom{Y_0}{n} ({}_T p_x)^n ({}_T q_x)^{Y_0-n} 1_{(Y_0 {}_T p_x^h < n)} \left(\Phi \left(\frac{\log\left(\frac{s_1^{\beta,n,*} S_0}{Z^2}\right) - \left(\alpha - r - \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}} \right) \right.$$
$$\left. + \left(\frac{Z}{S_0}\right)^{\frac{2(\alpha-r)}{\sigma^2}-1} \left(1 - \Phi \left(\frac{\log\left(\frac{s_1^{\beta,n,*} S_0}{Z^2}\right) - \left(\alpha - r - \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}} \right) \right) \right).$$

Proof of Proposition 2.5.8: See Section 2.10.4.

□

2.5.9 Constant relative amount δ in stocks until solvency is threatened

Consider the same set-up as in Section 2.5.8. The only difference is that the company invests in a strategy with constant relative portfolio weights until a possible intervention. Written in terms of the discounted value of the portfolio including risky investments the rebalancing takes place the first time

$$\mathcal{V}_t^*(\varphi) = e^{-rT} Y_0 G_T V_0^{ind} \frac{{}_T p_x^h}{{}_T p_x} (1+\beta) - E_0 \equiv \widetilde{Z}.$$

The result now follows from calculations similar to those in Section 2.5.8.

Proposition 2.5.9
In the situation with stop-loss the fair value of ρ must satisfy

$$E_0 = \sum_{n=0}^{Y_0} \binom{Y_0}{n} ({}_T p_x^h)^n ({}_T q_x^h)^{Y_0-n} \left(1^*_{\widetilde{Z}I} \left(1_{(v_1^n \le v_6 < v_2^n)} \min\left(e^{(r+\rho)T} E_0, v_6 - v_1^n\right) \right. \right.$$
$$+ 1_{(v_2^n \le v_6)} \min\left(e^{(r+\rho)T} E_0, v_6 - v_5^n\right)\Big) + e^{(r+\rho)T} E_0 \Big(BCC^*_{\widetilde{Z}O}\left(\max(v_2^{n,*}, v_4^{n,*})\right)$$
$$+ BCC^*_{\widetilde{Z}O}\left(\min\left(v_2^{n,*}, v_3^{n,*}\right)\right) - BCC^*_{\widetilde{Z}O}\left(v_2^{n,*}\right) \Big)$$
$$+ e^{rT} \left(C^*_{\widetilde{Z}O}(v_1^{\beta,n,*}) - C^*_{\widetilde{Z}O}\left(\min\left(v_2^{n,*}, v_3^{n,*}\right)\right) + C^*_{\widetilde{Z}O}(v_2^{n,*}) - C^*_{\widetilde{Z}O}(\max(v_2^{n,*}, v_4^{n,*})) \right)$$
$$+ (v_1^{\beta,n} - v_1^n) BCC^*_{\widetilde{Z}O}(v_1^{\beta,n,*}) - \left(\min(v_2^n, v_3^n) - v_1^n\right) BCC^*_{\widetilde{Z}O}\left(\min\left(v_2^{n,*}, v_3^{n,*}\right)\right)$$
$$\left. + (v_2^n - v_5^n) BCC^*_{\widetilde{Z}O}(v_2^{n,*}) - \left(\max(v_2^n, v_4^n) - v_5^n\right) BCC^*_{\widetilde{Z}O}\left(\max(v_2^{n,*}, v_4^{n,*})\right) \right),$$

where

$$v_1^{\beta,n} = nG_T V_0^{ind} \frac{1}{{}_T p_x}(1+\beta) - e^{rT} E_0.$$

Here, all option prices are calculated with initial value $V_0 + U_0$ and volatility $\delta\sigma$.

Now calculations similar to those leading to Proposition 2.5.8 give

Proposition 2.5.10
For a company following a strategy with constant relative portfolio weights with stop-loss the probability of insolvency is given by

$$p_{ins}(\varphi)$$

$$= \sum_{n=0}^{Y_0} \binom{Y_0}{n} ({}_T p_x)^n ({}_T q_x)^{Y_0-n} 1_{(Y_0 {}_T p_x^h < n)} \left(\Phi \left(\frac{\log\left(\frac{v_1^{\beta,n,*}(V_0+U_0)}{Z^2}\right) - \left(\delta(\alpha-r) - \frac{1}{2}(\delta\sigma)^2\right) T}{\delta\sigma\sqrt{T}} \right) \right.$$

$$\left. + \left(\frac{\widetilde{Z}}{V_0+U_0}\right)^{\frac{2\delta(\alpha-r)}{(\delta\sigma)^2}-1} \left(1 - \Phi\left(\frac{\log\left(\frac{v_1^{\beta,n,*}(V_0+U_0)}{\widetilde{Z}^2}\right) - \left(\delta(\alpha-r) - \frac{1}{2}(\delta\sigma)^2\right) T}{\delta\sigma\sqrt{T}} \right)\right)\right).$$

2.6 Numerical results

Since we obtain implicit equations for ρ only, we now resort to numerical techniques to obtain fair values of ρ. We rewrite the expressions for the fair value of ρ on the form $\rho = f(\rho)$ for some function f and use iterations to find fix points for f. For all numerical calculations we assume that time is measured in years and let $T = 1$. For an overview of the notation used in this chapter we refer to Table 2.6.1.

2.6.1 Dependence on investment strategy

In this section we fix the parameters $r = 0.06$, $\sigma = 0.2$, $G_T = 1.045$, $K_T = 1.06$ and $\gamma = 0.1$ and consider the dependence of ρ on the investment strategy.

For now we assume the initial capital is distributed as follows: $V_0 = 100$, $U_0 = 10$ and $E_0 = 10$. Figure 2.6.1 then shows the dependence of ρ on the relative initial investment in stocks for a buy and hold strategy and a constant relative portfolio. The relative initial investment in stocks is given by $\kappa = \vartheta S_0 / \mathcal{V}_0(\varphi)$ for the buy and hold strategy and by δ for the constant relative portfolio. The observations to be made from Figure 2.6.1 are twofold. Firstly, ρ is an increasing function of the relative initial investment in stocks for both investment strategies. This is not surprising, since ρ is a measure for the risk of the insurance company and investing in stocks increases the risk. Secondly, comparing

Symbol	Interpretation
V	Portfolio-wide deposit
U	Bonus reserve
E	Equity capital
S	Stock price
$\mathcal{V}(\varphi)$	Value of investment portfolio φ
ρ	Fair additional rate of return to equity capital
r	Riskfree interest rate
σ	Volatility of stock
G_T	Guaranteed accumulation factor
K_T	Announced accumulation factor
γ	Target for minimal bonus reserve per deposit
T	Length of accumulation period
β	Solvency requirement on equity capital per deposit
ϑ	Number of stocks held in a buy and hold strategy
δ	Constant proportion invested in stocks
Y	Number of survivors in insurance portfolio
h	Market attitude towards unsystematic mortality risk

Table 2.6.1: *Overview of notation*

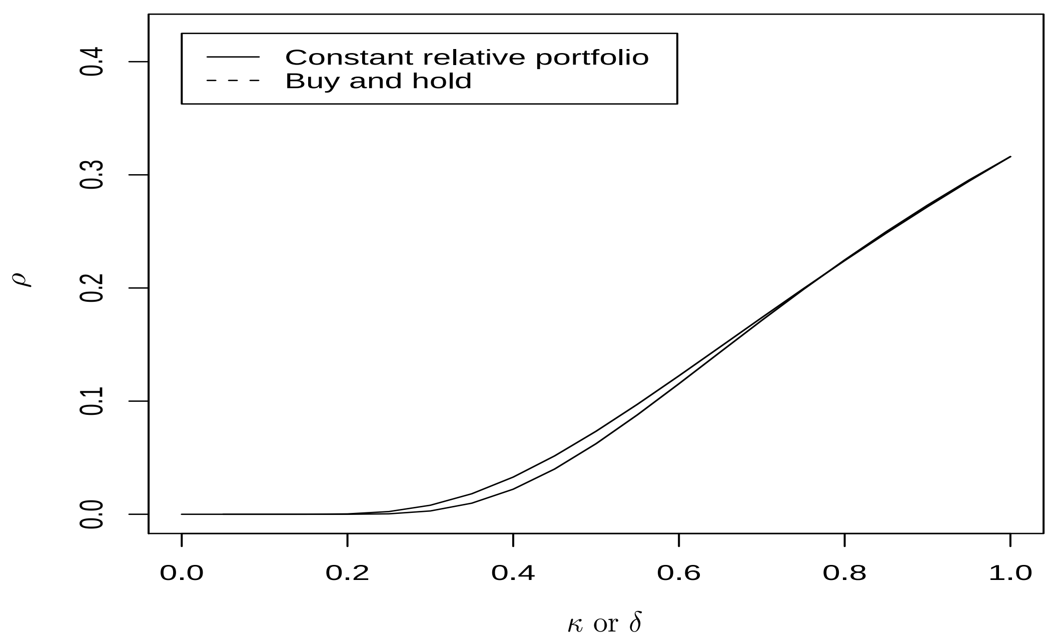

Figure 2.6.1: *ρ as a function of the relative initial investment in stocks.*

the two investment strategies, we observe that for a relative initial investment in stocks between 0.2 and 0.7 the fair value of ρ is slightly higher when investing in a constant relative portfolio rather than following a buy and hold strategy. This may be explained by the fact that when investing in a portfolio with constant relative portfolio weights a decrease in the stock price leads to additional investments in stocks and hence an increase in the capital at risk. Comparing the strategies we also note that the values of ρ coincide in the extremes where none or all capital is invested in stocks. This relies on the fact that the strategies coincide in these two situations.

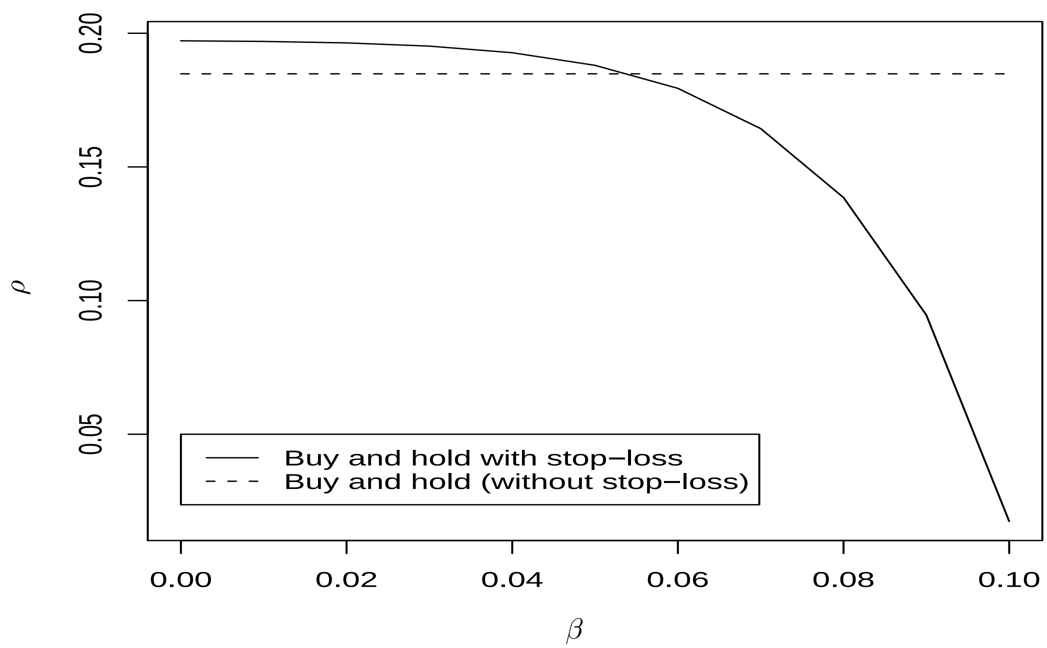

Figure 2.6.2: ρ *as a function of* β *for* $\kappa = 0.5$.

In order to investigate the dependence of β we consider a buy and hold strategy with stop-loss. The initial distribution of capital is changed such that $U_0 = 5$, since the dependence is more obvious in this case. The dependence of ρ on the required solvency margin β is now shown in Figure 2.6.2 for $\kappa = 0.5$. We observe that ρ is a decreasing function of β. This is also intuitively clear since increasing β, within the restrictions given in Section 2.4.5, increases the minimum payoff to the equity capital and hence decreases the risk of the company. For comparison Figure 2.6.2 also includes a horizontal line showing the fair value for an ordinary buy and hold strategy. Comparing the two strategies we observe that for low values of β the stop-loss strategy leads to higher values of ρ than the strategy without stop-loss. The reason for this is, that for low values of β the equity capital receives a payoff in case of intervention which is so low that at the time of a possible intervention the expected increase in the payoff from continuing the buy and hold strategy outweighes the risk of an even smaller payoff.

2.6.2 Dependence on parameters

For a company following a buy and hold strategy we now consider the dependence of ρ on the parameters r, σ, G_T, K_T and γ for a fixed initial distribution of capital. To study

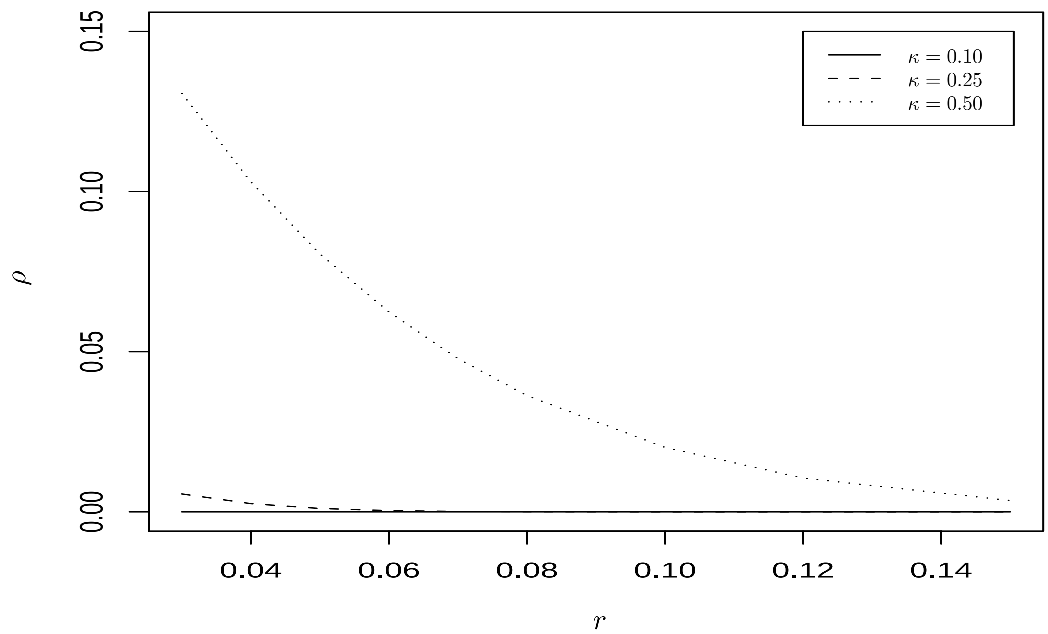

Figure 2.6.3: *ρ as a function of the short rate of interest.*

the dependence on r we let $\sigma = 0.2$, $G_T = 1.045$, $K_T = 1.06$, $\gamma = 0.1$, $V_0 = 100$, $U_0 = 10$ and $E_0 = 10$. Figure 2.6.3 then shows the dependence on r for $\kappa \in \{0.10, 0.25, 0.50\}$. The values of κ are chosen to illustrate a company with a conservative, a moderate and an aggressive investment strategy, respectively. We observe that ρ is a decreasing function of r for all values of κ. This is also expected since increasing the riskfree interest rate lowers the probability of investment returns below the guaranteed/announced accumulation factor, hence decreasing the risk of the insurance company.

Fixing $r = 0.06$ and letting $U_0 = 5$ and $E_0 = 5$, we now turn to the dependence on the guaranteed accumulation factor, G_T. The low values of E_0 and U_0 are chosen in order to observe a dependence on G_T for low values of κ. Figure 2.6.4 now shows the dependence on G_T for the same values of κ as above, i.e. $\kappa \in \{0.10, 0.25, 0.50\}$. We observe that ρ is an increasing function of G_T for all three values of κ. The positive dependence of ρ on G_T is intuitively clear, since the larger the guarantee to the insured, the more risky the contract is for the company.

For a company investing in a constant relative portfolio the constants δ and σ only enter the implicit equations for ρ as $\delta\sigma$, hence varying σ is identical to varying δ. Thus, we observe from Figure 2.6.1 that ρ is an increasing function of σ. This seems intuitively clear since

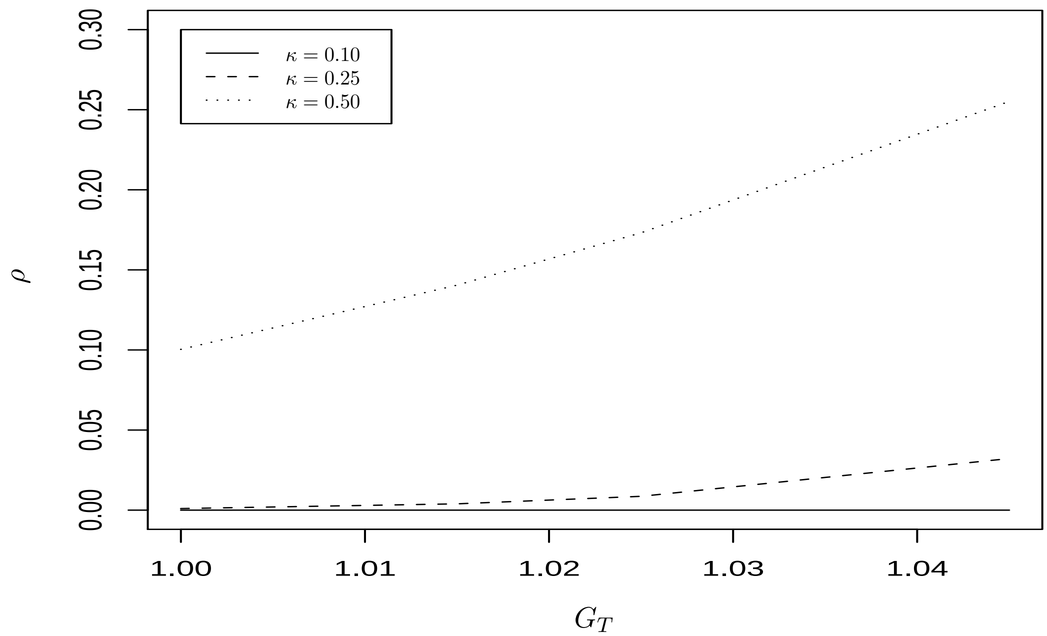

Figure 2.6.4: *ρ as a function of the guaranteed accumulation factor.*

increasing the volatility of the stocks increases the risk of the company. Investigating the dependence of ρ on γ, we find that ρ essentially is independent of γ. However, a slight negative dependence has been observed for high levels of volatility, low values of γ and an equity capital which is large compared to the bonus reserve. That ρ is a decreasing function of γ may be explained by the fact that increasing γ increases the probability of accumulating using G_T. Hence for some outcomes of the stock price there is a small increase in the payoff to the equity capital, whereas all other outcomes give the same payoff. Since the dependence is very small and in most cases non-existent, we have left out a figure illustrating this. Regarding the relationship between ρ and K_T we find that ρ only depends on K_T if V_0 and E_0 are large compared to U_0 and the investment strategy is quite risky. In this case plotting ρ as a function K_T shows a shape similar to a 2. order polynomial with branches pointing downwards. The dependence may be explained by the fact that, when increasing K_T the payoff to the insurance portfolio increases if K_T is used as accumulation factor, but at the same time the probability of accumulation with K_T decreases. Thus, the risk of the company is a tradeoff between two factors working in opposite directions, such that the value of K_T for which the maximum value of ρ is obtained depends on V_0 and U_0. Since the equity capital in practice is much smaller than the deposit, we conclude that ρ for practical purposes is independent of K_T, and doing so, we leave out a graph showing the uninteresting case where a dependence is found.

2.6.3 Dependence on initial distribution of capital

To study the dependence of ρ on the initial distribution of capital we fix the parameters $r = 0.06$, $\sigma = 0.20$, $G_T = 1.045$, $K_T = 1.06$ and $\gamma = 0.10$ and consider an insurance company investing according to a buy and hold strategy with $\kappa = 0.25$. Since the value of ρ is indifferent to scaling of the initial distribution of capital, we further fix $V_0 = 100$ and allow E_0 and U_0 to vary. Figure 2.6.5 now shows the dependence of ρ on U_0 for different

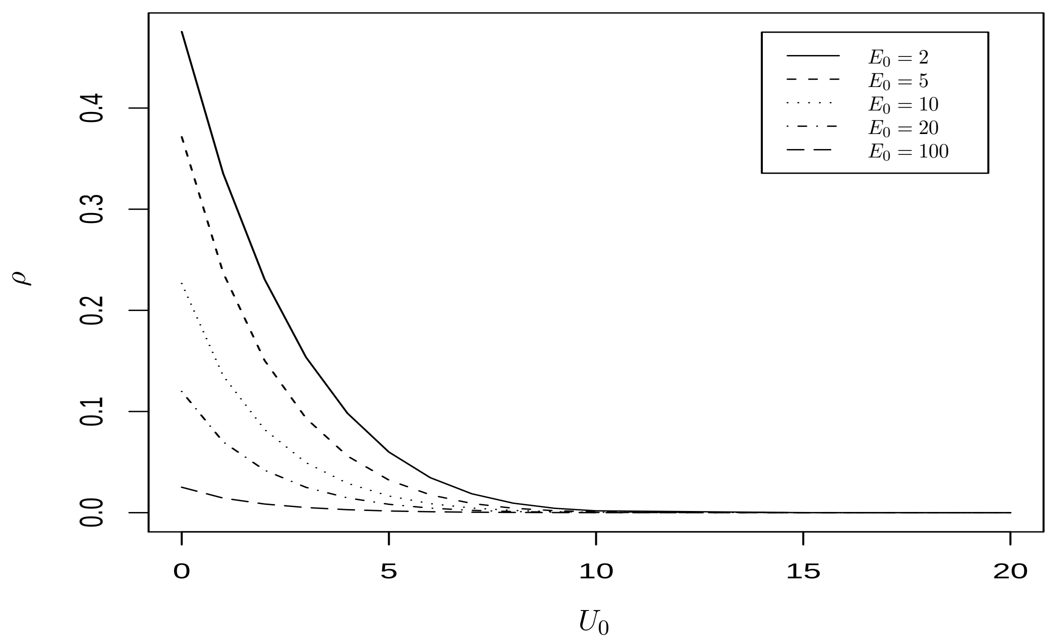

Figure 2.6.5: *ρ as a function of the initial bonus reserve for different values of the initial equity capital.*

values of E_0. Comparing the graphs for the different values of E_0, we observe that ρ is a decreasing function of the equity capital. However, since ρ is an additional interest rate to the entire equity capital, we still observe an increase in the nominal payment for the increased risk even though ρ is decreasing. A decrease in ρ should thus be interpreted as a decrease in the average risk of one unit of equity capital in the company. Furthermore, we observe that ρ is a decreasing function of U_0 for all values of E_0. In Section 2.10.2 it is shown that $\rho \to 0$ as $U_0 \to \infty$. Since the results are indifferent to scaling of the initial capital, then increasing V_0 is similar to decreasing E_0 and U_0. Hence, since ρ is a decreasing function of E_0 and U_0 we have that it, as expected, is an increasing function of V_0.

2.6.4 Effect from unsystematic mortality risk

Now consider an insurance company whose insurance portfolio consists of identical pure endowments for a group of persons of age 50. To model the possible deaths of the insured individuals we use a so-called Gompertz–Makeham form for the mortality intensity. Here, the mortality intensity can be written as

$$\mu_{x+t} = a + bc^{x+t}.$$

Here, the parameters, as in the Danish G82 mortality table for males, are given by $a = 0.0005$, $b = 0.000075858$ and $c = 1.09144$. To investigate the dependence on the number of insured and the choice of equivalent martingale measure we assume the company follows a buy and hold strategy with $\kappa = 0.25$ and keep the parameters and initial capital fixed as $r = 0.06$, $G_T = 1.045$, $K_T = 1.06$, $\sigma = 0.20$, $\gamma = 0.10$, $V_0 = 100$, $U_0 = 5$ and $E_0 = 5$. Recall that $V_0 = Y_0 V_0^{ind}$, so the total deposit is held constant while the number of insured individuals increases by decreasing the individual deposits accordingly. From

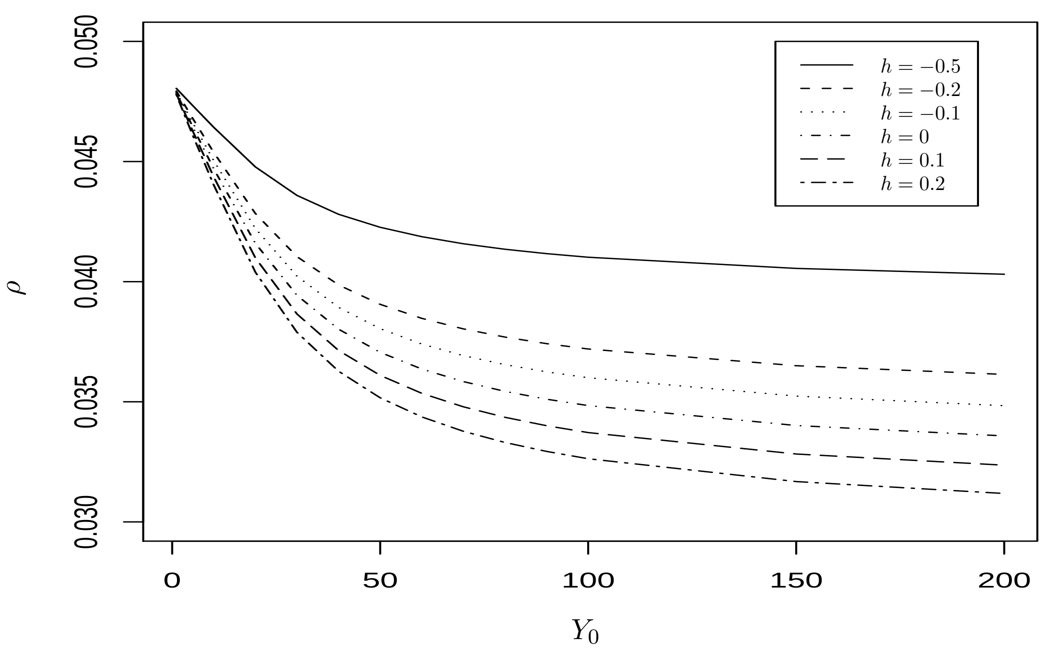

Figure 2.6.6: ρ *as a function of the number of insured for different values of* h.

Figure 2.6.6 we see that ρ is a decreasing function of the number of insured. This is in correspondence with our intuition, since increasing the size of the insurance portfolio decreases the unsystematic mortality risk. Furthermore, we observe that ρ is a decreasing function of h, and that the dependence on h is an increasing function of the number of insured. That ρ is a decreasing function of h is intuitively clear since decreasing h corresponds to decreasing the market mortality intensity, and hence increase the survival probability in the derivation of ρ. The increasing dependence on h can be explained by the strong law of large numbers, which says that as the number of insured increases the number of survivors concentrate increasingly around the mortality intensity. Hence the mortality intensity used to determine ρ becomes increasingly important as the size of the insured portfolio is increased. It can be shown, see Section 2.10.3, that if the number of insured tends to infinity then ρ converges downwards to the solution in case of capital insurances with G_T and K_T replaced by $G_T \frac{{}_Tp_x^h}{{}_Tp_x}$ and $K_T \frac{{}_Tp_x^h}{{}_Tp_x}$, respectively. Hence considering the case $h = 0$, we see that adding unsystematic mortality risk to a finite insurance portfolio leads to a fair value of ρ, which is higher than the fair value of ρ, 0.0322, obtained for capital insurances.

2.7 Impact of alternative distribution schemes

In this section we discuss how possible changes in the distribution scheme impact the results for the fair value of ρ.

A major possible change in the distribution scheme would be not to allow *any* transfer of capital from the bonus reserve to the equity capital. In the case where $G_T V_0 \leq A_T < K_T V_0(1+\gamma) + e^{rT} E_0$ this would lead to the following expression for the equity capital

$$E_T = \max\left(0, \min\left(e^{(r+\rho)T} E_0, A_T - V_T, A_T - (V_0 + U_0)\right)\right),$$

A similar change of course applies to the situation where $e^{rT} E_0 + K_T V_0(1+\gamma) \leq A_T$. Here the last term, which ensures that capital is not transferred from the bonus reserve to the equity capital, might be negative and hence the maximum operator is necessary to ensure that the equity capital is non-negative. Using this model increases the fair values of ρ, since the exposure of the equity capital to risk is larger. The increase is easily seen from the fact, that for a fixed ρ the new model would give an equity capital at time T which always is less or equal to the equity capital in the original model. Hence, a fair value of ρ must be higher. This model has been investigated in detail in the case where the solvency requirement applies to the sum of the equity capital and bonus reserve. Two important differences between the model above and the model considered in this chapter are: Firstly, as U_0 tends to infinity ρ converges to a strictly positive number, and secondly the dependence on the solvency parameter is more complex as the equity capital might receive the same negative payoff in case of intervention for different values of β.

Another possibility is to change the distribution scheme, such that the company use K_T to accumulate the deposit if $A_T \geq K_T V_0(1+\gamma)$. Thus, the company uses the accumulation factor K_T, providing that this leaves it with a minimum of $\gamma K_T V_0$ in the sum of the bonus reserve and equity capital. This criterion is closely related to a solvency requirement of βV_T on the sum of the equity capital and bonus reserve. Here, however the requirement on the sum of the bonus reserve and equity capital is set by the board of the company and not by legislation. Using this criterion in association with the model above we obtain a strange hump around $E_0 = 20$ for low values of U_0, when investigating the dependence on the size of the equity capital. This may be explained by the fact that with the proposed criterion the accumulation factor for the deposit depends on the initial equity capital. Hence, for some outcomes of the investments different values for the initial equity capital leads to different accumulation factors. Since applying a higher accumulation factor for the same investment return obviously increases the risk of the company, this leads to a positive dependence on the equity capital. The hump around $E_0 = 20$ for low values of U_0 shows that here this effect is more dominant than the otherwise predominant effect that increasing the equity capital decreases ρ. For $\gamma = 0$ the criterion corresponds to the case where the company views the announced accumulation factor as binding unless using K_T instead of G_T would bankrupt the company. In this case we would obviously expect an increase in the fair value of ρ.

If the company views the announced accumulation factor as legally binding the company is bankrupt if $A_T < K_T V_0$, and for $A_T \geq K_T V_0$ the deposit is accumulated using K_T. Applying the proper changes to the distribution scheme all necessary calculations are similar to those already presented. Since viewing K_T as binding obviously increases the risk for the equity capital, this change should lead to higher fair values of ρ.

2.8 On the realism and versatility of the model

In this section we comment on the chosen model. First we comment on the chosen probabilistic model and the requirement on the investment strategy. Then we discuss the advantages and versatility of the 1-period model. To end the section we discuss possible extensions.

The assumption that the financial market can be described by a Black–Scholes model is not very realistic, since both the interest rate and the volatility changes stochastically over time. However, if the accumulation period is relatively small the model is likely to be an acceptable approximation to reality. Hence, working with a more advanced financial model would make the results unnecessarily complicated. In the model we assume that the mortality intensity is deterministic, such that only the unsystematic mortality risk is considered. By unsystematic mortality risk we refer to the risk associated with the random development of an insured portfolio with known mortality intensity. Thus, the unsystematic mortality risk is the diversifiable part of the mortality risk. For a more realistic model we could introduce a stochastic mortality intensity as in Chapter 3. This would allow us to consider the systematic mortality risk, referring to the risk associated with changes in the underlying mortality intensity, as well. Since changes in the underlying mortality intensity affect all insured, the systematic mortality risk is non-diversifiable. On the contrary it increases as the number of similar contracts in the portfolio of insured increases. Hence, if we were to add systematic mortality risk to the model the impact on the fair value of ρ would increase as a function of the length of the accumulation period, T, and the number of insured, Y_0. Since we consider one accumulation period only, the assumption of deterministic mortality intensity is very close to reality and sufficient for our purpose.

Throughout chapter we assume that the company distinguishes between the investments belonging to the equity capital and the investments belonging to the insurance portfolio. Furthermore we assume that the assets belonging to the equity capital are invested in the savings account to keep possible risky investments on behalf of the owners aside from the risk associated with the insurance contracts. If the company does not make this distinction when investing, we may obtain the desired distinction by assuming that the equity capital is invested in the savings account and define the value of the risky portfolio residually as

$$\mathcal{V}_t(\varphi) = A_t - e^{rt} E_0.$$

Now the results in the chapter apply immediately for buy and hold strategies for A, whereas an investment strategy for A with constant relative portfolio weights would lead to minor modifications of the results.

Using a model with only one accumulation period has several advantages. Firstly, we, as seen above, can justify working with a relatively simple probabilistic model. Secondly, we are able to define a distribution scheme with only one endogenously given parameter, since we do not have to specify a formula used to anticipate how the company chooses K_T. This is of importance, since in practice the choice of K_T is widely influenced by the competition, and thus, it is difficult to model. As for the versatility of the model we are particularly interested in answers to the following two questions: Does repeated use of the 1-period model yield fairness in a multi-period setting? And if so, what insight does the company gain by repeated use of the model? To answer the first question we consider an arbitrary sequence of accumulation times $0 = T_0 < T_1 < \ldots < T_n$. For the distribution of the assets to be fair in the multi period model it must hold that

$$E^{Q^h}\left[e^{-rT_n} E_{T_n}\right] = E_0,$$

for an arbitrary, but fixed, equivalent martingale measure, Q^h. If we at each accumulation time, T_i, condition on the information $\mathcal{F}_{T_i}$ we obtain a string of 1-period models. Thus, if we determine the fair value of ρ in each 1-period model we obtain:

$$\begin{aligned} E^Q\left[e^{-rT_n} E_{T_n}\right] &= E^Q\left[e^{-r(T_n - T_{n-1})} E^Q\left[e^{-rT_{n-1}} E_{T_n}\middle|\, \mathcal{F}_{T_{n-1}}\right]\right] = E^Q[e^{-rT_{n-1}} E_{T_{n-1}}] \\ &= \ldots = E_0. \end{aligned}$$

Here, the only restriction is, that the initial distribution of capital in one period is the terminal distribution in the preceding period. Hence, it even holds if the model parameters r, σ, γ and β and the investment strategy varies for different accumulation periods. Thus, repeated use of the 1-period criterion for fairness yields fairness in a multi-period setting. Using the model in a multi-period setting the company can obtain confidence bands for the development of the balance sheet and long term ruin probabilities by simulating the development of the financial market and the insurance portfolio. However, using the model for simulation purposes we need to specify a formula, from which the company determines the announced accumulation factor in each period. Furthermore the assumptions about constant parameters in the financial market and a deterministic mortality intensity are less realistic on a long term basis. This however, could be remedied by applying stochastic models to determine the constant interest rate and volatility and deterministic mortality intensity for the next accumulation period.

Some possible extensions of the model are to include different types of insurance contracts, insured of different ages and payments during the accumulation period. However, extending the model to include different types of contracts and different age groups increases the possibility of a systematic redistribution of capital from one group of insured to another.

2.9 Conclusion

For a company issuing insurance contracts with guaranteed periodic accumulation factors we consider the problem of distributing the assets fairly between the accounts of the insured and the equity capital. To derive a fair distribution we consider a 1-period model representing one accumulation period. In the model the only free parameter in the distribution scheme is the interest rate ρ, paid to the equity capital in addition to the riskfree interest rate, when such an additional rate is possible. Using the principle of no arbitrage, we are able to derive an implicit equation for the fair value of ρ given one of four different investment strategies. Investigating the dependence of ρ on the investment strategy, we observe that a constant relative portfolio is slightly more risky than a buy and hold strategy, and that ρ is an increasing function of the relative initial investment in stocks. In the case of a solvency requirement and stop-loss strategies we find that ρ is a decreasing function of β. Considering the dependence of ρ on the parameters, we observe a positive dependence on the volatility and the guaranteed accumulation factor and a negative dependence on the riskfree interest rate. As for the announced deposit rate and the parameter γ we found that the dependence for practical purposes is non-existent. When considering the initial distribution of capital we find that ρ is an increasing function of the initial deposit and a decreasing function of the equity capital and the bonus reserve. Extending the model to include mortality obviously increases the fair value of ρ, since it adds more uncertainty to the model. As expected we observe that in the case of risk neutrality with respect to unsystematic mortality risk the fair value of ρ is a decreasing function of the number of insured tending to the fair value in the case without mortality. Furthermore we observe that the influence of the market attitude towards mortality risk on the fair value of ρ increases as the number of insured increases.

2.10 Proofs and technical calculations

2.10.1 Proof of Proposition 2.4.4

If the company invests in the savings account only, the value of the assets at time T is $A_T = e^{rT}A_0$. Since the value of the assets is deterministic, the distribution scheme is fair if and only if $E_T = e^{rT}E_0$. Considering the different intervals in the distribution scheme for the possible outcomes of A_T, we get

1. If $e^{rT}A_0 < G_T V_0$ then $E_T = 0$, so we cannot have $E_0 = e^{-rT}E_T$ if $E_0 > 0$ since

$r < \infty$. Thus, no value of ρ gives a fair distribution scheme.

2. If $G_T V_0 \leq e^{rT} A_0 < K_T V_0 (1+\gamma) + e^{rT} E_0$ then each of the two terms in the minimum operator may be the smallest, and we have to consider each of the possibilities.

   (a) If $e^{(r+\rho)T} E_0 \leq e^{rT} A_0 - G_T V_0$ then a fair value of ρ satisfies $E_0 = e^{-rT} e^{(r+\rho)T} E_0$, i.e. $\rho = 0$.

   (b) If $e^{rT} A_0 - G_T V_0 \leq e^{(r+\rho)T} E_0$ then we must have $E_0 = e^{-rT}(e^{rT} A_0 - G_T V_0)$, i.e. $G_T = e^{rT} \frac{V_0 + U_0}{V_0}$. Thus, fair values of ρ must satisfy $e^{rT} E_0 \leq e^{(r+\rho)T} E_0$, i.e. $\rho \geq 0$.

3. If $K_T V_0 (1+\gamma) + e^{rT} E_0 \leq e^{rT} A_0$ then each of the two terms in the minimum operator may be the smallest, and we have to consider each of the possibilities.

   (a) If $e^{(r+\rho)T} E_0 \leq e^{rT} A_0 - K_T V_0$ then a fair value of ρ satisfies $E_0 = e^{-rT} e^{(r+\rho)T} E_0$, i.e. $\rho = 0$.

   (b) If $e^{rT} A_0 - K_T V_0 \leq e^{(r+\rho)T} E_0$ then we must have $E_0 = e^{-rT}(e^{rT} A_0 - G_T V_0)$, i.e. $K_T = e^{rT} \frac{V_0 + U_0}{V_0}$. Thus, fair values of ρ must satisfy $e^{rT} E_0 \leq e^{(r+\rho)T} E_0$, i.e. $\rho \geq 0$.

2.10.2 Determining the limit as $U_0 \to \infty$

In this section we derive the fair value of ρ as the bonus reserve tends to ∞. For simplicity we consider the case of capital insurances. Taking the limit as $U_0 \to \infty$ in criterion (2.4.1) gives

$$
\begin{aligned}
E_0 = e^{-rT} \lim_{U_0 \to \infty} E^{Q^0} \Bigg[& 1_{(G_T V_0 \leq A_T < K_T V_0 (1+\gamma) + e^{rT} E_0)} \min\left(e^{(r+\rho)T} E_0, A_T - G_T V_0 \right) \\
& + 1_{(K_T V_0 (1+\gamma) \leq \mathcal{V}_T(\varphi))} \min\left(e^{(r+\rho)T} E_0, A_T - K_T V_0 \right) \Bigg]
\end{aligned}
\tag{2.10.1}
$$

Assuming $\rho < \infty$ dominated convergence allows us to interchange limit and expectation. Since we consider admissible investment strategies only, we have that $\lim_{U_0 \to \infty} \mathcal{V}_T = \infty$. Hence it holds that

$$\lim_{U_0 \to \infty} 1_{(G_T V_0 \leq A_T < K_T V_0 (1+\gamma) + e^{rT} E_0)} = 0$$

and

$$\lim_{U_0 \to \infty} 1_{(K_T V_0 (1+\gamma) < \mathcal{V}_T(\varphi))} = 1.$$

Furthermore we have for $\mathcal{E}_T \in \{G_T, K_T\}$ that

$$\lim_{U_0 \to \infty} \min\left(e^{(r+\rho)T} E_0, A_T - \mathcal{E}_T V_0 \right) = e^{(r+\rho)T} E_0 < \infty.$$

Hence, in the limit we obtain the following equation

$$E_0 = e^{-rT} e^{(r+\rho)T} E_0,$$

such that in the limit $\rho = 0$. This is also intuitively clear, since increasing the bonus reserve decreases the probability of the equity capital suffering a loss, and in the limit where the bonus reserve is infinitely large the equity capital bears no risk and obviously it should not receive an additional payment compared to the riskfree interest rate.

We end this section by noting that the assumption $\rho < \infty$ does not impose a restriction, since $\rho = \infty$ cannot be a solution to (2.10.1). In order to do so we assume $\rho = \infty$ solves (2.10.1). This in turn would lead to

$$\begin{aligned} E_0 &= e^{-rT} \lim_{U_0 \to \infty} E^{Q^0} \Bigg[1_{(G_T V_0 \le A_T < K_T V_0 (1+\gamma) + e^{rT} E_0)} (A_T - G_T V_0) \\ &\quad + 1_{(K_T V_0 (1+\gamma) \le \mathcal{V}_T(\varphi))} \left(A_T - K_T V_0 \right) \Bigg] \\ &\ge e^{-rT} \lim_{U_0 \to \infty} E^{Q^0} \left[1_{(K_T V_0 (1+\gamma) \le \mathcal{V}_T(\varphi))} \left(A_T - K_T V_0 \right) \right] \\ &= \infty, \end{aligned}$$

where we have used monotone convergence to interchange limit and integration in the last equality and considerations similar to those above to determine the limit. However, since $E_0 < \infty$ we have a contradiction, such that $\rho = \infty$ can not be the solution.

2.10.3 Determining the limit as $Y_0 \to \infty$

We now determine the convergence of ρ as Y_0 tends to ∞, while keeping $V_0 = Y_0 V_0^{ind}$ fixed. Taking the limit in (2.5.3) we get

$$\begin{aligned} E_0 = e^{-rT} &\lim_{Y_0 \to \infty} E^{Q^h} \Bigg[1_{(Y_T G_T V_0^{ind} \frac{1}{_T p_x} \le A_T < Y_T K_T V_0^{ind} \frac{1}{_T p_x} (1+\gamma) + e^{rT} E_0)} \\ &\times \min \left(e^{(r+\rho)T} E_0, A_T - Y_T G_T V_0^{ind} \frac{1}{_T p_x} \right) \\ &+ 1_{(Y_T K_T V_0^{ind} \frac{1}{_T p_x} (1+\gamma) \le \mathcal{V}_T(\varphi))} \min \left(e^{(r+\rho)T} E_0, A_T - Y_T K_T V_0^{ind} \frac{1}{_T p_x} \right) \Bigg] \end{aligned}$$

Assuming that $\rho < \infty$ we can use dominated convergence to interchange limit and integral. Using the strong law of large numbers we have for an arbitrary accumulation factor $\mathcal{E}_T$:

$$\lim_{Y_0 \to \infty} \left(Y_T \mathcal{E}_T V_0^{ind} \frac{1}{_T p_x} \right) = \lim_{Y_0 \to \infty} \left(\frac{Y_T}{Y_0} \mathcal{E}_T V_0 \frac{1}{_T p_x} \right) = \mathcal{E}_T V_0 \frac{_T p_x^h}{_T p_x}, \quad Q^h - a.s.$$

Since Q^h is identical to Q^0 with respect to the financial market for all h, we obtain the following equation in the limit

$$\begin{aligned}E_0 = e^{-rT} E^{Q^0} \Bigg[& 1_{\left(G_T V_0 \frac{{}_Tp_x^h}{{}_Tp_x} \le A_T < K_T V_0 \frac{{}_Tp_x^h}{{}_Tp_x}(1+\gamma) + e^{rT} E_0\right)} \min\left(e^{(r+\rho)T} E_0, A_T - G_T V_0 \frac{{}_Tp_x^h}{{}_Tp_x} \right) \\ & + 1_{\left(K_T V_0 \frac{{}_Tp_x^h}{{}_Tp_x}(1+\gamma) \le \mathcal{V}_T(\varphi)\right)} \min\left(e^{(r+\rho)T} E_0, A_T - K_T V_0 \frac{{}_Tp_x^h}{{}_Tp_x} \right) \Bigg].\end{aligned}$$

This is exactly the equation in the case of capital insurances with G_T replaced by $G_T \frac{{}_Tp_x^h}{{}_Tp_x}$ and K_T replaced by $K_T \frac{{}_Tp_x^h}{{}_Tp_x}$. Note in particular, that assuming risk neutrality with respect to unsystematic mortality risk, i.e. $h = 0$ gives the same results in the limit as in the case without mortality.

The calculations above are carried out for an arbitrary h. However in the limit the measures Q^h and P are singular rather than equivalent if $h \neq 0$. Thus, using a Q^h with $h \neq 0$ in an attempt to derive a fair value of ρ for an infinitely large insurance portfolio would thus result in introducing an arbitrage possibility in the model. However, even though the limit result for $h \neq 0$ has no economic interpretation, it still provides useful insight for the dependence of ρ on h for a large portfolio. Furthermore solving the limit equation gives an approximation to the fair value in the case of a large portfolio of pure endowments.

2.10.4 Proof of Proposition 2.5.8

In order to prove Proposition 2.5.8, we first note that the probability of insolvency can be written as:

$$\begin{aligned}p_{ins}(\varphi) &= P\left[E_T < Y_T \beta V_T^{ind}\right] \\ &= \sum_{n=0}^{Y_0} \binom{Y_0}{n} ({}_Tp_x)^n ({}_Tq_x)^{Y_0-n} P\left[E_T < n\beta V_T^{ind}\right] \\ &= \sum_{n=0}^{Y_0} \binom{Y_0}{n} ({}_Tp_x)^n ({}_Tq_x)^{Y_0-n} \Bigg(P\left[E_T < n\beta V_T^{ind}, \inf_{0\le t\le T} S_t^* > Z\right] \\ &\quad + P\left[E_T < n\beta V_T^{ind}, \inf_{0\le t\le T} S_t^* \le Z\right] \Bigg) \\ &= \sum_{n=0}^{Y_0} \binom{Y_0}{n} ({}_Tp_x)^n ({}_Tq_x)^{Y_0-n} \Bigg(P\left[A_T < n(1+\beta) G_T V_0^{ind} \frac{1}{{}_Tp_x}, \inf_{0\le t\le T} S_t^* > Z\right] \\ &\quad + P\left[A_T < n(1+\beta) G_T V_0^{ind} \frac{1}{{}_Tp_x}, \inf_{0\le t\le T} S_t^* \le Z\right] \Bigg).\end{aligned}$$

Here, we have use iterated expectations in the second equality, and in the third we split the probability according to whether the company intervenes or not. The fourth equality follows from the relationship $e^{rT}E_0 \geq \beta K_T Y_0 V_0^{ind}\frac{1}{{}_Tp_x}$, since this ensures that the company never is insolvent if the deposit is accumulated with the factor K_T or if the equity capital at time T is given by $E_T = e^{(r+\rho)T}E_0$. Now we insert $s_1^{\beta,n,*}$ and the deterministic value of A_T in case of intervention to obtain

$$\begin{aligned}
p_{ins}(\varphi) &= \sum_{n=0}^{Y_0}\binom{Y_0}{n}({}_Tp_x)^n({}_Tq_x)^{Y_0-n}\Bigg(P\left[S_T^* < s_1^{\beta,n,*}, \inf_{0\leq t\leq T} S_t^* > Z\right] \\
&\quad + P\left[Y_0 G_T V_0^{ind}\frac{{}_Tp_x^h}{{}_Tp_x}(1+\beta) < nG_T V_0^{ind}\frac{1}{{}_Tp_x}(1+\beta), \inf_{0\leq t\leq T} S_t^* \leq Z\right]\Bigg) \\
&= \sum_{n=0}^{Y_0}\binom{Y_0}{n}({}_Tp_x)^n({}_Tq_x)^{Y_0-n}\Bigg(P\left[S_T^* < s_1^{\beta,n,*}, \inf_{0\leq t\leq T} S_t^* > Z\right] \\
&\quad + 1_{(Y_0{}_Tp_x^h<n)}P\left[\inf_{0\leq t\leq T} S_t^* \leq Z\right]\Bigg).
\end{aligned} \tag{2.10.2}$$

From (2.10.2) we observe that if the number of survivors if greater that the Q^h expectation then the company is insolvent in case of intervention, whereas this is not necessarily the case in the situation without intervention. Calculations similar to those in the proof of Björk (2004, Theorem 18.8) give

$$\begin{aligned}
P\left[S_T^* < s_1^{\beta,n,*}, \inf_{0\leq t\leq T} S_t^* > Z\right] &= E^P\left[1_{(S_T^*<s_1^{\beta,n,*})}1_{(\inf_{0\leq t\leq T} S_t^*>Z)}\right] \\
&= E^P\left[1_{(Z<S_T^*<s_1^{\beta,n,*})}\right] - \left(\frac{Z}{S_0}\right)^{\frac{2(\alpha-r)}{\sigma^2}-1} E^P\left[1_{(Z<\widetilde{S}_T^*<s_1^{\beta,n,*})}\right],
\end{aligned}$$

where $\widetilde{S}^*$ is a process with the same dynamics as S^*, but with initial value $\widetilde{S}_0^* = \frac{Z^2}{S_0}$. Investigating each term separately we get

$$\begin{aligned}
&E^P\left[1_{(Z<S_T^*<s_1^{\beta,n,*})}\right] \\
&= 1_{(Z<s_1^{\beta,n,*})}\left(P\left[S_T^* < s_1^{\beta,n,*}\right] - P[S_T^* \leq Z]\right) \\
&= 1_{(Y_0{}_Tp_x^h<n)}\left(\Phi\left(\frac{\log\left(\frac{s_1^{\beta,n,*}}{S_0}\right) - \left(\alpha - r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) - \Phi\left(\frac{\log\left(\frac{Z}{S_0}\right) - \left(\alpha - r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)\right),
\end{aligned}$$

and

$$\begin{aligned}
&E^P\left[1_{(Z<\widetilde{S}_T^*<s_1^{\beta,n,*})}\right] \\
&= 1_{(Y_0{}_Tp_x^h<n)}\left(\Phi\left(\frac{\log\left(\frac{s_1^{\beta,n,*}S_0}{Z^2}\right) - \left(\alpha - r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) - \Phi\left(\frac{\log\left(\frac{S_0}{Z}\right) - \left(\alpha - r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)\right).
\end{aligned}$$

Similarly

$$
\begin{aligned}
P\left[\inf_{0\le t\le T} S_t^* \le Z\right] &= 1 - E^P\left[1_{(\inf_{0\le t\le T} S_t^* > Z)}\right] \\
&= 1 - E^P\left[1_{(Z<S_T^*)}\right] + \left(\frac{Z}{S_0}\right)^{\frac{2(\alpha-r-\frac{1}{2}\sigma^2)}{\sigma^2}} E^P\left[1_{(Z<\mathcal{S}_T^*)}\right] \\
&= \Phi\left(\frac{\log\left(\frac{Z}{S_0}\right) - \left(\alpha - r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) \\
&\quad + \left(\frac{Z}{S_0}\right)^{\frac{2(\alpha-r)}{\sigma^2}-1}\left(1 - \Phi\left(\frac{\log\left(\frac{S_0}{Z}\right) - \left(\alpha - r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)\right).
\end{aligned}
$$

Combining the results we get

$$
\begin{aligned}
p_{ins}(\varphi) = \sum_{n=0}^{Y_0}\binom{Y_0}{n}({}_Tp_x)^n({}_Tq_x)^{Y_0-n}1_{(Y_0\,{}_Tp_x^h<n)}&\left(\Phi\left(\frac{\log\left(\frac{s_1^{\beta,n,*}S_0}{Z^2}\right) - \left(\alpha - r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)\right. \\
&\left. + \left(\frac{Z}{S_0}\right)^{\frac{2(\alpha-r)}{\sigma^2}-1}\left(1 - \Phi\left(\frac{\log\left(\frac{s_1^{\beta,n,*}S_0}{Z^2}\right) - \left(\alpha - r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)\right)\right).
\end{aligned}
$$

Chapter 3

Stochastic Mortality in Life Insurance: Market Reserves and Mortality-Linked Insurance Contracts

(This chapter is an adapted version of Dahl (2004b))

In life insurance, actuaries have traditionally calculated premiums and reserves using a deterministic mortality intensity, which is a function of the age of the insured only. Here, we model the mortality intensity as a stochastic process. This allows us to capture two important features of the mortality intensity: Time dependency and uncertainty of the future development. The advantage of introducing a stochastic mortality intensity is two-fold. Firstly it gives more realistic premiums and reserves, and secondly it quantifies the risk of the insurance companies associated with the underlying mortality intensity. Having introduced a stochastic mortality intensity, we study possible ways of transferring the systematic mortality risk to other parties. One possibility is to introduce mortality-linked insurance contracts. Here the premiums and/or benefits are linked to the development of the mortality intensity, thereby transferring the systematic mortality risk to the insured. Alternatively the insurance company can transfer some or all of the systematic mortality risk to agents in the financial market by trading derivatives depending on the mortality intensity.

3.1 Introduction

Traditionally, actuaries have been calculating premiums and reserves using a deterministic mortality intensity, which is a function of the age only, and a constant interest rate (representing the payoff of the investments made by the companies). However, since neither the interest rate nor the mortality intensity is deterministic, life insurance companies are essentially exposed to three types of risk when issuing contracts: Financial risk, systematic mortality risk and unsystematic mortality risk. Here, we distinguish between systematic mortality risk, referring to the future development of the underlying mortality intensity, and unsystematic mortality risk, referring to a possible adverse development of the insured portfolio. So far the life insurance companies have dealt with the financial and (systematic) mortality risks by choosing both the interest rate and the mortality intensity to the safe side, as seen from the insurers' point of view. When the real mortality intensity and investment payoff are experienced over time, this usually leads to a surplus, which, by the so-called contribution principle, must be redistributed among the insured as bonus, see Norberg (1999). Since insurance contracts often run for 30 years or more, a mortality intensity or interest rate, which seems to be to the safe side at the beginning of the contract, might turn out not to be so. This phenomenon has in particular been observed for the interest rate during recent years, where we have experienced large drops in stock prices and low returns on bonds. However, the systematic mortality risk is of a different character than the financial risk. While the assets on the financial market are very volatile, changes in the mortality intensity seem to occur more slowly. Thus, the financial market poses an immediate problem, whereas the level of the mortality intensity poses a more long term, but also more permanent, problem. This difference could be the reason why emphasis so far has been on the financial markets. We hope to turn some of this attention towards the uncertainty associated with the mortality intensity by modelling it as a stochastic process.

In order to obtain a more accurate description of the liabilities of life insurance companies, market reserves have been introduced, see Steffensen (2000) and references therein. Here, the financial uncertainty as well as the uncertainty stemming from the development of an insurance portfolio with known mortality intensity is considered. By modelling the mortality intensity as a stochastic process, market reserves can be further extended to include the uncertainty associated with the future development of the mortality intensity. This should allow for an even more accurate assessment of future liabilities, since possible trends in the mortality intensity and the market attitude towards systematic mortality risk can be taken into account. In addition, a stochastic mortality intensity allows for a quantification of the systematic mortality risk of the insurance companies. Having quantified the systematic mortality risk, we investigate how the insurance companies could manage the risk. As a first possibility, we introduce a new type of contracts called *mortality-linked contracts*. The basic idea is to link and currently adapt benefits (and/or premiums) to the development of the mortality intensity in general, and thereby transfer the systematic mortality risk from the insurance company to the group of insured. A second possibility is to transfer the systematic mortality risk to other parties in the financial market. Here, the idea is to introduce certain traded assets, which depend on the development of the mortality intensity.

This chapter is organized as follows: Section 3.2 contains a review of existing literature on stochastic mortality. Section 3.3 deals with the modelling of the mortality intensity as a stochastic process, and Section 3.4 introduces the model considered in the rest of this chapter. An expression for the market reserve for a general payment stream is given in Section 3.5. In Section 3.6, we introduce the concept of a mortality-linked insurance contracts, whereas Section 3.7 includes a discussion of how the systematic mortality risk could be transferred to other agents in the financial market. Finally, the derivation of the dynamics of the benefit for a mortality-linked pure endowment in the case of risky investments is given in 3.8.

3.2 Existing literature on stochastic mortality

In this section we give a brief review on existing literature concerning the uncertainty associated with the future development of the mortality intensity. For further references see the referred papers.

Olivieri (2001) assumes that the insurance companies take possible trends in the mortality intensity into account by estimating a mortality intensity, which is a function of both time and age. Hence the companies obtain more realistic premiums and reserves than by using a function of age only. However, the estimated survival function, no matter how good it is, is only one possible future development. Thus, Olivieri uses the observed mortality intensities to generate two additional survival functions, which represent very high and very low future survival probabilities, respectively. Using these three possible scenarios for the future survival function, Olivieri illustrates the impact of systematic mortality risk by calculating variances of present values. Marocco and Pitacco (1998) model the yearly mortality rates via a beta distribution with age and time-dependent parameters. Hence, they are able to quantify the mortality risk inherent in an insurance portfolio, since the number of survivors follows a binomial-beta distribution. The approach in Olivieri and Pitacco (2002) is somewhat different. They describe the future survival function by a parameterized family of possible future survival functions. However, since the future is unknown, the parameter is a random variable. In order to obtain prices and assess the risk they apply Baysian methods to describe the distribution function for the parameter. Within this model they are able to distinguish between the unsystematic mortality risk stemming from the randomness for a given parameter (survival function), and the systematic mortality risk stemming from the uncertainty associated with the parameter (survival function).

In the above papers no explicit financial market has been introduced and all calculations are carried out using a constant interest rate. Models involving both interest rate risk and systematic mortality risk are proposed in Milevsky and Promislow (2001). For a fixed equivalent martingale measure they propose both a discrete and continuous time model for the mortality and interest rate. Within the proposed models they are able to obtain prices and determine hedging strategies for claims that are contingent on the mortality and interest rate.

The contribution of the present chapter is as follows: Inspired by interest rate modelling we model the mortality intensity by a fairly general diffusion model, which include the

Mean reverting Brownian Gompertz model proposed by Milevsky and Promislow (2001) as a special case. Taking the incomplete model comprised by the financial market, mortality intensity and insurance contract as a starting point, we then note that there exist infinitely many equivalent martingale measures corresponding to different market attitudes towards systematic and unsystematic mortality risk. Hence, contracts involving an insurance element cannot be priced uniquely using a no arbitrage approach. For a fixed but arbitrary equivalent martingale measure we derive integral expressions and partial differential equations for market reserves in the presence of stochastic mortality. These results show how market reserves depend on the expectation to the future mortality intensity and the market attitude towards systematic mortality risk. The latter seems to be a new result. Furthermore we introduce a new type of contracts called mortality-linked insurance contracts as a way to transfer the systematic mortality risk to the insured. Finally a general partial differential equation for mortality derivatives is derived, and it is shown how the company may use such derivatives to transfer the systematic mortality risk to the financial market.

3.3 Mortality intensity as a stochastic process

3.3.1 Stochastic versus deterministic mortality

In actuarial practice, statistical methods are usually used to estimate a mortality intensity, which is a function of the age, $x+t$, only. In Denmark, life insurance companies use a so-called Gompertz–Makeham model for the mortality intensity. Here, the mortality intensity can be written as

$$\mu_{x+t} = a + bc^{x+t}.$$

In this chapter more realism is added by viewing the mortality intensity as a stochastic process, which is adapted to some filtration $\mathbb{F}$. We model the mortality intensities as diffusion processes such that for every fixed $x \geq 0$ the mortality intensity has dynamics of the form

$$d\mu_{[x]+t} = \alpha^{\mu}(t, x, \mu_{[x]+t})dt + \sigma^{\mu}(t, x, \mu_{[x]+t})d\widetilde{W}_t, \tag{3.3.1}$$

where $\widetilde{W}$ is a Wiener process (standard Brownian motion) with respect to the filtration $\mathbb{F}$. Here and throughout we have borrowed the select mortality notation of Norberg (1988). Since the dynamics of μ depend on the present state of the process only, then μ is a Markov process. In (3.3.1), we have assumed that the person is of age x at time 0 (which is an arbitrary calendar time). The parameter t then describes the time that has passed since time 0. Results similar to those presented in this chapter can be obtained in the case, where the mortality intensity is driven by a finite state Markov process, see Dahl (2002).

Remark 3.3.1 By modelling mortality intensities by (3.3.1), we have made the following two rather unrealistic assumptions: Firstly, all sudden changes in the mortality intensities are of the same type and affect all ages/cohorts and secondly, the mortality intensity for each age is a Markov process. A more realistic model would recognize that the mortality intensities are affected by many different so-called mortality factors and that these factors affect the mortality intensities differently. Some mortality factor affect all ages/cohorts while others affect only some ages/cohorts. Furthermore even mortality factors affecting the same ages/cohorts may have a different impact on the mortality intensities. An appropriate model for the dependence on the different mortality factors could be a hierarchical model. Assume for example that the dynamics of the mortality intensities are given by the following extension of (3.3.1):

$$d\mu_{[x]+t} = \alpha^{\mu}(t, x, \mu_t)dt + (\sigma^{\mu}(t, x, \mu_t))^{tr} d\widetilde{W}_t^*,$$

where μ_t denotes the infinite dimensional vector containing the mortality intensities at time t for all x. Moreover, $\sigma^{\mu}(t, x, \mu_t)$ and $\widetilde{W}^*$ are d-dimensional column vectors, and a^{tr} denotes vector a transposed. Now a hierarchical structures is obtained, if we interpret each Wiener process as the impact of a specific mortality factor and define σ such that the Wiener process only affects the appropriate ages/cohorts. However, since we are working with one value of x only and because no further insight is gained from working with a multi-dimensional Wiener process, we restrict ourselves to the simple 1-dimensional case given by (3.3.1).

□

Remark 3.3.2 Instead of modelling the mortality intensity as a diffusion process of the form in (3.3.1), we could assume a Gompertz–Makeham structure and model the parameters a, b and c as stochastic processes. Dahl (2002) includes some examples, where a and b are modelled by stochastic processes.

□

In the case with known mortality intensity the survival probability from time t to T for a person of age x at time 0 is given by $e^{-\int_t^T \mu_{[x]+u} du}$. However, since we do not know the future development of the mortality intensity, this should be replaced by an expected value, conditioning on the known development up to time t, represented by $\mathcal{F}_t$. Here, $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ is the filtration for the model describing all the randomness observed, which in particular contains information about the development of the stochastic process μ. Informally, $\mathcal{F}_t$ is the information available to the insurer at time t. Note that we hereby assume that μ is observable, which corresponds to assuming that the portfolio of observed lives is sufficiently large, such that the mortality intensity can be estimated correctly. Using that μ is a Markov process and iterated expectations, we see that $\mathcal{S}(t, x, \mu_{[x]+t}, T)$, defined by

$$\mathcal{S}(t, x, \mu_{[x]+t}, T) := E^P \left[e^{-\int_t^T \mu_{[x]+\tau} d\tau} \middle| \mu_{[x]+t} \right],$$

is the *survival probability* from time t to T for a person of age $x + t$ given the information until time t.

Remark 3.3.3 Note that the mortality intensity, in contrast to the e.g. the interest rate, is modelled under the objective measure P. For the interest rate modelling usually takes place directly under some equivalent martingale measure Q.

□

For fixed x we now define a P-martingale M by

$$M(t,x) := E^P\left[e^{-\int_0^T \mu_{[x]+\tau} d\tau} \middle| \mathcal{F}_t \right] = e^{-\int_0^t \mu_{[x]+\tau} d\tau} \mathcal{S}(t,x,\mu_{[x]+t},T),$$

where μ is defined by (3.3.1). The quantity $M(t,x)$ can be interpreted as the probability of survival from time 0 to T for a person of age x at time 0 given the development of the mortality intensity until time t. Provided that $\mathcal{S}$ is continuously differentiable in t and twice continuously differentiable in μ, we can use Itô's formula on the martingale M, such that we for fixed x obtain the following partial differential equation (PDE) for $\mathcal{S}(t,x,\mu,T)$ on $[0,T] \times \mathbb{R}_+$:

$$\begin{aligned} 0 &= \partial_t \mathcal{S}(t,x,\mu,T) + \alpha^\mu(t,x,\mu)\partial_\mu \mathcal{S}(t,x,\mu,T) \\ &\quad + \frac{1}{2}(\sigma^\mu(t,x,\mu))^2 \partial_{\mu\mu}\mathcal{S}(t,x,\mu,T) - \mu\mathcal{S}(t,x,\mu,T), \end{aligned} \tag{3.3.2}$$

which should be solved with the boundary condition

$$\mathcal{S}(T,x,\mu,T) = 1.$$

Here, we have used the notation $\partial_t \mathcal{S} = \frac{\partial}{\partial t}\mathcal{S}$, $\partial_\mu \mathcal{S} = \frac{\partial}{\partial \mu}\mathcal{S}$ and $\partial_{\mu\mu}\mathcal{S} = \frac{\partial^2}{\partial \mu^2}\mathcal{S}$, which will be used throughout the chapter, whenever the derivatives exist. The differential equation (3.3.2) is analogous to the differential equation for zero coupon bonds obtained when working with a stochastic interest rate, see e.g. Björk (1997, Proposition 3.4).

3.3.2 Affine mortality structure

We now concentrate on a special mortality structure, which will be referred to as an *affine mortality structure*. The following definition of an affine mortality structure is almost analogous to the definition of an affine term structure, see e.g. Björk (1997, Definition 3.1):

Definition 3.3.4 (Affine mortality structure)
If, for fixed x, the survival probabilities are given by $\mathcal{S}(t,x,\mu_{[x]+t},T)$, where $\mathcal{S}$ has the form

$$\mathcal{S}(t,x,\mu_{[x]+t},T) = e^{A(t,x,T)-B(t,x,T)\mu_{[x]+t}}, \tag{3.3.3}$$

for deterministic functions $A(t,x,T)$ and $B(t,x,T)$, then the model for the mortality intensity is said to possess an affine mortality structure *for cohort x. If (3.3.3) holds for all x, then the model is simply said to possess an* affine mortality structure.

Affine mortality structures are of interest, since they allow survival probabilities to be expressed by the relatively simple expression in (3.3.3). However, explicit expressions for A and B may be quite complicated or even impossible to find.

Example 3.3.5 A natural question is, whether Definition 3.3.4 includes models with deterministic mortality intensity. This is indeed the case, as can be seen for example if we choose A and B by

$$\begin{aligned} A(t,x,T) &= -\int_t^T \mu_{[x]+\tau} d\tau, \\ B(t,x,T) &= 0, \end{aligned}$$

respectively, such that the survival probability is given by $\mathcal{S}(t,x,\mu_{[x]+t},T) = e^{-\int_t^T \mu_{[x]+\tau} d\tau}$. If the deterministic mortality intensity only depends on x and t through $x+t$, this is recognized as the traditional survival probability, ${}_{T-t}p_{x+t}$.

□

The definition of an affine mortality structure does not give a way to determine whether a given model for the mortality intensity possesses an affine structure. One has to find the expression for the survival probabilities and determine whether they can be written on the desired form in (3.3.3). This is not of much help, since the reason for checking whether we have an affine mortality structure (or at least an affine mortality structure for some cohorts x) exactly is, that it yields expression (3.3.3) for the probabilities. The following theorem, which also appears in Björk (1997) for zero coupon bond prices, gives sufficient conditions for a mortality structure to be affine for cohort x. In addition, it yields a set of differential equations for the functions A and B for fixed x.

Theorem 3.3.6 (Sufficient conditions for an affine mortality structure)
Assume that α^μ and σ^μ are of the form:

$$\begin{aligned} \alpha^\mu(t,x,\mu_{[x]+t}) &= \delta^\alpha(t,x)\mu_{[x]+t} + \zeta^\alpha(t,x), \\ \sigma^\mu(t,x,\mu_{[x]+t}) &= \sqrt{\delta^\sigma(t,x)\mu_{[x]+t} + \zeta^\sigma(t,x)}, \end{aligned}$$

for some deterministic functions δ^α, ζ^α, δ^σ and ζ^σ. Then the model has an affine mortality structure for cohort x, where A and B for fixed x satisfy the system

$$\begin{aligned} \partial_t B(t,x,T) &= -\delta^\alpha(t,x)B(t,x,T) + \frac{1}{2}\delta^\sigma(t,x)\left(B(t,x,T)\right)^2 - 1, \\ B(T,x,T) &= 0, \end{aligned} \tag{3.3.4}$$

and

$$\begin{aligned} \partial_t A(t,x,T) &= \zeta^\alpha(t,x)B(t,x,T) - \frac{1}{2}\zeta^\sigma(t,x)\left(B(t,x,T)\right)^2, \\ A(T,x,T) &= 0. \end{aligned} \tag{3.3.5}$$

Proof of Theorem 3.3.6.
The proof is analogous to the one given in Björk (1997, Proposition 3.5) for an affine term structure.

□

For fixed x an affine structure for α^μ and $(\sigma^\mu)^2$ in t is thus sufficient for an affine mortality structure for cohort x. If in addition α^μ and σ^μ are time independent, the condition is necessary as well, see Duffie (1992). Thus, provided we can solve (3.3.4) and (3.3.5), an affine mortality structure for cohort x gives a closed form expression for the survival probabilities for cohort x.

3.3.3 Model considerations

In this section, properties for the mortality intensity are discussed, and a specific model for the mortality intensity is considered. In interest rate modelling positivity of the interest rate is a desirable property. For the mortality intensity this is not only a desirable, but mandatory, property. While one could imagine having interest rate 0, the mortality intensity should be strictly positive, since a mortality intensity equal to 0 corresponds to a survival probability of 1, and this is not realistic for any time interval. One model which fulfills the requirement of a strictly positive mortality intensity, is the following analogue to the so-called extended Cox–Ingersoll–Ross model, which was first considered by Hull and White (1990) as a model for the interest rate. Applying the extended Cox–Ingersoll–Ross model for the modelling of mortality intensities leads to the following dynamics for fixed x

$$d\mu_{[x]+t} = (\beta^\mu(t,x) - \gamma^\mu(t,x)\mu_{[x]+t})dt + \rho^\mu(t,x)\sqrt{\mu_{[x]+t}}d\widetilde{W}_t, \tag{3.3.6}$$

where $\beta^\mu(t,x)$, $\gamma^\mu(t,x)$ and $\rho^\mu(t,x)$ are positive bounded functions. It can be shown that the extended Cox–Ingersoll–Ross model ensures strict positivity of the mortality intensity for cohort x provided that for fixed x we have $2\beta^\mu(t,x) \geq (\rho^\mu(t,x))^2$, for all $t \in [0,T]$, see Maghsoodi (1996). Furthermore, the model is mean reverting around the time and cohort dependent level $\frac{\beta^\mu(t,x)}{\gamma^\mu(t,x)}$. Theorem 3.3.6 shows that the mortality intensity given by (3.3.6) admits an affine mortality structure. Provided that we are able to solve the PDEs for A and B, we are thus able to find closed form expressions for the survival probabilities. It would thus be interesting to see whether statistical data supports modelling the dynamics of the mortality intensity by an extended Cox–Ingersoll–Ross model, such that the desirable properties for the mortality intensity are obtained.

3.3.4 Forward mortality intensities

When modelling interest rates, important quantities are forward rates defined by

$$f(t,T) := -\partial_T \log p(t,T), \quad 0 \leq t \leq T,$$

or equivalently

$$p(t,T) = e^{-\int_t^T f(t,u)du}.$$

Here $p(t,T)$ is the price at time t of a zero coupon bond maturing at time T. The forward rate $f(t,u)$ can thus be interpreted as the riskfree rate of interest, contracted at time t, over the infinitesimal interval $[u, u+du)$. Analogously to the concept of forward rates, we define the *forward mortality intensity* for cohort x at time t under the true measure P for the time T, by

$$f^{\mu}(t,x,T) := -\partial_T \log \mathcal{S}(t,x,\mu_{[x]+t},T).$$

Equivalently, we can express the relation between the forward mortality intensities and survival probabilities by

$$\mathcal{S}(t,x,\mu_{[x]+t},T) = e^{-\int_t^T f^{\mu}(t,x,u)du}.$$

Thus, the forward mortality intensity function for cohort x, $(f^{\mu}(t,x,u))_{t\leq u\leq T}$ is the adapted mortality (intensity) function, which makes the survival probability at time t for a $x+t$ year old equal to $e^{-\int_t^{\tau} f^{\mu}(t,x,u)du}$ for all $t < \tau \leq T$. Instead of modelling the mortality intensity directly, one could imagine that the life insurance companies would model the forward mortality intensity. This could be done by replacing the time homogeneous deterministic function, which they are using today, with a function of x, t, T and the observed mortality intensity $\mu_{[x]+t}$. Note that the forward mortality intensities are stochastic processes, since the forward mortality intensity for cohort x, $f^{\mu}(\tau,x,T)$, at τ is not known in general at t, if $t < \tau$.

3.4 The model

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ satisfying the usual conditions of right-continuity, i.e. $\mathcal{F}_t = \bigcap_{u>t} \mathcal{F}_u$, and completeness, i.e. $\mathcal{F}_0$ contains all P-null sets. Here, T is a fixed time horizon. Throughout, $\mathcal{F}_t$ describes the total information available at time t. Below we introduce the three components which constitute the model: The financial market, the mortality intensity and the insurance contract.

