Operator Algebraic Applications in Symbolic Dynamics

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Preface

This work is the result of the research I have carried out as a ph.d.-student at the Department of Mathematics at the University of Copenhagen from the first of May 2001 to the second of April 2004.

The subject of the thesis is application of operator algebra in symbolic dynamics. More specific, the thesis deals with C^*-algebras associated to symbolic dynamical systems and invariants of symbolic dynamical systems of a K-theoretically nature.

The starting point of the research is C^*-algebras associated with one-sided shift spaces, but we will also deal with C^*-algebras associated to two-sided shift spaces, to infinite topological Markov chains, k-graph and higher dimensional shifts on infinite alphabets.

A great effort is put into describing the K-theory of the C^*-algebra associated to a shift space, which is both a conjugacy and a flow invariant. Especially for substitutional dynamical systems, where it is shown that the K-theory contains information not captured by any other know invariant.

The thesis consists of 6 papers and one note. Each paper is precede and succeed by some remarks which place the paper in the context of the subject.

Each page in this thesis is numbered in succession. In addition to this, each paper has its own internal numbering which hopefully makes navigation easier.

Acknowledgment

I wish to thank the operator algebra group in Copenhagen for a friendly and stimulating environment for doing operator algebra in. I especially would like to express my gratitude for having had Søren Eilers as my advisor. It has been a fun and productive collaboration.

I also want to thank the staff at the math department at the University of Maryland, College Park, where I spend 11 month during my time as a ph.d.-student. Especially thanks to Mike Boyle for many stimulating

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Last, but certainly not least, I want to thanks Gert K. Pedersen, wherever he is, for being the personality he was.

Contents

Chapter 1

Cuntz-Pimsner C^*-algebras associated with subshifts

This first chapter consists of the preprint *Cuntz-Pimsner C^*-algebras associated with subshifts* in which a C^*-algebra $\mathcal{O}_{\mathsf{X}}$ is associated to every one-sided shift space (which is called a subshift in the paper).

Cuntz-Pimsner C^*-algebras associated with subshifts

Toke Meier Carlsen

Abstract

By using C^*-correspondences and Cuntz-Pimsner algebras, we associate to every subshift X a C^*-algebra $\mathcal{O}_{\mathsf{X}}$, which is a generalization of the Cuntz-Krieger algebra. We show that $\mathcal{O}_{\mathsf{X}}$ is the universal C^*-algebra generated by partial isometries satisfying relations given by X. We also show that $\mathcal{O}_{\mathsf{X}}$ is a conjugacy invariant of X.

KEYWORDS: C^*-algebras, subshifts, shift spaces, conjugacy, Cuntz-Krieger algebras, Cuntz-Pimsner algebras.
2000 MATHEMATICS SUBJECT CLASSIFICATION: Primary: 46L55, Secondary: 37B10.

1 Introduction

In [3] Cunzt and Krieger introduced a new class of C^*-algebras which in a natural way can be viewed as universal C^*-algebras associated with subshifts (also called shift spaces) of finite type. Frome the point of view of operator algebra these C^*-algebras were important examples of C^*-algebras with new properties and from the point of view of topological dynamics these C^*-algebras (or rather, the K-theory of these C^*-algebras) gave new invariants of subshifts of finite type.

In [12] Matsumoto tried to generalize this idea by constructing C^*-algebras associated with every subshift and he studied them in [7–11]. Unfortunately there is a mistake in [10] which makes many of the results in [7–11] wrong. This mistake has to do with the identification of an underlying compact space which among other things determine the K-theory of the C^*-algebras. It turned out that this compact space is not the space Matsumoto thought it was, and thus many of the results of [7–11] are wrong. To recover these results Matsumoto and the author introduced in [1] a new class of C^*-algebras associated with subshifts, which has the right underlying compact space and thus satisfies most of the results in [7–12], but these C^*-algebra don't have the universal property. Thus one could think of them as the reduced C^*-algebras associated with subshifts.

In this paper we will construct a new C^*-algebra $\mathcal{O}_\mathsf{X}$ by using C^*-correspondences (also called Hilbert bimodules) and Cunzt-Pimsner algebras, and this new C^*-algebra will both have the right underlying compact space and have the universal property and hence will satisfy all the results of [7–12] and has the C^*-algebra defined in [1] s a quotient. Thus is seems right to think of this C^*-algebra as the universal C^*-algebra associated to a subshift.

Matsumoto's original construction associated a C^*-algebra to every *two-sided* subshift, but it seems more natural to work with *one-sided* subshifts, so we will do that in this paper. We will show that $\mathcal{O}_\mathsf{X}$ is the universal C^*-algebra associated with partial isometries satisfying relations giving by X and which resemble the Cuntz-Krieger relations (Theorem 7.2). We will also show that $\mathcal{O}_\mathsf{X}$ is an invariant of X in the sense that if X and Y are conjugate one-sided subshifts, then $\mathcal{O}_\mathsf{X}$ and $\mathcal{O}_\mathsf{Y}$ are isomorphic (Theorem 8.5). This is a generalization of [12, Proposition 5.8] (see [9, Lemma 4.5] for a proof), where it is required that X and Y satisfy a certain condition (I).

2 Notation

Let $\mathfrak{a}$ be a finite set endowed with the discrete topology. We will call this set the alphabet. Let $\mathfrak{a}^{\mathbb{N}_0}$ be the infinite product spaces $\prod_{n=0}^\infty \mathfrak{a}$ endowed with the product topology. The transformation σ on $\mathfrak{a}^{\mathbb{N}_0}$ given by $(\sigma(x))_i = x_{i+1},\ i \in \mathbb{N}_0$ is called the shift. Let X be a shift invariant closed subset of $\mathfrak{a}^{\mathbb{N}_0}$ (by shift invariant we mean that $\sigma(\mathsf{X}) \subseteq \mathsf{X}$, not necessarily $\sigma(\mathsf{X}) = \mathsf{X}$). The topological dynamical system $(\mathsf{X}, \sigma_{|\mathsf{X}})$ is called a *subshift*. We will denote $\sigma_{|\mathsf{X}}$ by σ_X or σ for simplicity, and on occasion the alphabet $\mathfrak{a}$ by $\mathfrak{a}_\mathsf{X}$

A finite sequence $\mu = (\mu_1, \dots, \mu_k)$ of elements $\mu_i \in \mathfrak{a}$ is called a finite word. The length of μ is k and is denoted by $|\mu|$. We let for each $k \in \mathbb{N}_0$, $\mathfrak{a}^k$ be the set of all words with length k and we let $\mathsf{L}^k(\mathsf{X})$ be the set of all words with length k appearing in some $x \in \mathsf{X}$. We set $\mathsf{L}_l(\mathsf{X}) = \bigcup_{k=0}^l \mathsf{L}^k(\mathsf{X})$ and $\mathsf{L}(\mathsf{X}) = \bigcup_{k=0}^\infty \mathsf{L}^k(\mathsf{X})$ and likewise $\mathfrak{a}_l = \bigcup_{k=0}^l \mathfrak{a}^k$ and $\mathfrak{a}^* = \bigcup_{k=0}^\infty \mathfrak{a}^k$, where $\mathsf{L}^0(\mathsf{X}) = \mathfrak{a}^0$ denote the set consisting of the empty word ϵ. $\mathsf{L}(\mathsf{X})$ is called the *language* of X. Note that $\mathsf{L}(\mathsf{X}) \subseteq \mathfrak{a}^*$ for every subshift.

For a subshift X and a word $\mu \in \mathsf{L}(\mathsf{X})$ we denote by $C_\mathsf{X}(\mu)$ the *cylinder* *set*

$$C_{\mathsf{X}}(\mu) = \{x \in \mathsf{X} \mid (x_1, x_2, \ldots, x_{|\mu|}) = \mu\}.$$

It is easy to see that

$$\{C_{\mathsf{X}}(\mu) \mid \mu \in \mathsf{L}(\mathsf{X})\}$$

is a basis for the topology of X, and that $C_{\mathsf{X}}(\mu)$ is closed and hence compact for every $\mu \in \mathsf{L}(\mathsf{X})$. We will allow us self to write $C(\mu)$ instead of $C_{\mathsf{X}}(\mu)$ when it is clear which subshift space we are working with.

For a subshift X and words $\mu, \nu \in \mathsf{L}(\mathsf{X})$ we denote by $C(\mu, \nu)$ the set

$$C(\nu) \cap \sigma^{-|\nu|}(\sigma^{|\mu|}(C(\mu))) = \{\nu x \in \mathsf{X} \mid \mu x \in \mathsf{X}\}.$$

If X and Y are two subshifts and $\phi : \mathsf{X} \to \mathsf{Y}$ is a homeomorphism such that $\psi \circ \sigma_{\mathsf{X}} = \sigma_{\mathsf{Y}} \circ \phi$, then we say that ϕ is a *conjugacy* and that X and Y are *conjugate*.

3 Cunzt-Pimsner algebras

We will in this section give a short introduction to Cuntz-Pimsner algebras. We will follow the universal approach of [4] (see also [13] and [6]).

Let $\mathcal{A}$ be a C^*-algebra. A right *Hilbert $\mathcal{A}$-module* H is a Banach space with a right action of the C^*-algebra $\mathcal{A}$ and an $\mathcal{A}$-valued inner product $\langle \cdot, \cdot \rangle$ satisfying

1. $\langle \xi, \eta a \rangle = \langle \xi, \eta \rangle a$,
2. $\langle \xi, \eta \rangle = \langle \eta, \xi \rangle^*$,
3. $\langle \xi, \xi \rangle \geq 0$ and $\|\xi\| = \|\langle \xi, \xi \rangle\|^{1/2}$,

for $\xi, \eta \in \mathsf{H}$ and $a \in \mathcal{A}$.

For a Hilbert $\mathcal{A}$-module H, we denote by $\mathcal{L}(\mathsf{H})$ the C^*-algebra of all adjointable operators on H. For $\xi, \eta \in \mathsf{H}$, the operator $\theta_{\xi,\eta} \in \mathcal{L}(\mathsf{H})$ is defined by $\theta_{\xi,\eta}(\zeta) = \xi \langle \eta, \zeta \rangle$ for $\zeta \in \mathsf{H}$. We define $\mathcal{K}(\mathsf{H}) \subseteq \mathcal{L}(\mathsf{H})$ by

$$\mathcal{K}(\mathsf{H}) = \overline{\operatorname{span}}\{\theta_{\xi,\eta} \mid \xi, \eta \in \mathsf{H}\},$$

where $\overline{\operatorname{span}}\{\cdots\}$ means the closure of the linear span of $\{\cdots\}$.

Let $\phi : \mathcal{A} \to \mathcal{L}(\mathsf{H})$ be a $*$-homomorphism. Then $ax := \phi(a)x$ defines a left action of $\mathcal{A}$ on H, and we call H a *C^*-correspondence* over $\mathcal{A}$ (in [13] and [4] a C^*-correspondence is called a Hilbert bimodule, but it seems that the term C^*-correspondence has become the preferable).

A *Toeplitz representation* (ψ, π) of H in a C^*-algebra B consists of a linear map $\psi : \mathsf{H} \to B$ and a $*$-homomorphism $\pi : \mathcal{A} \to B$ such that

$$\psi(\xi a) = \psi(\xi)\pi(a),\ \ \psi(\xi)^*\psi(\eta) = \pi(\langle \xi, \eta \rangle),\ \text{and}\ \psi(a\xi) = \pi(a)\psi(\xi)$$

for $\xi, \eta \in \mathsf{H}$ and $a \in \mathcal{A}$. Given such a representation, there is a homomorphism $\pi^{(1)} : \mathcal{K}(\mathsf{H}) \to B$ which satisfies

$$\pi^{(1)}(\theta_{\xi,\eta}) = \psi(\xi)\psi(\eta)^*$$

for all $\xi, \eta \in \mathsf{H}$, and we then have

$$\pi^{(1)}(T)\psi(\xi) = \psi(T\xi)$$

for every $T \in \mathcal{K}(\mathsf{H})$ and $\xi \in \mathsf{H}$. If $\rho : B \to C$ is a $*$-homomorphism between C^*-algebras, then $(\rho \circ \psi, \rho \circ \pi)$ is a Toeplitz representation of H, and since

$$(\rho \circ \pi)^{(1)}(\theta_{\xi,\eta}) = (\rho \circ \psi(\xi))(\rho \circ \psi(\eta))^* = \rho \circ \pi^{(1)}(\theta_{\xi,\eta})$$

for all $\xi, \eta \in \mathsf{H}$, by linearity and continuity we have

$$(\rho \circ \pi)^{(1)} = \rho \circ \pi^{(1)}.$$

We denote by $\mathcal{J}(\mathsf{H})$ the closed two-sided ideal $\phi^{-1}(\mathcal{K}(\mathsf{H}))$ in $\mathcal{A}$, and we say that a Toeplitz representation (ψ, π) of H is *Cuntz-Pimsner coinvariant* if

$$\pi^{(1)}(\phi(a)) = \pi(a)$$

for all $a \in \mathcal{J}(\mathsf{H})$.

Theorem 3.1. *Let H be a C^*-correspondence over $\mathcal{A}$. Then there is a C^*-algebra $\mathcal{O}_{\mathsf{H}}$ and a Cuntz-Pimsner coinvariant Toeplitz representation $(k_{\mathsf{H}}, k_{\mathcal{A}}) : \mathsf{H} \to \mathcal{O}_{\mathsf{H}}$ which satisfies:*

1. *For every Cuntz-Pimsner coinvariant Toeplitz representation (ψ, π) of H, there is a homomorphism $\psi \times \pi$ of $\mathcal{O}_{\mathsf{H}}$ such that $(\psi \times \pi) \circ k_{\mathsf{H}} = \psi$ and $(\psi \times \pi) \circ k_{\mathcal{A}} = \pi$,*

2. *$\mathcal{O}_{\mathsf{H}}$ is generated as a C^*-algebra by $k_{\mathsf{H}}(\mathsf{H}) \cup k_{\mathcal{A}}(\mathcal{A})$.*

Remark 3.2. The triple $(\mathcal{O}_{\mathsf{H}}, k_x, k_{\mathcal{A}})$ is unique: if $(\mathcal{X}, k'_{\mathsf{H}}, k'_{\mathcal{A}})$ has similar properties, then there is an isomorphism $\theta : \mathcal{O}_{\mathsf{H}} \to \mathcal{X}$ such that $\theta \circ k_{\mathsf{H}} = k'_{\mathsf{H}}$ and $\theta \circ k_{\mathcal{A}} = k'_{\mathcal{A}}$. Thus there is a strongly continuous gauge action $\gamma : \mathbb{T} \to \operatorname{Aut} \mathcal{O}_{\mathsf{H}}$ which satisfies $\gamma_z(k_{\mathcal{A}}(a)) = k_{\mathcal{A}}(a)$ and $\gamma_z(k_{\mathsf{H}}(x)) = zk_{\mathsf{H}}(x)$ for $a \in \mathcal{A}$ and $x \in \mathsf{H}$.

4 C^*-correspondences associated with subshifts

We will now define the C^*-correspondence H_X that we associate to a subshift X.

We start by defining the C^*-algebra which H_X is a C^*-correspondence over.

Definition 4.1. For every subshift X we let $\mathfrak{B}(\mathsf{X})$ be the abelian C^*-algebra of all bounded functions on X, and $\widetilde{\mathcal{D}}_\mathsf{X}$ the C^*-subalgebra of $\mathfrak{B}(\mathsf{X})$ generated by $\{1_{C(\mu,\nu)} \mid \mu,\nu \in \mathfrak{a}^*\}$.

It turns out that the spectrum of $\widetilde{\mathcal{D}}_\mathsf{X}$ is the right underlying compact space for the C^*-algebra that we are going to associate with subshifts, but since we will not need an explicit description of this compact space we are not going to give one, but instead work with $\widetilde{\mathcal{D}}_\mathsf{X}$.

Definition 4.2. Let X be a subshift. For every $a \in \mathfrak{a}$ let $\widetilde{\mathcal{D}}_a$ be the ideal in $\widetilde{\mathcal{D}}_\mathsf{X}$ generated by $1_{\sigma(C(a))}$. Let H_X be the right Hilbert $\widetilde{\mathcal{D}}_\mathsf{X}$-module

$$\oplus_{a\in\mathfrak{a}}\widetilde{\mathcal{D}}_a$$

with the right action is given by $(f_a)_{a\in\mathfrak{a}}f = (f_af)_{a\in\mathfrak{a}}$ and the inner product by $\langle (f_a)_{a\in\mathfrak{a}}, (g_a)_{a\in\mathfrak{a}}\rangle = \sum_{a\in\mathfrak{a}} f_a^* g_a$ for $(f_a)_{a\in\mathfrak{a}}, (g_a)_{a\in\mathfrak{a}} \in \oplus_{a\in\mathfrak{a}}\widetilde{\mathcal{D}}_a$ and $f \in \widetilde{\mathcal{D}}_\mathsf{X}$.

Proposition 4.3. *Let X be a subshift and let $a \in \mathfrak{a}$. Define a $*$-homomorphism $\widetilde{\lambda}_a : \mathfrak{B}(\mathsf{X}) \to \mathfrak{B}(\mathsf{X})$ by letting*

$$\widetilde{\lambda}_a(f)(x) = \begin{cases} f(ax) & \text{if } ax \in \mathsf{X} \\ 0 & \text{if } ax \notin \mathsf{X} \end{cases}$$

for every $f \in \mathfrak{B}(\mathsf{X})$ and every $x \in \mathsf{X}$.

Then $\widetilde{\lambda}_a(\widetilde{\mathcal{D}}_\mathsf{X}) \subseteq \widetilde{\mathcal{D}}_a$.

Proof. Let $\mu,\nu \in \mathfrak{a}^*$ with $|\nu| \geq 1$. For every $x \in \mathsf{X}$ is

$$\begin{aligned}\widetilde{\lambda}_a\left(1_{C(\mu,\nu)}\right)(x) &= \begin{cases} 1_{C(\mu,\nu)}(ax) & \text{if } ax \in \mathsf{X} \\ 0 & \text{if } ax \notin \mathsf{X} \end{cases} \\ &= \begin{cases} 1 & \text{if } a = \nu_1,\ x_1 = \nu_2,\ \dots, x_{|\nu|-1} = \nu_{|\nu|},\ \mu\sigma^{|\nu|-1}(x),\ ax \in \mathsf{X} \\ 0 & \text{else.} \end{cases}\end{aligned}$$

So $\widetilde{\lambda}_a\left(1_{C(\mu,\nu)}\right) = 0$ if $a \neq \nu_1$, and

$$\widetilde{\lambda}_a\left(1_{C(\mu,\nu)}\right) = 1_{C(\mu,\nu_2\nu_3\cdots\nu_{|\nu|})}1_{\sigma(C(a))}$$

if $a = \nu_1$. Hence $\widetilde{\lambda}_a\left(1_{C(\nu,\mu)}\right) \in \widetilde{D}_a$. In a similar way, we see that $\widetilde{\lambda}_a\left(1_{C(\mu,\epsilon)}\right) = 1_{C(a\mu,\epsilon)}$, so $\widetilde{\lambda}_a\left(1_{C(\nu,\epsilon)}\right) \in \widetilde{D}_a$. Thus $\widetilde{\lambda}_a(\widetilde{\mathcal{D}}_{\mathsf{X}}) \subseteq \widetilde{\mathcal{D}}_a$, since $\widetilde{D}_{\mathsf{X}}$ is generated by $\{1_{C(\mu,\nu)} \mid \mu,\nu \in \mathfrak{a}^*\}$. □

Definition 4.4. Let X be a subshift. We let $\phi : \widetilde{\mathcal{D}}_{\mathsf{X}} \to \mathcal{L}(\mathsf{H}_{\mathsf{X}})$ be the $*$-homomorphism defined by

$$\phi(f)((f_a)_{a\in\mathfrak{a}}) = (\widetilde{\lambda}_a(f)f_a)_{a\in\mathfrak{a}}$$

for every $f \in \widetilde{\mathcal{D}}_{\mathsf{X}}$ and every $(f_a)_{a\in\mathfrak{a}} \in \mathsf{H}_{\mathsf{X}}$. With this H_{X} becomes a C^*-correspondence.

5 The C^*-algebra associated with a subshift

We are now ready to define the C^*-algebra $\mathcal{O}_{\mathsf{X}}$ associated with a subshift X.

Definition 5.1. Let X be a subshift. The C^*-algebra $\mathcal{O}_{\mathsf{X}}$ associated with X is the C^*-algebra $\mathcal{O}_{\mathsf{H}_{\mathsf{X}}}$ from Theorem 3.1, where H_{X} is the C^*-correspondence defined above.

We will now take a closer look at $\mathcal{O}_{\mathsf{X}}$. First, we show that $\mathcal{O}_{\mathsf{X}}$ is unital.

Lemma 5.2. *Let X be a subshift and let 1 be the unit of $\widetilde{D}_{\mathsf{X}}$. Then $k_{\widetilde{\mathcal{D}}_{\mathsf{X}}}(1)$ is a unit for $\mathcal{O}_{\mathsf{X}}$.*

Proof. We have that

$$k_{\mathsf{H}_{\mathsf{X}}}(\xi)k_{\widetilde{\mathcal{D}}_{\mathsf{X}}}(1) = k_{\mathsf{H}_{\mathsf{X}}}(\xi 1) = k_{\mathsf{H}_{\mathsf{X}}}(\xi),$$

and

$$k_{\widetilde{\mathcal{D}}_{\mathsf{X}}}(1)k_{\mathsf{H}_{\mathsf{X}}}(\xi) = k_{\mathsf{H}_{\mathsf{X}}}(\phi(1)\xi) = k_{\mathsf{H}_{\mathsf{X}}}(xi)$$

for every $\xi \in \mathsf{H}_{\mathsf{X}}$. Since we also have that

$$k_{\widetilde{\mathcal{D}}_{\mathsf{X}}}(1)k_{\widetilde{\mathcal{D}}_{\mathsf{X}}}(f) = k_{\widetilde{\mathcal{D}}_{\mathsf{X}}}(f)k_{\widetilde{\mathcal{D}}_{\mathsf{X}}}(1) = k_{\widetilde{\mathcal{D}}_{\mathsf{X}}}(f)$$

for every $f \in \widetilde{\mathcal{D}}_{\mathsf{X}}$, and $\mathcal{O}_{\mathsf{X}}$ is generated by $k_{\mathsf{H}_{\mathsf{X}}}(\mathsf{H}_{\mathsf{X}}) \cup k_{\widetilde{\mathcal{D}}_{\mathsf{X}}}(\widetilde{\mathcal{D}}_{\mathsf{X}})$, we have that $k_{\widetilde{\mathcal{D}}_{\mathsf{X}}}(1)$ is a unit for $\mathcal{O}_{\mathsf{X}}$. □

We will denote the unit of $\mathcal{O}_{\mathsf{X}}$ by I.

Definition 5.3. Let X be a shift. For every $a \in \mathfrak{a}$ let ξ_a be the element $(f_{a'})_{a'\in\mathfrak{a}} \in \mathsf{H}_{\mathsf{X}}$ where $f_a = 1_{\sigma(C(a))}$ and $f_{a'} = 0$ for $a' \neq a$, and let for every $\mu \in \mathfrak{a}^*$, S_μ be the product $k_{\mathsf{H}_{\mathsf{X}}}(\xi_{\mu_1})k_{\mathsf{H}_{\mathsf{X}}}(\xi_{\mu_2})\cdots k_{\mathsf{H}_{\mathsf{X}}}(\xi_{\mu_{|\mu|}}) \in \mathcal{O}_{\mathsf{X}}$ with the convention that $S_\epsilon = I$.