3.4.1 The financial market

We consider a financial market consisting of two traded assets only: A risky asset with price process S and a locally riskfree asset with price process B. The risky asset is usually referred to as a stock and the locally riskfree asset as a savings account. The price processes are defined on the above introduced probability space $(\Omega, \mathcal{F}, P)$, and the P-dynamics of the price processes are given by

$$dS_t = \alpha^s(t,S_t)S_t dt + \sigma^s(t,S_t)S_t dW_t, \quad S_0 > 0, \tag{3.4.1}$$

$$dB_t = r(t,S_t)B_t dt, \quad B_0 = 1, \tag{3.4.2}$$

where r is non-negative, σ^s is uniformly bounded away from 0 and $(W_t)_{0\leq t\leq T}$ is a Wiener process on the interval $[0,T]$ under P. Throughout the chapter we also use the shorthand notation exemplified by $r_t = r(t,S_t)$ for coefficient functions from stochastic differential equations that have already been introduced. The filtration $\mathbb{G} = (\mathcal{G}_t)_{0\leq t\leq T}$ is the P-augmentation of the natural filtration generated by (B, S), i.e. $\mathcal{G}_t = \mathcal{G}_t^+ \vee \mathcal{N}$, where $\mathcal{N}$ is the σ-algebra generated by all P-null sets and

$$\mathcal{G}_t^+ = \sigma\{(B_u, S_u), u \leq t\} = \sigma\{S_u, u \leq t\} = \sigma\{W_u, u \leq t\}, \tag{3.4.3}$$

since W accounts for all the randomness in the model defined by (3.4.1)–(3.4.2). We note that the last equality in (3.4.3) only holds if σ^s does not take the value 0, which is the case since we have assumed that σ^s is uniformly bounded away from 0. Assuming that α^s and σ^s fulfill certain regularity conditions, see Kloeden and Platen (1992, Theorem 4.5.3), the stochastic differential equation (3.4.1) has a unique solution. Henceforth it is assumed that these conditions are fulfilled and that $\int_0^T r_\tau d\tau$ exists and is finite almost surely, such that the function B_t is defined for all $t \in [0, T]$.

In the model given by (3.4.1)–(3.4.2), the process α^s is interpreted as the mean rate of return of the stock and σ^s as the standard deviation of the rate of return. The process r is known as the short rate of interest. Let further the process ν be defined by $\nu(t, S_t) = \frac{\alpha(t,S_t)-r(t,S_t)}{\sigma(t,S_t)}$. Hence, ν measures the excess return of the stock over the riskfree interest rate divided by the risk associated with the stock as measured by σ^s. In the literature, ν is known as the market price of risk associated with S. In the following we assume that ν satisfies the so-called Novikov condition

$$E^P\left[e^{\frac{1}{2}\int_0^T \nu^2(t,S_t)dt}\right] < \infty,$$

see Duffie (1992, Appendix D).

3.4.2 The mortality intensity

Let the mortality intensity process be defined on the above introduced probability space $(\Omega, \mathcal{F}, P)$. From here on all dependence on x is left out of the notation (except in $\mu_{[x]+t}$), since we only consider one fixed, but arbitrary, value of x. As in Section 3.3, we let the P-dynamics of the mortality intensity be given by

$$d\mu_{[x]+t} = \alpha^\mu(t, \mu_{[x]+t})dt + \sigma^\mu(t, \mu_{[x]+t})d\widetilde{W}_t, \tag{3.4.4}$$

where α^μ and σ^μ are non-negative and $(\widetilde{W}_t)_{0\leq t\leq T}$ is a Wiener process on the interval $[0, T]$ under P. Note that the coefficients are functions of the current value of the mortality intensity only, such that the mortality intensity is a Markov process. The filtration $\mathbb{I} = (\mathcal{I}_t)_{0\leq t\leq T}$ is the P-augmentation of the natural filtration generated by the mortality intensity. Thus, we have $\mathcal{I}_t = \mathcal{I}_t^+ \vee \mathcal{N}$, where

$$\mathcal{I}_t^+ = \sigma\{\mu_{[x]+u}, u \leq t\}.$$

In order to ensure the existence of a solution to (3.4.4), we assume that the coefficients fulfill the regularity conditions in Yamada and Watanabe (1971), see also Karatzas and Shreve (1991, Chapter 5, Proposition 2.13). This proposition is more general than Kloeden and Platen (1992, Theorem 4.5.3) since it allows for "square root diffusions".

3.4.3 The insurance contract

Let the development of the life insurance contract be described by an $\mathbb{F}$-adapted right-continuous Markov process $Z = (Z_t)_{0\leq t\leq T}$ on a finite state space $\mathcal{J} = \{0, 1, \ldots, J\}$. We assume that Z has at most a finite number of jumps, and let 0 be the initial state of the process, i.e. $Z_0 = 0$ a.s. For example, $\mathcal{J}$ could consist of two states describing whether the insured is alive or dead. The associated indicator functions I^j are defined by $I_t^j = 1_{\{Z_t=j\}}$. In addition, we introduce the multivariate counting process $N = (N^{jk})_{j\neq k}$ defined by

$$N_t^{jk} = \#\{u|u \in (0,t], Z_{u-} = j, Z_u = k\}.$$

The process N^{jk} counts the number of transitions directly from state j to k. Moreover, assume that the Markov process admits transition rates λ^{jk} given by

$$\lambda_t^{jk} = I_{t-}^j \mu_{[x]+t}^{jk},$$

where $\mu_{[x]+t}^{jk}$ are stochastic processes. In this chapter we restrict ourselves to models where the transition intensities depend on the mortality intensity only, i.e. we restrict ourselves to the situation

$$\mu_{[x]+t}^{jk} = R^{jk}(t, \mu_{[x]+t}),$$

where R^{jk} is a deterministic function. However, we could equally well have worked with a multi-dimensional process $\mu = (\mu^{jk})_{j\neq k}$. We obtain the following martingales with respect to P

$$M_t^{jk} = N_t^{jk} - \int_0^t \lambda_u^{jk} du = N_t^{jk} - \int_0^t I_{u-}^j \mu_{[x]+u}^{jk} du, \quad 0 \leq t \leq T.$$

By construction, the processes N^{jk} do not have simultaneous jumps, hence the martingales M^{jk} are orthogonal. The filtration $\mathbb{H} = (\mathcal{H}_t)_{0\leq t\leq T}$ is defined as the P-augmentation of the natural filtration generated by the insurance contract, i.e. $\mathcal{H}_t = \mathcal{H}_t^+ \vee \mathcal{N}$, where

$$\mathcal{H}_t^+ = \sigma\{Z_u, u \leq t\} = \sigma\{N_u, u \leq t\}.$$

Note that the above model can be used to describe both the development of the insurance contract for one insured individual and for a whole portfolio of insured individuals of the same age x at time 0.

3.4.4 The combined model

We assume that the filtration $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ introduced earlier is given by

$$\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t \vee \mathcal{I}_t.$$

Thus, $\mathbb{F}$ is the filtration for the combined model of the economy, the mortality intensity and the insurance contract. Moreover, we assume that the economy is stochastically independent of the development of the insurance contract and the mortality intensity, i.e. $\mathcal{G}_t$ and $(\mathcal{H}_t, \mathcal{I}_t)$ are independent.

We note that the combined model is on the general index-form studied in Steffensen (2000). However, Steffensen (2000) contains no explicit remarks or calculations regarding a stochastic mortality intensity.

3.4.5 Change of measure

In this section, we discuss the choice of equivalent martingale measure in the combined model. An equivalent martingale measure fulfills three requirements. Firstly, it is equivalent to P. Secondly, all discounted price processes on the financial market are martingales under the new measure and lastly it is a probability measure.

To construct a new measure Q we define a likelihood process by

$$\begin{aligned} d\Lambda_t &= \Lambda_{t-}\left(h_t^s dW_t + h_t^\mu d\widetilde{W}_t + \sum_{j,k:j\neq k} g_t^{jk} dM_t^{jk}\right), \\ \Lambda_0 &= 1. \end{aligned}$$

Here h^s and h^μ are adapted processes, and $g = (g^{jk})_{j\neq k}$ is a predictable process. We assume that h^s, h^μ and g are chosen such that $E^P[\Lambda_T] = 1$ and such that $g^{jk} > -1$ for all $j \neq k$. Here, h^s changes the drift term of S, h^μ changes the drift term of μ, and g^{jk} changes the intensity for a transition from j to k for Z. We can now define a measure Q by

$$\frac{dQ}{dP} = \Lambda_T. \tag{3.4.5}$$

Remark 3.4.1 We emphasize that Q defined above only changes measure for one value of x. If we were to consider a portfolio including different ages, we would model the mortality intensity by a d-dimensional Wiener process as proposed in Remark 3.3.1. Hence changing measure for the mortality intensity requires h^μ to be a d-dimensional Girsanov kernel. In addition we note that the martingales M^{jk} implicitly depends on x. Thus, we would need a different martingale M^{jk} and hence a new g^{jk} for each value of x in the portfolio. However, since we only consider one value of x, this is not necessary here.

□

Girsanov's theorem shows that under the measure Q defined by (3.4.5), $W_t^Q = W_t - \int_0^t h_u^s du$ and $\widetilde{W}_t^Q = \widetilde{W}_t - \int_0^t h_u^\mu du$ are independent Q-Wiener processes. If we consider the financial model only, it is well-known that the discounted price process of the stock is a Q-martingale if and only if

$$h_t^s = \frac{r(t,S_t) - \alpha^s(t,S_t)}{\sigma^s(t,S_t)} = -\nu(t,S_t), \tag{3.4.6}$$

see e.g. Duffie (1992, Chapter 7). In our model, the value of h_t^s in (3.4.6) still allows us to express the dynamics of the discounted price process of the stock under Q in terms of the Q-martingale W_t^Q, such that Q indeed is a martingale measure for the combined model. We see that h^s is a function of time and the present value of the stock only, such that the price process of the stock is a Markov process under Q as well. Note that *all* discounted price processes of assets tradeable on the market have to be martingales under the equivalent martingale measure. However, since contracts contingent on the mortality intensity or transitions of Z are not traded on the financial market, this requirement does not give further conditions on h^μ and g^{jk} than the ones already given by $E^P[\Lambda_T] = 1$ and $g > -1$. We do, however, impose the further condition, that h^μ and g must be of the form $h^\mu(t, \mu_{[x]+t})$ and $g^{jk}(t, \mu_{[x]+t})$. This preserves the independence between $\mathcal{G}_t$ and $(\mathcal{H}_t, \mathcal{I}_t)$ under Q, and ensures that μ and Z are Markov processes under Q. The dynamics of μ under Q are given by

$$\begin{aligned} d\mu_{[x]+t} &= \left(\alpha^\mu(t, \mu_{[x]+t}) + \sigma^\mu(t, \mu_{[x]+t}) h^\mu(t, \mu_{[x]+t})\right) dt + \sigma^\mu(t, \mu_{[x]+t}) d\widetilde{W}_t^Q \\ &= \alpha^{\mu,Q}(t, \mu_{[x]+t}) dt + \sigma^\mu(t, \mu_{[x]+t}) d\widetilde{W}_t^Q, \end{aligned} \tag{3.4.7}$$

where we have defined

$$\alpha^{\mu,Q}(t, \mu_{[x]+t}) := \alpha^\mu(t, \mu_{[x]+t}) + \sigma^\mu(t, \mu_{[x]+t}) h^\mu(t, \mu_{[x]+t}).$$

Using Girsanov's theorem for point processes, see e.g. Andersen et al. (1993), we find that the transition intensity of Z from j to k under Q is given by $\tilde{\lambda}_t^{jk} = (1 + g_t^{jk})\lambda_t^{jk}$. Hence, the above assumption $g^{jk} > -1$ is needed in order to ensure that $\tilde{\lambda}^{jk} > 0$. Changing the measure from P to Q yields some new natural Q-martingales:

$$M_t^{jk,Q} = N_t^{jk} - \int_0^t \tilde{\lambda}_u^{jk} du = N_t^{jk} - \int_0^t \left(1 + g_u^{jk}\right) I_{u-}^j \mu_{[x]+u}^{jk} du.$$

This shows that the P-martingales M^{jk} coincide with the corresponding Q-martingales $M^{jk,Q}$ if and only if $g^{jk} = 0$.

Remark 3.4.2 The sign of g^{jk} does not have to be the same for all t. In the model, where we only observe whether one insured individual is alive or dead, represented by states 0 and 1, respectively, we could for example expect $g_t^{01} > 0$ for low ages (low values of $x + t$) and $g_t^{01} < 0$ at large ages (large values of $x + t$). This leads to a mortality intensity which is too high at low ages and too low at high ages, such that the mortality intensity at all times is chosen to the safe side as seen from the insurance companies' point of view, if the insurance companies sell term insurance coverage at low ages and life annuities starting at large ages.

□

Remark 3.4.3 In the rest of the chapter we will be working under some arbitrary, but fixed martingale measure Q, and therefore it is of importance to be able to find expressions for the market survival probabilities, see Section 3.4.7 for a definition. Modelling the mortality intensity by an extended Cox–Ingersoll–Ross model under P, we are interested in choices of h^μ that lead to an extended Cox–Ingersoll–Ross model under Q as well. We thus need the dynamics under Q to be of the form

$$d\mu_{[x]+t} = \left(\beta^{\mu,Q}(t) - \gamma^{\mu,Q}(t)\mu_{[x]+t}\right) dt + \rho^{\mu,Q}(t)\sqrt{\mu_{[x]+t}}d\widetilde{W}_t^Q,$$

where $\beta^{\mu,Q}$, $\gamma^{\mu,Q}$ and $\rho^{\mu,Q}$ are functions of t, which satisfy the conditions given in Section 3.3.3 in order to ensure strict positivity of the mortality intensity. A comparison of the dynamics under P and Q shows that h^μ must be of the form

$$h^\mu(t, \mu_{[x]+t}) = \delta(t)\sqrt{\mu_{[x]+t}} + \frac{\delta^*(t)}{\sqrt{\mu_{[x]+t}}} \tag{3.4.8}$$

for some deterministic functions δ and δ^*. Since $\mu_{[x]+t} > 0$, this leads to the following equations

$$\rho^{\mu,Q}(t) = \rho^\mu(t), \tag{3.4.9}$$
$$\beta^{\mu,Q}(t) = \beta^\mu(t) + \rho^\mu(t)\delta^*(t), \tag{3.4.10}$$
$$\gamma^{\mu,Q}(t) = \gamma^\mu(t) - \rho^\mu(t)\delta(t). \tag{3.4.11}$$

This shows that given an extended Cox–Ingersoll–Ross model under P and a Girsanov kernel h^μ of the form (3.4.8), then the Q-dynamics are in accordance with an extended Cox–Ingersoll–Ross model, with coefficients given by (3.4.9), (3.4.10) and (3.4.11). Moreover, if we have strict positivity of the mortality intensity under P, then the condition $\delta^*(t) \geq 0$ ensures strict positivity under Q as well.

□

3.4.6 A brief review of financial concepts

In this section some concepts from the financial literature are introduced within the present framework. Under some equivalent martingale measure Q introduced in Section 3.4.5, the dynamics of the price processes under Q are given by

$$\begin{aligned} dS_t &= r(t, S_t)S_t dt + \sigma^s(t, S_t)S_t dW_t^Q, \quad S_0 > 0, \\ dB_t &= r(t, S_t)B_t dt, \quad B_0 = 1, \end{aligned}$$

where $(W_t^Q)_{0\leq t\leq T}$ is a Wiener process on the interval $[0, T]$ under Q.

A trading strategy is an adapted process $\varphi = (\vartheta, \eta)$ satisfying certain integrability conditions. The pair $\varphi_t = (\vartheta_t, \eta_t)$ is interpreted as the portfolio held at time t. Here, ϑ_t and η_t, respectively, denote the number of stocks and the discounted deposit on the savings account in the portfolio at time t. The value process $\mathcal{V}(\varphi)$ associated with φ is given by

$$\mathcal{V}_t(\varphi) = \vartheta_t S_t + \eta_t B_t.$$

A strategy φ is called self-financing if

$$\mathcal{V}_t(\varphi) = \mathcal{V}_0(\varphi) + \int_0^t \vartheta_u dS_u + \int_0^t \eta_u dB_u.$$

Thus, the value at any time t of a self-financing strategy is the initial value added interest on the savings account and trading gains; withdrawals and deposits are not allowed during $(0, T)$. A contingent claim (or a derivative) with maturity T is an $\mathcal{F}_T$-measurable, Q-square integrable random variable H. Hence, the class of contingent claims depends on the equivalent martingale measure Q. If H can be written as $\Phi(S_T, \mu_{[x]+T}, Z_T)$ for some function $\Phi : \mathbb{R}_+^2 \times \mathcal{J} \to \mathbb{R}$, it is called a simple contingent claim. A contingent claim is called attainable if there exists a self-financing strategy such that $\mathcal{V}_T(\varphi) = H$, P-a.s. An attainable claim can thus be replicated perfectly by investing $\mathcal{V}_0(\varphi)$ at time 0 and investing during the interval $[0, T]$ according to the self-financing strategy φ. Hence, at any time t, there is no difference between holding the claim H and the portfolio φ_t. In this sense, the claim H is redundant in the market and from the assumption of no arbitrage it follows that the price of H must be $\mathcal{V}_t(\varphi)$ at any time t. Thus, the initial investment $\mathcal{V}_0(\varphi)$ is the unique arbitrage free price of H. Note that if $\varphi = (\vartheta, \eta)$ is a self-financing portfolio replicating the contingent claim H, then H has the following representation

$$H = \mathcal{V}_T(\varphi) = \mathcal{V}_0(\varphi) + \int_0^T \vartheta_t dS_t + \int_0^T \eta_t dB_t.$$

If all contingent claims are attainable, the model is called complete and otherwise it is called incomplete. A self-financing strategy is a so-called arbitrage if $\mathcal{V}_0(\varphi) = 0$ and $\mathcal{V}_T(\varphi) \geq 0$ P-a.s. with $P(\mathcal{V}_T(\varphi) > 0) > 0$. It is well known from the financial literature, see e.g. Björk (2004), that the model $(B, S, \mathbb{G})$ is complete and arbitrage free under the assumptions on the coefficients given in Section 3.4.1. Thus, a contingent claim specifying the amount $\Phi(S_T)$ to be paid out at time T has a unique arbitrage free price process $(\pi(t, S_t))_{0 \leq t \leq T}$, which can be characterized by the following PDE on $[0, T] \times \mathbb{R}_+$:

$$\partial_t \pi(t,s) + r(t,s) s \partial_s \pi(t,s) + \frac{1}{2}(\sigma^s(t,s))^2 s^2 \partial_{ss}\pi(t,s) - r(t,s)\pi(t,s) = 0, \tag{3.4.12}$$

with boundary condition $\pi(T, s) = \Phi(s)$.

When we introduce other sources of randomness in the model, which are not tradeable on the market, we get an incomplete market. This will be the case for $(B, S, \mathbb{F})$. Here, we are still able to replicate claims which only depend on the randomness from (B, S), whereas claims containing an element of insurance are not replicable. Thus, insurance contracts cannot be priced uniquely by a no arbitrage argument. However, for each admissible choice of h^μ and g, we get an equivalent martingale measure Q, which can be used to derive possible prices for contingent claims, which are consistent with absence of arbitrage. One possible choice of Q is obtained by letting $h^\mu = 0$ and $g^{jk} = 0$. Here, the market is said to be risk neutral with respect to systematic and unsystematic mortality risk.

3.4.7 Market survival probabilities

Here, we derive a PDE for the market survival probabilities. Let the dynamics of μ be given by (3.4.7). Now consider the case where $\mathcal{J} = \{0, 1\}$, with 0 corresponding to the policyholder being alive and 1 to the policyholder being dead. Using the notation $\mu_{[x]+t}$

and g_t instead of $\mu^{01}_{[x]+t}$ and g^{01}_t we can define the *market survival probability* from time t to T for an $x+t$ year old by

$$\mathcal{S}^Q(t,\mu_{[x]+t},T) := E^Q\left[I^0_T \middle| Z_t = 0, \mu_{[x]+t}\right] = E^Q\left[e^{-\int_t^T (1+g_\tau)\mu_{[x]+\tau} d\tau} \middle| \mu_{[x]+t}\right].$$

Using Itô's formula and the fact that M_t given by

$$M_t = E^Q\left[e^{-\int_0^T (1+g_\tau)\mu_{[x]+\tau} d\tau} \middle| \mathcal{F}_t\right],$$

is a Q-martingale, we can obtain the following PDE on $[0,T]\times\mathbb{R}_+$:

$$\begin{aligned} 0 &= \partial_t \mathcal{S}^Q(t,\mu,T) + \alpha^{\mu,Q}(t,\mu)\partial_\mu \mathcal{S}^Q(t,\mu,T) \\ &\quad + \frac{1}{2}\left(\sigma^\mu(t,\mu)\right)^2 \partial_{\mu\mu}\mathcal{S}^Q(t,\mu,T) - (1+g(t,\mu))\mu\mathcal{S}^Q(t,\mu,T), \end{aligned} \tag{3.4.13}$$

with boundary condition

$$\mathcal{S}^Q(T,\mu,T) = 1.$$

This PDE differs from the one given in (3.3.2) for the survival probabilities by the coefficient $\alpha^{\mu,Q}$ and the loading factor g appearing in the last term only.

Remark 3.4.4 Analogously to the forward mortality intensity we can now define the *market forward mortality intensity* by

$$f^{\mu,Q}(t,T) := -\partial_T \log \mathcal{S}^Q(t,\mu_{[x]+t},T).$$

□

3.5 Market Reserves

In traditional literature on life insurance the (prospective) reserve is determined as the expected value of future discounted benefits less premiums under a technical probability measure, which is subjectively chosen and in general different from P and Q. In the present context, we are working with a market reserve, which is the price at which the insurance contract could be sold on the financial market. In order to exclude arbitrage possibilities, the market reserve is the expected value of discounted future benefits less premiums under some arbitrary, but fixed, market measure Q.

Consider a general payment stream A, where payments are allowed to depend on the development of the financial market. More precisely, A is assumed to be of the form

$$dA_t = \sum_k \left(I^k_t dA^k_t + \sum_{\ell:\ell\neq k} a^{k\ell}_t dN^{k\ell}_t \right), \tag{3.5.1}$$

where

$$dA^k_t = a^k_t dt + (A^k_t - A^k_{t-}) = a^k_t dt + \Delta A^k_t. \tag{3.5.2}$$

We thus consider a payment stream, where payments are contingent on the development of the underlying insurance contract as described by the Markov process Z, see Section 3.4.3. According to (3.5.1)–(3.5.2), we allow for 3 different types of payments, all of which may be linked to the stock S. Firstly, there are amounts $a_t^{jk} = a^{jk}(t, S_t)$ payable immediately upon transition from state j to state k. These are called general life insurances. Secondly, there are general state-wise annuities payable continuously at rate $a_t^j = a^j(t, S_t)$ at time t, contingent on the policy sojourning in state j. Lastly, we allow for lump sum payments $\Delta A^j(t, S_t)$. However, for notational convenience, we restrict lump sum payments to the initial time 0 and the terminal time T only, i.e. $\Delta A_t^k = 0$ if $t \notin \{0, T\}$. We note that since all payments are assumed to be functions of the current value of the stock only, we exclude path dependent payment functions, such as so-called Asian and Russian options; for a treatment of these and other exotic options see Musiela and Rutkowski (1997). Assume that $(t, s) \mapsto a^j(t, s)$, $(t, s) \mapsto a^{jk}(t, s)$ and $s \mapsto \Delta A^j(T, s)$ are measurable functions, and that

$$E^Q \left[\int_0^T \left| B_u^{-1} a_u^{jk} \right| \tilde{\lambda}_u^{jk} du \right] < \infty, \quad \forall j \neq k.$$

Then the processes $\int B^{-1} a^{jk} dM^{jk,Q}$ are Q-martingales, see Brémaud (1981, Lemma L3 p. 24).

We use the convention, which is standard in actuarial literature, that the reserve at time t is the value of future payments *after* payments due at time t. Let positive amounts represent benefits and negative amounts represent premiums. The market reserve for a contract with payment stream A described above and termination at time T can then be written as

$$V(t, S_t, \mu_{[x]+t}, Z_t) = E^Q \left[\int_t^T e^{-\int_t^\tau r_u du} dA_\tau \middle| \mathcal{F}_t \right], \quad 0 \leq t < T,$$

and

$$V(T, S_T, \mu_{[x]+T}, Z_T) = 0.$$

Note that V is a function of the state of the insurance portfolio and the current value of the stock and mortality intensity only. This is due to the restrictions on h^μ and g, which ensure that the processes are Markov under Q, and the fact that the payment functions are restricted to depend on the present value of the stock only. Since the present state of the policy is known, the relevant quantities are the state-wise market reserves. Using that S and μ are Markov processes under Q and inserting the definition of A from (3.5.1)–(3.5.2), we get the following expression for the state-wise reserves for $0 \le t < T$:

$$
\begin{aligned}
&V^j(t, S_t, \mu_{[x]+t}) \\
&\quad = E^Q\left[\int_t^T e^{-\int_t^\tau r_u du} dA_\tau \,\middle|\, Z_t = j, S_t, \mu_{[x]+t}\right] \\
&\quad = E^Q\left[\int_t^T e^{-\int_t^\tau r_u du} \sum_{k\in\mathcal{J}} \left(I_\tau^k a_\tau^k d\tau + \sum_{\ell:\ell\neq k} a_\tau^{k\ell} dN_\tau^{k\ell}\right) \,\middle|\, Z_t = j, S_t, \mu_{[x]+t}\right] \\
&\qquad + E^Q\left[e^{-\int_t^T r_u du} \sum_{k\in\mathcal{J}} I_T^k \Delta A_T^k \,\middle|\, Z_t = j, S_t, \mu_{[x]+t}\right] \\
&\quad = \sum_{k\in\mathcal{J}} \int_t^T E^Q\left[e^{-\int_t^\tau r_u du} a_\tau^k \,\middle|\, S_t\right] E^Q\left[I_\tau^k \,\middle|\, Z_t = j, \mu_{[x]+t}\right] d\tau \\
&\qquad + \sum_{k\in\mathcal{J}} \sum_{\ell:\ell\neq k} \int_t^T E^Q\left[e^{-\int_t^\tau r_u du} a_\tau^{k\ell} \,\middle|\, S_t\right] E^Q\left[I_{\tau-}^k \left(1 + g_\tau^{k\ell}\right) \mu_{[x]+\tau}^{k\ell} \,\middle|\, Z_t = j, \mu_{[x]+t}\right] d\tau \\
&\qquad + \sum_{k\in\mathcal{J}} E^Q\left[e^{-\int_t^T r_u du} \Delta A_T^k \,\middle|\, S_t\right] E^Q\left[I_T^k \,\middle|\, Z_t = j, \mu_{[x]+t}\right].
\end{aligned}
$$

Here, we have used the Q-compensators for N^{jk} and the fact that $\int B^{-1} a^{jk} dM^{jk,Q}$ are Q-martingales in the second equality. Moreover, we have used the Q-independence between S and (Z, μ). Disregarding the random course of the policy, the quantities $a^j(u, S_u)$, $a^{jk}(u, S_u)$ and $\Delta A^j(T, S_T)$ are simple contingent claims in the financial market given by $(B, S, \mathbb{G})$. Since this market is complete, see Section 3.4.6, the claims can be uniquely priced. Using that S is a Markov process the corresponding unique arbitrage free price processes are for $0 \le t \le u \le T$ given by

$$
\begin{aligned}
F^j(t, S_t, u) &= E^Q\left[e^{-\int_t^u r_\tau d\tau} a^j(u, S_u) \,\middle|\, S_t\right], \\
F^{jk}(t, S_t, u) &= E^Q\left[e^{-\int_t^u r_\tau d\tau} a^{jk}(u, S_u) \,\middle|\, S_t\right], \\
F^{\Delta j}(t, S_t, T) &= E^Q\left[e^{-\int_t^T r_\tau d\tau} \Delta A^j(T, S_T) \,\middle|\, S_t\right].
\end{aligned}
$$

Defining the functions $H^k(t, j, \mu_{[x]+t}, u)$ and $H^{k\ell}(t, j, \mu_{[x]+t}, u)$ by

$$
H^k(t, j, \mu_{[x]+t}, u) := E^Q\left[I_u^k \,\middle|\, Z_t = j, \mu_{[x]+t}\right], \quad 0 \le t \le u \le T,
$$

and

$$
H^{k\ell}(t, j, \mu_{[x]+t}, u) := E^Q\left[I_u^k \left(1 + g_u^{k\ell}\right) \mu_{[x]+u}^{k\ell} \,\middle|\, Z_t = j, \mu_{[x]+t}\right], \quad 0 \le t \le u \le T,\ k \neq \ell,
$$

the state-wise market reserves for $0 \leq t < T$ can be written as

$$\begin{aligned} V^j(t, S_t, \mu_{[x]+t}) = & \sum_{k \in \mathcal{J}} \int_t^T F^k(t, S_t, \tau) H^k(t, j, \mu_{[x]+t}, \tau) d\tau \\ & + \sum_{k \in \mathcal{J}} \sum_{\ell : \ell \neq k} \int_t^T F^{k\ell}(t, S_t, \tau) H^{k\ell}(t, j, \mu_{[x]+t}, \tau) d\tau \\ & + \sum_{k \in \mathcal{J}} F^{\Delta k}(t, S_t, T) H^k(t, j, \mu_{[x]+t}, T). \end{aligned}$$

For similar calculations under deterministic transition intensities, see Møller (2001c).

Using either martingale methods as in Møller (2001c) or the generalized Thiele differential equation in Steffensen (2000), we obtain the following system of PDEs for the market reserves on $[0, T) \times \mathbb{R}_+^2$ for all $j \in \mathcal{J}$:

$$\begin{aligned} & a^j(t, s) - r(t, s) V^j(t, s, \mu) + \partial_t V^j(t, s, \mu) + \alpha^{\mu, Q}(t, \mu) \partial_\mu V^j(t, s, \mu) \\ & + \frac{1}{2} (\sigma^\mu(t, \mu))^2 \partial_{\mu\mu} V^j(t, s, \mu) + r(t, s) s \partial_s V^j(t, s, \mu) + \frac{1}{2} (\sigma^s(t, s))^2 s^2 \partial_{ss} V^j(t, s, \mu) \\ & + \sum_{k : k \neq j} \left(a^{jk}(t, s) + V^k(t, s, \mu) - V^j(t, s, \mu) \right) \left(1 + g^{jk}(t, \mu) \right) \mu^{jk} = 0, \end{aligned} \tag{3.5.3}$$

with boundary conditions

$$V^j(T-, s, \mu) = \Delta A^j(T, s).$$

Since the system of PDEs in general does not have an analytic solution, we have to resort to numerical techniques in order to solve for $V^j(t, s, \mu)$ in (3.5.3).

Example 3.5.1 We address the special case where the state space for Z is $\mathcal{J} = \{0, 1\}$, with 0 corresponding to the policyholder being alive and 1 corresponding to the policyholder being dead. In this case we have that $\mathcal{S}^Q(t, \mu_{[x]+t}, u) = H^0(t, 0, \mu_{[x]+t}, u)$ for $0 \leq t \leq u \leq T$. Consider a unit-linked endowment insurance paid by a lump sum premium at time 0. An endowment insurance pays out a specified amount in case the policyholder dies or survives to time T whichever happens first. Here the state-wise market reserves for $0 \leq t < T$ are given by

$$\begin{aligned} & V^0(t, S_t, \mu_{[x]+t}) = \int_t^T F^{01}(t, S_t, \tau) H^{01}(t, 0, \mu_{[x]+t}, \tau) d\tau + F^{\Delta 0}(t, S_t, T) \mathcal{S}^Q(t, \mu_{[x]+t}, T), \\ & V^1(t, S_t, \mu_{[x]+t}) = 0. \end{aligned}$$

□

3.6 Mortality-linked contracts

3.6.1 Motivation

In Danish life insurance practice, premiums or benefits, depending on whichever is chosen in the contract, are determined applying the principle of equivalence with a deterministic mortality intensity, known as the first order mortality intensity. The first order mortality intensity is chosen to be on the safe side as seen from the company's point of view. This is usually obtained by working with a mortality intensity which is believed to be too high at low ages and too low at large ages, see for example the Danish mortality table $G82$. Of course, the future mortality intensity is unknown, so this has to be based on the information available at the time of signing of the contract.

If the mortality intensity behaves as expected, and the insured portfolio behaves according to the general mortality intensity, this approach leads to a systematic surplus for the entire portfolio. If the portfolio is large enough, the strong law of large numbers applies, which implies that the portfolio behaves according to the general mortality intensity, provided that no selection mechanism has been applied. By the so-called contribution principle, the systematic surplus must be redistributed among the insured as bonus by taking into consideration to which extent the insured has taken part in generating the surplus. The companies typically use the bonus to buy additional insurance cover, similar to the one(s) already stipulated in the contract.

This procedure is unproblematic as long as the real mortality intensity does not behave worse, as seen from the insurer's point of view, than the chosen deterministic mortality intensity. However, no matter how safe the deterministic mortality intensity is chosen, there is always a risk that the mortality intensity behaves worse, even though this risk may be very small. According to the insurance contracts, the companies cannot allocate negative bonus to the insured, i.e. they cannot reduce benefits (or, equivalently, increase premiums). Thus, the companies are subject to a *systematic mortality risk* related to the future development of the mortality intensity. One possible way for the insurance companies to reduce this risk is to transfer some or all of it to the insured or other agents or companies. For example, one could currently adapt premiums or benefits to the development of the mortality intensity. We shall refer to such contracts as *mortality-linked contracts*. More precisely, one could link the premiums or benefits to the development of some large group of reference individuals. This group might consist of the entire Danish population, the entire portfolio of the insurance company or a mixed portfolio from all Danish insurance companies. One could then agree on a specific estimation procedure from which the "true" mortality intensity is determined in order to avoid misuse from the companies and possible mistrust from the insured. In this sense, one can view the true mortality intensity as an observable quantity.

The main idea with mortality-linked insurance contracts is that equivalence between premiums and benefits is established by using the information available at time 0. At time t, the state-wise retrospective and prospective reserves are calculated using the information currently available. In order for the expected state-wise retrospective and prospective reserves to be equal there are two adjustment possibilities: The premiums and the benefits. Adjustment of the premium is only a possibility if the contract is not entirely paid by a lump sum premium. This approach would reduce the companies' mortality risk to the risk associated with changes in the mortality intensity that have occurred after the last adjustment and to unsystematic risk. If the adjustment is done sufficiently often, the systematic mortality risk can be considered negligible.

In the following section we present the idea of mortality-linked contracts by means of a simple example. More general results concerning mortality-linked contracts will be presented elsewhere.

3.6.2 Pure endowment

The simplest non-trivial contract is a pure endowment paid by a single premium π_0 at time 0 for some policy-holder aged x. Benefits are described by an adapted stochastic process $(K_t)_{0\leq t\leq T}$, which determines the sum to be paid out at time T in case of survival until age $x+T$. In particular K_0 is the sum insured calculated at time 0. At time t the sum insured is given by K_t, which may be smaller or bigger than K_0. The exact size of K_t will depend on the development of the underlying mortality intensity and the financial market in a specified way described below. In the following, we use the notation $\mu_{[x]+t}$ and g_t instead of $\mu^{01}_{[x]+t}$ and g^{01}_t, indicating that we are working with a two state Markov model for the insurance contract and one policyholder. The principle of equivalence under Q gives the premium π_0 for the pure endowment with sum insured K_0:

$$\pi_0 = K_0\, p(0,T)\, \mathcal{S}^Q(0,\mu_{[x]},T), \tag{3.6.1}$$

where the processes $\mathcal{S}^Q(t,\mu_{[x]+t},T)$ and $p(t,T)$ are defined by $E^Q[e^{-\int_t^T(1+g_u)\mu_{[x]+u}du}|\mu_{[x]+t}]$ and $E^Q[e^{-\int_t^T r_u du}|\mathcal{G}_t]$ respectively. Thus, the premium is the market price of benefits.

Remark 3.6.1 Using the market forward mortality intensity, the premium can be written as

$$\pi_0 = K_0\, p(0,T)\, e^{-\int_0^T f^{\mu,Q}(0,u)du}.$$

The advantage of rewriting the premium this way is twofold. Firstly, we observe that calculating premiums using the equivalence principle under the market measure is done using a known mortality intensity. Secondly, working with $f^{\mu,Q}$ instead of f^{μ} shows that the “fair” premium is determined by using a measure, which reflects the market’s attitude towards both the systematic and the unsystematic mortality risk.

□

Inspired by Norberg (1991) we work with state-wise retrospective reserves at time t, $0\leq t\leq T$, defined by

$$V_t^{i,retro} = \pi_0\frac{U_t}{U_0} - K_T 1_{\{i=0\}}1_{\{t=T\}}, \tag{3.6.2}$$

where $(U_t)_{0\leq t\leq T}$ is a stochastic process with Q-dynamics of the form

$$dU_t = \alpha_t^U dt + \sigma_t^U dW_t^Q.$$

Here α_t^U and σ_t^U are $\mathbb{G}$-adapted (and thus $\mathbb{F}$-adapted) processes. In order for the contract to be “fair” the process U must fulfill a condition, which will be given later. The state-wise retrospective reserves are thus the accumulated value of premiums less benefits in $[0,t]$, given the present state of the policy. Note that the accumulation is done with an arbitrary accumulation factor $\frac{U_t}{U_0}$, $0 \leq t$. For different choices of α_t^U and σ_t^U we thus have different state-wise retrospective reserves. One possibility is to choose $\alpha_t^U = r_t^* U_t$ and $\sigma_t^U = 0$, where r^* is some rate of return. Possible choices of r^* are the actual interest rate or some rate averaging out the true interest rate (or the investment return of the company) over time. If all individual contracts are fair, as measured by the no arbitrage principle, this can be thought of as the deposit rate used in practice. We recall that the prospective reserves are equal to the market reserves defined in Section 3.5. Let $V_t^{i,pro}$ denote the prospective reserve at time t given the insured is in state i and the sum insured is K_t

$$V_t^{i,pro} = E^Q\left[e^{-\int_t^T r_u du} K_t I_T^0 \middle| Z_t = i, \mathcal{I}_t \vee \mathcal{G}_t \right], \quad 0 \leq t < T. \tag{3.6.3}$$

As a criterion in order to calculate the adapted benefits, K_t, we use

$$E^Q\left[V_t^{Z_t,pro} \middle| \mathcal{G}_t \vee \mathcal{I}_t \right] = E^Q\left[V_t^{Z_t,retro} \middle| \mathcal{G}_t \vee \mathcal{I}_t \right]. \tag{3.6.4}$$

We note that the expectation operator only refers to the possible states of the insurance contract, i.e. whether Z_t is 0 or 1.

For the contract to be fair the expected discounted value under Q of the actual payments should be 0, i.e.

$$E^Q\left[\pi_0 - I_T^0 e^{-\int_0^T r_u du} K_T \right] = 0, \tag{3.6.5}$$

which means that the principle of equivalence under Q should apply. As we shall see below, this leads to a condition on the accumulation process U. First, we express K_T in terms of U: At time T we have by definition that

$$V_T^{0,pro} = V_T^{1,pro} = 0,$$

whereas (3.6.2) gives

$$V_T^{0,retro} = \pi_0 \frac{U_T}{U_0} - K_T$$

and

$$V_T^{1,retro} = \pi_0 \frac{U_T}{U_0}.$$

Criterion (3.6.4) applied at time T thus gives

$$K_T = \pi_0 \frac{U_T}{U_0} \frac{1}{e^{-\int_0^T (1+g_u)\mu_{[x]+u} du}}.$$

Inserting this into (3.6.5) and using iterated expectations we find that the process U must fulfill:

$$\begin{aligned}
0 &= E^Q\left[\pi_0 - I_T^0 e^{-\int_0^T r_u du} K_T \right] \\
&= E^Q\left[\pi_0 - I_T^0 e^{-\int_0^T r_u du} \pi_0 \frac{U_T}{U_0} \frac{1}{e^{-\int_0^T (1+g_u)\mu_{[x]+u} du}} \right] \\
&= \pi_0 \left(1 - E^Q\left[e^{-\int_0^T r_u du} \frac{U_T}{U_0} \right] \right),
\end{aligned}$$

which is equivalent to

$$E^Q\left[e^{-\int_0^T r_u du}\frac{U_T}{U_0}\right] = 1. \tag{3.6.6}$$

Hence the expected discounted value of accumulation factor from 0 to T should be 1. As expected, we note that if $U_t = e^{\int_0^t r_u du}$, i.e. if we accumulate with the real interest rate, then U fulfills (3.6.6).

Having resolved the problem of defining a "fair" contract, we now turn our attention towards the development of benefits. Applying criterion (3.6.4) at time $t < T$ gives

$$e^{-\int_0^t (1+g_u)\mu_{[x]+u}du} K_t p(t,T)\mathcal{S}^Q(t,\mu_{[x]+t},T) = \pi_0\frac{U_t}{U_0}.$$

Inserting the expression for the premium from (3.6.1) we find the following relationship between the benefits decided at time 0 and time t:

$$\frac{K_t}{K_0} = \frac{p(0,T)\mathcal{S}^Q(0,\mu_{[x]},T)\frac{U_t}{U_0}}{e^{-\int_0^t (1+g_u)\mu_{[x]+u}du} p(t,T)\mathcal{S}^Q(t,\mu_{[x]+t},T)}. \tag{3.6.7}$$

We see that the ratio between the new sum insured and the old sum insured is the ratio between the market value at time 0 of a pure endowment contract with expiration T accumulated to time t using the accumulation factor $\frac{U_t}{U_0}$, and the market value at time t of a pure endowment with expiration T multiplied by the (at time t known) market survival probability from time 0 to t.

For simplicity we restrict ourselves to the situation where r is deterministic and $\frac{U_t}{U_0}$ is equal to $e^{\int_0^t r_u du}$. We can thus consider the impact of the mortality intensity only. This implies that (3.6.7) reduces to

$$\frac{K_t}{K_0} = \frac{\mathcal{S}^Q(0,\mu_{[x]},T)}{e^{-\int_0^t (1+g_u)\mu_{[x]+u}du}\mathcal{S}^Q(t,\mu_{[x]+t},T)}.$$

To see how the benefit evolves in connection with changes in the mortality intensity, we derive the dynamics for K_t. First note that K_t can be written as

$$K_t = K_0\mathcal{S}^Q(0,\mu_{[x]},T)e^{\int_0^t (1+g_u)\mu_{[x]+u}du}\frac{1}{\mathcal{S}^Q(t,\mu_{[x]+t},T)}.$$

In the following we use the simplified notation $\mathcal{S}_t^Q = \mathcal{S}^Q(t,\mu_{[x]+t},T)$. Using the partial differential equation (3.4.13) for $\mathcal{S}_t^Q$ we find the dynamics of $\frac{1}{\mathcal{S}_t^Q}$:

$$d\left(\frac{1}{\mathcal{S}_t^Q}\right) = \frac{1}{\mathcal{S}_t^Q}\left(\left(-\left(1+g_t\right)\mu_{[x]+t} + \left(\sigma_t^\mu\frac{\partial_\mu\mathcal{S}_t^Q}{\mathcal{S}_t^Q}\right)^2\right)dt - \sigma_t^\mu\frac{\partial_\mu\mathcal{S}_t^Q}{\mathcal{S}_t^Q}d\widetilde{W}_t^Q\right). \tag{3.6.8}$$

Using (3.6.8) we arrive at the following dynamics of K_t under P:

$$dK_t = \left(\sigma_t^\mu \frac{\partial_\mu \mathcal{S}_t^Q}{\mathcal{S}_t^Q} h_t^\mu + \left(\sigma_t^\mu \frac{\partial_\mu \mathcal{S}_t^Q}{\mathcal{S}_t^Q} \right)^2 \right) K_t dt - \sigma_t^\mu \frac{\partial_\mu \mathcal{S}_t^Q}{\mathcal{S}_t^Q} K_t d\widetilde{W}_t. \tag{3.6.9}$$

The benefit thus increases or decreases by a fraction which is proportional to the current benefit. This proportion factor consists of two terms: The first term comes from changing measure with respect to the mortality intensity. This term is the risk associated with the relative change in market survival probability multiplied by h^μ, which is minus 1 times the market price of systematic mortality risk. The second drift term is the squared relative change in market survival probability associated with a change in mortality intensity. The last term in the expression is the product of the relative change in market survival probability, the present benefit and the change in the Wiener process driving the mortality intensity. Since $\int \sigma^\mu \frac{\partial_\mu \mathcal{S}^Q}{\mathcal{S}^Q} d\widetilde{W}$ is a (local) P-martingale, the stochastic exponential formula gives that the differential equation (3.6.9) has the solution

$$K_t = K_0 \exp\left(\int_0^t \left(\sigma_u^\mu \frac{\partial_\mu \mathcal{S}_u^Q}{\mathcal{S}_u^Q} h_u^\mu + \frac{1}{2} \left(\sigma_u^\mu \frac{\partial_\mu \mathcal{S}_u^Q}{\mathcal{S}_u^Q} \right)^2 \right) du - \int_0^t \sigma_u^\mu \frac{\partial_\mu \mathcal{S}_u^Q}{\mathcal{S}_u^Q} d\widetilde{W}_u \right)$$

Note that the dynamics are expressed under P instead of Q, since these are the dynamics to be observed by the insured and the insurer.

Development of the benefit when allowing for risky investments

Steffensen (2001) works with the value of past contractual payments accumulated by the development of the investment portfolio of the insurance company. Using this idea, we assume that the insurance company invests in a self-financing portfolio $\varphi = (\vartheta, \eta)$, which leads to the strictly positive value process $\mathcal{V}(\varphi)$. Choosing $U_t = \mathcal{V}_t(\varphi)$ the retrospective reserves are calculated using the accumulation factor obtained by the risky investments of the company, i.e. we have

$$V_t^{i,retro} = \pi_0 \frac{\mathcal{V}_t(\varphi)}{\mathcal{V}_0(\varphi)} - K_T 1_{\{i=0\}} 1_{\{t=T\}}. \tag{3.6.10}$$

Using the retrospective reserve in (3.6.10) together with criterion (3.6.4) leaves the insured with both the risk associated with the development of the financial market and the systematic mortality risk, hence leaving the insurance company with the unsystematic mortality risk only. In Section 3.8 we show that in this case the P-dynamics of K_t are given by

$$
\begin{aligned}
dK_t = \Bigg(& \left(\sigma_t^s S_t \frac{\partial_s p(t,T)}{p(t,T)}\right)^2 + \left(\sigma_t^\mu \frac{\partial_\mu \mathcal{S}_t^Q}{\mathcal{S}_t^Q}\right)^2 - \left(\sigma_t^s \frac{\vartheta_t S_t}{\mathcal{V}_t(\varphi)} \sigma_t^s S_t \frac{\partial_s p(t,T)}{p(t,T)}\right) \\
& + h_t^s \left(\sigma_t^s S_t \frac{\partial_s p(t,T)}{p(t,T)} - \sigma_t^s \frac{\vartheta_t S_t}{\mathcal{V}_t(\varphi)}\right) + h_t^\mu \left(\sigma_t^\mu \frac{\partial_\mu \mathcal{S}_t^Q}{\mathcal{S}_t^Q}\right)\Bigg) K_t dt \\
& + \left(\sigma_t^s \frac{\vartheta_t S_t}{\mathcal{V}_t(\varphi)} - \sigma_t^s S_t \frac{\partial_s p(t,T)}{p(t,T)}\right) K_t dW_t - \left(\sigma_t^\mu \frac{\partial_\mu \mathcal{S}_t^Q}{\mathcal{S}_t^Q}\right) K_t d\widetilde{W}_t. \qquad (3.6.11)
\end{aligned}
$$

The drift thus consists of five terms. The second and last drift term relate to the mortality intensity and are recognized as the drift in (3.6.9). The first term is the square of the relative change in zero coupon prices, and the third term comes from the correlation between the investment portfolio and the zero coupon prices. This term is positive (negative) if the interest rate has a positive (negative) dependence of the stock price. The fourth term is minus the market price of financial risk, h_t^s, multiplied by the sum of the relative change in zero coupon prices and minus the relative change in the value of the investment portfolio. The last two terms in the dynamics are related to the Wiener processes driving the stock and mortality intensity, respectively.

3.7 Securitization of systematic mortality risk

As a way to control the mortality risk inherent in an insurance portfolio the company may purchase reinsurance cover. Reinsurance contracts usually consider the specific insurance portfolio of the company, and hence provide coverage for both systematic and unsystematic mortality risk. An example of a mortality dependent reinsurance contract sold in practice is a so-called mortality swap. Prices of reinsurance contracts concerning both systematic and unsystematic mortality risk can be found using the methods already established in Section 3.5. However it seems that many life insurance companies are hesitant to buy long term reinsurance coverage. One reason could be that the riskiness of the reinsurance business would leave the insurance companies with a substantial credit risk.

As an alternative to reinsurance we consider securitization. Here, the company trades contracts on the financial market, which depend on the development of the mortality intensity. An important difference between reinsurance and securitization is, that mortality contracts sold on the financial market depend on the general development of the mortality intensity, and hence only offer protection for the systematic mortality risk. Introducing products contingent on the mortality intensity naturally raises questions regarding the estimation of the mortality intensity. Since, these questions are similar to those in the case of mortality-linked contracts, we refer to the discussion in Section 3.6. The advantages of securitization over traditional reinsurance is the possible lower cost when standardizing products and the larger capacity of the financial market. More details on securitization of mortality risk can be found in Lin and Cox (2005). For treatments of securitization of catastrophe losses, which seems to be the most developed area of securitization, see Christensen (2000), Cox, Fairchild and Pedersen (2000) and references therein.

In this section we first derive a PDE for the price process of a wide class of derivatives on the mortality intensity. Then we examine different possibilities for an insurance company, which is interested in hedging a pure endowment, and finally we investigate contracts with a risk premium.

3.7.1 Pricing mortality derivatives

Inspired by Björk (2004, Chapter 8) we consider derivatives of the mortality intensity with a payoff of the form

$$\Phi(T, \mu_{[x]+T}, \Psi_T^1, \Psi_T^2),$$

where the processes Ψ^i, $i = 1, 2$, are given by

$$\Psi_t^i = \int_0^t q^i(\tau, \mu_{[x]+\tau}) d\tau,$$

for positive functions q^i. The notation above indicates that the derivative is payable at time T, and that it may depend on the mortality intensity at expiration time T and on the integral over $(0, T]$ of two different functions of the mortality intensity. This type of contract covers standard European and Asian options, and thus includes most contracts. Using the independence between the financial market and the mortality intensity, the price process can be written as

$$\pi(t, S_t, \mu_{[x]+t}, \Psi_t^1, \Psi_t^2) = p(t, T) E^Q \left[\Phi(T, \mu_{[x]+T}, \Psi_T^1, \Psi_T^2) \middle| \mathcal{I}_t \right].$$

Given an expression for $p(t, T)$ it is thus sufficient to derive a PDE for the Q-martingale Υ defined by

$$\Upsilon(t, \mu_{[x]+t}, \Psi_t^1, \Psi_t^2) = E^Q \left[\Phi(T, \mu_{[x]+T}, \Psi_T^1, \Psi_T^2) \middle| \mathcal{I}_t \right].$$

Using Itô's formula and the product rule, we can now find the dynamics of Υ. Since Υ is a Q-martingale, the drift term must be 0, such that we get the following PDE on $[0, T] \times \mathbb{R}_+^3$:

$$\begin{aligned} 0 &= \partial_t \Upsilon(t, \mu, \psi^1, \psi^2) + \alpha^{\mu,Q}(t, \mu) \partial_\mu \Upsilon(t, \mu, \psi^1, \psi^2) + q^1(t, \mu) \partial_{\psi^1} \Upsilon(t, \mu, \psi^1, \psi^2) \\ &\quad + q^2(t, \mu) \partial_{\psi^2} \Upsilon(t, \mu, \psi^1, \psi^2) + \frac{1}{2} \left(\sigma^\mu(t, \mu)\right)^2 \partial_{\mu\mu} \Upsilon(t, \mu, \psi^1, \psi^2), \end{aligned} \tag{3.7.1}$$

with boundary condition

$$\Upsilon(T, \mu, \psi^1, \psi^2) = \Phi(T, \mu, \psi^1, \psi^2).$$

3.7.2 Possible ways of hedging

The fair premium for a pure endowment contract with sum insured K can be written as

$$\pi_0 = KE^Q\left[e^{-\int_0^T (1+g_u)\mu_{[x]+u}du}\right]p(0,T).$$

In the following we examine some possibilities for hedging/controlling the systematic mortality risk associated with a pure endowment on the financial market. One possibility is to buy a derivative with payout $Ke^{-\int_0^T \mu_{[x]+u}du}$ at time T. The price for such a derivative at time 0 is

$$\pi(0,\mu_{[x]},\Psi_0) = KE^Q\left[e^{-\int_0^T \mu_{[x]+u}du}\right]p(0,T), \tag{3.7.2}$$

where the process $\Psi = (\Psi_t)_{0\le t\le T}$ is given by $\Psi_t = \int_0^t \mu_{[x]+u}$, i.e. $q(t,\mu_{[x]+t}) = \mu_{[x]+t}$. This derivative hedges the financial risk and systematic mortality risk and leaves the company with the unsystematic mortality risk only. From (3.7.2), we see that the price of the derivative is larger than the premium obtained from the insured if and only if $E^Q[e^{-\int_0^T \mu_{[x]+u}du}] > E^Q[e^{-\int_0^T (1+g_u)\mu_{[x]+u}du}]$. Since the companies want a premium in order to carry a risk, the above hedging possibility only becomes interesting if the price of the derivative is less than the premium paid by the insured.

Often the companies are interested in carrying parts of the systematic mortality risk themselves. In this case the companies can buy a call option on the survival probability with strike C. The payoff from the call option is given by

$$\Phi(T,\mu_{[x]+T},\Psi_T) = \left(e^{-\Psi_T} - C\right)^+. \tag{3.7.3}$$

Here, as in (3.7.2), the process $\Psi = (\Psi_t)_{0\le t\le T}$ is defined by $\Psi_t = \int_0^t \mu_{[x]+u}du$. The derivative with payoff (3.7.3) leads to a payment if the real survival probability is above some predefined level C. This leaves the insurance company with the systematic mortality risk up to a certain level. Here, the strike C could be the survival probability calculated by using some known mortality intensity, for example the market forward mortality intensity. The price process $\pi(t,S_t,\mu_{[x]+t},\Psi_t)$ for the call option can be found by solving (3.7.1) with boundary condition $\Upsilon(T,\mu,\psi) = \Phi(T,\mu,\psi)$ and multiplying by $p(t,T)$.

3.7.3 Contracts with a risk premium

Assume that the company calculates the premium of a pure endowment with sum insured K using some specified mortality intensity $(\mu^*_{[x]+u})_{0\le u\le T}$, which satisfies

$$e^{-\int_0^T \mu^*_{[x]+u}du} > e^{-\int_0^T f^{\mu,Q}(0,u)du}.$$

The mortality intensity $(\mu^*_{[x]+u})_{0\le u\le T}$ can be interpreted as the first order mortality intensity used in practice. Using $(\mu^*_{[x]+u})_{0\le u\le T}$ the company charges a premium π_0^*, which is larger than the fair premium π_0, given by the market price under Q. This is similar to charging a risk premium. The systematic surplus generated by pricing with $(\mu^*_{[x]+u})_{0\leq u\leq T}$ instead of $(f^{\mu,Q}(0,u))_{0\leq u\leq T}$ must be returned to the policyholders, and this could be obtained by increasing benefits if the mortality intensity behaves as expected. For example, the company could pay

$$K_T = K\left(1 + a\left(e^{-\int_0^T \mu^*_{[x]+u}du} - e^{-\int_0^T \mu_{[x]+u}du}\right)^+\right), \tag{3.7.4}$$

if the person survives. Here, $a \in (0,1)$ is the proportion of the surplus which is paid to the policyholder. A natural restriction for contracts of the form (3.7.4) is that they are fair as measured by the market measure. This gives the following equation

$$\begin{aligned}
\pi_0^* &= E^Q\left[I_T^0 K_T e^{-\int_0^T r_u du}\right] \\
&= E^Q\left[I_T^0 K e^{-\int_0^T r_u du}\right] + E^Q\left[I_T^0 a K e^{-\int_0^T r_u du}\left(e^{-\int_0^T \mu^*_{[x]+u}du} - e^{-\int_0^T \mu_{[x]+u}du}\right)^+\right] \\
&= \pi_0 + aKp(0,T)E^Q E^Q\left[I_T^0\left(e^{-\int_0^T \mu^*_{[x]+u}du} - e^{-\int_0^T \mu_{[x]+u}du}\right)^+ \middle| \mathcal{I}_T\right] \\
&= \pi_0 + aKp(0,T)E^Q\left[e^{-\int_0^T (1+g_u)\mu_{[x]+u}du}\left(e^{-\int_0^T \mu^*_{[x]+u}du} - e^{-\int_0^T \mu_{[x]+u}du}\right)^+\right].
\end{aligned}$$

Here,

$$p(0,T)E^Q\left[e^{-\int_0^T (1+g_u)\mu_{[x]+u}du}\left(e^{-\int_0^T \mu^*_{[x]+u}du} - e^{-\int_0^T \mu_{[x]+u}du}\right)^+\right]$$

is the price at time 0, henceforth denoted $\pi(0, S_0, \mu_{[x]}, 0, 0)$, for a derivative with the following payoff at time T

$$\Phi(T, \mu_{[x]+T}, \Psi_T^1, \Psi_T^2) = e^{-\Psi_T^1}\left(e^{-\int_0^T \mu^*_{[x]+u}du} - e^{-\Psi_T^2}\right)^+,$$

where

$$\Psi_t^1 = \int_0^t (1+g_u)\mu_{[x]+u}du \quad \text{and} \quad \Psi_t^2 = \int_0^t \mu_{[x]+u}du.$$

Hence, the price at time 0 can be found by solving (3.7.1) with the boundary condition $\Upsilon(T,\mu,\psi^1,\psi^2) = \Phi(T,\mu,\psi^1,\psi^2)$ and multiply by $p(0,T)$. We obtain the following expression for the "fair" value of a

$$a = \frac{\pi_0^* - \pi_0}{K\pi(0, S_0, \mu_{[x]}, 0, 0)}.$$

This formula can be interpreted in the following way: The benefit is increased with a number of put options on the survival probability, which corresponds to the excess premium over the fair premium divided by the price of the put option.

3.8 Dynamics of the benefit with risky investments

In this section we derive (3.6.11). We assume that the insurance company invests in a self-financing portfolio $\varphi = (\vartheta, \eta)$, which leads to the value process $\mathcal{V}(\varphi)$ given by

$$\mathcal{V}_t(\varphi) = \vartheta_t S_t + \eta_t B_t.$$

We require that $\mathcal{V}_t(\varphi) > 0$ for all t. For $0 \le u \le t \le T$, the ratio $\frac{\mathcal{V}_t(\varphi)}{\mathcal{V}_u(\varphi)}$ describes the value at time t of one unit deposited at time u. In the present case, the state-wise retrospective reserves are given by

$$V_t^{i,retro} = \pi_0 \frac{\mathcal{V}_t(\varphi)}{\mathcal{V}_0(\varphi)} - K_T 1_{\{i=0\}} 1_{\{t=T\}}.$$

The state-wise prospective reserve given the insured is alive is still given by (3.6.3). At time $t < T$ the benefit must satisfy the following equation

$$\begin{aligned} K_t &= \pi_0 e^{\int_0^t (1+g_u)\mu_{[x]+u} du} \frac{\mathcal{V}_t(\varphi)}{\mathcal{V}_0(\varphi) p(t,T) \mathcal{S}^Q(t, \mu_{[x]+t}, T)} \\ &= \frac{\pi_0}{\mathcal{V}_0(\varphi)} e^{\int_0^t (1+g_u)\mu_{[x]+u} du} \mathcal{V}_t(\varphi) \frac{1}{p(t,T)} \frac{1}{\mathcal{S}^Q(t, \mu_{[x]+t}, T)}. \end{aligned} \tag{3.8.1}$$

Note that π_0 and $\mathcal{V}_0(\varphi)$ are determined at time 0, and thus they are independent of t. In order to find the dynamics for K_t, we need to find the dynamics of each of the last four factors and possible quadratic covariations. Since φ is self-financing, the dynamics of the value process are

$$\begin{aligned} d\mathcal{V}_t(\varphi) &= \vartheta_t dS_t + \eta_t dB_t \\ &= \vartheta_t \left(r_t S_t dt + \sigma_t^s S_t dW_t^Q \right) + \eta_t r_t B_t dt \\ &= r_t \mathcal{V}_t(\varphi) dt + \sigma_t^s \vartheta_t S_t dW_t^Q. \end{aligned}$$

For $\frac{1}{p(t,T)}$ we obtain

$$d\left(\frac{1}{p(t,T)} \right) = -\frac{1}{(p(t,T))^2} dp(t,T) + \frac{1}{(p(t,T))^3} d\langle p(t,T) \rangle. \tag{3.8.2}$$

First we find $dp(t,T)$

$$\begin{aligned} dp(t,T) &= \left(\partial_t p(t,T) + r_t S_t \partial_s p(t,T) + \frac{1}{2} (\sigma_t^s)^2 S_t^2 \partial_{ss} p(t,T) \right) dt + \sigma_t^s S_t \partial_s p(t,T) dW_t^Q \\ &= r_t p(t,T) dt + \sigma_t^s S_t \partial_s p(t,T) dW_t^Q, \end{aligned} \tag{3.8.3}$$

where we have used (3.4.12). The predictable quadratic variation is given by

$$d\langle p(t,T) \rangle = \left(\sigma_t^s S_t \partial_s p(t,T) \right)^2 dt. \tag{3.8.4}$$

Inserting (3.8.3) and (3.8.4) into (3.8.2), we get

$$
\begin{aligned}
d\left(\frac{1}{p(t,T)}\right) &= -\frac{1}{(p(t,T))^2}\left(r_t p(t,T)dt + \sigma_t^s S_t \partial_s p(t,T) dW_t^Q\right) \\
&\quad + \frac{1}{(p(t,T))^3}\left(\sigma_t^s S_t \partial_s p(t,T)\right)^2 dt \\
&= \frac{1}{p(t,T)}\left(-r_t dt + \left(\sigma_t^s S_t \frac{\partial_s p(t,T)}{p(t,T)}\right)^2\right) dt \\
&\quad - \frac{1}{p(t,T)}\left(\sigma_t^s S_t \frac{\partial_s p(t,T)}{p(t,T)}\right) dW_t^Q.
\end{aligned}
$$

The dynamics of $\frac{1}{\mathcal{S}^Q(t,\mu_{[x]+t},T)}$ are given in (3.6.8). Since W^Q and $\widetilde{W}^Q$ are independent, we only need to find

$$
d\left[\mathcal{V}_t(\varphi), \frac{1}{p(t,T)}\right] = -\vartheta_t \sigma_t^s S_t \frac{1}{p(t,T)}\left(\sigma_t^s S_t \frac{\partial_s p(t,T)}{p(t,T)}\right) dt.
$$

Itô's formula gives the following dynamics of K_t:

$$
\begin{aligned}
dK_t &= \frac{\pi_0}{\mathcal{V}_0(\varphi)}\mathcal{V}_t(\varphi)\frac{1}{p(t,T)}\frac{1}{\mathcal{S}^Q(t,\mu_{[x]+t},T)} d\left(e^{\int_0^t (1+g_u)\mu_{[x]+u}du}\right) \\
&\quad + \frac{\pi_0}{\mathcal{V}_0(\varphi)} e^{\int_0^t (1+g_u)\mu_{[x]+u}du}\frac{1}{p(t,T)}\frac{1}{\mathcal{S}^Q(t,\mu_{[x]+t},T)} d\mathcal{V}_t(\varphi) \\
&\quad + \frac{\pi_0}{\mathcal{V}_0(\varphi)} e^{\int_0^t (1+g_u)\mu_{[x]+u}du}\mathcal{V}_t(\varphi)\frac{1}{\mathcal{S}^Q(t,\mu_{[x]+t},T)} d\left(\frac{1}{p(t,T)}\right) \\
&\quad + \frac{\pi_0}{\mathcal{V}_0(\varphi)} e^{\int_0^t (1+g_u)\mu_{[x]+u}du}\mathcal{V}_t(\varphi)\frac{1}{p(t,T)} d\left(\frac{1}{\mathcal{S}^Q(t,\mu_{[x]+t},T)}\right) \\
&\quad + \frac{\pi_0}{\mathcal{V}_0(\varphi)} e^{\int_0^t (1+g_u)\mu_{[x]+u}du}\frac{1}{\mathcal{S}^Q(t,\mu_{[x]+t},T)} d\left[\mathcal{V}_t(\varphi), \frac{1}{p(t,T)}\right].
\end{aligned}
$$

Inserting the above expressions and using (3.8.1), we get

$$
\begin{aligned}
dK_t &= K_t(1+g_t)\mu_{[x]+t}dt + K_t r_t dt + K_t \sigma_t^s \frac{\vartheta_t S_t}{\mathcal{V}_t(\varphi)} dW_t^Q \\
&\quad + K_t\left(-r_t dt + \left(\sigma_t^s S_t \frac{\partial_s p(t,T)}{p(t,T)}\right)^2\right) dt - K_t\left(\sigma_t^s S_t \frac{\partial_s p(t,T)}{p(t,T)}\right) dW_t^Q \\
&\quad + K_t\left(-(1+g_t)\mu_{[x]+t} + \left(\sigma_t^\mu \frac{\partial_\mu \mathcal{S}^Q(t,\mu_{[x]+t},T)}{\mathcal{S}^Q(t,\mu_{[x]+t},T)}\right)^2\right) dt \\
&\quad - K_t\left(\sigma_t^\mu \frac{\partial_\mu \mathcal{S}^Q(t,\mu_{[x]+t},T)}{\mathcal{S}^Q(t,\mu_{[x]+t},T)}\right) d\widetilde{W}_t^Q - K_t\left(\sigma_t^s \frac{\vartheta_t S_t}{\mathcal{V}_t(\varphi)}\sigma_t^s S_t \frac{\partial_s p(t,T)}{p(t,T)}\right) dt.
\end{aligned}
$$

Simplifying, rearranging terms and changing to P-martingales now gives (3.6.11).

Chapter 4

Valuation and Hedging of Life Insurance Liabilities with Systematic Mortality Risk

This chapter considers the problem of valuating and hedging life insurance contracts that are subject to systematic mortality risk in the sense that the mortality intensity of all policy-holders is affected by some underlying stochastic processes. In particular, this implies that the insurance risk cannot be eliminated by increasing the size of the portfolio and appealing to the law of large numbers. We propose to apply techniques from incomplete markets in order to hedge and valuate these contracts. We consider a special case of the affine mortality structures considered in Chapter 3, where the underlying mortality process is driven by a time-inhomogeneous Cox-Ingersoll-Ross (CIR) model. Within this model, we study a general set of equivalent martingale measures, and determine market reserves by applying these measures. In addition, we derive risk-minimizing strategies and mean-variance indifference prices and hedging strategies for the life insurance liabilities considered. Numerical examples are included, and the use of the stochastic mortality model is compared with deterministic models.

4.1 Introduction

During the past years, expected lifetimes have increased considerably in many countries. This has forced life insurers to adjust expectations towards the underlying mortality laws used to determine reserves. Since the future mortality is unknown, a correct description requires a stochastic model, as it has already been proposed by several authors, see e.g. Marocco and Pitacco (1998), Milevsky and Promislow (2001), Dahl (2004b) (see Chapter 3), Cairns et al. (2004), Biffis and Millossovich (2004) and references therein. For a survey on current developments in the literature and their relation to our results, we refer the reader to Section 3.2. The main contribution of the present chapter is not the introduction of a specific model for the mortality intensity, but rather the study of the problem of valuating and hedging life insurance liabilities that are subject to systematic changes in the underlying mortality intensity.

In Chapter 3, a general class of Markov diffusion models are considered for the mortality intensity, and the affine mortality structures are recognized as a class with particular nice properties. Here, we study a special case of the general affine mortality structures and demonstrate how such models could be applied in practice. As starting point we take some smooth initial mortality intensity curve, which is estimated by standard methods. We then assume that the mortality intensity at a given future point in time at a given age is obtained by correcting the initial mortality intensity by the outcome of some underlying mortality improvement process, which is modelled via a time-inhomogeneous Cox-Ingersoll-Ross (CIR) model. Our model implies that the mortality intensity itself is described by a time-inhomogeneous CIR model as well. As noted in Chapter 3, the survival probability can now be determined by using standard results for affine term structures.

Within this setting, we consider an insurance portfolio and assume that the individual lifetimes are affected by the same stochastic mortality intensity. In particular, this implies that the lifetimes are not stochastically independent. Hence, the insurance company is exposed to systematic as well as unsystematic mortality risk. Here, as in Chapter 3, systematic mortality risk refers to the risk associated with changes in the underlying mortality intensity, whereas unsystematic mortality risk refers to the risk associated with the randomness of deaths in a portfolio with known mortality intensity. The systematic mortality risk is a non-diversifiable risk, which does not disappear when the size of the portfolio is increased, whereas the unsystematic mortality risk is diversifiable. Since the systematic mortality risk typically cannot be traded efficiently in the financial markets or in the reinsurance markets, this leaves open the problem of pricing insurance contracts. Here, we follow Chapter 3 and apply financial theories for pricing the contracts, and study a fairly general set of martingale measures for the model. We work with a simple financial market, consisting of a savings account and a zero coupon bond and derive market reserves for general life insurance liabilities. These market reserves depend on the market's attitude towards systematic and unsystematic mortality risk. Based on an investigation of some Danish mortality data, we propose some pragmatic parameter values and calculate market reserves by solving appropriate versions of Thiele's differential equation.

Furthermore, we investigate methods for hedging and valuating general insurance liabilities in incomplete financial markets. One possibility is to apply risk-minimization, which has been suggested by Föllmer and Sondermann (1986) and applied for the handling of insurance risks by Møller (1998, 2001a, 2001c). We demonstrate how risk-minimizing hedging strategies may be determined in the presence of systematic mortality risk. These results generalize the results in Møller (1998, 2001c), where risk-minimizing strategies were obtained without allowing for systematic mortality risk. In addition, this can be viewed as an extension of the work in Section 3, where market reserves were derived in the presence of systematic mortality risk, but without considering the hedging aspect.

Utility indifference valuation and hedging has gained considerable interest over the last years as a method for valuation and hedging in incomplete markets, see e.g. Schweizer (2001b) and Becherer (2003) and references therein. These methods have been applied for the handling of insurance contracts by e.g. Becherer (2003), who worked with exponential utility functions, and by Møller (2001b, 2003a, 2003b), who worked with mean-variance indifference principles. We derive mean-variance indifference prices within our model and compare the results with the ones obtained in Møller (2001b).

The present chapter is organized as follows. Section 4.2 contains a brief analysis of some Danish mortality data. In Section 4.3, we introduce the model for the underlying mortality intensity and derive the corresponding survival probabilities and forward mortality intensities. The financial market used for the calculation of market reserves, hedging strategies and indifference prices is introduced in Section 4.4, and the insurance portfolio is described in Section 4.5. Section 4.6 presents the combined model, the insurance payment process and the associated market reserves. Risk-minimizing hedging strategies are determined in Section 4.7, and mean-variance indifference prices and hedging strategies are obtained in Section 4.8. Numerical examples are provided in Section 4.9, and Section 4.10 contains proofs and calculations of some technical results.

4.2 Motivation and empirical evidence

We briefly describe typical empirical findings related to the development in the mortality during the last couple of decades. The results in this section are based on Danish mortality data, which have been compiled and analyzed by Andreev (2002). A more detailed statistical study is carried out in Fledelius and Nielsen (2002), who applied kernel hazard estimation. From the data material, we have determined the exposure times

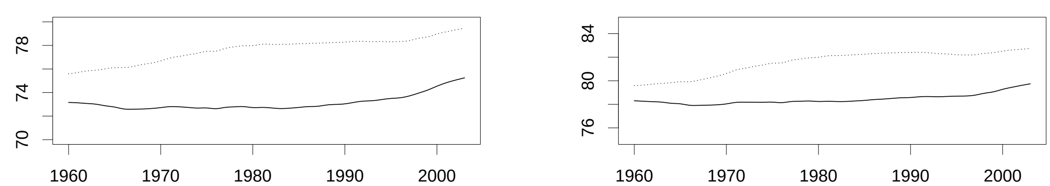

Figure 4.2.1: *To the left: Development of the expected lifetime of 30 year old females (dotted line at the top) and 30 year old males (solid line at the bottom) from year 1960 to 2003. To the right: Expected lifetimes for age 65. Estimates are based on the last 5 years of data available at calendar time.*

$W^*_{y,x}$ and number of deaths $N^*_{y,x}$ for each calendar year y, and age x, and calculated the occurrence-exposure rates $\mu^*_{y,x} = N^*_{y,x}/W^*_{y,x}$. For each fixed y, we have determined a smooth Gompertz-Makeham curve $\widetilde{\mu}_{y,x} = \alpha_y + \beta_y(c_y)^x$ based on the last 5 years of data available at calendar time by using standard methods as described in Norberg (2000).

We have visualized in Figure 4.2.1 the development in the total expected lifetime of 30-

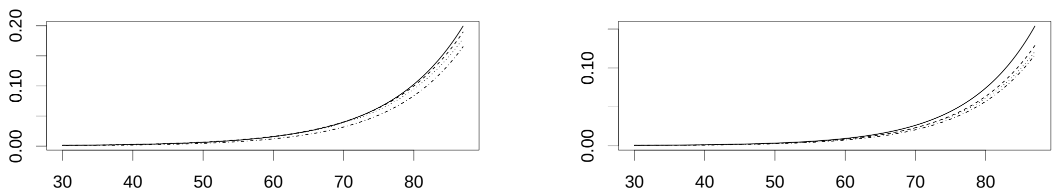

Figure 4.2.2: *Estimated mortality intensities for males (to the left) and females (to the right). Solid lines are 1970-estimates, dashed lines correspond to 1980, dotted lines 1990 and dot-dashed lines 2003.*

and 65-year old males and females based on the historical observations from the year 1960 to 2003. These numbers are based on raw occurrence-exposure rates. The figure shows that this method leads to an increase in the remaining lifetime from 1980 to 2003 of approximately 2.5 years for males and 1.5 years for females aged 30. Using this method, the expected lifetime in 2003 is about 75.3 years for 30 year old males and 79.5 for 30 year old females. If we alternatively use only one year of data we see an increase from 72.5 to 75.5 for males and from 77.9 to 79.9 for females. Figure 4.2.2 contains the estimated Gompertz-Makeham mortality intensities $\widetilde{\mu}_{y,x}$ for males and females, respectively, for 1970, 1980, 1990 and 2003. These figures show how the mortality intensities have decreased during this period. A closed study of the parameters (α_y, β_y, c_y) indicate that α_y has decreased. The estimates for β_y increase from 1960 to 1990, where the estimates for c_y decrease. In contrast, β_y decreases and c_y increases from 1990 to 2003. This approach

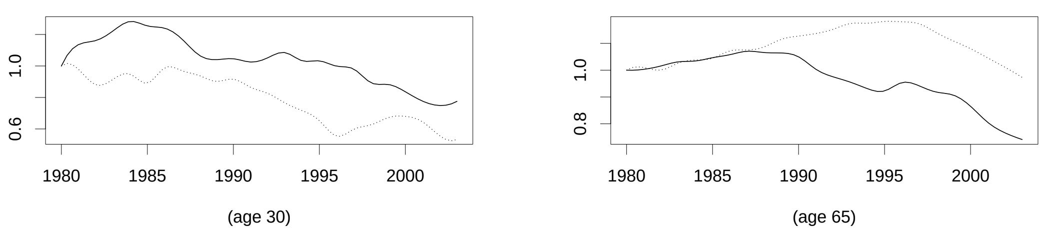

Figure 4.2.3: *Changes in the mortality intensity from 1980 to 2003 for males (solid lines) and females (dotted lines) at fixed ages. The numbers have been normalized with the 1980 mortality intensities and are based directly on the occurrence-exposure rates.*

does not involve a model that takes changes in the underlying mortality patterns into consideration. Another way to look at the mortality intensities is to consider changes in the mortality intensities at fixed ages, for example age 30 and 65, see Figure 4.2.3. For both ages, we see periods where the mortality increases and periods where it decreases. However, the general trend seems to be that mortality decreases. Moreover, we see that the mortality behaves differently for different ages and for males and females. Finally, we consider the situation, where we fix the initial age and compare the fitted Gompertz-Makeham curve for 1980 with the subsequent ones as the age increases with calendar

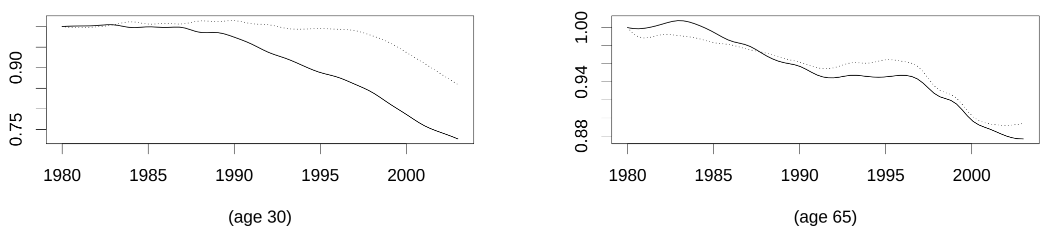

Figure 4.2.4: *Changes in the mortality intensity from 1980 to 2003 for males (solid lines) and females (dotted lines) as age increases. The numbers have been normalized with the 1980 mortality intensities and are based on the estimated Gompertz-Makeham curves.*

time. This is presented in Figure 4.2.4. Again, we see periods where the ratio between the current mortality and the 1980-estimate increases and periods where it decreases.

4.3 Modelling the mortality

4.3.1 The general model

We take as starting point an initial curve for the mortality intensity (at all ages) $\mu^{\circ,g}(x)$ for age $x \geq 0$ and gender g =male, female. It is assumed that $\mu^{\circ,g}(x)$ is continuously differentiable as a function of x. We neglect the gender aspect in the following, and simply write $\mu^{\circ}(x)$. For an individual aged x at time 0, the future mortality intensity is viewed as a stochastic process $\mu(x) = (\mu(x,t))_{t\in[0,T]}$ with the property that $\mu(x,0) = \mu^{\circ}(x)$. (Here, T is a fixed, finite time horizon.) In principle, one can view $\mu = (\mu(x))_{x\geq 0}$ as an infinitely dimensional process.

We model changes in the mortality intensity via a strictly positive infinite dimensional process $\zeta = (\zeta(x,t))_{x\geq 0, t\in[0,T]}$ with the property that $\zeta(x,0) = 1$ for all x. Here and in the following, we take all processes and random variables to be defined on some probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $\mathbb{F} = (\mathcal{F}(t))_{t\in[0,T]}$, which contains all available information. In addition, we work with several sub-filtrations. In particular, the filtration $\mathbb{I} = (\mathcal{I}(t))_{t\in[0,T]}$ is the natural filtration of the underlying process ζ. The mortality intensity process is then modelled via:

$$\mu(x,t) = \mu^{\circ}(x+t)\zeta(x,t). \tag{4.3.1}$$

Thus, $\zeta(x,t)$ describes the change in the mortality from time 0 to t for a person of age $x+t$. The true survival probability is defined by

$$\mathcal{S}(x,t,T) = E^P\left[e^{-\int_t^T \mu(x,\tau)d\tau}\middle|\mathcal{I}(t)\right] = E^P\left[e^{-\int_t^T \mu^{\circ}(x+\tau)\,\zeta(x,\tau)d\tau}\middle|\mathcal{I}(t)\right], \tag{4.3.2}$$

and it is related to the martingale

$$\mathcal{S}^M(x,t,T) = E^P\left[e^{-\int_0^T \mu(x,\tau)d\tau}\middle|\mathcal{I}(t)\right] = e^{-\int_0^t \mu(x,\tau)d\tau}\mathcal{S}(x,t,T). \tag{4.3.3}$$

In general, we can consider survival probabilities under various equivalent probability measures. This is discussed in more detail in section 4.6.1.

4.3.2 Deterministic changes in mortality intensities

As a special case, assume that $\zeta(x,t) = e^{-\widetilde{\gamma}(x)t}$, where $\widetilde{\gamma}(x)$ is fixed and constant. Thus, the mortality intensity at time t of an $x+t$ year old is defined by changing the known mortality intensity at time 0 of an $x+t$-year old by the factor $\exp(-\widetilde{\gamma}(x)t)$. If $\widetilde{\gamma}(x) > 0$, this model implies that the mortality improves by the factor $\exp(-\widetilde{\gamma}(x))$ each year. In particular, taking all $\widetilde{\gamma}(x)$ equal to one fixed $\widetilde{\gamma}$ means that all intensities improve/increase by the same factor.

If μ° corresponds to a Gompertz-Makeham mortality law, i.e.

$$\mu^\circ(x+t) = \alpha + \beta\, c^{x+t}, \tag{4.3.4}$$

then the mortality intensity μ is given by

$$\mu(x,t) = \alpha e^{-\widetilde{\gamma}(x)t} + \beta\, c^x (ce^{-\widetilde{\gamma}(x)})^t, \tag{4.3.5}$$

which no longer is a Gompertz-Makeham mortality law.

4.3.3 Time-inhomogeneous CIR models

The empirical findings in Section 4.3.1 indicate that the deterministic type of model considered above is too simple to capture the true nature of the mortality. We propose instead to model the underlying mortality improvement process via

$$d\zeta(x,t) = (\gamma(x,t) - \delta(x,t)\zeta(x,t))dt + \sigma(x,t)\sqrt{\zeta(x,t)}dW^\mu(t), \tag{4.3.6}$$

where W^μ is a standard Brownian motion under P. This is similar to a so-called time-inhomogeneous CIR model, originally proposed by Hull and White (1990) as an extension of the short rate model in Cox et al. (1985), see also Rogers (1995). We assume that $2\gamma(x,t) \geq (\sigma(x,t))^2$ such that ζ is strictly positive, see Maghsoodi (1996). Here, γ, δ and σ are assumed to be known, continuous functions. It now follows via Itô's formula that

$$d\mu(x,t) = (\gamma^\mu(x,t) - \delta^\mu(x,t)\mu(x,t))dt + \sigma^\mu(x,t)\sqrt{\mu(x,t)}dW^\mu(t), \tag{4.3.7}$$

where

$$\gamma^\mu(x,t) = \gamma(x,t)\mu^\circ(x+t), \tag{4.3.8}$$

$$\delta^\mu(x,t) = \delta(x,t) - \frac{\frac{d}{dt}\mu^\circ(x+t)}{\mu^\circ(x+t)}, \tag{4.3.9}$$

$$\sigma^\mu(x,t) = \sigma(x,t)\sqrt{\mu^\circ(x+t)}. \tag{4.3.10}$$

This shows that μ also follows an time-inhomogeneous CIR model, a property which was also noted by Rogers (1995). In particular, we note that $\gamma^\mu(x,t)/(\sigma^\mu(x,t))^2 = \gamma(x,t)/(\sigma(x,t))^2$, such that μ is strictly positive as well. If $\gamma(x,t)/(\sigma(x,t))^2$, and thus $\gamma^\mu(x,t)/(\sigma^\mu(x,t))^2$, is independent of t, then numerical calculations can be simplified considerably, see Jamshidian (1995). The following proposition regarding the survival probability follows e.g. from Björk (2004, Proposition 22.2); see also Chapter 3.

Proposition 4.3.1 (Affine mortality structure)
The survival probability $\mathcal{S}(x,t,T)$ is given by

$$\mathcal{S}(x,t,T) = e^{A^\mu(x,t,T) - B^\mu(x,t,T)\mu(x,t)},$$

where

$$\frac{\partial}{\partial t} B^\mu(x,t,T) = \delta^\mu(x,t)B^\mu(x,t,T) + \frac{1}{2}(\sigma^\mu(x,t))^2(B^\mu(x,t,T))^2 - 1, \tag{4.3.11}$$

$$\frac{\partial}{\partial t} A^\mu(x,t,T) = \gamma^\mu(x,t)B^\mu(x,t,T), \tag{4.3.12}$$

with $B^\mu(x,T,T) = 0$ and $A^\mu(x,T,T) = 0$. The dynamics of the survival probability are given by

$$d\mathcal{S}(x,t,T) = \mathcal{S}(x,t,T)\left(\mu(x,t)dt - \sigma^\mu(x,t)\sqrt{\mu(x,t)}B^\mu(x,t,T)dW^\mu(t)\right).$$

Forward mortality intensities
Inspired by interest rate theory we introduced the concept of forward mortality intensities in Chapter 3. In an affine setting, the forward mortality intensities are given by

$$f^\mu(x,t,T) = -\frac{\partial}{\partial T}\log \mathcal{S}(x,t,T) = \mu(x,t)\frac{\partial}{\partial T}B^\mu(x,t,T) - \frac{\partial}{\partial T}A^\mu(x,t,T). \tag{4.3.13}$$

The importance of forward mortality intensities is underlined by writing the survival probability on the form

$$\mathcal{S}(x,t,T) = e^{-\int_t^T f^\mu(x,t,u)du}.$$

4.4 The financial market

In this section, we introduce the financial market used for the calculations in the following sections. The financial market is essentially assumed to exist of two traded assets: A savings account and a zero coupon bond with maturity T. The price processes are given by B and $P(\cdot,T)$, respectively. The uncertainty in the financial market is described via a time-homogeneous affine model for the short rate. Hence, the short rate dynamics under P are

$$dr(t) = \alpha^r(r(t))dt + \sigma^r(r(t))dW^r(t), \tag{4.4.1}$$

where

$$\alpha^r(r(t)) = \gamma^{r,\alpha} - \delta^{r,\alpha} r(t),$$
$$\sigma^r(r(t)) = \sqrt{\gamma^{r,\sigma} + \delta^{r,\sigma} r(t)}.$$

Here, W^r is a standard Brownian motion under P and $\gamma^{r,\alpha}$, $\delta^{r,\alpha}$, $\gamma^{r,\sigma}$ and $\delta^{r,\sigma}$ are constants. Denote by $\mathbb{G} = (\mathcal{G}(t))_{t\in[0,T]}$ the natural filtration generated by W^r. The dynamics under P of the price processes are given by

$$dB(t) = r(t)B(t)dt, \tag{4.4.2}$$
$$dP(t,T) = (r(t) + \rho(t, r(t)))P(t,T)dt + \sigma^p(t, r(t))P(t,T)dW^r(t), \tag{4.4.3}$$

where

$$\rho(t, r(t)) = \sigma^p(t, r(t)) \left(\frac{\widetilde{c}}{\sigma^r(r(t))} + c\sigma^r(r(t)) \right). \tag{4.4.4}$$

Here, c and $\widetilde{c}$ are constants satisfying certain conditions given in Remark 4.4.1. With this choice of ρ, σ^p is uniquely determined from standard theory for affine short rate models, see (4.4.11).

If we restrict the model to the filtration $\mathbb{G}$, the unique equivalent martingale measure for the financial market is

$$\frac{dQ}{dP} = \widehat{\Lambda}(T), \tag{4.4.5}$$

where $d\widehat{\Lambda}(t) = \widehat{\Lambda}(t)h^r(t)dW^r(t)$, $\widehat{\Lambda}(0) = 1$, and where

$$h^r(t) = -\frac{\rho(t, r(t))}{\sigma^p(t, r(t))} = -\left(\frac{\widetilde{c}}{\sigma^r(r(t))} + c\sigma^r(r(t)) \right). \tag{4.4.6}$$

Under Q given by (4.4.5) the dynamics of the short rate are given by

$$dr(t) = \left(\gamma^{r,\alpha,Q} - \delta^{r,\alpha,Q} r(t)\right) dt + \sqrt{\gamma^{r,\sigma} + \delta^{r,\sigma} r(t)}dW^{r,Q}(t), \tag{4.4.7}$$

where $W^{r,Q}$ is a standard Brownian motion under Q and

$$\gamma^{r,\alpha,Q} = \gamma^{r,\alpha} - c\gamma^{r,\sigma} - \widetilde{c},$$
$$\delta^{r,\alpha,Q} = \delta^{r,\alpha} + c\delta^{r,\sigma}.$$

Since the drift and squared diffusion terms in (4.4.7) are affine in r, we have an affine term structure, see Björk (2004, Proposition 22.2). Thus, the bond price is given by

$$P(t,T) = e^{A^r(t,T) - B^r(t,T)r(t)},$$

where $A^r(t,T)$ and $B^r(t,T)$ solves

$$\frac{\partial}{\partial t} B^r(t,T) = \delta^{r,\alpha,Q} B^r(t,T) + \frac{1}{2}\delta^{r,\sigma}(B^r(t,T))^2 - 1, \tag{4.4.8}$$
$$\frac{\partial}{\partial t} A^r(t,T) = \gamma^{r,\alpha,Q} B^r(t,T) - \frac{1}{2}\gamma^{r,\sigma}(B^r(t,T))^2, \tag{4.4.9}$$

with $B^r(T,T) = 0$ and $A^r(T,T) = 0$. The bond price dynamics under Q can be determined by applying Itô's formula:

$$dP(t,T) = r(t)P(t,T)dt - \sigma^r(r(t))B^r(t,T)P(t,T)dW^{r,Q}(t), \tag{4.4.10}$$

which in turn gives that

$$\sigma^p(t,r(t)) = -\sigma^r(r(t))B^r(t,T). \tag{4.4.11}$$

Remark 4.4.1 Recall that if $\delta^{r,\sigma} \neq 0$ and

$$\frac{\gamma^{r,\alpha}}{\delta^{r,\sigma}} + \frac{\delta^{r,\alpha}\gamma^{r,\sigma}}{(\delta^{r,\sigma})^2} < \frac{1}{2}, \tag{4.4.12}$$

then $P(r(t) = 0) > 0$. Hence, we immediately get from (4.4.4) that $\widetilde{c} = 0$ in this case. If $\delta^{r,\sigma} \neq 0$ and (4.4.12) does not hold, then, exploiting the results of Cheridito, Filipović and Kimmel (2003), gives that (4.4.5) defines an equivalent martingale measure if

$$\widetilde{c} \leq \gamma^{r,\alpha} + \frac{\delta^{r,\alpha}\gamma^{r,\sigma}}{\delta^{r,\sigma}} - \frac{\delta^{r,\sigma}}{2}. \tag{4.4.13}$$

No restrictions apply to c in any case or to $\widetilde{c}$ if $\delta^{r,\sigma} = 0$.

□

Remark 4.4.2 If $\delta^{r,\sigma} = 0$ the short rate is described by a Vasiček model, see Vasiček (1977). In this case the functions A^r and B^r are given by

$$\begin{aligned}
B^r(t,T) &= \frac{1}{\delta^{r,\alpha,Q}}\left(1 - e^{-\delta^{r,\alpha,Q}(T-t)}\right), \\
A^r(t,T) &= \frac{(B^r(t,T) - T + t)(\gamma^{r,\alpha,Q}\delta^{r,\alpha,Q} - \frac{1}{2}\gamma^{r,\sigma})}{(\delta^{r,\alpha,Q})^2} - \frac{\gamma^{r,\sigma}(B^r(t,T))^2}{4\delta^{r,\alpha,Q}}.
\end{aligned}$$

Letting $\gamma^{r,\sigma} = 0$, we get a time-homogeneous CIR model (for the short rate), see Cox et al. (1985), which gives the following expressions for A^r and B^r

$$\begin{aligned}
B^r(t,T) &= \frac{2\left(e^{\xi^{r,Q}(T-t)} - 1\right)}{(\xi^{r,Q} + \delta^{r,\alpha,Q})(e^{\xi^{r,Q}(T-t)} - 1) + 2\xi^{r,Q}}, \\
A^r(t,T) &= \frac{2\gamma^{r,\alpha,Q}}{\delta^{r,\sigma}} \log\left(\frac{2\xi^{r,Q}e^{\left(\xi^{r,Q}+\delta^{r,\alpha,Q}\right)\frac{T-t}{2}}}{\left(\xi^{r,Q} + \delta^{r,\alpha,Q}\right)\left(e^{\xi^{r,Q}(T-t)} - 1\right) + 2\xi^{r,Q}}\right),
\end{aligned}$$

where $\xi^{r,Q} = \sqrt{(\delta^{r,\alpha,Q})^2 + 2\delta^{r,\sigma}}$. For both models, the functions A^r and B^r depend on t and T via the difference $T - t$, only.

□

4.5 The insurance portfolio

Consider an insurance portfolio consisting of n insured lives of the same age x. We assume that the individual remaining lifetimes at time 0 of the insured are described by a sequence $T_1, \ldots, T_n$ of identically distributed non-negative random variables. Moreover, we assume that

$$P(T_1 > t|\mathcal{I}(T)) = e^{-\int_0^t \mu(x,s)ds}, \;\; 0 \leq t \leq T,$$

and that the censored lifetimes $T_i^* = T_i 1_{\{T_i \leq T\}} + T 1_{\{T_i > T\}}$, $i = 1, \ldots, n$, are i.i.d. given $\mathcal{I}(T)$. Thus, given the development of the underlying process ζ, the mortality intensity at time s is simply $\mu(x, s)$.

Now define a counting process $N(x) = (N(x,t))_{0 \leq t \leq T}$ by

$$N(x,t) = \sum_{i=1}^{n} 1_{(T_i \leq t)},$$

which keeps track of the number of deaths in the portfolio of insured lives. We denote by $\mathbb{H} = (\mathcal{H}(t))_{0 \leq t \leq T}$ the natural filtration generated by $N(x)$. It follows that $N(x)$ is an $\mathbb{H} \vee \mathbb{I}$-Markov process, and the stochastic intensity process $\lambda(x) = (\lambda(x,t))_{0 \leq t \leq T}$ of $N(x)$ under P can be informally defined by

$$\lambda(x,t)dt \equiv E^P \left[dN(x,t) | \, \mathcal{H}(t-) \vee \mathcal{I}(t) \right] = (n - N(x,t-))\mu(x,t)dt, \tag{4.5.1}$$

which is proportional to the product of the number of survivors and the mortality intensity. It is well-known, that the process $M(x) = (M(x,t))_{0 \leq t \leq T}$ defined by

$$dM(x,t) = dN(x,t) - \lambda(x,t)dt, \quad 0 \leq t \leq T, \tag{4.5.2}$$

is an $(\mathbb{H} \vee \mathbb{I}, P)$-martingale.

4.6 The combined model

The filtration $\mathbb{F} = (\mathcal{F}(t))_{0 \leq t \leq T}$ introduced earlier is given by $\mathcal{F}(t) = \mathcal{G}(t) \vee \mathcal{H}(t) \vee \mathcal{I}(t)$. Thus, $\mathbb{F}$ is the filtration for the combined model of the financial market, the mortality intensity and the insurance portfolio. Moreover, we assume that the financial market is stochastically independent of the insurance portfolio and the mortality intensity, i.e. $\mathcal{G}(T)$ and $(\mathcal{H}(T), \mathcal{I}(T))$ are independent. In particular, this implies that the properties of the underlying processes are preserved. For example, $M(x)$ is also an $(\mathbb{F}, P)$-martingale, and the $(\mathbb{F}, P)$-intensity process is identical to the $(\mathbb{H} \vee \mathbb{I}, P)$-intensity process $\lambda(x)$. We note that the combined model is on the general index-form studied in Steffensen (2000). However, Steffensen (2000) contains no explicit remarks or calculations regarding a stochastic mortality intensity.

4.6.1 A class of equivalent martingale measures

If we consider the financial market only, i.e. if we restrict ourselves to the filtration $\mathbb{G}$, we found in Section 4.4 that (given some regularity conditions) there exists a unique equivalent martingale measure. This is not the case when analyzing the combined model of the financial market and the insurance portfolio, see e.g. Møller (1998, 2001c) for a discussion of this problem. In the present model, we can also perform a change of measure for the counting process $N(x)$ and for the underlying mortality intensity; we refer to Chapter 3 for a more detailed treatment of these aspects. Consider a likelihood process on the form

$$d\Lambda(t) = \Lambda(t-)\Big(h^r(t)dW^r(t) + h^\mu(t)dW^\mu(t) + g(t)dM(x,t)\Big), \tag{4.6.1}$$

with $\Lambda(0) = 1$. We assume that $E^P[\Lambda(T)] = 1$ and define an equivalent martingale measure Q via $\frac{dQ}{dP} = \Lambda(T)$. In the following, we describe the terms in (4.6.1) in more detail. The process h^r, which is defined in (4.4.6), is related to the change of measure for the underlying bond market. It is uniquely determined by requiring that the discounted bond price process is a Q-martingale.

The term involving h^μ leads to a change of measure for the Brownian motion which drives the mortality intensity process μ. Hence, $dW^{\mu,Q}(t) = dW^\mu(t) - h^\mu(t)dt$ defines a standard Brownian motion under Q. Here, we restrict ourselves to h^μ's of the form

$$h^\mu(t,\zeta(x,t)) = -\beta(x,t)\frac{\sqrt{\zeta(x,t)}}{\sigma(x,t)} + \frac{\beta^*(x,t)}{\sigma(x,t)\sqrt{\zeta(x,t)}} \tag{4.6.2}$$

for some continuous functions β and β^*. In this case, the Q-dynamics of $\zeta(x,t)$ are given by

$$d\zeta(x,t) = \left(\gamma^Q(x,t) - \delta^Q(x,t)\zeta(x,t)\right)dt + \sigma(x,t)\sqrt{\zeta(x,t)}dW^{\mu,Q}(t),$$

where

$$\gamma^Q(x,t) = \gamma(x,t) + \beta^*(x,t), \tag{4.6.3}$$

$$\delta^Q(x,t) = \delta(x,t) + \beta(x,t). \tag{4.6.4}$$

Hence, ζ also follows a time-inhomogeneous CIR model under Q. A necessary condition for the equivalence between P and Q is that ζ is strictly positive under Q. Thus, we observe from (4.6.3) that we must require that $\beta^*(x,t) \geq (\sigma(x,t))^2/2 - \gamma(x,t)$. The Q-dynamics of $\mu(x)$ are now given by

$$d\mu(x,t) = (\gamma^{\mu,Q}(x,t) - \delta^{\mu,Q}(x,t)\mu(x,t))dt + \sigma^\mu(x,t)\sqrt{\mu(x,t)}dW^{\mu,Q}(t), \tag{4.6.5}$$

where $\gamma^{\mu,Q}(x,t)$ and $\delta^{\mu,Q}(x,t)$ are given by (4.3.8) and (4.3.9) with $\gamma(x,t)$ and $\delta(x,t)$ replaced by $\gamma^Q(x,t)$ and $\delta^Q(x,t)$, respectively. If $h^\mu = 0$, i.e. if the dynamics of ζ (and thus μ) are identical under P and Q, we say the market is risk-neutral with respect to systematic mortality risk.

The last term in (4.6.1) involves a predictable process $g > -1$. This term affects the intensity for the counting process. More precisely, it can be shown, see e.g. Andersen et al. (1993), that the intensity process under Q is given by $\lambda^Q(x,t) = (1+g(t))\lambda(x,t)$. Using (4.5.1), we see that

$$\lambda^Q(x,t) = (n - N(x,t-))(1+g(t))\mu(x,t),$$

such that $\mu^Q(x,t) = (1+g(t))\mu(x,t)$ can be viewed as the mortality intensity under Q. Hence the process $M^Q(x) = (M^Q(x,t))_{0\le t\le T}$ defined by

$$dM^Q(x,t) = dN(x,t) - \lambda^Q(x,t)dt, \quad 0 \le t \le T, \tag{4.6.6}$$

is an $(\mathbb{F}, Q)$-martingale. If $g = 0$, the market is said to be risk-neutral with respect to unsystematic mortality risk. This choice of g can be motivated by the law of large numbers. In this chapter, we restrict the analysis to the case, where g is a deterministic, continuously differentiable function. Combined with the definition of h^r in (4.4.6) and the restricted form of h^μ in (4.6.2), this implies that the independence between $\mathcal{G}(T)$ and $(\mathcal{H}(T), \mathcal{I}(T))$ is preserved under Q.

Now define the Q-survival probability and the associated Q-martingale by

$$\mathcal{S}^Q(x,t,T) = E^Q\left[e^{-\int_t^T \mu^Q(x,\tau)d\tau} \middle| \zeta(x,t)\right]$$

and

$$\mathcal{S}^{Q,M}(x,t,T) = E^Q\left[e^{-\int_0^T \mu^Q(x,\tau)d\tau} \middle| \zeta(x,t)\right] = e^{-\int_0^t \mu^Q(x,\tau)d\tau}\mathcal{S}^Q(x,t,T).$$

Calculations similar to those in Section 4.3.3 give the following Q-dynamics of $\mu^Q(x)$

$$d\mu^Q(x,t) = (\gamma^{\mu,Q,g}(x,t) - \delta^{\mu,Q,g}(x,t)\mu^Q(x,t))dt + \sigma^{\mu,Q,g}(x,t)\sqrt{\mu^Q(x,t)}dW^{\mu,Q},$$

where

$$\begin{aligned}
\gamma^{\mu,Q,g}(x,t) &= (1+g(t))\gamma^{\mu,Q}(x,t),\\
\delta^{\mu,Q,g}(x,t) &= \delta^{\mu,Q}(x,t) - \frac{\frac{d}{dt}g(t)}{1+g(t)},\\
\sigma^{\mu,Q,g}(x,t) &= \sqrt{1+g(t)}\sigma^\mu(x,t).
\end{aligned}$$

Since the drift and squared diffusion terms for $\mu^Q(x,t)$ are affine in $\mu^Q(x,t)$, we have the following proposition

Proposition 4.6.1 (Affine mortality structure under Q)
The Q-survival probability $\mathcal{S}^Q(x,t,T)$ is given by

$$\mathcal{S}^Q(x,t,T) = e^{A^{\mu,Q}(x,t,T) - B^{\mu,Q}(x,t,T)(1+g(t))\mu(x,t)},$$

where $A^{\mu,Q}$ and $B^{\mu,Q}$ are determined from (4.3.11) and (4.3.12) with $\gamma^\mu(x,t)$, $\delta^\mu(x,t)$ and $\sigma^\mu(x,t)$ replaced by $\gamma^{\mu,Q,g}(x,t)$, $\delta^{\mu,Q,g}(x,t)$ and $\sigma^{\mu,Q,g}(x,t)$, respectively. The dynamics of the Q-martingale associated with the Q-survival probability are given by

$$d\mathcal{S}^{Q,M}(x,t,T) = -(1+g(t))\sigma^\mu(x,t)\sqrt{\mu(x,t)}B^{\mu,Q}(x,t,T)\mathcal{S}^{Q,M}(x,t,T)dW^{\mu,Q}(t). \tag{4.6.7}$$

Similarly to the forward mortality intensities, the Q-forward mortality intensities are given by

$$f^{\mu,Q}(x,t,T) = -\frac{\partial}{\partial T}\log \mathcal{S}^Q(x,t,T) = \mu^Q(x,t)\frac{\partial}{\partial T}B^{\mu,Q}(x,t,T) - \frac{\partial}{\partial T}A^{\mu,Q}(x,t,T). \tag{4.6.8}$$

4.6.2 The payment process

The total benefits less premiums on the insurance portfolio is described by a payment process A. Thus, $dA(t)$ are the net payments to the policy-holders during an infinitesimal interval $[t, t+dt)$. We take A of the form

$$\begin{aligned} dA(t) &= -n\pi(0)dI_{\{t\geq 0\}} + (n - N(x,\overline{T}))\Delta A_0(\overline{T})dI_{\{t\geq \overline{T}\}} \\ &\quad + a_0(t)(n - N(x,t))dt + a_1(t)dN(x,t), \end{aligned} \tag{4.6.9}$$

for $0 \leq t \leq T$. The first term, $n\pi(0)$ is the single premium paid at time 0 by all policy-holders. The second term involves a fixed time $\overline{T} \leq T$, which represents the retirement time of the insured lives. This term states that each of the surviving policy-holders receive the fixed amount $\Delta A_0(\overline{T})$ upon retirement. The third term involves a piecewise continuous function

$$a_0(t) = -\pi^c(t)1_{\{0\leq t<\overline{T}\}} + a^p(t)1_{\{\overline{T}\leq t\leq T\}},$$

where $\pi^c(t)$ are continuous premiums paid by the policy-holders (as long as they are alive) and $a^p(t)$ corresponds to a life annuity benefit received by the policy-holders. Finally, the last term in (4.6.9) represents payments immediately upon a death, and we assume that a_1 is some piecewise continuous function.

4.6.3 Market reserves

In the following we consider an arbitrary, but fixed, equivalent martingale measure Q from the class of measures introduced in Section 4.6.1 and define the process

$$V^{*,Q}(t) = E^Q\left[\left.\int_{[0,T]} e^{-\int_0^\tau r(u)du}dA(\tau)\right| \mathcal{F}(t)\right], \tag{4.6.10}$$

which is the conditional expected value, calculated at time t, of discounted benefits less premiums, where all payments are discounted to time 0. Using that the processes A and r are adapted, and introducing the discounted payment process A^* defined by

$$dA^*(t) = e^{-\int_0^t r(u)du}dA(t),$$

we see that

$$\begin{aligned} V^{*,Q}(t) &= \int_{[0,t]} e^{-\int_0^\tau r(u)du}dA(\tau) + e^{-\int_0^t r(u)du}E^Q\left[\left.\int_{(t,T]} e^{-\int_t^\tau r(u)du}dA(\tau)\right| \mathcal{F}(t)\right] \\ &= A^*(t) + e^{-\int_0^t r(u)du}\widetilde{V}^Q(t). \end{aligned} \tag{4.6.11}$$

In the literature, the process $V^{*,Q}$ is called the intrinsic value process, see Föllmer and Sondermann (1986) and Møller (2001c). The process $\widetilde{V}^Q(t)$ introduced in (4.6.11) represents the conditional expected value at time t, of future payments. We shall refer to this quantity as the market reserve. We have the following result:

Proposition 4.6.2
The market reserve $\widetilde{V}^Q$ is given by

$$\widetilde{V}^Q(t) = (n - N(x,t))V^Q(t, r(t), \mu(x,t)), \tag{4.6.12}$$

where

$$\begin{aligned} V^Q(t, r(t), \mu(x,t)) = \int_t^T & P(t,\tau)\mathcal{S}^Q(x,t,\tau)\left(a_0(\tau) + a_1(\tau) f^{\mu,Q}(x,t,\tau)\right) d\tau \\ &+ P(t,\overline{T})\mathcal{S}^Q(x,t,\overline{T})\Delta A_0(\overline{T}). \end{aligned} \tag{4.6.13}$$

This can be verified by using methods similar to the ones used in Møller (2001c) and Chapter 3. A sketch of proof is given below.

Some comments on this results: The quantity $V^Q(t, r(t), \mu(x,t))$ is the market reserve at time t for one policy-holder who is alive, given the current level for the short rate and the mortality intensity. The market reserve has the same structure as standard reserves. However, the usual discount factor has been replaced by a zero coupon bond price $P(t,T)$ and the usual (deterministic) survival probability of the form $\exp(-\int_t^\tau \mu^\circ(x,u)du)$ has been replaced by the term $\mathcal{S}^Q(x,t,\tau)$. In addition, the Q-forward mortality intensity, $f^{\mu,Q}(x,t,\tau)$, now appears instead of the deterministic mortality intensity $\mu^\circ(x,\tau)$ in connection with the sum $a_1(\tau)$ payable upon a death.

Sketch of proof of Proposition 4.6.2: The proposition follows by exploiting the independence between the financial market and the insured lives. In addition, we use that for any predictable, sufficiently integrable process $\widetilde{g}$,

$$\int_0^t \widetilde{g}(s)(dN(x,s) - \lambda^Q(x,s)ds) \tag{4.6.14}$$

is an $(\mathbb{F}, Q)$-martingale. For example, this implies that

$$\begin{aligned} & E^Q\left[\int_t^T e^{-\int_t^\tau r(u)du} a_1(\tau) dN(x,\tau) \middle| \mathcal{F}(t)\right] \\ &= E^Q\left[\int_t^T e^{-\int_t^\tau r(u)du} a_1(\tau)\lambda^Q(x,\tau)d\tau \middle| \mathcal{F}(t)\right] \\ &= \int_t^T P(t,\tau)a_1(\tau)E^Q\left[(n - N(x,\tau))\mu^Q(x,\tau)\middle| \mathcal{F}(t)\right] d\tau. \end{aligned} \tag{4.6.15}$$

Here, the second equality follows by changing the order of integration and by using the independence between r and (N,μ). By iterated expectations, we get that

$$\begin{aligned} E^Q\left[(n-N(x,\tau))\middle|\,\mathcal{F}(t)\right] &= E^Q\left[E^Q\left[(n-N(x,\tau))\middle|\,\mathcal{F}(t)\vee\mathcal{I}(T)\right]\middle|\,\mathcal{F}(t)\right] \\ &= E^Q\left[(n-N(x,t))e^{-\int_t^\tau \mu^Q(x,u)du}\middle|\,\mathcal{F}(t)\right] \\ &= (n-N(x,t))\mathcal{S}^Q(x,t,\tau), \end{aligned} \tag{4.6.16}$$

where the second equality follows by using that, given $\mathcal{I}(T)$, the lifetimes are i.i.d. under Q with mortality intensity $\mu^Q(x)$, and the third equality is the definition of the Q-survival probability. Similarly, we have that

$$\begin{aligned} &E^Q\left[(n-N(x,\tau))\mu^Q(x,\tau)\middle|\,\mathcal{F}(t)\right] \\ &= E^Q\left[E^Q\left[(n-N(x,\tau))\mu^Q(x,\tau)\middle|\,\mathcal{F}(t)\vee\mathcal{I}(T)\right]\middle|\,\mathcal{F}(t)\right] \\ &= E^Q\left[(n-N(x,t))\mu^Q(x,\tau)e^{-\int_t^\tau \mu^Q(x,u)du}\middle|\,\mathcal{F}(t)\right] \\ &= (n-N(x,t))\frac{\partial}{\partial\tau}\mathcal{S}^Q(x,t,\tau) \\ &= (n-N(x,t))\mathcal{S}^Q(x,t,\tau)f^{\mu,Q}(x,t,\tau). \end{aligned} \tag{4.6.17}$$

Here, the third equality follows by differentiating $\mathcal{S}^Q(x,t,\tau)$ under the integral. The result now follows by using (4.6.15)–(4.6.17).

□

We emphasize that the market reserve depends on the choice of equivalent martingale measure Q.

In the remaining of the paper we work under the following assumption

Assumption 4.6.3 $V^Q(t,r,\mu)\in C^{1,2,2}$, *i.e.* $V^Q(t,r,\mu)$ *is continuously differentiable with respect to* t *and twice differentiable with respect to* r *and* μ.

□

4.7 Risk-minimizing strategies

The discounted insurance payment process A^* is subject to both financial and mortality risk. This implies that the insurance liabilities typically cannot be hedged and priced uniquely by trading on the financial market. Møller (1998) applied the criterion of risk-minimization suggested by Föllmer and Sondermann (1986) for the handling of this combined risk for unit-linked life insurance contracts. This analysis led to so-called risk-minimizing hedging strategies, that essentially minimized the variance of the insurance liabilities calculated with respect to some equivalent martingale measure. Here, we follow Møller (2001c), who extended the approach of Föllmer and Sondermann (1986) to the case of a payment process. Further applications of the criterion of risk-minimization to insurance contracts can be found in Møller (2001a, 2002).

4.7.1 A review of risk-minimization

Consider the financial market introduced in Section 4.4 consisting of a zero coupon bond expiring at T and a savings account. We denote by $X(t) = P^*(t,T)$ the discounted price process of the zero coupon bond. A strategy is a process $\varphi = (\xi, \eta)$, where ξ is the number of zero coupon bonds held and η is the discounted deposit on the savings account. The discounted value process $V(\varphi)$ associated with φ is defined by $V(t,\varphi) = \xi(t)\,X(t) + \eta(t)$, and the cost process $C(\varphi)$ is defined by

$$C(t,\varphi) = V(t,\varphi) - \int_0^t \xi(u)dX(u) + A^*(t). \tag{4.7.1}$$

The accumulated costs $C(t,\varphi)$ at time t are the discounted value $V(t,\varphi)$ of the portfolio reduced by discounted trading gains (the integral) and added discounted net payments to the policy-holders. A strategy is called risk-minimizing, if it minimizes

$$R(t,\varphi) = E^Q\left[(C(T,\varphi) - C(t,\varphi))^2 \middle| \mathcal{F}(t) \right] \tag{4.7.2}$$

for all t with respect to all strategies, and a strategy φ with $V(T,\varphi) = 0$ is called 0-admissible. The process $R(\varphi)$ is called the risk process. Föllmer and Sondermann (1986) realized that the risk-minimizing strategies are related to the so-called Galtchouk-Kunita-Watanabe decomposition,

$$V^{*,Q}(t) = E^Q\left[A^*(T) \middle| \mathcal{F}(t) \right] = V^{*,Q}(0) + \int_0^t \xi^Q(u)dX(u) + L^Q(t), \tag{4.7.3}$$

where ξ^Q is a predictable process and where L^Q is a zero-mean Q-martingale orthogonal to X. It now follows by Møller (2001c, Theorem 2.1) that there exists a unique 0-admissible risk-minimizing strategy $\varphi^* = (\xi^*, \eta^*)$ given by

$$\varphi^*(t) = (\xi^*(t), \eta^*(t)) = \left(\xi^Q(t),\, V^{*,Q}(t) - \xi^Q(t)X(t) - A^*(t) \right). \tag{4.7.4}$$

In particular, it follows that the cost process associated with the risk-minimizing strategy is given by

$$C(t,\varphi^*) = V^{*,Q}(0) + L^Q(t). \tag{4.7.5}$$

The risk process associated with the risk-minimizing strategy, the so-called intrinsic risk process, is given by

$$R(t,\varphi^*) = E^Q\left[(L^Q(T) - L^Q(t))^2 \middle| \mathcal{F}(t) \right]. \tag{4.7.6}$$

It follows from (4.7.4) that $V(t,\varphi^*) = V^{*,Q}(t) - A^*(t)$, i.e. the discounted value process associated with the risk-minimizing strategy coincides with the intrinsic value process reduced by the discounted payments.

Note that the risk-minimizing strategy depends on the choice of martingale measure Q. In the literature, the minimal martingale measure has been applied for determining risk-minimizing strategies, since this essentially corresponds to the criterion of local risk-minimization, which is a criterion in terms of P, see Schweizer (2001a).

4.7.2 Risk-minimizing strategies for the insurance payment process

As noted in Section 4.6.3, the intrinsic value process $V^{*,Q}$ associated with the payment process A is given by

$$V^{*,Q}(t) = A^*(t) + (n - N(x,t))B(t)^{-1}V^Q(t, r(t), \mu(x,t)), \tag{4.7.7}$$

where $V^Q(t, r(t), \mu(x,t))$ is defined by (4.6.13). The Galtchouk-Kunita-Watanabe decomposition of $V^{*,Q}$ is determined by the following lemma:

Lemma 4.7.1
The Galtchouk-Kunita-Watanabe decomposition of $V^{*,Q}$ is given by

$$V^{*,Q}(t) = V^{*,Q}(0) + \int_0^t \xi^Q(\tau)dP^*(\tau,T) + L^Q(t), \tag{4.7.8}$$

where

$$V^{*,Q}(0) = -n\pi(0) + nV^Q(0, r(0), \mu(x,0)), \tag{4.7.9}$$

$$L^Q(t) = \int_0^t \nu^Q(\tau)dM^Q(x,\tau) + \int_0^t \kappa^Q(\tau)d\mathcal{S}^{Q,M}(x,\tau,T), \tag{4.7.10}$$

and

$$\begin{aligned}\xi^Q(t) = (n - N(x,t-))\Bigg(&\int_t^T \frac{B^r(t,\tau)P^*(t,\tau)}{B^r(t,T)P^*(t,T)}\mathcal{S}^Q(x,t,\tau)\left(a_0(\tau) + a_1(\tau)f^{\mu,Q}(x,t,\tau)\right)d\tau \\ &+ \frac{B^r(t,\overline{T})P^*(t,\overline{T})}{B^r(t,T)P^*(t,T)}\mathcal{S}^Q(x,t,\overline{T})\Delta A_0(\overline{T})\Bigg),\end{aligned} \tag{4.7.11}$$

$$\nu^Q(t) = B(t)^{-1}\left(a_1(t) - V^Q(t, r(t), \mu(x,t))\right), \tag{4.7.12}$$

$$\begin{aligned}\kappa^Q(t) = (n - N(x,t-))\Bigg(&\int_t^T P^*(t,\tau)\frac{B^{\mu,Q}(x,t,\tau)\mathcal{S}^Q(x,t,\tau)}{B^{\mu,Q}(x,t,T)\mathcal{S}^{Q,M}(x,t,T)} \\ &\times\left(a_0(\tau) + a_1(\tau)\left(f^{\mu,Q}(x,t,\tau) - \frac{\frac{\partial}{\partial\tau}B^{\mu,Q}(x,t,\tau)}{B^{\mu,Q}(x,t,\tau)}\right)\right)d\tau \\ &+ P^*(t,\overline{T})\frac{B^{\mu,Q}(x,t,\overline{T})\mathcal{S}^Q(x,t,\overline{T})}{B^{\mu,Q}(x,t,T)\mathcal{S}^{Q,M}(x,t,T)}\Delta A_0(\overline{T})\Bigg).\end{aligned} \tag{4.7.13}$$

Proof of Lemma 4.7.1: See Section 4.10.1.

□

In the decomposition obtained in Lemma 4.7.1, the integrals with respect to the compensated counting process $M^Q(x)$ and the Q-martingale $\mathcal{S}^{Q,M}(x,\cdot,T)$ associated with the Q-survival probability comprise the non-hedgeable part of the payment process. The factor $\nu^Q(t)$ appearing in the integral with respect to $M^Q(x)$ in (4.7.10) represents the discounted extra cost for the insurer associated with a death within the portfolio of insured lives. It consists of the discounted value of the amount $a_1(t)$ to be paid out immediately upon death, reduced by the discounted market reserve of one policy-holder $B(t)^{-1}V^Q(t, r(t), \mu(x,t))$. In traditional life insurance, $\nu^Q(t)$ is known as the (discounted) sum at risk associated with a death in the insured portfolio at time t, see e.g. Norberg (2001); in Møller (1998), a similar result is obtained with deterministic mortality intensities.

Changes in the mortality intensity lead to new Q-survival probabilities, and this affects the expected present value under Q of future payments. This sensitivity is described by the process $\kappa^Q(t)$ appearing in (4.7.10), which can be interpreted as the change in the discounted value of expected future payments associated with a change in the Q-martingale associated with the Q-survival probability. It follows from (4.7.6) that the intrinsic risk process is given by

$$\begin{aligned} R(t,\varphi) &= E^Q\left[\left(\int_t^T \nu^Q(u)dM^Q(x,u) + \kappa^Q(u)d\mathcal{S}^{Q,M}(x,u,T)\right)^2 \middle| \mathcal{F}(t)\right] \\ &= E^Q\left[\int_t^T \left(\nu^Q(u)\right)^2 d\langle M^Q\rangle(x,u) + \left(\kappa^Q(u)\right)^2 d\langle \mathcal{S}^{Q,M}(x,u,T)\rangle \middle| \mathcal{F}(t)\right] \\ &= E^Q\Big[\int_t^T \left(\nu^Q(u)\right)^2 (n - N(x,u-))\,(1+g(u))\mu(x,u)du \\ &\quad + \left(\kappa^Q(u)(1+g(u))\sigma^\mu(x,u)\sqrt{\mu(x,u)}B^{\mu,Q}(x,u,T)\mathcal{S}^{Q,M}(x,u,T)\right)^2 du \Big| \mathcal{F}(t)\Big]. \end{aligned}$$

Here, we have used the square bracket processes, that $M^Q(x)$ is an adapted process with finite variation, and that $\mathcal{S}^{Q,M}(x,\cdot,T)$ is continuous (hence predictable), such that the martingales are orthogonal.

Using the general results on risk-minimization and Lemma 4.7.1, we get the following result.

Theorem 4.7.2
The unique 0-admissible risk-minimizing strategy for the payment process (4.6.9) is

$$(\xi^*(t), \eta^*(t)) = \left(\xi^Q(t), (n - N(x,t))B(t)^{-1}V^Q(t, r(t), \mu(x,t)) - \xi^Q(t)P^*(t,T)\right),$$

where ξ^Q is given by (4.7.11).

This result is similar to the risk-minimizing hedging strategy obtained in Møller (2001c, Theorem 3.4). However, our results differ from the ones obtained there in that the market reserves depend on the current value of the mortality intensity. The fact that the strategies are similar is reasonable, since we are essentially adding a stochastic mortality to the model of Møller, and this does not change the market in which the hedger is allowed to trade. As in Møller (2001c), the discounted value process associated with the risk-minimizing strategy φ^* is

$$V(t,\varphi^*) = (n - N(x,t))B(t)^{-1}V^Q(t, r(t), \mu(x,t)), \tag{4.7.14}$$

where $V^Q(t, r(t), \mu(x,t))$ is given by (4.6.13). This shows that the portfolio is currently adjusted, such that the value at any time t is exactly the market reserve. Inserting (4.7.9) and (4.7.10) in (4.7.5) gives

$$C(t,\varphi^*) = nV^Q(0, r(0), \mu(x,0)) - n\pi(0) + \int_0^t \nu^Q(\tau) dM^Q(x,\tau) + \int_0^t \kappa^Q(\tau) d\mathcal{S}^{Q,M}(x,\tau,T).$$

Hence the hedger's loss is driven by $M^Q(x)$ and $\mathcal{S}^{Q,M}(x,\cdot,T)$. The first three terms are similar to the ones obtained by Møller (2001c). The last term, which accounts for costs associated with changes in the mortality intensity, did not appear in his model, since he worked with deterministic mortality intensities.

Example 4.7.3 Consider the case where $\overline{T} = T$, and where all n insured purchase a pure endowment of $\Delta A_0(T)$ paid by a single premium at time 0. In this case, the Galtchouk-Kunita-Watanabe decomposition (4.7.8) of $V^{*,Q}$ is determined via

$$\begin{aligned}
\xi^Q(t) &= (n - N(x,t-))\mathcal{S}^Q(x,t,T)\Delta A_0(T),\\
\nu^Q(t) &= -P^*(t,T)\mathcal{S}^Q(x,t,T)\Delta A_0(T),\\
\kappa^Q(t) &= (n - N(x,t-))P^*(t,T)e^{\int_0^t \mu^Q(x,u)du}\Delta A_0(T),
\end{aligned}$$

since $V^Q(t, r(t), \mu(x,t)) = P(t,T)\mathcal{S}^Q(x,t,T)\Delta A_0(T)$, and

$$\frac{\mathcal{S}^Q(x,t,T)}{\mathcal{S}^{Q,M}(x,t,T)} = e^{\int_0^t \mu^Q(x,u)du}.$$

This gives the 0-admissible risk-minimizing strategy

$$\begin{aligned}
\xi^*(t) &= (n - N(x,t-))\mathcal{S}^Q(x,t,T)\Delta A_0(T),\\
\eta^*(t) &= (N(x,t-) - N(x,t))\, P^*(t,T)\mathcal{S}^Q(x,t,T)\Delta A_0(T).
\end{aligned}$$

The risk-minimizing strategy has the following interpretation: The number of bonds held at time t is equal to the Q-expected number of bonds needed in order to cover the benefits at time T, conditional on the information available at time $t-$. The investments in the savings account only differ from 0 if a death occurs at time t, and in this case it consists of a withdrawal (loan) equal to the market reserve for one insured individual who is alive.

□

4.8 Mean-variance indifference pricing

Methods developed for incomplete markets have been applied for the handling of the combined risk inherent in a life insurance contract in Møller (2001b, 2002, 2003a, 2003b) with focus on the mean-variance indifference pricing principles of Schweizer (2001b). In this section, these results are reviewed and indifference prices and optimal hedging strategies are derived.

4.8.1 A review of mean-variance indifference pricing

Denote by K^* the discounted wealth of the insurer at time T and consider the mean-variance utility-functions

$$u_i(K^*) = E^P[K^*] - a_i(\text{Var}^P[K^*])^{\beta_i}, \tag{4.8.1}$$

$i = 1, 2$, where $a_i > 0$ are so-called risk-loading parameters and where we take $\beta_1 = 1$ and $\beta_2 = 1/2$. It can be shown that the equations $u_i(K^*) = u_i(0)$ indeed lead to the classical actuarial variance (i=1) and standard deviation principle (i=2), respectively, see e.g. Møller (2001b).

Schweizer (2001b) proposes to apply the mean-variance utility functions (4.8.1) in an indifference argument which takes into consideration the possibilities for trading in the financial markets. Denote by Θ the space of admissible strategies and let $G_T(\Theta)$ be the space of discounted trading gains, i.e. random variables of the form $\int_0^T \xi(u)dX(u)$, where X is the price process associated with the discounted traded asset. Denote by c the insurer's initial capital at time 0. The u_i-indifference price v_i associated with the liability H is defined via

$$\sup_{\vartheta\in\Theta} u_i\left(c + v_i + \int_0^T \vartheta(u)dX(u) - H^*\right) = \sup_{\widetilde{\vartheta}\in\Theta} u_i\left(c + \int_0^T \widetilde{\vartheta}(u)dX(u)\right), \tag{4.8.2}$$

where H^* is the discounted liability. The strategy ϑ^* which maximizes the left side of (4.8.2) will be called the optimal strategy for H. In order to formulate the main result, some more notation is needed. We denote by $\widetilde{P}$ the variance optimal martingale measure and let $\widetilde{\Lambda}(T) = \frac{d\widetilde{P}}{dP}$. In addition, we let $\pi(\cdot)$ be the projection in $L^2(P)$ on the space $G_T(\Theta)^\perp$ and write $1 - \pi(1) = \int_0^T \widetilde{\beta}(u)dX(u)$. It follows via the projection theorem that any discounted liability H^* allows for a unique decomposition on the form

$$H^* = c^H + \int_0^T \vartheta^H(u)dX(u) + N^H, \tag{4.8.3}$$

where $\int_0^T \vartheta^H dX$ is an element of $G_T(\Theta)$ and where N^H is in the space $(\mathbb{R} + G_T(\Theta))^\perp$.

From Schweizer (2001b) and Møller (2001b) we have that the indifference prices for H are:

$$v_1(H) = E^{\widetilde{P}}[H^*] + a_1 \text{Var}^P[N^H], \tag{4.8.4}$$

$$v_2(H) = E^{\widetilde{P}}[H^*] + a_2\sqrt{1 - \text{Var}^P[\widetilde{\Lambda}(T)]/a_2^2}\sqrt{\text{Var}^P[N^H]}, \tag{4.8.5}$$

where (4.8.5) is only defined if $a_2^2 \geq \text{Var}^P[\widetilde{\Lambda}(T)]$. The optimal strategies associated with these two principles are:

$$\vartheta_1^*(t) = \vartheta^H(t) + \frac{1 + \text{Var}^P[\widetilde{\Lambda}(T)]}{2a_1}\widetilde{\beta}(t), \tag{4.8.6}$$

$$\vartheta_2^*(t) = \vartheta^H(t) + \frac{1 + \text{Var}^P[\widetilde{\Lambda}(T)]}{a_2\sqrt{1 - \text{Var}^P[\widetilde{\Lambda}(T)]/a_2^2}}\sqrt{\text{Var}^P[N^H]}\widetilde{\beta}(t), \tag{4.8.7}$$

where (4.8.7) is only well-defined if $a_2^2 > \mathrm{Var}^P[\widetilde{\Lambda}(T)]$. For more details, see Møller (2001b, 2003a, 2003b).

4.8.2 The variance optimal martingale measure

In order to determine the variance optimal martingale measure $\widetilde{P}$ we first turn our attention to the minimal martingale measure, which loosely speaking is "the equivalent martingale measure which disturbs the structure of the model as little as possible", see Schweizer (1995). The minimal martingale measure is obtained by letting $h^\mu = 0$ and $g = 0$. Hence, we have from Section 4.4 that the Radon-Nikodym derivative $\widehat{\Lambda}(T)$ for the minimal martingale measure is given by

$$\widehat{\Lambda}(T) = \exp\left(\int_0^T h^r(u)dW^r(u) - \frac{1}{2}\int_0^T (h^r(u))^2 du\right),$$

where h^r is defined by (4.4.6).

In general, the variance optimal martingale measure $\widetilde{P}$ and the minimal martingale measure $\hat{P}$ differ. However, we find below that they coincide in our model. Since $h^r(t)$ is $\mathcal{G}(t)$-measurable, the density $\widehat{\Lambda}(T)$ is $\mathcal{G}(T)$-measurable, and therefore it can be represented by a constant D and a stochastic integral with respect to $P^*(\cdot, T)$, see e.g. Pham, Rheinländer and Schweizer (1998, Section 4.3). Thus, we have the following representation of $\widehat{\Lambda}(T)$

$$\widehat{\Lambda}(T) = D + \int_0^T \widetilde{\zeta}(u)dP^*(u,T). \tag{4.8.8}$$

Schweizer (1996, Lemma 1) gives that $\widehat{\Lambda}(T)$ is the density for the variance optimal martingale measure as well, i.e.

$$\frac{d\widetilde{P}}{dP} = \widehat{\Lambda}(T),$$

such that $\hat{P} = \widetilde{P}$. Hence, under the equivalent martingale measure $\widetilde{P}$, the dynamics of the mortality intensity and the intensity of the counting process $N(x)$ are unaltered. For later use, we introduce the $\widetilde{P}$-martingale

$$\widetilde{\Lambda}(t) := E^{\widetilde{P}}[\widehat{\Lambda}(T)|\mathcal{F}(t)] = E^{\widetilde{P}}[\widehat{\Lambda}(T)|\mathcal{G}(t)].$$

Note that $\widehat{\Lambda}(T) = \widetilde{\Lambda}(T)$. If h^r is constant, calculations similar to those in Møller (2003b) for the Black-Scholes case give that

$$\frac{\widehat{\Lambda}(t)}{\widetilde{\Lambda}(t)} = e^{-(h^r)^2(T-t)}.$$

4.8.3 Mean-variance indifference pricing for pure endowments

Let $\overline{T} = T$ and consider a portfolio of n individuals of the same age x each purchasing a pure endowment of $\Delta A_0(T)$ paid by a single premium at time 0. Thus, the discounted liability is given by

$$H^* = (n - N(x,T))B(T)^{-1}\Delta A_0(T).$$

Explicit expressions for the mean-variance indifference prices can be obtained under additional integrability conditions. More precisely, we need that certain local $\widetilde{P}$-martingales considered in the calculation of $\mathrm{Var}^P[N^H]$ are (true) $\widetilde{P}$-martingales. In this case we have the following proposition.

Proposition 4.8.1
The indifference prices are given by inserting the following expressions for $E^{\widetilde{P}}[H^*]$ and $\mathrm{Var}^P(N^H)$ in (4.8.4) and (4.8.5):

$$E^{\widetilde{P}}[H^*] = nP(0,T)\mathcal{S}(x,0,T)\Delta A_0(T), \tag{4.8.9}$$

and

$$\mathrm{Var}^P[N^H] = n\int_0^T \Upsilon_1(t)\Upsilon_2(t)dt + n^2\int_0^T \Upsilon_1(t)\Upsilon_3(t)dt, \tag{4.8.10}$$

where

$$\Upsilon_1(t) = E^P\left[\frac{\widehat{\Lambda}(t)}{\widetilde{\Lambda}(t)}(P^*(t,T)\Delta A_0(T))^2\right],$$
$$\Upsilon_2(t) = E^P\left[(\mathcal{S}(x,t,T))^2 e^{-\int_0^t \mu(x,u)du}\mu(x,t)\left(1 + (\sigma^\mu(x,t)B^\mu(x,t,T))^2(1 - e^{-\int_0^t \mu(x,u)du})\right)\right],$$
$$\Upsilon_3(t) = E^P\left[\left(\sigma^\mu(x,t)\sqrt{\mu(x,t)}B^\mu(x,t,T)\mathcal{S}(x,t,T)e^{-\int_0^t \mu(x,u)du}\right)^2\right].$$

Idea of proof of Proposition 4.8.1: The independence between r and (N,μ) under $\widetilde{P}$ immediately gives (4.8.9). The expression for the variance of N^H in (4.8.10) follows from calculations similar to those in Møller (2001b). For completeness the calculations are carried out in Section 4.10.2 under certain additional integrability conditions.

□

We see from (4.8.10) that the variance of N^H can be split into two terms. The first term, which is proportional to the number of insured, stems from both the systematic and unsystematic mortality risk. Møller (2001b) also obtained a term proportional to the number of insured in the case with deterministic mortality intensity and hence only unsystematic mortality risk. The second term which is proportional to the squared number of survivors stems solely from the systematic mortality risk. Hence, the uncertainty associated with the future mortality intensity becomes increasingly important, when determining indifference prices for a portfolio of pure endowments, as the size of the portfolio increases.

There are two reasons for this. Firstly, changes in the mortality intensity, as opposed to the randomness associated with the deaths within the portfolio, are non-diversifiable; in particular they affect all insured individuals in the same way. Secondly, this risk is not hedgeable in the market.

Proposition 4.8.2
The optimal strategies are given by inserting (4.8.10) and the following expression for ϑ^H *in (4.8.6) and (4.8.7):*

$$\vartheta^H(t) = \xi^{\widetilde{P}}(t) - \widetilde{\zeta}(t) \int_0^{t-} \frac{1}{\widetilde{\Lambda}(u)} \left(\nu^{\widetilde{P}}(u) dM(x,u) + \kappa^{\widetilde{P}}(u) d\mathcal{S}^M(x,u,T) \right), \tag{4.8.11}$$

where

$$\begin{aligned} \xi^{\widetilde{P}}(t) &= (n - N(x,t-))\mathcal{S}(x,t,T)\Delta A_0(T), \\ \nu^{\widetilde{P}}(t) &= -P^*(t,T)\mathcal{S}(x,t,T)\Delta A_0(T), && (4.8.12) \\ \kappa^{\widetilde{P}}(t) &= (n - N(x,t-))P^*(t,T)e^{\int_0^t \mu(x,u)du}\Delta A_0(T). && (4.8.13) \end{aligned}$$

Proof of Proposition 4.8.2: Expression (4.8.11) follows from Schweizer (2001a, Theorem 4.6) (Theorem 4.10.1), which relates the decomposition in (4.8.3) to the Galtchouk-Kunita-Watanabe decomposition of the $\widetilde{P}$-martingale $V^{*,\widetilde{P}}(t) = E^{\widetilde{P}}[H^*|\mathcal{F}(t)]$ given in Example 4.7.3.

□

4.8.4 Mean-variance hedging

We now briefly mention the principle of mean-variance hedging used for hedging and pricing in incomplete financial markets. This short review follows a similar review in Møller (2001b). With mean-variance hedging, the aim is to determine the self-financing strategy $\hat{\varphi} = (\hat{\vartheta}, \hat{\eta})$ which minimizes

$$E^P\left[(H^* - V(T,\varphi))^2\right].$$

The main idea is thus to approximate the discounted claim H^* as closely as possible in $L^2(P)$ by the discounted terminal value of a self-financing portfolio φ. Since we consider self-financing portfolios only, the optimal portfolio is uniquely determined by the pair $(V(0,\hat{\varphi}), \hat{\vartheta})$, where $V(0,\hat{\varphi})$ is known as the approximation price for H and $\hat{\vartheta}$ is the mean-variance optimal hedging strategy. Schweizer (2001a, Theorem 4.6) gives that $V(0,\hat{\varphi}) = E^{\widetilde{P}}[H^*]$ and $\hat{\vartheta} = \vartheta^H$. Thus, we recognize the approximation price and the mean-variance hedging strategy as the first part of the mean-variance indifference prices and optimal hedging strategies, respectively. Note that even though the minimization criterion is in terms of P, the solution is given (partly) in terms of $\widetilde{P}$.

4.9 Numerical examples

In this section, we present some numerical examples with calculations of the market reserves of Proposition 4.6.2. Furthermore, we investigate two different parameterizations within the class of time-inhomogeneous CIR models and compare these to the 2003 mortality intensities and a deterministic projection for the mortality intensities.

Calculation method
A useful way of evaluating the expression (4.6.13) is to define auxiliary functions

$$\begin{aligned}\widehat{V}^Q(t,t') = &\int_t^T e^{-\int_t^\tau (f^r(t',u)+f^{\mu,Q}(x,t',u))du}\left(a_0(\tau)+a_1(\tau)f^{\mu,Q}(x,t',\tau)\right)d\tau \\ &+ e^{-\int_t^{\overline{T}} (f^r(t',u)+f^{\mu,Q}(x,t',u))du}\Delta A_0(\overline{T}),\end{aligned} \tag{4.9.1}$$

where the zero coupon bond price and the Q-survival probabilities are expressed in terms of the relevant forward rates and Q-forward mortality intensities. Note moreover, that we have introduced the additional parameter t'. We note that $V^Q(t,r(t),\mu(x,t)) = \widehat{V}^Q(t,t)$, whereas these two quantities differ if $t \neq t'$. It follows immediately, that on the set $(0,\overline{T}) \cup (\overline{T},T)$, $\widehat{V}^Q(t,t')$ satisfies for fixed t' the differential equation

$$\frac{\partial}{\partial t}\widehat{V}^Q(t,t') = (f^r(t',t)+f^{\mu,Q}(x,t',t))\widehat{V}^Q(t,t') - a_0(t) - a_1(t)f^{\mu,Q}(x,t',t), \tag{4.9.2}$$

subject to the terminal condition $\widehat{V}^Q(T,t') = 0$ and with

$$\widehat{V}^Q(\overline{T}-,t') = \Delta A_0(\overline{T}) + \widehat{V}^Q(\overline{T},t').$$

Alternatively, the expression (4.6.13) can be determined by solving the following partial differential equation on $(0,\overline{T}) \cup (\overline{T},T) \times I\!R \times I\!R_+$

$$\begin{aligned}0 = &\frac{\partial}{\partial t}V^Q(t,r,\mu) + (\gamma^{\mu,Q}(x,t) - \delta^{\mu,Q}(x,t)\mu)\frac{\partial}{\partial \mu}V^Q(t,r,\mu) + \frac{1}{2}(\sigma^\mu(x,t))^2\mu\frac{\partial^2}{\partial \mu^2}V^Q(t,r,\mu) \\ &+ \left(\gamma^{r,\alpha,Q} - \delta^{r,\alpha,Q}r\right)\frac{\partial}{\partial r}V^Q(t,r,\mu) + \frac{1}{2}(\gamma^{r,\sigma}+\delta^{r,\sigma}r)\frac{\partial^2}{\partial r^2}V^Q(t,r,\mu) - rV^Q(t,r,\mu) \\ &+ a_0(t) + (1+g(t))\mu(a_1(t) - V^Q(t,r,\mu)),\end{aligned}$$

with terminal condition $V^Q(T,r,\mu) = 0$ and with

$$V^Q(\overline{T}-,r,\mu) = \Delta A_0(\overline{T}) + V^Q(\overline{T},r,\mu).$$

The partial differential equation follows either as a byproduct from the proof of Lemma 4.7.1 in Section 4.10.1 or as a special case of the generalized Thiele's differential equation in Steffensen (2000). A similar partial differential equation can be found in Chapter 3.

Parameters for financial market
We now present the parameters which will be used in the numerical examples. The financial market will be described via a standard Vasiček model with parameters $\gamma^{r,\alpha} = 0.008$, $\delta^{r,\alpha} = 0.2$, $\gamma^{r,\sigma} = 0.0001$, $\delta^{r,\sigma} = 0$, $\widetilde{c} = -0.003$ and $r_0 = 0.025$. Given these

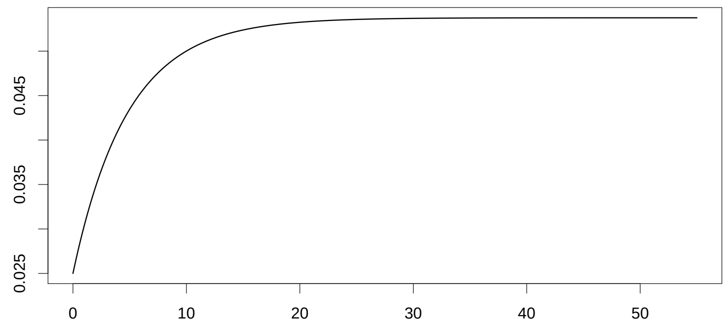

Figure 4.9.1: *Forward rate curve for the Vasiček model.*

parameters, the mean reversion level for the short rate is $\gamma^{r,\alpha}/\delta^{r,\alpha} = 0.04$ under P and $(\gamma^{r,\alpha} - \widetilde{c})/\delta^{r,\alpha} = 0.055$ under Q. The short rate volatility is given by $\sqrt{\gamma^{r,\sigma}} = 0.01$ and the speed of mean reversion is $\delta^{r,\alpha} = 0.2$. The parameter $-\widetilde{c}/\delta^{r,\alpha} = 0.015$ can be interpreted as the typical difference between the long and short term zero coupon yield, see Poulsen (2003) for more details. The initial short rate is given by $r_0 = 0.025$, which corresponds to the present short rate level. The forward rate curve $f^r(0,\tau)$ can be found in Figure 4.9.1.

Parameters for insurance portfolio
We have fitted the parameters for the underlying Gompertz-Makeham distributions at various time. In Table 4.9.1 below, we present the numbers for 1980 and 2003 which have been obtained by standard methods. We now list some parameters for the under-

	Males			Females		
Year	α	β	c	α	β	c
1980	0.000233	0.0000658	1.0959	0.000220	0.0000197	1.1063
2003	0.000134	0.0000353	1.1020	0.000080	0.0000163	1.1074

Table 4.9.1: *Estimated Gompertz-Makeham parameters for 1980 and 2003.*

lying mortality improvement process ζ defined by (4.3.6), which is supposed to capture the variation present in Figure 4.2.4 from Section 4.2. We consider two different parameterizations, Case I and Case II, see Table 4.9.2. Case I: We take $\delta(x,t) = \widetilde{\delta}$ constant

	$\delta(x,t)$	$\gamma(x,t)$	$\sigma(x,t)$
Case I	$\widetilde{\delta}$	$\widetilde{\delta}e^{-\widetilde{\gamma}t}$	$\widetilde{\sigma}$
Case II	$\widetilde{\gamma}$	$\frac{1}{2}\widetilde{\sigma}^2$	$\widetilde{\sigma}$

Table 4.9.2: *Parametrization for the underlying process ζ.*

and assume that $\gamma(x,t) = \widetilde{\delta}e^{-\widetilde{\gamma}t}$, where $\log(1+\widetilde{\gamma})$ represents the expected yearly relative decline in the mortality intensity. Thus, $e^{-\widetilde{\gamma}t}$ is the (time-dependent) level to which the process ζ adapts and $\widetilde{\delta}$ controls how fast it adapts to this level. Finally, we propose to let $\sigma(x,t) = \widetilde{\sigma}$, which describes the "noise". Case II: Here, we let $\delta(x,t) = \widetilde{\gamma}$, $\gamma(x,t) = \frac{1}{2}\widetilde{\sigma}^2$ and $\sigma(x,t) = \widetilde{\sigma}$. This means that we expect a relative yearly decline in ζ of approximately $\widetilde{\gamma}$. Note that Case II has one parameter less than case I. The choice $\gamma(x,t) = \frac{1}{2}\widetilde{\sigma}^2$ in Case II ensures that ζ remains strictly positive. Quantiles for the mortality improvement

	$\widetilde{\delta}$	$\widetilde{\gamma}$	$\widetilde{\sigma}$	5%	25%	50%	75%	95%
Case I	0.2	0.008	0.02	0.838	0.867	0.887	0.907	0.937
	1	0.008	0.02	0.837	0.850	0.859	0.868	0.881
	0.2	0.008	0.03	0.814	0.856	0.886	0.917	0.962
	1	0.008	0.03	0.827	0.846	0.859	0.872	0.892
Case II		0.008	0.02	0.726	0.801	0.854	0.909	0.990

Table 4.9.3: *Quantiles for the mortality improvement process ζ for time horizon 20 years. (Numbers are based on 100000 simulations with 100 steps per year in an Euler scheme. The results are indistinguishable if we altenatively use a Milstein scheme).*

process ζ for the two parameterizations can be found in Table 4.9.3. In Case II, the mean reversion level is $\gamma(t,x)/\delta(t,x) = \frac{1}{2}\widetilde{\sigma}^2/\widetilde{\gamma}$. With $\widetilde{\gamma} = 0.008$ and $\widetilde{\sigma} = 0.02$, this leads to the mean reversion level of 0.025, which is negligible. A comparison of the mortality for 2003

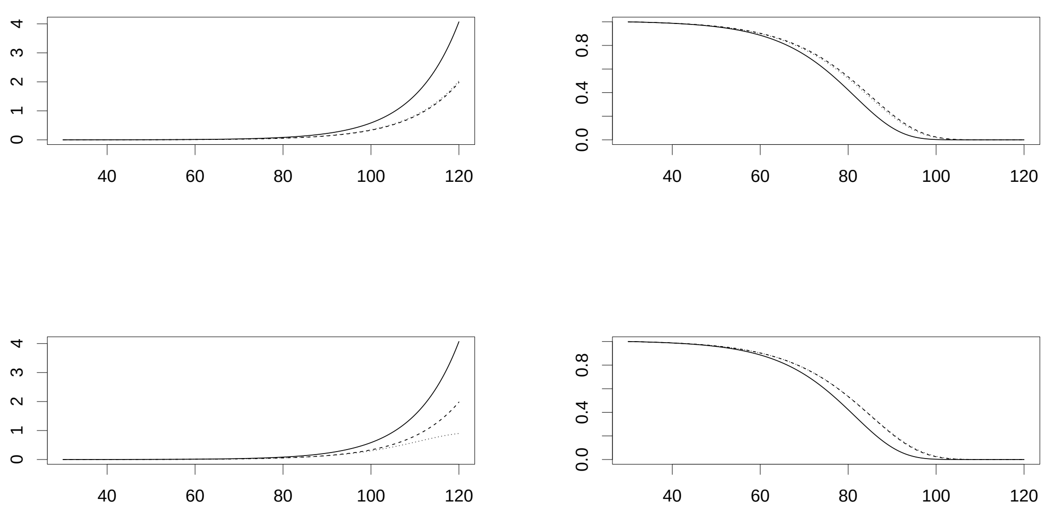

Figure 4.9.2: *Top pictures are Case I and bottom pictures are Case II. To the left: Mortality intensity curve for 30 year old males for 2003 (solid line), exponentially corrected with factor* $\exp(-\widetilde{\gamma}t)$ *(dashed line) and forward mortality intensities (dotted line). To the right: The corresponding survival probabilities.*

for males and the corresponding forward mortality intensities in Case I with parameters $(\widetilde{\delta}, \widetilde{\gamma}, \widetilde{\sigma}) = (0.2, 0.008, 0.02)$ can be found at the top of Figure 4.9.2. The figure shows a rather limited difference between the forward mortality intensities and the exponentially corrected intensities (they essentially coincide), whereas there is a big difference between these two curves and the 2003 estimate for the mortality intensities. For Case II, there is a more substantial difference between the forward mortality intensities and the exponentially corrected mortality intensities at very high ages.

Expected lifetimes
Figure 4.9.3(a) shows the histogram for the expected lifetime of a policyholder aged 30 for case I with parameters $(0.2, 0.008, 0.03)$. As a comparison, the expected lifetime for a male policyholder aged 30 is 75.8, 79.0 and 78.6 if we use the 2003 estimate, the exponentially corrected mortality and the forward mortality intensities, respectively. The variability in the figure reflects the uncertainty related to changes in the future mortality intensities. The histogram shows that there is a relatively small uncertainty associated with the expected lifetime in Case I. This is explained by the fact that the model for the mortality improvement process is mean-reverting with a relatively small volatility. If we instead

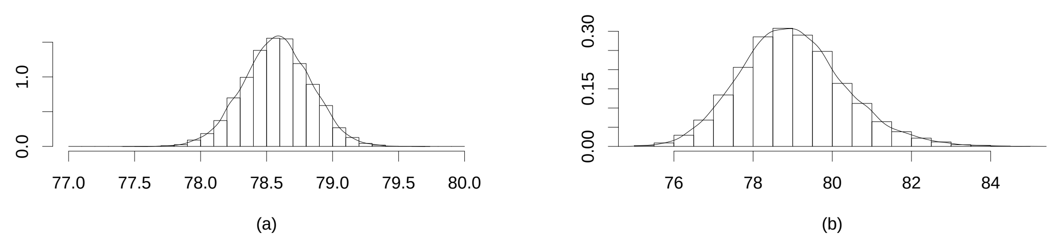

Figure 4.9.3: *Histograms for the expected lifetime for a policy-holder aged 30 for Case I with parameters* $(0.2, 0.008, 0.03)$ *(figure a) and Case II (figure b). (Histograms are based on 10000 simulations with 100 steps per year in an Euler scheme.)*

consider case II, the expected lifetime changes to 79.2, and we now get substantially bigger variation into the expected lifetimes, see the histogram for the expected lifetime in Figure 4.9.3(b).

Market reserves
In Figure 4.9.4, we have plotted the functions $\widehat{V}^Q(t, t'|x)$ for fixed $t = t' = 0$ as a function of age x in the case where Q is the minimal martingale measure. (Here, we have added an x to the function $\widehat{V}^Q$ in order to underline its dependence on the initial age x.) We have considered Case I with parameters $(0.2, 0.008, 0.03)$ and studied a life annuity starting at age 65. Moreover, we have compared this with the reserves obtained by using the 2003 estimate without any correction for future mortality improvements, and the mortality intensities obtained by reducing the mortality intensities exponentially. For each initial age x, we have calculated the relevant forward mortalities and solved the differential equation for $\widehat{V}^Q(t, t')$. We see only very little difference between the reserves obtained by using the forward mortality intensities and the exponentially corrected mortality intensities.