Lemma 5.4. *Let* X *be a subshift. Let* 1 *be the unit of* $\widetilde{D}_{\mathsf{X}}$*, and* Id *the unit of* $\mathcal{L}(\mathsf{H}'_{\mathsf{X}})$*. Then*

$$k_{\widetilde{\mathcal{D}}_{\mathsf{X}}}(1) = k^{(1)}_{\widetilde{\mathcal{D}}_{\mathsf{X}}}(\mathrm{Id}) = \sum_{a\in\mathfrak{a}} S_a S_a^*$$

is the unit of $\mathcal{O}_{\mathsf{X}}$*.*

Proof. It is easy to check that

$$\phi(1) = \mathrm{Id} = \sum_{a\in\mathfrak{a}} \theta_{\xi_a,\xi_a^*},$$

so since $(k_{\mathsf{H}_{\mathsf{X}}}, k_{\widetilde{\mathcal{D}}_{\mathsf{X}}})$ is a Cuntz-Pimsner coinvariant representation,

$$k_{\widetilde{\mathcal{D}}_{\mathsf{X}}}(1) = k^{(1)}_{\widetilde{\mathcal{D}}_{\mathsf{X}}}(\mathrm{Id}) = \sum_{a\in\mathfrak{a}} S_a S_a^*,$$

and we know from Lemma 5.2 that $k_{\widetilde{\mathcal{D}}_{\mathsf{X}}}(1)$ is the unit of $\mathcal{O}_{\mathsf{X}}$. □

Lemma 5.5. *Let* X *be a shift. Then*

$$k_{\widetilde{\mathcal{D}}_{\mathsf{X}}}\left(1_{C(\mu,\nu)}\right) = S_\nu S_\mu^* S_\mu S_\nu^*$$

for every $\mu, \nu \in \mathfrak{a}^*$*.*

Proof. Since

$$\begin{aligned} S_a^* S_a &= k_{\mathsf{H}_{\mathsf{X}}}(\xi_a)^* k_{\mathsf{H}_{\mathsf{X}}}(\xi_a) \\ &= k_{\widetilde{\mathcal{D}}_{\mathsf{X}}}(\langle \xi_a, \xi_a\rangle) \\ &= k_{\widetilde{\mathcal{D}}_{\mathsf{X}}}\left(1_{\sigma(C(a))}\right), \end{aligned}$$

and

$$\begin{aligned} S_a^* k_{\widetilde{\mathcal{D}}_{\mathsf{X}}}\left(1_{\sigma^{|\mu'|}(C(\mu'))}\right) S_a &= k_{\mathsf{H}_{\mathsf{X}}}(\xi_a)^* k_{\mathsf{H}_{\mathsf{X}}}\left(\phi'\left(1_{\sigma^{|\mu'|}(C(\mu'))}\right)\xi_a\right) \\ &= k_{\mathsf{H}_{\mathsf{X}}}(\xi_a)^* k_{\mathsf{H}_{\mathsf{X}}}\left(\xi_a \widetilde{\lambda}_a\left(1_{\sigma^{|\mu'|}(C(\mu'))}\right)\right) \\ &= k_{\mathsf{H}_{\mathsf{X}}}(\xi_a)^* k_{\mathsf{H}_{\mathsf{X}}}(\xi_a) k_{\widetilde{\mathcal{D}}_{\mathsf{X}}}\left(\widetilde{\lambda}_a\left(1_{\sigma^{|\mu'|}(C(\mu'))}\right)\right) \\ &= k_{\widetilde{\mathcal{D}}_{\mathsf{X}}}\left(1_{\sigma(C(a))}\widetilde{\lambda}_a\left(1_{\sigma^{|\mu'|}(C(\mu'))}\right)\right) \\ &= k_{\widetilde{\mathcal{D}}_{\mathsf{X}}}\left(1_{\sigma^{|\mu' a|}(C(\mu' a))}\right) \end{aligned}$$

for every $a \in \mathfrak{a}$ and every $\mu' \in \mathfrak{a}^*$, we have that

$$S_\mu^* S_\mu = k_{\widetilde{\mathcal{D}}_{\mathsf{X}}}\left(1_{\sigma^{|\mu|}(C(\mu))}\right)$$

for every $\mu \in \mathfrak{a}^*$.

It is easy to check that for every $f \in \widetilde{\mathcal{D}}_{\mathsf{X}}$ is

$$\phi(f) = \sum_{a \in \mathfrak{a}} \theta_{\xi_a \widetilde{\lambda}_a(f), \xi_a^*}.$$

Let $\mu, \nu \in \mathfrak{a}^*$ with $|\nu| \geq 1$ and $a \in \mathfrak{a}$. Then as proved in the proof of Proposition 4.3

$$\widetilde{\lambda}_a \left(1_{(C(\mu,\nu))}\right) = 0$$

if $a \neq \nu_1$, and

$$\widetilde{\lambda}_a \left(1_{C(\mu,\nu)}\right) = 1_{C(\mu,\nu_2,\ldots\nu_{|\nu|})} 1_{\sigma(C(a))}$$

if $a = \nu_1$.

So

$$\begin{aligned} k_{\widetilde{\mathcal{D}}_{\mathsf{X}}}\left(1_{C(\mu,\nu)}\right) &= k^{(1)}_{\widetilde{\mathcal{D}}_{\mathsf{X}}}\left(\phi\left(1_{C(\mu,\nu)}\right)\right) \\ &= k^{(1)}_{\widetilde{\mathcal{D}}_{\mathsf{X}}}\left(\theta_{\xi_{\nu_1} 1_{C(\mu,\nu_2,\ldots\nu_{|\nu|})}, \xi_{\nu_1}}\right) \\ &= k_{\mathsf{H}_{\mathsf{X}}}(\xi_{\nu_1} 1_{C(\mu,\nu_2,\ldots\nu_{|\nu|})}) k_{\mathsf{H}_{\mathsf{X}}}(\xi_{\nu_1})^* \\ &= S_{\nu_1} k_{\widetilde{\mathcal{D}}_{\mathsf{X}}}(1_{C(\mu,\nu_2,\ldots\nu_{|\nu|})}) S_{\nu_1}^*. \end{aligned}$$

Hence

$$k_{\widetilde{\mathcal{D}}_{\mathsf{X}}}\left(1_{C(\mu,\nu)}\right) = S_\nu S_\mu^* S_\mu S_\nu^*$$

for all $\mu, \nu \in \mathfrak{a}^*$. □

Proposition 5.6. *Let X be a subshift. Then $\mathcal{O}_{\mathsf{X}}$ is generated by $\{S_a\}_{a \in \mathfrak{a}}$.*

Proof. $\mathcal{O}_{\mathsf{X}}$ is by Theorem 3.1 generated by $k_{\mathsf{H}_{\mathsf{X}}}(\mathsf{H}_{\mathsf{X}}) \cup k_{\widetilde{\mathcal{D}}_{\mathsf{X}}}(\widetilde{\mathcal{D}}_{\mathsf{X}})$.

First notice that $k_{\widetilde{\mathcal{D}}_{\mathsf{X}}}(1) = \sum_{a \in \mathfrak{a}} S_a S_a^*$ is in the C^*-algebra generated by $\{S_a\}_{a \in \mathfrak{a}}$. Since

$$k_{\widetilde{\mathcal{D}}_{\mathsf{X}}}\left(1_{C(\mu,\nu)}\right) = S_\nu S_\mu^* S_\mu S_\nu^*$$

for all $\mu, \nu \in \mathfrak{a}^*$, and $\widetilde{\mathcal{D}}_{\mathsf{X}}$ is generated by $\{1_{C(\mu,\nu)} \mid \mu, \nu \in \mathfrak{a}^*\}$, we have that $k_{\widetilde{\mathcal{D}}_{\mathsf{X}}}(\widetilde{\mathcal{D}}_{\mathsf{X}})$ is in the C^*-algebra generated by $\{S_a\}_{a \in \mathfrak{a}}$.

Let $(f_a)_{a \in \mathfrak{a}} \in \mathsf{H}_{\mathsf{X}}$. Then

$$(f_a)_{a \in \mathfrak{a}} = \sum_{a \in \mathfrak{a}} \xi_a f_a,$$

so $k_{\mathsf{H}_{\mathsf{X}}}((f_a)_{a \in \mathfrak{a}}) = \sum_{a \in \mathfrak{a}} S_a k_{\widetilde{\mathcal{D}}_{\mathsf{X}}}(f_a)$, and $k_{\mathsf{H}_{\mathsf{X}}}((f_a)_{a \in \mathfrak{a}})$ is in the C^*-algebra generated by $\{S_a\}_{a \in \mathfrak{a}}$. Hence $\mathcal{O}_{\mathsf{X}}$ is generated by $\{S_a\}_{a \in \mathfrak{a}}$. □

6 The structure of C^*-algebras generated by partial isometries

We have now established that $\mathcal{O}_{\mathsf{X}}$ is a unital C^*-algebras generated by partial isometries $\{S_a\}_{a\in\mathfrak{a}}$, which by Lemma 5.4 and 5.5 satisfy

$$\begin{aligned}\sum_{a\in\mathfrak{a}} S_aS_a^* &= I,\\ S_\mu^*S_\mu S_\nu S_\nu^* &= S_\nu S_\nu^* S_\mu^* S_\mu,\\ S_\mu^*S_\mu S_\nu^* S_\nu &= S_\nu^* S_\nu S_\mu^* S_\mu,\end{aligned}$$

where $S_\mu = S_{\mu_1}\cdots S_{\mu_{|\mu|}}$ and $S_\nu = S_{\nu_1}\cdots S_{\nu_{|\nu|}}$, for every $\mu,\nu\in\mathfrak{a}^*$.

We will now take a closer look at unital C^*-algebras generated by partial isometries $\{S_a\}_{a\in\mathfrak{a}}$ that satisfy the 3 relations above.

Let $\mathfrak{a}$ be an alphabet. In the following we let $\mathcal{O}$ be a unital C^*-algebra generated by partial isometries $\{S_a\}_{a\in\mathfrak{a}}$, such that

$$\sum_{a\in\mathfrak{a}} S_aS_a^* = I, \tag{1}$$
$$S_\mu^*S_\mu S_\nu S_\nu^* = S_\nu S_\nu^* S_\mu^* S_\mu, \tag{2}$$
$$S_\mu^*S_\mu S_\nu^* S_\nu = S_\nu^* S_\nu S_\mu^* S_\mu, \tag{3}$$

where $S_\mu = S_{\mu_1}\cdots S_{\mu_{|\mu|}}$ and $S_\nu = S_{\nu_1}\cdots S_{\nu_{|\nu|}}$, for every $\mu,\nu\in\mathfrak{a}^*$.

Lemma 6.1. *For every $\mu\in\mathfrak{a}^*$, S_μ is a partial isometry.*

Proof. We will prove the lemma by induction over the length of $|\mu|$. If $|\mu|=1$, then S_μ is a partial isometry by definition. Assume now that S_ν is a partial isometry and $a\in\mathfrak{a}$. Then

$$\begin{aligned} S_{\nu a}S_{\nu a}^*S_{\nu a} &= S_\nu S_a S_a^* S_\nu^* S_\nu S_a\\ &= S_\nu S_\nu^* S_\nu S_a S_a^* S_a\\ &= S_\nu S_a\\ &= S_{\nu a}.\end{aligned}$$

So $S_{\nu a}$ is a partial isometry. Hence S_μ is a partial isometry for every $\mu\in\mathfrak{a}^*$. ∎

For $\mu\in\mathfrak{a}^*$ we set $A_\mu = S_\mu^*S_\mu$. We notice that since $\sum_{a\in\mathfrak{a}} S_aS_a^* = I$, the projections $\{S_aS_a^*\}_{a\in\mathfrak{a}}$ are mutually orthogonal, so $S_aS_a^*S_bS_b^* = 0$ for $a\neq b$.

Lemma 6.2. *Let $\mu,\ \nu \in \mathfrak{a}^*$ with $|\mu| = |\nu|$. If $S_\mu^* S_\nu \neq 0$, then $\mu = \nu$ and $S_\mu^* S_\nu = A_\mu$.*

Proof. We will prove the lemma by induction over the length of μ and ν. If the length is 1 and $\mu \neq \nu$, then

$$\begin{aligned} S_\mu^* S_\nu &= S_\mu^* S_\mu S_\mu^* S_\nu S_\nu^* S_\nu \\ &= 0 \end{aligned}$$

since $S_\mu S_\mu^* S_\nu S_\nu^* = 0$. So since $S_\mu^* S_\nu \neq 0$, we have that $\mu = \nu$ and $S_\mu^* S_\nu = A_\mu$.

Now assume that we have proved the lemma in case $|\mu| = |\nu| = n$, and assume that $|\mu'| = |\nu'| = n + 1$ and $S_{\mu'}^* S_{\nu'} \neq 0$. Set $\mu = (\mu'_1, \dots, \mu'_n)$ and $\nu = (\nu'_1, \dots, \nu'_n)$. Then $S_\mu^* S_\nu \neq 0$, so $\mu = \nu$. Since

$$\begin{aligned} 0 &\neq S_{\mu'}^* S_{\nu'} \\ &= S_{\mu'_{n+1}}^* S_\mu^* S_\nu S_{\nu'_{n+1}} \\ &= S_{\mu'_{n+1}}^* S_{\mu'_{n+1}} S_{\mu'_{n+1}}^* S_\mu^* S_\mu S_{\nu'_{n+1}} S_{\nu'_{n+1}}^* S_{\nu'_{n+1}} \\ &= S_{\mu'_{n+1}}^* S_{\mu'_{n+1}} S_{\mu'_{n+1}}^* S_{\nu'_{n+1}} S_{\nu'_{n+1}}^* S_\mu^* S_\mu S_{\nu'_{n+1}}, \end{aligned}$$

we have that $\mu'_{n+1} = \nu'_{n+1}$, and hence $\mu' = \nu'$. So the lemma is true. □

For each $l \in \mathbb{N}_0$ we denote by $\mathcal{A}_l(\mathcal{O})$ the C^*-subalgebra of $\mathcal{O}$ generated by $\{A_\mu\}_{\mu \in \mathfrak{a}_l}$. Since $\mathcal{A}_l(\mathcal{O})$ is generated by a finite number of mutually commuting projection, there exist a finite number of mutually orthogonal projections $E_i^l,\ i = 1, \dots m(l)$, such that $(E_i^l)_{i=1,\dots m(l)}$ is a basis for $\mathcal{A}_l(\mathcal{O})$. We have that $\overline{\bigcup_{l \in \mathbb{N}_0} \mathcal{A}_l(\mathcal{O})}$ is the C^*-algebra generated by $\{A_\mu\}_{\mu \in \mathfrak{a}^*}$. We denoted this C^*-algebra by $\mathcal{A}(\mathcal{O})$. Since $\mathcal{A}_l(\mathcal{O})$ is finite dimensional and $\mathcal{A}_l(\mathcal{O}) \subseteq \mathcal{A}_{l+1}(\mathcal{O})$ for every $l \in \mathbb{N}_0$, $\mathcal{A}(\mathcal{O})$ is an AF-algebra.

Lemma 6.3. *For $1 \leq k \leq l,\ \mu \in \mathfrak{a}^k$ and $i \in \{1, 2, \dots, m(l)\}$, the following two conditions are equivalent:*

a) $S_\mu E_i^l S_\mu^* \neq 0$,

b) $A_\mu E_i^l \neq 0$.

Proof. Since

$$S_\mu E_i^l S_\mu^* = S_\mu A_\mu E_i^l S_\mu^*,$$

and

$$A_\mu E_i^l = S_\mu^* S_\mu E_i^l S_\mu^* S_\mu,$$

we have that

$$S_\mu E_i^l S_\mu^* \neq 0 \Leftrightarrow A_\mu E_i^l \neq 0.$$

□

Lemma 6.4. *Let $l \geq k \geq 1$. Then*

a) For $i, i' \in \{1, 2, \dots m(l)\}$ and $\mu, \mu' \in \mathfrak{a}^k$ is

$$S_\mu E_i^l S_\mu^* S_{\mu'} E_{i'}^l S_{\mu'}^* = \begin{cases} S_\mu E_i^l S_\mu^* & \text{if } \mu = \mu' \text{ and } i = i' \\ 0 & \text{if } \mu \neq \mu' \text{ or } i \neq i'. \end{cases}$$

b) $(S_\mu E_i^l S_\mu^*)^* = S_\mu E_i^l S_\mu^*$ for $i \in \{1, 2, \dots, m(l)\}$ and $\mu \in \mathfrak{a}^k$.

Proof. a): By Lemma 6.2

$$\begin{aligned} S_\mu E_i^l S_\mu^* S_{\mu'} E_{i'}^l S_{\mu'}^* &= \begin{cases} S_\mu E_i^l A_\mu E_{i'}^l S_{\mu'}^* & \text{if } \mu = \mu' \\ 0 & \text{if } \mu \neq \mu' \end{cases} \\ &= \begin{cases} S_\mu A_\mu E_i^l E_{i'}^l S_{\mu'}^* & \text{if } \mu = \mu' \\ 0 & \text{if } \mu \neq \mu' \end{cases} \\ &= \begin{cases} S_\mu E_i^l S_{\mu'}^* & \text{if } \mu = \mu' \text{ and } i = i' \\ 0 & \text{if } \mu \neq \mu' \text{ or } i \neq i'. \end{cases} \end{aligned}$$

b): Obviously. □

7 The universal property of $\mathcal{O}_\mathsf{X}$

We let $\widetilde{\mathcal{A}}_\mathsf{X}$ be the C^*-subalgebra of $\widetilde{\mathcal{D}}_\mathsf{X}$ generated by $\{1_{\sigma^{|\mu|}(C(,\mu))} \mid \mu \in \mathfrak{a}^*\}$.

Lemma 7.1. *Let X be a shift, $\mathcal{X}$ a C^*-algebra, $\psi : \widetilde{\mathcal{A}}_\mathsf{X} \to \mathcal{X}$ a $*$-homomorphism and $\{S_a\}_{a\in\mathfrak{a}}$ partial isometries in $\mathcal{X}$ such that*

a) $\sum_{a\in\mathfrak{a}} S_a S_a^* = \psi(1)$,

b) $S_\mu^* S_\mu S_\nu S_\nu^* = S_\nu S_\nu^* S_\mu^* S_\mu$,

c) $S_\mu^* S_\mu = \psi\left(1_{\sigma^{|\mu|}(C(\mu))}\right)$,

where $S_\mu = S_{\mu_1} S_{\mu_2} \cdots S_{\mu_{|\mu|}}$ and $S_\nu = S_{\nu_1} S_{\nu_2} \cdots S_{\nu_{|\nu|}}$, for every $\mu, \nu \in \mathfrak{a}^*$.

Then ψ extends to a $*$-homomorphism from $\widetilde{\mathcal{D}}_\mathsf{X}$ to $\mathcal{X}$, such that

$$\psi\left(1_{C(\mu,\nu)}\right) = S_\nu S_\mu^* S_\mu S_\nu^*$$

for every $\mu, \nu \in \mathfrak{a}^*$.

Proof. Let $\mathcal{O}$ be the C^*-subalgebra of $\mathcal{X}$ generated by $\{S_a\}_{a\in\mathfrak{a}}$. Since $S_\mu^* S_\mu = \psi\left(1_{\sigma^{|\mu|}(C(\mu))}\right)$ and $S_\nu^* S_\nu = \psi\left(1_{\sigma^{|\nu|}(C(\nu))}\right)$, we have that $S_\mu^* S_\mu S_\nu^* S_\nu = S_\nu^* S_\nu S_\mu^* S_\mu$ for every $\mu, \nu \in \mathfrak{a}^*$. Since

$$\begin{aligned} S_a\psi(1) &= S_a S_a^* S_a \psi(1) \\ &= S_a \psi\left(1_{C(a)}\right)\psi(1) \\ &= S_a\psi\left(1_{C(a)}\right) \\ &= S_a S_a^* S_a \\ &= S_a \end{aligned}$$

and

$$\begin{aligned} \psi(1)S_a &= \psi(1)S_a S_a^* S_a \\ &= \sum_{a'\in\mathfrak{a}} S_{a'}S_{a'}^* S_a S_a^* S_a \\ &= S_a S_a^* S_a \\ &= S_a \end{aligned}$$

for every $a \in \mathfrak{a}$, $\psi(1)$ is a unit for $\mathcal{O}$. Hence $\mathcal{O}$ and $\{S_a\}_{a\in\mathfrak{a}}$, satisfy (1), (2) and (3) of section 6.

For each $l \in \mathbb{N}_0$, denote by $\widetilde{\mathcal{A}}_l$ the C^*-subalgebra of $\widetilde{\mathcal{A}}_{\mathsf{X}}$ generated by $\{1_{\sigma^{|\mu|}(C(\mu))} \mid \mu \in \mathfrak{a}_l\}$. Since $\widetilde{\mathcal{A}}_l$ is generated by a finite number of mutually commuting projections, there exists a finite number $m(l)$ of mutually disjoint subsets $\mathcal{E}_i^l,\ i = 1, 2, \ldots, m(l)$ of X such that

$$\left\{1_{\mathcal{E}_i^l} \mid i \in \{1, 2, \ldots, m(l)\}\right\}$$

is a basis for $\widetilde{\mathcal{A}}_l$.

Then $\psi\left(1_{\mathcal{E}_i^l}\right),\ i = 1, 2, \ldots, m(l)$ are mutually orthogonal projections in $\mathcal{O}$ and span$\left\{\psi(1_{\mathcal{E}_i^l}) \mid i \in \{1, 2, \ldots, m(l)\}\right\} = \mathcal{A}_l(\mathcal{O})$. So by Lemma 6.4 we have that for $1 \le k \le l$ are $S_\nu\psi(1_{\mathcal{E}_i^l})S_\nu^*,\ i = 1, 2, \ldots m(l)$ mutually orthogonal projections in $\mathcal{O}$.

For each $1 \le k \le l$ denote by $\widetilde{\mathcal{D}}_k^l$ the C^*-subalgebra of $\widetilde{\mathcal{D}}_{\mathsf{X}}$ generated by $\{1_{C(\mu,\nu)} \mid \nu \in \mathfrak{a}^k, \mu \in \mathfrak{a}_l\}$. It is easy to check that

$$1_{C(\nu)\cap\sigma^{-|\nu|}(\mathcal{E}_i^l)},\ \nu \in \mathfrak{a}^k,\ i = 1, 2, \ldots, m(l)$$

are mutually orthogonal projections in $\widetilde{\mathcal{D}}_k^l$, and since

$$\begin{aligned} 1_{C(\nu)\cap\sigma^{-|\nu|}(\mathcal{E}_i^l)} = 0 \;&\Rightarrow\; 1_{\sigma^{|\nu|}(C(\nu))}1_{\mathcal{E}_i^l} = 0 \\ &\Rightarrow\; S_\nu^* S_\nu \psi(1_{\mathcal{E}_i^l}) = 0 \\ &\Rightarrow\; S_\nu \psi(1_{\mathcal{E}_i^l}) S_\nu^* = 0, \end{aligned}$$

there exists a $*$-homomorphism $\psi_k^l : \widetilde{\mathcal{D}}_k^l \to \mathcal{X}$ such that $\psi_k^l\left(1_{C(\nu)\cap\sigma^{-|\nu|}(\mathcal{E}_i^l)}\right) = S_\nu \psi(1_{\mathcal{E}_i^l}) S_\nu^*$ for every $\nu \in \mathfrak{a}^k$ and every $i \in \{1, 2, \dots, m(l)\}$ and hence $\psi_k^l\left(1_{C(\mu,\nu)}\right) = S_\nu S_\mu^* S_\mu S_\nu^*$ for every $\nu \in \mathfrak{a}^k$ and every $\mu \in \mathfrak{a}_l$.