Risk-minimizing strategies and mean-variance indifference pricing
The risk-minimizing strategies and mean-variance indifference hedging strategies obtained

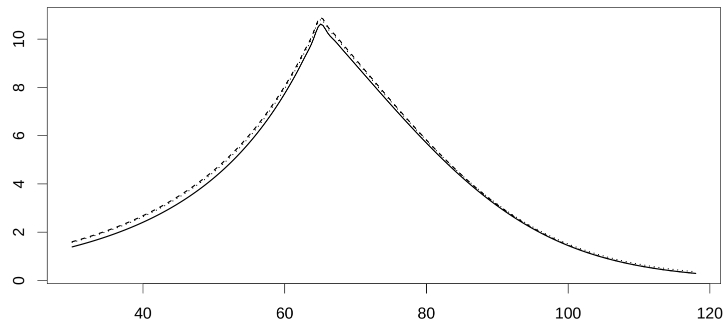

Figure 4.9.4: *Reserves for a life annuity starting at age 65. Reserve based on 2003-estimate for males (solid line), exponentially corrected mortality intensities (dashed line) and forward mortalities (dotted line).*

in Section 4.7 and 4.8 can also be determined numerically. Møller (2001b) contains a section with numerical examples for a similar contract (without systematic mortality risk), where the strategies have been determined for a couple of simulations. In addition, the methods listed there may be used for determining the mean-variance indifference prices of Proposition 4.8.1.

4.10 Proofs and technical calculations

4.10.1 Proof of Lemma 4.7.1

Recall from (4.7.7) that the Q-martingale $V^{*,Q}$ can be written as

$$V^{*,Q}(t) = A^*(t) + (n - N(x,t))B(t)^{-1}V^Q(t, r(t), \mu(x,t)).$$

Differentiating under the integral gives

$$\begin{aligned} \frac{\partial}{\partial r}V^Q(t,r,\mu) = &- \int_t^T B^r(t,\tau)P(t,\tau)\mathcal{S}^Q(x,t,\tau)\left(a_0(\tau) + a_1(\tau)f^{\mu,Q}(x,t,\tau)\right)d\tau \\ &- B^r(t,\overline{T})P(t,\overline{T})\mathcal{S}^Q(x,t,\overline{T})\Delta A_0(\overline{T}), \end{aligned} \tag{4.10.1}$$

and

$$\begin{aligned} \frac{\partial}{\partial \mu}V^Q(t,r,\mu) = -(1+g(t))\Bigg(&\int_t^T P(t,\tau)B^{\mu,Q}(x,t,\tau)\mathcal{S}^Q(x,t,\tau) \\ &\times \left(a_0(\tau) + a_1(\tau)\left(f^{\mu,Q}(x,t,\tau) - \frac{\frac{\partial}{\partial\tau}B^{\mu,Q}(x,t,\tau)}{B^{\mu,Q}(x,t,\tau)}\right)\right)d\tau \\ &+ P(t,\overline{T})B^{\mu,Q}(x,t,\overline{T})\mathcal{S}^Q(x,t,\overline{T})\Delta A_0(\overline{T})\Bigg), \end{aligned} \tag{4.10.2}$$

where we have used

$$\frac{\partial}{\partial \mu} f^{\mu,Q}(x,t,\tau) = (1+g(t))\frac{\partial}{\partial \tau} B^{\mu,Q}(x,t,\tau).$$

Integration by parts used on $(n - N(x,t))B(t)^{-1}V^Q(t, r(t), \mu(x,t))$ yields

$$\begin{aligned}
V^{*,Q}(t) &= A^*(t) + nV^Q(0, r(0), \mu(x,0)) \\
&\quad + \int_0^t (n - N(x,u))V^Q(u, r(u), \mu(x,u))dB(u)^{-1} \\
&\quad + \int_0^t B(u)^{-1}(n - N(x,u-))dV^Q(u, r(u), \mu(x,u)) \\
&\quad - \int_0^t B(u)^{-1}V^Q(u, r(u), \mu(x,u))dN(x,u).
\end{aligned} \tag{4.10.3}$$

In order to calculate the fourth term in (4.10.3), we need to find $dV^Q(u, r(u), \mu(x,u))$. Recall from (4.4.7) and (4.6.5) that the dynamics of r and $\mu(x)$ under Q are given by

$$\begin{aligned}
dr(t) &= \alpha^{r,Q}(r(t))dt + \sigma^r(t, r(t))dW^{r,Q}(t), \\
d\mu(x,t) &= \alpha^{\mu,Q}(t, \mu(x,t))dt + \sigma^\mu(t, \mu(x,t))\sqrt{\mu(x,t)}dW^{\mu,Q}(t),
\end{aligned}$$

where

$$\begin{aligned}
\alpha^{r,Q}(r(t)) &= \gamma^{r,\alpha,Q} - \delta^{r,\alpha,Q} r(t), \\
\alpha^{\mu,Q}(t, \mu(x,t)) &= \gamma^{\mu,Q}(x,t) - \delta^{\mu,Q}(x,t)\mu(x,t).
\end{aligned}$$

In the rest of the proof we use the shorthand notation $V^Q(u) = V^Q(u, r(u), \mu(x,u))$. Furthermore we only include explicitly the time argument in the coefficient functions. The assumption $V^Q(t) \in C^{1,2,2}$ allows us to apply Itô's formula. We obtain

$$\begin{aligned}
dV^Q(u) &= \Bigg(\frac{\partial}{\partial u}V^Q(u) + \alpha^{\mu,Q}(u)\frac{\partial}{\partial \mu}V^Q(u) + \frac{1}{2}(\sigma^\mu(u))^2\mu(x,u)\frac{\partial^2}{\partial \mu^2}V^Q(u) \\
&\quad + \alpha^{r,Q}(u)\frac{\partial}{\partial r}V^Q(u) + \frac{1}{2}(\sigma^r(u))^2\frac{\partial^2}{\partial r^2}V^Q(u)\Bigg)du + \sigma^r(u)\frac{\partial}{\partial r}V^Q(u)dW^{r,Q}(u) \\
&\quad + \sigma^\mu(u)\sqrt{\mu(x,u)}\frac{\partial}{\partial \mu}V^Q(u)dW^{\mu,Q}(u) \\
&= \Bigg(\frac{\partial}{\partial u}V^Q(u) + \alpha^{\mu,Q}(u)\frac{\partial}{\partial \mu}V^Q(u) + \frac{1}{2}(\sigma^\mu(u))^2\mu(x,u)\frac{\partial^2}{\partial \mu^2}V^Q(u) \\
&\quad + \alpha^{r,Q}(u)\frac{\partial}{\partial r}V^Q(u) + \frac{1}{2}(\sigma^r(u))^2\frac{\partial^2}{\partial r^2}V^Q(u)\Bigg)du - \frac{\frac{\partial}{\partial r}V^Q(u)}{B^r(u,T)P^*(u,T)}dP^*(u,T) \\
&\quad - \frac{\frac{\partial}{\partial \mu}V^Q(u)}{(1+g(u))B^{\mu,Q}(x,u,T)\mathcal{S}^{Q,M}(x,u,T)}d\mathcal{S}^{Q,M}(x,u,T).
\end{aligned}$$

In the first equality we have used the dynamics of r and $\mu(x)$ and that the Brownian motions $W^{r,Q}$ and $W^{\mu,Q}$ are independent, such that we do not get any mixed second order terms. In the second equality we use (4.10.1) and (4.10.2) together with the dynamics of

$\mathcal{S}^{Q,M}(x,\cdot,T)$ and $P^*(\cdot,T)$ given in (4.6.7) and (4.4.10), respectively. Rewriting A^* in terms of the Q-martingale $M^Q(x)$ we get

$$\begin{aligned} A^*(t) = -n\pi(0) &+ \int_0^t B(\tau)^{-1}\left(a_0(\tau)(n-N(x,\tau)) + a_1(\tau)(n-N(x,\tau-))\mu^Q(x,\tau)\right)d\tau \\ &+ \int_0^t B(\overline{T})^{-1}(n-N(x,\overline{T}))\Delta A_0(\overline{T})dI_{\{\tau\geq\overline{T}\}} + \int_0^t B(\tau)^{-1}a_1(\tau)dM^Q(x,\tau). \end{aligned}$$

Collecting the terms from (4.10.3) involving integrals with respect to $P^*(\cdot,T)$, $\mathcal{S}^{Q,M}(x,\cdot,T)$ and $M^Q(x)$, respectively, we get the last three terms in (4.7.8). Since these three terms and $V^{*,Q}$ are Q-martingales, the remaining terms constitute a Q-martingale as well. Since this process is continuous (hence predictable) and of finite variation, it is constant. Inserting $t=0$ we immediately get that $V^{*,Q}(0) = -n\pi(0) + nV^Q(0,r(0),\mu(x,0))$. Thus, we have proved the decomposition in (4.7.8).

4.10.2 Calculation of $\mathbf{Var}^P[N^H]$

The following theorem due to Schweizer (2001a, Theorem 4.6) relates the decomposition in (4.8.3) to the Galtchouk-Kunita-Watanabe decomposition of the $\widetilde{P}$-martingale $V^{*,\widetilde{P}}(t) = E^{\widetilde{P}}[H^*|\mathcal{F}(t)]$; see also Møller (2000).

Theorem 4.10.1
Assume that $H^* \in L^2(\mathcal{F}(T),P)$ and consider the Galtchouk-Kunita-Watanabe decomposition of $V^{*,\widetilde{P}}(t)$ given by

$$V^{*,\widetilde{P}}(t) = E^{\widetilde{P}}[H^*] + \int_0^t \xi^{\widetilde{P}}(u)dP^*(u,T) + L^{\widetilde{P}}(t), \quad 0\leq t\leq \overline{T}. \tag{4.10.4}$$

We can now express c^H, ϑ^H and N^H from (4.8.3) in terms of decomposition (4.10.4) by

$$\begin{aligned} c^H &= E^{\widetilde{P}}[H^*], \\ \vartheta^H(t) &= \xi^{\widetilde{P}}(t) - \widetilde{\zeta}(t)\int_0^{t-}\frac{1}{\widetilde{\Lambda}(u)}dL^{\widetilde{P}}(u), \\ N^H &= \widetilde{\Lambda}(\overline{T})\int_0^{\overline{T}}\frac{1}{\widetilde{\Lambda}(u)}dL^{\widetilde{P}}(u). \end{aligned}$$

Since

$$L^{\widetilde{P}}(t) = \int_0^t \nu^{\widetilde{P}}(u)dM(x,u) + \int_0^t \kappa^{\widetilde{P}}(u)d\mathcal{S}(x,u,T),$$

where $\nu^{\widetilde{P}}$ and $\kappa^{\widetilde{P}}$ are given by (4.8.12) and (4.8.13), respectively, Theorem 4.10.1 gives the following expression for N^H:

$$N^H = \widetilde{\Lambda}(T)\int_0^T\frac{1}{\widetilde{\Lambda}(t)}dL^{\widetilde{P}}(t) = \widetilde{\Lambda}(T)\int_0^T\frac{1}{\widetilde{\Lambda}(t)}\left(\nu^{\widetilde{P}}(t)dM(x,t) + \kappa^{\widetilde{P}}(t)d\mathcal{S}^M(x,t,T)\right).$$

Since $E^P[N^H] = 0$, we first note that

$$\begin{aligned}\mathrm{Var}^P[N^H] &= E^P[(N^H)^2] = E^{\widetilde{P}}\left[\widetilde{\Lambda}(T)\left(\widetilde{L}(T)+\widetilde{R}(T)\right)^2\right]\\ &= E^{\widetilde{P}}\left[\widetilde{\Lambda}(T)(\widetilde{L}(T))^2 + 2\widetilde{\Lambda}(T)\widetilde{L}(T)\widetilde{R}(T) + \widetilde{\Lambda}(T)(\widetilde{R}(T))^2\right], \qquad (4.10.5)\end{aligned}$$

where we have defined $\widetilde{L}(t) = \int_0^t \frac{\nu^{\widetilde{P}}(u)}{\widetilde{\Lambda}(u)} dM(x,u)$ and $\widetilde{R}(t) = \int_0^t \frac{\kappa^{\widetilde{P}}(u)}{\widetilde{\Lambda}(u)} d\mathcal{S}^M(x,u,T)$. The three terms appearing in (4.10.5) can be rewritten using Itô's formula. For the first term we get

$$\widetilde{\Lambda}(T)(\widetilde{L}(T))^2 = \int_0^T (\widetilde{L}(t-))^2 d\widetilde{\Lambda}(t) + 2\int_0^T \widetilde{\Lambda}(t)\widetilde{L}(t-)d\widetilde{L}(t) + \int_0^T \widetilde{\Lambda}(t)\left(\frac{\nu^{\widetilde{P}}(t)}{\widetilde{\Lambda}(t)}\right)^2 dN(x,t),$$

and for the last term we find that

$$\begin{aligned}\widetilde{\Lambda}(T)(\widetilde{R}(T))^2 &= \int_0^T \widetilde{R}(t)^2 d\widetilde{\Lambda}(t) + 2\int_0^T \widetilde{\Lambda}(t)\widetilde{R}(t)d\widetilde{R}(t) + \int_0^T \widetilde{\Lambda}(t)d\langle\widetilde{R}\rangle(t)\\ &= \int_0^T \widetilde{R}(t)^2 d\widetilde{\Lambda}(t) + 2\int_0^T \widetilde{\Lambda}(t)\widetilde{R}(t)d\widetilde{R}(t)\\ &\quad + \int_0^T \widetilde{\Lambda}(t)\left(\frac{\kappa^{\widetilde{P}}(t)}{\widetilde{\Lambda}(t)}\sigma^\mu(x,t)\sqrt{\mu(x,t)}B^\mu(x,t,T)\mathcal{S}^M(x,t,T)\right)^2 dt.\end{aligned}$$

The mixed term becomes

$$\begin{aligned}\widetilde{\Lambda}(T)\widetilde{R}(T)\widetilde{L}(T) &= \int_0^T \widetilde{\Lambda}(t)\widetilde{R}(t)d\widetilde{L}(t) + \int_0^T \widetilde{L}(t)\widetilde{R}(t)d\widetilde{\Lambda}(t) + \int_0^T \widetilde{\Lambda}(t)\widetilde{L}(t)d\widetilde{R}(t)\\ &\quad + \int_0^T \widetilde{L}(t)d[\widetilde{R},\widetilde{\Lambda}](t).\end{aligned}$$

Assuming all the local martingales are martingales, and using that the Brownian motions driving r and μ are independent, we get

$$\begin{aligned}\mathrm{Var}^P\left[N^H\right] &= E^{\widetilde{P}}\left[\int_0^T \frac{(\nu^{\widetilde{P}}(t))^2}{\widetilde{\Lambda}(t)} dN(x,t)\right]\\ &\quad + E^{\widetilde{P}}\left[\int_0^T \frac{\left(\kappa^{\widetilde{P}}(t)\sigma^\mu(x,t)\sqrt{\mu(x,t)}B^\mu(x,t,T)\mathcal{S}^M(x,t,T)\right)^2}{\widetilde{\Lambda}(t)} dt\right]. \qquad (4.10.6)\end{aligned}$$

We now investigate the two terms in (4.10.6) separately. The first term can be rewritten as

$$\begin{aligned}&E^{\widetilde{P}}\left[\int_0^T \frac{(\nu^{\widetilde{P}}(t))^2}{\widetilde{\Lambda}(t)} dN(x,t)\right]\\ &= \int_0^T E^{\widetilde{P}}\left[\frac{(P^*(t,T)\Delta A_0(T))^2}{\widetilde{\Lambda}(t)}\right] E^{\widetilde{P}}\left[(\mathcal{S}(x,t,T))^2\,(n - N(x,t-))\mu(x,t)\right] dt\\ &= \int_0^T E^P\left[\frac{\widehat{\Lambda}(t)}{\widetilde{\Lambda}(t)}\,(P^*(t,T)\Delta A_0(T))^2\right] E^P\left[(\mathcal{S}(x,t,T))^2\,(n - N(x,t-))\mu(x,t)\right] dt,\end{aligned}$$

where we have used the expression for $\nu^{\widetilde{P}}(t)$ from (4.8.12) and the independence between r and (N,μ). The second term is given by

$$
\begin{aligned}
& E^{\widetilde{P}}\left[\int_0^T \frac{\left(\kappa^{\widetilde{P}}(t)\sigma^\mu(x,t)\sqrt{\mu(x,t)}B^\mu(x,t,T)\mathcal{S}^M(x,t,T)\right)^2}{\widetilde{\Lambda}(t)}dt\right] \\
&= E^{\widetilde{P}}\left[\int_0^T \frac{\left((n-N(x,t-))P^*(t,T)\sigma^\mu(x,t)\sqrt{\mu(x,t)}B^\mu(x,t,T)\mathcal{S}(x,t,T)\Delta A_0(T)\right)^2}{\widetilde{\Lambda}(t)}dt\right] \\
&= \int_0^T E^P\left[\frac{\Lambda(t)}{\widetilde{\Lambda}(t)}(P^*(t,T)\Delta A_0(T))^2\right] \\
&\qquad \times E^P\left[\left((n-N(x,t-))\sigma^\mu(x,t)\sqrt{\mu(x,t)}B^\mu(x,t,T)\mathcal{S}(x,t,T)\right)^2\right]dt,
\end{aligned}
$$

where we have used the expression for $\kappa^{\widetilde{P}}(t)$ from (4.8.13) and once again the independence between r and (N,μ). Using that conditioned on $\mathcal{I}(t)$ the number of survivors at time t is binomially distributed with parameters $(n, e^{-\int_0^t \mu(x,u)du})$ under P, we get

$$
\begin{aligned}
& E^P\left[\left((n-N(x,t-))\sigma^\mu(x,t)\sqrt{\mu(x,t)}B^\mu(x,t,T)\mathcal{S}(x,t,T)\right)^2\right] \\
&= E^P E^P\left[\left((n-N(x,t-))\sigma^\mu(x,t)\sqrt{\mu(x,t)}B^\mu(x,t,T)\mathcal{S}(x,t,T)\right)^2\middle|\mathcal{I}(t)\right] \\
&= E^P\left[\left(\sigma^\mu(x,t)\sqrt{\mu(x,t)}B^\mu(x,t,T)\mathcal{S}(x,t,T)\right)^2 E^P\left[(n-N(x,t-))^2\middle|\mathcal{I}(t)\right]\right] \\
&= E^P\left[\left(\sigma^\mu(x,t)\sqrt{\mu(x,t)}B^\mu(x,t,T)\mathcal{S}(x,t,T)\right)^2\right. \\
&\qquad \left.\times\left(ne^{-\int_0^t \mu(x,u)du}\left(1-e^{-\int_0^t \mu(x,u)du}\right)+n^2\left(e^{-\int_0^t \mu(x,u)du}\right)^2\right)\right],
\end{aligned}
$$

and

$$
\begin{aligned}
E^P\left[(\mathcal{S}(x,t,T))^2(n-N(x,t-))\mu(x,t)\right] &= E^P E^P\left[(\mathcal{S}(x,t,T))^2(n-N(x,t-))\mu(x,t)\middle|\mathcal{I}(t)\right] \\
&= E^P\left[(\mathcal{S}(x,t,T))^2\mu(x,t)E^P\left[n-N(x,t-)\middle|\mathcal{I}(t)\right]\right] \\
&= nE^P\left[(\mathcal{S}(x,t,T))^2\mu(x,t)e^{-\int_0^t \mu(x,u)du}\right].
\end{aligned}
$$

Collecting the terms proportional to n and n^2, respectively, we arrive at (4.8.10).

Chapter 5

A Discrete-Time Model for Reinvestment Risk in Bond Markets

(This chapter is an adapted version of Dahl (2005b))

In this chapter we propose a discrete-time model with fixed maximum time *to* maturity of traded bonds. At each trading time, a bond matures and a new bond is introduced in the market, such that the number of traded bonds is constant. The entry price of the newly issued bond depends on the prices of the bonds already traded and a stochastic term independent of the existing bond prices. Hence, we obtain a bond market model for the reinvestment risk, which is present in practice, when hedging long term contracts. In order to determine optimal hedging strategies we consider the criteria of super-replication and risk-minimization.

5.1 Introduction

In the literature, bond markets are usually assumed to include all bonds with time of maturity less than or equal to time of maturity of the considered claim. However, this is in contrast to practice, where only bonds with a limited (sufficiently short) time to maturity are traded. Hence, a standard model is the correct framework for pricing and hedging so-called short term contracts, where the payoff depends on bonds with time to maturity less than or equal to the longest traded bond. However, when considering long term contracts, i.e. contracts, whose payoffs depend on bonds with longer time to maturity than the longest traded bond, the bond market does not in general include bonds, which at all times allow for a perfect hedge of the contract. Thus, in practice, an agent interested in pricing and hedging a long term contract, such as a life insurance contract, where the payments may be due 50 years or more into the future, is in general not able to eliminate the reinvestment risk associated with the contract. Since the reinvestment risk is ignored in standard bond market models, they do not seem to be the right framework for pricing and hedging long term contracts. Here, we propose a model, which behaves similarly to a standard model when hedging and pricing short term contracts, and at the same time it includes reinvestment risk, when hedging long term contracts.

A first idea in order to introduce reinvestment risk would be to consider a standard binomial model for the bond prices and restrict the investment strategies to bonds with a limited time to maturity only. However, this simple approach does not introduce reinvestment risk, since a long term bond can be perfectly replicated by a dynamic trading strategy, where we at all times invest in two short term bonds. Hence, we have to extend the standard model to include an additional unhedgeable stochastic term, whose uncertainty determines the reinvestment risk.

To describe the reinvestment risk, we propose a discrete-time bond market model, where the traded bonds have a fixed maximum time *to* maturity, $\widetilde{T}$. Hence, at time 0 all bonds with time of maturity v, $v \in \{1, \dots, \widetilde{T}\}$ are traded. At any time t, the bond with maturity t matures and a new bond with time to maturity $\widetilde{T}$ is introduced in the market. Thus, after the issue of the new bond, the model is similar to the one at time 0. At any time t, the entry price of the new bond depends on all past information, current prices of bonds already traded and an additional stochastic term. In this model the class of attainable claims depends on time. Hence, a claim which is unattainable at time t may be attainable at time $t+1$. Consider for example a claim of 1 at time $t+\widetilde{T}+1$, which is unattainable at time t, whereas it is clearly attainable at time $t+1$, where a bond with time of maturity $t+\widetilde{T}+1$ is issued. At time $t+1$ the unique arbitrage free price is equal to the price of the bond with maturity $t+\widetilde{T}+1$, and the replicating strategy consists of purchasing exactly one such bond. The idea of fixing the maximum time to maturity of the traded assets and introducing new assets as time passes can also be found in Neuberger (1999), who considers a market for futures on oil prices. To model the initial price of the new future, Neuberger assumes that it is a linear function of the prices of traded futures and a normally distributed error term.

To the author's knowledge, the only other papers to consider the problem of modelling the prices of newly issued bonds are Sommer (1997) and Dahl (2005a) (see Chapter 6), who both consider models in continuous time. In Sommer (1997), new bonds are issued continuously, whereas in Dahl (2005a) new bonds are issued at fixed times only, since this is the case in practice. In order to control and quantify the reinvestment risk, both authors consider the criterion of risk-minimization.

The chapter is organized as follows: In Section 5.2, a bond market model including reinvestment risk is introduced. This is done in two steps: First we describe a complete and arbitrage free standard bond market model. Then we extend the model to include reinvestment risk. Since the extended model is incomplete, there exist infinitely many equivalent martingale measures. We identify the equivalent martingale measures for the extended model and define the considered price processes, which for notational convenience are different from the bond prices. Given the considered price processes we review the relevant financial terminology. Optimal hedging strategies with respect to the criteria of super-replication and risk-minimization are determined in Section 5.3. Here, we also remark on the relationship between the criterion of super-replication and the maximal guarantees for which the shortfall risk can be eliminated. The chapter is concluded by a numerical illustration in Section 5.4. The numerical illustration includes a comparison with practice in Danish life insurance, where long term contracts are common.

5.2 A bond market model

Let $\widehat{T} \in \mathbb{N}$ be a fixed time horizon and $(\Omega, \mathcal{F}, P)$ a probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in\{0,1,\ldots,\widehat{T}\}}$ satisfying the usual condition of completeness, i.e. $\mathcal{F}_0$ contains all P-null sets.

5.2.1 A standard bond market model

Prior to the introduction of the bond market model with reinvestment risk, we now describe a standard discrete-time bond market model. For a thorough description of discrete-time bond market models we refer to Jarrow (1996).

Consider a bond market where trading takes place at times $t = 0, 1, \ldots, \widetilde{T}$, for a fixed time horizon $\widetilde{T} \in \mathbb{N}$, $\widetilde{T} < \widehat{T}$. At time t we assume that all zero coupon bonds with maturity $v = t, \ldots, \widetilde{T}$ are traded in the bond market. For $t \in \{0, \ldots, \widetilde{T}\}$ and $v \in \{t, \ldots, \widetilde{T}\}$ we denote by $P(t, v)$ the price at time t of a zero coupon bond maturing at time v. To avoid arbitrage we assume $P(t, v)$ is strictly positive and $P(t, t) = 1$ for all t. For non-negative interest rates $P(t, v)$ is a decreasing function of v for fixed t. An important quantity when modelling bond prices is the forward rate, $f_{t,v}$, contracted at time t for the period $[v, v+1]$ defined by

$$f_{t,v} = \frac{P(t,v)}{P(t,v+1)} - 1, \quad t \in \{0, \ldots, \widetilde{T}-1\} \text{ and } v \in \{t, \ldots, \widetilde{T}-1\}, \tag{5.2.1}$$

or, stated differently,

$$P(t,v) = \frac{1}{\prod_{i=t}^{v-1}(1+f_{t,i})}, \quad t \in \{0, \ldots, \widetilde{T}-1\} \text{ and } v \in \{t+1, \ldots, \widetilde{T}\}. \tag{5.2.2}$$

The forward rate $f_{t,v}$ can be interpreted as the riskfree interest rate contracted at time t for the interval $[v, v+1]$. Now introduce the short rate process $r = (r_t)_{t\in\{0,1,\ldots,\widetilde{T}-1\}}$ given by $r_t = f_{t,t}$. Since (5.2.1) and (5.2.2) establish a one-to-one correspondence between forward rates and bond prices, modelling the development of the bond prices and the forward rates is equivalent. As it is standard in the literature, we model the forward rates. Let $f_t = (r_t, f_{t,t+1}, \ldots, f_{t,\widetilde{T}-1})$ denote the $(\widetilde{T}-t)$-dimensional forward rate vector at time t. To model the development of the forward rate vector we assume

$$f_t = g_t(f_0, \ldots, f_{t-1}, \rho_t), \quad t \in \{1, \ldots, \widetilde{T}-1\}, \tag{5.2.3}$$

for some function $g_t : \mathbb{R}^{\widetilde{T}} \times \mathbb{R}^{\widetilde{T}-1} \times \cdots \times \mathbb{R}^{\widetilde{T}-(t-1)} \times \{u, d\} \mapsto \mathbb{R}^{\widetilde{T}-t}$ and an i.i.d. sequence $\rho_1, \ldots, \rho_{\widetilde{T}-1}$ of random variables with distribution $P(\rho_1 = u) = 1 - P(\rho_1 = d) = p$, $p \in (0, 1)$. A natural restriction would be to consider strictly positive forward rates, only. In this case we would have $g_t : \mathbb{R}_+^{\widetilde{T}} \times \mathbb{R}_+^{\widetilde{T}-1} \times \cdots \times \mathbb{R}_+^{\widetilde{T}-(t-1)} \times \{u, d\} \mapsto \mathbb{R}_+^{\widetilde{T}-t}$. We observe from (5.2.3) that contingent on the development of the forward rates until time $t-1$, the forward rate vector at time t takes one of two possible values: $g_t(f_0, \ldots, f_{t-1}, u)$ or $g_t(f_0, \ldots, f_{t-1}, d)$. If $f_t = g_t(f_0, \ldots, f_{t-1}, u)$ we say that the forward rates have moved *up*, and likewise, if $f_t = g_t(f_0, \ldots, f_{t-1}, d)$ we say they have moved *down*. We note from (5.2.2) that the bond prices move in the opposite direction of the forward rates. The development of the forward rates (and bond prices) can be represented by non-recombining binomial tree, see Figure 5.2.1 for a visualization of the first three possible values of the forward rates.

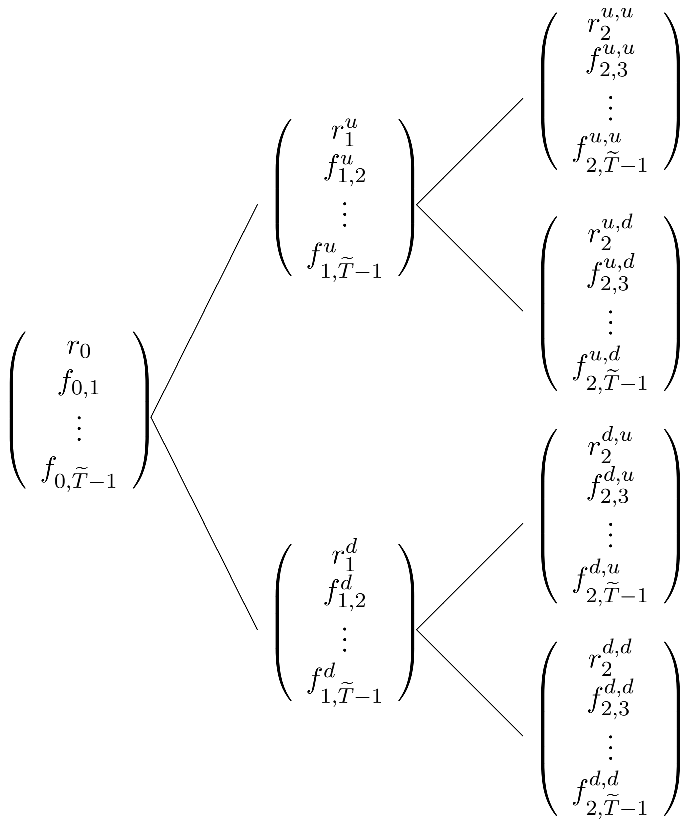

Figure 5.2.1: *Development of the forward rate vector.*

Remark 5.2.1 If g_t only depends on $(f_0, \ldots, f_{t-1})$ through f_{t-1}, then the forward rate vector is a discrete time-inhomogeneous Markov chain.

$\square$

The natural filtration $\mathbb{G} = (\mathcal{G}_t)_{t\in\{0,\ldots,\widetilde{T}\}}$ generated by the forward rates is given by

$$\mathcal{G}_0 = \{\emptyset, \Omega\} \text{ and } \mathcal{G}_t = \sigma\{\rho_1, \ldots, \rho_{t\wedge(\widetilde{T}-1)}\}, \ t \in \{1, \ldots, \widetilde{T}\}.$$

Introduce the notation ξ_t, $\xi_t \in \Xi_t = \{$all possible sequences of u's and d's of length $t\}$. This allows us to denote the generic value of for instance the forward rate vector at time t by $f_t^{\xi_t}$ and the forward rate vector at time $t+1$ given $\rho_{t+1} = d$ by $f_{t+1}^{\xi_t,d}$.

Risk-neutral probabilities

It is well known that the bond market model described above is arbitrage free if there exists a so-called equivalent martingale measure Q. Recall that an equivalent martingale measure is a probability measure equivalent to P, such that all discounted bond prices are martingales. The discounted bond prices are Q-martingales if for $t \in \{0, \ldots, \widetilde{T}-1\}$ and $v \in \{t+1, \ldots, \widetilde{T}\}$ it holds that

$$P(t, v) = \frac{1}{1+r_t} E^Q \left[P(t+1, v) | \, \mathcal{G}_t \right]. \tag{5.2.4}$$

If further the equivalent martingale measure Q is unique, the model is complete; see also Section 5.2.3 for the definition of arbitrage and completeness. Denote by $q_{t+1}^{\xi_t}$ the Q-probability of the event $\rho_{t+1} = u$ given the present information ξ_t. Since (5.2.4) is trivially fulfilled for $v = t+1$, we have $\widetilde{T} - (t+1)$ equations for $q_{t+1}^{\xi_t}$, $t \in \{0, \ldots, \widetilde{T}-2\}$. Thus, if a solution exists, it is unique, provided there exists a $v \in \{t+2, \ldots, \widetilde{T}\}$, such that $P^{\xi_t,d}(t+1, v) \neq P^{\xi_t,u}(t+1, v)$. For $t \in \{0, \ldots, \widetilde{T}-2\}$, solving (5.2.4) gives the following expressions for $q_{t+1}^{\xi_t}$:

$$q_{t+1}^{\xi_t} = \frac{P^{\xi_t,d}(t+1, v) - (1+r_t^{\xi_t})P^{\xi_t}(t, v)}{P^{\xi_t,d}(t+1, v) - P^{\xi_t,u}(t+1, v)}, \quad v \in \{t+2, \ldots, \widetilde{T}\}. \tag{5.2.5}$$

Here, we have used the notation $r_t^{\xi_t}$ and $P^{\xi_t}(t, v)$ to denote explicitly the dependence on the past. From (5.2.5) we observe that the Q-probability of $\rho_{t+1} = u$ depends on ξ_t and hence in general differs for different outcomes of $(\rho_1, \ldots, \rho_t)$. Furthermore, (5.2.5) gives that the Q-probability of an upward movement is small (large) if the difference $P^{\xi_t,d}(t+1, v) - (1+r_t^{\xi_t})P^{\xi_t}(t, v)$ is small (large) compared to the difference $P^{\xi_t,d}(t+1, v) - P^{\xi_t,u}(t+1, v)$. The measure Q given by (5.2.5) for all t ensures that all discounted bond price processes are Q-martingales. If further $q_{t+1}^{\xi_t} \in (0, 1)$ for all t and ξ_t, then P and Q are equivalent measures, such that Q indeed is an equivalent martingale measure. From (5.2.5) we get that Q and P are equivalent if for all $t \in \{0, \ldots, \widetilde{T}-2\}$ and ξ_t it holds that

$$P^{\xi_t,u}(t+1, v) < (1+r_t^{\xi_t})P^{\xi_t,u}(t, v) < P^{\xi_t,d}(t+1, v), \quad v \in \{t+2, \ldots, \widetilde{T}\}. \tag{5.2.6}$$

Here, we have used that $P^{\xi_t,u}(t+1, v) < P^{\xi_t,d}(t+1, v)$, since an upward movement of the forward rates corresponds to a downward move of the bond prices. Using (5.2.2) one can alternatively express (5.2.5) and (5.2.6) in terms of the forward rates. Condition (5.2.6) can be interpreted as follows: No bond with time to maturity larger than one must dominate or be dominated by the 1-period bond. If this was the case we could make arbitrage by trading in the particular bond and the 1-period bond.

5.2.2 A bond market model with reinvestment risk

Now, we extend the standard model in Section 5.2.1 to include reinvestment risk. The idea is as follows: Assume that at any time t only bonds with time to maturity less than or equal to $\widetilde{T}$ are traded, and the development of the bond prices from time t to $t+1$ can be described by a binomial model. Hence, at time t the one period development of the traded bonds is the same as in the standard model introduced in Section 5.2.1. At time $t+1$ the bond with maturity $t+1$ matures and a new bond with time *to* maturity $\widetilde{T}$ is issued, such that after the introduction of the new bond the model considered is similar to the one at time t. To model the reinvestment risk we assume that conditional on the past and the prices at time $t+1$ of the bonds traded at time t, the entry price at time $t+1$ of the new bond with time to maturity $\widetilde{T}$ can take two different values.

Consider a bond market where trading takes place at times $t = 0, 1, \ldots, \widehat{T}$. In this bond market not all zero coupon bonds with maturity less than or equal to $\widehat{T}$ are traded at all times $t = 0, 1, \ldots, \widehat{T}$. Instead we fix the maximum time to maturity, $\widetilde{T}$, for bonds traded in the market. Hence, the zero coupon bond prices $P(t, v)$ are defined for $t \in \{0, \ldots, \widehat{T}\}$ and $v \in \{t, \ldots, (t+\widetilde{T}) \wedge \widehat{T}\}$. In addition to $\widehat{T}$ and $\widetilde{T}$ we introduce the fixed time horizon $T \in \mathbb{N}$, which is the time of maturity of the considered contract. Figure 5.2.2 shows the possible orderings of T, $\widetilde{T}$ and $\widehat{T}$. Without loss of generality we assume that $\widehat{T} = T + \widetilde{T}$, such that

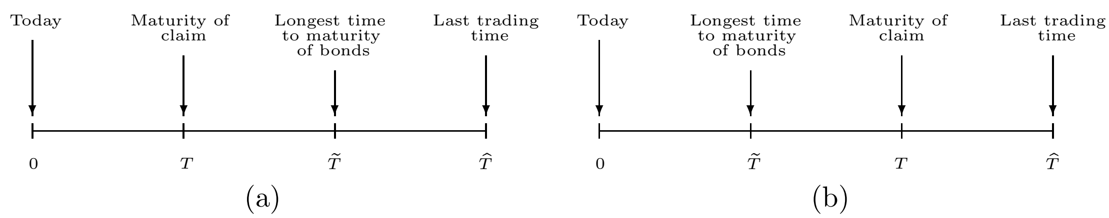

Figure 5.2.2: *Possible orderings of T, $\widetilde{T}$ and $\widehat{T}$. In case (a) all fixed claims with maturity T are attainable. In (b) they are unattainable.*

the bond market at all times t, $t \in \{0, \ldots, T\}$, includes the $\widetilde{T}$ bonds with time of maturity v, $v \in \{t+1, \ldots, t+\widetilde{T}\}$. From (5.2.1) we observe that the forward rates are defined for $t \in \{0, \ldots, \widehat{T}-1\}$ and $v \in \{t, \ldots, (t+\widetilde{T}-1) \wedge (\widehat{T}-1)\}$, so the forward rate vector at time t (which we still denote by f_t) is given by $f_t = (r_t, f_{t,t+1}, \ldots, f_{t,(t+\widetilde{T}-1)\wedge(\widehat{T}-1)})$. Define the $((u-t+1) \wedge \widetilde{T})$-dimensional vector $\widetilde{f}_{t,u}$ of forward rates defined at time t with time of maturity less than or equal to u. Now assume that

$$\widetilde{f}_{t,(t+\widetilde{T}-2)\wedge(\widehat{T}-1)} = \tilde{g}_t(f_0, \ldots, f_{t-1}, \rho_t), \tag{5.2.7}$$

where $\tilde{g}_t : \mathbb{R}^{\widetilde{T}} \times \mathbb{R}^{\widetilde{T}\wedge(\widehat{T}-1)} \times \ldots \times \mathbb{R}^{\widetilde{T}\wedge(\widehat{T}-(t-1))} \times \{u, d\} \mapsto \mathbb{R}$, and $\rho_1, \ldots, \rho_{\widehat{T}-1}$, similarly to Section 5.2.1, is an i.i.d. sequence of random variables with distribution $P(\rho_1 = u) = 1 - P(\rho_1 = d) = p$, $p \in (0,1)$. The filtration $\mathbb{G} = (\mathcal{G}_t)_{t\in\{0,\ldots,\widehat{T}\}}$ is now given by

$$\mathcal{G}_0 = \{\emptyset, \Omega\} \text{ and } \mathcal{G}_t = \sigma\{\rho_1, \ldots, \rho_{t\wedge(\widehat{T}-1)}\}, \ t \in \{1, \ldots, \widehat{T}\}.$$

At time t the maturities of the forward rates given by (5.2.7) are those where a forward rate with the same maturity is defined at time $t-1$. Hence, the forward rates at time t given by (5.2.7) determine the bond prices at time t for bonds with time of maturity v, $v \in \{t+1, \dots, (t+\widetilde{T}-1) \wedge \widehat{T}\}$, which are the bonds traded at time $t-1$ (when disregarding the bond maturing at time t). Thus, the uncertainty associated with the development of the forward rates (bond prices) from time $t-1$ to t is described by ρ_t. However, the uncertainty associated with the price of the new bond with time to maturity $\widetilde{T}$ introduced in the market at time t, $t \in \{1, \dots, T\}$, cannot be described entirely by ρ_t; it depends on an additional source of risk. In order to model this additional uncertainty we assume that at time t, $t \in \{1, \dots, T\}$, the $(\widetilde{T}-1)$-period forward rate, $f_{t,t+\widetilde{T}-1}$, is given by

$$f_{t,t+\widetilde{T}-1} = c_t(f_0, \dots, f_{t-1}, \tilde{f}_{t,t+\widetilde{T}-2}, \varepsilon_t) \tag{5.2.8}$$

for some function $c_t : (\mathbb{R}^{\widetilde{T}})^t \times \mathbb{R}^{\widetilde{T}-1} \times \{h, \ell\} \mapsto \mathbb{R}$ and an i.d.d. sequence $\varepsilon_1, \dots, \varepsilon_T$ of random variables independent of $(\rho_t)_{t \in \{1,\dots,\widehat{T}-1\}}$. The distribution of ε_1 is given by $P(\varepsilon_1 = h) = 1 - P(\varepsilon_1 = \ell) = \tilde{p}$, $\tilde{p} \in (0,1)$. Hence, for $t \in \{1, \dots, T\}$ it holds that given the past forward rates and the $(\widetilde{T}-1)$-dimensional forward rate vector $\tilde{f}_{t,t+\widetilde{T}-2} = (r_t, f_{t,t+1}, \dots, f_{t,t+\widetilde{T}-2})$ at time t, the $(\widetilde{T}-1)$-period forward rate, $f_{t,t+\widetilde{T}-1}$, can attain two different values: $c_t(f_0, \dots, f_{t-1}, \tilde{f}_{t,t+\widetilde{T}-2}, h)$ and $c_t(f_0, \dots, f_{t-1}, \tilde{f}_{t,t+\widetilde{T}-2}, \ell)$. We refer to these values as the *high* and *low* value, respectively. Analogously to ξ_t we now introduce λ_t, $\lambda_t \in \Lambda_t = \{$all possible sequences of h's and ℓ's of length $t\}$ for all $t = 0, \dots, T$. Thus, λ_t keeps track of whether the past values of the $(\widetilde{T}-1)$-period forward rate has attained the high or the low value. Hence, ξ_t and $\lambda_{t \wedge T}$ determine the development of the entire forward rate vector until time t, $t \in \{1, \dots, \widehat{T}-1\}$, such that we can denote the generic value of the forward rate vector at time t by $f_t^{\xi_t, \lambda_{t \wedge T}}$. Now introduce the filtration $\mathbb{H} = (\mathcal{H}_t)_{t \in \{0,\dots,\widehat{T}\}}$ by

$$\mathcal{H}_0 = \{\emptyset, \Omega\} \text{ and } \mathcal{H}_t = \sigma\{\varepsilon_1, \dots, \varepsilon_{t \wedge T}\}, \; t \in \{1, \dots, \widehat{T}\}.$$

We now assume the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in \{0,1,\dots,\widehat{T}\}}$ introduced earlier is given by

$$\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t.$$

Hence, $\mathbb{F}$ is the filtration for the extended bond market. We note that it is sufficient to consider the state space for ω given by $\Omega = \{u, d\}^{\widehat{T}-1} \times \{h, \ell\}^T$ and the σ-algebra $\mathcal{F} = \mathcal{F}_{\widehat{T}} = \mathcal{F}_{\widehat{T}-1}$.

At time t, $t \in \{T+1, \dots, \widehat{T}\}$ the development of the bond market is essentially identical to the binomial model in Section 5.2.1, whereas the model is non-standard at time t, $t \in \{1, \dots, T\}$. Here, we have that contingent on ξ_{t-1} and λ_{t-1} there are four possible forward rate vectors at time t and hence 4^t possible states at time t. Thus, for $t \le T$ the development of the forward rate vector can be represented using a non-recombining quadrinomial tree, see Figure 5.2.3 for a visualization of the forward rates with $\widetilde{T} = 2$.

From (5.2.7) we observe that the forward rate at time t with maturity τ, $\tau \in \{t, \dots, (t + \widetilde{T}-2) \wedge (\widehat{T}-1)\}$, is allowed to depend on all past forward rates, such that the $(\widetilde{T}-1)$-period forward rate at time $t-1$ may influence the entire forward rate vector at time t. Hence, the forward rate $f_{t,\tau}$, $\tau \in \{t, \ldots, (t+\widetilde{T}-2) \wedge (\widehat{T}-1)\}$, is $(\mathcal{F}_{t-1} \vee \mathcal{G}_t)$-measurable, which in turn gives that $P(t,\tau)$ also is $(\mathcal{F}_{t-1} \vee \mathcal{G}_t)$-measurable for $\tau \in \{t+1, \ldots, (t+\widetilde{T}-1) \wedge \widehat{T}\}$. For an illustration of the dependence of the forward rates on the ρ's and ε's we again refer to Figure 5.2.3, where the dependence is shown explicitly.

Note that if we contingent on the outcome of the vector $(\varepsilon_1, \ldots, \varepsilon_T)$, the development can be described by a binomial model, and hence the conditional model is complete, such that in the conditional model zero coupon bonds with all maturities have unique prices (even before they are traded). Hence, in the conditional model we, at all times, have a forward rate vector for all maturities. However, in the unconditional model the future values of $\varepsilon_{t+1}, \ldots, \varepsilon_T$ are unknown at time t, such that it is uncertain which of the conditional forward rate vectors in retrospect will turn out to have been "the correct one", when $\varepsilon_{t+1}, \ldots, \varepsilon_T$ have been observed at time T. Thus, the reinvestment risk can be interpreted as the uncertainty associated with which of the conditional forward rate vectors in retrospect has turned out to have been "the correct one". This in turn gives that the magnitude of the reinvestment risk is related to how much the conditional forward rate vectors differ.

Example 5.2.2 Consider the case where $\widetilde{T}=2$ and $T=3$. Hence, the time to maturity of the longest bond in the market is 2 and the time of maturity of the considered claim is 3. The development of the forward rate vector can be visualized by Figure 5.2.3. Here, the superscripts denote the dependence of the forward rates on the outcome of the variables ρ and ε. As an example the notation $r_2^{uu,\ell}$ denotes the short rate in period 2 if $\rho_1 = u$, $\rho_2 = u$ and $\varepsilon_1 = \ell$. We end the example by noting that all examples in this chapter are one continuing example.

□

Risk-neutral probabilities

We now aim at determining the equivalent martingale measures in the extended model. Here, the uncertainty is generated by $(\rho_t)_{t\in\{1,\ldots,\widehat{T}-1\}}$ and $(\varepsilon_t)_{t\in\{1,\ldots,T\}}$, such that the measure Q is uniquely determined by $(q_t^{\xi_{t-1},\lambda_{(t-1)\wedge T}})_{t\in\{1,\ldots,\widehat{T}-1\}}$ and $(\tilde{q}_t^{\xi_t,\lambda_{t-1}})_{t\in\{1,\ldots,T\}}$, where $q_t^{\xi_{t-1},\lambda_{(t-1)\wedge T}}$ denotes the probability of $\rho_t = u$ given ξ_{t-1} and $\lambda_{(t-1)\wedge T}$, and $\tilde{q}_t^{\xi_t,\lambda_{t-1}}$ denotes the probability of $\varepsilon_t = h$ given ξ_t and λ_{t-1}. Recall that for $t \in \{0, \ldots, \widehat{T}-1\}$ and $v \in \{t+1, \ldots, (t+\widetilde{T}) \wedge \widehat{T}\}$, a necessary condition for Q to be an equivalent martingale measure is

$$P(t,v) = \frac{1}{1+r_t} E^Q \left[P(t+1,v) \middle| \mathcal{F}_t \right], \tag{5.2.9}$$

such that the discounted bond prices are martingales. Now note that (5.2.9) is trivially fulfilled if $v = t+1$. Since $P(t+1,v)$ in (5.2.9) is $\mathcal{F}_t \vee \mathcal{G}_{t+1}$-measurable, i.e. independent of ε_{t+1}, then (5.2.9) yields $(\widetilde{T}-2) \wedge (\widehat{T}-t-1)$ equations for $q_{t+1}^{\xi_t,\lambda_{t\wedge T}}$. Hence, if there exists an equivalent martingale measure, then $q_{t+1}^{\xi_t,\lambda_{t\wedge T}}$ is unique for all $t \in \{0, \ldots, \widehat{T}-2\}$,

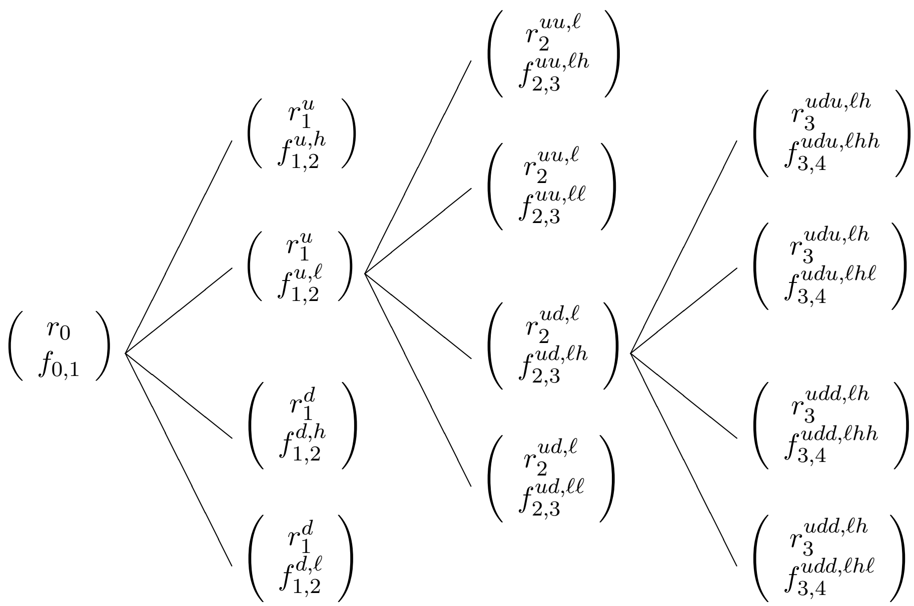

Figure 5.2.3: *Development of the forward rates in the extended model with* $\widetilde{T} = 2$.

provided there for each t exists a $v \in \{t+2, \ldots, (t+\widetilde{T}) \wedge \widetilde{T}\}$, such that $P^{\xi_t,\lambda_t,d}(t+1,v) \neq P^{\xi_t,\lambda_t,u}(t+1,v)$. For $t \in \{0, \ldots, T-1\}$ no information regarding $\tilde{q}_{t+1}^{\xi_{t+1},\lambda_t}$ can be derived from (5.2.9), so any Q for which $q_{t+1}^{\xi_t,\lambda_{t\wedge T}}$ fulfills (5.2.9) for all $t \in \{0, \ldots, \widehat{T}-2\}$ ensures that the discounted bond prices are martingales. If further both $q_{t+1}^{\xi_t,\lambda_{t\wedge T}}$ and $\tilde{q}_{t+1}^{\xi_{t+1},\lambda_t}$ lie in the interval $(0,1)$, then Q is an equivalent martingale measure. If $\tilde{q}_{t+1}^{\xi_{t+1},\lambda_t} = \tilde{p}$, we say the market is risk-neutral with respect to reinvestment risk. This measure is known as the minimal martingale measure for the extended model, i.e. the equivalent martingale measure which “disturbs the structure of the model as little as possible”, see Schweizer (1995). Here, we restrict ourselves to Q's given by $\tilde{q}_{t+1}^{\xi_{t+1},\lambda_t} = \tilde{q}_{t+1}^{\lambda_t}$, such that under Q the distribution of the ε's is independent of the realization of the ρ's. Henceforth we consider a fixed, but arbitrary, equivalent martingale measure Q.

Remark 5.2.3 Note that for $t \in \{0, \ldots, \widehat{T}-1\}$ and $v \in \{t+1, \ldots, (t+\widetilde{T}) \wedge \widehat{T}\}$ repeated use of (5.2.9) gives the equation

$$P(t,v) = \frac{1}{1+r_t} E^Q \left[\frac{1}{1+r_{t+1}} E^Q \left[P(t+2,v) \middle| \mathcal{F}_{t+1} \right] \middle| \mathcal{F}_t \right]. \tag{5.2.10}$$

Hence, since $P(t+2,v)$ is $(\mathcal{F}_{t+1} \vee \mathcal{G}_{t+2})$-measurable it seems as if (5.2.10) gives an equation from which to determine $\tilde{q}_{t+1}^{\xi_{t+1},\lambda_t}$ for $t \in \{0, \ldots, T-1\}$. However, this is not the case, since the $(\mathcal{F}_t \vee \mathcal{G}_{t+1})$-measurability of $P(t+1,v)$ ensures that $E^Q[P(t+2,v)|\mathcal{F}_{t+1}]$ is independent of ε_{t+1}.

□

Model considerations

Consider the case where we for fixed t model the forward rate $f_{t,u}$, $u \in \{t, \ldots, (t + \widetilde{T} - 2) \wedge (\widehat{T} - 1)\}$ by

$$f_{t,u} = g_{t,u}(\widetilde{f}_{0,u}, \ldots, \widetilde{f}_{t-1,u}, \rho_t), \quad t \in \{1, \ldots, \widehat{T} - 1\}, \tag{5.2.11}$$

where $g_{t,u} : \mathbb{R}^{(u+1)\wedge\widetilde{T}} \times \cdots \times \mathbb{R}^{(u-t+2)\wedge\widetilde{T}} \times \{u, d\} \mapsto \mathbb{R}$. Hence, the development of the forward rates given by (5.2.11) is a special case of (5.2.7), where we have restricted the possible dependence on the past forward rates. Here, the forward rate at time t with maturity τ, $\tau \in \{t, \ldots, (t + \widetilde{T} - 2) \wedge (\widehat{T} - 1)\}$, is allowed to depend on the past forward rates with maturity less than or equal to τ only. This in turn gives that $f_{t,\tau}$ is independent of ε_v for $v > ((\tau - \widetilde{T} + 1) \vee 0)$. With this restriction we have that the price at time t of a bond with time of maturity $\tau \in \{t, \ldots, (t+\widetilde{T}-1)\wedge\widehat{T}\}$ is $(\mathcal{F}_{\tau-\widetilde{T}} \vee \mathcal{G}_t)$-measurable. Here, and throughout the chapter, we adopt the convention that $\mathcal{F}_{-\tau} = \mathcal{F}_0$ for $\tau \in \mathbb{N}$. Within this model we have that once a bond is introduced in the market, the development of the price process is entirely described by the outcome of the ρ's. Hence, at time t we essentially are in the complete and arbitrage free model from Section 5.2.1 when considering the filtration $\mathbb{G}$ and the time horizon $(t + \widetilde{T} - 1) \wedge \widehat{T}$.

5.2.3 Discrete-time trading

Since the bonds traded in the bond market depend on the time considered, it is inconvenient to define trading strategies in terms of the bonds. Hence, we define $\widetilde{T}$ new price processes $(S^k)_{k=1,\ldots,\widetilde{T}}$, which are defined for all $t = 0, 1, \ldots, T$, by

$$S_0^k = 1 \text{ and } S_t^k = \frac{P(t, t-1+k)}{P(t-1, t-1+k)} S_{t-1}^k = \prod_{i=0}^{t-1} \frac{P(i+1, i+k)}{P(i, i+k)}, \;\; t \in \{1, \ldots, T\}. \tag{5.2.12}$$

We note that until time T these price processes include exactly the same information as the original bond prices. The price process S^k is generated by investing 1 unit at time 0 in bonds with time to maturity k and at times $t = 1, \ldots, T$ selling the bonds with time to maturity $k - 1$ purchased at time $t - 1$ and reinvesting the money in bonds with time to maturity k. Hence, for $k \in \{1, \ldots, \widetilde{T}\}$ the price process S^k is the value process generated by a roll-over strategy in bonds with time to maturity k. Such a value process is usually referred to as a rolling-horizon bond, see Rutkowski (1999). Recall that in a discrete-time model, the 1-period bond is equal to a savings account, so the price process S^1 corresponds to investing in a savings account with a locally riskfree interest rate. Note that given $\mathcal{F}_{t-1}$, the future value of the price process vector at time t, $(S_t^k)_{k\in\{1,\ldots,\widetilde{T}\}}$, depends on ρ_t only, such that it is sufficient for hedging purposes to consider *any* two of the rolling-horizon bonds defined by (5.2.12). Here, we consider the savings account, henceforth denoted B, and $S^{\widetilde{T}}$, henceforth simply denoted S. We shall refer to the asset with price process S as the risky asset. Note that the measurability conditions on the bond prices give that B_t and S_t, respectively, are $(\mathcal{F}_{t-2} \vee \mathcal{G}_{t-1})$- and $(\mathcal{F}_{t-1} \vee \mathcal{G}_t)$-measurable.

A trading strategy with respect to (B, S) is an adapted two-dimensional process $\varphi = (\vartheta, \eta)$. Hence, ϑ_t and η_t are $\mathcal{F}_t$-measurable for all t. The pair $\varphi_t = (\vartheta_t, \eta_t)$ is interpreted as the portfolio established at time t and held until time $t+1$. Here, ϑ_t denotes the number of risky assets held in the portfolio, and η_t is the discounted deposit in the savings account. The value process associated with the trading strategy φ is denoted $\mathcal{V}(\varphi)$. Here, $\mathcal{V}_t(\varphi)$, which is the value at time t of holding the portfolio (ϑ_t, η_t), is given by

$$\mathcal{V}_t(\varphi) = \vartheta_t S_t + \eta_t B_t. \tag{5.2.13}$$

With the definition of ϑ and η above the value process is seen to be the value after any in- or outflow of capital at time t. A trading strategy is called self-financing if for all t

$$\mathcal{V}_t(\varphi) = \mathcal{V}_0(\varphi) + \sum_{u=0}^{t-1} \vartheta_u \Delta S_{u+1} + \sum_{u=0}^{t-1} \eta_u \Delta B_{u+1}, \tag{5.2.14}$$

where we have introduced the notation exemplified by $\Delta S_u = S_u - S_{u-1}$. Thus, the value at time t of a self-financing strategy is the initial value added interest and investment gains from trading in the bond market. Hence, withdrawals or deposits are not allowed at intermediate times $t = 1, \ldots, T-1$. A self-financing strategy is a so-called arbitrage if $\mathcal{V}_0(\varphi) = 0$ and $\mathcal{V}_T(\varphi) \geq 0$ P-a.s. with $P(\mathcal{V}_T(\varphi) > 0) > 0$. A contingent claim (or a derivative) with maturity T is an $\mathcal{F}_T$-measurable random variable H. A contingent claim is called attainable if there exists a self-financing strategy φ such that $\mathcal{V}_T(\varphi) = H$ P-a.s. An attainable claim can thus be replicated perfectly by investing $\mathcal{V}_0(\varphi)$ at time 0 and adjusting the portfolio at times $t = 1, \ldots, T-1$, according to the self-financing strategy φ. Hence, at any time t, there is no difference between holding the claim H and the portfolio φ_t. In this sense, the claim H is redundant in the market and from the assumption of no arbitrage it follows that the price of H at time t must be $\mathcal{V}_t(\varphi)$. Thus, the initial investment $\mathcal{V}_0(\varphi)$ is the unique arbitrage free price of H. Note that if φ is a self-financing portfolio replicating the contingent claim H, then (5.2.14) gives the following representation for H:

$$H = \mathcal{V}_T(\varphi) = \mathcal{V}_0(\varphi) + \sum_{u=0}^{T-1} \vartheta_u \Delta S_{u+1} + \sum_{u=0}^{T-1} \eta_u \Delta B_{u+1}. \tag{5.2.15}$$

If all contingent claims are attainable, the model is called complete and otherwise it is called incomplete. Throughout the chapter, we denote by S^*, $\mathcal{V}^*(\varphi)$ and H^* the discounted price process of the risky asset, the discounted value process and the discounted claim, respectively.

Remark 5.2.4 The definition of trading strategies in discrete time is not uniform in the literature. Harrison and Kreps (1979), Jarrow (1996) and Musiela and Rutkowski (1997) define trading strategies as adapted processes, whereas Harrison and Pliska (1981) and Björk (2004) consider predictable processes. The different measurability conditions lead to one significant difference, namely, whether the value process defined by (5.2.13) denotes the value before or after a possible withdrawal or deposit. A third possibility is the definition in Föllmer and Schweizer (1988). They consider a predictable process ϑ and an adapted process η. Hence, the portfolio at time t is given by the number of risky assets held in the portfolio from time $t-1$ to t and the discounted deposit in the savings account after a possible withdrawal or deposit. Since they define the value process *after* a possible withdrawal or deposit their value process coincides with the value process in the present chapter. Hence, we have the following connection between our definition of trading strategies and the Föllmer–Schweizer definition:

$$\vartheta_t = \vartheta_{t+1}^{FS} \tag{5.2.16}$$
$$\eta_t = \eta_t^{FS} + \left(\vartheta_t^{FS} - \vartheta_{t+1}^{FS}\right) S_t^*. \tag{5.2.17}$$

Here, $(\vartheta_t^{FS}, \eta_t^{FS})$ denotes the portfolio at time t using the Föllmer–Schweizer definition.

□

Example 5.2.5 If $\widetilde{T} = 2$ and $T = 3$ then the price processes for the savings account and the risky asset are given by

$$B_0 = 1,\ B_t = \prod_{i=0}^{t-1}(1 + r_i),\ \ t \in \{1,2,3\} \ \text{ and } \ S_0 = 1,\ S_t = \prod_{i=0}^{t-1} \frac{P(i+1, i+2)}{P(i, i+2)},\ \ t \in \{1,2,3\},$$

respectively. Here, one easily observes that, as noted above, B_t is $(\mathcal{F}_{t-2} \vee \mathcal{G}_{t-1})$-measurable and S_t is $(\mathcal{F}_{t-1} \vee \mathcal{G}_t)$-measurable.

□

5.3 Hedging strategies

Consider a company interested in hedging the claim H with maturity T. If H only depends on bonds with time of maturity at time $\widetilde{T}$ or earlier, it has a unique arbitrage free price and can be replicated perfectly leaving the company without any risk. However, if H depends on bonds maturing after time $\widetilde{T}$, then H does in general not have a perfect replicating strategy, and hence in general it does not have a unique arbitrage free price. For unattainable claims we determine the optimal hedging strategies for the criteria of super-replication and risk-minimization.

5.3.1 Super-replication

A strategy φ is called super-replicating for the claim H with maturity T if the value process is of the form

$$\mathcal{V}_t(\varphi) = \mathcal{V}_0(\varphi) + \sum_{u=0}^{t-1} \vartheta_u \Delta S_{u+1} + \sum_{u=0}^{t-1} \eta_u \Delta B_{u+1} - U_t, \tag{5.3.1}$$

where U is a non-decreasing process, and the terminal value of the value process satisfies $\mathcal{V}_T(\varphi) \geq H$ P-a.s. Here, the process U is the accumulated outflow of capital when using the strategy φ. Thus, when following a super-replicating strategy no inflow of capital is needed in addition to the initial investment in order to guarantee that at time of maturity, the value of the portfolio is at least as large as the considered claim. Hence, following a super-replicating strategy allows the hedger to eliminate the risk of falling short of the claim. The smallest initial value needed at time t to construct a super-replicating strategy is referred to as the *super-replicating price at time t*, henceforth denoted $\hat{\pi}_t(H)$. Hence, the super-replicating price at time t is the smallest initial investment at time t allowing the company to hedge the considered claim without any risk of falling short. For more details on super-replication see El Karoui and Quenez (1995) and Föllmer and Schied (2002). Now define the *super-replicating price process* as the process of the super-replicating prices, i.e. the value of the super-replicating price process at time t is exactly the super-replicating price at time t. At any time t the *optimal super-replicating strategy* is defined as the super-replicating strategy corresponding to the super-replicating price process. Prior to the general result for the super-replicating price process and optimal super-replicating strategy for a claim H with maturity T, we first consider super-replication in a 1-period model.

Lemma 5.3.1
At time t, $t \in \{0, \ldots, T-1\}$, the optimal super-replicating strategy, $\hat{\varphi}_t = (\hat{\vartheta}_t, \hat{\eta}_t)$, for a claim H with time of maturity $t+1$ is given by

$$\hat{\vartheta}_t = \frac{\widehat{H}(d) - \widehat{H}(u)}{S^d_{t+1} - S^u_{t+1}} \text{ and } \hat{\eta}_t = \frac{\widehat{H}(u)S^d_{t+1} - \widehat{H}(d)S^u_{t+1}}{B_{t+1}(S^d_{t+1} - S^u_{t+1})},$$

where

$$\widehat{H}(\rho_{t+1}) = \max(H(\rho_{t+1}, h), H(\rho_{t+1}, \ell)), \quad \rho_{t+1} \in \{u, d\}.$$

The super-replicating price is

$$\hat{\pi}_t(H) = \frac{1}{1+r_t}\left(q_{t+1}\widehat{H}(u) + (1 - q_{t+1})\widehat{H}(d)\right).$$

Proof of Lemma 5.3.1: Consider an agent holding the portfolio (ϑ_t, η_t) at time t. Before any adjustments at time $t+1$ the value of the portfolio can take one of two values: $\vartheta_t S^u_{t+1} + \eta_t B_{t+1}$ or $\vartheta_t S^d_{t+1} + \eta_t B_{t+1}$. Hence, the value of the portfolio is the same in the states (u, h) and (u, ℓ), as well as in (d, h) and (d, ℓ). Thus, for (ϑ_t, η_t) to be super-replicating it must hold that

$$\vartheta_t S^u_{t+1} + \eta_t B_{t+1} \geq \max(H(u, h), H(u, \ell)),$$
$$\vartheta_t S^d_{t+1} + \eta_t B_{t+1} \geq \max(H(d, h), H(d, \ell)),$$

where the $H(i, j)$ denotes the payoff from H if $\rho_{t+1} = i$ and $\varepsilon_{t+1} = j$, where $i \in \{u, d\}$ and $j \in \{h, \ell\}$. Define the contingent claim $\widehat{H}$ with payoff

$$\widehat{H}(\rho_{t+1}) = \max(H(\rho_{t+1}, h), H(\rho_{t+1}, \ell)), \quad \rho_{t+1} \in \{u, d\},$$

and note that the strategy

$$\vartheta_t = \frac{\widehat{H}(d) - \widehat{H}(u)}{S^d_{t+1} - S^u_{t+1}} \text{ and } \eta_t = \frac{\widehat{H}(u)S^d_{t+1} - \widehat{H}(d)S^u_{t+1}}{B_{t+1}(S^d_{t+1} - S^u_{t+1})}$$

replicates $\widehat{H}$. A no arbitrage argument now gives that the replicating strategy and the unique arbitrage free price for $\widehat{H}$ is the optimal super-replicating strategy and super-replicating price, respectively.

$\square$

Lemma 5.3.1 has the following interpretation: The dependence of H on ε_{t+1} is unhedgeable. Hence, for each outcome of ρ_{t+1} we assume the outcome of ε_{t+1} which leads to the highest value of H and replicate this claim. The replicating strategy and the unique arbitrage free price of this "worst scenario" claim are then equal to the optimal super-replicating strategy and super-replicating price, respectively.

Remark 5.3.2 The main result in Aliprantis, Polyrakis and Tourky (2002) states that in a 1-period model the optimal super-replicating strategy shall be found among the replicating strategies in the complete sub-models arising from eliminating states of the world. Hence, Lemma 5.3.1 can be seen as a special case, where the optimal super-replicating strategy is easily identifiable.

$\square$

Theorem 5.3.3
Consider a claim H with time of maturity T. For $t \in \{0, \dots, T-1\}$ the portfolio, $\hat{\varphi}_t = (\hat{\vartheta}_t, \hat{\eta}_t)$ held in the optimal super-replicating strategy is given by

$$\hat{\vartheta}_t^{\xi_t,\lambda_t} = \frac{\hat{\pi}_{t+1}^{\xi_t,d,\lambda_t}(H) - \hat{\pi}_{t+1}^{\xi_t,u,\lambda_t}(H)}{S_{t+1}^{\xi_t,d,\lambda_t} - S_{t+1}^{\xi_t,u,\lambda_t}} \quad \text{and} \quad \hat{\eta}_t^{\xi_t,\lambda_t} = \frac{\hat{\pi}_{t+1}^{\xi_t,u,\lambda_t} S_{t+1}^{\xi_t,d,\lambda_t} - \hat{\pi}_{t+1}^{\xi_t,d,\lambda_t} S_{t+1}^{\xi_t,u,\lambda_t}}{B_{t+1}^{\xi_t,\lambda_{t-1}}\left(S_{t+1}^{\xi_t,d,\lambda_t} - S_{t+1}^{\xi_t,u,\lambda_t}\right)},$$

where

$$\hat{\pi}_{t+1}^{\xi_t,\rho_{t+1},\lambda_t}(H) = \max\left(\hat{\pi}_{t+1}^{\xi_t,\rho_{t+1},\lambda_t,h}(H), \hat{\pi}_{t+1}^{\xi_t,\rho_{t+1},\lambda_t,\ell}(H)\right), \quad \rho_{t+1} \in \{u, d\}.$$

Starting with the terminal value

$$\hat{\pi}_T^{\xi_T,\lambda_T}(H) = H,$$

the super-replicating price process at time t, $t \in \{0, \dots, T-1\}$, is given by the following recursive formula

$$\hat{\pi}_t^{\xi_t,\lambda_t}(H) = \frac{1}{1 + r_t^{\xi_t,\lambda_{(t-1)\vee 0}}}\left(q_{t+1}^{\xi_t,\lambda_t}\hat{\pi}_{t+1}^{\xi_t,u,\lambda_t}(H) + (1 - q_{t+1}^{\xi_t,\lambda_t})\hat{\pi}_{t+1}^{\xi_t,d,\lambda_t}(H)\right).$$

Proof of Theorem 5.3.3: First note that at time T the super-replicating price is trivial and equal to H. At time t, $t \in \{0, \dots, T-1\}$ we may consider the super-replicating price at time $t+1$, $\hat{\pi}_{t+1}(H)$, as the payoff from at contingent claim with maturity $t+1$. Thus, Lemma 5.3.1 gives the super-replicating price and optimal super-replicating strategy at time t in terms of the super-replicating price at time $t+1$.

$\square$

Note that we in Theorem 5.3.3 explicitly denote the dependence on the past through ξ_t and λ_t in order to emphasize the dependence of the optimal super-replicating strategy and super-replicating price process on the past.

For sufficiently nice claims, such as fixed claims, the following corollary allows for an easy calculation of the super-replicating price process and the optimal super-replicating strategy.

Corollary 5.3.4
If for each t, $t \in \{0, \dots, T-1\}$, it holds, for fixed $k_{t+1} \in \{h, \ell\}$ that

$$\hat{\pi}_{t+1}^{\xi_t,\rho_{t+1},\lambda_t}(H) = \hat{\pi}_{t+1}^{\xi_t,\rho_{t+1},\lambda_t,k_{t+1}}(H)$$

for all ξ_t, ρ_{t+1} and λ_t, then the super-replicating price and optimal super-replicating strategy at time τ are, respectively, the unique arbitrage free price and the replicating strategy in the conditional model given $(\varepsilon_{\tau+1}, \dots, \varepsilon_T) = (k_{\tau+1}, \dots, k_T)$.

Denote by $\widehat{U}$ the process U from (5.3.1) associated with the optimal super-replicating strategy. Hence, $\widehat{U}$ denotes the accumulated outflow of capital, when using the optimal super-replicating strategy. Combining (5.3.1) and Theorem 5.3.3 gives the following explicit expression for the change in $\widehat{U}$ at time t

$$\Delta\widehat{U}_t^{\xi_{t-1},\rho_t,\lambda_{t-1},\varepsilon_t} = \hat{\pi}_t^{\xi_{t-1},\rho_t,\lambda_{t-1}}(H) - \hat{\pi}_t^{\xi_{t-1},\rho_t,\lambda_{t-1},\varepsilon_t}(H). \tag{5.3.2}$$

Investigating (5.3.2) we observe that the withdrawal is the difference between the value at time t of the optimal super-replicating portfolio purchased at time $t-1$ and the super-replicating price at time t. Hence, when using the optimal super-replicating strategy the withdrawal from the portfolio at time t depends on the outcome of the two random variables observed at time t, ρ_t and ε_t. Given (5.3.2), one easily derives the conditional expectation under P of $\Delta\widehat{U}_t^{\xi_{t-1},\rho_t,\lambda_{t-1},\varepsilon_t}$ given $\mathcal{F}_{t-1}$, namely,

$$\begin{aligned}
E^P & \left[\Delta\widehat{U}_t^{\xi_{t-1},\rho_t,\lambda_{t-1},\varepsilon_t} \middle| \mathcal{F}_{t-1} \right] \\
&= E^P \left[\hat{\pi}_t^{\xi_{t-1},\rho_t,\lambda_{t-1}}(H) - \hat{\pi}_t^{\xi_{t-1},\rho_t,\lambda_{t-1},\varepsilon_t}(H) \middle| \mathcal{F}_{t-1} \right] \\
&= p \left(\hat{\pi}_t^{\xi_{t-1},u,\lambda_{t-1}}(H) - \left(\tilde{p}\hat{\pi}_t^{\xi_{t-1},u,\lambda_{t-1},h}(H) + (1-\tilde{p})\hat{\pi}_t^{\xi_{t-1},u,\lambda_{t-1},\ell}(H) \right) \right) \\
&\quad + (1-p) \left(\hat{\pi}_t^{\xi_{t-1},d,\lambda_{t-1}}(H) - \left(\tilde{p}\hat{\pi}_t^{\xi_{t-1},d,\lambda_{t-1},h}(H) + (1-\tilde{p})\hat{\pi}_t^{\xi_{t-1},d,\lambda_{t-1},\ell}(H) \right) \right).
\end{aligned}$$

Thus, the expected withdrawal from the optimal super-replicating portfolio is the probability of an upward jump multiplied by the expected withdrawal contingent on an upward jump added the probability of a downward jump multiplied by the expected withdrawal in this case.

Example 5.3.5 Let $\widetilde{T} = 2$, $T = 3$ and $H = 1$. Since H is attainable at time 1 and $\hat{\pi}_1^{\rho_1} = \hat{\pi}_1^{\rho_1,\ell}$, we have from Corollary 5.3.4 that the super-replicating price and the super-replicating strategy at time 0 corresponds to using the conditional forward rate vector given $\varepsilon_1 = \ell$. Hence, we obtain the following super-replicating price process, expressed in terms of bond prices:

$$\begin{aligned}
\hat{\pi}_3^{\xi_3,\lambda_3}(1) &= 1, \\
\hat{\pi}_2^{\xi_2,\lambda_2}(1) &= \frac{1}{1+r_2^{\xi_2,\lambda_1}}\left(q_3^{\xi_2,\lambda_2} + \left(1 - q_3^{\xi_2,\lambda_2}\right)\right) = \frac{1}{1+r_2^{\xi_2,\lambda_1}} = P^{\xi_2,\lambda_1}(2,3), \\
\hat{\pi}_1^{\xi_1,\lambda_1}(1) &= \frac{1}{1+r_1^{\xi_1}}\left(q_2^{\xi_1,\lambda_1} P^{\xi_1,u,\lambda_1}(2,3) + \left(1 - q_2^{\xi_1,\lambda_1}\right) P^{\xi_1,d,\lambda_1}(2,3)\right) = P^{\xi_1,\lambda_1}(1,3), \\
\hat{\pi}_0(1) &= \frac{1}{1+r_0}\left(q_1 P^{u,\ell}(1,3) + (1-q_1)\, P^{d,\ell}(1,3)\right).
\end{aligned}$$

The optimal super-replicating strategy is given by

$$\left(\hat{\vartheta}_0, \hat{\eta}_0\right) = \left(\frac{P(0,2)(P^{d,\ell}(1,3) - P^{u,\ell}(1,3))}{P^d(1,2) - P^u(1,2)}, \frac{P^d(1,2)P^{u,\ell}(1,3) - P^u(1,2)P^{d,\ell}(1,3)}{(1+r_0)(P^d(1,2) - P^u(1,2))}\right),$$

$$\left(\hat{\vartheta}_1^{\xi_1,\lambda_1}, \hat{\eta}_1^{\xi_1,\lambda_1}\right) = \left(\frac{P(0,2)P^{\xi_1,\lambda_1}(1,3)}{P^{\xi_1}(1,2)}, 0\right) \text{ and } \left(\hat{\vartheta}_2^{\xi_2,\lambda_2}, \hat{\eta}_2^{\xi_2,\lambda_2}\right) = \left(0, \frac{1}{B_3^{\xi_2,\lambda_1}}\right).$$

□

Relation to guarantees

Apart from the nice property of allowing the hedger to eliminate the shortfall risk the super-replicating price process relates to the maximal possible guarantees for which the risk of falling short can be eliminated. Here, we consider two types of guarantees: Maturity guarantees and periodic interest rate guarantees. Given a deposit at time t the maturity guarantee is the minimal possible payoff at time T, whereas the periodic interest guarantee is the minimum interest earned on the deposit in each period until time T. We shall refer to the maximal guarantees for which the short fall risk can be eliminated as *the maximal riskfree maturity guarantee* and *maximal riskfree periodic interest rate guarantee.*

Proposition 5.3.6
Given an initial deposit of 1 at time t, the maximal riskfree maturity guarantee, G_t^T, at time T is given by

$$G_t^T = \frac{1}{\hat{\pi}_t(1)}. \tag{5.3.3}$$

The maximal riskfree periodic interest rate guarantee is

$$g_t^T = \left(\frac{1}{\hat{\pi}_t(1)}\right)^{\frac{1}{T-t}} - 1. \tag{5.3.4}$$

Proof of Proposition 5.3.6: At time t the super-replicating price of 1 unit at time T is given by $\hat{\pi}_t(1)$. Hence, by investing 1 at time t we may purchase $1/\hat{\pi}_t(1)$ units of the super-replicating strategy. This guarantees a payoff at time T of at least $1/\hat{\pi}_t(1)$. Hence, since the super-replicating price per definition is the lowest initial deposit for which a certain payoff is guaranteed, the maximal riskfree maturity guarantee at time T is given by (5.3.3). Now, the maximal riskfree periodic interest rate guarantee is the constant short rate which gives a payoff of G_t^T at time T, when depositing 1 unit at time t. Hence, g_t^T is the unique solution greater than -1 to

$$(1 + g_t^T)^{T-t} = G_t^T.$$

Inserting (5.3.3) and isolating g_t^T now gives (5.3.4).

□

Proposition 5.3.6 is of importance to for instance life insurance companies, since it gives the maximal guarantees, which the companies should promise the insured at initiation of the contract.

5.3.2 Risk-minimizing strategies

As an alternative to the hedging criterion of super-replication we now consider risk-minimization. Here, we give a brief review of risk-minimization and determine risk-minimizing strategies in the presence of reinvestment risk. We note that since we define trading strategies differently than Föllmer and Schweizer (1988) and Møller (2001a) our results cannot be compared directly to their results.

A brief review of risk-minimization

In this section we review the criterion of risk-minimization introduced in discrete time by Föllmer and Schweizer (1988). The presentation is based on Møller (2001a).

The idea of risk-minimization is closely related to the introduction of the *cost process* defined by

$$C_t(\varphi) = \mathcal{V}_t^*(\varphi) - \sum_{u=0}^{t-1} \vartheta_u \Delta S_{u+1}^*. \tag{5.3.5}$$

Thus, the cost process is the discounted value of the portfolio reduced by discounted trading gains. The cost process measures the accumulated discounted cost of an agent following the strategy φ. Comparing (5.2.15) and (5.3.5) we note that the cost process is constant P-a.s. if and only if the strategy φ is self-financing. To measure the risk associated with the strategy φ we introduce the *risk process* defined by

$$R_t(\varphi) = E^Q \left[(C_T(\varphi) - C_t(\varphi))^2 \middle| \mathcal{F}_t \right]. \tag{5.3.6}$$

Hence, the risk process is the conditional expected value of the future costs associated with the strategy φ. A trading strategy φ is called risk-minimizing for the contingent claim H if for all $t \in \{0, \ldots, T\}$ it minimizes $R_t(\varphi)$ over all trading strategies with $\mathcal{V}_T^*(\varphi) = H^*$.

The construction of risk-minimizing strategies is based on the Q-martingale

$$V_t^* = E^Q\left[H^* | \mathcal{F}_t\right],$$

known as *the intrinsic value process*. Using the so-called Kunita–Watanabe decomposition for martingales, V^* can be uniquely decomposed as

$$V_t^* = V_0^* + \sum_{u=1}^{t} \vartheta_u^H \Delta S_u^* + L_t^H, \tag{5.3.7}$$

where ϑ^H is predictable, and L^H is a zero-mean Q-martingale orthogonal to S^*, i.e. $S^* L^H$ is a Q-martingale as well. For more details on the Kunita–Watanabe decomposition we refer to Föllmer and Schied (2002). Shifting the index in (5.3.7) and defining the adapted process $\tilde{\vartheta}^H$ by $\tilde{\vartheta}_u^H = \vartheta_{u+1}^H$ we have the following decomposition

$$V_t^* = V_0^* + \sum_{u=0}^{t-1} \tilde{\vartheta}_u^H \Delta S_{u+1}^* + L_t^H. \tag{5.3.8}$$

Comparing (5.2.15) and (5.3.8) we observe that H is attainable if and only if $L_T^H = 0$ Q-a.s. Using (5.2.16) and (5.2.17) we obtain the following theorem, due to Föllmer and Schweizer (1988), which relates the Kunita–Watanabe decomposition to the risk-minimizing strategy.

Theorem 5.3.7
There exists a unique risk-minimizing strategy, φ^*, with $\mathcal{V}_T^*(\varphi) = H^*$ given by

$$(\vartheta_t^*, \eta_t^*) = \left(\vartheta_{t+1}^H, V_t^* - \vartheta_{t+1}^H S_t^*\right). \tag{5.3.9}$$

Inserting (5.3.9) in (5.3.5) and using the Kunita-Watanabe decomposition from (5.3.7) we obtain the following expression for the cost process associated with the risk-minimizing strategy:

$$C_t(\varphi^*) = V_t^* - \sum_{u=0}^{t-1} \vartheta_{u+1}^H \Delta S_{u+1}^* = V_0^* + L_t^H. \tag{5.3.10}$$

Combining (5.3.10) and (5.3.6) now gives the following expression for the so-called *intrinsic risk process*, which is the risk process associated with the risk-minimizing strategy:

$$R_t(\varphi^*) = E^Q\left[\left(L_T^H - L_t^H\right)^2 \Big| \mathcal{F}_t\right]. \tag{5.3.11}$$

Note that when determining the risk-minimizing strategy we consider *all* admissible strategies. This is in contrast to many other quadratic hedging criteria, such as mean-variance hedging, where only self-financing strategies are allowed. From (5.3.11) we observe that risk-minimizing strategies are not self-financing for non-attainable claims. However, they are *mean-self-financing*, i.e. the corresponding cost processes are Q-martingales.

Since (5.3.6) involves an expectation with respect to Q, the risk-minimizing strategy depends on the chosen equivalent martingale measure. Furthermore we observe from (5.3.6) that the criterion of risk-minimization, like other quadratic hedging criteria, penalizes gains and losses equally. This is of course disadvantageous, however, when using a criterion penalizing only losses, explicit results are hard to obtain; see the discussion in Møller (2001a) and references therein.

In general, the risk-minimizing strategy is given by the (predictable) Q-expectation of the replicating strategy given the unhedgeable uncertainty, see Schweizer (1994) for a proof in a continuous-time setup. A particular simple risk-minimizing strategy is obtained in Møller (2001a), since he considers an unhedgeable risk, which is stochastically independent of the financial market. As we shall see below in Theorem 5.3.9, the expression for the risk-minimizing strategy is slightly more complicated in the present model than in Møller (2001a), since the unhedgeable risk is in the financial market.

Risk-minimizing strategies in the presence of reinvestment risk

We now turn to the derivation of risk-minimizing strategies in the present model including reinvestment risk. In order to determine the Kunita–Watanabe decomposition of V^* we introduce the Q-martingales

$$M_t^{\lambda_T} = E^Q\left[1_{((\varepsilon_1,\ldots,\varepsilon_T)=\lambda_T)}\big|\,\mathcal{F}_t\right] = E^Q\left[1_{((\varepsilon_1,\ldots,\varepsilon_T)=\lambda_T)}\big|\,\mathcal{H}_t\right] \tag{5.3.12}$$

for all $\lambda_T \in \Lambda_T$. Here, we have used that under Q the distribution of the ε's is independent of the filtration $\mathbb{G}$. Using the quantities defined in (5.3.12), we get the following expression for V_t^*:

$$V_t^* = E^Q\left[H^*\big|\,\mathcal{F}_t\right] = E^Q\left[E^Q\left[H^*\big|\,\mathcal{F}_t \vee \mathcal{H}_T\right]\big|\,\mathcal{F}_t\right] = \sum_{\lambda_T\in\Lambda_T} M_t^{\lambda_T}\pi_t^{\lambda_T,*}(H), \tag{5.3.13}$$

where $\pi_t^{\lambda_T,*}(H)$ is the unique discounted arbitrage free price for H given $(\varepsilon_1,\ldots,\varepsilon_T)=\lambda_T$. Using (5.3.13) we obtain the following expression for the development of V^* from time $t-1$ to t,

$$\begin{aligned}
\Delta V_t^* &= V_t^* - V_{t-1}^* \\
&= \sum_{\lambda_T\in\Lambda_T} M_t^{\lambda_T}\pi_t^{\lambda_T,*}(H) - \sum_{\lambda_T\in\Lambda_T} M_{t-1}^{\lambda_T}\pi_{t-1}^{\lambda_T,*}(H) \\
&= \sum_{\lambda_T\in\Lambda_T}\left(M_t^{\lambda_T}\pi_t^{\lambda_T,*}(H) - M_{t-1}^{\lambda_T}\pi_{t-1}^{\lambda_T,*}(H)\right) \\
&= \sum_{\lambda_T\in\Lambda_T}\left(\left(M_t^{\lambda_T} - M_{t-1}^{\lambda_T}\right)\pi_t^{\lambda_T,*}(H) + M_{t-1}^{\lambda_T}\left(\pi_t^{\lambda_T,*}(H) - \pi_{t-1}^{\lambda_T,*}(H)\right)\right) \\
&= \sum_{\lambda_T\in\Lambda_T}\left(\pi_t^{\lambda_T,*}(H)\Delta M_t^{\lambda_T} + M_{t-1}^{\lambda_T}\Delta\pi_t^{\lambda_T,*}(H)\right) \\
&= \sum_{\lambda_T\in\Lambda_T}\left(\pi_t^{\lambda_T,*}(H)\Delta M_t^{\lambda_T} + M_{t-1}^{\lambda_T}\vartheta_{t-1}^{\lambda_T}\Delta S_t^*\right),
\end{aligned}$$

where ϑ^{λ_T} is the number of risky assets in the replicating strategy in the complete model given $(\varepsilon_1, \ldots, \varepsilon_T) = \lambda_T$. Hence, we have the following decomposition of V^*:

$$V_t^* = V_0^* + \sum_{u=1}^{t} \left(\sum_{\lambda_T \in \Lambda_T} M_{u-1}^{\lambda_T} \vartheta_{u-1}^{\lambda_T} \right) \Delta S_u^* + \sum_{u=1}^{t} \sum_{\lambda_T \in \Lambda_T} \pi_u^{\lambda_T,*}(H) \Delta M_u^{\lambda_T}. \tag{5.3.14}$$

In order to show that (5.3.14) actually is the Kunita–Watanabe decomposition of V^*, we first note that $\sum_{\lambda_T \in \Lambda_T} M_{t-1}^{\lambda_T} \vartheta_{t-1}^{\lambda_T}$ is $\mathcal{F}_{t-1}$-measurable, such that the process ϑ^H defined by

$$\vartheta_t^H = \sum_{\lambda_T \in \Lambda_T} M_{t-1}^{\lambda_T} \vartheta_{t-1}^{\lambda_T}$$

is predictable. Now define the process L^H by

$$L_t^H = \sum_{u=1}^{t} \sum_{\lambda_T \in \Lambda_T} \pi_u^{\lambda_T,*}(H) \Delta M_u^{\lambda_T}. \tag{5.3.15}$$

Using the law of iterated expectations we see that

$$\begin{aligned} E^Q\left[\Delta L_t^H \middle| \mathcal{F}_{t-1}\right] &= E^Q\left[\sum_{\lambda_T \in \Lambda_T} \pi_t^{\lambda_T,*}(H) \Delta M_t^{\lambda_T} \middle| \mathcal{F}_{t-1} \right] \\ &= \sum_{\lambda_T \in \Lambda_T} E^Q\left[\pi_t^{\lambda_T,*}(H) \Delta M_t^{\lambda_T} \middle| \mathcal{F}_{t-1} \right] \\ &= \sum_{\lambda_T \in \Lambda_T} E^Q\left[\pi_t^{\lambda_T,*}(H) E^Q\left[\Delta M_t^{\lambda_T} \middle| \mathcal{G}_t \vee \mathcal{H}_{t-1} \right] \middle| \mathcal{F}_{t-1} \right] \\ &= 0, \end{aligned}$$

since M^{λ_T} is a martingale stochastically independent of the filtration $\mathbb{G}$. Hence, L^H is a Q-martingale. To show that that $L^H S^*$ is a Q-martingale we first observe that

$$\Delta\left(L^H S^*\right)_t = L_t^H S_t^* - L_{t-1}^H S_{t-1}^* = L_{t-1}^H \Delta S_t^* + S_{t-1}^* \Delta L_t^H + \Delta L_t^H \Delta S_t^*.$$

Thus, since L^H and S^* are Q-martingales, it is sufficient to show that

$$\begin{aligned} E^Q[\Delta L_t \Delta S_t^* | \mathcal{F}_{t-1}] &= E^Q\left[\sum_{\lambda_T \in \Lambda_T} \pi_t^{\lambda_T,*}(H) \Delta M_t^{\lambda_T} \Delta S_t^* \middle| \mathcal{F}_{t-1} \right] \\ &= \sum_{\lambda_T \in \Lambda_T} E^Q\left[\pi_t^{\lambda_T,*}(H) \Delta M_t^{\lambda_T} \Delta S_t^* \middle| \mathcal{F}_{t-1} \right] \\ &= \sum_{\lambda_T \in \Lambda_T} E^Q\left[\pi_t^{\lambda_T,*}(H) \Delta S_t^* E^Q\left[\Delta M_t^{\lambda_T} \middle| \mathcal{G}_t \vee \mathcal{H}_{t-1} \right] \middle| \mathcal{F}_{t-1} \right] \\ &= 0. \end{aligned}$$

Hence, we have proved the following.

Lemma 5.3.8
For a claim H with time of maturity T the Kunita–Watanabe decomposition is given by

$$V_t^* = V_0^* + \sum_{u=1}^{t} \left(\sum_{\lambda_T \in \Lambda_T} M_{u-1}^{\lambda_T} \vartheta_{u-1}^{\lambda_T} \right) \Delta S_u^* + \sum_{u=1}^{t} \sum_{\lambda_T \in \Lambda_T} \pi_u^{\lambda_T,*}(H) \Delta M_u^{\lambda_T}.$$

Combining Lemma 5.3.8, Theorem 5.3.7 and the expression for the intrinsic risk process in (5.3.11) we obtain the following theorem regarding the risk-minimizing strategy and the intrinsic risk process.

Theorem 5.3.9
The risk-minimizing strategy, φ^*, for H is given by

$$(\vartheta_t^*, \eta_t^*) = \left(\sum_{\lambda_T \in \Lambda_T} M_t^{\lambda_T} \vartheta_t^{\lambda_T}, \sum_{\lambda_T \in \Lambda_T} M_t^{\lambda_T} \pi_t^{\lambda_T,*}(H) - \left(\sum_{\lambda_T \in \Lambda_T} M_t^{\lambda_T} \vartheta_t^{\lambda_T} \right) S_t^* \right).$$

The intrinsic risk process is given by

$$R_t(\varphi^*) = E^Q \left[\left(\sum_{u=t+1}^{T} \sum_{\lambda_T \in \Lambda_T} \pi_u^{\lambda_T,*}(H) \Delta M_u^{\lambda_T} \right)^2 \middle| \mathcal{F}_t \right].$$

Thus, the number of risky assets held in the risk-minimizing strategy at time t is the average under Q of the replicating strategies for H in the conditional models given the outcome of $(\varepsilon_1, \ldots, \varepsilon_T)$. The deposit in the savings account is adjusted each period according to the realization of the unhedgeable variables, such that the discounted value process is equal to the intrinsic value process.

Inserting (5.3.15) in (5.3.10) gives the following expression for the cost process associated with φ^*:

$$C_t(\varphi^*) = V_0^* + \sum_{u=1}^{t} \sum_{\lambda_T \in \Lambda_T} \pi_u^{\lambda_T,*}(H) \Delta M_u^{\lambda_T}. \tag{5.3.16}$$

From (5.3.16) we see that the change in the cost process at time t for an agent following the risk-minimizing strategy depends on the change in the Q-martingales M^{λ_T} associated with the outcome of ε_t. If the claim is attainable at some time t prior to T, then the cost process is constant P-a.s. from time t, and hence, the intrinsic risk process is zero from time t.

Example 5.3.10 Let $\widetilde{T} = 2$, $T = 3$ and $H = 1$. For the fixed Q-measure given by $\tilde{q}^{\lambda_{t-1}} = \tilde{q} \in (0,1)$ we now obtain the risk-minimizing strategy from Theorem 5.3.9. At time 0 the risk-minimizing strategy is given by

$$\begin{aligned}
\vartheta_0^* &= P(0,2)\left(\tilde{q}\frac{P^{d,h}(1,3)-P^{u,h}(1,3)}{P^d(1,2)-P^u(1,2)}+(1-\tilde{q})\frac{P^{d,\ell}(1,3)-P^{u,\ell}(1,3)}{P^d(1,2)-P^u(1,2)}\right),\\
\eta_0^* &= \tilde{p}\frac{P^d(1,2)P^{u,h}(1,3)-P^u(1,2)P^{d,h}(1,3)}{(1+r_0)(P^d(1,2)-P^u(1,2))}\\
&\quad+(1-\tilde{p})\frac{P^d(1,2)P^{u,\ell}(1,3)-P^u(1,2)P^{d,\ell}(1,3)}{(1+r_0)(P^d(1,2)-P^u(1,2))},
\end{aligned}$$

whereas it at time 1 and 2 is given by

$$\left(\vartheta_1^{\xi_1,\lambda_1,*},\eta_1^{\xi_1,\lambda_1,*}\right)=\left(\frac{P(0,2)P^{\xi_1,\lambda_1}(1,3)}{P^{\xi_1}(1,2)},0\right) \quad\text{and}\quad \left(\vartheta_2^{\xi_2,\lambda_2,*},\eta_2^{\xi_2,\lambda_2,*}\right)=\left(0,\frac{1}{B_3^{\xi_2,\lambda_1}}\right).$$

We note that since the claim is attainable from time 1, the risk-minimizing strategy and super-replicating strategies coincide at times 1 and 2. Moreover, since the strategies coincide so do the super-replicating price and the value of the portfolio held in the risk-minimizing strategy.

□

5.4 A numerical illustration

Here, the purpose is to provide some numbers in the continuing example considered in Sections 5.2 and 5.3. Hence, $\widetilde{T}=2$, $T=3$ and $H=1$. Now assume that given the initial forward rate vector $(r_0, f_{0,1})$ the forward rates at time t, $t\in\{1,2,3\}$, are given by

$$\begin{aligned}
r_t^{\xi_t,\lambda_{t-1}} &= r_0\prod_{i=1}^{t}\left(a_1 1_{(\rho_i=u)}+a_2 1_{(\rho_i=d)}\right)\prod_{i=1}^{t-1}\left(a_3 1_{(\varepsilon_i=h)}+a_4 1_{(\varepsilon_i=\ell)}\right),\\
f_{t,t+1}^{\xi_t,\lambda_t} &= r_t^{\xi_t,\lambda_{t-1}}\left(b_1 1_{(\varepsilon_t=h)}+b_2 1_{(\varepsilon_t=\ell)}\right),
\end{aligned}$$

where $a_1,\ldots,a_4,b_1,b_2$ are positive constants, and $\prod_{i=1}^{u}$ is interpreted as 1 if $u=0$. The constants a_1 and a_2 describe the movement of the forward rate vector due to the outcome of the ρ's, whereas a_3 and a_4 describe the dependence of the forward rate vector on past values of the ε's, and finally b_1 and b_2 describe the unhedgeable uncertainty associated with the newly issued bonds. In this simple model the dependence on ξ_t is given by the number of u's and not by the ordering of the u's. Hence, the number of states at time 2 is reduced from 16 to 12. However, this is still a large number of states compared to the 4 in a binomial model (3 if the binomial model is recombining). In contrast to an additive structure, the multiplicative structure above ensures that the forward rates are strictly positive. Now let the initial forward rate curve and the constants be given by $r_0=0.03$, $f_{0,1}=0.031$, $a_1=1.25$, $a_2=0.8$, $a_3=1.01$, $a_4=0.99$, $b_1=1.0325$ and $b_2=1.015$.

Recall from Examples 5.3.5 and 5.3.10 that the optimal super replicating and risk-minimizing strategies for $H=1$ depend on ξ_2 and λ_1, only. Thus, Figure 5.4.1 shows the forward rates relevant for determining the hedging strategies. Furthermore, Example 5.3.5 gives that the super-replicating price at time 0 corresponds to the zero coupon bond price in the conditional model given $\varepsilon_1 = \ell$, which in turn corresponds to a 2-period forward rate of 0.03142. Here, and in the remaining of the section, all numbers are given with 4 significant digits.

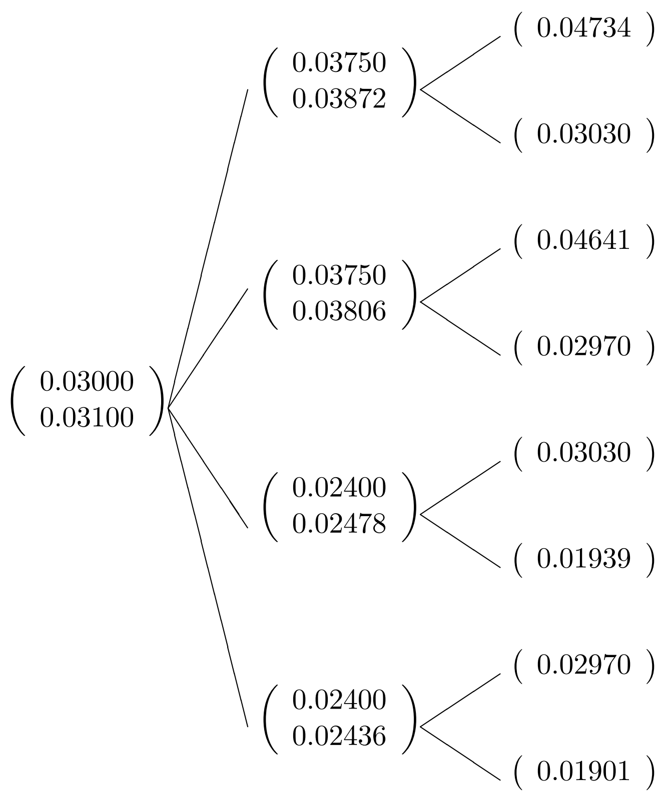

Figure 5.4.1: *Relevant forward rates at time 0, 1 and 2. At time 0 and 1 the vector shows the short rate and 1-period forward rate, whereas at time 2 only the short rate is relevant.*

From Example 5.3.10 we furthermore note that the risk-minimizing strategy depends on $\tilde{q}$. Thus, to obtain some numbers we have to specify Q. Henceforth, we let $\tilde{p} = 0.5$ and consider risk-minimization under the minimal martingale measure, i.e. $\tilde{q} = \tilde{p}$. The optimal super-replicating and risk-minimizing strategies and the corresponding prices are illustrated in Figure 5.4.2. Here, the first column gives the super-replicating price, $\hat{\vartheta}$ and $\hat{\eta}$, and the second column shows the risk-minimizing price, ϑ^* and η^*. Here, and henceforth we refer to the value of the risk-minimizing strategy as a price, since it is the arbitrage free price under the chosen equivalent martingale measure. Since the super-replicating price is an upper bound for the interval of arbitrage free prices, the price using the criterion of risk-minimization is obviously lower than or equal to the super-replicating price. In particular it is strictly lower if there is a reinvestment risk, i.e. if $b_1 \neq b_2$. In addition to the hedging strategies we may apply Proposition 5.3.6 to obtain the maximal riskfree maturity guarantee $G_0^3 = 1.095$ and the maximal riskfree periodic interest rate guarantee $g_0^3 = 0.03081$.

$$\begin{pmatrix} 0.9130 & 0.9128 \\ 1.8397 & 1.8470 \\ -0.9267 & -0.9342 \end{pmatrix}$$

$$\begin{pmatrix} 0.9279 & 0.9279 \\ 0.9068 & 0.9068 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0.9548 & 0.9548 \\ 0 & 0 \\ 0.8935 & 0.8935 \end{pmatrix} \quad \begin{pmatrix} 0.9706 & 0.9706 \\ 0 & 0 \\ 0.9083 & 0.9083 \end{pmatrix}$$

$$\begin{pmatrix} 0.9285 & 0.9285 \\ 0.9353 & 0.9353 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0.9557 & 0.9557 \\ 0 & 0 \\ 0.8943 & 0.8943 \end{pmatrix} \quad \begin{pmatrix} 0.9712 & 0.9712 \\ 0 & 0 \\ 0.9088 & 0.9088 \end{pmatrix}$$

$$\begin{pmatrix} 0.9529 & 0.9529 \\ 0.9474 & 0.9474 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0.9706 & 0.9706 \\ 0 & 0 \\ 0.9202 & 0.9202 \end{pmatrix} \quad \begin{pmatrix} 0.9810 & 0.9810 \\ 0 & 0 \\ 0.9301 & 0.9301 \end{pmatrix}$$

$$\begin{pmatrix} 0.9533 & 0.9533 \\ 0.9478 & 0.9478 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0.9712 & 0.9712 \\ 0 & 0 \\ 0.9208 & 0.9208 \end{pmatrix} \quad \begin{pmatrix} 0.9813 & 0.9813 \\ 0 & 0 \\ 0.9304 & 0.9304 \end{pmatrix}$$

Figure 5.4.2: *Hedging strategies and associated prices. First column: Super-replicating price, $\hat{\vartheta}$ and $\hat{\eta}$. Second column: Price using risk-minimization, ϑ^* and η^*.*

Now we are interested in how the prices using the criteria of super-replication and risk-minimization are affected by changing b_1 and b_2, which determine the shape of the forward rate curve at time t, $t \in \{1, 2, 3\}$. Investigating Table 5.4.1 we observe that the price at time 0 using risk-minimization is decreasing in both b_1 and b_2. This is intuitively clear since a steeper positive slope leads to lower bond prices and hence a smaller initial investment. The super-replicating price is also decreasing in b_2, however, in contrast to the risk-minimizing price, it is independent of b_1. The independence can be explained by the fact that the criterion of super-replication considers the "worst scenario" only. Furthermore, we observe that, as anticipated above, the risk-minimizing and super-replicating prices coincide when $b_1 = b_2$, i.e. when there is no reinvestment risk.

A comparison with practice in Danish life insurance

The Danish life insurance companies are forced by legislation to disregard the reinvestment risk and value their long term liabilities using a yield curve, which is level beyond 30 years. Here, we consider the similar principle of a level yield curve beyond the time of maturity of the longest traded bond. We shall refer to this approach as the principle of a *level long*

b_1	b_2	Risk-minimization	Super-replication
1.05	1.015	0.9125	0.9130
1.0325	1.015	0.9128	0.9130
1.015	1.015	0.9130	0.9130
1.0325	1.015	0.9128	0.9130
1.0325	1	0.9130	0.9134
1.0325	0.99	0.9131	0.9137
1.0325	0.98	0.9132	0.9140

Table 5.4.1: *A comparison of prices at time 0 using risk-minimization and super-replication. Top: Dependence on b_1. Bottom: Dependence on b_2.*

term yield curve (even though we in discrete time have a yield vector rather than a yield curve). In this setting with discrete compounding the yield at time 0 of a zero coupon bond with maturity t is defined by

$$y_{0,t} = \left(\frac{1}{P(0,t)}\right)^{\frac{1}{t}} - 1.$$

Here, the yield vector at time 0 is given by $(y_{0,1}, y_{0,2}) = (0.03000, 0.03050)$. Thus, the principle of a level long term yield curve corresponds to assuming $y_{0,3} = y_{0,2} = 0.03050$, which leads to a price of 0.9138. In addition to the level long term yield curve principle we introduce the analogous principle of a *level long term forward rate curve*, where we price using a forward rate curve, which is level beyond the time of maturity of the longest traded bond. Here, this leads to the price 0.9134. We note that both principles only depend on the present forward rate curve, and thus they are independent of the possible future developments. Furthermore none of the principles are based on the no arbitrage principle.

We now turn to the relationship between the yield vector and the forward rate vector. When the yield vector is increasing (decreasing) the forward rate vector lies above (below) the yield vector. Thus, if we have a level long term yield vector, the long term forward rate vector is level and equal to the yield vector. On the other hand an increasing (decreasing) forward rate vector which is level for long times to maturity corresponds to a yield vector which increases (decreases) and tends towards the forward rate vector as the time to maturity increases. The increase (decrease) in the yield vector on the interval, where the forward rate vector is level, is given by

$$y_{0,t} - y_{0,t-1} = (1 + y_{0,t-1})\left(\left(\frac{1+f_{0,t-1}}{1+y_{0,t-1}}\right)^{\frac{1}{t}} - 1\right).$$

In this example the forward rate vector at time 0 is increasing, such that the principle of a level long term forward rate curve leads to a lower price than the level long term yield curve principle.

Now we are interested in whether the principles lead to prices in the interval of arbitrage free prices. In this simple example, where we consider a fixed claim and the time horizons

$\widetilde{T} = 2$ and $T = 3$ a principle leads to a price in the interval of arbitrage free prices if and only if the value of $f_{0,2}$ implied by the principle lies above the 2-period forward rate implied by the super-replicating price and below the 2-period forward rate implied by the "best scenario" price (which is a lower bound for the interval of arbitrage free prices). From Table 5.4.2 we observe that if we allow for increasing forward rate vectors only, both

b_1	b_2	"Best scenario"	Super-replication	Level forward	Level yield
1.05	1.015	0.03250	0.03142	0.03100	0.03050
1.0325	1.015	0.03196	0.03142	0.03100	0.03050
1.015	1.015	0.03142	0.03142	0.03100	0.03050
1.0325	1.015	0.03196	0.03142	0.03100	0.03050
1.0325	1	0.03196	0.03096	0.03100	0.03050
1.0325	0.99	0.03196	0.03065	0.03100	0.03050
1.0325	0.98	0.03196	0.03034	0.03100	0.03050

Table 5.4.2: *Values of $f_{0,2}$ implied by, respectively, the "best scenario" price, the super-replicating price and the principles of a level long term forward rate/yield curve. Top: Dependence on b_1. Bottom: Dependence on b_2.*

principles lead to a 2-period forward rate below the one implied by the super-replicating price, and hence they lead to a price higher than the super-replicating price. Thus, in this case both principles clearly overestimate the price. If we allow for a level or decreasing forward rate vector, the 2-period forward rate implied by the super-replicating price is lower than the one implied by a the principle of a level long term forward rate curve, and if the possible decrease is sufficiently large also lower than the one implied by using a level long term yield curve, such that the principles lead to prices, which lie in the interval of arbitrage free prices. However, especially the price obtained using a level long term yield curve is in the high end of the interval of arbitrage free prices. Note that the same information also could have been observed from Table 5.4.1. Based on the discussion above we conclude that the principles should not be used in situations where a decreasing forward rate curve is very unlikely. If one uses one of the principles anyhow, we recommend using the level long term forward rate principle and at the same time to keep in mind that the price (most likely) is overestimated. In situations where a decreasing forward rate vector is more likely, the principles are more likely to be accurate. The accuracy depends heavily on the situation and in particular on the correspondence between the present forward rate vector and the conditional forward rate vectors. The conclusion regarding the principles is that even though they are easy to use, their results should be used as guidelines only.

Chapter 6

A Continuous-Time Model for Reinvestment Risk in Bond Markets

(This chapter is an adapted version of Dahl (2005a))

We propose a bond market model, where, as in practice, only bonds with a limited time *to* maturity are traded in the market. As time passes, new bonds with stochastic initial prices are introduced in the market. Hence, we are able to model the reinvestment risk present in practice, when considering long term contracts. To quantify and control the reinvestment risk we apply the criterion of risk-minimization.

6.1 Introduction

In the literature, bond markets are usually assumed to include all bonds with time of maturity less than or equal to the time of maturity of the considered claim. However, in practice only bonds with a limited (sufficiently short) time to maturity are traded. Hence, standard models are only adequate to describe pricing and hedging of so-called short term contracts, where the payoff depends on bonds with time to maturity less than or equal to the longest traded bond. When considering long term contracts, where the payoff depends on bonds with longer time to maturity than the longest traded bond, the bond market does not in general include bonds which at all times allow for a perfect hedge of the contract. Thus, in practice, an agent interested in pricing and hedging long term contracts is exposed to a reinvestment risk, which is ignored in standard bond market models. Here, the reinvestment risk refers to the uncertainty associated with the obtainable rate of return, when reinvesting in bonds not yet traded in the market. An example of long term contracts sold in practice are life insurance contracts, where the liabilities of the insurance companies often extend 50 years, or more, into the future. In this chapter, we propose a model, where pricing and hedging of short term contracts is similar to a standard bond market model, whereas the model includes reinvestment risk, when considering long term contracts.

In order to describe the reinvestment risk, we initially consider a standard continuous-time bond market model with some fixed finite time horizon, which is less than (or equal to) the time horizon of the considered payment process. At fixed times new bonds are issued in the market, such that we immediately after the issue of new bonds consider a standard model identical to the initial one. The entry prices of the new bonds depend on the prices of the bonds already traded and a stochastic term. As is standard in bond market literature, we model the forward rates rather than the bond prices themselves. Between the times of issue, the forward rates follow a standard Heath–Jarrow–Morton model, see Heath et al. (1992). When new bonds are issued, the forward rate curve is extended. We assume that at each time of issue the extension is continuous and depends on a single random variable. The idea of fixing the maximum time to maturity of the traded assets and introducing new assets as times passes can also be found in Neuberger (1999), who considers a market for futures on oil prices. Neuberger (1999) models the initial price of the new future as a linear function of prices on traded futures and a normally distributed error term.

To the author's knowledge the only other papers considering the problem of modelling the prices of newly issued bonds are Sommer (1997) and Dahl (2005b) (see Chapter 5). Dahl (2005b) considers a discrete-time model for the reinvestment risk, whereas Sommer (1997) considers a continuous-time bond market. A major difference between Sommer (1997) and this chapter is the way new bonds are issued and priced in the market. While Sommer (1997) considers the case where new bonds are issued continuously, this chapter, as is the case in practice, considers a set of fixed times, where new bonds are issued. Hence, the present model should be more apt to describe practice. Within his setup Sommer derives conditions on the forward rate dynamics in order to have sufficiently smooth forward rate curves and risk-minimizing strategies.

To quantify and control the reinvestment risk associated with long term contracts, we apply the criterion of risk-minimization introduced by Föllmer and Sondermann (1986) for contingent claims and extended in Møller (2001c) to the case of payment processes. The derivation of the risk-minimizing strategies are based on the ideas of Schweizer (1994) regarding risk-minimization under restricted information. Hence, the risk-minimizing strategies are given in terms of the replicating strategies in the case without reinvestment risk.

The chapter is organized as follows: In Section 6.2 a bond market model including reinvestment risk is introduced. This is done in two steps: First we describe a standard bond market model, and then the model is extended to include reinvestment risk. In this section we also introduce the considered class of equivalent martingale measures and the relevant financial terminology. Risk-minimizing strategies are derived in Section 6.3, and we conclude the chapter by describing a possible implementation of the model in Section 6.4.

6.2 The bond market model

Let $\widehat{T}$ be a fixed finite time horizon and $(\Omega, \mathcal{F}, P)$ a probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le \widehat{T}}$ satisfying the usual conditions of right-continuity, i.e. $\mathcal{F}_t = \bigcap_{u>t} \mathcal{F}_u$, and completeness, i.e. $\mathcal{F}_0$ contains all P-null sets.

6.2.1 A standard model

Consider another fixed time horizon $\widetilde{T}$, $\widetilde{T} \le \widehat{T}$, and a bond market, where at time t, $0 \le t \le \widetilde{T}$ all zero coupon bonds with maturity τ, $t \le \tau \le \widetilde{T}$ are traded. Let $P(t,\tau)$ denote the price at time t of a zero coupon bond maturing at time τ. To avoid arbitrage we assume that $P(t,\tau)$ is strictly positive and $P(t,t) = 1$ for all t. For non-negative interest rates the price $P(t,\tau)$ is a decreasing function of τ for fixed t. An important quantity when modelling bond prices is the (instantaneous) forward rate with maturity τ contracted at time t defined by

$$f(t,\tau) = -\frac{\partial \log P(t,\tau)}{\partial \tau}, \tag{6.2.1}$$

or, stated differently,

$$P(t,\tau) = e^{-\int_t^\tau f(t,u)du}. \tag{6.2.2}$$

The forward rate $f(t,\tau)$ can be interpreted as the riskfree interest rate, contracted at time t over the infinitesimal interval $[\tau, \tau + d\tau)$. The short rate process $(r_t)_{0 \le t \le \widetilde{T}}$ is defined as $r_t = f(t,t)$. Since it is inconvenient to model the dynamics of bond prices directly, the common approach in the literature is to model interest rates. Here, we take the approach of Heath et al. (1992) where the dynamics of not only the short rate but the entire forward rate curve are modelled. The connection between the forward rates and bond prices established in (6.2.1) and (6.2.2) then gives the dynamics of the bond prices. For fixed τ, $0 \le t \le \tau \le \widetilde{T}$, the P-dynamics of the forward rates are given by

$$df(t,\tau) = \alpha^P(t,\tau)dt + \sigma(t,\tau)dW_t^P, \tag{6.2.3}$$

where W^P is a Wiener process under P. For simplicity W^P is assumed to be 1-dimensional. The processes α^P and σ are adapted to the filtration $\mathbb{G} = (\mathcal{G}_t)_{0 \le t \le \widetilde{T}}$, which is the P-augmentation of the natural filtration generated by the Wiener process, i.e. $\mathcal{G}_t = \mathcal{G}_t^+ \vee \mathcal{N}$, where $\mathcal{N}$ is the σ-algebra generated by all P-null sets and

$$\mathcal{G}_t^+ = \sigma\left\{W_u^P, u \le t\right\}.$$

Using Björk (2004, Proposition 20.5) we obtain the following P-dynamics for the price process of a bond with maturity τ:

$$\begin{aligned} dP(t,\tau) = &\left(r_t - \int_t^\tau \alpha^P(t,u)du + \frac{1}{2}\left(\int_t^\tau \sigma(t,u)du\right)^2\right) P(t,\tau)dt \\ &- \int_t^\tau \sigma(t,u)du P(t,\tau)dW_t^P, \;\; 0 \le t \le \tau \le \widetilde{T}. \end{aligned} \tag{6.2.4}$$

In addition to the bonds, we assume that the financial market includes a savings account earning the short rate r. The dynamics of the savings account are

$$dB_t = r_t B_t dt, \quad B_0 = 1. \tag{6.2.5}$$

Remark 6.2.1 The existence of a savings account with drift r can be proven if we allow for investments in infinitely many different bonds. In this case, investing in a roll-over strategy in just-maturing bonds produces a value process, whose dynamics are given by (6.2.5), see Björk, Kabanov and Runggaldier (1997).

□

For any $\mathbb{G}$-adapted process h we may define a likelihood process $(\Lambda_t)_{0\leq t\leq \widetilde{T}}$ by

$$\Lambda_t = e^{-\frac{1}{2}\int_0^t h_u^2 du + \int_0^t h_u dW_u^P}.$$

It is well known, see e.g. Musiela and Rutkowski (1997) and Björk (2004), that if there exists an h such that $E^P[\Lambda_{\widetilde{T}}] = 1$ and, for all $0 \leq t \leq \tau \leq \widetilde{T}$, the Heath–Jarrow–Morton (HJM) drift condition

$$\alpha^P(t,\tau) = \sigma(t,\tau)\left(\int_t^\tau \sigma(t,u)du - h_t\right) \tag{6.2.6}$$

holds, then there exists a unique equivalent martingale measure Q given by

$$\frac{dQ}{dP} = \Lambda_{\widetilde{T}}. \tag{6.2.7}$$

Here, it is important that for fixed t, (6.2.6) holds simultaneously for all τ, $0 \leq t \leq \tau \leq \widetilde{T}$. Recall that an equivalent martingale measure fulfills three requirements: Firstly, it is equivalent to P. Secondly, all discounted price processes are martingales under the new measure and lastly, it is a probability measure. If there exists a unique equivalent martingale measure the model is arbitrage free and complete, see e.g. Björk (2004, Chapter 10).

6.2.2 Extending the standard model to include reinvestment risk

We now extend the standard model in Section 6.2.1 to include reinvestment risk. The idea is as follows: Assume that at time 0 the bond market can be described by the standard model introduced in Section 6.2.1. At some predetermined times new bonds are issued in the market, such that immediately after the issue the bond market is given by a standard model identical to the one at time 0. To introduce reinvestment risk in the model the initial prices of the new bonds issued depend on a random variable independent of the observable bond prices.

In order to extend the bond market we introduce the fixed time horizon T, $\widetilde{T} \leq T \leq \widehat{T}$. The interpretation of the time horizons is as follows: $\widehat{T}$ is the last time where trading is possible in the bond market, i.e. $\widehat{T}$ may be thought of as "the end of the world", T is the last time at which we allow payments and $\widetilde{T}$ is the upper limit for the time to maturity of a bond traded in the market. Hence, at any time t the time to maturity of the longest traded bond is less than or equal to $\widetilde{T}$. Now define the sequence $0 = T_0 < T_1 < \ldots < T_n \leq T$ of times, where new bonds are issued in the market. At time T_i new bonds are issued such that all bonds with time *to* maturity less than or equal to $\widetilde{T}$ are traded. To ensure that at any time, bonds are traded in the market, we assume that $\widetilde{T} \geq \max_{i=1,\ldots,n}(T_i - T_{i-1})$ and $\widehat{T} = T_n + \widetilde{T}$.

The illustration in Figure 6.2.1 shows one possible ordering of $T_1, \ldots, T_n$, T and $\widetilde{T}$ in the case $n = 3$.

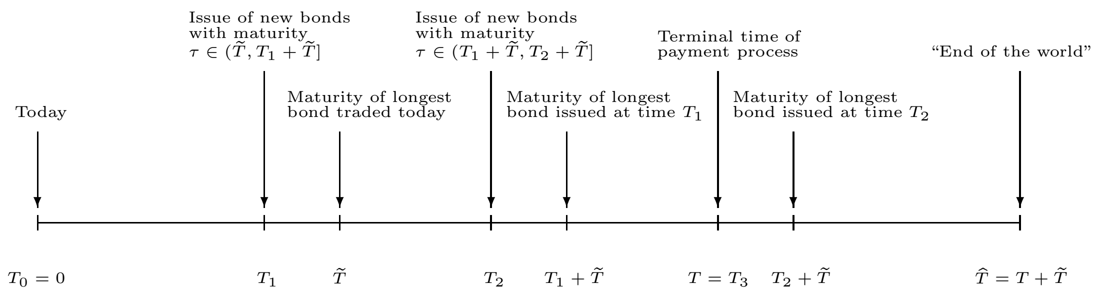

Figure 6.2.1: *Illustration of* T_1, T_2, T_3, $\widetilde{T}$, T *and* $\widehat{T}$.

For fixed t we define

$$i_t = \sup \{0 \leq i \leq n | T_i \leq t\},$$

such that T_{i_t} is the last time new bonds are issued prior to time t (time t included). Thus, at time t the time of maturity, τ, of the bonds traded in the bond market satisfies $t \leq \tau \leq T_{i_t} + \widetilde{T}$. For an illustration of T_{i_t} see Figure 6.2.2.

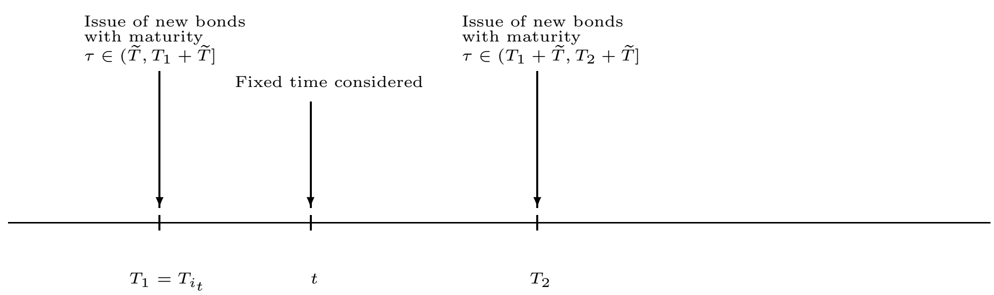

Figure 6.2.2: *Illustration of* T_{i_t}.

When the forward rates are defined, i.e. for $0 \leq t \leq \tau \leq T_{i_t} + \widetilde{T}$, their dynamics are given by

$$df(t,\tau) = \alpha^P(t,\tau)dt + \sigma(t,\tau)dW_t^P,$$

where the processes α^P and σ are $\mathbb{F}$-adapted, and W^P is a 1-dimensional Wiener process under P. As in Section 6.2.1 the filtration $\mathbb{G} = (\mathcal{G}_t)_{0\le t\le \widehat{T}}$ is the P-augmentation of the natural filtration generated by the Wiener process. Note that $\mathbb{G}$ and the short rate process $(r_t)_{0\le t\le \widehat{T}}$ are defined until time $\widehat{T}$. As noted above, forward rates for all maturities are not defined at time 0. They are introduced at the times of issue of new bonds. To model the initial value of the new forward rates at time T_i, $i \in \{1,\ldots,n\}$, we introduce a sequence $Y = (Y_i)_{i=1,\ldots,n}$ of mutually independent random variables with distribution functions $(F_i^P)_{i=1,\ldots,n}$. Assume that Y and W^P are independent (as discussed below this is no restriction). Here, Y_i, which is revealed at time T_i, describes the uncertainty independent of the observed bond prices associated with the initial prices of bonds issued at time T_i. The filtration $\mathbb{H} = (\mathcal{H}_t)_{0\le t\le \widehat{T}}$ is defined as the P-augmentation of the natural filtration generated by the random variables $(Y_i)_{i=1,\ldots,n}$, i.e. $\mathcal{H}_t = \mathcal{H}_t^+ \vee \mathcal{N}$, where

$$\mathcal{H}_t^+ = \sigma\{(Y_i)_{i=1,\ldots,i_t}\}.$$

We now assume that $\mathbb{F}$ is the total filtration generated by the bond market, such that

$$\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t.$$

For $T_{i-1} + \widetilde{T} < \tau \le T_i + \widetilde{T}$ we model the forward rates by

$$f(T_i,\tau) = f(T_i, T_{i-1} + \widetilde{T}) + \int_{T_{i-1}+\widetilde{T}}^{\tau} \gamma_u^i du, \tag{6.2.8}$$

where γ^i is an $\mathcal{F}_{T_i}$-measurable function, i.e. each γ_u^i is $\mathcal{F}_{T_i}$-measurable for $u \in (T_{i-1}+\widetilde{T}, T_i+\widetilde{T}]$. The interpretation is that the new forward rates introduced at time T_i depend on the past forward rates and some noise represented by the random variable Y_i. The assumed independence between W^P and Y is no restriction, since we otherwise could define a vector of random variables $\widetilde{Y} = (\widetilde{Y}_i)_{i=1,\ldots,n}$ independent of W^P and functions $\widetilde{\gamma}^1,\ldots,\widetilde{\gamma}^n$, such that $\widetilde{f}(T_i,\tau)$ given by (6.2.8) with γ and Y replaced by $\widetilde{\gamma}$ and $\widetilde{Y}$, respectively, has the same distribution as $f(T_i,\tau)$ for $T_{i-1} + \widetilde{T} < \tau \le T_i + \widetilde{T}$. We note from (6.2.8) that the forward rate curve is continuous at all times. In addition to $\mathbb{F}$ we consider the filtrations $\mathbb{F}^{T_i}$, $i \in \{0,1,\ldots,n\}$, given by

$$\mathbb{F}^{T_i} = (\mathcal{F}_t^{T_i})_{0\le t\le \widehat{T}} = (\mathcal{H}_{T_i} \vee \mathcal{F}_t)_{0\le t\le \widehat{T}}.$$

We immediately note that $\mathbb{F} = \mathbb{F}^{T_0} = \mathbb{F}^0$. For $i \ge 1$ the interpretation of the filtration $\mathbb{F}^{T_i}$ is that the sequence $(Y_j)_{j=1,\ldots,i}$ is known at time 0. When considering $\mathbb{F}^{T_n}$ the entire vector Y is known at time 0, so the model is complete. Furthermore, we note that if we for $i \in \{0,\ldots,n-1\}$ consider the filtration $\mathbb{F}^{T_i}$ and the time interval $[0, T_{i+1})$, then the model is complete.

In the extended bond market we have that for any i the outcome of the random variable Y_i affects the initial prices of bonds issued at time T_i, and once it is realized it may affect the drift and the volatility of the forward rates. Thus, prior to time T_i, where Y_i is realized, we are unable to trade in assets depending on the outcome of Y_i. Hence, the vector Y is unhedgeable. Once the forward rates are introduced, the dynamics of the bonds are driven solely by W^P. Thus, the model can be viewed as a series of complete models on $[T_i, T_{i+1})$ and a vector of independent random variables realized at times T_i, $i \in \{1, \ldots, n\}$.

Remark 6.2.2 As noted above the model is complete when considering $\mathbb{F}^{T_n}$. Hence, contingent on the outcome of Y, all zero coupon bonds have unique prices at all times (even before they are traded). Thus, at time t, $0 \leq t < T_n$, where the unconditional model is incomplete, we have a forward rate curve for all maturities in the conditional model. Here, we note that all conditional forward rate curves, of which there may be infinitely many, are identical until time $T_{i_t} + \widetilde{T}$. However, in the unconditional model, the future values of $Y_{i_t+1}, \ldots, Y_n$ are unknown. Hence, it is uncertain which of the conditional forward rate curves will turn out in retrospect to have been "the correct one" when $Y_{i_t+1}, \ldots, Y_n$ have been observed at time T. Thus, we can interpret the reinvestment risk as the uncertainty associated with which of the conditional forward rate curves in retrospect has turned out to have been "the correct one". This in turn gives that the magnitude of the reinvestment risk is related to how much the conditional forward rate curves differ.

□

We now derive an expression for future forward rates, and in particular future short rates, in terms of the present forward rates and the future uncertainty. For fixed τ we define

$$\bar{i}_\tau = \inf \left\{ 0 \leq i \leq n \,\middle|\, T_i + \widetilde{T} \geq \tau \right\},$$

such that $T_{\bar{i}_\tau}$ is the first time a bond with maturity τ is traded. Hence, the initial time a forward rate with maturity τ is defined. For $0 \leq t \leq u \leq \tau \leq T_{i_t} + \widetilde{T}$ we have the well-known relation

$$f(u,\tau) = f(t,\tau) + \int_t^u df(s,\tau). \tag{6.2.9}$$

However, as can be seen from the following proposition, the relationship between the forward rates is in general more involved, since the future forward rates depend on the entry prices of bonds yet to be issued. In the proposition, and throughout the chapter, we interpret $\sum_{k=j}^{\ell}$ as 0 if $\ell < j$.

Proposition 6.2.3
For $0 \leq t \leq u \leq \tau$ we have the following relation between the forward rates:

$$\begin{aligned} f(u,\tau) = f(t,(T_{i_t} + \widetilde{T}) \wedge \tau) + \sum_{k=i_t}^{\bar{i}_\tau - 1} \left(\int_{t \vee T_k}^{T_{k+1}} df(s, T_k + \widetilde{T}) + \int_{T_k + \widetilde{T}}^{\tau \wedge (T_{k+1} + \widetilde{T})} \gamma_s^{k+1} ds \right) \\ + \int_{T_{\bar{i}_\tau} \vee t}^{u} df(s,\tau). \end{aligned} \tag{6.2.10}$$

In particular it holds for the short rate that

$$r_u = f(t, (T_{i_t} + \widetilde{T}) \wedge u) + \sum_{k=i_t}^{\bar{i}_u - 1} \left(\int_{t \vee T_k}^{T_{k+1}} df(s, T_k + \widetilde{T}) + \int_{T_k + \widetilde{T}}^{u \wedge (T_{k+1} + \widetilde{T})} \gamma_s^{k+1} ds \right) + \int_{T_{\bar{i}_u} \vee t}^{u} df(s, u). \tag{6.2.11}$$

Proof of Proposition 6.2.3: Formula (6.2.10) follows by repeated use of relations (6.2.8) and (6.2.9), whereas the expression for the short rate in (6.2.11) is obtained by setting $\tau = u$ in (6.2.10).

□

A class of equivalent martingale measures

In this section we introduce the considered class of equivalent martingale measures. First we determine the unique Girsanov kernel with respect to the Wiener process and define the equivalent martingale measure corresponding to a change of measure with respect to the Wiener process only. We then consider a change of measure with respect to Y as well. Since Y is unhedgeable there exist infinitely many equivalent martingale measures. Here, we consider a class of measures with particular nice properties.

Similarly to the standard model we observe that the existence of an equivalent martingale measure depends on the existence of an $\mathbb{F}$-adapted process h, such that for all $0 \leq t \leq \tau \leq T_{i_t} + \widetilde{T}$ the HJM drift condition

$$\alpha^P(t, \tau) = \sigma(t, \tau) \left(\int_t^\tau \sigma(t, u) du - h_t \right)$$

is satisfied, and the likelihood process $\Lambda = (\Lambda_t)_{0 \leq t \leq \widehat{T}}$ defined by

$$\Lambda_t = e^{-\frac{1}{2} \int_0^t h_u^2 du + \int_0^t h_u dW_u^P}$$

fulfills $E^P[\Lambda_{\widehat{T}}] = 1$. Hence, if such an h exists, we may define an equivalent martingale measure Q^0 by

$$\frac{dQ^0}{dP} = \Lambda_{\widehat{T}}. \tag{6.2.12}$$

However, the equivalent martingale measure Q^0 is not unique. In particular we can define another likelihood process $U = (U_t)_{0 \leq t \leq \widehat{T}}$ by

$$U_t = \prod_{j=1}^{i_t} (1 + u_j(Y_j)),$$

for some functions u_j, $j \in \{1, \ldots, n\}$, satisfying $u_j(y) > -1$ for all y in the support of Y_j and $E^P[u_j(Y_j)] = 0$. Here, and henceforth, $\prod_{j=k}^{\ell}$ is interpreted as 1 if $k > \ell$. If

$E^{Q^0}[U_{\widehat{T}}] = 1$ (or equivalently $E^P[\Lambda_{\widehat{T}} U_{\widehat{T}}] = 1$), we can define an equivalent martingale measure Q by

$$\frac{dQ}{dQ^0} = U_{\widehat{T}}. \tag{6.2.13}$$

Girsanov's theorem gives that for any Q of the form (6.2.13), the process

$$W_t^Q = W_t^P - \int_0^t h_u du \tag{6.2.14}$$

is a Wiener process. Moreover, the distribution function of Y_i, $i \in \{1, \dots, n\}$, under Q, F_i^Q, is given by

$$F_i^Q(y) = \int_{-\infty}^y (1 + u_i(z)) dF_i^P(z).$$

Here, we restrict ourselves to the case, where h is $\mathbb{G}$-adapted, such that the measures considered are particularly simple, since Y and W^Q are independent under Q and the mutual independence of the Y_i's is preserved under Q. Using (6.2.14) we find that the dynamics of the forward rates under Q are given by

$$df(t,\tau) = \alpha^Q(t,\tau)dt + \sigma(t,\tau)dW_t^Q, \tag{6.2.15}$$

where we have defined

$$\alpha^Q(t,\tau) = \sigma(t,\tau)\int_t^\tau \sigma(t,u)du. \tag{6.2.16}$$

Now, Björk (2004, Proposition 20.5) gives the following bond price dynamics for $0 \le t \le \tau \le T_{i_t} + \widetilde{T}$ under Q:

$$dP(t,\tau) = r_t P(t,\tau)dt - \int_t^\tau \sigma(t,u)du P(t,\tau)dW_t^Q. \tag{6.2.17}$$

Remark 6.2.4 It can be shown that Q^0 defined by (6.2.12) is the so-called minimal martingale measure for the extended model, i.e. the equivalent martingale measure which "disturbs the structure of the model as little as possible", see Schweizer (1995).

□

6.2.3 Model considerations

In this section we comment on the model specification in Section 6.2.2. At any time t the prices of bonds with maturity τ, $t < \tau \le T_{i_t} + \widetilde{T}$, must satisfy both

$$P(t,\tau) = e^{-\int_t^\tau f(t,u)du}$$

and

$$P(t,\tau) = E^Q\left[e^{-\int_t^\tau r_u du} \middle| \mathcal{F}_t \right]. \tag{6.2.18}$$

Furthermore inserting (6.2.15) and (6.2.16) in Proposition 6.2.3 gives the following expression for the short rate at at time u, $t \le u \le T_{i_t} + \widetilde{T}$:

$$\begin{aligned} r_u &= f(t,u) + \int_t^u df(s,u) \\ &= f(t,u) + \int_t^u \sigma(s,u) \int_s^u \sigma(s,v)dv\, ds + \int_t^u \sigma(s,u) dW_s^Q. \end{aligned}$$

Hence, since σ is $\mathbb{F}$-adapted, we have that in addition to the present information $\mathcal{F}_t$ and the future development of W^Q the future short rate at time u, r_u, may depend on the future outcome of Y_j, $j \in \{i_t+1, \dots, i_u\}$. Thus, at a first glance it seems as if the expectation in (6.2.18) depends on the distribution of Y_j, $j \in \{i_t+1, \dots, i_\tau\}$ under Q, such that the distribution of Y_j, $j \in \{i_t+1, \dots, i_{T_{i_t}+\widetilde{T}}\}$ under Q may be (partly) given at time t by (6.2.18). However, as we shall see below, this is not the case. First observe that

$$\begin{aligned} \int_t^\tau \int_t^u \sigma(s,u) \int_s^u \sigma(s,v)dv\, ds\, du &= \int_t^\tau \int_t^u \frac{1}{2}\frac{\partial}{\partial u}\left(\int_s^u \sigma(s,v)dv\right)^2 ds\, du \\ &= \frac{1}{2}\int_t^\tau \int_s^\tau \frac{\partial}{\partial u}\left(\int_s^u \sigma(s,v)dv\right)^2 du\, ds \\ &= \frac{1}{2}\int_t^\tau \left(\int_s^\tau \sigma(s,v)dv\right)^2 ds \end{aligned}$$

and

$$\int_t^\tau \int_t^u \sigma(s,u) dW_s^Q du = \int_t^\tau \int_s^\tau \sigma(s,u) du\, dW_s^Q.$$

Now, use that provided $\int_s^\tau \sigma(s,u)du$ is sufficiently integrable it holds for fixed τ that

$$E^Q\left[e^{-\frac{1}{2}\int_t^\tau \left(\int_s^\tau \sigma(s,u)du\right)^2 ds - \int_t^\tau \int_s^\tau \sigma(s,u)du\, dW_s^Q} \,\middle|\, \mathcal{F}_t \right] = 1,$$

such that

$$E^Q\left[e^{-\int_t^\tau r_u du} \,\middle|\, \mathcal{F}_t \right] = e^{-\int_t^\tau f(t,u)du}.$$

Hence, no undesirable restrictions on the class of equivalent martingale measures occur when σ is $\mathbb{F}$-adapted.

Since σ is $\mathbb{F}$-adapted the volatility (and hence the drift under Q) of all forward rates at time t may depend on Y_i if $T_i < t$. Hence, the initial bond prices of the newly issued bonds at time T_i may influence the future prices of not only the newly issued bonds, but also bonds with shorter time to maturity. As an alternative model consider the case where the dependence on Y_i is restricted to the volatility of the forward rates (bond prices) introduced at time T_i (and later). In this case the future development of the bond prices depend on the information at the time of issue and the future development of W^Q only. As a last example we mention the (quite restrictive) case where σ is $\mathbb{G}$-adapted, such that the information gathered from the issue of new bonds does not influence the volatility of the forward rates (bond prices).

6.2.4 Trading in the bond market

When trading in the extended bond market introduced in Section 6.2.2 two problems arise: Firstly, at any time infinitely many bonds are traded in the bond market and secondly, the bonds traded at time t depend on the time considered. Regarding the first problem we note that since the forward rates are driven by a 1-dimensional Wiener process only, it is sufficient if we at all times are allowed to invest in two assets, which are not linearly dependent. Furthermore, we note that the second problem may be overcome by considering a new set of price processes defined for all t including the same information as the original price processes. Thus, both problems are solved by considering the following two assets: A savings account with dynamics given by (6.2.5) and an asset, with price process X, generated by investing 1 unit at time 0 and at time t, $0 \leq t \leq \widehat{T}$, investing in the longest bond traded in the market. The dynamics of X are given by

$$dX_t = X_t \frac{dP(t, T_{i_t} + \widetilde{T})}{P(t, T_{i_t} + \widetilde{T})}, \quad X_0 = 1.$$

Inserting the bond price dynamics from (6.2.17) we get the following Q-dynamics of X

$$dX_t = r_t X_t dt - \int_t^{T_{i_t}+\widetilde{T}} \sigma(t,u) du X_t dW_t^Q. \tag{6.2.19}$$

The price process X can be seen as the best available approximation to the value process generated by a roll-over strategy in bonds with time to maturity $\widetilde{T}$. Such a value process is usually referred to as a rolling-horizon bond, see Rutkowski (1999). The idea of roll-over strategies is closely related to the Musiela parametrization of forward rates, see Musiela (1993), where the forward rates are parameterized by time *to* maturity instead of time *of* maturity. In a continuous-time setting where bonds with all maturities are traded, a rolling-horizon bond requires investments in infinitely many different bonds. However, here we only adjust the portfolio, when new bonds are issued, such that X requires a finite number of bonds, $n + 1$, only.

Following the ideas of Møller (2001c) we now define trading in the presence of payment processes. Henceforth fix an arbitrary equivalent martingale measure Q for the model $(B, X, \mathbb{F})$, that is, we are working with the probability space $(\Omega, \mathcal{F}, Q)$ and the filtrations $(\mathbb{F}^{T_i})_{i \in \{0,\ldots,n\}}$. We note that for all i, $i \in \{0, \ldots, n\}$, the discounted price process X^* is a $(Q, \mathbb{F}^{T_i})$-martingale. Here, and throughout the chapter, we use an asterisk (*) to denote discounted price processes. Let $\langle X^* \rangle$ denote the predictable quadratic variation process for X^* associated with Q and $\mathbb{F}^{T_n}$, i.e. the unique predictable process such that $(X^*)^2 - \langle X^* \rangle$ is a $(Q, \mathbb{F}^{T_n})$-martingale. For any i we now introduce the space $\mathcal{L}^2(Q_{X^*}, \mathbb{F}^{T_i})$ of $\mathbb{F}^{T_i}$-predictable processes ϑ satisfying

$$E^Q \left[\int_0^{\widehat{T}} \vartheta_u^2 d\langle X^* \rangle_u \right] < \infty.$$

An $\mathbb{F}^{T_i}$-trading strategy is any process $\varphi = (\vartheta, \eta)$, where $\vartheta \in \mathcal{L}^2(Q_{X^*}, \mathbb{F}^{T_i})$ and η is $\mathbb{F}^{T_i}$-adapted such that the value process $\mathcal{V}(\varphi)$ defined by

$$\mathcal{V}_t(\varphi) = \vartheta_t X_t + \eta_t B_t, \quad 0 \leq t \leq \widehat{T},$$

is RCLL (Right Continuous with Left Limits) and $\mathcal{V}_t(\varphi) \in \mathcal{L}^2(Q)$ for all $t \in [0, \widehat{T}]$. The pair $\varphi_t = (\vartheta_t, \eta_t)$ is interpreted as the portfolio held at time t. Here, ϑ denotes the number of assets with price process X, and η denotes the discounted deposit in the savings account.

A *payment process* is an $\mathbb{F}$-adapted process $A = (A_t)_{0 \le t \le T}$ describing the liabilities of the seller of a contract towards the buyer. Note that A is defined on $[0, T]$ only, such that no payments take place after time T. Moreover, we note that since A is $\mathbb{F}$-adapted, it is $\mathbb{F}^{T_i}$-adapted for all $i \in \{1, \ldots, n\}$. We assume that A is square integrable, i.e. that $E^Q[A_t^2] < \infty$ for all t, and RCLL. For $0 \le s \le t \le T$, we let $A_t - A_s$ be the total outgoes less incomes in the interval $(s, t]$. In the following we shall consider the discounted payment process A^* defined by

$$dA_t^* = e^{-\int_0^t r_u du} dA_t.$$

The *cost process* associated with the pair (φ, A) is given by

$$C_t(\varphi) = \mathcal{V}_t^*(\varphi) - \int_0^t \vartheta_u dX_u^* + A_t^*. \tag{6.2.20}$$

Thus, the cost process is the discounted value of the portfolio reduced by discounted trading gains and added the total discounted outgoes less incomes of the payment process. The cost process is interpreted as the seller's accumulated discounted costs during $[0, t]$. The cost process is square integrable due to the square integrability of the payment process A and the assumptions on the strategy φ and X. Furthermore the cost process is adapted to the same filtration as the trading strategy.

We say that a strategy φ is $\mathbb{F}^{T_i}$-self-financing for the payment process A, if the cost process is constant Q-a.s. with respect to $\mathbb{F}^{T_i}$. In contrast to the classical definition of self-financing strategies, we thus allow for exogenous deposits and withdrawals as represented by A. The two definitions of self-financing strategies are equivalent if and only if the payment process is constant Q-a.s. with respect to the considered filtration. The interpretation of a self-financing strategy in the presence of payment processes is that all fluctuations of the value process are either trading gains/losses or due to the payment process. A payment process is called $\mathbb{F}^{T_i}$-attainable, if there exists an $\mathbb{F}^{T_i}$-self-financing strategy φ for A such that $\mathcal{V}_T^*(\varphi) = 0$ Q-a.s. with respect to $\mathbb{F}^{T_i}$. A payment process is thus $\mathbb{F}^{T_i}$-attainable, if investing the initial amount $C_0(\varphi)$ according to the trading strategy φ leaves us with a portfolio value of 0 after the settlement of all liabilities. Hence, the unique arbitrage free price in $(B, X, \mathbb{F}^{T_i})$ of an $\mathbb{F}^{T_i}$-attainable payment process is $C_0(\varphi)$. At any time t, there is no difference between receiving the *future* payments of the $\mathbb{F}^{T_i}$-attainable payment process A and holding the portfolio φ_t and investing according to the $\mathbb{F}^{T_i}$-replicating strategy φ. Thus, a no arbitrage argument gives that at any time t the price of future payments from A in $(B, X, \mathbb{F}^{T_i})$ must be $\mathcal{V}_t(\varphi)$. It can be shown that the payment process A is attainable if and only if the contingent claim $H = A_T$ with maturity T is (classically) attainable. If all contingent claims, and hence all payment processes, are attainable, the model is called complete and otherwise it is called incomplete.

6.3 Risk-minimization

As noted above, an $\mathbb{F}^{T_i}$-attainable payment process has a unique arbitrage free price $C_0(\varphi)$ in $(B, X, \mathbb{F}^{T_i})$. However, for a non-attainable payment process, we do not have a unique arbitrage free price. Thus, for non-attainable processes, quantifying and controlling the risk becomes important. Here, we apply the criterion of risk-minimization. We give a review of risk-minimization and determine risk-minimizing strategies in the presence of reinvestment risk.

6.3.1 A review of risk-minimization for payment processes

In this section we review the concept of risk-minimization introduced by Föllmer and Sondermann (1986) for contingent claims, and further developed in Møller (2001c) to cover payment processes. For more details we refer to Møller (2001c). Throughout this section, we consider a fixed but arbitrary filtration $\mathbb{F}^{T_i}$, such that we are working with the filtered probability space $(\Omega, \mathcal{F}, Q, \mathbb{F}^{T_i})$.

For a given payment process A we define the $\mathbb{F}^{T_i}$-*risk process* associated with φ by

$$R_t^{T_i}(\varphi) = E^Q\left[(C_T(\varphi) - C_t(\varphi))^2 \middle| \mathcal{F}_t^{T_i}\right], \tag{6.3.1}$$

where the cost process is defined in (6.2.20). Thus, the risk process is the conditional expectation of the discounted squared future costs given the current available information. We will use this quantity to measure the risk associated with (φ, A). An $\mathbb{F}^{T_i}$-trading strategy $\varphi = (\vartheta, \eta)$ is called $\mathbb{F}^{T_i}$-risk-minimizing if for any $t \in [0, T]$ it minimizes $R_t^{T_i}(\varphi)$ over all $\mathbb{F}^{T_i}$-trading strategies with the same value at time T. With the interpretation of the cost process in mind, we note that $\mathcal{V}_t^*(\varphi)$ is the discounted value of the portfolio φ_t after possible payments at time t. In particular, $\mathcal{V}_T^*(\varphi)$ is the discounted value of the portfolio φ_T upon settlement of all liabilities. Thus, a natural restriction is to consider so-called 0-*admissible* strategies which satisfy

$$\mathcal{V}_T^*(\varphi) = 0, \quad Q\text{-a.s.}$$

The construction of risk-minimizing strategies is based on the so-called Galtchouk–Kunita–Watanabe decomposition for martingales. Define the $(Q, \mathbb{F}^{T_i})$-martingale $V^{T_i,*}$ by

$$V_t^{T_i,*} = E^Q\left[A_T^* \middle| \mathcal{F}_t^{T_i}\right], \quad 0 \le t \le T. \tag{6.3.2}$$

The process $V^{T_i,*}$, which is known as the *intrinsic value process* with respect to $\mathbb{F}^{T_i}$, can now be uniquely decomposed using the Galtchouk–Kunita–Watanabe decomposition

$$V_t^{T_i,*} = V_0^{T_i,*} + \int_0^t \vartheta_u^{T_i,A} dX_u^* + L_t^{T_i,A}. \tag{6.3.3}$$

Here, $L^{T_i,A}$ is a zero-mean square integrable $(Q, \mathbb{F}^{T_i})$-martingale which is orthogonal to X^*, i.e. the process $X^* L^{T_i,A}$ is a $(Q, \mathbb{F}^{T_i})$-martingale, and $\vartheta^{T_i,A}$ is an $\mathbb{F}^{T_i}$-predictable process in $\mathcal{L}^2(Q_{X^*}, \mathbb{F}^{T_i})$. We note that if A is $\mathbb{F}^{T_i}$-attainable, then $V_t^{T_i,*}$ is the discounted unique arbitrage free price in $(B, X, \mathbb{F}^{T_i})$ at time t of the future payments specified by the payment process A and $L^{T_i,A} = 0$ Q-a.s. with respect to $\mathbb{F}^{T_i}$. The following theorem relates the risk-minimizing strategy and the associated risk process to the Galtchouk–Kunita–Watanabe decomposition.

Theorem 6.3.1 (Møller (2001c))
There exists a unique 0-admissible $\mathbb{F}^{T_i}$-risk-minimizing strategy $\varphi^{T_i} = (\vartheta^{T_i}, \eta^{T_i})$ for A given by

$$(\vartheta_t^{T_i}, \eta_t^{T_i}) = \left(\vartheta_t^{T_i,A}, V_t^{T_i,*} - A_t^* - \vartheta_t^{T_i,A} X_t^*\right), \quad 0 \le t \le T.$$

The associated $\mathbb{F}^{T_i}$-risk process is given by

$$R_t^{T_i}(\varphi^{T_i}) = E^Q\left[\left(L_T^{T_i,A} - L_t^{T_i,A}\right)^2 \middle| \mathcal{F}_t^{T_i}\right]. \tag{6.3.4}$$

When determining the risk-minimizing strategy, we minimize over all admissible strategies. This is in contrast to many other quadratic hedging criteria such as mean-variance indifference principles and mean-variance hedging, where only self-financing strategies are allowed. For more details on and a comparison of these criteria see Møller (2001b). As noted earlier, risk-minimizing strategies are not self-financing for non-attainable payment processes. However, they can be shown to be *mean-self-financing*, i.e. the corresponding cost processes are Q-martingales with respect to the considered filtration, see Møller (2001c, Lemma A.4).

Note that the risk-minimizing strategy depends on the choice of equivalent martingale measure Q. In the literature, the minimal martingale measure has been applied for determining risk-minimizing strategies, since this, in the case where X^* is continuous, essentially corresponds to the criterion of *local risk-minimization*, which is a criterion in terms of P, see Schweizer (2001a).

Remark 6.3.2 We note that since $\mathcal{F}_t^{T_i}$ and $\mathcal{F}_t^{T_j}$ coincide for $t \ge \max(T_j, T_i)$ so do the intrinsic value processes. This is also intuitively clear, since for $t \ge \max(T_j, T_i)$ the additional information at time 0 in the larger of the filtrations has been revealed, and thus is included in the σ-algebra in the smaller filtration as well.

□

6.3.2 Risk-minimization in the presence of reinvestment risk

From Section 6.3.1 it follows that if we determine the Galtchouk–Kunita–Watanabe decomposition of $V^{T_i,*}$, then the unique 0-admissible $\mathbb{F}^{T_i}$-risk-minimizing strategy, $i \in \{0, \ldots, n\}$, is given by Theorem 6.3.1. However, since it is often difficult to determine the Galtchouk–Kunita–Watanabe decomposition we take a different approach. We apply the main result in Schweizer (1994) regarding risk-minimization under restricted information in order to obtain the following theorem, allowing us to determine the $\mathbb{F}^{T_i}$-risk-minimizing strategy in terms of the $\mathbb{F}^{T_n}$-risk-minimizing strategy.