For every $k \in \mathbb{N}_0$ denote by $\widetilde{\mathcal{D}}_k$ the C^*-subalgebra of $\widetilde{\mathcal{D}}_{\mathsf{X}}$ generated by $\{1_{C(\mu,\nu)} \mid \nu \in \mathfrak{a}^k, \mu \in \mathfrak{a}^*\}$. Then $\widetilde{\mathcal{D}}_k = \overline{\bigcup_{l\geq k} \widetilde{\mathcal{D}}_k^l}$. Let ι_k^l denote the inclusion of $\widetilde{\mathcal{D}}_k^l$ into $\widetilde{\mathcal{D}}_k^{l+1}$. Since $\psi_k^{l+1} \circ \iota_k^l = \psi_k^l$ for every $l \geq k$, the ψ_k^l's induce a $*$-homomorphism $\psi_k : \widetilde{\mathcal{D}}_k \to \mathcal{O}$ such that $\psi_k\left(1_{C(\mu,\nu)}\right) = S_\nu S_\mu^* S_\mu S_\nu^*$ for every $\nu \in \mathfrak{a}^k$ and every $\mu \in \mathfrak{a}^*$.

Since

$$C(\mu,\nu) = \bigcup_{a\in\mathfrak{a}} C(\mu a, \nu a)$$

for every $\mu, \nu \in \mathfrak{a}^*$, $\widetilde{\mathcal{D}}_k \subseteq \widetilde{\mathcal{D}}_{k+1}$ for every $k \in \mathbb{N}_0$ and the inclusion ι_k of $\widetilde{\mathcal{D}}_k$ into $\widetilde{\mathcal{D}}_{k+1}$ is given by

$$\iota_k\left(1_{C(\mu,\nu)}\right) = \sum_{a\in\mathfrak{a}} 1_{C(\mu a,\nu a)}.$$

Hence $\psi_{k+1} \circ \iota_k = \psi_k$ and since $\widetilde{\mathcal{D}}_{\mathsf{X}} = \overline{\bigcup_{k\in\mathbb{N}_0} \widetilde{\mathcal{D}}_k}$, the ψ_k's induce a $*$-homomorphism $\psi : \widetilde{\mathcal{D}}_{\mathsf{X}} \to \mathcal{O} \subseteq \mathcal{X}$ such that

$$\psi\left(1_{C(\mu,\nu)}\right) = S_\nu S_\mu^* S_\mu S_\nu^*$$

for every $\mu, \nu \in \mathfrak{a}^*$. □

We are now ready to state and prove the universel property of $\mathcal{O}_{\mathsf{X}}$.

Theorem 7.2. *Let X be a subshift. Then $\mathcal{O}_{\mathsf{X}}$ is the universel unital C^*-algebra generatede by partial isometries $\{S_a\}_{a\in\mathfrak{a}}$ satisfying*

a) $\sum_{a\in\mathfrak{a}} S_a S_a^* = I$,

b) $S_\mu^* S_\mu S_\nu S_\nu^* = S_\nu S_\nu^* S_\mu^* S_\mu$,

c) the map $1_{C(\mu)} \mapsto S_\mu^* S_\mu$ extends to a unital $*$-homomorphism from $\widetilde{\mathcal{A}}_{\mathsf{X}}$ to the C^*-algebra generated by $\{S_a\}_{a\in\mathfrak{a}}$, *where* $S_\mu = S_{\mu_1}\cdots S_{\mu_{|\mu|}}$ *and* $S_\nu = S_{\nu_1}\cdots S_{\nu_{|\nu|}}$ *for every* $\mu,\nu\in\mathfrak{a}^*$.

Proof. It follows from Lemma 5.4 and 5.5 and Proposition 5.6 together with the fact that $\widetilde{\mathcal{A}}_{\mathsf{X}}$ is a C^*-subalgebra of $\widetilde{\mathcal{D}}_{\mathsf{X}}$, that $\mathcal{O}_{\mathsf{X}}$ is generated by partial isometries $\{S_a\}_{a\in\mathfrak{a}}$ satisfying a), b) and c).

Assume now that $\mathcal{X}$ is a unital C^*-algebra generated by partial isometries $\{T_a\}_{a\in\mathfrak{a}}$ and that $\pi:\widetilde{\mathcal{A}}_{\mathsf{X}}\to\mathcal{X}$ is a unital $*$-homomorphism such that

a) $\sum_{a\in\mathfrak{a}} T_aT_a^* = I$,

b) $T_\mu^*T_\mu T_\nu T_\nu^* = T_\nu T_\nu^* T_\mu^* T_\mu$,

c) $T_\mu^*T_\mu = \pi\left(1_{\sigma^{|\mu|}(C(,\mu))}\right)$,

where $T_\mu = T_{\mu_1}T_{\mu_2}\cdots T_{\mu_{|\mu|}}$ and $T_\nu = T_{\nu_1}T_{\nu_2}\cdots T_{\nu_{|\nu|}}$, for every $\mu,\nu\in\mathfrak{a}^*$.

By Lemma 7.1, π extends to a $*$-homomorphism from $\widetilde{\mathcal{D}}_{\mathsf{X}}$ to $\mathcal{X}$, such that

$$\pi\left(1_{C(\mu,\nu)}\right) = T_\nu T_\mu^* T_\mu T_\nu^*$$

for every $\mu,\nu\in\mathfrak{a}^*$. Let

$$\psi((f_a)_{a\in\mathfrak{a}}) = \sum_{a\in\mathfrak{a}} T_a\pi(f_a)$$

for every $(f_a)_{a\in\mathfrak{a}}\in\mathsf{H}_{\mathsf{X}}$. We will show that (ψ,π) is a Cuntz-Pimsner coinvariant representation of H_{X}.

Then $\alpha\psi(\xi)+\beta\psi(\zeta) = \psi(\alpha\xi+\beta\zeta)$ for every $\alpha,\beta\in\mathbb{C}$ and every $\xi,\zeta\in\mathsf{H}_{\mathsf{X}}$, and $\psi(\xi)\pi(f) = \psi(\xi f)$ for every $\xi\in\mathsf{H}_{\mathsf{X}}$ and every $f\in\widetilde{\mathcal{A}}_{\mathsf{X}}$.

Recall from the proof of Proposition 4.3 that for $\mu,\nu\in\mathfrak{a}^*$ with $|\nu|\geq 1$ is $\widetilde{\lambda}_a\left(1_{C(\mu,\nu)}\right) = 0$ if $a\neq\nu_1$, and $\widetilde{\lambda}_a\left(1_{C(\mu,\nu)}\right) = 1_{C(\mu,\nu_2\nu_3\cdots\nu_{|\nu|})}1_{\sigma(C(a))}$ if $a=\nu_1$. Thus

$$\begin{aligned}
\pi\left(1_{C(\mu,\nu)}\right)\psi((f_a)_{a\in\mathfrak{a}}) &= \pi\left(1_{C(\mu,\nu)}\right)\sum_{a\in\mathfrak{a}} T_a\pi(f_a)\\
&= \sum_{a\in\mathfrak{a}} T_\nu T_\mu^* T_\mu T_\nu^* T_a\pi(f_a)\\
&= T_\nu T_\mu^* T_\mu T_\nu^* T_{\nu_1}\pi(f_{\nu_1})\\
&= T_{\nu_1}T_{\nu_2\nu_3\cdots\nu_{|\nu|}} T_\mu^* T_\mu T^*_{\nu_2\nu_3\cdots\nu_{|\nu|}}\pi(f_{\nu_1})\\
&= T_{\nu_1}\pi\left(1_{C(\mu,\nu_2\nu_3\cdots\nu_{|\nu|})}\right)\pi(f_{\nu_1})\\
&= \psi\left(\xi_{\nu_1}1_{C(\mu,\nu_2\nu_3\cdots\nu_{|\nu|})}f_{\nu_1}\right)\\
&= \psi\left((\widetilde{\lambda}_a(1_{C(\mu,\nu)})f_a)_{a\in\mathfrak{a}}\right)\\
&= \psi\left(\phi(1_{C(\mu,\nu)})(f_a)_{a\in\mathfrak{a}}\right),
\end{aligned}$$

for $(f_a)_{a\in\mathfrak{a}} \in \mathsf{H}_{\mathsf{X}}$. We also have that

$$\begin{aligned}
\pi\left(1_{\sigma^{|\mu|}(C(\mu))}\right)\psi((f_a)_{a\in\mathfrak{a}}) &= \pi\left(1_{\sigma^{|\mu|}(C(\mu))}\right)\sum_{a\in\mathfrak{a}} T_a\pi(f_a) \\
&= \sum_{a\in\mathfrak{a}} T_\mu^* T_\mu T_a \pi(f_a) \\
&= \sum_{a\in\mathfrak{a}} T_a T_{\mu a}^* T_{\mu a}\pi(f_a) \\
&= \sum_{a\in\mathfrak{a}} T_a \pi\left(1_{\sigma^{|\mu a|}(C(\mu a))}\right)\pi(f_a) \\
&= \psi\left((1_{\sigma^{|\mu a|}(C(\mu a))} f_a)_{a\in\mathfrak{a}}\right) \\
&= \psi\left((\widetilde{\lambda}_a(1_{\sigma^{|\mu|}(C(\mu))}) f_a)_{a\in\mathfrak{a}}\right) \\
&= \psi\left(\phi(1_{\sigma^{|\mu|}(C(\mu))})(f_a)_{a\in\mathfrak{a}}\right).
\end{aligned}$$

Since $\widetilde{\mathcal{D}}_{\mathsf{X}}$ is generated by $1_{C(\mu,\nu)},\ \mu,\nu\in\mathfrak{a}^*$, we have that

$$\pi(f)\psi((f_a)_{a\in\mathfrak{a}}) = \psi(\phi(f)(f_a)_{a\in\mathfrak{a}})$$

for every $f \in \widetilde{\mathcal{D}}_{\mathsf{X}}$ and every $(f_a)_{a\in\mathfrak{a}} \in \mathsf{H}_{\mathsf{X}}$.

Since $\sum_{a\in\mathfrak{a}} T_a T_a^* = I$, the projections $\{T_a T_a^*\}_{a\in\mathfrak{a}}$ are mutually orthogonal, so

$$\begin{aligned}
T_a^* T_{a'} &= T_a^* T_a T_a^* T_{a'} T_{a'}^* T_{a'} \\
&= 0
\end{aligned}$$

if $a \neq a'$. Thus

$$\begin{aligned}
\psi((f_a)_{a\in\mathfrak{a}})^*\psi((g_a)_{a\in\mathfrak{a}}) &= \sum_{a\in\mathfrak{a}} \pi(f_a^*) T_a^* \sum_{a'\in\mathfrak{a}} T_{a'}\pi(g_{a'}) \\
&= \sum_{a\in\mathfrak{a}} \pi(f_a^*) T_a^* T_a \pi(g_a) \\
&= \sum_{a\in\mathfrak{a}} \pi(f_a^*)\pi\left(1_{\sigma(C(a))}\right)\pi(g_a) \\
&= \pi(\langle (f_a)_{a\in\mathfrak{a}}, (g_a)_{a\in\mathfrak{a}}\rangle)
\end{aligned}$$

for every $(f_a)_{a\in\mathfrak{a}}, (g_a)_{a\in\mathfrak{a}} \in \mathsf{H}_{\mathsf{X}}$.

Finally we see that for every $f \in \widetilde{\mathcal{D}}_{\mathsf{X}}$ is $\phi(f) = \sum_{a\in\mathfrak{a}} \theta_{\xi_a \widetilde{\lambda}_a(f),\xi_a}$, so

$$\begin{aligned}\pi^{(1)}(\phi(f)) &= \sum_{a\in\mathfrak{a}} \psi(\xi_a \widetilde{\lambda}_a(f))\psi(\xi_a)^* \\ &= \sum_{a\in\mathfrak{a}} T_a \pi(\widetilde{\lambda}_a(f))T_a^* \\ &= \sum_{a\in\mathfrak{a}} \pi(f) T_a T_a^* \\ &= \pi(f).\end{aligned}$$

Thus (ψ, π) is a Cuntz-Pimsner coinvariant representation of H_{X}, so it follows from Theorem 3.1 that there exists a $*$-homomorphism $\psi \times \pi$ from $\mathcal{O}_{\mathsf{X}}$ to $\mathcal{X}$ such that $\psi \times \pi(k_{\mathsf{H}_{\mathsf{X}}}((f_a)_{a\in\mathfrak{a}})) = \psi((f_a)_{a\in\mathfrak{a}})$ for every $(f_a)_{a\in\mathfrak{a}} \in \mathsf{H}_{\mathsf{X}}$ and hence

$$\psi \times \pi(S_a) = \psi \times \pi(k_{\mathsf{H}_{\mathsf{X}}}(\xi_a))) = \psi(\xi_a) = T_a$$

for every $a \in \mathfrak{a}$. □

Remark 7.3. Condition *b*) can be replaced by

b') $S_\mu^* S_\mu S_\nu = S_\nu S_{\mu\nu}^* S_{\mu\nu}$,

because b') implies that $S_\mu^* S_\mu S_\nu S_\nu^* = S_\nu S_{\mu\nu}^* S_{\mu\nu} S_\nu^* = S_\nu S_\nu^* S_\mu^* S_\mu S_\nu S_\nu^*$ and $S_\nu S_\nu^* S_\mu^* S_\mu = S_\nu (S_\mu^* S_\mu S_\nu)^* = S_\nu (S_\nu S_{\mu\nu}^* S_{\nu\mu})^* = S_\nu S_\nu^* S_\mu^* S_\mu S_\nu S_\nu^*$, and thus $S_\mu^* S_\mu S_\nu S_\nu^* = S_\nu S_\nu^* S_\mu^* S_\mu$, and *b*) implies that $S_\mu^* S_\mu S_\nu = S_\mu^* S_\mu S_\nu S_\nu^* S_\nu = S_\nu S_\nu^* S_\mu^* S_\mu S_\nu = S_\nu S_{\mu\nu}^* S_{\mu\nu}$. Thus $\mathcal{O}_{\mathsf{X}}$ has the universal property [12, Theorem 4.9] and also has the right underlying compact space (cf. [10, Lemma 3.1]) and thus satisfy all of the results of [7–12].

Remark 7.4. It follows from Lemma 7.1 that $\mathcal{O}_{\mathsf{X}}$ also can be characterized as the universal C^*-algebra generated by partial isometries $\{S_a\}_{a\in\mathfrak{a}}$ such that the map $1_{C(\mu,\nu)} \mapsto S_\nu S_\mu^* S_\mu S_\nu^*$ extends to a $*$-homomorphism from $\widetilde{\mathcal{D}}_{\mathsf{X}}$ to C^*-algebra generated by $\{S_a\}_{a\in\mathfrak{a}}$, where $S_\mu = S_{\mu_1} \cdots S_{\mu_{|\mu|}}$ and $S_\nu = S_{\nu_1} \cdots S_{\nu_{|\nu|}}$ for every $\mu, \nu \in \mathfrak{a}^*$.

8 $\mathcal{O}_{\mathsf{X}}$ is an invariant

We will now show that $\mathcal{O}_{\mathsf{X}}$ is an invariant for subshifts. We will do that by showing that if two shift spaces X and Y are conjugate, then H_{X} and H_{Y} are isomorphic as C^*-correspondences, and it then follows that $\mathcal{O}_{\mathsf{X}}$ and $\mathcal{O}_{\mathsf{Y}}$ are isomorphic.

Definition 8.1. Let $\mathcal{X}$ and $\mathcal{X}'$ be C^*-algebras, (H,ϕ) a C^*-correspondence over $\mathcal{X}$ and (H,ϕ') a C^*-correspondence over $\mathcal{X}'$. If there exist a $*$-isomorphism $\psi:\mathcal{X}\to\mathcal{X}'$ and a bijective map $T:\mathsf{H}\to\mathsf{H}$ such that

$$\langle T\xi,\zeta\rangle=\psi(\langle\xi,T^{-1}\zeta\rangle),$$

and

$$T(\phi(X)\xi)=\phi'(\psi(X))(T\xi)$$

for all $\xi\in\mathsf{H},\ \zeta\in\mathsf{H}',\ X\in\mathcal{X}$; then we say that (T,ψ) is an C^*-correspondence isomorphism, (H,ϕ) and (H',ϕ') are isomorphic and we write $\mathsf{H}\cong\mathsf{H}'$.

It easily follows from Theorem 3.1 that if (T,ψ) is an C^*-correspondence isomorphism from (H,ϕ) to (H',ϕ'), then there exists a $*$-isomorphism $T\times\psi$ from $\mathcal{O}_{\mathsf{H}}$ to $\mathcal{O}_{\mathsf{H}'}$ such that $T\times\psi\circ k_{\mathsf{H}}=k_{\mathsf{H}'}\circ T$ and $T\times\psi\circ\ k_{\mathcal{X}}=k_{\mathcal{X}'}\circ\psi$.

Lemma 8.2. *Let X be a one-sided shift space. Define a $*$-homomorphism $\widetilde{\phi}_{\mathsf{X}}:\mathfrak{B}(\mathsf{X})\to\mathfrak{B}(\mathsf{X})$ by letting*

$$\widetilde{\phi}_{\mathsf{X}}(f)(x)=f(\sigma(x))$$

for every $f\in\mathfrak{B}(\mathsf{X})$ and every $x\in\mathsf{X}$.

Then $\widetilde{\phi}_{\mathsf{X}}(\widetilde{\mathcal{D}}_{\mathsf{X}})\subseteq\widetilde{\mathcal{D}}_{\mathsf{X}}$.

Proof. Let $\mu,\nu\in\mathfrak{a}^*$. Then

$$\sigma^{-1}(C(\mu,\nu))=\bigcup_{a\in\mathfrak{a}}C(\mu,a\nu),$$

so

$$\begin{aligned}\widetilde{\phi}_{\mathsf{X}}\left(1_{C(\mu,\nu)}\right)&=1_{\sigma^{-1}(C(\mu,\nu))}\\&=1_{\bigcup_{a\in\mathfrak{a}}C(\mu,a\nu)}\\&=\sum_{a\in\mathfrak{a}}1_{C(\mu,a\nu)}\in\widetilde{\mathcal{D}}_{\mathsf{X}}.\end{aligned}$$

Thus, since $\widetilde{\mathcal{D}}_{\mathsf{X}}$ is generated by $\{1_{C(\mu,\nu)}\mid\mu,\nu\in\mathfrak{a}^*\}$ and $\widetilde{\phi}_{\mathsf{X}}$ is a $*$-homomorphism, it follows that $\widetilde{\phi}_{\mathsf{X}}(\widetilde{\mathcal{D}}_{\mathsf{X}})\subseteq\widetilde{\mathcal{D}}_{\mathsf{X}}$. □

Lemma 8.3. *Let X be a one-sided shift space. Then we have:*

a) If $\mathcal{E}_1,\mathcal{E}_2$ are subsets of X such that $1_{\mathcal{E}_1},1_{\mathcal{E}_2}\in\widetilde{\mathcal{D}}_{\mathsf{X}}$, then $1_{\mathcal{E}_1\cup\mathcal{E}_2}\in\widetilde{\mathcal{D}}_{\mathsf{X}}$.

b) If $\mathcal{E}$ is a subset of X such that $1_{\mathcal{E}}\in\widetilde{\mathcal{D}}_{\mathsf{X}}$, then $1_{\sigma(\mathcal{E})}\in\widetilde{\mathcal{D}}_{\mathsf{X}}$.

c) If $\mathcal{E}$ is a subset of X such that $1_{\mathcal{E}} \in \widetilde{\mathcal{D}}_{\mathsf{X}}$, then $1_{\sigma^{-1}(\mathcal{E})} \in \widetilde{\mathcal{D}}_{\mathsf{X}}$.

Proof. *a*) Let $\mathcal{E}_1, \mathcal{E}_2$ be subsets of X such that $1_{\mathcal{E}_1}, 1_{\mathcal{E}_2} \in \widetilde{\mathcal{D}}_{\mathsf{X}}$, then

$$1_{\mathcal{E}_1 \cup \mathcal{E}_2} = 1_{\mathcal{E}_1} + 1_{\mathcal{E}_2} - 1_{\mathcal{E}_1} 1_{\mathcal{E}_2} \in \widetilde{\mathcal{D}}_{\mathsf{X}}.$$

b) Let $\mathcal{E}$ be a subset of X such that $1_{\mathcal{E}} \in \widetilde{\mathcal{D}}_{\mathsf{X}}$. Set for each $a \in \mathfrak{a}$,

$$\mathcal{E}_a = \{x \in \mathsf{X} \mid ax \in \mathcal{E}\}.$$

It is easy to check that

$$\sigma(\mathcal{E}) = \bigcup_{a \in \mathfrak{a}} \mathcal{E}_a.$$

Since $1_{\mathcal{E}_a} = \widetilde{\lambda}_a(1_{\mathcal{E}}) \in \widetilde{\mathcal{D}}_{\mathsf{X}}$ (cf. Proposition 4.3), it follows from a) that $1_{\sigma(\mathcal{E})} \in \widetilde{\mathcal{D}}_{\mathsf{X}}$.

c) Let $\mathcal{E}$ be a subset of X such that $1_{\mathcal{E}} \in \widetilde{\mathcal{D}}_{\mathsf{X}}$. It is easy to check that $1_{\sigma^{-1}(\mathcal{E})} = \widetilde{\phi}_{\mathsf{X}}(1_{\mathcal{E}})$, so $1_{\sigma^{-1}(\mathcal{E})} \in \widetilde{\mathcal{D}}_{\mathsf{X}}$ by Lemma 8.2. □

Proposition 8.4. *If two subshifts X and Y are conjugate, then $\widetilde{\mathcal{D}}_{\mathsf{X}} \cong \widetilde{\mathcal{D}}_{\mathsf{Y}}$ and $\mathsf{H}_{\mathsf{X}} \cong \mathsf{H}_{\mathsf{Y}}$.*

Proof. Let $\psi : \mathsf{X} \to \mathsf{Y}$ be a conjugacy. Then we can define a $*$-isomorphism $\Psi : \mathfrak{B}(\mathsf{Y}) \to \mathfrak{B}(\mathsf{X})$ by setting $\Psi(f)(x) = f(\psi(x))$ for every $f \in \mathfrak{B}(\mathsf{Y})$ and every $x \in \mathsf{X}$.

Let $\mu \in \mathsf{L}(\mathsf{Y})$. Since $C_{\mathsf{Y}}(\mu)$ is clopen and ψ is continuous, $\psi^{-1}(C_{\mathsf{Y}}(\mu))$ is clopen and hence compact. So since $C_{\mathsf{X}}(\nu)$, $\nu \in \mathsf{L}(\mathsf{X})$ is a basis for the topology of X, there exist a finite number of words $\mu_1, \mu_2, \ldots, \mu_r \in \mathsf{L}(\mathsf{X})$ such that

$$\psi^{-1}(C_{\mathsf{Y}}(\mu)) = \bigcup_{k=1}^{r} C_{\mathsf{X}}(\mu_k).$$

Let $\mu, \nu \in \mathsf{L}(\mathsf{Y})$ and let $\mu_1, \ldots \mu_r, \nu_1, \ldots, \nu_s \in \mathsf{L}(\mathsf{X})$ such that

$$\psi^{-1}(C_{\mathsf{Y}}(\mu)) = \bigcup_{k=1}^{r} C_{\mathsf{X}}(\mu_k)$$

and

$$\psi^{-1}(C_{\mathsf{Y}}(\nu)) = \bigcup_{k=1}^{s} C_{\mathsf{X}}(\nu_k).$$

Since both $\psi\circ\sigma_{\mathsf{X}}=\sigma_{\mathsf{Y}}\circ\psi$, we have that

$$\begin{aligned}\psi^{-1}(C_{\mathsf{Y}}(\mu,\nu)) &= \psi^{-1}(C_{\mathsf{Y}}(\nu))\cap\sigma_{\mathsf{X}}^{-|\nu|}(\sigma_{\mathsf{X}}^{|\mu|}(\psi^{-1}(C_{\mathsf{Y}}(\mu)))) \\ &= \left(\bigcup_{k=1}^{s} C_{\mathsf{X}}(\nu_k)\right)\bigcap\left(\bigcup_{k=1}^{r}\sigma_{\mathsf{X}}^{-|\nu|}(\sigma_{\mathsf{X}}^{|\mu|}(C_{\mathsf{X}}(\mu_j)))\right),\end{aligned}$$

so it follows from Lemma 8.3 that

$$\Psi\left(1_{C_{\mathsf{Y}}(\mu,\nu)}\right) = 1_{\psi^{-1}(C_{\mathsf{Y}}(\mu,\nu))}\in\widetilde{\mathcal{D}}_{\mathsf{X}}.$$

Hence $\Psi(\widetilde{\mathcal{D}}_{\mathsf{Y}})\subseteq\widetilde{\mathcal{D}}_{\mathsf{X}}$. In the same way we can prove that $\Psi^{-1}(\widetilde{\mathcal{D}}_{\mathsf{X}})\subseteq\widetilde{\mathcal{D}}_{\mathsf{Y}}$, so $\Psi(\widetilde{\mathcal{D}}_{\mathsf{Y}})=\widetilde{\mathcal{D}}_{\mathsf{X}}$, and thus $\Psi_{|\widetilde{\mathcal{D}}_{\mathsf{Y}}}:\widetilde{\mathcal{D}}_{\mathsf{Y}}\to\widetilde{\mathcal{D}}_{\mathsf{X}}$ is a $*$-isomorphism.

Define $T:\mathsf{H}_{\mathsf{Y}}\to\mathsf{H}_{\mathsf{X}}$ by

$$T(f_a)_{a\in\mathfrak{a}_{\mathsf{Y}}}=\left(\sum_{a\in\mathfrak{a}_{\mathsf{Y}}}\widetilde{\lambda}_b\left(\Psi\left(1_{C_{\mathsf{Y}}(a)}\right)\right)\Psi(f_a)\right)_{b\in\mathfrak{a}_{\mathsf{X}}}$$

and $S:\mathsf{H}_{\mathsf{X}}\to\mathsf{H}_{\mathsf{Y}}$ by

$$S(g_b)_{b\in\mathfrak{a}_{\mathsf{X}}}=\left(\sum_{b\in\mathfrak{a}_{\mathsf{X}}}\widetilde{\lambda}_a\left(\Psi^{-1}\left(1_{C_{\mathsf{X}}(b)}\right)\right)\Psi^{-1}(g_b)\right)_{a\in\mathfrak{a}_{\mathsf{Y}}}.$$

Let $a\in\mathfrak{a}_{\mathsf{Y}},\ b\in\mathfrak{a}_{\mathsf{X}}$ and $x\in\mathsf{Y}$. If $ax\in\mathsf{Y}$ and $(\psi^{-1}(ax))_1=b$, then

$$\begin{aligned}\psi^{-1}(ax) &= (\psi^{-1}(ax))_1\sigma(\psi^{-1}(ax)) \\ &= b\psi^{-1}(\sigma(ax)) \\ &= b\psi^{-1}(x)\end{aligned}$$

and thus $b\psi^{-1}(x)\in\mathsf{X}$ and $(\psi(b\psi^{-1}(x)))_1=a$.

If $b\psi^{-1}(x)\in\mathsf{X}$ and $(\psi(b\psi^{-1}(x)))_1=a$, then

$$\begin{aligned}\psi(b\psi^{-1}(x)) &= (\psi(b\psi^{-1}(x)))_1\sigma(\psi(b\psi^{-1}(x))) \\ &= a\psi(\sigma(b\psi^{-1}(x))) \\ &= a\psi(\psi^{-1}(x)) \\ &= ax\end{aligned}$$

and thus $ax\in\mathsf{Y}$ and $(\psi^{-1}(ax))_1=b$.

Hence $(ax \in \mathsf{Y} \wedge (\psi^{-1}(ax))_1 = b) \Leftrightarrow (b\psi^{-1}(x) \in \mathsf{X} \wedge (\psi(b\psi^{-1}(x)))_1 = a)$. So

$$\begin{aligned}
\Psi^{-1}\left(\widetilde{\lambda}_b\left(\Psi\left(1_{C_{\mathsf{Y}}(a)}\right)\right)\right)(x) &= \begin{cases} \Psi\left(1_{C_{\mathsf{Y}}(a)}\right)(b\psi^{-1}(x)) & \text{if } b\psi^{-1}(x) \in \mathsf{X} \\ 0 & \text{if } b\psi^{-1}(x) \notin \mathsf{X} \end{cases} \\
&= \begin{cases} 1 & \text{if } b\psi^{-1}(x) \in \mathsf{X} \wedge (\psi(b\psi^{-1}(x)))_1 = a \\ 0 & \text{else} \end{cases} \\
&= \begin{cases} 1 & \text{if } ax \in \mathsf{Y} \wedge (\psi^{-1}(ax))_1 = b \\ 0 & \text{else} \end{cases} \\
&= \begin{cases} \Psi^{-1}\left(1_{C_{\mathsf{X}}(b)}\right)(ax) & \text{if } ax \in \mathsf{Y} \\ 0 & \text{if } ax \notin \mathsf{Y} \end{cases} \\
&= \widetilde{\lambda}_a\left(\Psi^{-1}\left(1_{C_{\mathsf{X}}(b)}\right)\right)(x)
\end{aligned}$$

and hence $\Psi^{-1}\left(\widetilde{\lambda}_b\left(\Psi\left(1_{C_{\mathsf{Y}}(a)}\right)\right)\right) = \widetilde{\lambda}_a\left(\Psi^{-1}\left(1_{C_{\mathsf{X}}(b)}\right)\right)$ for all $a \in \mathfrak{a}_{\mathsf{Y}}$ and $b \in \mathfrak{a}_{\mathsf{X}}$, and thus

$$\begin{aligned}
\langle T(f_a)_{a\in\mathfrak{a}_{\mathsf{Y}}}, (g_b)_{b\in\mathfrak{a}_{\mathsf{X}}}\rangle &= \left\langle \left(\sum_{a\in\mathfrak{a}_{\mathsf{Y}}} \widetilde{\lambda}_b\left(\Psi\left(1_{C_{\mathsf{Y}}(a)}\right)\right)\Psi(f_a)\right)_{b\in\mathfrak{a}_{\mathsf{X}}}, (g_b)_{b\in\mathfrak{a}_{\mathsf{X}}}\right\rangle \\
&= \sum_{b\in\mathfrak{a}_{\mathsf{X}}}\sum_{a\in\mathfrak{a}_{\mathsf{Y}}} \widetilde{\lambda}_b\left(\Psi\left(1_{C_{\mathsf{Y}}(a)}\right)\right)\Psi(f_a^*)g_b \\
&= \sum_{a\in\mathfrak{a}_{\mathsf{Y}}} \Psi(f_a^*)\sum_{b\in\mathfrak{a}_{\mathsf{X}}} \widetilde{\lambda}_b\left(\Psi\left(1_{C_{\mathsf{Y}}(a)}\right)\right)g_b \\
&= \Psi\left(\left\langle (f_a)_{a\in\mathfrak{a}_{\mathsf{Y}}}, \left(\sum_{b\in\mathfrak{a}_{\mathsf{X}}} \widetilde{\lambda}_a\left(\Psi^{-1}\left(1_{C_{\mathsf{X}}(b)}\right)\right)\Psi^{-1}(g_b)\right)_{a\in\mathfrak{a}_{\mathsf{Y}}}\right\rangle\right) \\
&= \Psi(\langle (f_a)_{a\in\mathfrak{a}_{\mathsf{Y}}}, S(g_b)_{b\in\mathfrak{a}_{\mathsf{X}}}\rangle)
\end{aligned}$$

for all $(f_a)_{a\in\mathfrak{a}_{\mathsf{Y}}} \in \mathsf{H}_{\mathsf{Y}}$ and all $(g_b)_{b\in\mathfrak{a}_{\mathsf{X}}} \in \mathsf{H}_{\mathsf{X}}$.

Let $a \in \mathfrak{a}_{\mathsf{Y}}$, $b \in \mathfrak{a}_{\mathsf{X}}$ and $y \in \mathsf{X}$. If $by \in \mathsf{X}$ and $(\psi(by))_1 = a$, then

$$\begin{aligned}
\psi(by) &= (\psi(by))_1\sigma(\psi(by)) \\
&= a\psi(\sigma(by)) \\
&= a\psi(y),
\end{aligned}$$

and thus $a\psi(y) \in \mathsf{Y}$.

So for every $f \in \widetilde{\mathcal{D}}_{\mathsf{Y}}$ is

$$\begin{aligned}
\widetilde{\lambda}_b\left(\Psi\left(1_{C_{\mathsf{Y}}(a)}\right)\right)\Psi(\widetilde{\lambda}_a(f))(y) &= \begin{cases} \Psi\left(1_{C_{\mathsf{Y}}(a)}\right)(by)\Psi(\widetilde{\lambda}_a(f))(y) & \text{if } by \in \mathsf{X} \\ 0 & \text{if } by \notin \mathsf{X} \end{cases} \\
&= \begin{cases} 1_{C_{\mathsf{Y}}(a)}(\psi(by))\widetilde{\lambda}_a(f)(\psi(y)) & \text{if } by \in \mathsf{X} \\ 0 & \text{if } by \notin \mathsf{X} \end{cases} \\
&= \begin{cases} f(a\psi(y)) & \text{if } by \in \mathsf{X},\ (\psi(by))_1 = a \text{ and } a\psi(y) \in \mathsf{Y} \\ 0 & \text{else} \end{cases} \\
&= \begin{cases} f(\psi(by)) & \text{if } by \in \mathsf{X} \text{ and } (\psi(by))_1 = a \\ 0 & \text{else} \end{cases} \\
&= \widetilde{\lambda}_b\left(\Psi\left(1_{C_{\mathsf{Y}}(a)}\right)\right)\widetilde{\lambda}_b(\Psi(f))(y),
\end{aligned}$$

and hence $\widetilde{\lambda}_b\left(\Psi\left(1_{C_{\mathsf{Y}}(a)}\right)\right)\Psi(\widetilde{\lambda}_a(f)) = \widetilde{\lambda}_b\left(\Psi\left(1_{C_{\mathsf{Y}}(a)}\right)\right)\widetilde{\lambda}_b(\Psi(f))$ for all $f \in \widetilde{\mathcal{D}}_{\mathsf{Y}}$, $a \in \mathfrak{a}_{\mathsf{Y}}$ and $b \in \mathfrak{a}_{\mathsf{X}}$. Thus

$$\begin{aligned}
T(\phi'(f)(f_a)_{a\in\mathfrak{a}_{\mathsf{Y}}}) &= T(\widetilde{\lambda}_a(f)f_a)_{a\in\mathfrak{a}_{\mathsf{Y}}} \\
&= \left(\sum_{a\in\mathfrak{a}_{\mathsf{Y}}} \widetilde{\lambda}_b\left(\Psi\left(1_{C_{\mathsf{Y}}(a)}\right)\right)\Psi(\widetilde{\lambda}_a(f)f_a)\right)_{b\in\mathfrak{a}_{\mathsf{X}}} \\
&= \left(\widetilde{\lambda}_b(\Psi(f))\sum_{a\in\mathfrak{a}_{\mathsf{Y}}} \widetilde{\lambda}_b\left(\Psi\left(1_{C_{\mathsf{Y}}(a)}\right)\right)\Psi(f_a)\right)_{b\in\mathfrak{a}_{\mathsf{X}}} \\
&= \phi'(\Psi(f))T(f_a)_{a\in\mathfrak{a}_{\mathsf{Y}}}
\end{aligned}$$

for all $(f_a)_{a\in\mathfrak{a}_{\mathsf{Y}}} \in \mathsf{H}_{\mathsf{Y}}$ and all $f \in \widetilde{\mathcal{D}}_{\mathsf{Y}}$.

Since $\Psi^{-1}\left(\widetilde{\lambda}_b\left(\Psi\left(1_{C_{\mathsf{Y}}(a)}\right)\right)\right) = \widetilde{\lambda}_a\left(\Psi^{-1}\left(1_{C_{\mathsf{X}}(b)}\right)\right)$ for all $a \in \mathfrak{a}_{\mathsf{Y}}$ and $b \in \mathfrak{a}_{\mathsf{X}}$, Ψ, Ψ^{-1} and $\widetilde{\lambda}_a$ are $*$-homomorphisms and

$$1_{C_{\mathsf{X}}(b)}1_{C_{\mathsf{X}}(b')} = \begin{cases} 1_{C_{\mathsf{X}}(b)} & \text{if } b = b' \\ 0 & \text{if } b \neq b' \end{cases}$$

for $b, b' \in \mathfrak{a}_{\mathsf{X}}$, we have that

$$\widetilde{\lambda}_b\left(\Psi\left(1_{C_{\mathsf{Y}}(a)}\right)\right)\widetilde{\lambda}_{b'}\left(\Psi\left(1_{C_{\mathsf{Y}}(a)}\right)\right) = \begin{cases} \widetilde{\lambda}_b\left(\Psi\left(1_{C_{\mathsf{Y}}(a)}\right)\right) & \text{if } b = b' \\ 0 & \text{if } b \neq b' \end{cases}$$

for all $a \in \mathfrak{a}_{\mathsf{Y}}$ and all $b, b' \in \mathfrak{a}_{\mathsf{X}}$; and hence

$$\begin{aligned}
TS(g_b)_{b\in\mathfrak{a}_{\mathsf{X}}} &= T\left(\sum_{b\in\mathfrak{a}_{\mathsf{X}}} \widetilde{\lambda}_a\left(\Psi^{-1}\left(1_{C_{\mathsf{X}}(b)}\right)\right)\Psi^{-1}(g_b)\right)_{a\in\mathfrak{a}_{\mathsf{Y}}} \\
&= \left(\sum_{a\in\mathfrak{a}_{\mathsf{Y}}} \widetilde{\lambda}_{b'}\left(\Psi\left(1_{C_{\mathsf{Y}}(a)}\right)\right)\sum_{b\in\mathfrak{a}_{\mathsf{X}}}\Psi\left(\widetilde{\lambda}_a\left(\Psi^{-1}\left(1_{C_{\mathsf{Y}}(b)}\right)\right)\right) g_b\right)_{b'\in\mathfrak{a}_{\mathsf{X}}} \\
&= \left(\sum_{a\in\mathfrak{a}_{\mathsf{Y}}} \widetilde{\lambda}_{b'}\left(\Psi\left(1_{C_{\mathsf{X}}(a)}\right)\right)\sum_{b\in\mathfrak{a}_{\mathsf{X}}}\widetilde{\lambda}_b\left(\Psi\left(1_{C_{\mathsf{X}}(a)}\right)\right) g_b\right)_{b'\in\mathfrak{a}_{\mathsf{X}}} \\
&= \left(\sum_{a\in\mathfrak{a}_{\mathsf{Y}}} \widetilde{\lambda}_b\left(\Psi\left(1_{C_{\mathsf{Y}}(a)}\right)\right) g_b\right)_{b\in\mathfrak{a}_{\mathsf{X}}} \\
&= \left(\widetilde{\lambda}_b\left(\Psi\left(\sum_{a\in\mathfrak{a}_{\mathsf{Y}}} 1_{C_{\mathsf{Y}}(a)}\right)\right) g_b\right)_{b\in\mathfrak{a}_{\mathsf{X}}} \\
&= (\widetilde{\lambda}_b(1)g_b)_{b\in\mathfrak{a}_{\mathsf{X}}} \\
&= \left(1_{\sigma(C_{\mathsf{X}}(b))}g_b\right)_{b\in\mathfrak{a}_{\mathsf{X}}} \\
&= (g_b)_{b\in\mathfrak{a}_{\mathsf{X}}}
\end{aligned}$$

for all $(g_b)_{b\in\mathfrak{a}_{\mathsf{X}}} \in \mathsf{H}_{\mathsf{X}}$.

In the same way one can prove that $ST(f_a)_{a\in\mathfrak{a}_{\mathsf{Y}}} = (f_a)_{a\in\mathfrak{a}_{\mathsf{Y}}}$ for all $(f_a)_{a\in\mathfrak{a}_{\mathsf{Y}}} \in \mathsf{H}_{\mathsf{Y}}$.

Hence (T, Ψ) is a C^*-correspondence isomorphism and $\mathsf{H}_{\mathsf{X}} \cong \mathsf{H}_{\mathsf{Y}}$. □

Theorem 8.5. *If two subshifts* X *and* Y *are conjugate, then there exists a* $*$*-isomorphism* ρ *from* $\mathcal{O}_{\mathsf{X}}$ *to* $\mathcal{O}_{\mathsf{Y}}$ *such that* $\gamma_z \circ \rho = \rho \circ \gamma_z$ *for every* $z \in \mathbb{T}$.

Proof. It follows from Theorem 8.4 that there exists a C^*-correspondence isomorphism (T, Ψ) from H_{X} to H_{Y}. Thus there exists a $*$-isomorphism $\rho : \mathcal{O}_{\mathsf{X}} \to \mathcal{O}_{\mathsf{Y}}$ such that $\rho(k_{\mathsf{H}_{\mathsf{X}}}(\xi)) = k_{\mathsf{H}_{\mathsf{Y}}}(T\xi)$ for every $\xi \in \mathsf{H}_{\mathsf{X}}$ and $\rho(k_{\widetilde{\mathcal{D}}_{\mathsf{X}}}(f)) = k_{\widetilde{\mathcal{D}}_{\mathsf{Y}}}(\Psi(f))$ for every $f \in \widetilde{\mathcal{D}}_{\mathsf{X}}$. Hence

$$\begin{aligned}
\gamma_z(\rho(k_{\mathsf{H}_{\mathsf{X}}}(\xi))) &= \gamma_z(k_{\mathsf{H}_{\mathsf{Y}}}(T\xi)) \\
&= zk_{\mathsf{H}_{\mathsf{Y}}}(T\xi) \\
&= k_{\mathsf{H}_{\mathsf{Y}}}(Tz\xi) \\
&= \rho(k_{\mathsf{H}_{\mathsf{X}}}(z\xi)) \\
&= \rho(zk_{\mathsf{H}_{\mathsf{X}}}(\xi)) \\
&= \rho(\gamma_z(k_{\mathsf{H}_{\mathsf{X}}}(\xi)))
\end{aligned}$$

for every $\xi \in \mathsf{H}_\mathsf{X}$ and every $z \in \mathbb{T}$, and

$$\begin{aligned}\gamma_z(\rho(k_{\widetilde{\mathcal{D}}_\mathsf{X}}(f))) &= \gamma_z(k_{\widetilde{\mathcal{D}}_\mathsf{Y}}(\Psi(f)))\\ &= zk_{\widetilde{\mathcal{D}}_\mathsf{Y}}(\Psi(f))\\ &= k_{\widetilde{\mathcal{D}}_\mathsf{Y}}(\Psi(zf))\\ &= \rho(k_{\widetilde{\mathcal{D}}_\mathsf{X}}(zf))\\ &= \rho(zk_{\widetilde{\mathcal{D}}_\mathsf{X}}(f))\\ &= \rho(\gamma_z(k_{\widetilde{\mathcal{D}}_\mathsf{X}}(f)))\end{aligned}$$

for every $f \in \widetilde{\mathcal{D}}_\mathsf{X}$ and every $z \in \mathbb{T}$. Since $\mathcal{O}_\mathsf{X}$ is generated by $k_{\mathsf{H}_\mathsf{X}}(\mathsf{H}_\mathsf{X}) \cup k_{\widetilde{\mathcal{D}}_\mathsf{X}}(\widetilde{\mathcal{D}}_\mathsf{X})$, it follows that $\gamma_z \circ \rho = \rho \circ \gamma_z$ for every $z \in \mathbb{T}$. □

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1.10 Closing remarks

Matsumoto has in [7–10, 12–15] made a thorough investigation of the properties of the C^*-algebra $\mathcal{O}_{\mathsf{X}}$ and among other things shown that $\mathcal{O}_{\mathsf{X}}$ is nuclear and satisfies the universal coefficient theorem, given sufficient conditions for when it is purely infinite, described its ideal structure and given formulas for computing its K-theory. The author has in [2] shown that $\mathcal{O}_{\mathsf{X}}$ is a Cuntz-Krieger algebra when X is a sofic shift.

The invariants Matsumoto introduced in [11] are all based on K-groups associated with $\mathcal{O}_{\mathsf{X}}$.

The C^*-algebra $\mathcal{O}_{\mathsf{X}}$ can, as most C^*-algebras associated to dynamical systems, also be constructed in other ways than by C^*-correspondence, for example by Exel's crossed product (cf. [5]). In the next chapter we will construct $\mathcal{O}_{\mathsf{X}}$ as a groupoid C^*-algebra.

Matsumoto has in [6] generalized the construction of $\mathcal{O}_{\mathsf{X}}$ to λ-graph systems.

Chapter 2

A groupoid construction

A groupoid construction

We will in this chapter construct the C^*-algebra $\mathcal{O}_{\mathsf{X}}$ defined in the [1] as a groupoid C^*-algebra. We will stick to the notation used in [1].

A natural approach is to use the groupoid $G(X,T)$ that Renault in [6] has associated to a *singly generated dynamical system* (X,T), and then look at the C^*-algebra has defined in [5]. However, this approach has its obstacles, because a one-sided shift $\sigma : \mathsf{X} \to \mathsf{X}$ is not in generally a local homeomorphism which is required for it to be a singly generated dynamical system (in fact, σ is a local homeomorphism if and only if X is a shift of finite type). We surmount this obstacle by associating to each one-sided shift space (X,σ) a cover $(\widetilde{\mathsf{X}},\widetilde{\sigma})$ which is a singly generated dynamical system ((X,σ) is a generalization of the left Krieger cover cf. [2, 3]).

1 The cover $(\widetilde{\mathsf{X}},\widetilde{\sigma})$

Let (X,σ) be a one-sided shift space, and $\mathfrak{a}$ its alphabet. Set for every $x \in \mathsf{X}$ and every $k \in \mathbb{N}_0$

$$\mathcal{P}_k(x) = \{\mu \in \mathsf{L}(\mathsf{X}) \mid \mu x \in \mathsf{X}, |\mu| = k\}.$$

Following Matsumoto [4], we define for every $l \in \mathbb{N}_0$ an equivalence relation $\sim_l$ on X called *l-past equivalence* by

$$x \sim_l y \Leftrightarrow \forall k \leq l : \mathcal{P}_k(x) = \mathcal{P}_k(y),$$

and we denote by $[x]_l$ the l-past equivalence class of x.

Let $\mathcal{I} = \{(k,l) \in \mathbb{N}_0^2 \mid k \leq l\}$. We define an order $\leq$ on $\mathcal{I}$ by

$$(k_1,l_1) \leq (k_2,l_2) \Leftrightarrow k_1 \leq k_2 \wedge l_1 - k_1 \leq l_2 - k_2.$$

For $(k,l) \in \mathcal{I}$ we define an equivalence relation ${}_k{\sim}_l$ on X by

$$x \; {}_k{\sim}_l \; y \Leftrightarrow x_{[0,k[} = y_{[0,k[} \wedge \mathcal{P}_l(x_{[k,\infty[}) = \mathcal{P}_l(y_{[k,\infty[}).$$

We denote the equivalence class of x by ${}_k[x]_l$ and we let ${}_k\mathsf{X}_l$ be the quotient of X by ${}_k{\sim}_l$. Notice that ${}_k\mathsf{X}_l$ is finite. We endow ${}_k\mathsf{X}_l$ with the discrete topology.