Theorem 6.3.3
The unique 0-admissible $\mathbb{F}^{T_i}$-risk-minimizing strategy $\varphi^{T_i} = (\vartheta^{T_i}, \eta^{T_i})$ for A given by

$$\left(\vartheta_t^{T_i}, \eta_t^{T_i}\right) = \left(E^Q\left[\vartheta_t^{T_n}\middle|\mathcal{F}_{t-}^{T_i}\right], V_t^{T_i,*} - A_t^* - \vartheta_t^{T_i} X_t^*\right), \quad 0 \le t \le T,$$

and the process $L^{T_i,A}$ is given by

$$L_t^{T_i,A} = V_0^{T_i \vee T_{i_t},*} - V_0^{T_i,*} + E^Q\left[\int_0^t \vartheta_u^{T_n} dX_u^* \middle| \mathcal{F}_t^{T_i}\right] - \int_0^t \vartheta_u^{T_i} dX_u^*. \tag{6.3.5}$$

Proof of Theorem 6.3.3: Since $\mathcal{F}_t^{T_i} \subseteq \mathcal{F}_t^{T_n}$ for all t and X^* is $\mathbb{F}$-adapted with X_T^* being $\mathcal{F}_T^{T_i}$-measurable, Schweizer (1994, Theorem 3.1) gives the $\mathbb{F}^{T_i}$-risk-minimizing strategy for any $\mathcal{F}_T^{T_i}$-measurable contingent claim in terms of the $\mathbb{F}^{T_n}$-risk-minimizing strategy. Since both X^* and A are $\mathbb{F}^{T_i}$-adapted for all i, the result for contingent claims carries over to the present framework with payment processes. Using the fact that X^* furthermore is $\mathbb{F}^{T_i}$-predictable for all i, we have from Schweizer (1994, Section 4), that the $\mathbb{F}^{T_i}$-risk-minimizing strategy is given by

$$\vartheta_t^{T_i} = E^Q\left[\vartheta_t^{T_n}\middle|\mathcal{F}_{t-}^{T_i}\right]$$

and

$$\begin{aligned}
\eta_t^{T_i} &= E^Q\left[V_t^{T_n,*} - A_t^* - \vartheta_t^{T_i} X_t^* \middle| \mathcal{F}_t^{T_i}\right] \\
&= E^Q\left[V_t^{T_n,*}\middle|\mathcal{F}_t^{T_i}\right] - A_t^* - \vartheta_t^{T_i} X_t^* \\
&= V_t^{T_i,*} - A_t^* - \vartheta_t^{T_i} X_t^*.
\end{aligned}$$

Here, we have used that ϑ^{T_i} is $\mathbb{F}^{T_i}$-predictable, and that A^* and X^* are $\mathbb{F}^{T_i}$-adapted in the second equality, and iterated expectations in the third. To derive an expression for $L^{T_i,A}$ we first note that $V_T^{T_n,*} = V_T^{T_i,*}$, see Remark 6.3.2. Inserting the expressions from (6.3.3) and isolating $L_T^{T_i,A}$ we obtain

$$L_T^{T_i,A} = V_0^{T_n,*} - V_0^{T_i,*} + \int_0^T \left(\vartheta_u^{T_n} - \vartheta_u^{T_i}\right) dX_u^*.$$

Using that $L^{T_i,A}$ is a $(Q, \mathbb{F}^{T_i})$-martingale we get

$$\begin{aligned}
L_t^{T_i,A} &= E^Q\left[L_T^{T_i,A}\middle|\mathcal{F}_t^{T_i}\right] \\
&= E^Q\left[V_0^{T_n,*} - V_0^{T_i,*} + \int_0^T \left(\vartheta_u^{T_n} - \vartheta_u^{T_i}\right) dX_u^* \middle| \mathcal{F}_t^{T_i}\right] \\
&= V_0^{T_i \vee T_{i_t},*} - V_0^{T_i,*} + E^Q\left[\int_0^t \vartheta_u^{T_n} dX_u^* \middle| \mathcal{F}_t^{T_i}\right] - \int_0^t \vartheta_u^{T_i} dX_u^*.
\end{aligned}$$

Here, we have used that X^* is a martingale and ϑ^{T_j} lies in $\mathcal{L}^2(Q_{X^*}, \mathbb{F}^{T_j})$ such that the integral $E^Q[\int_t^T \vartheta_u^{T_j} dX_u^* | \mathcal{F}_t^{T_i}] = 0$ for all j. Furthermore, we have used that $\mathcal{F}_t^{T_i} = \mathcal{F}_t^{T_i \vee T_{i_t}}$

to obtain

$$E^Q\left[V_0^{T_n,*}\middle|\mathcal{F}_t^{T_i}\right] = E^Q\left[V_0^{T_n,*}\middle|\mathcal{F}_t^{T_i\vee T_{i_t}}\right] = E^Q\left[V_0^{T_n,*}\middle|\mathcal{F}_0^{T_i\vee T_{i_t}}\right] = V_0^{T_i\vee T_{i_t},*}.$$

□

From Theorem 6.3.3 we get that the $\mathbb{F}^{T_i}$-risk-minimizing strategy is the predictable conditional expectation of the risk-minimizing strategy in the complete model $(B, X, \mathbb{F}^{T_n})$ given the present information, $\mathcal{F}_t^{T_i}$. Thus, Theorem 6.3.3 provides an alternative to Theorem 6.3.1 when determining the $\mathbb{F}^{T_i}$-risk-minimizing strategy. The advantage of Theorem 6.3.3 is that, since $(B, X, \mathbb{F}^{T_n})$ is a complete model, the $\mathbb{F}^{T_n}$-risk-minimizing strategy for any payment process coincides with the $\mathbb{F}^{T_n}$-replicating strategy. Hence, using Theorem 6.3.3 to determine the $\mathbb{F}^{T_i}$-risk-minimizing strategy requires the derivation of a replicating strategy in a complete model and a conditional expectation instead of the derivation of a Galtchouk–Kunita–Watanabe decomposition for a non-attainable payment process.

Remark 6.3.4 Investigating the risk minimizing strategies in Theorem 6.3.3 we observe that since $\mathcal{F}_t^{T_i}$ and $\mathcal{F}_t^{T_j}$ coincide for $t \geq \max(T_i, T_j)$, then the $\mathbb{F}^{T_i}$- and $\mathbb{F}^{T_j}$-risk-minimizing strategies coincide for $t > \max(T_i, T_j)$. The intuitive interpretation is that the strategies are based on the same information, and hence they are identical.

□

Corollary 6.3.5
If we restrict ourselves to payment processes for which ϑ^{T_n} is uniformly bounded then $L_t^{T_i,A}$ is given by

$$L_t^{T_i,A} = V_0^{T_i\vee T_{i_t},*} - V_0^{T_i,*} + \int_0^t \left(\vartheta_u^{T_i\vee T_{i_t}} - \vartheta_u^{T_i}\right) dX_u^*.$$

Proof of Corollary 6.3.5: Since ϑ^{T_n} is uniformly bounded we may use stochastic Fubini, see Protter (2004, Chapter IV, Theorem 64), to interchange the order of integration in (6.3.5).

□

The expression for $L^{T_i,A}$ in Corollary 6.3.5 has the following nice interpretation: At any time the unhedgeable part of $V^{T_i,*}$ consists of two terms. The first term is the difference between the initial deposit given the information at time 0 and the current information, respectively, whereas the second term is the difference between the trading gains generated by the risk-minimizing strategy given the present information regarding Y and the $\mathbb{F}^{T_i}$-risk-minimizing strategy. In particular we note from Corollary 6.3.5 that for $t < T_{i+1}$, we have that $L_t^{T_i,A} = 0$. This is also intuitively clear, since no additional information concerning the Y_j's has been revealed.

Corollary 6.3.6
In the case where ϑ^{T_n} is uniformly bounded we have the following alternative expression for the process $L^{T_i,A}$:

$$L_t^{T_i,A} = \sum_{j=i+1}^{i_t} \left(V_{T_j}^{T_j,*} - E^Q\left[V_{T_j}^{T_n,*}\middle|\mathcal{F}_{T_j-}^{T_{j-1}}\right]\right).$$

This leads to the following expression for the $\mathbb{F}^{T_i}$-risk process associated with φ^{T_i}

$$R_t^{T_i}(\varphi^{T_i}) = E^Q\left[\left(\sum_{j=i_t+1}^{T}\left(V_{T_j}^{T_j,*} - E^Q\left[V_{T_j}^{T_n,*}\middle|\mathcal{F}_{T_j-}^{T_{j-1}}\right]\right)\right)^2\middle|\mathcal{F}_t^{T_i}\right]. \tag{6.3.6}$$

Proof of Corollary 6.3.6: From Corollary 6.3.5 we have the following expression for $L_t^{T_i,A}$:

$$L_t^{T_i,A} = V_0^{T_i\vee T_{i_t},*} - V_0^{T_i,*} + \int_0^t \left(\vartheta_u^{T_i\vee T_{i_t}} - \vartheta_u^{T_i}\right) dX_u^*.$$

Now, write $V_0^{T_i\vee T_{i_t},*} - V_0^{T_i,*}$ and $\vartheta_u^{T_i\vee T_{i_t}} - \vartheta_u^{T_i}$ as telescoping sums and use that, as noted above, the $\mathbb{F}^{T_i}$- and $\mathbb{F}^{T_j}$-risk-minimizing strategies coincide for $t > \max(T_i, T_j)$ to obtain

$$\begin{aligned} L_t^{T_i,A} &= \sum_{j=i+1}^{i_t}\left(V_0^{T_j,*} - V_0^{T_{j-1},*} + \int_0^t \left(\vartheta_u^{T_j} - \vartheta_u^{T_{j-1}}\right) dX_u^*\right) \\ &= \sum_{j=i+1}^{i_t}\left(V_0^{T_j,*} - V_0^{T_{j-1},*} + \int_0^{T_j} \left(\vartheta_u^{T_j} - \vartheta_u^{T_{j-1}}\right) dX_u^*\right). \end{aligned}$$

Using iterated expectations in order to express all quantities as expectations of the respective $\mathbb{F}^{T_n}$-quantities, we get

$$\begin{aligned} L_t^{T_i,A} = \sum_{j=i+1}^{i_t}\Bigg(& E^Q\left[V_0^{T_n,*}\middle|\mathcal{F}_0^{T_j}\right] - E^Q\left[V_0^{T_n,*}\middle|\mathcal{F}_0^{T_{j-1}}\right] \\ &+ \int_0^{T_j}\left(E^Q\left[\vartheta_u^{T_n}\middle|\mathcal{F}_{u-}^{T_j}\right] - E^Q\left[\vartheta_u^{T_n}\middle|\mathcal{F}_{u-}^{T_{j-1}}\right]\right) dX_u^*\Bigg). \end{aligned}$$

The result now follows from

$$\begin{aligned} L_t^{T_i,A} &= \sum_{j=i+1}^{i_t}\Bigg(E^Q\left[V_0^{T_n,*}\middle|\mathcal{F}_{T_j}^{T_j}\right] - E^Q\left[V_0^{T_n,*}\middle|\mathcal{F}_{T_j-}^{T_{j-1}}\right] \\ &\quad + \int_0^{T_j}\left(E^Q\left[\vartheta_u^{T_n}\middle|\mathcal{F}_{T_j}^{T_j}\right] - E^Q\left[\vartheta_u^{T_n}\middle|\mathcal{F}_{T_j-}^{T_{j-1}}\right]\right) dX_u^*\Bigg) \\ &= \sum_{j=i+1}^{i_t}\left(E^Q\left[V_0^{T_n,*} + \int_0^{T_j}\vartheta_u^{T_n} dX_u^*\middle|\mathcal{F}_{T_j}^{T_j}\right] - E^Q\left[V_0^{T_n,*} + \int_0^{T_j}\vartheta_u^{T_n} dX_u^*\middle|\mathcal{F}_{T_j-}^{T_{j-1}}\right]\right) \\ &= \sum_{j=i+1}^{i_t}\left(E^Q\left[V_{T_j}^{T_n,*}\middle|\mathcal{F}_{T_j}^{T_j}\right] - E^Q\left[V_{T_j}^{T_n,*}\middle|\mathcal{F}_{T_j-}^{T_{i-1}}\right]\right) \\ &= \sum_{j=i+1}^{i_t}\left(V_{T_j}^{T_j,*} - E^Q\left[V_{T_j}^{T_n,*}\middle|\mathcal{F}_{T_j-}^{T_{j-1}}\right]\right). \end{aligned}$$

Here, we have used $E^Q[V_0^{T_n,*}|\mathcal{F}_0^{T_j}] = E^Q[V_0^{T_n,*}|\mathcal{F}_t^{T_j}]$ for $t < T_{j+1}$ and $E^Q[\vartheta_u^{T_n}|\mathcal{F}_{u-}^{T_j}] = E^Q[\vartheta_u^{T_n}|\mathcal{F}_t^{T_j}]$ for $u \le t < T_{j+1}$ in the first equality. The uniform boundedness of ϑ^{T_n}

allows us to use stochastic Fubini, see Protter (2004, Chapter IV, Theorem 64), to interchange the order of integration in the second equality. Furthermore we have used that φ^{T_n} is self-financing in the third equality and the definition of $V^{T_j,*}$ in the last equality. The expression for the $\mathbb{F}^{T_i}$-risk process associated with φ^{T_i} in (6.3.6) is now obtained by inserting the expression for $L^{T_i,A}$ in (6.3.4).

□

From Corollary 6.3.6 we observe that $L^{T_i,A}$, which measures the deposits or withdrawals to/from the risk-minimizing portfolio in addition to those generated by the payment process, only changes value at times T_j, $j > i$. Hence, the risk-minimizing strategy is self-financing between the times of issue. At time T_j the information revealed by the issued bonds, i.e. the observed value of Y_j, affects the weights given to the different outcomes of Y, and hence it leads to a change in $L^{T_i,A}$.

6.3.3 $\mathbb{F}$-risk-minimizing strategies

We now derive the $\mathbb{F}$-risk-minimizing strategy for a general payment process of the form

$$dA_t = \Delta A_0 d1_{(t\geq 0)} + a_t dt + \Delta A_T d1_{(t\geq T)}.$$

Here, ΔA_0 is a constant, whereas a_t is $\mathcal{F}_t$-measurable for all t, and ΔA_T is $\mathcal{F}_T$-measurable. We note that the payment process is $\mathbb{F}$- and $\mathbb{F}^{T_n}$-adapted. In order to derive the risk-minimizing strategy we consider the discounted payment process

$$A^*_t = A_0 + \int_0^t e^{-\int_0^s r_u du} dA_s = A_0 + \int_0^t e^{-\int_0^s r_u du} a_s ds + e^{-\int_0^T r_s ds} \Delta A_T 1_{(t=T)}.$$

Since the model $(B, X, \mathbb{F}^{T_n})$ is complete, we have the following expression for A^*:

$$A^*_t = A_0 + \int_0^t \left(F_0^{T_n,s} + \int_0^s \vartheta_u^{T_n,s} dX^*_u \right) ds + \left(F_0^{T_n,\Delta T} + \int_0^T \vartheta_u^{T_n,\Delta T} dX^*_u \right) 1_{(t=T)},$$

where $(\vartheta^{T_n,s})_{0\leq s\leq T}$ and $\vartheta^{T_n,\Delta T}$ are the replicating strategies in $(B, X, \mathbb{F}^{T_n})$ for $(a_s)_{0\leq s\leq T}$ and ΔA_T, respectively, and we for $0 \leq t \leq s \leq T$ have defined

$$F_t^{T_n,s} = E^Q \left[e^{-\int_t^s r_u du} a_s \middle| \mathcal{F}_t^{T_n} \right] \text{ and } F_t^{T_n,\Delta T} = E^Q \left[e^{-\int_t^T r_u du} \Delta A_T \middle| \mathcal{F}_t^{T_n} \right].$$

Hence, $F_t^{T_n,s}$ and $F_t^{T_n,\Delta T}$ are the unique arbitrage free prices at time t in the model $(B, X, \mathbb{F}^{T_n})$ for the claims a_s and ΔA_T, respectively. Now use that $(B, X, \mathbb{F}^{T_n})$ is complete to obtain

$$\begin{aligned}
V_t^{T_n,*} &= E^Q \left[A^*_T \middle| \mathcal{F}_t^{T_n} \right] \\
&= E^Q \left[A_0 + \int_0^T \left(F_0^{T_n,s} + \int_0^s \vartheta_u^{T_n,s} dX^*_u \right) ds + \left(F_0^{T_n,\Delta T} + \int_0^T \vartheta_u^{T_n,\Delta T} dX^*_u \right) \middle| \mathcal{F}_t^{T_n} \right] \\
&= A_0 + \int_0^T F_0^{T_n,s} ds + F_0^{T_n,\Delta T} + E^Q \left[\int_0^T \int_0^s \vartheta_u^{T_n,s} dX^*_u ds + \int_0^T \vartheta_u^{T_n,\Delta T} dX^*_u \middle| \mathcal{F}_t^{T_n} \right].
\end{aligned}$$

Here, we restrict ourselves to payment processes for which $(\vartheta^{T_n,s})_{0\leq s\leq T}$ are uniformly bounded, such that we may use stochastic Fubini, see Protter (2004, Chapter IV, Theorem 64), to interchange the order of integration above. Hence, we obtain the following Galtchouk–Kunita–Watanabe decomposition of $V^{T_n,*}$:

$$\begin{aligned} V_t^{T_n,*} &= V_0^{T_n,*} + E^Q\left[\int_0^T \int_u^T \vartheta_u^{T_n,s} ds\, dX_u^* + \int_0^T \vartheta_u^{T_n,\Delta T} dX_u^* \middle| \mathcal{F}_t^{T_n}\right] \\ &= V_0^{T_n,*} + \int_0^t \int_u^T \vartheta_u^{T_n,s} ds\, dX_u^* + \int_0^t \vartheta_u^{T_n,\Delta T} dX_u^* \\ &= V_0^{T_n,*} + \int_0^t \vartheta_u^{T_n,A} dX_u^*, \end{aligned}$$

where

$$\vartheta_u^{T_n,A} = \int_u^T \vartheta_u^{T_n,s} ds + \vartheta_u^{T_n,\Delta T}. \tag{6.3.7}$$

Recall that $\vartheta^{T_n} = \vartheta^{T_n,A}$. The $\mathbb{F}$-risk-minimizing strategy $\varphi^0 = (\vartheta^0, \eta^0)$ and the associated risk process are now given by inserting (6.3.7) in Theorem 6.3.3.

$\mathbb{F}$-risk-minimizing strategies when Y has finite support

Consider the case where Y_i, $i \in \{1, \dots, n\}$, has finite support, hence $Y_i \in \{y_1^i, \dots, y_{m_i}^i\}$. Let $K = \prod_{i=1}^n m_i$ denote the possible number of outcomes of the vector Y. To simplify the expression for the risk-minimizing strategies we introduce the notation

$$M_t^{\delta_k} = E^Q\left[1_{((Y_1,\dots,Y_n)=\delta_k)} \middle| \mathcal{F}_t\right] = E^Q\left[1_{((Y_1,\dots,Y_n)=\delta_k)} \middle| \mathcal{H}_t\right],$$

where $\delta_1, \dots, \delta_K$ are the possible outcomes of the vector $(Y_1, \dots, Y_n)$. Here, we have used the Q-independence between Y and W^Q in the second equality. If we further introduce the notation ϑ^{δ_k} and $V^{\delta_k,*}$ to denote, respectively, the replicating strategy and the intrinsic value process given $Y = \delta_k$, then the $\mathbb{F}$-risk-minimizing strategy is given by

$$\left(\vartheta_t^0, \eta_t^0\right) = \left(\sum_{k=1}^K M_{t-}^{\delta_k} \vartheta_t^{\delta_k}, \sum_{k=1}^K M_t^{\delta_k} V_t^{\delta_k,*} - A_t^* - \vartheta_t^0 X_t^*\right), \quad 0 \leq t \leq T. \tag{6.3.8}$$

In this case the expression for the process $L^{0,A}$ in Corollary 6.3.6 simplifies to

$$\begin{aligned}
L_t^{0,A} &= \sum_{j=1}^{i_t} \left(V_{T_j}^{T_j,*} - E^Q \left[V_{T_j}^{T_n,*} \middle| \mathcal{F}_{T_j-}^{T_{j-1}} \right] \right) \\
&= \sum_{j=1}^{i_t} \left(E^Q \left[V_{T_j}^{T_n,*} \middle| \mathcal{F}_{T_j}^{T_j} \right] - E^Q \left[V_{T_j}^{T_n,*} \middle| \mathcal{F}_{T_j-}^{T_{j-1}} \right] \right) \\
&= \sum_{j=1}^{i_t} \left(\sum_{k=1}^{K} M_{T_j}^{\delta_k} V_{T_j}^{\delta_k,*} - \sum_{k=1}^{K} M_{T_{j-1}}^{\delta_k} V_{T_j}^{\delta_k,*} \right) \\
&= \sum_{j=1}^{i_t} \sum_{k=1}^{K} V_{T_j}^{\delta_k,*} \left(M_{T_j}^{\delta_k} - M_{T_{j-1}}^{\delta_k} \right) \\
&= \sum_{k=1}^{K} \int_0^t V_u^{\delta_k,*} dM_u^{\delta_k}.
\end{aligned}$$

Here, we have used that the probabilities change at times T_i, $i = 1, \dots, n$, only, in the last equation, and that we are allowed to interchange summation and integration. Hence, in the case where Y has finite support we have the following simple Galtchouk–Kunita–Watanabe decomposition:

$$V_t^{0,*} = \sum_{k=1}^{K} M_0^{\delta_k} V_0^{\delta_k,*} + \int_0^t \sum_{k=1}^{K} M_{u-}^{\delta_k} \vartheta_u^{\delta_k} dX_u^* + \sum_{k=1}^{K} \int_0^t V_u^{\delta_k,*} dM_u^{\delta_k}. \tag{6.3.9}$$

Example 6.3.7 Consider the case where $Y = Y_1$ follows a binomial distribution, i.e. $Y \in \{0,1\}$ with $1 - P(Y=0) = P(Y=1) = p$, $p \in (0,1)$. Now the goal is to determine the risk-minimizing strategy under the minimal martingale measure, Q^0. In this simple example with just two possible outcomes of Y we have $\delta_i = i-1$, $i \in \{1,2\}$. We do not specify the payment process, the forward rate dynamics and γ. Here, the quantities M^{δ_k} simplify to

$$M_t^{\delta_1} = E^{Q^0} \left[1_{(Y=\delta_1)} \middle| \mathcal{F}_t \right] = (1-p) 1_{(0 \le t < T_1)} + (1-Y) 1_{(T_1 \le t \le T)}, \tag{6.3.10}$$
$$M_t^{\delta_2} = E^{Q^0} \left[1_{(Y=\delta_2)} \middle| \mathcal{F}_t \right] = p 1_{(0 \le t < T_1)} + Y 1_{(T_1 \le t \le T)}, \tag{6.3.11}$$

where we have used that the distribution of Y is unaffected by the change to the minimal martingale measure. Furthermore, the intrinsic value process $V^{0,*}$ given by

$$V_t^{0,*} = 1_{(0 \le t < T_1)} \left((1-p) V_t^{\delta_1,*} + p V_t^{\delta_2,*} \right) + 1_{(T_1 \le t \le T)} \left((1-Y) V_t^{\delta_1,*} + Y V_t^{\delta_2,*} \right). \tag{6.3.12}$$

Inserting (6.3.10) and (6.3.11) into (6.3.8) gives the following risk-minimizing strategy

$$\vartheta_t^0 = 1_{(0 \le t \le T_1)} \left((1-p) \vartheta_t^{\delta_1} + p \vartheta_t^{\delta_2} \right) + 1_{(T_1 < t \le T)} \left((1-Y) \vartheta_t^{\delta_1} + Y \vartheta_t^{\delta_2} \right), \tag{6.3.13}$$

and

$$\eta_t^0 = V_t^{0,*} - A_t^* - \left(1_{(0 \le t \le T_1)} \left((1-p) \vartheta_t^{\delta_1} + p \vartheta_t^{\delta_2} \right) + 1_{(T_1 < t \le T)} \left((1-Y) \vartheta_t^{\delta_1} + Y \vartheta_t^{\delta_2} \right) \right) X_t^*,$$

where $V_t^{0,*}$ is given by (6.3.12). Now, inserting (6.3.10)–(6.3.13) in (6.3.9) gives the following Galtchouk–Kunita–Watanabe decomposition

$$V_t^{0,*} = (1-p)V_0^{\delta_1,*} + pV_0^{\delta_2,*} + \int_0^t \left(1_{(0\le u\le T_1)}\left((1-p)\vartheta_u^{\delta_1} + p\vartheta_u^{\delta_2}\right)\right.$$
$$\left.+1_{(T_1<u\le T)}\left((1-Y)\vartheta_u^{\delta_1} + Y\vartheta_u^{\delta_2}\right)\right)dX_u^* + 1_{(T_1\le t\le T)}(Y-p)\left(V_{T_1}^{\delta_2,*} - V_{T_1}^{\delta_1,*}\right).$$

□

Example 6.3.8 We now extend Example 6.3.7 by specifying the payment process, the forward rate dynamics and the function γ. Hence, we still assume that $Y = Y_1 \in \{0,1\}$ with $1 - P(Y=0) = P(Y=1) = p$, $p \in (0,1)$. Consider a company, which at time 0 wants to hedge a claim of 1 at time T, i.e. $\Delta A_T = 1$. Without loss of generality we assume $T = \widetilde{T} + T_1$. To model the dynamics of the forward rates, we let σ be given by

$$\sigma(t,\tau) = ce^{-a(\tau-t)},$$

for some positive constants c and a. Here, as in practice, fluctuations of the forward rates dampen exponentially as a function of time to maturity. Using that

$$\int_t^\tau \sigma(t,u)du = \int_t^\tau ce^{-a(u-t)}du = \frac{c}{a}\left(1 - e^{-a(\tau-t)}\right),$$

we obtain the following forward rate dynamics under Q for $0 \le t \le \tau \le T_{i_t} + \widetilde{T}$:

$$df(t,\tau) = \frac{c^2}{a}e^{-a(\tau-t)}\left(1 - e^{-a(\tau-t)}\right)dt + ce^{-a(\tau-t)}dW_t^Q. \tag{6.3.14}$$

To model the extension of the forward rate curve at time T_1 we assume γ is given by

$$\gamma_s = \frac{1}{T-\widetilde{T}}\left(k_1Y + k_2(1-Y)\right), \tag{6.3.15}$$

for some constants k_1 and k_2. Thus, the forward rate curve is continued by a straight line with slope $k_1/(T-\widetilde{T})$ or $k_2/(T-\widetilde{T})$. In order to obtain the $\mathbb{F}$-risk-minimizing strategy under Q^0 we now consider the complete model $(B, S, \mathbb{F}^{T_1})$, where Y is known. Proposition 6.2.3 gives the following expression for the short rate

$$r_t = \begin{cases} f(0,t) + \int_0^t df(s,t), & t \le \widetilde{T}, \\ f(0,\widetilde{T}) + \int_0^{T_1} df(s,\widetilde{T}) + \int_{\widetilde{T}}^t \gamma_s ds + \int_{T_1}^t df(s,t), & t > \widetilde{T}. \end{cases} \tag{6.3.16}$$

We note from (6.3.16) that r_t depends on Y for $t > \widetilde{T}$. Inserting (6.3.14) and (6.3.15) in (6.3.16) gives

$$r_t = \begin{cases} m(t) + \int_0^t ce^{-a(t-s)}dW_s^Q, & t \le \widetilde{T}, \\ m(t,Y) + \int_{T_1}^t ce^{-a(t-s)}dW_s^Q, & t > \widetilde{T}, \end{cases}$$

where we have defined

$$m(t) = f(0,t) + \frac{c^2}{2a^2}\left(1 - e^{-at}\right)^2, \;\; t \le \widetilde{T},$$

and

$$m(t,Y) = f(0,\widetilde{T}) + \frac{c^2}{2a^2}\left(\left(1-e^{-a\widetilde{T}}\right)^2 - \left(1-e^{-a(\widetilde{T}-T_1)}\right)^2 + \left(1-e^{-a(t-T_1)}\right)^2\right)$$
$$+ \frac{t-\widetilde{T}}{T-\widetilde{T}}\left(k_1 Y + k_2(1-Y)\right) + \int_0^{T_1} ce^{-a(\widetilde{T}-s)}dW_s^Q, \quad t > \widetilde{T}.$$

Using Itô's formula we now obtain the short rate dynamics

$$dr_t = \begin{cases} (\phi(t) - ar_t)\,dt + cdW_t^Q, & t \le \widetilde{T}, \\ (\phi(t,Y) - ar_t)\,dt + cdW_t^Q, & t > \widetilde{T}, \end{cases} \tag{6.3.17}$$

where

$$\phi(t) = am(t) + \frac{\partial}{\partial t}m(t) \text{ and } \phi(t,Y) = am(t,Y) + \frac{\partial}{\partial t}m(t,Y).$$

Hence, given Y the short rate follows an extended Vasiček model under Q. The result is well-known for $t \le \widetilde{T}$, where r_t is independent of Y, see e.g. Musiela and Rutkowski (1997). From (6.3.17) we observe that the drift and the squared diffusion both are affine in r, such that an extended Vasiček model for the short rate leads to an affine term structure, see e.g. Björk (2004, Proposition 22.2). Thus, in the conditional model we have the following expression for the unique arbitrage free price at time t for 1 unit at time T:

$$P^{\delta_Y+1}(t,T) = \exp(A(t,T,Y) - B(t,T)r_t), \tag{6.3.18}$$

with $A(t,T,Y)$ and $B(t,T)$ given by

$$B(t,T) = \frac{1}{a}\left(1-e^{-a(T-t)}\right), \tag{6.3.19}$$
$$A(t,T,Y) = \int_t^T \frac{1}{2}c^2B^2(s,T)ds - \int_t^{\widetilde{T}} \phi(s)B(s,T)ds - \int_{\widetilde{T}}^T \phi(s,Y)B(s,T)ds.$$

Even though B in general is allowed to depend on Y, it is not the case here, so we have omitted Y in the notation for B. Note that we have used the notation $P^{\delta_Y+1}(t,T)$ even though the bond is not traded. Applying Itô's formula to (6.3.18) and using the differential equations for A and B from Björk (2004, Proposition 22.2), we obtain the following Q-dynamics for the price process $P^{\delta_Y+1}(t,T)$:

$$dP^{\delta_Y+1}(t,T) = r_tP^{\delta_Y+1}(t,T)dt - cB(t,T)P^{\delta_Y+1}(t,T)dW_t^Q. \tag{6.3.20}$$

Combining (6.2.19), (6.3.19) and (6.3.20) gives

$$dX_t = r_tX_tdt - cB(t,T_{i_t}+\widetilde{T})X_tdW_t^Q. \tag{6.3.21}$$

Note that at time t, $t > \widetilde{T}$, we have that X_t depends on Y through the short rate process. However, if we consider the discounted price process, X^*, the dynamics are given by

$$dX_t^* = -cB(t,T_{i_t}+\widetilde{T})X_t^*dW_t^Q,$$

such that X^* is independent of Y. Comparing (6.3.20) and (6.3.21) we find that given Y the replicating strategy is given by

$$\vartheta_t^{\delta_{Y+1}} = \frac{B(t,T)P^{\delta_{Y+1}}(t,T)}{B(t,T_{i_t}+\widetilde{T})X_t} = \frac{B(t,T)P^{\delta_{Y+1},*}(t,T)}{B(t,T_{i_t}+\widetilde{T})X_t^*}. \tag{6.3.22}$$

Since

$$V_t^{\delta_{Y+1},*} = e^{-\int_0^t r_u du} P^{\delta_{Y+1}}(t,T) = P^{\delta_{Y+1},*}(t,T),$$

we have

$$\begin{aligned} V_t^{0,*} &= 1_{(0\le t<T_1)} \Big((1-p)P^{\delta_1,*}(t,T) + pP^{\delta_2,*}(t,T)\Big) \\ &\quad + 1_{(T_1\le t\le T)} \Big((1-Y)P^{\delta_1,*}(t,T) + YP^{\delta_2,*}(t,T)\Big). \end{aligned} \tag{6.3.23}$$

Inserting (6.3.22) and (6.3.23) in the results from Example 6.3.7 gives the following risk-minimizing strategy:

$$\begin{aligned} \vartheta_t^0 &= \frac{1}{X_t^*}\Bigg(1_{(0\le t\le T_1)} \frac{B(t,T)}{B(t,T_{i_t}+\widetilde{T})} \Big((1-p)P^{\delta_1,*}(t,T) + pP^{\delta_2,*}(t,T)\Big) \\ &\quad + 1_{(T_1<t\le T)} \Big((1-Y)P^{\delta_1,*}(t,T) + YP^{\delta_2,*}(t,T)\Big)\Bigg), \end{aligned} \tag{6.3.24}$$

and

$$\eta_t^0 = \begin{cases} (1-p)P^{\delta_1,*}(t,T) + pP^{\delta_2,*}(t,T) - \Big((1-p)\vartheta_t^{\delta_1} + p\vartheta_t^{\delta_2}\Big)X_t^*, & 0<t<T_1, \\ (1-Y)P^{\delta_1,*}(t,T) + YP^{\delta_2,*}(t,T) - \Big((1-p)\vartheta_t^{\delta_1} + p\vartheta_t^{\delta_2}\Big)X_t^*, & t=T_1, \\ 0, & T_1<t\le T. \end{cases} \tag{6.3.25}$$

Investigating (6.3.24) and (6.3.25) we note that for $t>T_1$ the risk-minimizing strategy consists of $P^{\delta_{Y+1},*}(t,T)/X_t^*$ units of the risky asset, which at this time corresponds to investing in bonds with maturity T. Hence, holding $P^{\delta_{Y+1},*}(t,T)/X_t^*$ units of the risky asset is equivalent to holding one bond with maturity T, which in turn is the replicating strategy. The Galtchouk–Kunita–Watanabe decomposition is given by

$$\begin{aligned} V_t^{0,*} &= (1-p)P^{\delta_1,*}(0,T) + pP^{\delta_2,*}(0,T) \\ &\quad + \int_0^t \frac{1}{X_u^*}\Bigg(1_{(0\le u\le T_1)} \frac{B(u,T)}{B(u,T_{i_u}+\widetilde{T})} \Big((1-p)P^{\delta_1,*}(u,T) + pP^{\delta_2,*}(u,T)\Big) \\ &\quad + 1_{(T_1<u\le T)} \Big((1-Y)P^{\delta_1,*}(u,T) + YP^{\delta_2,*}(u,T)\Big)\Bigg) dX_u^* \\ &\quad + 1_{(T_1\le t\le T)}(Y-p)\Big(P^{\delta_2,*}(T_1,T) - P^{\delta_1,*}(T_1,T)\Big). \end{aligned}$$

□

6.4 A practical implementation of the model

In this section we discuss a possible implementation of the model. Without loss of generality we assume that new bonds are issued at time 0, such that at time 0 the time to maturity of the longest traded bond is $\widetilde{T}$.

At time 0 we observe the bond prices in the market. Assuming the forward rates are given by a parametric model, with parameter $\theta \in \Theta$, we estimate the value of θ, say θ_0, which gives the best correspondence with the observed bond prices. For a possible parametrization we refer to Svensson (1995), who considers an extension of the so-called Nelson–Siegel parametrization; see Nelson and Siegel (1987) for the original Nelson–Siegel parametrization. Now let the initial forward rate curve at time 0, $(f(0,\tau))_{0\le\tau\le\widetilde{T}}$, be given by the estimated forward rate curve $(f^{\theta_0}(0,\tau))_{0\le\tau\le\widetilde{T}}$. In addition to the initial forward rate curve we, for later purpose, use θ_0 to estimate $f^{\theta_0}(0,T_1+\widetilde{T})$. Given a model for the forward rate dynamics we simulate the forward rate vector $(f(T_1,\tau))_{T_1\le\tau\le\widetilde{T}}$ and the point $f^{\theta_0}(T_1,T_1+\widetilde{T})$. The forward rate $f(T_1,T_1+\widetilde{T})$ is now drawn from a distribution (estimated from historical data) with mean $f^{\theta_0}(T_1,T_1+\widetilde{T})$. To obtain the forward rate curve at time T_1 after the issue of new bonds we combine $f(T_1,\widetilde{T})$ and $f(T_1,T_1+\widetilde{T})$ by a method giving a smooth extension of the forward rate curve. One possibility is the method of cubic splines, see e.g. Press, Flannery, Teukolsky and Vetterling (1986). Using the parametric forward rate model, we now estimate the parameter θ_1, which gives the best correspondence with the forward rate curve at time T_1. The estimated parameter is only used to estimate $f^{\theta_1}(T_1,T_2+\widetilde{T})$. Starting from the forward rate curve at time T_1, $(f(T_1,T_1+\tau))_{0\le\tau\le\widetilde{T}}$, and $f^{\theta_1}(T_1,T_2+\widetilde{T})$ the procedure above is repeated to determine the forward rate curve at time T_2, and in turn the forward rate curve at any future time.

When implementing the model as described above we have the standard problems of estimating the initial forward rate curve from the observed bond prices and modelling the forward rates. In addition we have the model related problem of determining the distribution of $f(T_i,T_i+\widetilde{T})$, $i\in\{1,\dots,n\}$. We note, however, that we avoid a direct modelling of γ^i. Instead γ^i is given indirectly by $f^{\theta_{i-1}}(T_i,T_i+\widetilde{T})$, the estimated distribution with mean $f^{\theta_{i-1}}(T_i,T_i+\widetilde{T})$ and the chosen smoothing method.

Chapter 7

Valuation and Hedging of Unit-Linked Life Insurance Contracts Subject to Reinvestment and Mortality Risks

(This chapter is an adapted version of Dahl (2005d))

This paper considers the problem of valuating and hedging a portfolio of unit-linked life insurance contracts, which are subject to several hedgeable and unhedgeable sources of risk. In Chapter 4 we consider a portfolio of life insurance contracts with deterministic payoffs which are subject to hedgeable interest rate risk as well as unhedgeable systematic and unsystematic mortality risk. Here, we extend this setup by considering a portfolio of unit-linked life insurance contracts, which are subject to both hedgeable and unhedgeable financial risk, as well as unhedgeable systematic and unsystematic mortality risk. The unhedgeable financial risk is the reinvestment risk, described in Chapter 6, which is present in bond markets, where only bonds with a limited time to maturity are traded. In addition to the bond market, the financial market consists of a stock, whose price process is correlated to the bond prices. To model the underlying mortality intensity we apply a time-inhomogeneous Cox–Ingersoll–Ross model, as proposed in Chapter 4. Within the combined model, we study a general set of equivalent martingale measures and determine market reserves by applying these measures. As an alternative to the market reserves we derive mean-variance indifference prices. To quantify and control the risk of the insurance company, we derive risk-minimizing strategies and the optimal strategies associated with the mean-variance indifference prices.

7.1 Introduction

In Chapter 4 we derive optimal hedging strategies and market values for standard life insurance contracts with fixed payments, in the presence of systematic and unsystematic mortality risk. Here, we extend this work by considering unit-linked life insurance contracts, which in addition to hedgeable financial risk and unhedgeable systematic and unsystematic mortality risk, are subject to an unhedgeable financial risk. This unhedgeable financial risk is the so-called reinvestment risk present in bond markets, where only bonds with a limited time to maturity are traded. The extension to unit-linked life-insurance contracts without reinvestment risk is trivial, since the financial market remains complete after the addition of a stock. Hence, the main contribution of this paper is the inclusion of the unhedgeable reinvestment risk, such that we obtain a more refined model for the uncertainty associated with (unit-linked) life insurance contracts.

In order to model the reinvestment risk we apply the model proposed in Chapter 6. Hence, we initially consider a standard continuous-time bond market model with some fixed finite time horizon, which is smaller than the time horizon of the considered payment process. At fixed times new bonds are issued in the market, such that we immediately after the issue of new bonds consider a standard model similar to the initial one. The entry prices of the new bonds depend on the prices of existing bonds and some independent random variable, whose outcome determines the extension of the forward rate curve. In addition to the bonds, the financial market consists of a stock, which is correlated to the bonds. As in Chapter 4 we model the mortality intensity by a time-inhomogeneous Cox–Ingersoll–Ross (CIR) model, such that we obtain an affine mortality structure, see Chapter 3.

Within this setting, we apply financial theory for pricing and hedging the payment process generated by a portfolio of unit-linked life insurance contracts. We study a fairly general set of equivalent martingale measures for the model and derive market reserves, which depend on the market's attitude towards systematic and unsystematic mortality risk as well as reinvestment risk. Similarly to Chapter 4 we derive risk-minimizing strategies and mean-variance indifference prices and optimal hedging strategies. The derivation of risk-minimizing strategies consists of a two-step procedure. First we disregard the reinvestment risk and derive the risk-minimizing strategies in the case of a complete financial market. The strategies obtained here are essentially identical to the ones in Chapter 4. The second step is to apply the result of Schweizer (1994) for risk-minimization under restricted information to derive the risk-minimizing strategies in the case, where we also consider reinvestment risk. This two-step procedure has also been applied in Chapter 6.

The paper is organized as follows: In Section 7.2 we introduce the various sub-models. These include the financial market and the mortality and insurance portfolio. Section 7.3 introduces the combined model, the payment process, market reserves and the financial terminology necessary to define trading strategies in the present model. Risk-minimizing strategies are obtained in Section 7.4, and in Section 7.5 we derive mean-variance indifference prices and optimal hedging strategies for a portfolio of unit-linked pure endowments. Proofs and calculations of some technical results can be found in Section 7.6.

7.2 The sub-models

Let $\widehat{T}$ be a fixed time horizon and $(\Omega, \mathcal{F}, P)$ a probability space with a filtration $\mathbb{F} = (\mathcal{F}(t))_{0\le t\le \widehat{T}}$ satisfying the usual conditions of right-continuity and completeness. In addition to the filtration $\mathbb{F}$, which contains all available information, we shall consider several sub-filtrations.

7.2.1 The financial market

The model for the financial market consists of the bond market model in Chapter 6 with the inclusion of a stock. Here, we first give a brief introduction to the model. A detailed review of the bond market model is then given in Section 7.2.1.

Consider a financial market consisting of three traded assets: A savings account with price process B and two risky assets with price processes Z and S. Here, Z is a price process generated by investing in bonds (see Section 7.2.1 for more details), and S is the price process for a stock. The P-dynamics of the traded assets are

$$dB(t) = r(t)B(t)dt, \quad B(0) = 1, \tag{7.2.1}$$

$$dZ(t) = \Big(r(t) + h^f(t)\sigma^z(t)\Big)\, Z(t)dt - \sigma^z(t)Z(t)dW^f(t), \quad Z(0) = 1, \tag{7.2.2}$$

$$dS(t) = (r(t) + \rho^s(t))S(t)dt - \sigma^s(t)S(t)dW^f(t) + \beta^s(t)S(t)dW^s(t), \quad S(0) > 0, \tag{7.2.3}$$

where

$$\rho^s(t) = \sigma^s(t)h^f(t) - \beta^s(t)h^s(t). \tag{7.2.4}$$

Here, $(W^f(t))_{0\le t\le \widehat{T}}$ and $(W^s(t))_{0\le t\le \widehat{T}}$ are independent Wiener processes under P on the interval $[0, \widehat{T}]$. In (7.2.2)–(7.2.4) the process r is the stochastic rate of interest. The dynamics of r are assumed to be driven by W^f only. Furthermore we introduce the notation $X = (Z, S)^{tr}$, where a^{tr} denotes the vector a transposed, and let $\mathbb{G}^x = (\mathcal{G}^x(t))_{0\le t\le \widehat{T}}$ be the filtration generated by X , i.e. by W^f and W^s. In addition to the uncertainty generated by the traded assets, we observe a sequence $Y = (Y_i)_{i=1,\dots,m}$ of mutually independent random variables independent of W^f and W^s. The observation times of Y are given by the sequence $0 = T_0 < T_1 < \ldots < T_m \le T$, where T_i is the observation time for Y_i and T, $T \le \widehat{T}$, is the terminal time of the considered payment process, see Section 7.3.2. Let $\mathbb{G}^y$ be the natural filtration generated by Y, i.e.

$$\mathcal{G}^y(t) = \sigma\{(Y_i)_{i=1,\dots,m}, T_i \le t\}.$$

The outcome of the Y_i's influences the future values of the stochastic short rate r through the coefficient functions in the dynamics of r. We are now in a position to define $\mathbb{G}$, which is the total filtration generated by the financial market, i.e. $\mathcal{G}(t) = \mathcal{G}^y(t) \vee \mathcal{G}^x(t)$. In (7.2.2)–(7.2.4) the processes σ^z, σ^s and β^s are $\mathbb{G}$-adapted, whereas h^f and h^s are $\mathbb{G}^x$-adapted. In addition to the filtrations above we shall consider the enlarged filtrations

$$\mathbb{G}^{T_i} = (\mathcal{G}^{T_i}(t))_{0\le t\le \widehat{T}} = (\mathcal{G}(t) \vee \mathcal{G}^y(T_i))_{0\le t\le \widehat{T}}, \;\; i \in \{0, \dots, m\}.$$

We immediately note that $\mathbb{G} = \mathbb{G}^{T_0} = \mathbb{G}^0$. For $i \in \{1, \ldots, m\}$ the interpretation of the filtration $\mathbb{G}^{T_i}$ is that $(Y_j)_{j=1,\ldots,i}$ are known at time 0. Furthermore, we note that if we consider the filtration $\mathbb{G}^{T_i}$ and the time interval $[0, T_{i+1})$ then the financial market is complete. Hence in particular, when considering $\mathbb{G}^{T_m}$ the financial market is complete.

Remark 7.2.1 The fact that h^f and h^s are $\mathbb{G}^x$-adapted is an assumption, which we impose in order to simplify the calculations in Section 7.5.

□

The bond market

Here, we review the bond market model including reinvestment risk proposed in Chapter 6.

Let $P(t, \tau)$ denote the price at time t of a zero coupon bond maturing at time τ. To avoid arbitrage we assume $P(t, \tau)$ is strictly positive and $P(t, t) = 1$ for all t. An important quantity when modelling bond prices is the (instantaneous) forward rate with maturity date τ contracted at time t defined by

$$f(t, \tau) = -\frac{\partial \log P(t, \tau)}{\partial \tau}, \tag{7.2.5}$$

or, stated differently,

$$P(t, \tau) = e^{-\int_t^\tau f(t,u)du}. \tag{7.2.6}$$

The forward rate $f(t, \tau)$ can be interpreted as the riskfree interest rate, contracted at time t over the infinitesimal interval $[\tau, \tau + d\tau)$. The short rate process $(r(t))_{0 \le t \le \widehat{T}}$ is defined as $r(t) = f(t, t)$.

Now introduce two additional fixed time horizons $\widetilde{T}$ and T, where $\widetilde{T} \le T \le \widehat{T}$. Here, $\widetilde{T}$, T and $\widehat{T}$, respectively, describe the upper limit for the time *to* maturity of a bond traded in the market, the terminal time of the considered payment process and the last time where trading is possible in the bond market, i.e. "the end of the world". Thus, at any time t the time to maturity of the longest traded bond is less than or equal to $\widetilde{T}$. The sequence $0 = T_0 < T_1 < \ldots < T_m \le T$ describes the times, where new bonds are issued in the market. At time T_i new bonds are issued such that all bonds with time *to* maturity less than or equal to $\widetilde{T}$ are traded. To ensure that at all times, bonds are traded in the market, we assume that $\widetilde{T} \ge \max_{i=1,\ldots,m}(T_i - T_{i-1})$ and $\widehat{T} = T_m + \widetilde{T}$. The illustration in Figure 7.2.1 shows one possible ordering of $T_1, \ldots, T_m$, T and $\widetilde{T}$ in the case $m = 3$.

For fixed t we define

$$i_t = \sup \{0 \le i \le m | T_i \le t\}, \tag{7.2.7}$$

such that T_{i_t} is the last time new bonds are issued prior to time t (time t included). Thus, the time of maturity, τ, of the bonds traded in the bond market at time t satisfy $t \le \tau \le T_{i_t} + \widetilde{T}$.

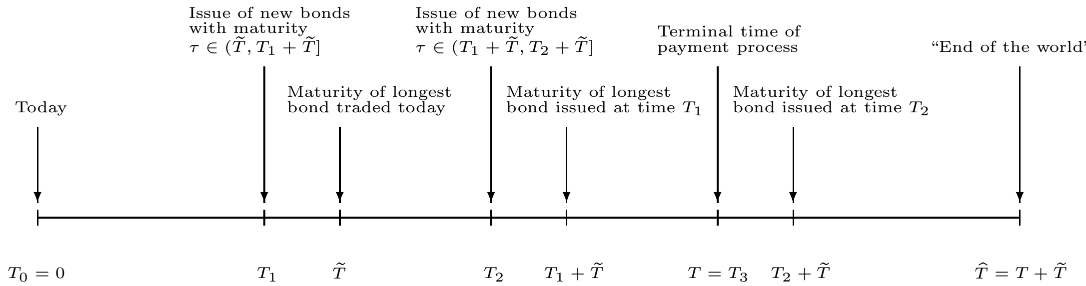

Figure 7.2.1: *Illustration of* T_1, T_2, T_3, $\widetilde{T}$, T *and* $\widehat{T}$.

Since it is inconvenient to model the bond prices directly, we model the forward rate dynamics, as proposed in Heath et al. (1992). The connection between forward rates and bond prices established in (7.2.5) and (7.2.6) then gives the dynamics of the bond prices. For $0 \le t \le \tau \le T_{i_t} + \widetilde{T}$ we assume that the forward rate dynamics under P are given by

$$df(t,\tau) = \sigma^f(t,\tau)\left(\int_t^\tau \sigma^f(t,u)du - h^f(t)\right)dt + \sigma^f(t,\tau)dW^f(t), \tag{7.2.8}$$

where σ^f is $\mathbb{G}$-adapted. As noted above forward rates for all maturities are not defined at time 0. They are introduced at the times of issue of new bonds. To model the initial value of the forward rates introduced at time T_i, $i \in \{1,\dots,m\}$, we assume that for $T_{i-1} + \widetilde{T} < \tau \le T_i + \widetilde{T}$ it holds that

$$f(T_i,\tau) = f(T_i, T_{i-1} + \widetilde{T}) + \int_{T_{i-1}+\widetilde{T}}^{\tau} \gamma^i(u)du. \tag{7.2.9}$$

Here, γ^i is an $\mathcal{G}(T_i)$-measurable function, i.e each $\gamma^i(u)$ is $\mathcal{G}(T_i)$-measurable for $u \in (T_{i-1} + \widetilde{T}, T_i + \widetilde{T}]$, and $Y = (Y_i)_{i=1,\dots,m}$ is sequence of mutually independent random variables independent of W^f with distribution functions $(F_i^P)_{i=1,\dots,m}$. Hence, Y_i describes the unhedgeable uncertainty associated with the initial prices of bonds issued at time T_i.

Using Björk (2004, Proposition 20.5) we obtain the following P-dynamics for the price process of a bond with maturity τ, $0 \le t \le \tau \le T_{i_t} + \widetilde{T}$:

$$dP(t,\tau) = \Big(r(t) + h^f(t)\sigma^p(t,\tau)\Big)P(t,\tau)dt - \sigma^p(t,\tau)P(t,\tau)dW^f(t), \tag{7.2.10}$$

where we have defined

$$\sigma^p(t,\tau) = \int_t^\tau \sigma^f(t,u)du.$$

When trading in the bond market it is sufficient to consider investments in a savings account with dynamics (7.2.1), and an asset with price process Z generated by investing 1 unit at time 0 and at times $0 \le t \le \widehat{T}$ investing in the longest bond traded in the market. The dynamics of Z are given by

$$dZ(t) = Z(t)\frac{dP(t, T_{i_t} + \widetilde{T})}{P(t, T_{i_t} + \widetilde{T})}, \quad Z(0) = 1.$$

Inserting the bond price dynamics from (7.2.10) and defining

$$\sigma^z(t) = \sigma^p(t, T_{i_t} + \widetilde{T}),$$

we get the dynamics in (7.2.2).

7.2.2 Modelling the mortality

In order to model the uncertainty associated with the future mortality intensities we use the model proposed in Chapter 4. Let $\mu^\circ = (\mu^\circ(x))_{x\geq 0}$ be a given initial curve for the mortality intensity at all ages. It is assumed that $\mu^\circ(x)$ is continuously differentiable as a function of x. Here, and in the following we neglect the gender aspect. For an individual aged x at time 0, the future mortality intensity is viewed as a stochastic process $\mu(x) = (\mu(x,t))_{0\leq t\leq \widehat{T}}$ with the property that $\mu(x,0) = \mu^\circ(x)$. In principle, one can view $\mu = (\mu(x))_{x\geq 0}$ as an infinitely dimensional process.

We model changes in the mortality intensity via a strictly positive infinite dimensional process $\zeta = (\zeta(x,t))_{x\geq 0, t\in[0,\widehat{T}]}$ with the property that $\zeta(x,0) = 1$ for all x. The filtration $\mathbb{I} = (\mathcal{I}(t))_{t\in[0,\widehat{T}]}$ is the natural filtration of the underlying process ζ. The mortality intensity process is then modelled via

$$\mu(x,t) = \mu^\circ(x+t)\zeta(x,t). \tag{7.2.11}$$

Thus, $\zeta(x,t)$ describes the relative change in the mortality intensity from time 0 to t for a person of age $x+t$. The true survival probability is defined by

$$\mathcal{S}(x,t,T) = E^P\left[e^{-\int_t^T \mu(x,\tau)d\tau} \middle| \mathcal{I}(t)\right], \tag{7.2.12}$$

and the related martingale is given by

$$\mathcal{S}^M(x,t,T) = E^P\left[e^{-\int_0^T \mu(x,\tau)d\tau} \middle| \mathcal{I}(t)\right] = e^{-\int_0^t \mu(x,\tau)d\tau}\mathcal{S}(x,t,T). \tag{7.2.13}$$

In general, we can consider survival probabilities under various equivalent probability measures. This is discussed in more detail in Section 7.3.1.

The process $\zeta(x)$ is modelled via a so-called time-inhomogeneous CIR model

$$d\zeta(x,t) = \left(\gamma^\zeta(x,t) - \delta^\zeta(x,t)\zeta(x,t)\right)dt + \sigma^\zeta(x,t)\sqrt{\zeta(x,t)}dW^\mu(t), \tag{7.2.14}$$

where γ^ζ, δ^ζ and σ^ζ are known functions and W^μ is a Wiener process under P on the interval $[0,\widehat{T}]$. Here, and in the following, we assume that $2\gamma^\zeta(x,t) \geq (\sigma^\zeta(x,t))^2$, such that ζ is strictly positive, see Maghsoodi (1996). It now follows via Itô's formula that

$$d\mu(x,t) = (\gamma^\mu(x,t) - \delta^\mu(x,t)\mu(x,t))\,dt + \sigma^\mu(x,t)\sqrt{\mu(x,t)}dW^\mu(t), \tag{7.2.15}$$

where

$$\gamma^\mu(x,t) = \gamma^\zeta(x,t)\mu^\circ(x+t), \tag{7.2.16}$$

$$\delta^\mu(x,t) = \delta^\zeta(x,t) - \frac{\frac{d}{dt}\mu^\circ(x+t)}{\mu^\circ(x+t)}, \tag{7.2.17}$$

$$\sigma^\mu(x,t) = \sigma^\zeta(x,t)\sqrt{\mu^\circ(x+t)}. \tag{7.2.18}$$

This shows that μ also follows an time-inhomogeneous CIR model. Furthermore μ is strictly positive as well. Since we have an affine mortality structure, see Theorem 3.3.6, the survival probability is given by

$$\mathcal{S}(x,t,T) = e^{A^\mu(x,t,T) - B^\mu(x,t,T)\mu(x,t)},$$

where

$$\frac{\partial}{\partial t}B^\mu(x,t,T) = \delta^\mu(x,t)B^\mu(x,t,T) + \frac{1}{2}(\sigma^\mu(x,t))^2(B^\mu(x,t,T))^2 - 1, \tag{7.2.19}$$

$$\frac{\partial}{\partial t}A^\mu(x,t,T) = \gamma^\mu(x,t)B^\mu(x,t,T), \tag{7.2.20}$$

with $B^\mu(x,T,T) = 0$ and $A^\mu(x,T,T) = 0$. In this case the forward mortality intensities are given by

$$f^\mu(x,t,T) = -\frac{\partial}{\partial T}\log\mathcal{S}(x,t,T) = \mu(x,t)\frac{\partial}{\partial T}B^\mu(x,t,T) - \frac{\partial}{\partial T}A^\mu(x,t,T). \tag{7.2.21}$$

7.2.3 The insurance portfolio

Consider an insurance portfolio consisting of n insured lives of the same age x. We assume that the individual remaining lifetimes at time 0 of the insured are described by a sequence $D_1, \ldots, D_n$ of identically distributed non-negative random variables. Moreover, we assume that

$$P(D_1 > t | \mathcal{I}(\widehat{T})) = e^{-\int_0^t \mu(x,s)ds}, \;\; 0 \le t \le \widehat{T},$$

and that the censored lifetimes $D_i^c = D_i 1_{(D_i \le \widehat{T})} + \widehat{T} 1_{(D_i > \widehat{T})}$, $i = 1, \ldots, n$, are i.i.d. given $\mathcal{I}(\widehat{T})$. Thus, given the development of the underlying process $\zeta(x)$, the mortality intensity at time s is $\mu(x,s)$.

Now define a counting process $N(x) = (N(x,t))_{0 \le t \le \widehat{T}}$ by

$$N(x,t) = \sum_{i=1}^n 1_{(D_i \le t)}.$$

Hence, $N(x)$ keeps track of the number of deaths in the portfolio of insured lives. We denote by $\mathbb{H} = (\mathcal{H}(t))_{0 \le t \le \widehat{T}}$ the natural filtration generated by $N(x)$. It follows that $N(x)$

is an $(\mathbb{H} \vee \mathbb{I})$-Markov process. The stochastic intensity process $\lambda(x) = (\lambda(x,t))_{0 \le t \le \widehat{T}}$ of $N(x)$ under P can now be informally defined by

$$\lambda(x,t)dt \equiv E^P\left[dN(x,t)|\,\mathcal{H}(t-) \vee \mathcal{I}(t)\right] = (n - N(x,t-))\mu(x,t)dt, \tag{7.2.1}$$

which is given by the product of the number of survivors and the mortality intensity. It is well-known that the compensated counting process $M(x) = (M(x,t))_{0 \le t \le \widehat{T}}$ defined by

$$dM(x,t) = dN(x,t) - \lambda(x,t)dt, \;\; 0 \le t \le \widehat{T}, \tag{7.2.2}$$

is an $(\mathbb{H} \vee \mathbb{I}, P)$-martingale.

7.3 The combined model

Assume that the filtration $\mathbb{F} = (\mathcal{F}(t))_{0 \le t \le \widehat{T}}$ introduced earlier is given by

$$\mathcal{F}(t) = \mathcal{G}(t) \vee \mathcal{H}(t) \vee \mathcal{I}(t).$$

Thus, $\mathbb{F}$ is the filtration for the combined model of the financial market, the mortality intensity and the insurance portfolio. Moreover, we assume that the financial market is stochastically independent of the development of the insurance portfolio and the mortality intensity, i.e. $\mathcal{G}(\widehat{T})$ and $(\mathcal{H}(\widehat{T}), \mathcal{I}(\widehat{T}))$ are independent. In particular, this implies that the properties of the underlying processes are preserved. For example, $M(x)$ is also an $(\mathbb{F}, P)$-martingale, and the $(\mathbb{F}, P)$-intensity process is identical to the $(\mathbb{H} \vee \mathbb{I}, P)$-intensity process $\lambda(x)$.

7.3.1 A class of equivalent martingale measures

The combined model allows for infinitely many equivalent martingale measures, such that the model is arbitrage free, but not complete, see e.g. Björk (2004, Chapter 10). In order to perform a simultaneous change of measure for the Wiener processes W^f, W^s and W^μ, and the counting process $N(x)$, we consider the likelihood process

$$d\Lambda(t) = \Lambda(t-)\Big(h^f(t)dW^f(t) + h^s(t)dW^s(t) + h^\mu(t)dW^\mu(t) + g(x,t)dM(x,t)\Big), \tag{7.3.1}$$

with $\Lambda(0) = 1$. In addition to Λ, we define the likelihood process O, which leads to a change of measure for Y, by

$$O(t) = \prod_{j=1}^{i_t} \left(1 + o_j(Y_j)\right), \tag{7.3.2}$$

for some functions o_j, $j \in \{1, \dots, m\}$, satisfying $o_j(y) > -1$ for all y in the support of Y_j and $E^P[o_j(Y_j)] = 0$. Here, i_t is defined in (7.2.7) and $\prod_{j=1}^{i_t}$ is interpreted as 1 if $t < T_1$

(and thus $i_t = 0$). We assume that $E^P[\Lambda_{\widehat{T}} O_{\widehat{T}}] = 1$ and define an equivalent martingale measure Q via

$$\frac{dQ}{dP} = \Lambda(\widehat{T})O(\widehat{T}). \tag{7.3.3}$$

In the following, we describe the terms in (7.3.1) and (7.3.2) in more detail. The processes h^f and h^s are related to the change of measure for the financial market. Girsanov's theorem gives that under Q defined by (7.3.1)–(7.3.3), $W^{f,Q}(t) = W^f(t) - \int_0^t h^f(u)du$ and $W^{s,Q}(t) = W^s(t) - \int_0^t h^s(u)du$ are independent Wiener processes, such that the Q-dynamics of Z and S are given by

$$dZ(t) = r(t)Z(t)dt - \sigma^z(t)Z(t)dW^{f,Q}(t), \tag{7.3.4}$$
$$dS(t) = r(t)S(t)dt - \sigma^s(t)S(t)dW^{f,Q}(t) + \beta^s(t)S(t)dW^{s,Q}(t). \tag{7.3.5}$$

Hence, the specification of the financial market ensures that under any Q given by (7.3.3) the discounted price processes are Q-martingales.

The term involving h^μ leads to a change of measure for the Wiener process which drives the mortality intensity process μ. Hence, $W^{\mu,Q}(t) = W^\mu(t) - \int_0^t h^\mu(u)du$ defines a Wiener process under Q. Here, as in Chapter 4, we restrict ourselves to h^μ's of the form

$$h^\mu(t, \zeta(x,t)) = -\beta(x,t)\frac{\sqrt{\zeta(x,t)}}{\sigma^\zeta(x,t)} + \frac{\beta^*(x,t)}{\sigma^\zeta(x,t)\sqrt{\zeta(x,t)}} \tag{7.3.6}$$

for some continuous functions β and β^*. In this case the Q-dynamics of $\zeta(x)$ are given by

$$d\zeta(x,t) = \left(\gamma^{\zeta,Q}(x,t) - \delta^{\zeta,Q}(x,t)\zeta(x,t)\right)dt + \sigma^\zeta(x,t)\sqrt{\zeta(x,t)}dW^{\mu,Q}(t),$$

where

$$\gamma^{\zeta,Q}(x,t) = \gamma^\zeta(x,t) + \beta^*(x,t), \tag{7.3.7}$$
$$\delta^{\zeta,Q}(x,t) = \delta^\zeta(x,t) + \beta(x,t). \tag{7.3.8}$$

Hence, ζ follows a time-inhomogeneous CIR under Q as well. For (7.3.1) and (7.3.2) to define an equivalent martingale measure it must hold that ζ is strictly positive under Q. Thus, we observe from (7.3.7) that a necessary condition is $\beta^*(x,t) \geq (\sigma^\zeta(x,t))^2/2 - \gamma^\zeta(x,t)$. The Q-dynamics of $\mu(x)$ are given by

$$d\mu(x,t) = \left(\gamma^{\mu,Q}(x,t) - \delta^{\mu,Q}(x,t)\mu(x,t)\right)dt + \sigma^\mu(x,t)\sqrt{\mu(x,t)}dW^{\mu,Q}(t), \tag{7.3.9}$$

where $\gamma^{\mu,Q}(x,t)$ and $\delta^{\mu,Q}(x,t)$ are given by (7.2.16) and (7.2.17) with $\gamma^\zeta(x,t)$ and $\delta^\zeta(x,t)$ replaced by $\gamma^{\zeta,Q}(x,t)$ and $\delta^{\zeta,Q}(x,t)$, respectively. If $h^\mu = 0$, i.e. if the dynamics of ζ (and μ) are identical under P and Q, we say the market is risk-neutral with respect to systematic mortality risk.

The last term in (7.3.1) involves a predictable process $g(x) > -1$. This term affects the intensity for the counting process. More precisely, it can be shown, see e.g. Andersen et al. (1993), that the intensity process under Q is given by $\lambda^Q(x,t) = (1+g(x,t))\lambda(x,t)$. Using (7.2.1), we see that

$$\lambda^Q(x,t) = (n - N(x,t-))(1+g(x,t))\mu(x,t),$$

such that $\mu^Q(x,t) = (1+g(x,t))\mu(x,t)$ can be interpreted as the mortality intensity under Q. Hence, the process $M^Q(x) = (M^Q(x,t))_{0\le t\le \widehat{T}}$ defined by

$$dM^Q(x,t) = dN(x,t) - \lambda^Q(x,t)dt, \quad 0 \le t \le \widehat{T}, \tag{7.3.10}$$

is an $(\mathbb{F},Q)$-martingale. If $g(x) = 0$, the market is said to be risk-neutral with respect to unsystematic mortality risk. This choice of g can be motivated by the law of large numbers. In this paper, we restrict the analysis to the case, where $g(x)$ is a deterministic, continuously differentiable function.

With O given by (7.3.2) the distribution function of Y_i, $i \in \{1,\dots,m\}$, under Q is given by

$$F_i^Q(y) = \int_{-\infty}^{y} (1+o_i(z))dF_i^P(z).$$

If $o_i = 0$ for all i the market is called risk-neutral with respect to reinvestment risk. The measures considered are particularly simple, since the independence between $\mathcal{G}^x(\widehat{T})$, $\mathcal{G}^y(\widehat{T})$ and $(\mathcal{H}(\widehat{T}),\mathcal{I}(\widehat{T}))$ as well as the mutual independence of the Y_i's are preserved under Q.

Now define the Q-survival probability and the associated Q-martingale by

$$\mathcal{S}^Q(x,t,T) = E^Q\left[e^{-\int_t^T \mu^Q(x,\tau)d\tau} \middle| \zeta(x,t) \right]$$

and

$$\mathcal{S}^{Q,M}(x,t,T) = E^Q\left[e^{-\int_0^T \mu^Q(x,\tau)d\tau} \middle| \zeta(x,t) \right] = e^{-\int_0^t \mu^Q(x,\tau)d\tau}\mathcal{S}^Q(x,t,T).$$

Calculations similar to those in Section 7.2.2 give the following Q-dynamics of $\mu^Q(x)$:

$$d\mu^Q(x,t) = \left(\gamma^{\mu,Q,g}(x,t) - \delta^{\mu,Q,g}(x,t)\mu^Q(x,t)\right)dt + \sigma^{\mu,g}(x,t)\sqrt{\mu^Q(x,t)}dW^{\mu,Q}, \tag{7.3.11}$$

where

$$\begin{aligned}
\gamma^{\mu,Q,g}(x,t) &= (1+g(x,t))\gamma^{\mu,Q}(x,t),\\
\delta^{\mu,Q,g}(x,t) &= \delta^{\mu,Q}(x,t) - \frac{\frac{\partial}{\partial t}g(x,t)}{1+g(x,t)},\\
\sigma^{\mu,g}(x,t) &= \sqrt{1+g(x,t)}\sigma^{\mu}(x,t).
\end{aligned}$$

Since the drift and squared diffusion terms in (7.3.11) are affine in $\mu^Q(x,t)$ we have an affine mortality structure under Q. Hence, we have the following expression for the Q-survival probability:

$$\mathcal{S}^Q(x,t,T) = e^{A^{\mu,Q}(x,t,T) - B^{\mu,Q}(x,t,T)(1+g(x,t))\mu(x,t)},$$

where $A^{\mu,Q}$ and $B^{\mu,Q}$ are determined from (7.2.19) and (7.2.20) with $\gamma^{\mu}(x,t)$, $\delta^{\mu}(x,t)$ and $\sigma^{\mu}(x,t)$ replaced by $\gamma^{\mu,Q,g}(x,t)$, $\delta^{\mu,Q,g}(x,t)$ and $\sigma^{\mu,g}(x,t)$, respectively. Furthermore, the dynamics of $\mathcal{S}^{Q,M}(x,\cdot,T)$ are given by

$$d\mathcal{S}^{Q,M}(x,t,T) = -(1+g(x,t))\sigma^{\mu}(x,t)\sqrt{\mu(x,t)}B^{\mu,Q}(x,t,T)\mathcal{S}^{Q,M}(x,t,T)dW^{\mu,Q}(t), \tag{7.3.12}$$

and the Q-forward mortality intensities by

$$f^{\mu,Q}(x,t,T) = -\frac{\partial}{\partial T}\log \mathcal{S}^{Q}(x,t,T) = \mu^{Q}(x,t)\frac{\partial}{\partial T}B^{\mu,Q}(x,t,T) - \frac{\partial}{\partial T}A^{\mu,Q}(x,t,T). \tag{7.3.13}$$

7.3.2 The payment process

The total benefits less premiums on the insurance portfolio is described by a payment process A, where $dA(t)$ are the net payments to the policy-holders during an infinitesimal interval $[t,t+dt)$. For $0 \leq t \leq T$ we let A be of the form

$$\begin{aligned} dA(t) &= -n\pi(0)d1_{(t\geq 0)} + (n - N(x,\overline{T}))\Delta A_0(\overline{T})d1_{(t\geq \overline{T})} \\ &\quad + a_0(t)(n-N(x,t))dt + a_1(t)dN(x,t), \end{aligned} \tag{7.3.14}$$

where $\pi(0)$ is a constant, a_0 and a_1 are $\mathbb{G}$-adapted processes and $\Delta A_0(\overline{T})$ is $\mathcal{G}(\overline{T})$-measurable. The first term, $n\pi(0)$ is the single premium paid at time 0 by all policy-holders. The second term involves a fixed time $\overline{T} \leq T$, which represents the retirement time of the insured. This term states that each of the surviving policy-holders receive the amount $\Delta A_0(\overline{T})$ upon retirement. The third term involves the process a_0 given by

$$a_0(t) = -\pi^{c}(t)1_{(0\leq t<\overline{T})} + a^{p}(t)1_{(\overline{T}\leq t\leq T)},$$

where $\pi^c(t)$ are continuous premiums paid by the policy-holders (as long as they are alive) and $a^p(t)$ corresponds to a life annuity benefit received by the policy-holders. Finally, the last term states that a_1 is paid immediately upon a death.

Henceforth we consider an arbitrary but fixed equivalent martingale measure Q from the class of measures introduced in Section 7.3.1. Since the payments $a_0(u)$ and $a_1(u)$ are $\mathcal{G}(u)$-measurable for all $u \in [0,T]$ and $\Delta A_0(\overline{T})$ is $\mathcal{G}(\overline{T})$-measurable we can define the arbitrage free prices with respect to the filtration $\mathbb{G}^{T_i}$ under the fixed equivalent martingale measure Q by

$$\begin{aligned} F^{T_i,0}(t,u) &= E^{Q}\left[e^{-\int_t^u r(\tau)d\tau} a_0(u) \middle| \mathcal{G}^{T_i}(t)\right], \\ F^{T_i,1}(t,u) &= E^{Q}\left[e^{-\int_t^u r(\tau)d\tau} a_1(u) \middle| \mathcal{G}^{T_i}(t)\right], \\ F^{T_i,\Delta}(t,\overline{T}) &= E^{Q}\left[e^{-\int_t^{\overline{T}} r(\tau)d\tau} \Delta A_0(\overline{T}) \middle| \mathcal{G}^{T_i}(t)\right]. \end{aligned}$$

We note that since the model $(B,X,\mathbb{G}^{T_m})$ is complete the functions $(F^{T_m,0}(t,u))_{t\leq u\leq T}$, $(F^{T_m,1}(t,u))_{t\leq u\leq T}$ and $F^{T_m,\Delta}(t,\overline{T})$ are unique for all t; in particular they are independent of the fixed equivalent martingale measure Q. Here, we restrict ourselves to payment processes, where $(F^{T_m,0}(t,u))_{t\leq u\leq T}$, $(F^{T_m,1}(t,u))_{t\leq u\leq T}$ and $F^{T_m,\Delta}(t,\overline{T})$ are functions of t and $X(t)$, only. Henceforth, we shall apply the notation

$$\vartheta^{T_i,c} = \left(\vartheta^{T_i,c,z}, \vartheta^{T_i,c,s}\right) = \left(\frac{\partial}{\partial z}F^{T_i,c}, \frac{\partial}{\partial s}F^{T_i,c}\right) = \frac{\partial}{\partial x}F^{T_i,c}$$

for $c \in \{0, 1, \Delta\}$.

7.3.3 Market reserves

For each i, $i \in \{0, \ldots, m\}$, we define the process $V^{T_i,*}$ by

$$V^{T_i,*}(t) = E^Q\left[\int_{[0,T]} e^{-\int_0^\tau r_u du} dA(\tau) \middle| \mathcal{F}^{T_i}(t)\right]. \tag{7.3.15}$$

Hence, $V^{T_i,*}(t)$ is the conditional expected value with respect to the filtration $\mathbb{F}^{T_i}$, calculated at time t, of discounted benefits less premiums, where all payments are discounted to time 0. Using that the processes A and r are $\mathbb{F}^{T_i}$-adapted for all i, and introducing the discounted payment process A^* defined by

$$dA^*(t) = e^{-\int_0^t r(u)du} dA(t),$$

we see that

$$\begin{aligned} V^{T_i,*}(t) &= \int_{[0,t]} e^{-\int_0^\tau r(u)du} dA(\tau) + e^{-\int_0^t r_u du} E^Q\left[\int_{(t,T]} e^{-\int_t^\tau r(u)du} dA(\tau) \middle| \mathcal{F}^{T_i}(t)\right] \\ &= A^*(t) + e^{-\int_0^t r(u)du} \widetilde{V}^{T_i}(t). \end{aligned} \tag{7.3.16}$$

In the literature, the process $V^{T_i,*}$ has been called the intrinsic value process (with respect to $\mathbb{F}^{T_i}$), see Föllmer and Sondermann (1986) and Møller (2001c). The process $\widetilde{V}^{T_i}$ represents the conditional expected value with respect to $\mathbb{F}^{T_i}$, of future discounted payments. We shall refer to this quantity as the $\mathbb{F}^{T_i}$-market reserve. Using methods similar to those in Møller (2001c) and Chapters 3 and 4, we obtain the following result.