Let $(k_1, l_1) \leq (k_2, l_2) \in \mathcal{I}$. Since

$$x \;{}_{k_2}{\sim}_{l_2}\; y \Rightarrow x \;{}_{k_1}{\sim}_{l_1}\; y,$$

there exists a map ${}_{(k_1,l_1)}\pi_{(k_2,l_2)} : {}_{k_1}\mathsf{X}_{l_1} \to {}_{k_2}\mathsf{X}_{l_2}$ such that

$${}_{(k_1,l_1)}\pi_{(k_2,l_2)}({}_{k_2}[x]_{l_2}) = {}_{k_1}[x]_{l_1}.$$

Let $\widetilde{\mathsf{X}}$ be the projective limit, $\lim_{(k,l)\in\mathcal{I}}({}_k\mathsf{X}_l, \pi)$. We will identify $\widetilde{\mathsf{X}}$ with the closed subset

$$\left\{({}_k[{}_kx_l]_l)_{(k,l)\in\mathcal{I}} \in \prod_{(k,l)\in\mathcal{I}} {}_k\mathsf{X}_l \;\middle|\; \forall (k_1, l_1) \leq (k_2, l_2) \in \mathcal{I} : {}_{k_2}x_{l_2} \;{}_{k_1}{\sim}_{l_1}\; {}_{k_1}x_{l_1}\right\}$$

of $\prod_{(k,l)\in\mathcal{I}} {}_k\mathsf{X}_l$, where $\prod_{(k,l)\in\mathcal{I}} {}_k\mathsf{X}_l$ is endowed with the product of the discrete topologies.

For $(k, l) \in \mathcal{I}$ and $x \in \mathsf{X}$, we let $U(x, k, l)$ be the set

$$\{({}_r[{}_rx_s]_s)_{(r,s)\in\mathcal{I}} \in \widetilde{\mathsf{X}} \mid {}_kx_l \;{}_k{\sim}_l\; x\}.$$

Then $U(x, k, l)$ is open and closed, and $\{U(x, k, l) \mid x \in \mathsf{X}, (k, l) \in \mathcal{I}\}$ generates the topology of $\widetilde{\mathsf{X}}$. Let $(k, l) \in \mathcal{I}$. Then

$$x \;{}_k{\sim}_l\; y \Rightarrow \sigma(x) \;{}_{k-1}{\sim}_l\; \sigma(y),$$

so there exists a map ${}_k\sigma_l : {}_k\mathsf{X}_l \to {}_{k-1}\mathsf{X}_l$ such that ${}_k\sigma_l({}_k[x]_l) = {}_{k-1}[\sigma(x)]_l$. We then have that the following diagram commutes for every $(k_1, l_1) \leq (k_2, l_2) \in \mathcal{I}$:

$$\begin{array}{ccc}
{}_{k_1}\mathsf{X}_{l_1} & \xrightarrow{\;{}_{k_1}\sigma_{l_1}\;} & {}_{k_1-1}\mathsf{X}_{l_1} \\
\Big\downarrow {\scriptstyle {}_{(k_1,l_1)}\pi_{(k_2,l_2)}} & & \Big\downarrow {\scriptstyle {}_{(k_1-1,l_1)}\pi_{(k_2-1,l_2)}} \\
{}_{k_2}\mathsf{X}_{l_2} & \xrightarrow{\;{}_{k_2}\sigma_{l_2}\;} & {}_{k_2-1}\mathsf{X}_{l_2}.
\end{array}$$

Thus we get a continuous map $\widetilde{\sigma} : \widetilde{\mathsf{X}} \to \widetilde{\mathsf{X}}$, such that $\widetilde{\sigma}(({}_k[{}_kx_l]_l)_{(k,l)\in\mathcal{I}}) = ({}_k[\sigma({}_{k_1}x_l)]_l)_{(k,l)\in\mathcal{I}}$.

Proposition 1. *$\widetilde{\sigma}$ is a local homeomorphism.*

Proof. We will first show that $\widetilde{\sigma}$ is open. $\{U(x, k, l) \mid x \in \mathsf{X}, (k, l) \in \mathcal{I}, k > 0\}$ generates the topology of $\widetilde{\mathsf{X}}$, so it is enough to show that

$$\widetilde{\sigma}\left(\bigcap_{i=1}^{n} U(x_i, k'_i, l'_i)\right)$$

is open for $x_1, x_2, \ldots, x_n \in \mathsf{X}$ and $(k'_1, l'_1), (k'_2, l'_2) \ldots (k'_n, l'_n) \in \mathcal{I}$, where $k'_1 > 0$ for every $i \in \{1, 2, \ldots, n\}$. We can assume that all the x_i's begin with the same letter, which we denote by a, because otherwise $\bigcap_{i=1}^n U(x_i, k_i, l_i) = \emptyset$. We claim that

$$\widetilde{\sigma}\left(\bigcap_{i=1}^n U(x_i, k_i', l_i')\right) = \bigcap_{i=1}^n U(\sigma(x_i), k_i' - 1, l_i')).$$

Clearly

$$\widetilde{\sigma}\left(\bigcap_{i=1}^n U(x_i, k_i', l_i')\right) \subseteq \bigcap_{i=1}^n U(\sigma(x_i), k_i' - 1, l_i')).$$

Assume that $({}_k[{}_kx_l]_l)_{(k,l)\in\mathcal{I}} \in \bigcap_{i=1}^n U(\sigma(x_i), k_i' - 1, l_i')$. Then ${}_{k_1'-1}x_{l_1'}\ {}_{k_1'-1}\!\sim_{l_1'}\ \sigma(x_1)$, so $a_{k_1'-1}x_{l_1'} \in \mathsf{X}$, which implies that $a_kx_l \in \mathsf{X}$ for every $(k,l) \in \mathcal{I}$. Let ${}_ky_l = a_kx_{l+1}$ for $(k,l) \in \mathcal{I}$. Then

$$\begin{aligned} {}_{(k_1,l_1)}\pi_{(k_2,l_2)}({}_{k_2}[{}_{k_2}y_{l_2}]_{l_2}) &= {}_{k_1}[{}_{k_2}y_{l_2}]_{l_1} \\ &= {}_{k_1}[a_{k_2}x_{l_2+1}]_{l_1} \\ &= {}_{k_1}[a_{k_1}x_{l_1+1}]_{l_1} \\ &= {}_{k_1}[{}_{k_1}y_{l_1}]_{l_1} \end{aligned}$$

for $(k_1, l_1) \leq (k_2, l_2)$. Thus $({}_k[{}_ky_l]_l)_{(k,l)\in\mathcal{I}} \in \widetilde{\mathsf{X}}$.

Since ${}_{k_i'}x_{l_i'}\ {}_{k_i'-1}\!\sim_{l_i'}\ \sigma(x_i)$,

$${}_{k_i'}y_{l_i'} = a_{k_i'}x_{l_i'+1}\ {}_{k_i'}\!\sim_{l_i'}\ x_i$$

for every $i \in \{1, 2, \ldots, n\}$, so $({}_k[{}_ky_l]_l)_{(k,l)\in\mathcal{I}} \in \bigcap_{i=1}^n U(x_i, k_i', l_i')$. We have that

$${}_k[\sigma({}_{k+1}y_l)]_l = {}_k[{}_{k+1}x_{l+1}]_l = {}_k[{}_kx_l]_l$$

for every $(k,l) \in \mathcal{I}$, so

$$({}_k[{}_kx_l]_l)_{(k,l)\in\mathcal{I}} = \widetilde{\sigma}(({}_k[{}_ky_l]_l)_{(k,l)\in\mathcal{I}}) \in \widetilde{\sigma}\left(\bigcap_{i=1}^n U(x_i, k_i', l_i')\right).$$

Thus

$$\widetilde{\sigma}\left(\bigcap_{i=1}^n U(x_i, k_i', l_i')\right) = \bigcap_{i=1}^n U(\sigma(x_i), k_i' - 1, l_i')).$$

We then claim that $\widetilde{\sigma}$ is injective on $U(x, 1, 1)$ for every $x \in \mathsf{X}$, which shows that $\widetilde{\sigma}$ is locally injective and thus a local homeomorphism.

Assume that $({}_k[{}_kx_l]_l)_{(k,l)\in\mathcal{I}}, ({}_k[{}_ky_l]_l)_{(k,l)\in\mathcal{I}} \in U(x, 1, 1)$ and

$$\widetilde{\sigma}(({}_k[{}_kx_l]_l)_{(k,l)\in\mathcal{I}}) = \widetilde{\sigma}(({}_k[{}_ky_l]_l)_{(k,l)\in\mathcal{I}}).$$

Then

$${}_k[{}_kx_l]_l = {}_k[x_0\sigma({}_kx_l)]_l = {}_k[x_0\sigma({}_ky_l)]_l = {}_k[{}_ky_l]_l$$

for every $(k,l) \in \mathcal{I}$, so $({}_k[{}_kx_l]_l)_{(k,l)\in\mathcal{I}} = ({}_k[{}_ky_l]_l)_{(k,l)\in\mathcal{I}}$. □

Remark 2. There exists a surjective map π from $\widetilde{\mathsf{X}}$ to X such that $\pi\circ\widetilde{\sigma} = \sigma\circ\pi$, but since we are not going to need it here, we will not go into details on this matter.

The dynamical system $(\widetilde{\mathsf{X}},\widetilde{\sigma})$ is a singly generated dynamical system as defined in Definition 2.3 of [6]. We also have that $\widetilde{\mathsf{X}}$ is Hausdorff, second countable and compact. In [6] Renault constructed for such a dynamical system a groupoid $G(\widetilde{\mathsf{X}},\widetilde{\sigma})$ which is Hausdorff locally compact étale groupoid.

We will show that the corresponding C^*-algebra $C^*(\widetilde{\mathsf{X}},\widetilde{\sigma})$ is isomorphic to $\mathcal{O}_{\mathsf{X}}$ by construction a $*$-homomorphism from $\mathcal{O}_{\mathsf{X}}$ to $C^*(\widetilde{\mathsf{X}},\widetilde{\sigma})$ and a $*$-homomorphism from $C^*(\widetilde{\mathsf{X}},\widetilde{\sigma})$ to $\mathcal{O}_{\mathsf{X}}$ and then show that there are each other inverse.

2 The $*$-homomorphism from $\mathcal{O}_{\mathsf{X}}$ to $C^*(\widetilde{\mathsf{X}},\widetilde{\sigma})$

For every $\nu\in\mathsf{L}(\mathsf{X})$ we let

$$U_\nu = \bigcup_{x\in C(\nu)} U(x,|\nu|,|\nu|).$$

Then U_ν is a open subset of $\widetilde{\mathsf{X}}$, $\bigcup_{a\in\mathfrak{a}} U_a = \widetilde{\mathsf{X}}$, and $(\widetilde{\sigma})_{|U_a}$ is injective.

Let $a\in\mathfrak{a}$. Since $(\widetilde{\sigma})_{|U_a}$ is injective, we can for every $f\in C(\widetilde{\mathsf{X}})$ define a map $\lambda_a(f):\widetilde{\mathsf{X}}\to\mathbb{C}$ by

$$\lambda_a(f)(x) = \begin{cases} (f((\widetilde{\sigma})_{|U_a})^{-1}(x)) & x\in\widetilde{\sigma}(U_a) \\ 0 & x\notin\widetilde{\sigma}(U_a), \end{cases}$$

and since σ is open and continuous, and U_a is open and compact, $\lambda_a(f)$ is continuous. Hence λ_a is a $*$-homomorphism from $C(\widetilde{\mathsf{X}})$ to $C(\widetilde{\mathsf{X}})$. For $\mu=\mu_1\mu_2\dots\mu_n\in\mathsf{L}(X)$ we let $\lambda_\mu = \lambda_{\mu_1}\circ\lambda_{\mu_2}\circ\cdots\circ\lambda_{\mu_n}$.

Proposition 3. *There exists a $*$-isomorphism $\psi:\widetilde{\mathcal{D}}_{\mathsf{X}}\to C(\widetilde{\mathsf{X}})$, such that $\psi\left(1_{C(\mu)\cap\sigma^{-|\mu|}([x]_l)}\right) = 1_{U(\mu x,|\mu|,l)}$ for every $\mu\in\mathsf{L}(\mathsf{X})$, every $l\in\mathbb{N}_0$ and every $x\in\mathsf{X}$, and such that $\psi\left(\widetilde{\phi}_{\mathsf{X}}(f)\right) = \psi(f)\circ\sigma$ and $\psi(\widetilde{\lambda}_a(f)) = \lambda_a(\psi(f))$ for every $a\in\mathfrak{a}$ and every $f\in\widetilde{\mathcal{D}}_{\mathsf{X}}$.*

Proof. Let $(k,l)\in\mathcal{I}$. It is obvious that there only are finitely many l-past equivalence classes. Denote these by

$$\mathcal{E}^l_1,\mathcal{E}^l_2,\dots\mathcal{E}^l_{m(l)},$$

and choose for each $i\in\{1,2,\dots,m(l)\}$ an $x^l_i\in\mathcal{E}^l_i$.

Denote by $\widetilde{\mathcal{D}}_k^l$ the C^*-subalgebra of $\widetilde{\mathcal{D}}_{\mathsf{X}}$ generated by $1_{C(\mu,\nu)},\ \mu,\nu \in \mathsf{L}(\mathsf{X}),\ |\mu| \leq l,\ |\nu| = k$. It is easy to check that

$$1_{C(\nu)\cap\sigma^{-|\nu|}(\mathcal{E}_i^l)},\ \nu \in \mathsf{L}(\mathsf{X}),\ |\nu| = k,\ i = 1,2,\ldots,m(l)$$

are mutually orthogonal projections that generate $\widetilde{\mathcal{D}}_k^l$, and that

$$1_{U(\nu x_i^l,k,l)},\ \nu \in \mathsf{L}(\mathsf{X}),\ |\nu| = k,\ i = 1,2,\ldots,m(l)$$

are mutually orthogonal projections in $C(\widetilde{\mathsf{X}})$, and since

$$\begin{aligned} 1_{C(\nu)\cap\sigma^{-|\nu|}(\mathcal{E}_i^l)} = 0 &\Leftrightarrow C(\nu)\cap\sigma^{-|\nu|}(\mathcal{E}_i^l) = \emptyset \\ &\Leftrightarrow U(\nu x_i^l,k,l) = \emptyset \\ &\Leftrightarrow 1_{U(\nu x_i^l,k,l)} = 0, \end{aligned}$$

there exists a $*$-monomorphism $\psi_k^l : \widetilde{\mathcal{D}}_k^l \to C(\widetilde{\mathsf{X}})$ such that

$$\psi_k^l\left(1_{C(\nu)\cap\sigma^{-|\nu|}(\mathcal{E}_i^l)}\right) = 1_{U(\nu x_i^l,k,l)}$$

for every $\nu \in \mathsf{L}(\mathsf{X})$ with $|\nu| = k$ and every $i \in \{1,2,\ldots,m(l)\}$.

For every $k \in \mathbb{N}_0$ denote by $\widetilde{\mathcal{D}}_k$ the C^*-subalgebra of $\widetilde{\mathcal{D}}_\Lambda$ generated by $1_{C(\mu,\nu)},\ \mu,\nu \in \mathsf{L}(\mathsf{X}),\ |\nu| = k$. Then $\widetilde{\mathcal{D}}_k = \overline{\bigcup_{l\geq k}\widetilde{\mathcal{D}}_k^l}$. Let $\widetilde{\iota}_k^l$ denote the inclusion of $\widetilde{\mathcal{D}}_k^l$ into $\widetilde{\mathcal{D}}_k^{l+1}$. Since $\psi_k^{l+1}\circ\widetilde{\iota}_k^l = \psi_k^l$ for every $l \geq k$, the ψ_k^l's induce a $*$-monomorphism $\psi_k : \widetilde{\mathcal{D}}_k \to C(\widetilde{\mathsf{X}})$ such that $\psi_k^l\left(1_{C(\nu)\cap\sigma^{-|\nu|}(\mathcal{E}_i^l)}\right) = 1_{U(\nu x_i^l,k,l)}$ for every $\nu \in \mathsf{L}(\mathsf{X})$ and every $i \in \{1,2,\ldots,m(l)\}$.

Denote for every $k \in \mathbb{N}_0$ by $\widetilde{\iota}_k$ the inclusion of $\widetilde{\mathcal{D}}_k$ into $\widetilde{\mathcal{D}}_{k+1}$. Since $\psi_{k+1}\circ\widetilde{\iota}_k = \psi_k$ for every $k \in \mathbb{N}_0$ and $\widetilde{\mathcal{D}}_{\mathsf{X}} = \overline{\bigcup_{k\in\mathbb{N}}\widetilde{\mathcal{D}}_k}$, the ψ_k's induce a $*$-monomorphism $\psi : \widetilde{\mathcal{D}}_{\mathsf{X}} \to C(\widetilde{\mathsf{X}})$ such that $\psi\left(1_{C(\nu)\cap\sigma^{-|\nu|}(\mathcal{E}_i^l)}\right) = 1_{U(\nu x_i^l,k,l)}$ for every $\mu \in \mathsf{L}(\mathsf{X})$, every $l \in \mathbb{N}_0$ and every $i \in \{1,2,\ldots,m(l)\}$ and hence $\psi\left(1_{C(\mu)\cap\sigma^{-|\mu|}([x]_l)}\right) = 1_{U(\mu x,|\mu|,l)}$ for every $\mu \in \mathsf{L}(\mathsf{X})$, every $l \in \mathbb{N}_0$ and every $x \in \mathsf{X}$. By the Stone-Weierstrass Theorem

$$\psi\left(1_{C(\mu)\cap\sigma^{-|\mu|}(\mathcal{E}_i^l)}\right) = 1_{U(|\mu|,l,\mu,\mathcal{E}_i^l)},\ \mu \in \mathsf{L}(\mathsf{X}),\ l \in \mathbb{N}_0,\ i \in \{1,2,\ldots,m(l)\}$$

generates $C(\widetilde{\mathsf{X}})$, so ψ is surjective.

Let $a' \in \mathfrak{a}$, $\mu \in \mathsf{L}(\mathsf{X})$, $l \in \mathbb{N}_0$ and $x \in \mathsf{X}$. Then

$$\begin{aligned}
\psi\left(\widetilde{\phi}_{\mathsf{X}}\left(1_{C(\mu)\cap\sigma^{-|\mu|}([x]_l)}\right)\right) &= \sum_{a\in\mathfrak{a}} \psi\left(1_{C(a\mu)\cap\sigma^{-|a\mu|}([x]_l)}\right) \\
&= \sum_{a\in\mathfrak{a}} 1_{U(a\mu|x,|\mu|+1,l)} \\
&= 1_{\cup_{a\in\mathfrak{a}}U(a\mu|x,|\mu|+1,l)} \\
&= 1_{(\widetilde{\sigma})^{-1}(U(\mu|x,|\mu|,l))} \\
&= 1_{U(\mu|x,|\mu|,l)} \circ \widetilde{\sigma} \\
&= \psi\left(1_{C(\mu)\cap\sigma^{-|\mu|}([x]_l)}\right) \circ \widetilde{\sigma},
\end{aligned}$$

and

$$\begin{aligned}
\psi\left(\widetilde{\lambda}_{a'}\left(1_{C(\mu)\cap\sigma^{-|\mu|}([x]_l)}\right)\right) &= \psi\left(1_{C(\sigma(\mu))\cap\sigma^{1-|\mu|}([x]_l)\cap\sigma(C(a'))}\right) \\
&= 1_{U(\sigma(\mu x),|\mu|-1,l)\cap\sigma(U_{a'})} \\
&= 1_{\sigma(U(x,|\mu|,l))\cap\sigma(U_{a'})} \\
&= \lambda_{a'}(1_{U(x,|\mu|,l)}) \\
&= \lambda_{a'}\left(\psi\left(1_{C(\mu)\cap\sigma^{-|\mu|}([x]_l)}\right)\right),
\end{aligned}$$

and since $\widetilde{\mathcal{D}}_{\mathsf{X}}$ is generated by $1_{C(\mu)\cap\sigma^{-|\mu|}([x]_l)}$, $\mu \in \mathsf{L}(\mathsf{X})$, $l \in \mathbb{N}_0$, $x \in \mathsf{X}$ it follows that $\psi(\widetilde{\phi}_{\mathsf{X}}(f)) = \psi(f)\circ\sigma$ and $\psi(\widetilde{\lambda}_{a'}(f)) = \lambda_{a'}(\psi(f))$ for every $a' \in \mathfrak{a}$ and every $f \in \widetilde{\mathcal{D}}_{\mathsf{X}}$. $\square$

We will identify every $f \in C(\widetilde{\mathsf{X}})$ with the function $\tilde{f} \in C_C(G(\widetilde{\mathsf{X}},\widetilde{\sigma}))$ given by

$$\widetilde{f}(x,n,y) = \begin{cases} f(x) & \text{if } x = y \text{ and } n = 0, \\ 0 & \text{if } x \neq y \text{ or } n \neq 0. \end{cases}$$

As in [6, Definition 2.4], we let for $m,n \in \mathbb{N}_0$ and open subsets U and V of $\widetilde{\mathsf{X}}$ such that $\widetilde{\sigma}^m$ is injective on U and $\widetilde{\sigma}^n$ is injective on V, $\mathcal{U}(U,m,n,V)$ be the open subset

$$\{(x,m-n,y) \mid (x,y) \in U \times V,\ \widetilde{\sigma}^m(x) = \widetilde{\sigma}^n(y)\}$$

of $G(\widetilde{\mathsf{X}},\widetilde{\sigma})$, and we let for every $\mu \in \mathsf{L}(\mathsf{X})$, $\mathcal{U}_\mu = \mathcal{U}(U_\mu,|\mu|,0,\widetilde{\sigma}^{|\mu|}(U_\mu))$.

Proposition 4. *There exists a $*$-homomorphism from $\mathcal{O}_{\mathsf{X}}$ to $C^*(\widetilde{\mathsf{X}},\widetilde{\sigma})$ sending S_a to $1_{\mathcal{U}_a}$ for every $a \in \mathfrak{a}$.*

Proof. Let $\widetilde{\psi} : \widetilde{\mathcal{D}}_{\mathsf{X}} \to C_C(G(\widetilde{\mathsf{X}},\widetilde{\sigma}))$ be the $*$-isomorphism from Proposition 3 composed with the inclusion of $C(\widetilde{\mathsf{X}})$ into $C_C(G(\widetilde{\mathsf{X}},\widetilde{\sigma}))$ and let for every $a \in \mathfrak{a}$, $\widetilde{T}_a = 1_{\mathcal{U}_a}$. Let for $\mu = \mu_1\mu_2\cdots\mu_n \in \mathsf{L}(\mathsf{X})$, $T_\mu = T_{\mu_1}T_{\mu_2}\cdots T_{\mu_n}$.

Since

$$\begin{aligned}\widetilde{T}_a 1_{\mathcal{U}_\mu} &= \widetilde{T}_a 1_{\mathcal{U}(U_\nu,|\nu|,0,\widetilde{\sigma}^{|\nu|}(U_\nu))}(x,n,y)\\ &= \sum 1_{\mathcal{U}(U_a,1,0,\widetilde{\sigma}(U_a))}(x,m,z)1_{\mathcal{U}(U_\nu,|\nu|,0,\widetilde{\sigma}^{|\nu|}(U_\nu))}(z,n-m,y)\\ &= 1_{\mathcal{U}(U_{a\nu},|\nu|+1,0,\widetilde{\sigma}^{|\nu|+1}(U_{a\nu}))}(x,n,y)\\ &= 1_{\mathcal{U}_{a\nu}}\end{aligned}$$

for every $a \in \mathfrak{a}$, $\nu \in \mathsf{L}(\mathsf{X})$ and every $(x,n,y) \in G(\widetilde{\mathsf{X}},\widetilde{\sigma})$, we have that

$$\widetilde{T}_\mu = 1_{\mathcal{U}_\mu}$$

for every $\mu \in \mathsf{L}(\mathsf{X})$.

Since

$$\begin{aligned}\widetilde{T}_\mu \widetilde{T}_\mu^*(x,n,y) &= \sum 1_{\mathcal{U}(U_\mu,|\mu|,0,\widetilde{\sigma}^{|\mu|}(U_\mu))}(x,m,z)1_{\mathcal{U}(U_\mu,|\mu|,0,\widetilde{\sigma}^{|\mu|}(U_\mu))}(y,m-n,z)\\ &= 1_{\mathcal{U}(U_\mu,0,0,U_\mu)}(x,n,y)\\ &= \widetilde{\psi}\left(1_{C(\mu)}\right)(x,n,y)\end{aligned}$$

and

$$\begin{aligned}\widetilde{T}_\mu^* \widetilde{T}_\mu(x,n,y) &= \sum 1_{\mathcal{U}(U_\mu,|\mu|,0,\widetilde{\sigma}^{|\mu|}(U_\mu))}(z,-m,x)1_{\mathcal{U}(U_\mu,|\mu|,0,\widetilde{\sigma}^{|\mu|}(U_\mu))}(z,n-m,y)\\ &= 1_{\mathcal{U}(\widetilde{\sigma}(U_\mu),0,0,\widetilde{\sigma}(U_\mu))}(x,n,y)\\ &= \widetilde{\psi}\left(1_{\sigma^{|\mu|}(C(\mu))}\right)(x,n,y)\end{aligned}$$

for every $\mu \in \mathsf{L}(\mathsf{X})$ and every $(x,n,y) \in G(\widetilde{\mathsf{X}},\widetilde{\sigma})$, we have that

a) $\sum_{a\in\mathfrak{a}} \widetilde{T}_a \widetilde{T}_a^* = \widetilde{\psi}(I)$,

b) $\widetilde{T}_\mu^* \widetilde{T}_\mu \widetilde{T}_\nu \widetilde{T}_\nu^* = \widetilde{T}_\nu \widetilde{T}_\nu^* \widetilde{T}_\mu^* \widetilde{T}_\mu$,

c) $\widetilde{T}_\mu^* \widetilde{T}_\mu = \widetilde{\psi}\left(1_{\sigma^{|\mu|}(C(\mu))}\right)$.

for every $\mu, \nu \in \mathsf{L}(\mathsf{X})$. So by Theorem 7.1 in [1], there exists a $*$-homomorphism from $\mathcal{O}_\mathsf{X}$ to $C^*(G(\widetilde{\mathsf{X}},\phi_\mathsf{X}))$ sending S_a to $\widetilde{T}_a = 1_{\mathcal{U}_a}$ for every $a \in \mathfrak{a}$. □

3 The $*$-homomorphism from $C^*(\widetilde{\mathsf{X}},\widetilde{\sigma})$ to $\mathcal{O}_\mathsf{X}$

Lemma 5. *Let K be a compact subset of $G(\widetilde{\mathsf{X}},\widetilde{\sigma})$. Then there exists a finite subset F of $\mathsf{L}(\mathsf{X})^2$ such that*

$$\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu) \cap \mathcal{U}(U_{\mu'},|\mu'|,|\nu'|,U_{\nu'}) = \emptyset$$

for $(\mu,\nu),(\mu',\nu') \in F$ with $(\mu,\nu) \neq (\mu',\nu')$, and such that

$$K \subseteq \bigcup_{(\mu,\nu)\in F} \mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu).$$

Proof. Let K be a compact subset of $G(\widetilde{\mathsf{X}},\widetilde{\sigma})$. Since

$$K \subseteq G(\widetilde{\mathsf{X}},\widetilde{\sigma}) = \bigcup_{(\mu,\nu)\in \mathsf{L}(\mathsf{X})^2} \mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu),$$

there exists a finite set F' of $\mathsf{L}(\mathsf{X})^2$ such that

$$K \subseteq \bigcup_{(\mu,\nu)\in F'} \mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu).$$

Let

$$m = \max\{|\mu| \mid \exists \nu \in \mathsf{L}(\mathsf{X}) : (\mu,\nu)\in F'\},$$

and let

$$F = \{(\mu\alpha,\nu\alpha) \mid (\mu,\nu)\in F,\ \alpha\in\mathsf{L}(\mathsf{X}),\ |\alpha| = m-|\mu|\}.$$

Then F is a finite subset of $\mathsf{L}(\mathsf{X})^2$. Since for every $(\mu,\nu)\in F$,

$$\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu) = \bigcup_{|\alpha|=m-|\mu|} \mathcal{U}(U_{\mu\alpha},|\mu\alpha|,|\nu\alpha|,U_{\nu\alpha})$$

and

$$\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu)\cap\mathcal{U}(U_{\mu'},|\mu'|,|\nu'|,U_{\nu'}) = \emptyset$$

if $|\mu| = |\mu'|$ and $(\mu,\nu)\neq(\mu',\nu')$, we have that

$$\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu)\cap\mathcal{U}(U_{\mu'},|\mu'|,|\nu'|,U_{\nu'}) = \emptyset$$

for $(\mu,\nu),(\mu',\nu')\in F$ with $(\mu,\nu)\neq(\mu',\nu')$, and

$$K \subseteq \bigcup_{(\mu,\nu)\in F'} \mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu) = \bigcup_{(\mu,\nu)\in F} \mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu).$$

□

Lemma 6. *Let $\mu,\nu\in\mathsf{L}(\mathsf{X})$. For every $f\in C_C(G(\widetilde{\mathsf{X}},\widetilde{\sigma}))$ with $\operatorname{supp} f \subseteq \mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu)$ is*

$$\operatorname{supp}(1^*_{\mathcal{U}_\mu} f 1_{\mathcal{U}_\nu}) \subseteq \mathcal{U}(\sigma^{|\mu|}(U_\mu)\cap\sigma^{|\nu|}(U_\nu),0,0,\sigma^{|\mu|}(U_\mu)\cap\sigma^{|\nu|}(U_\nu)) \subseteq G^0(\widetilde{\mathsf{X}},\widetilde{\sigma}).$$

Proof. This follows from the fact that

$$1^*_{\mathcal{U}_\mu} f 1_{\mathcal{U}_\nu}(x,n,y) = \begin{cases} f(((\widetilde{\sigma})_{|U_\mu})^{-|\mu|}(x), n+|\mu|-|\nu|, ((\widetilde{\sigma})_{|U_\nu})^{-|\nu|}(y)) & \text{if } x\in\widetilde{\sigma}^{|\mu|}(U_\mu) \\ & \text{and } y\in\widetilde{\sigma}^{|\nu|}(U_\nu), \\ 0 & \text{if } x\notin\widetilde{\sigma}^{|\mu|}(U_\mu) \\ & \text{or } y\notin\widetilde{\sigma}^{|\nu|}(U_\nu). \end{cases}$$

□

Proposition 7. *There exists a $*$-homomorphism from $C^*(\widetilde{\mathsf{X}},\widetilde{\sigma})$ to $\mathcal{O}_{\mathsf{X}}$ sending $1_{\mathcal{U}_a}$ to S_a for every $a\in\mathfrak{a}$.*

Proof. Let $f\in C_C(G(\widetilde{\mathsf{X}},\widetilde{\sigma}))$. By Lemma 5 there exists a finite subset F of $\mathsf{L}(\mathsf{X})^2$ such that

$$\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu)\cap\mathcal{U}(U_{\mu'},|\mu'|,|\nu'|,U_{\nu'})=\emptyset$$

for $(\mu,\nu),(\mu',\nu')\in F$ with $(\mu,\nu)\neq(\mu',\nu')$, and such that

$$\operatorname{supp} f\subseteq\bigcup_{(\mu,\nu)\in F}\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu).$$

Since $\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu)$ is open and compact, $f_{|\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu)}\in C_C(G(\widetilde{\mathsf{X}},\widetilde{\sigma}))$ for every $(\mu,\nu)\in F$, and

$$f=\sum_{(\mu,\nu)\in F}f_{|\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu)}.$$

By Lemma 6

$$\operatorname{supp}(1^*_{\mathcal{U}_\mu}f_{|\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu)}1_{\mathcal{U}_\nu})\subseteq G^0(\widetilde{\mathsf{X}},\widetilde{\sigma}).$$

Let ϕ be the isomorphism from $C(G^0(\widetilde{\mathsf{X}},\widetilde{\sigma}))$ to $C(\widetilde{\mathsf{X}})$ composed with the inverse of the isomorphism from Proposition 3.

We want to show that a $*$-homomorphism from $C_C(G(\widetilde{\mathsf{X}},\widetilde{\sigma}))$ to $\mathcal{O}_{\mathsf{X}}$ is defined by

$$f\mapsto\sum_{(\mu,\nu)\in F}S_\mu\phi(1^*_{\mathcal{U}_\mu}f_{|\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu)}1_{\mathcal{U}_\nu})S^*_\nu.$$

We first show that $\sum_{(\mu,\nu)\in F}S_\mu\phi(1^*_{\mathcal{U}_\mu}f_{|\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu)}1_{\mathcal{U}_\nu})S^*_\nu$ is independent of F. So let F' be another finite subset of $\mathsf{L}(\mathsf{X})^2$ such that

$$\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu)\cap\mathcal{U}(U_{\mu'},|\mu'|,|\nu'|,U_{\nu'})=\emptyset$$

for $(\mu,\nu),(\mu',\nu')\in F'$ with $(\mu,\nu)\neq(\mu',\nu')$, and such that

$$\operatorname{supp} f\subseteq\bigcup_{(\mu,\nu)\in F'}\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu).$$

Let

$$m=\max\{|\mu|\mid\exists\nu\in\mathsf{L}(\mathsf{X}):(\mu,\nu)\in F\vee(\mu,\nu)\in F'\},$$

and let

$$F''=\{(\mu\alpha,\nu\alpha)\mid(\mu,\nu)\in F\vee(\mu,\nu)\in F',\ \alpha\in\mathsf{L}(\mathsf{X}),\ |\alpha|=m-|\mu|\}.$$

We let for every $(\mu,\nu)\in F$,

$$F''_{(\mu,\nu)}=\{(\mu\alpha,\nu\alpha)\in F''\mid\alpha\in\mathsf{L}(\mathsf{X})\}.$$

Then

$$\begin{aligned}\sum_{(\mu',\nu')\in F''_{(\mu,\nu)}} & S_{\mu'}\phi\left(1^*_{\mathcal{U}_{\mu'}} f_{|\mathcal{U}(U_{\mu'},|\mu'|,|\nu'|,U_{\nu'})} 1_{\mathcal{U}_{\nu'}}\right) S^*_{\nu'} \\ &= \sum_{|\alpha|=m-|\mu|} S_{\mu\alpha}\phi\left(1^*_{\mathcal{U}_{\mu\alpha}} f_{|\mathcal{U}(U_{\mu\alpha},|\mu\alpha|,|\nu\alpha|,U_{\nu\alpha})} 1_{\mathcal{U}_{\nu\alpha}}\right) S^*_{\nu\alpha} \\ &= \sum_{|\alpha|=m-|\mu|} S_\mu S_\alpha \phi\left(\lambda_\alpha\left(1^*_{\mathcal{U}_\mu} f_{|\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu)} 1_{\mathcal{U}_\nu}\right)\right) S^*_\alpha S^*_\nu \\ &= \sum_{|\alpha|=m-|\mu|} S_\mu S_\alpha S^*_\alpha \phi\left(1^*_{\mathcal{U}_\mu} f_{|\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu)} 1_{\mathcal{U}_\nu}\right) S_\alpha S^*_\alpha S^*_\nu \\ &= S_\mu\phi\left(1^*_{\mathcal{U}_\mu} f_{|\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu)} 1_{\mathcal{U}_\nu}\right) S^*_\nu.\end{aligned}$$

Since

$$\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu)\cap\mathcal{U}(U_{\mu'},|\mu'|,|\nu'|,U_{\nu'})=\emptyset$$

for $(\mu,\nu),(\mu',\nu')\in F$ with $(\mu,\nu)\neq(\mu',\nu')$,

$$F''_{(\mu,\nu)}\cap F''_{(\mu',\nu')}=\emptyset$$

for $(\mu,\nu),(\mu',\nu')\in F$ with $(\mu,\nu)\neq(\mu',\nu')$. Let $(\mu,\nu)\in F''$. If

$$\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu)\cap\operatorname{supp} f\neq\emptyset,$$

then since

$$\operatorname{supp} f\subseteq\bigcup_{(\mu,\nu)\in F}\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu)$$

there exists $(\mu',\nu')\in F$ such that

$$\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu)\cap\mathcal{U}(U_{\mu'},|\mu'|,|\nu'|,U_{\nu'})\neq\emptyset,$$

and thus there exist a $\alpha\in\mathsf{L}(\mathsf{X})$ such that $\mu=\mu'\alpha$ and $\nu=\nu'\alpha$. Hence $(\mu,\nu)\in F''_{(\mu',\nu')}$. So

$$\begin{aligned}\sum_{(\mu,\nu)\in F''} & S_\mu\phi(1^*_{\mathcal{U}_\mu} f_{|\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu)} 1_{\mathcal{U}_\nu})S^*_\nu \\ &= \sum_{(\mu,\nu)\in F}\ \sum_{(\mu\alpha,\nu\alpha)\in F''_{(\mu,\nu)}} S_{\mu\alpha}\phi(1^*_{\mathcal{U}_{\mu\alpha}} f_{|\mathcal{U}(U_{\mu\alpha},|\mu\alpha|,|\nu\alpha|,U_{\nu\alpha})} 1_{\mathcal{U}_{\nu\alpha}})S^*_{\nu\alpha} \\ &= \sum_{(\mu,\nu)\in F} S_\mu\phi(1^*_{\mathcal{U}_\mu} f_{|\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu)} 1_{\mathcal{U}_\nu})S^*_\nu.\end{aligned}$$

In the same way we can prove that

$$\sum_{(\mu,\nu)\in F''} S_\mu\phi(1^*_{\mathcal{U}_\mu} f_{|\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu)} 1_{\mathcal{U}_\nu})S^*_\nu = \sum_{(\mu,\nu)\in F'} S_\mu\phi(1^*_{\mathcal{U}_\mu} f_{|\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu)} 1_{\mathcal{U}_\nu})S^*_\nu.$$

So the map $\psi : C_C(G(\widetilde{\mathsf{X}}, \widetilde{\sigma})) \to \mathcal{O}_{\mathsf{X}}$ given by

$$\psi(f) = \sum_{(\mu,\nu)\in F} S_\mu \phi(1^*_{\mathcal{U}_\mu} f_{|\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu)} 1_{\mathcal{U}_\nu}) S^*_\nu$$

is well defined and is clearly linear.

If $\operatorname{supp} g \subseteq \mathcal{U}(U_\mu, |\mu|, |\nu|, U_\nu)$, then $\operatorname{supp} g^* \subseteq \mathcal{U}(U_\nu, |\nu|, |\mu|, U_\mu)$, so if $f \in C_C(G(\widetilde{\mathsf{X}}, \widetilde{\sigma}))$ and F is a finite subset of $\mathsf{L}(\mathsf{X})^2$ such that

$$\mathcal{U}(U_\mu, |\mu|, |\nu|, U_\nu) \cap \mathcal{U}(U_{\mu'}, |\mu'|, |\nu'|, U_{\nu'}) = \emptyset$$

for $(\mu, \nu), (\mu', \nu') \in F$ with $(\mu, \nu) \neq (\mu', \nu')$, and such that

$$\operatorname{supp} f \subseteq \bigcup_{(\mu,\nu)\in F} \mathcal{U}(U_\mu, |\mu|, |\nu|, U_\nu),$$

then $F' = \{(\nu, \mu) \in \mathsf{L}(\mathsf{X})^2 \mid (\mu, \nu) \in F\}$ is a finite subset of $\mathsf{L}(\mathsf{X})^2$ such that

$$\mathcal{U}(U_\mu, |\mu|, |\nu|, U_\nu) \cap \mathcal{U}(U_{\mu'}, |\mu'|, |\nu'|, U_{\nu'}) = \emptyset$$

for $(\mu, \nu), (\mu', \nu') \in F'$ with $(\mu, \nu) \neq (\mu', \nu')$, and such that

$$\operatorname{supp} f^* \subseteq \bigcup_{(\mu,\nu)\in F'} \mathcal{U}(U_\mu, |\mu|, |\nu|, U_\nu).$$

So

$$\begin{aligned}
\psi(f)^* &= \left(\sum_{(\mu,\nu)\in F} S_\mu \phi\left(1^*_{\mathcal{U}_\mu} f_{|\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu)} 1_{\mathcal{U}_\nu}\right) S^*_\nu \right)^* \\
&= \sum_{(\mu,\nu)\in F} S_\nu \phi\left(1^*_{\mathcal{U}_\nu} \left(f_{|\mathcal{U}(U_\nu,|\nu|,|\mu|,U_\mu)}\right)^* 1_{\mathcal{U}_\mu}\right) S^*_\mu \\
&= \sum_{(\mu,\nu)\in F'} S_\mu \phi\left(1^*_{\mathcal{U}_\mu} f^*_{|\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu)} 1_{\mathcal{U}_\nu}\right) S^*_\nu \\
&= \psi(f^*).
\end{aligned}$$

Let $f, g \in C_C(G(\widetilde{\mathsf{X}}, \widetilde{\sigma}))$, and let F_1 be a finite subset of $\mathsf{L}(\mathsf{X})^2$ such that

$$\mathcal{U}(U_\mu, |\mu|, |\nu|, U_\nu) \cap \mathcal{U}(U_{\mu'}, |\mu'|, |\nu'|, U_{\nu'}) = \emptyset$$

for $(\mu, \nu), (\mu', \nu') \in F_1$ with $(\mu, \nu) \neq (\mu', \nu')$, and such that

$$\operatorname{supp} f \subseteq \bigcup_{(\mu,\nu)\in F_1} \mathcal{U}(U_\mu, |\mu|, |\nu|, U_\nu),$$

and let F_2 be a finite subset of $\mathsf{L}(\mathsf{X})^2$ such that

$$\mathcal{U}(U_\mu, |\mu|, |\nu|, U_\nu) \cap \mathcal{U}(U_{\mu'}, |\mu'|, |\nu'|, U_{\nu'}) = \emptyset$$

for $(\mu,\nu),(\mu',\nu')\in F_2$ with $(\mu,\nu)\neq(\mu',\nu')$, and such that

$$\operatorname{supp} g\subseteq\bigcup_{(\mu,\nu)\in F_2}\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu).$$

Let

$$m=\max\{|\mu|\mid\exists\nu\in\mathsf{L}(\mathsf{X}):(\nu,\mu)\in F_1\vee(\mu,\nu)\in F_2\},$$
$$F_1'=\{(\mu\alpha,\nu\alpha)\mid(\mu,\nu)\in F_1,\ \alpha\in\mathsf{L}(\mathsf{X}),\ |\alpha|=m-|\nu|\}$$

and

$$F_2'=\{(\mu\alpha,\nu\alpha)\mid(\mu,\nu)\in F_2,\ \alpha\in\mathsf{L}(\mathsf{X}),\ |\alpha|=m-|\mu|\}.$$

Then F_1' is a finite subset of $\mathsf{L}(\mathsf{X})^2$ such that

$$\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu)\cap\mathcal{U}(U_{\mu'},|\mu'|,|\nu'|,U_{\nu'})=\emptyset$$

for $(\mu,\nu),(\mu',\nu')\in F_1'$ with $(\mu,\nu)\neq(\mu',\nu')$, and such that

$$\operatorname{supp} f\subseteq\bigcup_{(\mu,\nu)\in F_1'}\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu),$$

and F_2' is a finite subset of $\mathsf{L}(\mathsf{X})^2$ such that

$$\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu)\cap\mathcal{U}(U_{\mu'},|\mu'|,|\nu'|,U_{\nu'})=\emptyset$$

for $(\mu,\nu),(\mu',\nu')\in F_2'$ with $(\mu,\nu)\neq(\mu',\nu')$, and such that

$$\operatorname{supp} g\subseteq\bigcup_{(\mu,\nu)\in F_2'}\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu).$$

Let

$$k=\max\{|\mu|\mid\exists\nu\in\mathsf{L}(\mathsf{X}):(\mu,\nu)\in F_1'\}$$

and set

$$F=\{(\mu\alpha,\nu\alpha)\mid\exists\gamma\in\mathsf{L}(\mathsf{X}):(\mu,\gamma)\in F_1'\wedge (\gamma,\nu)\in F_2',\alpha\in\mathsf{L}(\mathsf{X}),\ |\alpha|=k-|\mu|\}.$$

Then F is a finite subset of $\mathsf{L}(\mathsf{X})^2$ such that

$$\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu)\cap\mathcal{U}(U_{\mu'},|\mu'|,|\nu'|,U_{\nu'})=\emptyset$$

for $(\mu,\nu),(\mu',\nu')\in F$ with $(\mu,\nu)\neq(\mu',\nu')$. Since

$$fg(x,n,y)=\sum f(x,m,z)g(z,n-m,y)$$

for every $(x,n,y)\in G(\widetilde{\mathsf{X}},\widetilde{\sigma})$, we have that if $(x,n,y)\in\operatorname{supp} fg$, then there exist a $m\in\mathbb{N}$ and a $z\in\widetilde{\mathsf{X}}$ such that $(x,m,z)\in\operatorname{supp} f$, and $(z,n-$

$m, y) \in \operatorname{supp} g$, and thus there exist $(\mu,\nu) \in F_1'$ and $(\mu',\nu') \in F_2'$ such that $(x,m,z) \in \mathcal{U}(U_\mu, |\mu|, |\nu|, U_\nu)$, and $(z, n-m, y) \in \mathcal{U}(U_{\mu'}, |\mu'|, |\nu'|, U_{\nu'})$. Since $|\nu| = |\mu'|$ and $z \in U_\nu \cap U_{\mu'}$, we have that $\nu = \mu'$, and thus $(\mu\alpha, \nu'\alpha) \in F$, where α is the unique element in $\mathsf{L}(\mathsf{X})$ such that $|\alpha| = k - |\mu|$ and such that

$$\widetilde{\sigma}^{|\mu|}(x) = \widetilde{\sigma}^{|\nu|}(z) = \widetilde{\sigma}^{|\nu'|}(y) \in U_\alpha.$$

Hence

$$\operatorname{supp} fg \subseteq \bigcup_{(\mu,\nu)\in F} \mathcal{U}(U_\mu, |\mu|, |\nu|, U_\nu).$$