Proposition 7.3.1
The $\mathbb{F}^{T_i}$-market reserve, $\widetilde{V}^{T_i}$, is given by

$$\widetilde{V}^{T_i}(t) = (n - N(x,t))\widehat{V}^{T_i}(t), \tag{7.3.17}$$

where

$$\begin{aligned} \widehat{V}^{T_i}(t) = &\int_t^T \mathcal{S}^Q(x,t,\tau)\left(F^{T_i,0}(t,\tau) + f^{\mu,Q}(x,t,\tau)F^{T_i,1}(t,\tau)\right) d\tau \\ &+ \mathcal{S}^Q(x,t,\overline{T})F^{T_i,\Delta}(t,\overline{T}). \end{aligned} \tag{7.3.18}$$

Here, the process $\widehat{V}^{T_i}$ is interpreted as the individual $\mathbb{F}^{T_i}$-market reserve given the insured is alive.

In the remaining of the paper we work under the following assumption

Assumption 7.3.2 *$\widehat{V}^{T_m}(t,x,\mu) \in C^{1,2,2}$, i.e. $\widehat{V}^{T_m}(t,x,\mu)$ is continuously differentiable with respect to t and twice differentiable with respect to x and μ.*

□

7.3.4 Trading in the financial market

As in Chapter 6 we follow the ideas of Møller (2001c) and define trading in the presence of payment processes. Since we consider a fixed arbitrary equivalent martingale measure Q for the model $(B, X, \mathbb{F})$, we are working with the probability space $(\Omega, \mathcal{F}, Q)$ and the filtrations $(\mathbb{F}^{T_i})_{i\in\{0,\ldots,m\}}$.

An $\mathbb{F}^{T_i}$-trading strategy is a process $\varphi = (\vartheta, \eta)$ satisfying certain integrability conditions, where ϑ is $\mathbb{F}^{T_i}$-predictable and η is $\mathbb{F}^{T_i}$-adapted. The value process $\mathcal{V}(\varphi)$ associated with φ is defined by

$$\mathcal{V}(t,\varphi) = \vartheta(t)X(t) + \eta(t)B(t), \quad 0 \le t \le \widehat{T}.$$

The pair $\varphi(t) = (\vartheta(t), \eta(t))$ is interpreted as the portfolio held at time t. Here, $\vartheta = (\vartheta^z, \vartheta^s)$ is a vector denoting, respectively, the number of assets with price process Z and S, and η denotes the discounted deposit in the savings account.

The *cost process* associated with the pair (φ, A) is given by

$$C(t,\varphi) = \mathcal{V}^*(t,\varphi) - \int_0^t \vartheta(u)dX^*(u) + A^*(t). \tag{7.3.19}$$

Here, and throughout, we denote by $\mathcal{V}^*$ and X^*, respectively, the discounted value process and the discounted price process of the risky assets. Thus, the cost process is the discounted value of the investment portfolio reduced by discounted trading gains and added the total discounted net payments to the policy-holders. The cost process is interpreted as the company's accumulated discounted costs during $[0,t]$.

We say that a strategy φ is $\mathbb{F}^{T_i}$-self-financing for the payment process A, if the cost process is constant Q-a.s. with respect to $\mathbb{F}^{T_i}$. In contrast to the classical definition of self-financing strategies, we thus allow for exogenous deposits and withdrawals as represented by A. The two definitions of self-financing strategies are equivalent if and only if the payment process is constant Q-a.s. with respect to $\mathbb{F}^{T_i}$. The interpretation of a self-financing strategy in the presence of payment processes is that all fluctuations of the value process are either generated by the trading strategy or due to the payment process. The payment process A is called $\mathbb{F}^{T_i}$-attainable, if there exist an $\mathbb{F}^{T_i}$-self-financing strategy φ for A such that $\mathcal{V}^*(T,\varphi) = 0$ Q-a.s. with respect to $\mathbb{F}^{T_i}$. Thus, A is $\mathbb{F}^{T_i}$-attainable, if investing the initial amount $C(0,\varphi)$ according to the $\mathbb{F}^{T_i}$-trading strategy φ leaves us with a portfolio value of 0 after the settlement of all liabilities. Hence, if A is $\mathbb{F}^{T_i}$-attainable the unique arbitrage free price in $(B, X, \mathbb{F}^{T_i})$ is $C(0, \varphi)$. At any time t, there is no difference between receiving the *future* payments of the $\mathbb{F}^{T_i}$-attainable payment process A and holding the portfolio $\varphi(t)$ and investing according to the $\mathbb{F}^{T_i}$-replicating strategy φ. Thus, a no arbitrage argument gives that at any time t the price of future payments from A in $(B, X, \mathbb{F}^{T_i})$ must be $\mathcal{V}(t, \varphi)$. It can be shown that the payment process A is attainable if and only if the contingent claim $H = A(T)$ with maturity T is (classically) attainable. If all contingent claims, and hence all payment processes, are attainable, the model is called complete and otherwise it is called incomplete.

7.4 Risk-minimization for unit-linked insurance contracts

The insurance payment process A may be subject to both unhedgeable reinvestment and mortality risks. This implies that A typically cannot be replicated perfectly and priced uniquely by trading in the financial market. In order to quantify and control the risk associated with A we apply the criterion of risk-minimization proposed by Föllmer and Sondermann (1986) for contingent claims and extended in Møller (2001c) to payment processes. Here, we give a review of risk-minimization and derive risk-minimizing strategies in the present set-up. The derivation consists two steps. First, we derive risk-minimizing strategies in the case of a complete financial market and unhedgeable systematic and unsystematic mortality risk; a study which also was carried out in Chapter 4 in a slightly different financial market. Second, we extend to the case where A also is subject to unhedgeable reinvestment risk.

7.4.1 A review of risk-minimization

Throughout this section, we consider a fixed but arbitrary filtration $\mathbb{F}^{T_i}$, such that we are working with the filtered probability space $(\Omega, \mathcal{F}, Q, \mathbb{F}^{T_i})$.

The $\mathbb{F}^{T_i}$-*risk process* associated with φ is defined by

$$R^{T_i}(t, \varphi) = E^Q\left[(C(T, \varphi) - C(t, \varphi))^2 \,\middle|\, \mathcal{F}^{T_i}(t) \right], \tag{7.4.1}$$

where the cost process is defined in (7.3.19). Thus, the $\mathbb{F}^{T_i}$-risk process is the conditional expectation of the discounted squared future costs given the current available information given by $\mathbb{F}^{T_i}$. We shall use this quantity to measure the risk associated with (φ, A). An $\mathbb{F}^{T_i}$-trading strategy $\varphi = (\vartheta, \eta)$ is called $\mathbb{F}^{T_i}$-risk-minimizing if for any $t \in [0, T]$ it minimizes $R^{T_i}(t, \varphi)$ over all $\mathbb{F}^{T_i}$-trading strategies with the same value at time T. Since $\mathcal{V}^*(T, \varphi)$ is the discounted value of the portfolio $\varphi(T)$ upon settlement of all liabilities a natural restriction is to consider so-called 0-*admissible* strategies which satisfy

$$\mathcal{V}^*(T, \varphi) = 0, \quad Q\text{-a.s.}$$

The construction of risk-minimizing strategies is based on the Galtchouk–Kunita–Watanabe decomposition of the intrinsic value process

$$V^{T_i,*}(t) = V^{T_i,*}(0) + \int_0^t \vartheta^{T_i,A}(u)dX^*(u) + L^{T_i}(t). \tag{7.4.2}$$

Here, L^{T_i} is a zero-mean square integrable $(Q, \mathbb{F}^{T_i})$-martingale which is orthogonal to X^*, i.e. the process $X^* L^{T_i}$ is a $(Q, \mathbb{F}^{T_i})$-martingale, and $\vartheta^{T_i,A}$ is an $\mathbb{F}^{T_i}$-predictable process. We note that if A is $\mathbb{F}^{T_i}$-attainable, then $V^{T_i,*}(t)$ is the discounted unique arbitrage free price at time t in $(B, X, \mathbb{F}^{T_i})$ of future payments, and $L^{T_i} = 0$ Q-a.s. with respect to $\mathbb{F}^{T_i}$. The following theorem relates the risk-minimizing strategy and the associated risk process to the Galtchouk–Kunita–Watanabe decomposition.

Theorem 7.4.1 (Møller (2001c))
There exists a unique 0-admissible $\mathbb{F}^{T_i}$-risk-minimizing strategy $\varphi^{T_i} = (\vartheta^{T_i}, \eta^{T_i})$ for A given by

$$\left(\vartheta^{T_i}(t), \eta^{T_i}(t)\right) = \left(\vartheta^{T_i,A}(t), V^{T_i,*}(t) - A^*(t) - \vartheta^{T_i,A}(t)X^*(t)\right).$$

From Theorem 7.4.1 we immediately get that

$$\mathcal{V}^*(t, \varphi^{T_i}) = V^{T_i,*}(t) - A^*(t) \tag{7.4.3}$$

such that the discounted value process associated with the $\mathbb{F}^{T_i}$-risk-minimizing strategy coincides with the $\mathbb{F}^{T_i}$-intrinsic value process reduced by the discounted payments. Inserting (7.4.1) and (7.4.3) in (7.3.19) it follows that the cost process associated with the $\mathbb{F}^{T_i}$-risk-minimizing strategy is given by

$$C(t, \varphi^{T_i}) = V^{T_i,*}(0) + L^{T_i}(t). \tag{7.4.4}$$

Hence, the cost process associated with the $\mathbb{F}^{T_i}$-risk-minimizing strategy is an $(\mathbb{F}^{T_i}, Q)$-martingale. Inserting (7.4.4) in (7.4.1) we get that the so-called $\mathbb{F}^{T_i}$-intrinsic risk process, which is the risk process associated with the $\mathbb{F}^{T_i}$-risk-minimizing strategy, is given by

$$R^{T_i}(t, \varphi^{T_i}) = E^Q\left[\left(L^{T_i}(T) - L^{T_i}(t)\right)^2 \middle| \mathcal{F}^{T_i}(t)\right]. \tag{7.4.5}$$

Note that the risk-minimizing strategy depends on the equivalent martingale measure Q. In the literature, the so-called minimal martingale measure, see Section 7.5.2, has been applied in order to determine risk-minimizing strategies, since this essentially corresponds to the criterion of local risk-minimization, which is a criterion in terms of P, see Schweizer (2001a).

7.4.2 Unhedgeable mortality risk

In this section we consider risk-minimization with respect to $\mathbb{F}^{T_m}$. Hence, for now we disregard the reinvestment risk, such that the only unhedgeable risks are the systematic and unsystematic mortality risk. In this case we have the following result with respect to the Galtchouk–Kunita–Watanabe decomposition of $V^{T_m,*}$.

Lemma 7.4.2
The Galtchouk–Kunita–Watanabe decomposition of $V^{T_m,*}$ is given by

$$V^{T_m,*}(t) = V^{T_m,*}(0) + \int_0^t \vartheta^{T_m,A}(\tau)dX^*(\tau) + L^{T_m}(t), \tag{7.4.6}$$

where

$$V^{T_m,*}(0) = -n\pi(0) + n\widehat{V}^{T_m}(0), \tag{7.4.7}$$

$$L^{T_m}(t) = \int_0^t \nu^{T_m}(\tau)dM^Q(x,\tau) + \int_0^t \kappa^{T_m}(\tau)d\mathcal{S}^{Q,M}(x,\tau,T), \tag{7.4.8}$$

and

$$\begin{aligned}\vartheta^{T_m,A}(t) = (n - N(x,t-))\Bigg(&\int_t^T \mathcal{S}^Q(x,t,\tau)\left(\vartheta^{T_m,0}(t,\tau) + f^{\mu,Q}(x,t,\tau)\vartheta^{T_m,1}(t,\tau)\right)d\tau \\ &+ \mathcal{S}^Q(x,t,\overline{T})\vartheta^{T_m,\Delta}(t,\overline{T})\Bigg),\end{aligned} \tag{7.4.9}$$

$$\nu^{T_m}(t) = B(t)^{-1}\left(a_1(t) - \widehat{V}^{T_m}(t)\right), \tag{7.4.10}$$

$$\begin{aligned}\kappa^{T_m}(t) = (n - N(x,t-))B(t)^{-1}\Bigg(&\int_t^T \frac{B^{\mu,Q}(x,t,\tau)\mathcal{S}^Q(x,t,\tau)}{B^{\mu,Q}(x,t,T)\mathcal{S}^{Q,M}(x,t,T)} \\ &\times \left(F^{T_m,0}(t,\tau) + F^{T_m,1}(t,\tau)\left(f^{\mu,Q}(x,t,\tau) - \frac{\frac{\partial}{\partial\tau}B^{\mu,Q}(x,t,\tau)}{B^{\mu,Q}(x,t,\tau)}\right)\right)d\tau \\ &+ \frac{B^{\mu,Q}(x,t,\overline{T})\mathcal{S}^Q(x,t,\overline{T})}{B^{\mu,Q}(x,t,T)\mathcal{S}^{Q,M}(x,t,T)}F^{T_m,\Delta}(t,\overline{T})\Bigg).\end{aligned} \tag{7.4.11}$$

Proof of Lemma 7.4.2: The proof, which is similar to the proof of Lemma 4.7.1, is carried out in Section 7.6.1.

□

The Galtchouk–Kunita–Watanabe decomposition of $V^{T_m,*}$ in Lemma 7.4.2 is essentially the same as the one obtained in Lemma 4.7.1. The process L^{T_m} describes the unhedgeable risk associated with the payment process. The integral with respect to the Q-compensated counting process, $M^Q(x)$, is related to the unsystematic mortality risk, whereas the integral with respect to the martingale associated with the Q-survival probability, $\mathcal{S}^{Q,M}(x,\cdot,T)$, is related to the systematic mortality risk. Combining Theorem 7.4.1 and Lemma 7.4.2 we get the following result regarding the $\mathbb{F}^{T_m}$-risk-minimizing strategy for the payment process A in (7.3.14).

Theorem 7.4.3
For the payment process given by (7.3.14), the unique 0*-admissible $\mathbb{F}^{T_m}$-risk-minimizing strategy φ^{T_m} is*

$$\left(\vartheta^{T_m}(t), \eta^{T_m}(t)\right) = \left(\vartheta^{T_m,A}(t), (n - N(x,t))B(t)^{-1}\widehat{V}^{T_m}(t) - \vartheta^{T_m}(t)X^*(t)\right),$$

where $\vartheta^{T_m,A}$ is given by (7.4.9).

The importance of Lemma 7.4.2 and Theorem 7.4.3 is twofold. Firstly, they give the risk-minimizing strategy and the process L^{T_m} for the payment process A in the case of a complete financial market and unhedgeable systematic and unsystematic mortality risk. Secondly, they are of importance, since the $\mathbb{F}$-risk-minimizing strategies determined in Section 7.4.3 are given in terms of the $\mathbb{F}^{T_m}$-risk-minimizing strategies.

7.4.3 Unhedgeable mortality and reinvestment risks

Now consider the case where the company, in addition to the unhedgeable mortality risks, is exposed to reinvestment risk. We can now apply Schweizer (1994, Theorem 3.1) to the case of payment processes to obtain the $\mathbb{F}$-risk-minimizing strategies in terms of the $\mathbb{F}^{T_m}$-risk-minimizing strategies. Hence, calculations similar to those leading to Theorem 6.3.3 give the following theorem, which we state without given a proof.

Theorem 7.4.4
The unique 0-admissible $\mathbb{F}$-risk-minimizing strategy $\varphi^0 = (\vartheta^0, \eta^0)$ for A is given by

$$\big(\vartheta^0(t), \eta^0(t)\big) = \Big(E^Q\left[\vartheta^{T_m}(t) \middle| \mathcal{F}(t-)\right], (n - N(x,t))B(t)^{-1}\widehat{V}^0(t) - \vartheta^0(t)X^*(t)\Big).$$

In the following we shall use the quantities given by

$$\vartheta^{T_i}(t) = E^Q\left[\vartheta^{T_m}(t) \middle| \mathcal{F}^{T_i}(t-)\right], \tag{7.4.12}$$
$$\nu^{T_i}(t) = E^Q\left[\nu^{T_m}(\tau) \middle| \mathcal{F}^{T_i}(t)\right], \tag{7.4.13}$$
$$\kappa^{T_i}(t) = E^Q\left[\kappa^{T_m}(\tau) \middle| \mathcal{F}^{T_i}(t)\right], \tag{7.4.14}$$

and the notation exemplified by $\Delta_i\vartheta(t) = \vartheta^{T_i}(t) - \vartheta^{T_{i-1}}(t)$. The Galtchouk–Kunita–Watanabe decomposition of $V^{0,*}$ is given by the following proposition.

Proposition 7.4.5
If ϑ^{T_i}, ν^{T_i} and κ^{T_i} are sufficiently integrable for all $i \in \{0, \dots, m\}$ then the Galtchouk–Kunita–Watanabe decomposition of $V^{0,*}$ is given by

$$V^{0,*}(t) = V^{0,*}(0) + \int_0^t \vartheta^0(\tau)dX^*(\tau) + L^0(t), \tag{7.4.15}$$

where

$$V^{0,*}(0) = -n\pi(0) + n\widehat{V}^0(0),$$
$$L^0(t) = M^{y,Q}(t) + \int_0^t \nu^0(\tau)dM^Q(x,\tau) + \int_0^t \kappa^0(\tau)d\mathcal{S}^{Q,M}(x,\tau,T), \tag{7.4.16}$$

and

$$M^{y,Q}(t) = \sum_{i=1}^{i_t}\Bigg(\Delta_iV^*(0) + \int_0^{T_i}\Delta_i\vartheta(\tau)dX^*(\tau) + \int_0^{T_i}\Delta_i\nu(\tau)dM^Q(x,\tau) + \int_0^{T_i}\Delta_i\kappa(\tau)d\mathcal{S}^{Q,M}(x,\tau,T)\Bigg).$$

Proof of Proposition 7.4.5: See Section 7.6.2

□

Investigating the expression for L^0 in (7.4.16) we observe that the two integrals, which are associated with the mortality risks are similar to ones from Lemma 7.4.2. However, the sum, $M^{y,Q}$, which is related to the reinvestment risk, is new. It describes the additional in- or outflow to/from the investment strategy upon the realization of the prices of newly issued bonds bonds.

Example 7.4.6 Consider the case where $\overline{T} = T$ and all n insured have purchased a pure endowment of $\Delta A_0(T)$ paid by a single premium at time 0. In this case, the Galtchouk–Kunita–Watanabe decomposition of $V^{0,*}$ is determined via the processes

$$\vartheta^{T_i}(\tau) = (n - N(x,\tau-))\mathcal{S}^Q(x,\tau,T)\vartheta^{T_i,\Delta}(\tau), \tag{7.4.17}$$
$$\nu^{T_i}(\tau) = -B(\tau)^{-1}\mathcal{S}^Q(x,\tau,T)F^{T_i,\Delta}(\tau,T), \tag{7.4.18}$$
$$\kappa^{T_i}(\tau) = (n - N(x,\tau))e^{\int_0^\tau \mu^Q(x,u)du}B(\tau)^{-1}F^{T_i,\Delta}(\tau,T). \tag{7.4.19}$$

Using Theorem 7.4.4 we obtain the 0-admissible $\mathbb{F}$-risk-minimizing strategy

$$\vartheta^0(t) = (n - N(x,t-))\mathcal{S}^Q(x,t,T)\vartheta^{0,\Delta}(t),$$
$$\eta^0(t) = (n - N(x,t))\mathcal{S}^Q(x,t,T)B(t)^{-1}F^{0,\Delta}(t,T) - \vartheta^0(t)X^*(t).$$

Inserting (7.4.17)–(7.4.19) in Proposition 7.4.5 gives the following expressions for the terms in the Galtchouk–Kunita–Watanabe decomposition of $V^{0,*}$:

$$V^{0,*}(0) = -n\pi(0) + n\mathcal{S}^Q(x,0,T)F^{0,\Delta}(0,T)$$

and

$$\begin{aligned}L^0(t) = M^{y,Q}(t) &- \int_0^t B(\tau)^{-1}\mathcal{S}^Q(x,\tau,T)F^{0,\Delta}(\tau,T)dM^Q(x,\tau)\\ &+ \int_0^t (n - N(x,\tau))e^{\int_0^\tau \mu^Q(x,u)du}B(\tau)^{-1}F^{0,\Delta}(\tau,T)d\mathcal{S}^{Q,M}(x,\tau,T),\end{aligned}$$

where

$$\begin{aligned}M^{y,Q}(t) = \sum_{i=1}^{i_t}\Bigg(&n\mathcal{S}^Q(x,0,T)\Delta_i F^\Delta(0,T) + \int_0^{T_i}(n - N(x,\tau-))\mathcal{S}^Q(x,\tau,T)\Delta_i\vartheta^\Delta(\tau)dX^*(\tau)\\ &- \int_0^{T_i} B(\tau)^{-1}\mathcal{S}^Q(x,\tau,T)\Delta_i F^\Delta(\tau,T)dM^Q(x,\tau)\\ &+ \int_0^{T_i}(n - N(x,\tau))e^{\int_0^\tau \mu^Q(x,u)du}B(\tau)^{-1}\Delta_i F^\Delta(\tau,T)d\mathcal{S}^{Q,M}(x,\tau,T)\Bigg).\end{aligned} \tag{7.4.20}$$

□

Example 7.4.7 Assume each of the entries in the vector Y has finite support, i.e. Y_i takes values in $y_1^i, \dots, y_{c_i}^i$, $i = 1, \dots, m$, for some $c_i \in \mathbb{N}$. Now introduce the Q-martingales

$$M^{\delta_k,Q}(t) = E^Q\left[1_{((Y_1,\dots,Y_m)=\delta_k)}\middle|\mathcal{F}(t)\right] = E^Q\left[1_{((Y_1,\dots,Y_m)=\delta_k)}\middle|\mathcal{G}^y(t)\right],$$

where $\delta_1, \dots, \delta_K$ are the possible outcomes of the vector $(Y_1, \dots, Y_m)$, such that $K = \prod_{i=1}^m c_i$. Here, we have used that Y is Q-independent of all other sources of randomness in the second equality. Throughout this example we use the notation exemplified by ϑ^{δ_k} for ϑ^{T_m} in the case where $Y = \delta_k$. This allows us to write the $\mathbb{F}$-risk-minimizing strategy as

$$\vartheta^0(t) = \sum_{k=1}^K M^{\delta_k,Q}(t-)\vartheta^{\delta_k}(t), \tag{7.4.21}$$

$$\eta^0(t) = (n - N(x,t))B(t)^{-1}\sum_{k=1}^K M^{\delta_k,Q}(t)\widehat{V}^{\delta_k}(t) - \vartheta^0(t)X^*(t).$$

Since $dM_t^{\delta_k,Q} = 0$ for $t \notin \{T_1, \dots, T_m\}$ the expression for L^0 in Proposition 7.4.5 simplifies to

$$L^0(t) = \sum_{k=1}^K \int_0^t V^{\delta_k,*}(\tau)dM^{\delta_k,Q}(\tau) + \int_0^t \nu^0(\tau)dM^Q(x,\tau) + \int_0^t \kappa^0(\tau)d\mathcal{S}^{Q,M}(x,\tau,T).$$

□

7.5 Mean-variance indifference pricing

The mean-variance indifference pricing principles of Schweizer (2001b) have been applied for the handling of the combined risk inherent in life insurance contracts in Møller (2001b, 2002, 2003a, 2003b) and Dahl and Møller (2005) (see Chapter 4). In this section, we present a review of mean-variance indifference pricing (almost) identical to the one in Chapter 4 and derive indifference prices and optimal hedging strategies for a portfolio of unit-linked pure endowments.

7.5.1 A review of mean-variance indifference pricing

Denote by K^* the discounted wealth of the insurer at time T and consider the mean-variance utility functions

$$u_i(K^*) = E^P[K^*] - a_i(\mathrm{Var}^P[K^*])^{\beta_i}, \tag{7.5.1}$$

$i = 1, 2$, where $a_i > 0$ are so-called risk-loading parameters and where we take $\beta_1 = 1$ and $\beta_2 = 1/2$. It can be shown that the equations $u_i(K^*) = u_i(0)$ indeed lead to the classical actuarial variance (i=1) and standard deviation principle (i=2), respectively, see e.g. Møller (2001b).

Schweizer (2001b) proposes to apply the mean-variance utility functions in (7.5.1) in an indifference argument, which takes the possibility for trading in the financial market into consideration. Denote by Θ the space of admissible strategies and let $G(T,\Theta)$ be the space of discounted trading gains, i.e. random variables of the form $\int_0^T \vartheta(u)dX^*(u)$, where X^* is the discounted price process associated with the risky assets. Denote by c the insurer's initial capital at time 0. The u_i-indifference price v_i associated with the liability H is defined via

$$\sup_{\vartheta\in\Theta} u_i\left(c+v_i+\int_0^T \vartheta(u)dX^*(u)-H^*\right)=\sup_{\widetilde{\vartheta}\in\Theta} u_i\left(c+\int_0^T \widetilde{\vartheta}(u)dX^*(u)\right), \tag{7.5.2}$$

where H^* is the discounted liability. The strategy ϑ^* which maximizes the left side of (7.5.2) will be called the optimal strategy for H. In order to formulate the main result, some more notation is needed. We denote by $\widetilde{P}$ the variance optimal martingale measure and let $\widetilde{\Lambda}(T)=\frac{d\widetilde{P}}{dP}|_{\mathcal{F}_T}$. In addition, we let $\pi(\cdot)$ be the projection in $L^2(P)$ on the space $G(T,\Theta)^\perp$ and write $1-\pi(1)=\int_0^T \widetilde{\beta}(u)dX^*(u)$. It follows via the projection theorem that any discounted liability H^* allows for a unique decomposition on the form

$$H^*=c^H+\int_0^T \vartheta^H(u)dX^*(u)+N^H, \tag{7.5.3}$$

where $\int_0^T \vartheta^H dX^*$ is an element of $G(T,\Theta)$, and N^H lies in the space $(\mathbb{R}+G(T,\Theta))^\perp$.

From Schweizer (2001b) and Møller (2001b) we have that the indifference prices for H are:

$$v_1(H)=E^{\widetilde{P}}[H^*]+a_1\mathrm{Var}^P[N^H], \tag{7.5.4}$$

$$v_2(H)=E^{\widetilde{P}}[H^*]+a_2\sqrt{1-\mathrm{Var}^P[\widetilde{\Lambda}(T)]/a_2^2}\sqrt{\mathrm{Var}^P[N^H]}, \tag{7.5.5}$$

where (7.5.5) only is defined if $a_2^2\geq \mathrm{Var}^P[\widetilde{\Lambda}(T)]$. The optimal strategies associated with these two principles are:

$$\vartheta_1^*(t)=\vartheta^H(t)+\frac{1+\mathrm{Var}^P[\widetilde{\Lambda}(T)]}{2a_1}\widetilde{\beta}(t), \tag{7.5.6}$$

$$\vartheta_2^*(t)=\vartheta^H(t)+\frac{1+\mathrm{Var}^P[\widetilde{\Lambda}(T)]}{a_2\sqrt{1-\mathrm{Var}^P[\widetilde{\Lambda}(T)]/a_2^2}}\sqrt{\mathrm{Var}^P[N^H]}\widetilde{\beta}(t), \tag{7.5.7}$$

where (7.5.7) only is well-defined if $a_2^2>\mathrm{Var}^P[\widetilde{\Lambda}(T)]$. For more details, see Møller (2001b, 2003a, 2003b).

7.5.2 The variance optimal martingale measure

In order to determine the variance optimal martingale measure $\widetilde{P}$ we first turn our attention to the minimal martingale measure, $\widehat{P}$, which loosely speaking is "the equivalent martingale measure which disturbs the structure of the model as little as possible", see Schweizer (1995). It is easily seen that the minimal martingale measure is obtained by letting $h^\mu = 0$, $g = 0$ and $o_i = 0$ for all i. Hence, the likelihood process for the change of measure to the minimal martingale measure is given by

$$\widehat{\Lambda}(t) = \exp\left(\int_0^t h^f(u)dW^f(u) + \int_0^t h^s(u)dW^s(u) - \frac{1}{2}\int_0^t ((h^f(u))^2 + (h^s(u))^2)du\right).$$

Since $h^f(u)$ and $h^s(u)$ are assumed to be $\mathcal{G}^x(u)$-measurable, the density $\widehat{\Lambda}(t)$ is $\mathcal{G}^x(t)$-measurable, and therefore it can be represented by a constant C and a stochastic integral with respect to X^*, see e.g. Pham et al. (1998, Section 4.3). Thus, we have the following representation of $\widehat{\Lambda}(t)$

$$\widehat{\Lambda}(t) = C + \int_0^t \widetilde{\zeta}(u)dX^*(u). \tag{7.5.8}$$

Schweizer (1996, Lemma 1) now gives that $\widehat{\Lambda}(\widehat{T})$ is the density for the variance optimal martingale measure as well, such that $\widehat{P} = \widetilde{P}$ and $\widetilde{\Lambda}(T) = \widehat{\Lambda}(T)$. For later use, we introduce the $\widetilde{P}$-martingale $\widetilde{\Lambda}$ by

$$\bar{\Lambda}(t) = E^{\widetilde{P}}\left[\widehat{\Lambda}(\widehat{T})\middle|\mathcal{F}(t)\right] = E^{\widetilde{P}}\left[\widehat{\Lambda}(\widehat{T})\middle|\mathcal{G}^x(t)\right].$$

Hence, even though the variance optimal martingale measure $\widetilde{P}$ and the minimal martingale measure $\widehat{P}$ in general differ, they coincide in our model.

7.5.3 Mean-variance indifference pricing for pure endowments

Let $\overline{T} = T$ and consider a portfolio of n individuals of the same age x each purchasing a unit-linked pure endowment of $\Delta A_0(T)$ paid by a single premium at time 0. Thus, the discounted liability of the company is given by

$$H^* = (n - N(x,T))B(T)^{-1}\Delta A_0(T).$$

Explicit expressions for the mean-variance indifference prices can be obtained under additional integrability conditions. More precisely, we need that certain local $\widetilde{P}$-martingales considered in the calculation of $\mathrm{Var}^P[N^H]$ are (true) $\widetilde{P}$-martingales. In this case we have the following proposition.

Proposition 7.5.1
The indifference prices are given by inserting the following expressions for $E^{\widetilde{P}}[H^*]$ and $\mathrm{Var}^P(N^H)$ in (7.5.4) and (7.5.5):

$$E^{\widetilde{P}}[H^*] = n\mathcal{S}(x,0,T)F^{0,\Delta}(0,T), \tag{7.5.9}$$

and

$$\begin{aligned}\mathrm{Var}^P[N^H] = {} & n\int_0^T \Upsilon_1(t)\Upsilon_2(t)dt + n^2\int_0^T \Upsilon_1(t)\Upsilon_3(t)dt \\ & + n\sum_{i=1}^m \Upsilon_4(T_i) + n^2\sum_{i=1}^m \Upsilon_5(T_i),\end{aligned} \tag{7.5.10}$$

where

$$\Upsilon_1(t) = E^P\left[\frac{\widehat{\Lambda}(t)}{\bar{\Lambda}(t)}(B(t)^{-1}F^{0,\Delta}(t,T))^2\right],$$

$$\Upsilon_2(t) = E^P\Big[(\mathcal{S}(x,t,T))^2\, e^{-\int_0^t \mu(x,u)du}\mu(x,t)$$
$$\times\Big(1+(\sigma^\mu(x,t)B^\mu(x,t,T))^2\Big(1-e^{-\int_0^t \mu(x,u)du}\Big)\Big)\Big],$$

$$\Upsilon_3(t) = E^P\Big[\mu(x,t)\left(\sigma^\mu(x,t)B^\mu(x,t,T)\mathcal{S}^M(x,t,T)\right)^2\Big],$$

$$\Upsilon_4(T_i) = E^P\left[\frac{\widehat{\Lambda}(T_i)}{\bar{\Lambda}(T_i)}\int_0^{T_i} e^{-\int_0^\tau \mu(x,u)du}\,(\mathcal{S}(x,\tau,T))^2\Bigg(\left(B(\tau)^{-1}\Delta_i F^\Delta(\tau,T)\right)^2\right.$$
$$\times\,\mu(x,\tau)\Big(1+\Big(1-e^{-\int_0^\tau \mu(x,\tau)}\Big)(\sigma^\mu(x,\tau)B^\mu(x,\tau,T))^2\Big)$$
$$+\Big(1-e^{-\int_0^\tau \mu(x,\tau)}\Big)\Bigg(2\Delta_i\vartheta^{\Delta,z}(\tau)\Delta_i\vartheta^{\Delta,s}(\tau)\sigma^z(\tau)\sigma^s(\tau)Z^*(\tau)S^*(\tau)$$
$$\left.+\left((\sigma^s(\tau))^2+(\beta^s(\tau))^2\right)\left(\Delta_i\vartheta^{\Delta,s}(\tau)S^*(\tau)\right)^2+\left(\Delta_i\vartheta^{\Delta,z}(\tau)\sigma^z(\tau)Z^*(\tau)\right)^2\Bigg)\Bigg)d\tau\right],$$

$$\Upsilon_5(T_i) = E^P\left[\frac{\widehat{\Lambda}(T_i)}{\bar{\Lambda}(T_i)}\Bigg(\left(\mathcal{S}(x,0,T)\Delta_i F^\Delta(0,T)\right)^2+\int_0^{T_i}\left(\mathcal{S}^M(x,\tau,T)\right)^2\right.$$
$$\times\Bigg(\left(\Delta_i\vartheta^{\Delta,z}(\tau)\sigma^z(\tau)Z^*(\tau)\right)^2+\left((\sigma^s(\tau))^2+(\beta^s(\tau))^2\right)\left(\Delta_i\vartheta^{\Delta,s}(\tau)S^*(\tau)\right)^2$$
$$+\,2\Delta_i\vartheta^{\Delta,z}(\tau)\Delta_i\vartheta^{\Delta,s}(\tau)\sigma^z(\tau)\sigma^s(\tau)Z^*(\tau)S^*(\tau)$$
$$\left.+\,\mu(x,\tau)\left(B(\tau)^{-1}\Delta_i F^\Delta(\tau,T)\sigma^\mu(x,\tau)B^\mu(x,\tau,T)\right)^2\Bigg)d\tau\Bigg)\right].$$

Idea of proof of Proposition 7.5.1: The $\widetilde{P}$-independence between the financial market and the insurance elements immediately gives (7.5.9). The expression for the variance of N^H in (7.5.10) follows from calculations similar to those in Møller (2001b) and Chapter 4. For completeness the calculations are carried out in Section 7.6.3 under certain additional integrability conditions.

□

The first two terms in (7.5.10) are essentially the same as those obtained in Proposition 4.8.1. The first term, which is proportional to the number of insured, stems from both the systematic and unsystematic mortality risk, whereas the second term, which is proportional to the squared number of survivors, stems solely from the systematic mortality risk. The last two terms are related to the reinvestment risk. Each of these terms involve a sum measuring adjustments of the intrinsic value process upon the realization of the Y_i's. The reinvestment risk at time T_i is the difference between the intrinsic value process before and after the observation of Y_i, as measured by $\Delta M^{\widetilde{P},y}(T_i)$. Investigating the terms in $M^{\widetilde{P},y}$ we find that the difference in the initial investments, the difference in trading gains/losses due to different trading strategies and the difference in gains/losses generated by changes in the survival probability all contribute to the term proportional to n^2. The last two sources along with the difference in gains/losses generated by the development of the compensated counting process $M(x)$ give rise to the terms proportional to n. The reason for collecting the terms with respect to n and n^2, respectively, is that it enables us to distinguish the importance of the terms as the size of the portfolio of insured increases.

Concerning the optimal strategies we have the following proposition.

Proposition 7.5.2
The optimal strategies are given by inserting (7.5.10) and the following expression for ϑ^H *in (7.5.6) and (7.5.7):*

$$\vartheta^H(t) = \vartheta^0(t) - \widetilde{\zeta}(t) \int_0^{t-} \frac{1}{\bar{\Lambda}(u)} \left(dM^{\widetilde{P},y}(u) + \nu^0(u) dM(x,u) + \kappa^0(u) d\mathcal{S}^M(x,u,T) \right), \tag{7.5.11}$$

where

$$\nu^0(\tau) = -B(\tau)^{-1} \mathcal{S}(x,\tau,T) F^{0,\Delta}(\tau,T), \tag{7.5.12}$$

$$\kappa^0(\tau) = (n - N(x,\tau)) e^{\int_0^\tau \mu(x,u)du} B(\tau)^{-1} F^{0,\Delta}(\tau,T), \tag{7.5.13}$$

and $M^{\widetilde{P},y}$ *is given by*

$$\begin{aligned} M^{\widetilde{P},y}(t) = \sum_{i=1}^{i_t} \Bigg(& n\mathcal{S}(x,0,T)\Delta_i F^\Delta(0,T) + \int_0^{T_i} (n - N(x,\tau-))\mathcal{S}(x,\tau,T)\Delta_i \vartheta^\Delta(\tau) dX^*(\tau) \\ & - \int_0^{T_i} B(\tau)^{-1} \mathcal{S}(x,\tau,T) \Delta_i F^\Delta(\tau,T) dM(x,\tau) \\ & + \int_0^{T_i} (n - N(x,\tau)) e^{\int_0^\tau \mu(x,u)du} B(\tau)^{-1} \Delta_i F^\Delta(\tau,T) d\mathcal{S}^M(x,\tau,T) \Bigg). \end{aligned} \tag{7.5.14}$$

Proof of Proposition 7.5.2: Expression (7.5.11) follows from Schweizer (2001a, Theorem 4.6) (Theorem 7.6.1), which relates the decomposition in (7.5.3) to the Galtchouk–Kunita–Watanabe decomposition of the $\widetilde{P}$-martingale $V^{0,*}(t) = E^{\widetilde{P}}[H^*|\mathcal{F}(t)]$ given in Example 7.4.6.

□

7.6 Proofs and technical calculations

7.6.1 Proof of Lemma 7.4.2

Recall from (7.3.16) and (7.3.17) that the Q-martingale $V^{T_m,*}$ can be written as

$$V^{T_m,*}(t) = A^*(t) + (n - N(x,t))B(t)^{-1} \widehat{V}^{T_m}(t, X(t), \mu(x,t)). \tag{7.6.1}$$

Here, we explicitly denote the dependence of $\widehat{V}^{T_m}$ on X and μ. Differentiating under the integral gives

$$\frac{\partial}{\partial x}\widehat{V}^{T_m}(t,x,\mu) = \int_t^T \mathcal{S}^Q(x,t,\tau)\left(\frac{\partial}{\partial x}F^{T_m,0}(t,\tau) + f^{\mu,Q}(x,t,\tau)\frac{\partial}{\partial x}F^{T_m,1}(t,\tau)\right)d\tau \\ + \mathcal{S}^Q(x,t,\overline{T})\frac{\partial}{\partial x}F^{T_m,\Delta}(t,\overline{T}), \tag{7.6.2}$$

and

$$\frac{\partial}{\partial \mu}\widehat{V}^{T_m}(t,x,\mu) = -(1+g(x,t))\Bigg(\int_t^T B^{\mu,Q}(x,t,\tau)\mathcal{S}^Q(x,t,\tau) \\ \times\left(F^{T_m,0}(t,\tau) + F^{T_m,1}(t,\tau)\left(f^{\mu,Q}(x,t,\tau) - \frac{\frac{\partial}{\partial\tau}B^{\mu,Q}(x,t,\tau)}{B^{\mu,Q}(x,t,\tau)}\right)\right)d\tau \\ + B^{\mu,Q}(x,t,\overline{T})\mathcal{S}^Q(x,t,\overline{T})F^{T_m,\Delta}(t,\overline{T})\Bigg), \tag{7.6.3}$$

where we have used

$$\frac{\partial}{\partial\mu}f^{\mu,Q}(x,t,\tau) = (1+g(x,t))\frac{\partial}{\partial\tau}B^{\mu,Q}(x,t,\tau).$$

Using integration by parts on $(n-N(x,t))B(t)^{-1}\widehat{V}^{T_m}(t,X(t),\mu(x,t))$ allows us to write (7.6.1) as

$$\begin{aligned} V^{T_m,*}(t) = {} & A^*(t) + n\widehat{V}^{T_m}(0,X(0),\mu(x,0)) \\ & + \int_0^t (n-N(x,u))\widehat{V}^{T_m}(u,X(u),\mu(x,u))dB(u)^{-1} \\ & + \int_0^t B(u)^{-1}(n-N(x,u-))d\widehat{V}^{T_m}(u,X(u),\mu(x,u)) \\ & - \int_0^t B(u)^{-1}\widehat{V}^{T_m}(u,X(u),\mu(x,u))dN(x,u). \end{aligned} \tag{7.6.4}$$

In order to calculate the fourth term in (7.6.4), we need to find $d\widehat{V}^{T_m}(u,X(u),\mu(x,u))$. Recall from (7.3.9) that the dynamics of $\mu(x,t)$ under Q are given by

$$d\mu(x,t) = \alpha^{\mu,Q}(t,\mu(x,t))dt + \sigma^{\mu}(t,\mu(x,t))\sqrt{\mu(x,t)}dW^{\mu,Q}(t),$$

where

$$\alpha^{\mu,Q}(t,\mu(x,t)) = \gamma^{\mu,Q}(x,t) - \delta^{\mu,Q}(x,t)\mu(x,t).$$

In the rest of the proof we return to the shorthand notation $\widehat{V}^{T_m}(u)$. Furthermore in the coefficient functions we explicitly include the time argument only. The assumption

$\widehat{V}^{T_m}(t) \in C^{1,2,2}$ allows us to apply Itô's formula to obtain

$$\begin{aligned}
d\widehat{V}^{T_m}(u) = &\Big(\frac{\partial}{\partial u}\widehat{V}^{T_m}(u) + \alpha^{\mu,Q}(u)\frac{\partial}{\partial \mu}\widehat{V}^{T_m}(u) + r(u)S(u)\frac{\partial}{\partial s}\widehat{V}^{T_m}(u) + r(u)Z(u)\frac{\partial}{\partial z}\widehat{V}^{T_m}(u) \\
&+ \frac{1}{2}(\sigma^{\mu}(u))^2\mu(x,u)\frac{\partial^2}{\partial \mu^2}\widehat{V}^{T_m}(u) + \frac{1}{2}\left((\sigma^s(u))^2 + (\beta^s(u))^2\right)(S(u))^2\frac{\partial^2}{\partial s^2}\widehat{V}^{T_m}(u) \\
&+\frac{1}{2}(\sigma^z(u)Z(u))^2\frac{\partial^2}{\partial z^2}\widehat{V}^{T_m}(u) + \sigma^s(u)\sigma^z(u)S(u)Z(u)\frac{\partial^2}{\partial z\partial s}\widehat{V}^{T_m}(u)\Big)\,du \\
&+ \sigma^{\mu}(u)\sqrt{\mu(x,u)}\frac{\partial}{\partial \mu}\widehat{V}^{T_m}(u)dW^{\mu,Q}(u) - \sigma^z(u)Z(u)\frac{\partial}{\partial z}\widehat{V}^{T_m}(u)dW^{f,Q}(u) \\
&- \sigma^s(u)S(u)\frac{\partial}{\partial s}\widehat{V}^{T_m}(u)dW^{f,Q}(u) + \beta^s(u)S(u)\frac{\partial}{\partial s}\widehat{V}^{T_m}(u)dW^{s,Q}(u) \\
= &\Big(\frac{\partial}{\partial u}\widehat{V}^{T_m}(u) + \alpha^{\mu,Q}(u)\frac{\partial}{\partial \mu}\widehat{V}^{T_m}(u) + r(u)S(u)\frac{\partial}{\partial s}\widehat{V}^{T_m}(u) + r(u)Z(u)\frac{\partial}{\partial z}\widehat{V}^{T_m}(u) \\
&+ \frac{1}{2}(\sigma^{\mu}(u))^2\mu(x,u)\frac{\partial^2}{\partial \mu^2}\widehat{V}^{T_m}(u) + \frac{1}{2}\left((\sigma^s(u))^2 + (\beta^s(u))^2\right)(S(u))^2\frac{\partial^2}{\partial s^2}\widehat{V}^{T_m}(u) \\
&+\frac{1}{2}(\sigma^z(u)Z(u))^2\frac{\partial^2}{\partial z^2}\widehat{V}^{T_m}(u) + \sigma^s(u)\sigma^z(u)S(u)Z(u)\frac{\partial^2}{\partial z\partial s}\widehat{V}^{T_m}(u)\Big)\,du \\
&+ B(u)\frac{\partial}{\partial z}\widehat{V}^{T_m}(u)dZ^*(u) + B(u)\frac{\partial}{\partial s}\widehat{V}^{T_m}(u)dS^*(u) \\
&- \frac{\frac{\partial}{\partial \mu}\widehat{V}^{T_m}(u)}{(1+g(x,u))B^{\mu,Q}(x,u,T)\mathcal{S}^{Q,M}(x,u,T)}d\mathcal{S}^{Q,M}(x,u,T).
\end{aligned}$$

In the first equality we have used the dynamics of Z, S and μ, whereas we in the second use (7.6.2) and (7.6.3) together with the dynamics of Z^*, S^* and $\mathcal{S}^{Q,M}(x,\cdot,T)$. Rewriting A^* in terms of the Q-martingale $M^Q(x)$ we get

$$\begin{aligned}
A^*(t) = &-n\pi(0) + \int_0^t B(\tau)^{-1}\left(a_0(\tau)(n-N(x,\tau)) + a_1(\tau)(n-N(x,\tau-))\mu^Q(x,\tau)\right)d\tau \\
&+ \int_0^t B(\overline{T})^{-1}(n-N(x,\overline{T}))\Delta A_0(\overline{T})d1_{(\tau\geq\overline{T})} + \int_0^t B(\tau)^{-1}a_1(\tau)dM^Q(x,\tau).
\end{aligned}$$

Now collect the terms from (7.6.4) involving integrals with respect to X^*, $\mathcal{S}^{Q,M}(x,\cdot,T)$ and $M^Q(x)$, respectively. Since these three terms and $V^{T_m,*}$ are Q-martingales, the remaining terms constitute a Q-martingale as well. Since this process is continuous (hence predictable) and of finite variation, it is constant. Inserting $t=0$ we immediately get that $V^{T_m,*}(0) = -n\pi(0) + n\widehat{V}^{T_m}(0, X(0), \mu(x,0))$. Thus, we have proved the decomposition in (7.4.6).

7.6.2 Proof of Proposition 7.4.5

The expression for $V^{*,0}(0)$ follows immediately from (7.3.16) and (7.3.17). Now, use that $V^{T_m,*}(T) = V^{0,*}(T)$ and the Galtchouk–Kunita–Watanabe decomposition of $V^{T_m,*}$ in

Lemma 7.4.2 to get

$$
\begin{aligned}
L^0(T) &= V^{T_m,*}(0) + \int_0^T \vartheta^{T_m}(\tau)dX^*(\tau) + \int_0^T \nu^{T_m}(\tau)dM^Q(x,\tau) \\
&\quad + \int_0^T \kappa^{T_m}(\tau)d\mathcal{S}^{Q,M}(x,\tau,T) - \left(V^{0,*}(0) + \int_0^T \vartheta^0(\tau)dX^*(\tau)\right).
\end{aligned}
$$

Since L^0 is an $(\mathbb{F}, Q)$-martingale we get

$$
\begin{aligned}
L^0(t) &= E^Q\Bigg[V^{T_m,*}(0) - V^{0,*}(0) + \int_0^T (\vartheta^{T_m}(\tau) - \vartheta^0(\tau))dX^*(\tau) + \int_0^T \nu^{T_m}(\tau)dM^Q(x,\tau) \\
&\quad + \int_0^T \kappa^{T_m}(\tau)d\mathcal{S}^{Q,M}(x,\tau,T)\Bigg|\,\mathcal{F}(t)\Bigg] \\
&= \sum_{i=1}^m E^Q\Bigg[V^{T_i,*}(0) - V^{T_{i-1},*}(0) + \int_0^T (\vartheta^{T_i}(\tau) - \vartheta^{T_{i-1}}(\tau))dX^*(\tau) \\
&\quad + \int_0^T \left(\nu^{T_i}(\tau) - \nu^{T_{i-1}}(\tau)\right)dM^Q(x,\tau) + \int_0^T \left(\kappa^{T_i}(\tau) - \kappa^{T_{i-1}}(\tau)\right)d\mathcal{S}^{Q,M}(x,\tau,T) \\
&\quad + \int_0^T \nu^0(\tau)dM^Q(x,\tau) + \int_0^T \kappa^0(\tau)d\mathcal{S}^{Q,M}(x,\tau,T)\Bigg|\,\mathcal{F}(t)\Bigg] \\
&= \sum_{i=1}^m \Bigg(V^{T_i\wedge T_{i_t},*}(0) - V^{T_{i-1}\wedge T_{i_t},*}(0) + \int_0^t (\vartheta^{T_i\wedge T_{i_t}}(\tau) - \vartheta^{T_{i-1}\wedge T_{i_t}}(\tau))dX^*(\tau) \\
&\quad + \int_0^t \left(\nu^{T_i\wedge T_{i_t}}(\tau) - \nu^{T_{i-1}\wedge T_{i_t}}(\tau)\right)dM^Q(x,\tau) \\
&\quad + \int_0^t \left(\kappa^{T_i\wedge T_{i_t}}(\tau) - \kappa^{T_{i-1}\wedge T_{i_t}}(\tau)\right)d\mathcal{S}^{Q,M}(x,\tau,T)\Bigg) \\
&\quad + \int_0^t \nu^0(\tau)dM^Q(x,\tau) + \int_0^t \kappa^0(\tau)d\mathcal{S}^{Q,M}(x,\tau,T).
\end{aligned}
$$

Here, the second equality follows by writing the differences as telescoping sums using the quantities defined in (7.4.12)–(7.4.14). In the third equality we use the assumption that ϑ^{T_i}, ν^{T_i} and κ^{T_i} are sufficiently integrable for all $i \in \{0, \ldots, m\}$ to ensure that all the considered local Q-martingales are Q-martingales. Furthermore, we use iterated expectations together with the structure of the filtrations $\mathbb{F}^{T_i}$. The result now follows by observing that for $i > i_t$ all terms in the sum are zero and that $\mathcal{F}^{T_i}(\tau) = \mathcal{F}^{T_{i-1}}(\tau)$ for $\tau \geq T_i$.

7.6.3 Calculation of $\mathbf{Var}^P[N^H]$

The following theorem due to Schweizer (2001a, Theorem 4.6) relates the decomposition in (7.5.3) to the Galtchouk-Kunita-Watanabe decomposition of the $\widetilde{P}$-martingale $V^{0,*}(t) = E^{\widetilde{P}}[H^*|\mathcal{F}(t)]$; see also Møller (2000).

Theorem 7.6.1
Assume that $H^* \in L^2(\mathcal{F}(T), P)$ and consider the Galtchouk–Kunita–Watanabe decomposition of $V^{0,*}$ given by

$$V^{0,*}(t) = E^{\widetilde{P}}[H^*] + \int_0^t \vartheta^0(u)dX^*(u) + L^0(t), \quad 0 \leq t \leq T. \tag{7.6.5}$$

Then c^H, ϑ^H and N^H from (7.5.3) are given in terms of decomposition (7.6.5) by

$$\begin{aligned} c^H &= E^{\widetilde{P}}[H^*], \\ \vartheta^H(t) &= \vartheta^0(t) - \widetilde{\zeta}(t) \int_0^{t-} \frac{1}{\bar{\Lambda}(u)} dL^0(u), \\ N^H &= \bar{\Lambda}(T) \int_0^T \frac{1}{\bar{\Lambda}(u)} dL^0(u). \end{aligned}$$

Here, we have

$$L^0(t) = \int_0^t dM^{\widetilde{P},y}(u) + \int_0^t \nu^0(u)dM(x,u) + \int_0^t \kappa^0(u)d\mathcal{S}^M(x,u,T),$$

where ν^0, κ^0 and $M^{\widetilde{P},y}$ are given by (7.5.12)–(7.5.14). Thus, Theorem 7.6.1 gives the following expression for N^H

$$N^H = \bar{\Lambda}(T) \int_0^T \frac{1}{\bar{\Lambda}(t)} \left(dM^{\widetilde{P},y}(t) + \nu^0(t)dM(x,t) + \kappa^0(t)d\mathcal{S}^M(x,t,T) \right).$$

Since $E^P[N^H] = 0$, we first note that

$$\begin{aligned} \mathrm{Var}^P[N^H] &= E^P\left[(N^H)^2\right] = E^{\widetilde{P}}\left[\bar{\Lambda}(T)\left(\widetilde{M}^y(T) + \widetilde{L}(T) + \widetilde{R}(T)\right)^2\right] \\ &= E^{\widetilde{P}}\Big[\bar{\Lambda}(T)(\widetilde{M}^y(T))^2 + \bar{\Lambda}(T)(\widetilde{L}(T))^2 + \bar{\Lambda}(T)(\widetilde{R}(T))^2 + 2\bar{\Lambda}(T)\widetilde{M}^y(T)\widetilde{R}(T) \\ &\quad + 2\bar{\Lambda}(T)\widetilde{L}(T)\widetilde{R}(T) + 2\bar{\Lambda}(T)\widetilde{M}^y(T)\widetilde{R}(T)\Big], \end{aligned} \tag{7.6.6}$$

where we have defined $\widetilde{M}^y(t) = \int_0^t \frac{1}{\bar{\Lambda}(u)} dM^{\widetilde{P},y}(u)$, $\widetilde{L}(t) = \int_0^t \frac{\nu^0(u)}{\bar{\Lambda}(u)} dM(x,u)$ and $\widetilde{R}(t) = \int_0^t \frac{\kappa^0(u)}{\bar{\Lambda}(u)} d\mathcal{S}^M(x,u,T)$. The six terms appearing in (7.6.6) can be rewritten using Itô's formula, see Jacod and Shiryaev (2003) for a version that applies in this setting. For the first term we get

$$\begin{aligned} \bar{\Lambda}(T)(\widetilde{M}^y(T))^2 &= \int_0^T (\widetilde{M}^y(t-))^2 d\bar{\Lambda}(t) + \sum_{i=1}^m \bar{\Lambda}(T_i)\left(\left(\frac{M^{\widetilde{P},y}(T_i)}{\bar{\Lambda}(T_i)}\right)^2 - \left(\frac{M^{\widetilde{P},y}(T_{i-1})}{\bar{\Lambda}(T_i)}\right)^2\right) \\ &= \int_0^T (\widetilde{M}^y(t-))^2 d\bar{\Lambda}(t) + 2\sum_{i=1}^m \bar{\Lambda}(T_i)\widetilde{M}^y(T_{i-1})\Delta\widetilde{M}^y(T_i) \\ &\quad + \sum_{i=1}^m \bar{\Lambda}(T_i)\left(\frac{\Delta M^{\widetilde{P},y}(T_i)}{\bar{\Lambda}(T_i)}\right)^2, \end{aligned}$$

where the second equality follows from rearranging the terms. For the second term similar rearrangements gives

$$\bar{\Lambda}(T)(\widetilde{L}(T))^2 = \int_0^T (\widetilde{L}(t-))^2 d\bar{\Lambda}(t) + 2\int_0^T \bar{\Lambda}(t)\widetilde{L}(t-)d\widetilde{L}(t) + \int_0^T \bar{\Lambda}(t)\left(\frac{\nu^0(t)}{\bar{\Lambda}(t)}\right)^2 dN(x,t),$$

whereas we for the last term find that

$$\begin{aligned}
\bar{\Lambda}(T)(\widetilde{R}(T))^2 &= \int_0^T \widetilde{R}(t)^2 d\bar{\Lambda}(t) + 2\int_0^T \bar{\Lambda}(t)\widetilde{R}(t)d\widetilde{R}(t) + \int_0^T \bar{\Lambda}(t)d\langle\widetilde{R}\rangle(t)\\
&= \int_0^T \widetilde{R}(t)^2 d\bar{\Lambda}(t) + 2\int_0^T \bar{\Lambda}(t)\widetilde{R}(t)d\widetilde{R}(t)\\
&\quad + \int_0^T \bar{\Lambda}(t)\left(\frac{\kappa^0(t)}{\bar{\Lambda}(t)}\sigma^\mu(x,t)\sqrt{\mu(x,t)}B^\mu(x,t,T)\mathcal{S}^M(x,t,T)\right)^2 dt.
\end{aligned}$$

Assuming all the local martingales are martingales, and using that the Wiener processes are independent, we get

$$\begin{aligned}
\mathrm{Var}^P\left[N^H\right] &= E^{\widetilde{P}}\left[\sum_{i=1}^m \frac{(M^{\widetilde{P},y}(T_i))^2}{\bar{\Lambda}(T_i)}\right] + E^{\widetilde{P}}\left[\int_0^T \frac{(\nu^0(t))^2}{\bar{\Lambda}(t)}dN(x,t)\right]\\
&\quad + E^{\widetilde{P}}\left[\int_0^T \frac{\left(\kappa^0(t)\sigma^\mu(x,t)\sqrt{\mu(x,t)}B^\mu(x,t,T)\mathcal{S}^M(x,t,T)\right)^2}{\bar{\Lambda}(t)}dt\right]. \qquad (7.6.7)
\end{aligned}$$

Note that given $\mathcal{I}(T)$ the number of survivors at time t, $n-N(x,t)$, follows a binomial distribution under P (and $\widetilde{P}$) with parameters $(n, e^{-\int_0^t \mu(x,u)du})$. Hence, calculations similar to those in Chapter 4 give

$$\begin{aligned}
&E^{\widetilde{P}}\left[\int_0^T \frac{(\nu^0(t))^2}{\bar{\Lambda}(t)}dN(x,t)\right]\\
&= n\int_0^T E^P\left[\frac{\widehat{\Lambda}(t)}{\bar{\Lambda}(t)}(B(t)^{-1}F^{0,\Delta}(t,T))^2\right] E^P\left[(\mathcal{S}(x,t,T))^2 e^{-\int_0^t \mu(x,u)du}\mu(x,t)\right] dt
\end{aligned}$$

and

$$\begin{aligned}
&E^{\widetilde{P}}\left[\int_0^T \frac{\left(\kappa^0(t)\sigma^\mu(x,t)\sqrt{\mu(x,t)}B^\mu(x,t,T)\mathcal{S}^M(x,t,T)\right)^2}{\bar{\Lambda}(t)}dt\right]\\
&= n\int_0^T E^P\left[\frac{\widehat{\Lambda}(t)}{\bar{\Lambda}(t)}(B(t)^{-1}F^{0,\Delta}(t,T))^2\right]\\
&\quad \times E^P\left[(\mathcal{S}(x,t,T))^2 e^{-\int_0^t \mu(x,u)du}\mu(x,t)\left(\sigma^\mu(x,t)B^\mu(x,t,T)\right)^2\left(1-e^{-\int_0^t \mu(x,u)du}\right)\right] dt\\
&\quad + n^2\int_0^T E^P\left[\frac{\widehat{\Lambda}(t)}{\bar{\Lambda}(t)}(B(t)^{-1}F^{0,\Delta}(t,T))^2\right] E^P\left[\mu(x,t)\left(\sigma^\mu(x,t)B^\mu(x,t,T)\mathcal{S}^M(x,t,T)\right)^2\right] dt.
\end{aligned}$$

The first term, which relates to the reinvestment risk is new, so we investigate this term in more detail. First we introduce the notation

$$M_1^y(T_i) = n\mathcal{S}(x,0,T)\Delta_i F^{\Delta}(0,T),$$
$$M_2^y(T_i) = \int_0^{T_i} (n - N(x,\tau-))\mathcal{S}(x,\tau,T)\Delta_i \vartheta^{\Delta}(\tau)dX^*(\tau),$$
$$M_3^y(T_i) = -\int_0^{T_i} B(\tau)^{-1}\mathcal{S}(x,\tau,T)\Delta_i F^{\Delta}(\tau,T)dM(x,\tau),$$
$$M_4^y(T_i) = \int_0^{T_i} (n - N(x,\tau))e^{\int_0^\tau \mu(x,u)du}B(\tau)^{-1}\Delta_i F^{\Delta}(\tau,T)d\mathcal{S}^M(x,\tau,T).$$

Thus, the first term in (7.6.7) can be written as

$$\begin{aligned}
&E^{\widetilde{P}}\left[\sum_{i=1}^m \frac{(M^{\widetilde{P},y}(T_i))^2}{\bar{\Lambda}(T_i)}\right] \\
&= \sum_{i=1}^m E^P\left[\frac{\widehat{\Lambda}(T_i)}{\bar{\Lambda}(T_i)}(M^{\widetilde{P},y}(T_i))^2\right] \\
&= \sum_{i=1}^m E^P\left[\frac{\widehat{\Lambda}(T_i)}{\bar{\Lambda}(T_i)}(M_1^{\widetilde{P},y}(T_i) + M_2^{\widetilde{P},y}(T_i) + M_3^{\widetilde{P},y}(T_i) + M_4^{\widetilde{P},y}(T_i))^2\right] \\
&= \sum_{i=1}^m E^P\left[\frac{\widehat{\Lambda}(T_i)}{\bar{\Lambda}(T_i)}\Big((M_1^{\widetilde{P},y}(T_i))^2 + (M_2^{\widetilde{P},y}(T_i))^2 + (M_3^{\widetilde{P},y}(T_i))^2 + (M_4^{\widetilde{P},y}(T_i))^2\Big)\right].
\end{aligned}$$

Here, the third equality follows by assuming that all local martingales are martingales. Hence, we can investigate the four terms separately. For the first term we immediately obtain

$$E^P\left[\frac{\widehat{\Lambda}(T_i)}{\bar{\Lambda}(T_i)}\left(n\mathcal{S}(x,0,T)\Delta_i F^{\Delta}(0,T)\right)^2\right] = n^2(\mathcal{S}(x,0,T))^2 E^P\left[\frac{\widehat{\Lambda}(T_i)}{\bar{\Lambda}(T_i)}\left(\Delta_i F^{\Delta}(0,T)\right)^2\right].$$

For the second term we first use the process $\langle X^*\rangle$, which makes the process $(X^*)^2 - \langle X^*\rangle$ a Q-martingale, to obtain

$$\begin{aligned}
&E^P\left[\frac{\widehat{\Lambda}(T_i)}{\bar{\Lambda}(T_i)}\left(\int_0^{T_i}(n - N(x,\tau-))\mathcal{S}(x,\tau,T)\Delta_i\vartheta^{\Delta}(\tau)dX^*(\tau)\right)^2\right] \\
&= E^P\Bigg[\frac{\widehat{\Lambda}(T_i)}{\bar{\Lambda}(T_i)}\int_0^{T_i}((n - N(x,\tau-))\mathcal{S}(x,\tau,T))^2\Bigg(\left((\sigma^s(\tau))^2 + (\beta^s(\tau))^2\right)\left(\Delta_i\vartheta^{\Delta,s}(\tau)S^*(\tau)\right)^2 \\
&\quad + \left(\Delta_i\vartheta^{\Delta,z}(\tau)\sigma^z(\tau)Z^*(\tau)\right)^2 + 2\Delta_i\vartheta^{\Delta,z}(\tau)\Delta_i\vartheta^{\Delta,s}(\tau)\sigma^z(\tau)\sigma^s(\tau)Z^*(\tau)S^*(\tau)\Bigg)d\tau\Bigg].
\end{aligned}$$

Now the following result follows from the use of iterated expectations

$$
\begin{aligned}
& E^P\left[\frac{\widehat{\Lambda}(T_i)}{\bar{\Lambda}(T_i)}\left(\int_0^{T_i}(n-N(x,\tau-))\mathcal{S}(x,\tau,T)\Delta_i\vartheta^{\Delta}(\tau)dX^*(\tau)\right)^2\right] \\
& = nE^P\left[\frac{\widehat{\Lambda}(T_i)}{\bar{\Lambda}(T_i)}\int_0^{T_i} e^{-\int_0^\tau \mu(x,u)du}\left(1-e^{-\int_0^\tau \mu(x,u)du}\right)(\mathcal{S}(x,\tau,T))^2\right. \\
& \left(\left((\sigma^s(\tau))^2+(\beta^s(\tau))^2\right)\left(\Delta_i\vartheta^{T_i,\Delta,s}(\tau)S^*(\tau)\right)^2+\left(\Delta_i\vartheta^{\Delta,z}(\tau)\sigma^z(\tau)Z^*(\tau)\right)^2\right. \\
& \left.\left.+2\Delta_i\vartheta^{\Delta,z}(\tau)\Delta_i\vartheta^{\Delta,s}(\tau)\sigma^z(\tau)\sigma^s(\tau)Z^*(\tau)S^*(\tau)\right)d\tau\right] \\
& + n^2E^P\left[\frac{\widehat{\Lambda}(T_i)}{\bar{\Lambda}(T_i)}\int_0^{T_i}\left(\mathcal{S}^M(x,\tau,T)\right)^2\left(\left((\sigma^s(\tau))^2+(\beta^s(\tau))^2\right)\left(\Delta_i\vartheta^{\Delta,s}(\tau)S^*(\tau)\right)^2\right.\right. \\
& \left.\left.+\left(\Delta_i\vartheta^{\Delta,z}(\tau)\sigma^z(\tau)Z^*(\tau)\right)^2+2\Delta_i\vartheta^{\Delta,z}(\tau)\Delta_i\vartheta^{\Delta,s}(\tau)\sigma^z(\tau)\sigma^s(\tau)Z^*(\tau)S^*(\tau)\right)d\tau\right].
\end{aligned}
$$

Similar calculations give

$$
\begin{aligned}
& E^P\left[\frac{\widehat{\Lambda}(T_i)}{\bar{\Lambda}(T_i)}\left(\int_0^{T_i}(n-N(x,\tau))e^{\int_0^\tau \mu(x,u)du}B(\tau)^{-1}\Delta_iF^{\Delta}(\tau,T)d\mathcal{S}^M(x,\tau,T)\right)^2\right] \\
& = E^P\left[\frac{\widehat{\Lambda}(T_i)}{\bar{\Lambda}(T_i)}\int_0^{T_i}\left((n-N(x,\tau))e^{\int_0^\tau \mu(x,u)du}B(\tau)^{-1}\Delta_iF^{\Delta}(\tau,T)\right.\right. \\
& \left.\left.\times\sigma^\mu(x,\tau)\sqrt{\mu(x,\tau)}B^\mu(x,\tau,T)\mathcal{S}^M(x,\tau,T)\right)^2 d\tau\right] \\
& = nE^P\left[\frac{\widehat{\Lambda}(T_i)}{\bar{\Lambda}(T_i)}\int_0^{T_i}\mu(x,\tau)e^{-\int_0^\tau \mu(x,u)du}\left(1-e^{-\int_0^\tau \mu(x,u)du}\right)\right. \\
& \left.\times\left(B(\tau)^{-1}\Delta_iF^{\Delta}(\tau,T))\sigma^\mu(x,\tau)B^\mu(x,\tau,T)\mathcal{S}(x,\tau,T)\right)^2 d\tau\right] \\
& + n^2E^P\left[\frac{\widehat{\Lambda}(T_i)}{\bar{\Lambda}(T_i)}\int_0^{T_i}\mu(x,\tau)\left(B(\tau)^{-1}\Delta_iF^{\Delta}(\tau,T)\sigma^\mu(x,\tau)B^\mu(x,\tau,T)\mathcal{S}^M(x,\tau,T)\right)^2 d\tau\right],
\end{aligned}
$$

and

$$
\begin{aligned}
& E^P\left[\frac{\widehat{\Lambda}(T_i)}{\bar{\Lambda}(T_i)}\left(\int_0^{T_i}B(\tau)^{-1}\mathcal{S}(x,\tau,T)\Delta_iF^{\Delta}(\tau,T)dM(x,\tau)\right)^2\right] \\
& = E^P\left[\frac{\widehat{\Lambda}(T_i)}{\bar{\Lambda}(T_i)}\int_0^{T_i}\left(B(\tau)^{-1}\mathcal{S}(x,\tau,T)\Delta_iF^{\Delta}(\tau,T)\right)^2(n-N(x,\tau-))\mu(x,\tau)d\tau\right] \\
& = nE^P\left[\frac{\widehat{\Lambda}(T_i)}{\bar{\Lambda}(T_i)}\int_0^{T_i}\left(B(\tau)^{-1}\mathcal{S}(x,\tau,T)\Delta_iF^{\Delta}(\tau,T)\right)^2 e^{-\int_0^\tau \mu(x,u)du}\mu(x,\tau)d\tau\right].
\end{aligned}
$$

Collecting the terms in the sum with respect to n and n^2, respectively, completes the proof.

Chapter 8

A Numerical Study of Reserves and Risk Measures in Life Insurance

(This chapter is an adapted version of Dahl (2005c))

In this chapter we study different methods for calculating reserves for life insurance contracts with deterministic benefits in a slight simplification of the model in Section 7. Hence, the model considered includes the equity, standard interest rate and reinvestment risks on the financial side and the systematic and unsystematic mortality risks on the insurance side. We consider market reserves calculated by the no arbitrage principle, only. Furthermore, we consider the following alternative approaches to pricing the dependence on the reinvestment risk: Super-replication and the principles of a level long term yield/forward rate curve. Combined with the no arbitrage principle for the remaining risks, these principles give reserves, which can be compared to the market reserves. Moreover, the risk measures of Value at Risk and tail conditional expectation are considered. These different reservation principles and the relationship to the risk measures are compared numerically.

8.1 Introduction

In recent years legislation has forced life insurance companies to value both assets and liabilities at market value. Here, the value of the assets is easily obtained from the financial market. Life insurance contracts, on the other hand, are not traded in the financial market, so determining market values for the liabilities represents a greater problem.

We consider a model including a large number of risks faced by a life insurance company. The model, which is a simplification of the model in Chapter 7 to the case of deterministic coefficient functions, consists of two independent parts: A financial market and an insurance portfolio. In the financial market the company is allowed to invest in a savings account, bonds with a limited time to maturity and a stock. Hence, the following three financial risks are included in the model: Equity risk, standard interest rate risk and reinvestment risk. In the insurance portfolio the mortality intensity is modelled as a stochastic process, so we consider both systematic and unsystematic mortality risk, see Chapter 3 for an explanation of the different types of mortality risk. Within this model we study different reservation principles. As a first approach we apply the no arbitrage principle from financial mathematics in order to obtain market reserves. Since the insurance contracts are not traded in the financial market the market reserve depends on the market's attitude towards reinvestment risk as well as systematic and unsystematic mortality risk. Danish legislation force the life insurance companies to disregard the reinvestment risk and value their long term liabilities using a yield curve, which is level beyond 30 years. Here, we consider the similar principle of a level yield curve beyond the time of maturity of the longest traded bond. Combined with the no arbitrage principle for the remaining risks this principle yields a (semi) market reserve. Moreover, we consider two alternatives to the principle of a level long term yield curve. The first alternative is the principle of a level long term forward rate curve, which was first considered in Dahl (2005b) (see Chapter 5) in discrete time. The second alternative is to super-replicate the reinvestment risk.

In addition to determining reserves the life insurance companies are concerned with measuring the riskiness of their business. Here, we consider two measures for the riskiness of the insurance portfolio. Firstly, we consider the Value at Risk, which for a given investment strategy describes the initial capital necessary to meet the liability with a certain probability, and secondly, we consider the tail conditional expectation which measures the average necessary initial capital given it exceeds a certain threshold.

We emphasize that the main focus of this chapter is not of theoretical nature. On the contrary we keep all technicalities to an absolute minimum to improve the readability. Hence, the aim is to provide an easily readable overview of different reservation principles and risk measures in the presence of a large number of risks. A main part of the insight is gained through a numerical section, where we compare the different principles and illustrate the impact of the market's attitude toward the different unhedgeable risks.

The chapter is organized as follows: Section 8.2 contains an introduction of the model. In Section 8.3 the different reservation principles are considered. The risk measures Value at Risk and tail conditional expectation are considered in Section 8.4. Section 8.5 contains the numerical results. Furthermore, this section contains an explanation of the simulation procedure used to calculate the risk measures, and an overview of and motivation for the parameters used in the numerical calculations.

8.2 The Model

Since the model considered in this chapter is a slight simplification of the one in Chapter 7, we refer the reader to that chapter for details. In general the simplification consists of restricting all parameter processes to deterministic functions and to consider a specific model for the development of the mortality intensity. Furthermore, the present exposition deviates from Chapter 7 by a particularly simple approach to the modelling of bond prices at the time of issue. An approach, which is suitable for numerical calculations.

8.2.1 The financial market

Let $P(t,u)$ denote the price at time t of a zero coupon bond maturing at time u, $f(t,u)$ the forward rate with maturity date u contracted at time t and $r(t) = f(t,t)$ the short rate of interest.

Now consider two fixed time horizons $\widetilde{T}$ and T, where $\widetilde{T} \leq T$. Here, $\widetilde{T}$ and T, respectively, describe the upper limit for the time *to* maturity of traded bonds and the terminal time of the considered payment process. Thus, at any time t the time to maturity of the longest traded bond is less than or equal to $\widetilde{T}$. In addition to $\widetilde{T}$ and T, we consider the sequence $0 = T_0 < T_1 < \ldots < T_m = T - \widetilde{T}$, which describes the times, where new bonds are issued in the market. At time T_i new bonds are issued, such that all bonds with time *to* maturity less than or equal to $\widetilde{T}$ are traded. To ensure that at all times, bonds are traded in the market, we assume that $\widetilde{T} \geq \max_{i=1,\ldots,m}(T_i - T_{i-1})$. Now introduce a sequence $Y = (Y_i)_{i=1,\ldots,m}$ of mutually independent and identically distributed random variables with finite support. Without loss of generality we assume the support is given by $\{1,\ldots,b\}$ and let $p_j = P(Y_1 = j) \in (0,1)$. To model the initial price of new bonds issued at time T_i, we assume the forward rate curve (and hence the zero coupon bond price curve) generated by existing bonds is continued in a nice continuous fashion. Here, the outcome of Y_i determines the continuation at time T_i. Instead of modelling the continuation directly as in Chapters 6 and 7, we present an indirect approach, which is particularly suitable for obtaining numerical results.