So

$$\begin{aligned}
\psi(f)\psi(g) &= \sum_{(\mu,\nu)\in F_1'} S_\mu \phi\left(1^*_{\mathcal{U}_\mu} f_{|\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu)} 1_{\mathcal{U}_\nu}\right) S_\nu^* \\
&\quad \sum_{(\mu',\nu')\in F_2'} S_{\mu'} \phi\left(1^*_{\mathcal{U}_{\mu'}} g_{|\mathcal{U}(U_{\mu'},|\mu'|,|\nu'|,U_{\nu'})} 1_{\mathcal{U}_{\nu'}}\right) S_{\nu'}^* \\
&= \sum_{\substack{(\mu,\nu)\in F_1' \\ (\nu,\nu')\in F_2'}} S_\mu \phi\left(1^*_{\mathcal{U}_\mu} f_{|\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu)} 1_{\mathcal{U}_\nu}\right) S_\nu^* \\
&\quad S_\nu \phi\left(1^*_{\mathcal{U}_\nu} g_{|\mathcal{U}(U_\nu,|\nu|,|\nu'|,U_{\nu'})} 1_{\mathcal{U}_{\nu'}}\right) S_{\nu'}^* \\
&= \sum_{\substack{(\mu,\nu)\in F_1' \\ (\nu,\nu')\in F_2'}} S_\mu \phi\left(1^*_{\mathcal{U}_\mu} f_{|\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu)} 1_{\mathcal{U}_\nu}\right) \phi\left(1_{\widetilde{\sigma}^{|\nu|}(U_\nu)}\right) \\
&\quad \phi\left(1^*_{\mathcal{U}_\nu} g_{|\mathcal{U}(U_\nu,|\nu|,|\nu'|,U_{\nu'})} 1_{\mathcal{U}_{\nu'}}\right) S_{\nu'}^* \\
&= \sum_{\substack{(\mu,\nu)\in F_1' \\ (\nu,\nu')\in F_2'}} S_\mu \phi\left(1^*_{\mathcal{U}_\mu} f_{|\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu)} g_{|\mathcal{U}(U_\nu,|\nu|,|\nu'|,U_{\nu'})} 1_{\mathcal{U}_{\nu'}}\right) S_{\nu'}^* \\
&= \sum_{\substack{(\mu,\nu)\in F_1' \\ (\nu,\nu')\in F_2' \\ |\alpha|=k-n}} S_\mu S_\alpha S_\alpha^* \phi\left(1^*_{\mathcal{U}_\mu} f_{|\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu)} g_{|\mathcal{U}(U_\nu,|\nu|,|\nu'|,U_{\nu'})} 1_{\mathcal{U}_{\nu'}}\right) S_\alpha S_\alpha^* S_{\nu'}^* \\
&= \sum_{\substack{(\mu,\nu)\in F_1' \\ (\nu,\nu')\in F_2' \\ |\alpha|=k-n}} S_\mu S_\alpha \phi\left(1^*_{\mathcal{U}_\alpha} 1^*_{\mathcal{U}_\mu} f_{|\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu)} g_{|\mathcal{U}(U_\nu,|\nu|,|\nu'|,U_{\nu'})} 1_{\mathcal{U}_{\nu'}} 1_{\mathcal{U}_\alpha}\right) S_\alpha^* S_{\nu'}^* \\
&= \sum_{(\mu,\nu)\in F} S_\mu \phi\left(1^*_{\mathcal{U}_\mu} fg_{|\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu)} 1_{\mathcal{U}_\nu}\right) S_\nu^* \\
&= \psi(fg)
\end{aligned}$$

Hence ψ is a $*$-homomorphism, and since $G(\widetilde{\mathsf{X}}, \widetilde{\sigma})$ is a second countable locally compact r-discrete groupoid, it follows from Corollary II,1.22 of [5] that ψ is $||\cdot||_I$-bounded and hence extends to the C^*-algebra $C^*(\widetilde{\mathsf{X}}, \widetilde{\sigma})$. □

4 $\mathcal{O}_{\mathsf{X}}$ is isomorphic to $C^*(\widetilde{\mathsf{X}},\widetilde{\sigma})$

Lemma 8. *Let $\mathcal{X}$ be the $*$-subalgebra of $C^*(\widetilde{\mathsf{X}},\widetilde{\sigma})$ generated by $1_{\mathcal{U}_a},\ a\in\mathfrak{a}$. Then $1_{U(\mu x,|\mu|,l)}\in\mathcal{X}$ for every $\mu\in\mathsf{L}(\mathsf{X}),\ l\in\mathbb{N}_0$ and every $x\in\mathsf{X}$.*

Proof. We know that if $\mu=\mu_1\mu_2\cdots\mu_m\in\mathsf{L}(\mathsf{X})$, then $1_{\mathcal{U}_\mu}=1_{\mathcal{U}_{\mu_1}}1_{\mathcal{U}_{\mu_2}}\cdots 1_{\mathcal{U}_{\mu_m}}$, so $1_{\mathcal{U}_\mu}\in\mathcal{X}$ for every $\mu\in\mathsf{L}(\mathsf{X})$.

Let $\mu\in\mathsf{L}(\mathsf{X})$, $l\in\mathbb{N}_0$ and $x\in\mathsf{X}$. Since

$$U(\mu x,|\mu|,l)=$$
$$U_\mu\cap\widetilde{\sigma}^{-|\mu|}\left(\left[\bigcap_{\nu\in\cup_{k=0}^l\mathcal{P}_k(x)}\widetilde{\sigma}^{|\nu|}(U_\nu)\right]\bigcap\left[\bigcap_{\nu\in\cup_{k=0}^l\mathfrak{a}^k\setminus\mathcal{P}_k(x)}\widetilde{\mathsf{X}}\setminus\widetilde{\sigma}^{|\nu|}(U_\nu)\right]\right).$$

we have that

$$1_{U(\mu x,|\mu|,l)}=\left[\prod_{\nu\in\cup_{k=0}^l\mathcal{P}_k(x)}1_{\mathcal{U}_\mu}1_{\mathcal{U}_\nu}^*1_{\mathcal{U}_\nu}1_{\mathcal{U}_\mu}\right]$$
$$\left[\prod_{\nu\in\cup_{k=0}^l\mathfrak{a}^k\setminus\mathcal{P}_k(x)}(1_{\mathcal{U}_\mu}1_{\mathcal{U}_\mu}^*-1_{\mathcal{U}_\mu}1_{\mathcal{U}_\nu}^*1_{\mathcal{U}_\nu}1_{\mathcal{U}_\mu}^*)\right]\in\mathcal{X}.$$

□

Lemma 9. *$C^*(\widetilde{\mathsf{X}},\widetilde{\sigma})$ is generated by $1_{\mathcal{U}_a},\ a\in\mathfrak{a}$.*

Proof. Let $\mathcal{X}$ be the $*$-subalgebra of $C^*(\widetilde{\mathsf{X}},\widetilde{\sigma})$ generated by $1_{\mathcal{U}_a},\ a\in\mathfrak{a}$. We then have to show that $\mathcal{X}$ is dense in $C^*(\widetilde{\mathsf{X}},\widetilde{\sigma})$. Since $C_C(G(\widetilde{\mathsf{X}},\widetilde{\sigma}))$ is dense in $C^*(\widetilde{\mathsf{X}},\widetilde{\sigma})$, and the C^*-norm is dominated by $||\cdot||_I$, it is enough to show that $\mathcal{X}$ is $||\cdot||_I$-dense in $C_C(G(\widetilde{\mathsf{X}},\widetilde{\sigma}))$.

From Lemma 5 we know that every $f\in C_C(G(\widetilde{\mathsf{X}},\widetilde{\sigma}))$ can be written as a finite sum $\sum_{i=1}^n f_i$, where for each $i\in\{1,2,\dots,n\}$, $\operatorname{supp} f_i\subseteq\mathcal{U}(U_{\mu_i},|\mu_i|,|\nu_i|,U_{\nu_i})$ for some $\mu_i,\nu_i\in\mathsf{L}(\mathsf{X})$. So we just have to show that $\{f\in\mathcal{X}\mid\operatorname{supp} f\subseteq\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu)\}$ is $||\cdot||_I$-dense in

$$\{f\in C_C(G(\widetilde{\mathsf{X}},\widetilde{\sigma}))\mid\operatorname{supp} f\subseteq\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu)\}$$

for every $\mu,\nu\in\mathsf{L}(\mathsf{X})$. Since $\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu)$ is a $\mathcal{G}$-set (i.e. r and s are injective on $\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu)$), $||\cdot||_I$ is dominated by the uniform norm on $\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu)$. Hence we just have to show that every

$$f\in C(\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu))$$

can be approximated by elements of

$$\{g_{|\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu)}\mid g\in\mathcal{X},\ \operatorname{supp} g\subseteq\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu)\}.$$

$C(\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu)$ is of course a C^*-algebra with pointwise operations and the uniform norm. Let $\mathcal{X}_{(\mu,\nu)}$ be the $*$-subalgebra of $C(\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu)$ generated by

$$1_{\mathcal{U}(U(\mu\alpha x,|\mu\alpha|,l),|\mu|,|\nu|,U(\nu\alpha x,|\nu\alpha|,l))},\ \alpha\in\mathsf{L}(\mathsf{X}),\ l\in\mathbb{N}_0,\ x\in\mathsf{X}.$$

For $\mu=\mu_1\mu_2\cdots\mu_m\in\mathsf{L}(\mathsf{X})$, we let $\eta(\mu)=\mu_1\mu_2\cdots\mu_{m-1}$. Since the product of

$$1_{\mathcal{U}(U(\mu\alpha_1 x_1,|\mu\alpha|,l_1),|\mu|,|\nu|,U(\nu\alpha_1 x_1,|\nu\alpha_1|,l_1))}$$

and

$$1_{\mathcal{U}(U(\mu\alpha_2 x_2,|\mu\alpha_2|,l_2),|\mu|,|\nu|,U(\nu\alpha_2 x_2,|\nu\alpha_2|,l_2))}$$

in $C(\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu)$ is

$$\begin{cases}1_{\mathcal{U}(U(\mu\alpha_1 x_1,|\mu\alpha_1|,l_1),|\mu|,|\nu|,U(\nu\alpha_1 x_1,|\nu\alpha_1|,l_1))} & \text{if } |\alpha_1|\geq|\alpha_2|,\\ & \eta^{(|\alpha_1|-|\alpha_2|)}(\alpha_1)=\alpha_2,\ l_1\geq l_2\\ & \text{and } [x_1]_{l_2}=[x_2]_{l_2},\\ 1_{\mathcal{U}(U(\mu\alpha_1 x_2,|\mu\alpha_1|,l_2),|\mu|,|\nu|,U(\nu\alpha_1 x_2,|\nu\alpha_1|,l_2))} & \text{if } |\alpha_1|\geq|\alpha_2|,\\ & \eta^{(|\alpha_1|-|\alpha_2|)}(\alpha_1)=\alpha_2,\ l_2\geq l_1\\ & \text{and } [x_2]_{l_1}=[x_1]_{l_1},\\ 1_{\mathcal{U}(U(\mu\alpha_2 x_1,|\mu\alpha_2|,l_1),|\mu|,|\nu|,U(\nu\alpha_2 x_1,|\nu\alpha_2|,l_1))} & \text{if } |\alpha_2|\geq|\alpha_1|,\\ & \eta^{(|\alpha_2|-|\alpha_1|)}(\alpha_2)=\alpha_1, l_1\geq l_2\\ & \text{and } [x_1]_{l_2}=[x_2]_{l_2},\\ 1_{\mathcal{U}(U(\mu\alpha_2 x_2,|\mu\alpha_2|,l_2),|\mu|,|\nu|,U(\nu\alpha_2 x_2,|\nu\alpha_2|,l_2))} & \text{if } |\alpha_2|\geq|\alpha_1|,\\ & \eta^{(|\alpha_2|-|\alpha_1|)}(\alpha_2)=\alpha_1, l_2\geq l_1\\ & \text{and } [x_2]_{l_1}=[x_1]_{l_1},\\ 0 & \text{else.}\end{cases}$$

and

$$\overline{1_{\mathcal{U}(U(\mu\alpha x,|\mu\alpha|,l),|\mu|,|\nu|,U(\nu\alpha x,|\nu\alpha|,l))}}=1_{\mathcal{U}(U(\mu\alpha x,|\mu\alpha|,l),|\mu|,|\nu|,U(\nu\alpha x,|\nu\alpha|,l))}$$

we have that

$$\mathcal{X}_{(\mu,\nu)}=\\ \operatorname{span}\{1_{\mathcal{U}(U(\mu\alpha x,|\mu\alpha|,l),|\mu|,|\nu|,U(\nu\alpha x,|\nu\alpha|,l))}\mid\alpha\in\mathsf{L}(\mathsf{X}),\ l\in\mathbb{N},\ x\in\mathsf{X}\}.$$

By Lemma 8, $1_{U(\mu\alpha x,|\mu\alpha|,l)}\in\mathcal{X}$ for every $\alpha\in\mathsf{L}(\mathsf{X}),\ l\in\mathbb{N},\ x\in\mathsf{X}$, so

$$1_{\mathcal{U}(U(\mu\alpha x,|\mu\alpha|,l),|\mu|,|\nu|,U(\nu\alpha x,|\nu\alpha|,l))}=1_{\mathcal{U}_\mu}1_{U(|\alpha|,l,U_\alpha,[x]_l)}1^*_{\mathcal{U}_\nu}\in\mathcal{X}$$

and hence $\mathcal{X}_{(\mu,\nu)}\subseteq\{g_{|\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu)}\mid \operatorname{supp} g\subseteq\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu)\}$. Since $\mathcal{X}_{(\mu,\nu)}$ contain the constant functions and separate points it follows from the Stone-Weierstrass Theorem that $\mathcal{X}_{(\mu,\nu)}$ and hence

$$\{g_{|\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu)}\mid g\in\mathcal{X},\ \operatorname{supp} g\subseteq\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu)\}$$

is uniform dense in $C(\mathcal{U}(U_\mu,|\mu|,|\nu|,U_\nu))$, and we are done. □

Theorem 10. *There exists a $*$-isomorphism between $\mathcal{O}_{\mathsf{X}}$ and $C^*(\widetilde{\mathsf{X}},\widetilde{\sigma})$ sending S_a to $1_{\mathcal{U}_a}$ for every $a \in \mathfrak{a}$.*

Proof. We know from Proposition 4 that there exists a $*$-homomorphism from $\mathcal{O}_{\mathsf{X}}$ to $C^*(\widetilde{\mathsf{X}},\phi_{\mathsf{X}})$ sending S_a to $1_{\mathcal{U}_a}$ for every $a \in \mathfrak{a}$ and from Proposition 7 that there exists a $*$-homomorphism from $C^*(\widetilde{\mathsf{X}},\phi_{\mathsf{X}})$ to $\mathcal{O}_{\mathsf{X}}$ sending $1_{\mathcal{U}_a}$ to S_a for every $a \in \mathfrak{a}$, and since $\mathcal{O}_{\mathsf{X}}$ is generated by $S_a,\ a \in \mathfrak{a}$ and $C^*(\widetilde{\mathsf{X}},\widetilde{\sigma})$ is generated by $1_{\mathcal{U}_a},\ a \in \mathfrak{a}$ these $*$-homomorphisms are each other inverse. □

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Chapter 3

Symbolic dynamics, partial dynamical systems, Boolean algebras and C^*-algebras generated by partial isometries

This chapter consists of the preprint *Symbolic dynamics, partial dynamical systems, Boolean algebras and C^*-algebras generated by partial isometries* which is an attempt to relate the C^*-algebra associated to one-sided shift spaces to other C^*-algebras associated to symbolic dynamical systems and to unify all these C^*-algebras into a single construction which also applies to classes of symbolic dynamics to which, as far as I know, no C^*-algebras have previous been associated.

The preprint also includes a results which connects the crossed product of a two-sided shift space having a certain property $(*)$ and the C^*-algebra associated to the corresponding one-sided shift space. This result is the foundation for the work presented in the remaining 4 papers.

SYMBOLIC DYNAMICS, PARTIAL DYNAMICAL SYSTEMS, BOOLEAN ALGEBRAS AND C^*-ALGEBRAS GENERATED BY PARTIAL ISOMETRIES

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ABSTRACT. We associate to each partial dynamical system a universal C^*-algebra generated by partial isometries satisfying relations given by a Boolean algebra. We show that for symbolic dynamical systems like one-sided and two-sided shift spaces, topological Markov chains with an arbitrary state space and higher rank graph, the C^*-algebras usually associated to them, can be obtained in this way.

As a consequence of this, we get that for two-sided shift spaces having a certain property, the crossed product of the two-sided shift space is a quotient of the C^*-algebra associated to the corresponding one-sided shift space.

We also suggest how to associate C^*-algebras to higher dimensional shifts over an infinite alphabet.

1. INTRODUCTION

One-sided and two-sided shift spaces (also called subshifts), topological Markov chains with an arbitrary state space and higher rank graph are all symbolic dynamical systems to which C^*-algebras have been associated. We will in this paper show that all of these C^*-algebras can be obtained as crossed product of C^*-partial dynamical systems.

The symbolic dynamical systems mentioned here all come with a natural partial dynamical system, but this system can not in general be turned in to a C^*-partial dynamical systems in a straight forward way. We surmount this problem by turning every discrete partial dynamical system into a C^*-partial dynamical systems by using Boolean algebras and thus associate to every discrete partial dynamical system a universal C^*-algebra generated by partial isometries satisfying relations given by a Boolean algebra.

This allows us to associated to shifts with an infinite alphabet and to higher dimensional shifts C^*-algebras which are generalizations of the C^*-algebra associated to one-sided shift spaces, the C^*-algebra associated to higher rank graph and the C^*-algebra associated to 0-1 matrices.

We will also to each discrete partial dynamical system associate a reduced C^*-algebra, and we will also show that for two-sided shift spaces having a certain property, the crossed product of the two-sided shift space is a factor of the C^*-algebra associated to the corresponding one-sided shift space. This

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lays the ground for a description of the K-theory of the C^*-algebra associated to the one-sided shift space which is explained in [2, 3].

2. Preliminaries and notation

Throughout this paper e will denote the neutral element of a given group and Id_X will denote the identity map on the set X.

If θ is a map defined on a subset A of some set X, then we will for another subset B of X by $\theta(B)$ mean $\theta(A \cap B)$. We use $\overline{\mathrm{span}}(A)$ to denote the closure of the linear span of A. For a subset B of a C^*-algebra we denote the C^*-sub algebra generated by B, by $C^*(B)$.

For partial isometries S, T we say that S extends T and write $S \geq T$ if $TT^* = TS^*$ (cf. [10, Lemma 1.6]).

3. Discrete partial dynamical systems

Definition 1. Given a group G and a set X, a *partial action* θ of G on X is a pair

$$(\{D_t\}_{t \in G}, \{\theta_t\}_{t \in G}),$$

where for each $t \in G$, D_t is a subset of X and θ_t is a bijective map from $D_{t^{-1}}$ to D_t, satisfying for all r and s in G

(1) $D_e = X$ and θ_e is the identity map on X,
(2) $\theta_r(D_s) = D_r \cap D_{rs}$,
(3) $\theta_r(\theta_s(x)) = \theta_{rs}(x)$ for $x \in D_{s^{-1}} \cap D_{s^{-1}r^{-1}}$.

The triple (X, G, θ) is called a *discrete partial dynamical system.*

Let us look at some examples of some symbolic dynamical systems which all come with a natural discrete partial dynamical systems.

Example 1. Let (X, σ) be a two-sided shift space over the finite alphabet $\mathfrak{a}$ (cf. [8]). Let for every $a \in \mathfrak{a}$, $D_a = \{x \in \mathsf{X} \mid x_0 = a\}$, $D_{a^{-1}} = \{x \in \mathsf{X} \mid x_{-1} = a\}$,

$$\theta_a = \sigma^{-1}_{|D_{a^{-1}}} : D_{a^{-1}} \to D_a$$

and

$$\theta_{a^{-1}} = \sigma_{|D_a} : D_a \to D_{a^{-1}}.$$

Let $\mathbb{F}_{\mathfrak{a}}$ be the free group generated by $\mathfrak{a}$ and let for every $t \in \mathbb{F}_{\mathfrak{a}}$ written on reduced form $u_1 u_2 \cdots u_n$, where $u_1, u_2, \ldots, u_n \in \mathfrak{a} \cup \{a^{-1} \mid a \in \mathfrak{a}\}$,

$$D_t = \theta_{u_1} \circ \theta_{u_2} \circ \cdots \circ \theta_{u_n}(\mathsf{X})$$

and

$$\theta_t = \theta_{u_1} \circ \theta_{u_2} \circ \cdots \circ \theta_{u_n}.$$

Then $\mathbf{X} = (\mathsf{X}, \mathbb{F}_{\mathfrak{a}}, \theta)$ is a discrete partial dynamical system.

Example 2. Let (X^+, σ_+) be a one-sided shift space over the finite alphabet $\mathfrak{a}$ (cf. [8, §13.8]). Let for every $a \in \mathfrak{a}$, $D_a = \{x \in \mathsf{X}^+ \mid x_0 = a\}$, $D_{a^{-1}} = \sigma_+(D_a)$, $\theta_a : D_{a^{-1}} \to D_a$ the map

$$x \mapsto ax,$$

and $\theta_{a^{-1}} : D_a \to D_{a^{-1}}$ the map

$$x \mapsto \sigma_+(x).$$

Let $\mathbb{F}_{\mathfrak{a}}$ be the free group generated by $\mathfrak{a}$ and let for every $t \in \mathbb{F}_{\mathfrak{a}}$ written on reduced form $u_1 u_2 \cdots u_n$, where $u_1, u_2, \ldots, u_n \in \mathfrak{a} \cup \{a^{-1} \mid a \in \mathfrak{a}\}$,

$$D_t = \theta_{u_1} \circ \theta_{u_2} \circ \cdots \circ \theta_{u_n}(\mathsf{X}^+)$$

and

$$\theta_t = \theta_{u_1} \circ \theta_{u_2} \circ \cdots \circ \theta_{u_n}.$$

Then $\mathbf{X}^+ = (\mathsf{X}^+, \mathbb{F}_{\mathfrak{a}}, \theta)$ is a discrete partial dynamical system.

Example 3. Let $\mathcal{I}$ be an arbitrary index set and let $A = A(i,j)_{i,j\in\mathcal{I}}$ be a matrix with entries in $\{0,1\}$ and having no identically zero rows. Let X_A^+ be the set

$$\{(x_n)_{n\in\mathbb{N}_0} \in \prod_{n\in\mathbb{N}_0} \mathcal{I} \mid \forall n \in \mathbb{N}_0 : A(x_n, x_{n+1}) = 1\},$$

and let $\sigma_+ : \mathsf{X}_A^+ \to \mathsf{X}_A^+$ be the shift mapping defined by

$$\sigma_+((x_n)_{n\in\mathbb{N}_0})_n = x_{n+1}$$

for $(x_n)_{n\in\mathbb{N}_0}\mathsf{X}_A^+$ and $n \in \mathbb{N}_0$.

Let for every $i \in \mathcal{I}$, $D_i = \{x \in \mathsf{X}_A^+ \mid x_0 = i\}$ and $D_{i^{-1}} = \sigma_+(D_i)$, and let $\theta_i : D_{i^{-1}} \to D_i$ be the map

$$x \mapsto ix,$$

and $\theta_{i^{-1}} : D_i \to D_{i^{-1}}$ the map

$$x \mapsto \sigma_+(x).$$

Let $\mathbb{F}_{\mathcal{I}}$ be the free group generated by $\mathcal{I}$ and let for every $t \in \mathbb{F}_{\mathcal{I}}$ written on reduced form $u_1 u_2 \cdots u_n$, where $u_1, u_2, \ldots, u_n \in \mathcal{I} \cup \{i^{-1} \mid i \in \mathcal{I}\}$,

$$D_t = \theta_{u_1} \circ \theta_{u_2} \circ \cdots \circ \theta_{u_n}(\mathsf{X}_A^+)$$

and

$$\theta_t = \theta_{u_1} \circ \theta_{u_2} \circ \cdots \circ \theta_{u_n}.$$

Then $\mathbf{X}_{\mathbf{A}}^+ = (\mathsf{X}_A^+, \mathbb{F}_{\mathcal{I}}, \theta)$ is a discrete partial dynamical system.

Example 4. Let Λ be a k-graph (cf. [7]) and let G be the quotient of $\mathbb{F}_{\Lambda\setminus\Lambda^0}$ where we for $\lambda, \mu \in \Lambda \setminus \Lambda^0$ with $s(\lambda) = r(\lambda)$ identify the product of λ and μ with $\lambda\mu \in \Lambda$.