Let $\delta_1, \ldots, \delta_{b^m}$ denote the b^m different possible outcomes of the vector Y. Given $Y = \delta_k$ the bond market is complete (since the development of the bond prices is assumed to be driven by one Wiener process, see (8.2.1)). Hence, we are able to obtain zero coupon bond prices $P^{\delta_k}(0,t)$ and forward rates $f^{\delta_k}(0,t)$ for *all* maturities $0 \leq t \leq T$. Since all bonds with time of maturity less than or equal to $\widetilde{T}$ are traded at time 0, all conditional forward rate curves are identical for $0 \leq t \leq \widetilde{T}$. Furthermore, all conditional forward rate curves conditioned on the same values of $Y_1, \ldots, Y_i$ are identical for $0 \leq t \leq T_i + \widetilde{T}$. Now assume that the dynamics under P of the conditional forward rate curves are given by

$$df^{\delta_k}(t,\tau) = \sigma^f(t,\tau)\left(\int_t^\tau \sigma^f(t,u)du - h^f(t)\right)dt + \sigma^f(t,\tau)dW^f(t), \tag{8.2.1}$$

where

$$\sigma^f(t,\tau) = ce^{-a(\tau-t)} \tag{8.2.2}$$

for some constants a and c. Here, W^f is a Wiener process under P independent of Y. Hence, the dynamics of the conditional forward rate curves are identical, such that the only difference between the conditional forward rate curves is the initial long term forward rates. With σ^f given by (8.2.2) the fluctuations of the forward rates, as in practice, dampen exponentially as a function of time to maturity. Furthermore, it is well known that the conditional short rates follow an extended Vasiček model, see Musiela and Rutkowski (1997). At time T_i the value of Y_i is observed and the extension of the forward rate curve is given by $(f^{\delta_k}(T_i,u))_{T_{i-1}+\widetilde{T}\le u\le T_i+\widetilde{T}}$, where the observed outcome of $(Y_1,\dots,Y_i)$ are the first i values in the vector δ_k. We note that all b^{m-i} values of k for which the observed outcome of $(Y_1,\dots,Y_i)$ are the first i values give the same extension of the forward rate curve at time T_i.

One way to interpret this model is that an investor knows that the initial price of long term bonds should be calculated with one of b^m different forward rate vectors. At times T_i more information is revealed regarding which curve initially was the correct one. This additional information uniquely determines part of the initial forward rate curve and narrows the possibilities for the remaining part of the initial forward rate curve.

For fixed t we define

$$i_t = \sup\left\{0 \le i \le m | T_i \le t\right\},$$

such that T_{i_t} is the last time new bonds are issued prior to time t (time t included). Thus, the time of maturity, τ, of the bonds traded in the bond market at time t satisfy $t \le \tau \le T_{i_t} + \widetilde{T}$.

When trading in the bond market it is sufficient to consider investments in a savings account with price process B earning the stochastic rate of interest r, and an asset with price process Z generated by investing 1 unit at time 0 and at times $0 \le t \le T$ investing in the longest bond traded in the market. Now assume that the financial market includes a stock with price process S, whose development is correlated with the development of the bond market. The P-dynamics of the traded assets are

$$\begin{aligned}
dB(t) &= r(t)B(t)dt, \quad B(0) = 1,\\
dZ(t) &= \left(r(t) + h^f(t)\sigma^z(t)\right) Z(t)dt - \sigma^z(t)Z(t)dW^f(t), \quad Z(0) = 1,\\
dS(t) &= (r(t) + \rho^s(t))S(t)dt - \sigma^s(t)S(t)dW^f(t) + \beta^s(t)S(t)dW^s(t), \quad S(0) = 1,
\end{aligned}$$

where

$$\begin{aligned}
\rho^s(t) &= \sigma^s(t)h^f(t) - \beta^s(t)h^s(t),\\
\sigma^z(t) &= \int_t^{T_{i_t}+\widetilde{T}} \sigma^f(t,u)du = \frac{c}{a}\left(1 - e^{-a(T_{i_t}+\widetilde{T}-t)}\right),
\end{aligned}$$

and h^f, h^s, σ^s and β^s are known functions. Here, W^s is a Wiener process under P independent of W^f and Y.

A trading strategy is a three-dimensional vector process $\varphi = (\vartheta^s, \vartheta^z, \eta)$. The triplet $\varphi(t) = (\vartheta^s(t), \vartheta^z(t), \eta(t))$ is interpreted as the portfolio held at time t. Here, ϑ^s and ϑ^z

denote, respectively, the number of assets with price process S and Z, whereas η is the discounted deposit in the savings account. The value process $\mathcal{V}(\varphi)$ associated with the strategy φ is defined by

$$\mathcal{V}(t,\varphi) = \vartheta^s(t)S(t) + \vartheta^z(t)Z(t) + \eta(t)B(t), \quad 0 \le t \le T.$$

In this chapter, we restrict ourselves to so-called self-financing strategies, where no in- or outflow of capital to/from the portfolio is allowed.

8.2.2 Modelling the mortality

Let $\mu(x,t)$ denote the mortality intensity at time t of an insured of age x at time 0. If we further let μ° describe the initial mortality curve for all ages, then it holds that $\mu(x,0) = \mu^\circ(x)$. For a fixed initial age x we assume the mortality intensity follows the following time-inhomogeneous Cox–Ingersoll–Ross model

$$d\mu(x,t) = \left(\gamma^\mu(x,t) - \delta^\mu(x,t)\mu(x,t)\right)dt + \sigma^\mu(x,t)\sqrt{\mu(x,t)}dW^\mu(t), \tag{8.2.3}$$

where

$$\begin{aligned}
\gamma^\mu(x,t) &= \frac{1}{2}\tilde{\sigma}^2\mu^\circ(x+t),\\
\delta^\mu(x,t) &= \tilde{\delta} - \frac{\frac{d}{dt}\mu^\circ(x+t)}{\mu^\circ(x+t)},\\
\sigma^\mu(x,t) &= \tilde{\sigma}\sqrt{\mu^\circ(x+t)}.
\end{aligned}$$

Here, $\tilde{\sigma}$ and $\tilde{\delta}$ are non-negative constants, W^μ is a Wiener process under P independent of the financial market, $\delta^\mu(x,t)$ is the time-dependent speed of mean-reversion and $\gamma^\mu(x,t)/\delta^\mu(x,t)$ is the time-dependent level of mean-reversion. Note that $2\gamma^\mu(x,t) \ge (\sigma^\mu(x,t))^2$, such that the mortality intensity is strictly positive, see Maghsoodi (1996). Now define the survival probability by

$$\mathcal{S}(x,t,T) = E^P\left[e^{-\int_t^T \mu(x,u)du} \,\middle|\, \mu(x,t)\right].$$

From Proposition 4.3.1 we have the following expression for the survival probability

$$\mathcal{S}(x,t,T) = e^{A^\mu(x,t,T) - B^\mu(x,t,T)\mu(x,t)},$$

where

$$\frac{\partial}{\partial t}B^\mu(x,t,T) = \delta^\mu(x,t)B^\mu(x,t,T) + \frac{1}{2}(\sigma^\mu(x,t))^2(B^\mu(x,t,T))^2 - 1, \tag{8.2.4}$$

$$\frac{\partial}{\partial t}A^\mu(x,t,T) = \gamma^\mu(x,t)B^\mu(x,t,T), \tag{8.2.5}$$

with $B^\mu(x,T,T) = 0$ and $A^\mu(x,T,T) = 0$. The forward mortality intensities are given by

$$f^\mu(x,t,T) = \mu(x,t)\frac{\partial}{\partial T}B^\mu(x,t,T) - \frac{\partial}{\partial T}A^\mu(x,t,T).$$

8.2.3 The insurance portfolio

Consider an insurance portfolio consisting of n insured lives of the same age x, and let the mortality intensity in Section 8.2.2 describe the probability of the death of an insured in a small time interval. The insured lives are obviously not independent, since the survival/death of all insured depend on the development of the mortality intensity. However, conditioned on the development of the mortality intensity, we assume the insured lives are independent. To keep track of the number of deaths in the insurance portfolio we introduce the counting process N. The stochastic intensity process λ of N under P, which describes the probability of experiencing a death in the portfolio within the next small time interval, is given by the number of survivors multiplied by the mortality intensity. The independence between the mortality intensity and the financial market then ensures that the insurance portfolio is independent of the financial market as well.

8.2.4 A class of equivalent martingale measures

In the model described in Sections 8.2.1–8.2.3 there exists infinitely many equivalent martingale measures, such that the model is arbitrage free, but not complete, see e.g. Björk (2004, Chapter 10). Here, we only consider a specific class of equivalent martingale measures. The class is particularly nice, since any independence under P is preserved under Q and the Q-properties of N, μ and Y are closely related to the P-properties.

For all equivalent martingale measures it holds that the discounted price processes of traded assets are Q-martingales.

To account for the unsystematic mortality risk we let the intensity process for N under Q be given by $\lambda^Q(x,t) = (1+g)\lambda(x,t)$, for some constant $g > -1$. This essentially corresponds to changing the mortality intensity to $\mu^Q(x,t) = (1+g)\mu(x,t)$. If $g = 0$, the market is called risk-neutral with respect to unsystematic mortality risk. This choice of g can be motivated by the law of large numbers.

Now introduce the constants β and β^*, which affect the market price of systematic mortality risk, and let the Q-dynamics of $\mu^Q(x)$ be given by

$$d\mu^Q(x,t) = \left(\gamma^{\mu,Q,g}(x,t) - \delta^{\mu,Q,g}(x,t)\mu^Q(x,t)\right) dt + \sigma^{\mu,g}(x,t)\sqrt{\mu^Q(x,t)}dW^{\mu,Q}, \quad (8.2.6)$$

where

$$\gamma^{\mu,Q,g}(x,t) = (1+g)\left(\left(\frac{1}{2}\tilde{\sigma}^2 + \beta^*\right)\mu^\circ(x+t)\right), \quad (8.2.7)$$

$$\delta^{\mu,Q,g} = \tilde{\delta} + \beta - \frac{\frac{d}{dt}\mu^\circ(x+t)}{\mu^\circ(x+t)},$$

$$\sigma^{\mu,g}(x,t) = \sqrt{1+g}\ \tilde{\sigma}\sqrt{\mu^\circ(x+t)}. \quad (8.2.8)$$

Hence, β^* affects the level of mean-reversion, whereas β affects both the level and speed of mean-reversion. If β and β^* are equal to zero, we say the market is risk-neutral with respect to systematic mortality risk. From (8.2.7) and (8.2.8) we get that μ^Q is strictly positive under Q if and only if $\beta^* \geq 0$. Now define the Q-survival probability by

$$\mathcal{S}^Q(x,t,T) = E^Q\left[e^{-\int_t^T \mu^Q(x,u)du} \middle| \mu^Q(x,t) \right].$$

Since (8.2.6) has the same form as (8.2.3) we have an affine mortality structure under Q as well. Hence, we have the following expression for the Q-survival probability:

$$\mathcal{S}^Q(x,t,T) = e^{A^{\mu,Q}(x,t,T) - B^{\mu,Q}(x,t,T)\mu^Q(x,t)},$$

where $A^{\mu,Q}$ and $B^{\mu,Q}$ are determined from (8.2.4) and (8.2.5) with $\gamma^\mu(x,t)$, $\delta^\mu(x,t)$ and $\sigma^\mu(x,t)$ replaced by $\gamma^{\mu,Q,g}(x,t)$, $\delta^{\mu,Q,g}(x,t)$ and $\sigma^{\mu,g}(x,t)$, respectively. Furthermore, the Q-forward mortality intensities are given by

$$f^{\mu,Q}(x,t,T) = \mu^Q(x,t)\frac{\partial}{\partial T}B^{\mu,Q}(x,t,T) - \frac{\partial}{\partial T}A^{\mu,Q}(x,t,T).$$

Under all considered equivalent martingale measures it still holds that $Y_1, \ldots, Y_m$ are identically distributed. For any $j \in \{1, \ldots, b\}$ it holds that under the equivalent martingale measure the probability of $Y_1 = j$ is changed to from p_j to q_j, where $q_j \in (0,1)$. If $q_j = p_j$ for all j the market is called risk-neutral with respect to reinvestment risk.

8.2.5 The payment process

The total benefits less premiums on the insurance portfolio is described by a payment process A. Thus, $dA(t)$ are the net payments to the policy-holders during an infinitesimal interval $[t, t+dt)$. We take A of the form

$$\begin{aligned} dA(t) = & -n\pi(0)d1_{(t\geq 0)} + (n - N(\overline{T}))\Delta A_0(\overline{T})d1_{(t\geq \overline{T})} \\ & + a_0(t)(n - N(t))dt + a_1(t)dN(t), \end{aligned} \tag{8.2.9}$$

for $0 \leq t \leq T$. The first term $n\pi(0)$ is the single premium paid at time 0 by all policy-holders. The second term involves a fixed time $\overline{T} \leq T$, which represents the retirement time of the insured. This term states that each of the surviving policy-holders receive the fixed amount $\Delta A_0(\overline{T})$ upon retirement. Hence, $\Delta A_0(\overline{T})$ corresponds to a pure endowment. The third term involves a piecewise continuous function

$$a_0(t) = -\pi^c(t)1_{(0\leq t<\overline{T})} + a^p(t)1_{(\overline{T}\leq t\leq T)},$$

where $\pi^c(t)$ are continuous premiums paid by the policy-holders (as long as they are alive), and $a^p(t)$ corresponds to a life annuity benefit received by the policy-holders. Finally, the last term in (8.2.9) represents payments immediately upon a death, and we assume that a_1 is some piecewise continuous function.

8.3 Reserving

In this section we consider different reservation principles applicable in life insurance. We note that all reserves calculated at time 0 are calculated *after* a possible initial premium, so an obtained reserve at time 0 can be interpreted as the initial premium implied by the considered criterion.

8.3.1 Market reserves

For any equivalent martingale measure Q from the class of measures considered in Section 8.2.4 we can define a market reserve by

$$V^Q(t) = E^Q\left[\int_t^T e^{-\int_t^u r(s)ds} dA(u) \middle| \mathcal{F}(t)\right],$$

where $\mathcal{F}(t)$ represents all information available at time t. Calculations similar to those in Chapter 4 give the following simplification of the market reserve in Chapter 7:

$$V^Q(t) = (n - N(t))V^{Q,i}(t),$$

where

$$\begin{aligned} V^{Q,i}(t) = \sum_{k=1}^{b^m} Q(Y = \delta_k)\Bigg(&\int_t^T P^{\delta_k}(t,u)\mathcal{S}^Q(x,t,u)\left(a_0(u) + a_1(u)f^{\mu,Q}(x,t,u)\right) du \\ &+ P^{\delta_k}(t,\overline{T})\mathcal{S}^Q(x,t,\overline{T})\Delta A_0(\overline{T})\Bigg). \end{aligned}$$

Here, the quantity $V^{Q,i}(t)$ is the individual market reserve at time t for a policy-holder who is alive.

8.3.2 Super-replication

The super-replicating (super-hedging) price for a liability is the minimal initial capital necessary to guarantee the existence of a trading strategy, which always leaves the company with sufficient capital to cover the liability. Hence, for a contingent claim H the super-replicating price, $F^{sr}(0,H)$, is given by

$$F^{sr}(0,H) = \inf_{\mathcal{V}(0,\varphi)} (P(\mathcal{V}(T,\varphi) \geq H) = 1).$$

In the case of a payment process the super-replicating price is given by

$$F^{sr}(0,A) = \inf_{\mathcal{V}(0,\varphi)} \left(P\left(\mathcal{V}(T,\varphi) \geq \int_0^T \frac{\mathcal{V}(T,\varphi)}{\mathcal{V}(u,\varphi)} dA(u)\right) = 1\right). \tag{8.3.1}$$

Here, it is sufficient to consider the terminal time only, since an initial capital and a trading strategy, which ensures sufficient capital to cover the accumulated payments at terminal time T also is sufficient at time t to cover the accumulated benefits and the time t super-replicating price. In (8.3.1) the payments are accumulated with the rate of return obtained by the super-replicating strategy, since they can be interpreted as in- or outflow to/from the portfolio.

Note that the super-replicating price is upper boundary for the open interval of arbitrage free prices. Hence, the super-replicating price can be interpreted as the lowest price which allows the seller of the contract to make arbitrage.

In the considered model with deterministic benefits and a finite number of conditional forward rate curves the super-replicating prices are particularly simple to calculate, if the benefits only are contingent on either survival or death.

Proposition 8.3.1
The super-replicating price for a pure endowment of $\Delta A_0(\overline{T})$ at time $\overline{T}$ and a temporary life annuity with a continuous time-dependent rate a^p from time $\overline{T}$ to T is given by

$$F^{sr}(0, A^{pe,a}) = \max_{k\in\{1,\ldots,b^m\}} \left(P^{\delta_k}(0,\overline{T})\Delta A_0(\overline{T}) + \int_{\overline{T}}^{T} P^{\delta_k}(0,u)a^p(u)du \right).$$

For a term insurance of a constant amount a_1 payable upon death prior to time $\overline{T}$ the super-replicating price is

$$F^{sr}(0, A^{ti}) = a_1.$$

The super-replicating prices above can be interpreted as follows: If the benefits are contingent on survival we assume the insured never dies, and if the benefits are contingent on death, we assume the insured dies immediately. This corresponds to using a survival probability over any time interval of 1 or 0, respectively. The resulting purely financial claim is now priced in the conditional models without reinvestment risk. The super-replicating price is then the maximum of these conditional prices.

In general the criterion of super-replication is not suitable to determine reserves for life insurance contracts, and hence it shall not be pursued further in this chapter. However, for a financial risk, such as the reinvestment risk, the criterion of super-replication may provide valuable information. Hence, this idea is pursued in Section 8.3.3.

8.3.3 Alternative approaches to the reinvestment risk

Instead of calculating reserves by the no arbitrage principle only, we now consider three alternative approaches to handling the reinvestment risk. Combined with the no arbitrage principle for the remaining risks these principles yield reserves, which serve as alternatives to the market reserves determined in Section 8.3.1. We note that the reserves in this section also depend on the market's attitude towards systematic and unsystematic mortality risk, and hence the considered equivalent martingale measure Q.

Super-replication of reinvestment risk

Consider the case where the company determines reserves using the criterion of super-replicating for the reinvestment risk. In this case, the company for a fixed Q determines the market reserves in the conditional models without reinvestment risk. The reserve based on the criterion super-replication of the reinvestment risk is then the maximum of the conditional market reserves. Mathematically this corresponds to

$$\begin{aligned} F^{srr}(0,A) = \max_{k \in \{1,\ldots,b^m\}} n\Bigg(& \int_t^T P^{\delta_k}(0,u)\mathcal{S}^Q(x,0,u)\left(a_0(u) + a_1(u) f^{\mu,Q}(x,0,u)\right) du \\ & + P^{\delta_k}(0,\overline{T})\mathcal{S}^Q(x,0,\overline{T})\Delta A_0(\overline{T})\Bigg). \end{aligned}$$

Level long term yield and forward rate curves

In order to handle the reinvestment risk we, inspired by Danish legislation, consider the principle of a *level long term yield curve*. Here, the companies value their liabilities using a yield curve, which is level after time of maturity of the longest bond currently traded in the market. In addition we also consider the related principle of a *level long term forward rate curve*, where reserves are obtained using a forward rate curve which is level beyond the time of maturity of the longest traded bond. This principle was introduced in discrete time in Dahl (2005b) (see Chapter 5). Denote the bond prices using a level long term yield and forward rate curve by $P^y(0,\cdot)$ and $P^f(0,\cdot)$, respectively. For a fixed Q the reserves using these principles are given by

$$\begin{aligned} F^c(0,A) = n\Bigg(& \int_t^T P^c(0,u)\mathcal{S}^Q(x,0,u)\left(a_0(u) + a_1(u) f^{\mu,Q}(x,0,u)\right) du \\ & + P^c(0,\overline{T})\mathcal{S}^Q(x,0,\overline{T})\Delta A_0(\overline{T})\Bigg), \end{aligned}$$

where $c \in \{y,f\}$. Note that the reserves calculated by these principles not necessarily lie in the interval of arbitrage free prices.

Connection between yield and forward rate curves
In order to compare the principles of a level long term yield curve and a level long term forward rate curve, we study the shape of the forward rate (yield) curve implied by a level long term yield (forward rate) curve.

The yield from time u to t is defined as the constant rate of interest, $y(u,t)$, implied by the price at time u of a zero coupon bond maturing at time t. Hence, $y(u,t)$ is given by

$$y(u,t) = -\frac{\log P(u,t)}{t-u},$$

or stated differently

$$P(u,t) = e^{-y(u,t)(t-u)}.$$

Björk (2004) refers to the yield for the period $[u,t]$ as the continuously compounded spot rate for the period $[u,t]$. For $0 \leq u \leq t$ we have the following connection between yields and forward rates

$$e^{-y(u,t)(t-u)} = e^{-\int_u^t f(u,s)ds}. \tag{8.3.2}$$

Since we are interested in applying the principles of a level long term yield or forward rate curve to obtain reserves at time 0, we henceforth restrict ourselves to the case $u = 0$.

First consider the yield curve corresponding to a level long term forward rate curve. To be more specific we assume the forward rate curve is level from time $\widetilde{T}$, such that we for any $t \geq \widetilde{T}$ have

$$e^{-y(0,t)t} = e^{-\int_0^{\widetilde{T}} f(0,s)ds - f(0,\widetilde{T})(t-\widetilde{T})}.$$

Using (8.3.2) with $t = \widetilde{T}$ and isolating $y(0,t)$ gives

$$y(0,t) = f(0,\widetilde{T}) + \frac{(y(0,\widetilde{T}) - f(0,\widetilde{T}))\widetilde{T}}{t}. \tag{8.3.3}$$

From (8.3.3) we observe that if $y(0,\widetilde{T})$ is smaller (larger) than $f(0,\widetilde{T})$ then $y(0,t)$ converges upwards (downwards) to $f(0,\widetilde{T})$ as $t \to \infty$.

Now consider the implications on the forward rate curve of a level long term yield curve. Assuming the yield curve is level from time $\widetilde{T}$ gives the following equation for any $t \geq \widetilde{T}$:

$$e^{-y(0,\widetilde{T})t} = e^{-\int_0^t f(0,s)ds}.$$

Again we apply (8.3.2) with $t = \widetilde{T}$ to obtain

$$e^{-y(0,\widetilde{T})(t-\widetilde{T})} = e^{-\int_{\widetilde{T}}^t f(0,s)ds}. \tag{8.3.4}$$

Since (8.3.4) holds for all $t \geq \widetilde{T}$ we get $f(0,t) = y(0,\widetilde{T})$. Hence, the long term forward rate curve is level as well and equal to the yield curve. We note that if $y(0,\widetilde{T}) \neq f(0,\widetilde{T})$ then the forward rate curve is discontinuous at $\widetilde{T}$, which is counter intuitive and in contrast to standard financial literature. Figure 8.3.1 contains an illustration of the principles of a level long term forward rate curve and a level long term yield curve. In this example the principle of a level long term yield curve would lead to a discontinuity in the forward rate curve.

Regarding the general relationship between the yield curve and the forward rate curve it holds that when the yield curve is increasing (decreasing) the forward rate curve lies above (below) the yield curve. Furthermore it holds that if the forward rate curve is increasing (decreasing) for all maturities then it lies above (below) the yield curve.

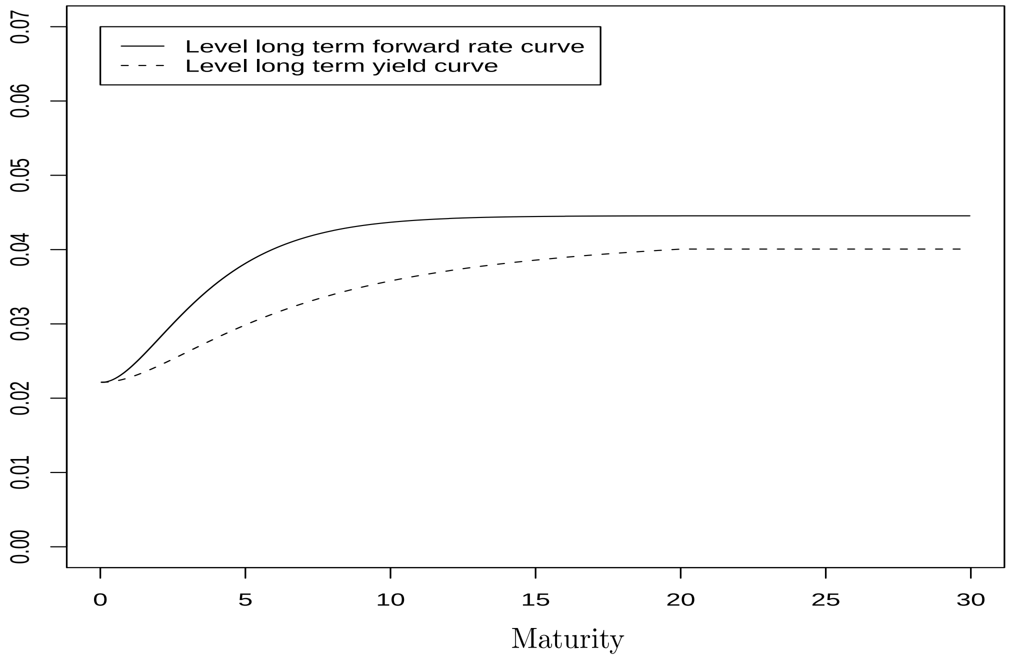

Figure 8.3.1: *An illustration of the principles of a level long term forward rate curve and a level long term yield curve.*

8.4 Risk measures

In order to measure the risk of the company associated with the insurance portfolio we consider the risk measures of Value at Risk and tail conditional expectation. We note that both risk measures are calculated *after* a possible initial premium. In this exposition we follow Artzner, Delbaen, Eber and Heath (1999) and define the risk measures in terms of the initial capital instead of the terminal capital. The mean excess function from actuarial literature (the expected short fall in the financial literature) measures the overshoot for a given level. However, since we consider the necessary initial capital and not just the additional necessary initial capital exceeding a specific level we prefer the term tail conditional expectation, which stems from Artzner et al. (1999). For an overview of Value at Risk written especially for practitioners we refer to Duffie and Pan (1997).

8.4.1 Value at Risk

Given a trading strategy the terminal Value at Risk at confidence level κ, VaR_κ^t, is the initial capital necessary to meet a given liability with probability κ. Hence, for a contingent claim H with maturity T we have

$$VaR_\kappa^t(\varphi, H) = \inf_{\mathcal{V}(0,\varphi)} \left(P\left(\mathcal{V}(T,\varphi) \geq H\right) \geq \kappa \right). \tag{8.4.1}$$

However, it is not enough for a company to hold sufficient funds at the time of maturity of the claim, it should hold sufficient funds throughout the course of the contract. Hence, in addition to the terminal Value at Risk given by (8.4.1) we consider the barrier Value at Risk, VaR^b_κ, given by

$$VaR^b_\kappa(\varphi, H) = \sup_{0\le t\le T} \inf_{\mathcal{V}(0,\varphi)} \left(P\left(\mathcal{V}(t,\varphi) \ge V(t,H)\right) \ge \kappa\right). \tag{8.4.2}$$

Here, $V(t,H)$ describes a capital requirement at time t for the claim H. Hence, $V(t,H)$ could be a solvency requirement or a market reserve. From (8.4.1) and (8.4.2) we observe that $VaR^b_\kappa(\varphi,H) \ge VaR^t_\kappa(\varphi,H)$, which also is intuitively clear since the barrier Value at Risk should be sufficient to meet requirements at any time prior to (and including) maturity, while the terminal Value at Risk is sufficient to fulfill a requirement at maturity only.

Since we consider a payment process we need slightly different formulas than those in (8.4.1) and (8.4.2) to calculate the Value at Risk. For a payment process the terminal and barrier Value at Risk are given by

$$VaR^t_\kappa(\varphi, A) = \inf_{\mathcal{V}(0,\varphi)} \left(P\left(\mathcal{V}(T,\varphi) \ge \int_0^T \frac{\mathcal{V}(T,\varphi)}{\mathcal{V}(t,\varphi)} dA(t)\right) \ge \kappa\right), \tag{8.4.3}$$

and

$$VaR^b_\kappa(\varphi, A) = \sup_{0\le t\le T} \inf_{\mathcal{V}(0,\varphi)} \left(P\left(\mathcal{V}(t,\varphi) \ge \left(\int_0^t \frac{\mathcal{V}(t,\varphi)}{\mathcal{V}(u,\varphi)} dA(u) + V(t,A)\right)\right) \ge \kappa\right), \tag{8.4.4}$$

respectively. Here, a natural idea is to let

$$V(t,A) = E^Q\left[\int_t^T e^{-\int_t^u r(s)ds} dA(u)\right]$$

for some equivalent martingale measure Q. Note that, as in (8.3.1), the payments in (8.4.3) and (8.4.4) are accumulated with the relative return on the investments. In practice companies calculate the short term Value at Risk for their liabilities. Here, the fixed short term time-horizon usually lies between one day and one year. Hence, in addition to (8.4.3) and (8.4.4) we define the time s terminal and barrier Value at Risk for $s < T$, by

$$VaR^{t,s}_\kappa(\varphi, A) = \inf_{\mathcal{V}(0,\varphi)} \left(P\left(\mathcal{V}(s,\varphi) \ge \int_0^s \frac{\mathcal{V}(s,\varphi)}{\mathcal{V}(t,\varphi)} dA(t) + V(s,A)\right) \ge \kappa\right),$$

and

$$VaR^{b,s}_\kappa(\varphi, A) = \sup_{0\le t\le s} \inf_{\mathcal{V}(0,\varphi)} \left(P\left(\mathcal{V}(t,\varphi) \ge \left(\int_0^t \frac{\mathcal{V}(t,\varphi)}{\mathcal{V}(u,\varphi)} dA(u) + V(t,A)\right)\right) \ge \kappa\right).$$

Example 8.4.1 Consider the case, where a company follows either a buy and hold strategy or a strategy with constant relative portfolio weights. In a buy and hold strategy no adjustments are made to the initial portfolio during the considered time-period, whereas a company following a strategy with constant relative portfolio weights continuously adjusts the investment portfolio, such that at all times, $t \in [0,T]$ the same proportion of the value process is invested in the different assets. We observe that in both cases the dynamics under P of the value process are of the form

$$d\mathcal{V}(t,\varphi) = \big(r(t) + \rho^{\mathcal{V}}(t)\big)\, \mathcal{V}(t,\varphi)dt + \sigma^{\mathcal{V}}(t)\mathcal{V}(t,\varphi)dW^f(t) + \beta^{\mathcal{V}}(t)\mathcal{V}(t,\varphi)dW^s(t),$$

for some functions $\rho^{\mathcal{V}}$, $\sigma^{\mathcal{V}}$ and $\beta^{\mathcal{V}}$. Now we are interested in calculating VaR^t_κ for a capital insurance of K at time T. Hence, we have to determine

$$\begin{aligned} &\inf_{\mathcal{V}(0,\varphi)} \left(P(\mathcal{V}(T,\varphi) \geq K) \geq \kappa\right) \\ \Leftrightarrow &\inf_{\mathcal{V}(0,\varphi)} \sum_{k=1}^{b^m} P(Y=\delta_k)\left(P(\mathcal{V}^{\delta_k}(T,\varphi) \geq K) \geq \kappa\right), \end{aligned} \tag{8.4.5}$$

where $\mathcal{V}^{\delta_k}(\varphi)$ is the value process in the conditional model given $Y=\delta_k$. In order to determine $P(\mathcal{V}^{\delta_k}(T,\varphi) \geq K)$ we note that the conditional model given $Y=\delta_k$ the short rate is given by

$$r^{\delta_k}(u) = f^{\delta_k}(0,u) + \frac{c^2}{2a^2}(1-e^{-au})^2 + \int_0^u ce^{-a(u-s)}dW^f(s).$$

Hence, we have

$$\begin{aligned} \log \mathcal{V}^{\delta_k}(t,\varphi) &= \log \mathcal{V}(0,\varphi) + \int_0^t \left(f^{\delta_k}(0,u) + \alpha^{\mathcal{V}}(u)\right) du \\ &\quad + \int_0^t \int_0^u ce^{-a(u-s)}dW^f(s)du + \int_0^t \sigma^{\mathcal{V}}(u)dW^f(u) + \int_0^t \beta^{\mathcal{V}}(u)dW^s(u) \\ &= \log \mathcal{V}(0,\varphi) + \int_0^t \left(f^{\delta_k}(0,u) + \alpha^{\mathcal{V}}(u)\right) du \\ &\quad + \int_0^t \frac{c}{a}\left(1-e^{-a(t-u)}\right) dW^f(u) + \int_0^t \sigma^{\mathcal{V}}(u)dW^f(u) + \int_0^t \beta^{\mathcal{V}}(u)dW^s(u), \end{aligned}$$

where we have defined $\alpha^{\mathcal{V}}(u) = \frac{c^2}{2a^2}(1-e^{-au})^2 + \rho^{\mathcal{V}}(u) - \frac{1}{2}(\sigma^{\mathcal{V}}(u))^2 - \frac{1}{2}(\beta^{\mathcal{V}}(u))^2$ and used Fubini's theorem for stochastic processes to interchange integrals in the double integral. This gives

$$\log \mathcal{V}^{\delta_k}(t,\varphi) \sim N(\log \mathcal{V}(0,\varphi) + \alpha^{\delta_k}(t), (\sigma(t))^2),$$

where

$$\begin{aligned} \alpha^{\delta_k}(t) &= \int_0^t \left(f^{\delta_k}(0,u) + \frac{c^2}{2a^2}(1-e^{-au})^2 + \rho^{\mathcal{V}}(u) - \frac{1}{2}(\sigma^{\mathcal{V}}(u))^2 - \frac{1}{2}(\beta^{\mathcal{V}}(u))^2\right) du, \\ \sigma(t) &= \sqrt{\int_0^t \left(\frac{c}{a}\left(1-e^{-a(t-u)}\right) + \sigma^{\mathcal{V}}(u)\right)^2 + (\beta^{\mathcal{V}}(u))^2\, du}. \end{aligned}$$

Thus, it holds that

$$P(\mathcal{V}^{\delta_k}(T,\varphi) \geq K) = P\left(\log \mathcal{V}^{\delta_k}(T,\varphi) \geq \log(K)\right)$$
$$= \Phi\left(\frac{\log \mathcal{V}(0,\varphi) + \alpha^{\delta_k}(T) - \log(K)}{\sigma(T)}\right), \tag{8.4.6}$$

where Φ is the standard normal distribution function. Inserting (8.4.6) in (8.4.5) we obtain the following implicit expression for VaR^t_κ for a capital insurance of K at time T

$$\inf_{\mathcal{V}(0,\varphi)} \left(\sum_{k=1}^{b^m} P(Y=\delta_k)\Phi\left(\frac{\log \mathcal{V}(0,\varphi) + \alpha^{\delta_k}(T) - \log(K)}{\sigma(T)}\right) \geq \kappa \right). \tag{8.4.7}$$

Similar arguments give that in the case without systematic mortality risk and reinvestment risk the VaR^t_κ for a portfolio of n identical pure endowments with sum insured $\Delta A_0(T)$ and maturity T is given by

$$\inf_{\mathcal{V}(0,\varphi)} \left(\sum_{\ell=0}^{n} \binom{n}{\ell} {}_Tp_x^{n-\ell} \left(1 - {}_Tp_x\right)^\ell \Phi\left(\frac{\log \mathcal{V}(0,\varphi) + \alpha(T) - \log\left((n-\ell)\Delta A_0(T)\right)}{\sigma(T)}\right) \geq \kappa. \right) \tag{8.4.8}$$

where ${}_Tp_x = \exp(-\int_0^T \mu(x,u)du)$.

We note that in these simple cases no simulation is necessary, since we (at least numerically) are able to calculate the value at risk explicitly from (8.4.7) and (8.4.8).

□

8.4.2 Tail conditional expectation

The VaR_κ's measure the initial capital necessary to cover a future liability (and possible intermediate requirements) with probability κ. However, the criterion provides no information regarding the magnitude of the necessary capital in the cases, which occur with probability $1-\kappa$, where this initial capital is insufficient. Thus, in addition to the Value at Risk we now consider the tail conditional expectation, which for a fixed trading strategy measures the average initial investment necessary to cover the liabilities provided that a given initial investment is insufficient.

Let ω denote a possible state of the world. For each ω we now define the initial capital necessary to cover the time s value of the benefits by

$$\mathcal{V}^{t,s}_{min}(0,\varphi,\omega) = \inf_{\mathcal{V}(0,\varphi)} \left(\mathcal{V}(s,\varphi) \geq \int_0^s \frac{\mathcal{V}(s,\varphi)}{\mathcal{V}(t,\varphi)} dA(t) + V(s,A) \right).$$

For a given initial investment, u, we are now able to define the time s terminal tail conditional expectation by

$$\mathcal{V}^{t,s}_u(0,\varphi) = E^P\left[\mathcal{V}^{t,s}_{min}(0,\varphi,\omega) \middle| \mathcal{V}^{t,s}_{min}(0,\varphi,\omega) > u \right].$$

If $s = T$ we simply refer to the value as the terminal tail conditional expectation. Similarly we can define

$$\mathcal{V}^{b,s}_{min}(0,\varphi,\omega) = \sup_{0\le t\le s} \inf_{\mathcal{V}(0,\varphi)} \left(\mathcal{V}(t,\varphi) \ge \int_0^t \frac{\mathcal{V}(t,\varphi)}{\mathcal{V}(u,\varphi)} dA(u) + V(t,A)\right)$$

and the time s barrier tail conditional expectation at level u by

$$\mathcal{V}^{b,s}_{u}(0,\varphi) = E^P\left[\mathcal{V}^{b,s}_{min}(0,\varphi,\omega)\middle|\, \mathcal{V}^{b,s}_{min}(0,\varphi,\omega) > u\right].$$

If we let u be equal to VaR_κ for some κ then we obtain the so-called tail Value at Risk.

Remark 8.4.2 Note that since all calculations for the Value at Risk and tail conditional expectation are carried out under P the only possible dependence on Q is through the capital requirement $V(t,A)$.

□

8.5 Numerics

8.5.1 Simulation of Value at Risk and tail conditional expectation

Here, we explain the simulation procedure used to calculate the Value at Risk and tail conditional expectation for a fixed trading strategy and payment process. At time-step j the necessary capital to cover the accumulated benefits and the reserve for the future liabilities is given by

$$\mathcal{V}(j\Delta t,\varphi) = \sum_{i=1}^{j} \Delta A(i\Delta t) \frac{\mathcal{V}(j\Delta t,\varphi)}{\mathcal{V}(i\Delta t,\varphi)} + V(j\Delta t, A) \tag{8.5.1}$$

where $\Delta A(i\Delta t) = A(i\Delta t) - A((i-1)\Delta t)$ and $V(j\Delta t, A)$ is the capital requirement. We assume the capital requirement is given by the market reserve calculated with risk-neutrality with respect to all unhedgeable sources of risk (reinvestment, systematic and unsystematic mortality risks), i.e. under the so-called minimal martingale measure, see Schweizer (1995). Now let $\mathcal{V}^1(t,\varphi)$ be the value of 1 invested at time 0. The necessary initial investment to cover the requirement at time-step j is then given by

$$\mathcal{V}_j(0,\varphi) = \frac{\sum_{i=1}^{j} \Delta A(i\Delta t)\frac{\mathcal{V}^1(j\Delta t,\varphi)}{\mathcal{V}^1(i\Delta t,\varphi)} + V(j\Delta t, A)}{\mathcal{V}^1(j\Delta t,\varphi)}. \tag{8.5.2}$$

Hence, in order to calculate (8.5.2) we keep track of the value process generated by investing 1 at time 0, and of the past benefits accumulated by the rate of return obtained by the investment strategy. In each simulation we can now for a fixed time-horizon s calculate

$$\mathcal{V}^{b,s}_{min}(0,\varphi) = \max_{j\in\{0,\ldots,s/\Delta t\}} \mathcal{V}_j(0,\varphi).$$

and

$$\mathcal{V}_{min}^{t,s}(0,\varphi) = \mathcal{V}_{s/\Delta t}(0,\varphi).$$

For any time horizon s the barrier and terminal Value at Risk and tail conditional expectation for different κ's and u's can now be calculated from the vectors containing $\mathcal{V}_{min}^{b,s}(0,\varphi)$ and $\mathcal{V}_{min}^{t,s}(0,\varphi)$ from each simulation.

Note that in the simple case of a pure endowment the first term on the right hand side in (8.5.1) disappears, such that the calculations simplify considerably.

8.5.2 Parameters

In this section we motivate the choice of parameters in the numerical calculations. It is important to note that we have no empirical ambitions, but the parameters should be reasonable.

All conditional short rates follow an extended Vasiček model with the same speed of mean-reversion, a, and volatility, c. Hence, we may use the values used in Poulsen (2003) for a standard Vasiček model. Furthermore, the market price of standard interest rate risk, h^f, is determined such that it is identical to the one in Poulsen (2003), namely $h^f = 0.3125$. Consider a so-called Nelson–Siegel parametrization, see Nelson and Siegel (1987), for the initial forward rate curve

$$f(0,t) = \alpha_0 + \alpha_1 \exp\left(-\frac{t}{\tau}\right) + \alpha_2 \frac{t}{\tau} \exp\left(-\frac{t}{\tau}\right).$$

Here, α_0, α_1, α_2 and τ are some constants of which α_0 and τ strictly positive. As a basis for the initial conditional forward rate curves we estimate the parameters in the Nelson–Siegel parametrization from the prices of Danish government bonds early 2005. Let $Y = \delta_1$ correspond to the conditional initial curve, where $Y_i = 1$ for all i. This curve is obtained by multiplying the estimated forward rates $f(0,t)$ by $(\psi^d)^i$ for $T_{i-1} + \widetilde{T} < t \leq T_i + \widetilde{T}$. We now restrict ourselves to the case $b = 2$ and obtain the remaining conditional initial forward rate curves by multiplying the forward rates $(f^{\delta_1}(0,t))_{T_{i-1}+\widetilde{T}<t\leq T_i+\widetilde{T}}$ by $(\psi^u)^j$ if j of the i'th first values in δ_k is equal to 2. To describe the probabilistic nature of the Y_i's, we let $p = P(Y_1 = 2) = 0.5$ and $q = Q(Y_1 = 2)$.

In general, empirical evidence shows that there is a positive correlation between bonds and stocks, such that increasing interest rates lead to a decrease in both bond and stock prices. We calibrate the parameters σ^s and β^s, such that the correlation, $\sigma^s/\sqrt{(\sigma^s)^2 + (\beta^s)^2}$, is equal to 0.5 and the volatility of the stock, $\sqrt{(\sigma^s)^2 + (\beta^s)^2}$, is 0.2. This gives $\sigma^s = 0.02$ and $\beta^s \approx 0.19$. Furthermore, we let $h^s = -0.2$, such that the additional expected rate of return on the stock compared to the long term bonds is approximately 0.03, which seems reasonable, see e.g. Graham and Harvey (2005).

The initial mortality curve is described by a so-called Gompertz–Makeham curve, where the mortality intensity is of the form $\mu^\circ(x+t) = a^\circ + b^\circ(c^\circ)^{x+t}$, for some constants

a°, b° and c°. We use the parameters estimated in Chapter 4 for males in year 2003. The stochastic model (including parameters) for the future development of the mortality intensity is identical to so-called "case II"-model in Chapter 4.

Table 8.5.1 provides an overview of the interpretation of the different parameters and the values used in the numerical calculations.

Parameter	Interpretation	Value
T	Terminal time of payments	30 (varies)
$\overline{T}$	Time of retirement	30
α_0	Parameter Nelson–Siegel parametrization of forward rates	0.044556
α_1	Parameter Nelson–Siegel parametrization of forward rates	-0.0224
α_2	Parameter Nelson–Siegel parametrization of forward rates	-0.0231
τ	Time-parameter Nelson–Siegel	1.97184
a	Speed of mean-reversion of r	0.25
c	Volatility of r	0.012
h^f	Market price of interest rate risk	0.3125
$\widetilde{T}$	Maximum time *to* maturity of traded bonds	20
ΔT_i	Time between issue of new bonds	2 (varies)
m	Number of issues of new bonds	2
p	Probability under P of $Y_i = 2$	0.5
q	Probability under Q of $Y_i = 2$	0.5 (varies)
σ^s	Volatility parameter stock	0.02
β^s	Volatility parameter stock	0.19
h^s	Related to market price of risk associated to the stock	-0.2
a°	Gompertz–Makeham parameter	0.000134
b°	Gompertz–Makeham parameter	0.0000353
c°	Gompertz–Makeham parameter	1.1020
x	Initial age	30
$\tilde{\delta}$	Speed of mean-reversion for μ under P	0.008
$\tilde{\sigma}$	Volatility parameter for μ	0.02
β	Affects level and speed of mean-reversion for μ under Q	0 (varies)
β^*	Affects level of mean-reversion for μ under Q	0
g	Affects market price of unsystematic mortality risk	0
ψ^u	Multiplication factor *up*	1.05 (varies)
ψ^d	Multiplication factor *down*	0.99 (varies)

Table 8.5.1: *Parameters used in the numerical calculations. A number followed by "(varies)" means that where nothing else is stated the parameter is equal to the value, and in at least one case this is not the case.*

8.5.3 Numerical results

In this section we, unless stated otherwise, consider a portfolio of pure endowments paid by a single initial premium. Furthermore, in order to ease comparison all numbers in this section, unless explicitly explained, are scaled by the market reserve under the minimal martingale measure, i.e. where $q = p$ and $\beta = \beta^* = g = 0$.

Dependence on market's attitude towards reinvestment risk

From Figure 8.5.1 we observe that the market reserve is a decreasing function of q. This relies on the fact that we consider a pure endowment paid by a single initial premium. In this case, the conditional reserve is an increasing function of the conditional price of a zero coupon bond maturing at time T, and since increasing q increases the weight assigned to the low conditional bond prices the market reserve is decreasing in q. The alternative reservation principles are independent of the choice of martingale measure with respect to reinvestment risk, so they are independent of q. Note that as $q \to 0$ the market reserve converges to the reserve calculated by the principle of reinvestment risk super-replication. This is also intuitively obvious since the weight assigned to the largest conditional zero coupon bond price, which is exactly the price used to calculate the reinvestment risk super-replicating reserve, approaches 1. The initial forward rate curve is increasing for all maturities, so the principle of a level long term yield curve gives a larger reserve than the reserve calculated with a level long term forward rate curve. Furthermore, we observe that in this case the principle of reinvestment risk super-replicating gives a larger reserve than the principle of a level long term forward rate curve. A necessary requirement for this is that (at least) one conditional forward rate curve is decreasing for some maturities.

Investigating Figure 8.5.2 we observe that the relative magnitude of the dependence on q depends on the number of issues of new bonds. This is also intuitively clear since increasing the number of issues for fixed ψ^u and ψ^d increases the diversity between the conditional bond prices. Hence, the weights assigned to the different conditional bond prices become increasingly important. We note that since the reserves depend on the number of issues the scaling factors differ for the three lines i Figure 8.5.2, so the figure can only be used to observe the impact of the number of issues on the relative dependence on q.

Dependence on market's attitude towards mortality risk

For the considered portfolio of pure endowments the reserves are given by the product of the number of insured, the Q-survival probability, the fixed benefit per insured and the price of a zero coupon bond maturing at time T under the assumptions imposed by the reservation principle. Hence, the relative impact of changing β is the same for all four reservation principles. This fact is easily observed form Figure 8.5.3. Furthermore Figure 8.5.3 shows that the reserves have a positive dependence on β. This is also intuitively

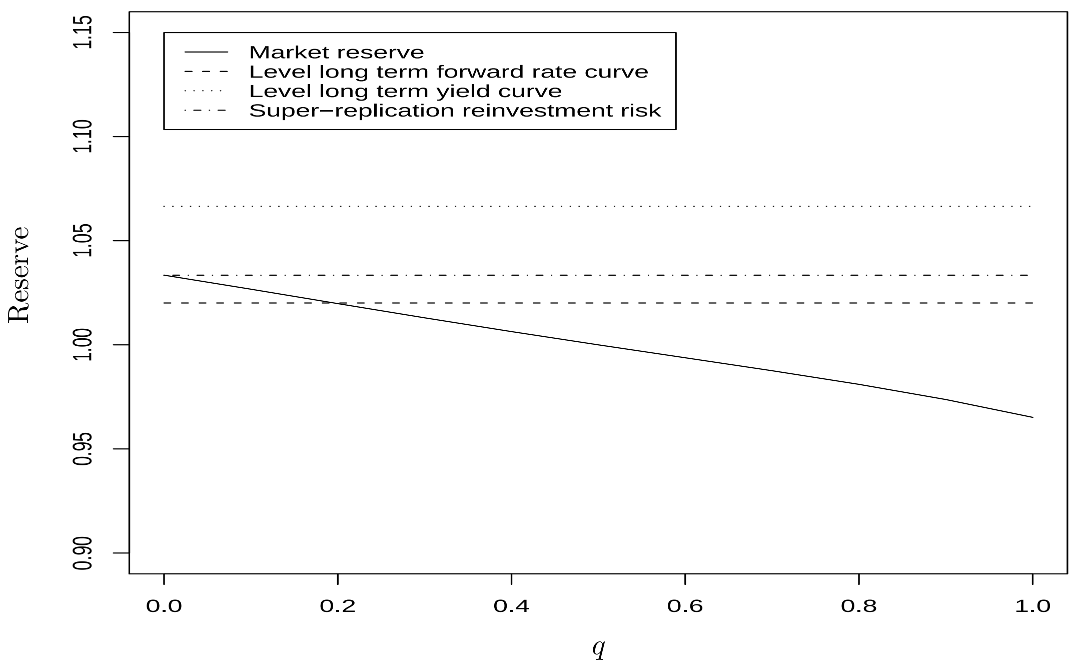

Figure 8.5.1: *Initial market reserve as a function of market's attitude to reinvestment risk. For comparison the reserves calculated by the principles of a level long term yield/forward rate curve and super-replication of reinvestment risk are plotted as well.*

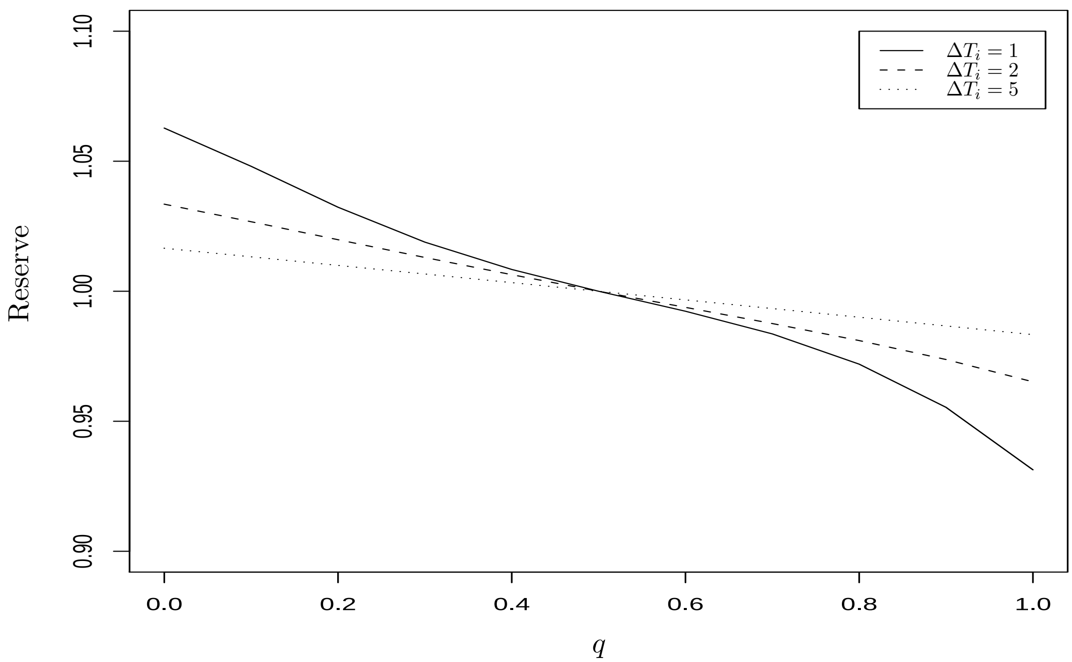

Figure 8.5.2: *Initial market reserves as a function of market's attitude to reinvestment risk for three different time intervals between the time of issue of new bonds.*

clear, since increasing β increases the speed of mean-reversion and decreases the level of mean-reversion for μ under Q, such that the Q-survival probability increases.

Similarly we have that increasing β^* increases the level of mean reversion of μ under Q, such that the reserves decrease. Since $\beta^* = 0$ is the lowest possible value ensuring a strictly positive mortality intensity under Q, other values of β^* would lead to lower reserves. The parameter g, which is related to the unsystematic mortality risk changes the level of mean-reversion and the volatility of μ under Q. However, since the level of mean-reversion is very small the effect of g (for reasonable values) is negligible.

Dependence on the multiplication factors

From Table 8.5.2 we observe that for a fixed Q the market reserve is a decreasing function of both ψ^u and ψ^d. This corresponds to our intuition, since increasing ψ^u and/or ψ^d decreases the conditional bond prices and hence the market reserve. Since the principle of super-replication of reinvestment risk only considers the largest conditional bond price, this reserve is independent of ψ^u and a decreasing function of ψ^d. The principles of a level long term yield/forward rate curve are independent of the long term conditional bond prices and thus of ψ^u and ψ^d. We observe that if ψ^d is small enough (0.96 in this case) then the principle of reinvestment risk super-replication gives a reserve larger than the principle of a level long term yield curve. Likewise we have that if ψ^d is large enough (1 in

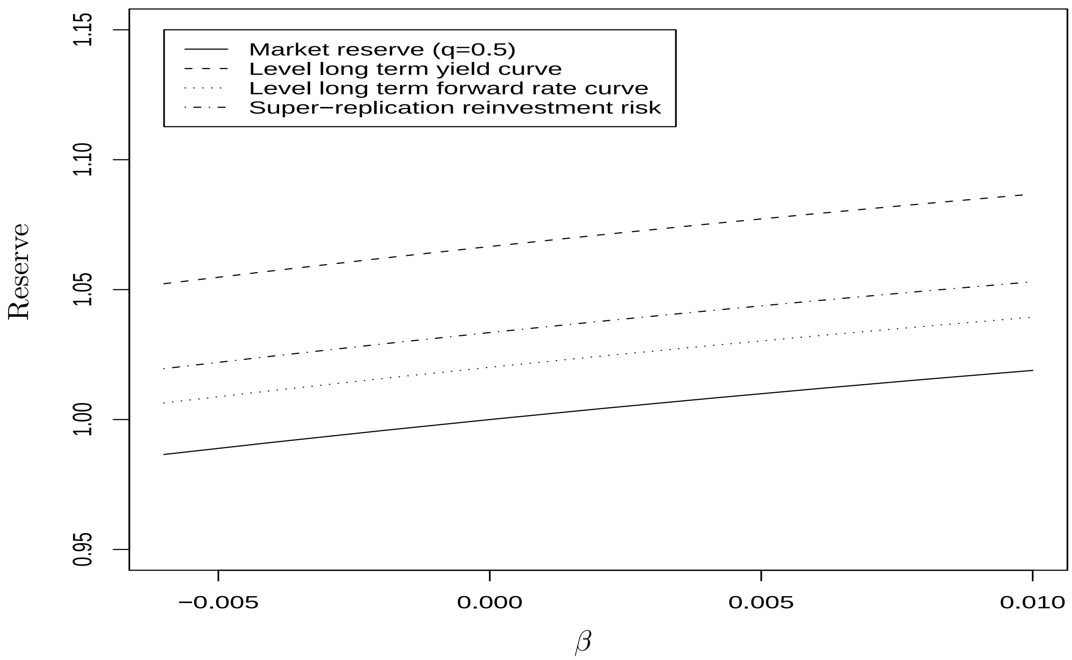

Figure 8.5.3: *Initial reserves as a function of β, which is associated with market's attitude to systematic mortality risk.*

ψ^u	ψ^d	Market reserve	Super-replication	Level forward	Level yield
1.05	0.99	254.63	263.16	259.75	271.59
1.04	0.99	256.35	263.16	259.75	271.59
1.03	0.99	258.07	263.16	259.75	271.59
1.02	0.99	259.77	263.16	259.75	271.59
1.05	1	251.01	259.73	259.75	271.59
1.05	0.99	254.63	263.16	259.75	271.59
1.05	0.98	258.21	266.54	259.75	271.59
1.05	0.97	261.74	269.87	259.75	271.59
1.05	0.96	265.22	273.15	259.75	271.59

Table 8.5.2: *Dependence of initial reserves on the multiplication factors ψ^u and ψ^d. Top: Dependence on ψ^u. Bottom: Dependence on ψ^d. All values with two decimals. No scaling is applied in this table.*

this case) the principle of reinvestment risk super-replicating gives a smaller reserve than the principle of a level long term forward rate curve.

Value at Risk and tail conditional expectation

We now turn to the risk measures of Value at Risk and tail conditional expectation. Since we only consider the tail conditional expectation at levels given by VaR_κ's, we, as explained in Section 8.4.2, refer to the considered tail conditional expectations as tail Value at Risk. All results in this section are based on investment strategies with constant relative portfolio weights and 1000 simulations using the Euler method.

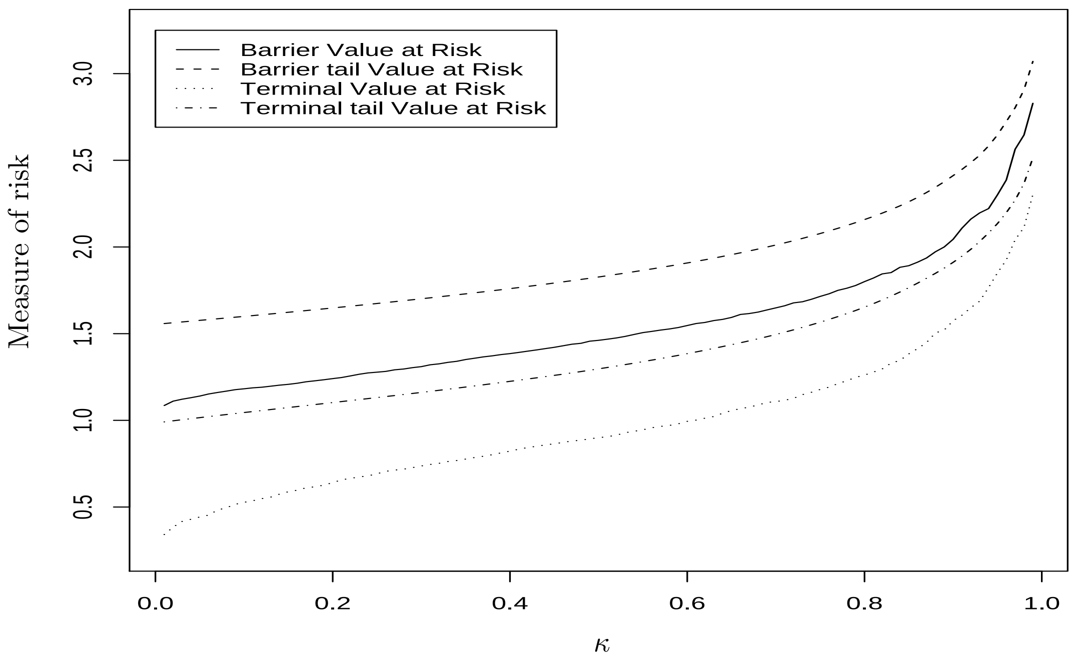

Figure 8.5.4: *Terminal/barrier (tail) Value at Risk as a function of* κ*.*

Figure 8.5.4 shows the dependence of the terminal and barrier Value at Risk and tail Value at Risk on κ for a fixed investment strategy with 40% invested in stocks and bonds, respectively, and 20% in the savings account. From Figure 8.5.4 we immediately note that for fixed κ the barrier (tail) Value at Risk is larger than the terminal (tail) Value at Risk and that the tail Value at Risk is larger than the corresponding Value at Risk. Furthermore, we observe that in this case the barrier Value at Risk is larger than the terminal tail Value at Risk. For any κ it holds that the barrier Value at Risk is greater than or equal to the market reserve under the minimal martingale measure, which corresponds to a horizon line at 1. This is due to the fact that the market reserve under the minimal martingale measure is equal to the barrier restriction. For all four risk measures we observe a steep slope for very large values of κ.

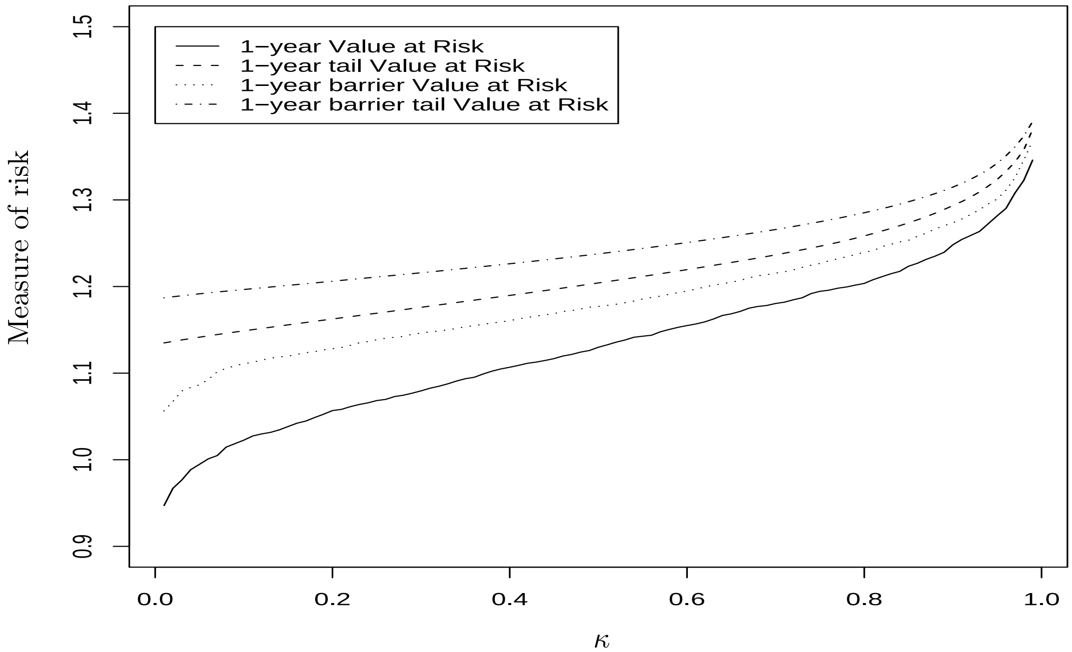

Figure 8.5.5: *1-year terminal/barrier (tail) Value at Risk as a function of* κ*.*

Figure 8.5.5 is essentially identical to Figure 8.5.4, except here the risk measures are considered on a one year time horizon. We observe that on a one year time-horizon the barrier Value at Risk lies below the terminal tail Value at Risk. Apart from the terminal Value at Risk for small κ's we observe a considerably smaller magnitude of the risk measures on a one year time horizon than when considering the time horizon of the contract. This is also intuitively clear since the amount of uncertainty on a one year scale is considerably smaller than on a long term scale. As an example we mention that on a one year time-horizon, both the barrier and terminal Value at Risk at level 0.99 are approximately 1.35 times the market reserve under the minimal martingale measure, whereas on a 30 year time-horizon this only corresponds to a barrier Value at Risk at level 0.35 and a terminal Value at Risk at level 0.85.

To consider the dependence of the (tail) Value at Risk on the investment strategy we investigate Tables 8.5.3 and 8.5.4. Here, we have restricted ourselves to strategies without short selling and borrowing. We observe that the proportion in the savings account is unchanged by moving one cell up and to the right.

Investigating Table 8.5.3 we find that for a fixed proportion invested in stocks, all risk measures decrease as the proportion in bonds increases. This is not surprising, since the bonds most closely resembles the financial nature of the benefits. Furthermore we observe that for fixed proportion in bonds, all risk measures, except the barrier (tail) Value at Risk, decrease, when the proportion in stocks increases. The barrier (tail) Value at Risk also indicate that it is better to hold some stocks than none. For a fixed proportion in the

Bonds	Stocks					
	0	0.2	0.4	0.6	0.8	1
0	4.84 (5.84) 4.54 (5.59) 3.01 (3.43) 3.01 (3.43)	3.10 (3.68) 2.66 (3.26) 2.12 (2.32) 2.15 (2.35)	2.64 (3.42) 1.97 (2.71) 1.69 (1.80) 1.73 (1.84)	2.44 (3.46) 1.58 (2.47) 1.42 (1.54) 1.48 (1.59)	2.52 (3.96) 1.44 (2.60) 1.24 (1.37) 1.32 (1.43)	2.81 (5.11) 1.44 (3.23) 1.14 (1.29) 1.22 (1.35)
0.2	3.16 (3.64) 2.95 (3.48) 2.15 (2.32) 2.15 (2.32)	2.30 (2.69) 1.95 (2.35) 1.69 (1.78) 1.71 (1.80)	2.15 (2.32) 1.97 (2.71) 1.69 (1.80) 1.73 (1.84)	2.44 (3.46) 1.58 (2.47) 1.42 (1.54) 1.48 (1.59)	2.19 (3.33) 1.15 (2.20) 1.10 (1.22) 1.16 (1.27)	
0.4	2.27 (2.49) 2.10 (2.35) 1.67 (1.74) 1.67 (1.75)	1.77 (2.00) 1.45 (1.72) 1.37 (1.43) 1.40 (1.45)	1.72 (2.08) 1.18 (1.57) 1.19 (1.25) 1.23 (1.27)	1.76 (2.38) 1.08 (1.67) 1.07 (1.15) 1.12 (1.19)		
0.6	1.75 (1.84) 1.61 (1.72) 1.37 (1.41) 1.38 (1.42)	1.48 (1.63) 1.19 (1.36) 1.17 (1.21) 1.19 (1.23)	1.44 (1.70) 0.97 (1.25) 1.05 (1.11) 1.08 (1.13)			
0.8	1.42 (1.46) 1.25 (1.30) 1.17 (1.20) 1.18 (1.21)	1.26 (1.38) 0.95 (1.09) 1.03 (1.07) 1.05 (1.08)				
1	1.23 (1.27) 1.01 (1.03) 1.03 (1.06) 1.04 (1.07)					

Table 8.5.3: *The risk measures as a function of the proportion invested in the different assets for $\kappa = 0.75$. First line for a fixed investment strategy: Barrier Value at Risk (barrier tail Value at Risk). Second line: Terminal Value at Risk (terminal tail Value at Risk). Third line: 1-year terminal Value at Risk (1-year terminal tail Value at Risk) and fourth line: 1-year barrier Value at Risk (1-year barrier tail Value at Risk). All values with two decimals.*

	Stocks					
Bonds	0	0.2	0.4	0.6	0.8	1
0	7.98 (8.90) 7.90 (8.72) 4.27 (4.44) 4.27 (4.44)	4.89 (5.15) 4.39 (4.82) 2.67 (2.76) 2.70 (2.77)	5.61 (6.33) 4.71 (5.56) 2.00 (2.08) 2.04 (2.12)	5.53 (7.27) 4.38 (5.67) 1.74 (1.80) 1.78 (1.83)	7.45 (12.60) 4.97 (9.75) 1.59 (1.69) 1.62 (1.72)	11.49 (18.05) 7.53 (12.95) 1.53 (1.65) 1.56 (1.71)
0.2	4.50 (5.00) 4.38 (4.82) 2.61 (2.71) 2.62 (2.71)	3.38 (3.78) 3.15 (3.55) 1.91 (1.99) 1.92 (2.01)	3.66 (4.39) 3.10 (3.62) 1.62 (1.66) 1.64 (1.67)	4.99 (6.62) 3.83 (5.51) 1.43 (1.49) 1.47 (1.52)	6.06 (9.90) 4.84 (8.50) 1.40 (1.52) 1.47 (1.55)	
0.4	2.98 (3.10) 2.86 (3.01) 1.88 (1.92) 1.89 (1.92)	2.47 (2.81) 2.25 (2.58) 1.53 (1.55) 1.54 (1.56)	2.83 (3.07) 2.30 (2.51) 1.35 (1.38) 1.37 (1.39)	3.76 (4.54) 3.03 (3.52) 1.28 (1.33) 1.30 (1.34)		
0.6	2.00 (2.05) 1.94 (1.99) 1.48 (1.51) 1.48 (1.52)	1.89 (1.97) 1.64 (1.73) 1.28 (1.30) 1.28 (1.30)	2.21 (2.36) 1.77 (2.02) 1.21 (1.23) 1.22 (1.24)			
0.8	1.52 (1.53) 1.38 (1.41) 1.25 (1.27) 1.15 (1.28)	1.62 (1.72) 1.36 (1.45) 1.12 (1.15) 1.13 (1.15)				
1	1.35 (1.38) 1.07 (1.08) 1.11 (1.12) 1.12 (1.13)					

Table 8.5.4: *The risk measures as a function of the proportion invested in the different assets for* $\kappa = 0.99$. *First line for a fixed investment strategy: Barrier Value at Risk (barrier tail Value at Risk). Second line: Terminal Value at Risk (terminal tail Value at Risk). Third line: 1-year terminal Value at Risk (1-year terminal tail Value at Risk) and fourth line: 1-year barrier Value at Risk (1-year barrier tail Value at Risk). All values with two decimals.*

savings account it is hard to say anything in general.

From Table 8.5.4 we observe that for a fixed proportion in bonds, the terminal and barrier (tail) Value at Risk indicate that holding some stocks usually lead to a lower risk measure than holding no stocks. However, a large investment in stocks is clearly more dangerous than none or few stocks. This is especially obvious from the tail Value at Risks. The 1-year risk measures are (almost) all decreasing as a function of the proportion in stocks. For a fixed proportion in stocks all risk measures are decreasing as a function of the proportion in bonds and for a fixed proportion in the savings account, they are decreasing as a function of the proportion in bonds.

Life annuities
Now consider a portfolio of life annuities, where the insured contingent on survival receives a continuous benefit from age 60 to 90. In order to illustrate the dependence of the risk measures on κ, we, as in the case of a pure endowment, consider an investment strategy with 40% invested in stocks and bonds, respectively, and 20% in the savings account. Investigating Figures 8.5.6 and 8.5.7 we observe the same behavior for the risk measures as in Figures 8.5.4 and 8.5.5 for the pure endowment. Comparing the relative magnitude of the risk measures for the two different contracts, we observe that for all risk measures the relative size is larger for the annuity than for the pure endowment. This indicates that the annuity is a more risky contract than the pure endowment, which corresponds to our intuition, since the longer time horizon of the annuity exposes the company to more risk.

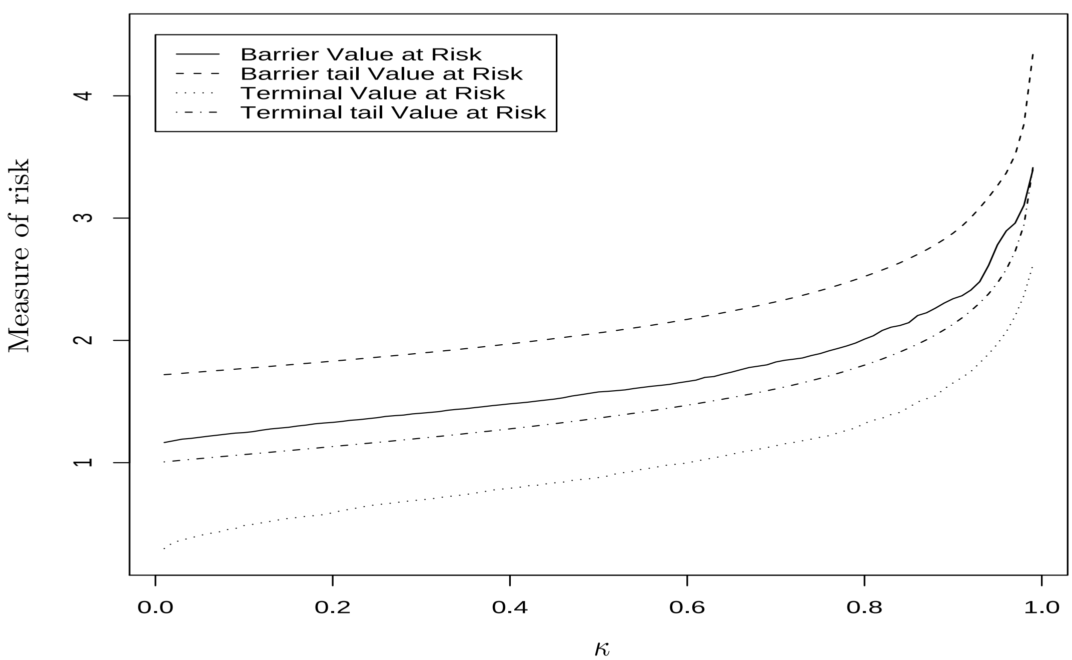

Figure 8.5.6: *Terminal/barrier (tail) Value at Risk for a life annuity as a function of κ.*

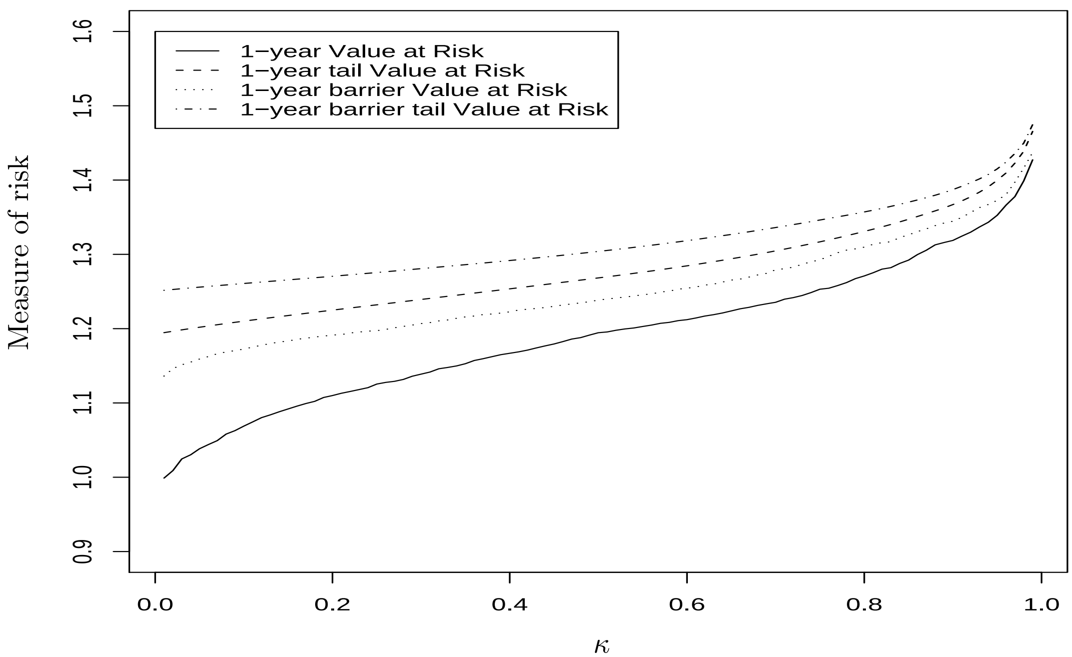

Figure 8.5.7: *1-year terminal/barrier (tail) Value at Risk for a life annuity as a function of κ.*

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