Let for $\lambda \in \Lambda \setminus \Lambda^0$,

$$D_\lambda = \{x \in \Lambda^\infty \mid x(0, d(\lambda)) = \lambda\}.$$

It follows from [7, Proposition 2.3] that $\sigma^{d(\lambda)}_{|D_\lambda}$ is injective. Let $D_{\lambda^{-1}} = \sigma^{d(\lambda)}_{|D_\lambda}(D_\lambda)$, $\theta_\lambda = \left(\sigma^{d(\lambda)}_{|D_\lambda}\right)^{-1}$ and $\theta_{\lambda^{-1}} = \sigma^{d(\lambda)}_{|D_\lambda}$. Let for every $t \in G$ written on reduced form $u_1 u_2 \cdots u_n$, where $u_1, u_2, \ldots, u_n \in \Lambda \setminus \Lambda^0 \cup \{\lambda^{-1} \mid \lambda \in \Lambda \setminus \Lambda^0\}$,

$$D_t = \theta_{u_1} \circ \theta_{u_2} \circ \cdots \circ \theta_{u_n}(\mathsf{X}^+)$$

and

$$\theta_t = \theta_{u_1} \circ \theta_{u_2} \circ \cdots \circ \theta_{u_n}.$$

Then $\mathbf{\Lambda} = (\Lambda, G, \theta)$ is a discrete partial dynamical system.

4. The reduced C^*-algebra

As mentioned before, the object of this paper is to associate to each discrete partial dynamical system $\mathbf{X} = (X, G, \theta)$ a C^*-algebra $\mathcal{O}_{\mathbf{X}}$.

Let us first try the naive approach and look at the Hilber space $l_2(X)$. Let $(e_x)_{x\in X}$ be an orthonormal basis for $l_2(X)$, and define for each $t \in G$ an operator S_t by setting

$$S_t\left(\sum_{x\in X} \lambda_x e_x\right) = \sum_{x\in D_t} \lambda_{\theta_{t^{-1}}(x)} e_x$$

for every $\sum_{x\in X} \lambda_x e_x \in l_2(X)$.

Definition 2. We let, for a discrete partial dynamical system $\mathbf{X} = (X, G, \theta)$, $\mathcal{O}_{\mathbf{X}}^{red}$ be the C^* generated by the operators $S_t,\ t \in G$ defined above.

We call $\mathcal{O}_{\mathbf{X}}^{red}$ the reduced C^*-algebra associated to $\mathbf{X}$.

In order to define the universal C^*-algebra $\mathcal{O}_{\mathbf{X}}$ associated to $\mathbf{X}$, we will first determine a reasonable set of conditions which the generators of $\mathcal{O}_{\mathbf{X}}$ should satisfy.

It is straight forward to check that the operators $\{S_t \in \mathcal{O}_{\mathbf{X}}^{red} \mid t \in G\}$ are partial isometries and satisfy that $S_e = 1$, $S_t^* = S_{t^{-1}}$, $S_tS_s \le S_{ts}$ and $S_tS_t^* = \operatorname{proj}(\overline{\operatorname{span}}\{e_x \mid x \in D_t\})$, where $\operatorname{proj}(\overline{\operatorname{span}}\{e_x \mid x \in D_t\})$ is the projection of $l_2(X)$ onto $\overline{\operatorname{span}}\{e_x \mid x \in D_t\}$, for every $t, s \in G$.

We will take these relations to be the conditions which the generators of $\mathcal{O}_{\mathbf{X}}$ should satisfy.

5. The universal C^*-algebra

Let $\mathbf{X} = (X, G, \theta)$ be a discrete partial dynamical system. We want $\mathcal{O}_{\mathbf{X}}$ to be the universal C^*-algebra generated by partial isometries $S_t,\ t \in G$ satisfying $S_e = 1$, $S_t^* = S_{t^{-1}}$, $S_tS_s \le S_{ts}$ and $S_tS_t^* = \operatorname{proj}(\overline{\operatorname{span}}\{e_x \mid x \in D_t\})$. One way of constructing $\mathcal{O}_{\mathbf{X}}$ is by using full crossed product of C^*-partial dynamical systems (cf. [5, 10]).

First we have to turn the discrete partial dynamical system $\mathbf{X}$ into a C^*-partial dynamical system. We will do that by using Boolean algebras (see [6] for a introduction to Boolean algebras). A Boolean algebra is an algebraic object with three operations $\cup, \cap$ and $\neg$ which act like union, intersection and complement. Examples are the power set of a set, the set of clopen subsets of a topological space, projections of a unital C^*-algebra, ideals of a C^*-algebra and the set $\{0, 1\}$.

A map ϕ between two Boolean algebras $\mathcal{B}$ and $\mathcal{B}'$ is called a Boolean homomorphism if $\phi(A \cup B) = \phi(A) \cup \phi(B)$, $\phi(A \cap B) = \phi(A) \cap \phi(B)$ and $\phi(\neg A) = \neg\phi(A)$ for every $A, B \in \mathcal{B}$.

Let $\mathcal{B}_X$ be the Boolean algebra generated by $D_t,\ t \in G$. Notice that $\theta_r(D_s) = D_r \cap D_{rs} \in \mathcal{B}_X$ for every $r, s \in G$, so $\theta_t(A) \in \mathcal{B}_X$ for every $A \in \mathcal{B}_X$.

Let for each $t \in G$, $\widetilde{D}_t$ be the ideal in $\mathcal{B}_X$ generated by D_t, i.e.

$$\widetilde{D}_t = \{A \in \mathcal{B}_X \mid A \subseteq D_t\}.$$

We define for every $t \in G$ a map $\widetilde{\theta}_t$ from $\widetilde{D}_t$ to $\widetilde{D}_{t^{-1}}$ by

$$\widetilde{\theta}_t(A) = \theta_t^{-1}(A).$$

It is easy to see that $\widetilde{\theta}_t$ is a bijective Boolean homomorphism.

Let $\hat{X}$ be the dual of $\mathcal{B}_X$, i.e. $\hat{X}$ is the closed subset

$$\{\phi \in \{0,1\}^{\mathcal{B}_X} \mid \phi \text{ is a Boolean homomorphism}\}$$

of the Cantor space $\{0,1\}^{\mathcal{B}_X}$ endowed with the product topology of the discrete topology on $\{0,1\}$. Then $\hat{X}$ is a totally disconnected compact Hausdorff space.

For each $A \in \mathcal{B}_X$ we let $\hat{A} = \{\phi \in \hat{X} \mid \phi(A) = 1\}$, and notice that $\hat{A}$ is a clopen subset of $\hat{X}$.

Lemma 3. *The system $\{\hat{D}_t \mid t \in G\}$ separates points in $\hat{X}$.*

Proof. Let $\phi_1, \phi_2 \in \hat{X}$ and let

$$\mathcal{A} = \{A \in \mathcal{B}_X \mid \phi_1(A) = \phi_2(A)\}.$$

Then $\mathcal{A}$ is a Boolean subalgebra of $\mathcal{B}_X$. Assume that

$$\phi_1 \in \hat{D}_t \Leftrightarrow \phi_2 \in \hat{D}_t$$

for every $t \in G$. This means that $D_t \in \mathcal{A}$ for every $t \in G$ and thus that $\mathcal{A} = \mathcal{B}_X$. But then ϕ_1 and ϕ_2 must be equal. □

For each $t \in G$, let $\hat{\theta}_t$ be the map given by

$$\hat{\theta}_t(\phi)(A) = \phi(\theta_t^{-1}(A))$$

for $A \in \mathcal{B}_X$ and $\phi \in \hat{D}_{t^{-1}}$. It is easy to check that $\hat{\theta}_t$ is a Boolean homomorphism from $\hat{D}_{t^{-1}}$ to $\hat{D}_t$.

Let $\bar{X} = C(\hat{X})$ and let for each $t \in G$,

$$\bar{D}_t = \{f \in \bar{X} \mid f_{|\hat{X} \setminus \hat{D}_t} = 0\}$$

and let $\bar{\theta}_t : \bar{D}_{t^{-1}} \to \bar{D}_t$ be defined by

$$\bar{\theta}_t(f)(\phi) = \begin{cases} f(\hat{\theta}_{t^{-1}}(\phi)) & \text{if } \phi \in \hat{D}_t, \\ 0 & \text{if } \phi \in \hat{X} \setminus \hat{D}_t, \end{cases}$$

for $f \in \bar{D}_{t^{-1}}$ and $\phi \in \hat{X}$. Then $\bar{\theta} = (\{\bar{D}_t\}_{t \in G}, \{\bar{\theta}_t\}_{t \in G})$ is a partial action of G on the C^*-algebra $\bar{X}$. Thus $(\bar{X}, G, \bar{\theta})$ is a C^*-partial dynamical system.

We define $\mathcal{O}_{\mathbf{X}}$ to be the full crossed product $\bar{X} \rtimes_{\bar{\theta}} G$. It is characterized by the following theorem:

Theorem 4. *Let $\mathbf{X} = (X, G, \theta)$ be a discrete partial dynamical system. Then $\mathcal{O}_{\mathbf{X}}$ is the universal C^*-algebra generated by partial isometries $\{S_t\}_{t \in G}$ satisfying*

(1) $S_e = 1$,
(2) $S_{t^{-1}} = S_t^*$ for every $t \in G$,
(3) S_{st} extends $S_s S_t$ for every $s, t \in G$,
(4) the map $D_t \mapsto S_t S_t^*$ extends to a Boolean homomorphism from $\mathcal{B}_X$ to the set of projections in $C^*(S_t \mid t \in G)$.

Proof. We will first show that if (π, u) is a covariant representation of $(A, G, \bar{\theta})$, then $S_t = u_t$ are partial isometries which satisfy (1), (2), (3) and (4), and we will then show if $\{S_t\}_{t\in G}$ are partial isometries on a Hilbert space H and $\{S_t\}_{t\in G}$ satisfy (1), (2), (3) and (4) and $\overline{\text{span}}\{S_t\xi \mid t \in G, \xi \in \mathsf{H}\} = \mathsf{H}$, then there is a covariant representation of $(A, G, \bar{\theta})$ such that $u_t = S_t$ for every $t \in G$.

Let (π, u) be a covariant representation of $(A, G, \bar{\theta})$. Let for every $t \in G$, $S_t = u_t$. Then clearly (1), (2) and (3) are satisfied.

Denote for every subset B of $\bar{X}$, the subspace

$$\overline{\text{span}}\{\pi(T)\xi \mid T \in B,\ \xi \in \mathsf{H}\}$$

of H by $[B]$.

Since the map sending an element A of $\mathcal{B}_X$ to $\hat{A}$, the map sending a clopen subset U of $\hat{X}$ to $\{f \in \bar{X} \mid f_{|\hat{X}\setminus U} = 0\}$, and the map sending an ideal I of $\bar{X}$ to $\text{proj}[I]$ all are Boolean homomorphism, so is the map sending an element A of $\mathcal{B}_X$ to

$$\text{proj}([\{f \in \bar{X} \mid f_{|\hat{X}\setminus\hat{A}} = 0\}]),$$

and since

$$\begin{aligned}\text{proj}([\{f \in \bar{X} \mid f_{|\hat{X}\setminus\hat{D}_t} = 0\}]) &= \text{proj}([\bar{D}_t]) \\ &= u_t u_t^* \\ &= S_t S_t^*\end{aligned}$$

for every $t \in G$, (4) is satisfied.

Now let $S_t,\ t \in G$ be partial isometries on a Hilbert space H such that $\{S_t\}_{t\in G}$ satisfy (1), (2), (3) and (4) and $\overline{\text{span}}\{S_t\xi \mid t \in G, \xi \in \mathsf{H}\} = \mathsf{H}$. Let for every $t \in G$, $u_t = S_t$. Then $t \mapsto u_t$ is a partial representation of G on H.

It follows from Lemma 3 and the Stone-Weierstrass Theorem that

$$\overline{\text{span}}\{1_{\hat{D}_t} \mid t \in G\} = \bar{X},$$

so since the map $D_t \mapsto S_t S_t^*$ extends to a Boolean homomorphism from $\mathcal{B}_X$ to the set of projections in $C^*(S_t \mid t \in G)$, the map

$$\sum_{t\in G} \lambda_t 1_{\hat{D}_t} \mapsto \sum_{t\in G} \lambda_t S_t S_t^*,\ \lambda_t \in \mathbb{C}$$

extends to a $*$-homomorphism π from $\bar{X}$ to $C^*(S_t \mid t \in G)$. Thus π is a nondegenerated representation of $\bar{X}$ on H, which satisfies that

$$\begin{aligned}\text{proj}([\bar{D}_t]) &= \text{proj}(\overline{\text{span}}\{S_t S_t^*\xi \mid \xi \in H\}) \\ &= S_t S_t^* \\ &= u_t u_t^*.\end{aligned}$$

Let $D_s \subseteq D_{t^{-1}}$. Then $1_{\hat{D}_s} \in \bar{D}_{t^{-1}}$, and $\bar{\theta}_t(1_{\hat{D}_s}) = 1_{\hat{D}_t \cap \hat{D}_{ts}}$. Thus

$$\begin{aligned}
\pi(\bar{\theta}_t(1_{\hat{D}_s})) &= \pi(1_{\hat{D}_t \cap \hat{D}_{ts}}) \\
&= S_t S_t^* S_{ts} S_{ts}^* \\
&= S_t S_s S_{ts}^* \\
&= S_t S_s S_s^* S_s S_{ts}^* \\
&= S_t S_s S_s^* S_t^* \\
&= u_t \pi(1_{\hat{D}_s}) u_{t^{-1}},
\end{aligned}$$

and since $\overline{\text{span}}\{1_{\hat{D}_s} \mid D_s \subseteq D_{t^{-1}}\} = \bar{D}_{t^{-1}}$, this shows that

$$\pi(\bar{\theta}(f)) = u_t(\pi(f)) u_{t^{-1}}$$

for every $f \in \bar{D}_{t^{-1}}$.

Thus (π, u) is a covariant representation of $(A, G, \bar{\theta})$ □

Proposition 5. *Let $\mathbf{X} = (X, G, \theta)$ be a discrete partial dynamical system. Then there is a $*$-homomorphism from $\mathcal{O}_{\mathbf{X}}$ to $\mathcal{O}_{\mathbf{X}}^{red}$, sending S_t to S_t for every $t \in G$.*

Proof. This follows from Theorem 4 since $\{S_t \in \mathcal{O}_{\mathbf{X}}^{red} \mid t \in G\}$ are partial isometries that satisfy that $S_e = 1$, $S_t^* = S_{t^{-1}}$, $S_t S_s \leq S_{ts}$ and $S_t S_t^* = \text{proj}(\overline{\text{span}}\{e_x \mid x \in D_t\})$ for all $s, t \in G$, and the map

$$A \mapsto \text{proj}(\overline{\text{span}}\{e_x \mid x \in A\})$$

is a Boolean homomorphism from $\mathcal{B}_X$ to the set of projections on $l_2(X)$. □

Lemma 6. *The Boolean homomorphism from $\mathcal{B}_X$ to the set of projections in $\mathcal{O}_{\mathbf{X}}$ that extends the map $D_t \mapsto S_t S_t^*$ is injective.*

Proof. Since the Boolean homomorphism

$$A \mapsto \text{proj}(\overline{\text{span}}\{e_x \mid x \in A\})$$

from $\mathcal{B}_X$ to the set of projections on $l_2(X)$ is injective, the lemma follows. □

6. C^*-algebras associated to symbolic dynamical systems

We will now return to the symbolic dynamical system we looked at in Section 3 and show that the C^*-algebras associated to them can be realized as universal C^*-algebras associated with discrete dynamical systems.

Example 1 continued. Let (X, σ) be a two-sided shift space over the finite alphabet, and let $\mathbf{X}$ be the discrete partial dynamical system defined in Example 1. Let $\sigma^\star$ be the automorphism on $C(\mathsf{X})$ defined by $f \mapsto f \circ \sigma$ and let $C(\mathsf{X}) \rtimes_{\sigma^\star} \mathbb{Z}$ be the full crossed product (cf. [9, 7.6.5]). Thus $C(\mathsf{X}) \rtimes_{\sigma^\star} \mathbb{Z}$ is the universal C^*-algebra generated by a copy of $C(\mathsf{X})$ and an unitary operator U which satisfies that $UfU^* = f \circ \sigma$ for every $f \in C(\mathsf{X})$. We then have:

Proposition 7. *$\mathcal{O}_{\mathbf{X}}$ is isomorphic to the crossed product $C(\mathsf{X}) \rtimes_{\sigma^\star} \mathbb{Z}$.*

Proof. We will first show that the Boolean algebra $\mathcal{B}_{\mathsf{X}}$ generated by $\{D_t \mid t \in \mathbb{F}_{\mathfrak{a}}\}$ is the Boolean algebra of clopen subsets of X. To see this, notice first that D_a and $D_{a^{-1}}$ are clopen for every $a \in \mathfrak{a}$. Since σ is a homeomorphism, D_t is clopen for every $t \in \mathbb{F}_{\mathfrak{a}}$, so every set in $\mathcal{B}_{\mathsf{X}}$ is clopen.

In the other direction, we have that

$$D_{(a_{-1}a_{-2}\cdots a_{-n})^{-1}} \cap D_{a_0a_1\cdots a_m} = \{x \in \mathsf{X} \mid x_{-n} = a_{-n}, x_{-n+1} = a_{-n+1}, \ldots, x_m = a_m\} \in \mathcal{B}_{\mathsf{X}}$$

for $a_{-n}, a_{-n+1}, \ldots a_m \in \mathfrak{a}$, and that the system consisting of sets of this form is a basis for the topology of X, so every clopen set is a finite union of sets of this form and thus belongs to $\mathcal{B}_{\mathsf{X}}$.

So it follows from the Stone-Weierstrass theorem that $C(\mathsf{X}) = \overline{\operatorname{span}}\{1_{D_t} \mid t \in \mathbb{F}_{\mathfrak{a}}\}$. Since the map $D_t \mapsto S_tS_t^*$ extends to a Boolean homomorphism from $\mathcal{B}_{\mathsf{X}}$ to the set of projections in $\mathcal{O}_{\mathbf{X}}$, the map

$$\sum_{t\in\mathbb{F}_{\mathfrak{a}}} \lambda_t 1_{\hat{D}_t} \mapsto \sum_{t\in\mathbb{F}_{\mathfrak{a}}} \lambda_t S_tS_t^*,\ \ \lambda_t \in \mathbb{C}$$

extends to a $*$-homomorphism ϕ from $C(\mathsf{X})$ to $\mathcal{O}_{\mathbf{X}}$. Let $U = \sum_{a\in\mathfrak{a}} S_a \in \mathcal{O}_{\mathbf{X}}$. Then

$$\begin{aligned}
U\phi(1_{D_t})U^* &= \sum_{a\in\mathfrak{a}} S_aS_tS_t^* \sum_{a'\in\mathfrak{a}} S_{a'}^* \\
&= \sum_{\substack{a\in\mathfrak{a}\\ a'\in\mathfrak{a}}} S_aS_tS_t^*S_a^*S_aS_{a'}^*S_{a'}S_{a'}^* \\
&= \sum_{a\in\mathfrak{a}} S_aS_tS_t^*S_a^* \\
&= \sum_{a\in\mathfrak{a}} S_aS_a^*S_{at}S_{at}^*S_aS_a^* \\
&= \sum_{a\in\mathfrak{a}} \phi(1_{D_a\cap D_{at}}) \\
&= \sum_{a\in\mathfrak{a}} \phi(1_{\theta_a(D_t)}) \\
&= \phi(1_{\cup_{a\in\mathfrak{a}}\theta_a(D_t)}) \\
&= \phi(1_{\sigma^{-1}(D_t)}) \\
&= \phi(1_{D_t} \circ \sigma)
\end{aligned}$$

for every $t \in \mathbb{F}_{\mathfrak{a}}$. Since $C(\mathsf{X}) = \overline{\operatorname{span}}\{1_{D_t} \mid t \in \mathbb{F}_{\mathfrak{a}}\}$, this shows that $U\phi(f)U^* = \phi(f\circ\sigma)$ for every $f \in C(\mathsf{X})$. Thus there is a $*$-homomorphism $\tilde{\phi}$ from $C(\mathsf{X}) \rtimes_{\sigma^\star} \mathbb{Z}$ to $\mathcal{O}_{\mathbf{X}}$ which is equal to ϕ on $C(\mathsf{X})$ and sends U to $\sum_{a\in\mathfrak{a}} S_a$.

Let for every $a \in \mathfrak{a}$, $S_a = 1_{D_a}U \in C(\mathsf{X}) \rtimes_{\sigma^\star} \mathbb{Z}$, $S_{a^{-1}} = S_a^*$ and let $S_e = 1$ and $S_t = S_{u_1}S_{u_2}\cdots S_{u_n}$, where $t = u_1u_2\cdots u_n \in \mathbb{F}_{\mathfrak{a}}$ is written in reduced form. Then (1), (2) and (3) of Theorem 4 hold.

We define a function $[\cdot] : \mathbb{F}_{\mathfrak{a}} \to \mathbb{Z}$ recursively by $[e] = 0$, $[at] = [t] + 1$ and $[a^{-1}t] = [t] - 1$ for $a \in \mathfrak{a}$ and $t \in \mathbb{F}_{\mathfrak{a}}$. For every $t \in \mathbb{F}_{\mathfrak{a}}$ and $a \in \mathfrak{a}$ are

$$\begin{aligned} S_a 1_{D_t} U^{[t]} &= 1_{D_a} U 1_{D_t} U^{[t]} \\ &= 1_{D_a} U 1_{D_t} U^* U^{[t]+1} \\ &= 1_{D_a \cap \sigma^{-1}(D_t)} U^{[t]+1} \\ &= 1_{D_{at}} U^{[t]+1} \end{aligned}$$

and if t written in reduced form does not begin with an a, then $D_{a^{-1}t} \subseteq D_{a^{-1}}$, so

$$\begin{aligned} S_{a^{-1}} 1_{D_t} U^{[t]} &= U^* 1_{D_a} 1_{D_t} U^{[t]} \\ &= 1_{\sigma(D_a \cap D_t)} U^{[t]-1} \\ &= 1_{\theta_{a^{-1}}(D_t)} U^{[t]-1} \\ &= 1_{D_{a^{-1}} \cap D_{a^{-1}t}} U^{[t]-1} \\ &= 1_{D_{a^{-1}t}} U^{[t]-1}. \end{aligned}$$

This shows that $S_t = 1_{D_t} U^{[t]}$ and thus $S_t S_t^* = 1_{D_t}$ for every $t \in \mathbb{F}_{\mathfrak{a}}$. Thus (4) holds. Hence there is a $*$-homomorphism ψ from $\mathcal{O}_{\mathbf{X}}$ to $C(\mathsf{X}) \rtimes_{\sigma^\star} \mathbb{Z}$ such that $\psi(S_t) = 1_{D_t} U^{[t]}$ for every $t \in \mathbb{F}_{\mathfrak{a}}$.

We have that $\psi(\tilde{\phi}(U)) = \psi(\sum_{a \in \mathfrak{a}} S_a) = \sum_{a \in \mathfrak{a}} 1_{D_a} U = U$, and $\psi(\tilde{\phi}(1_{D_t})) = \psi(S_t S_t^*) = 1_{D_t} U^{[t]} U^{-[t]} 1_{D_t} = 1_{D_t}$ for every $t \in \mathbb{F}_{\mathfrak{a}}$, and since $C(\mathsf{X}) \rtimes_{\sigma^\star} \mathbb{Z}$ is generated by U and 1_{D_t}, $t \in \mathbb{F}_{\mathfrak{a}}$, this shows that $\psi \circ \tilde{\phi} = \mathrm{Id}_{\mathcal{O}_{C(\mathsf{X}) \rtimes_{\sigma^\star} \mathbb{Z}}}$.

We also have that $\tilde{\phi}(\psi(S_t)) = \tilde{\phi}(1_{D_t} U) = S_t S_t^* \left(\sum_{a \in \mathfrak{a}} S_a\right)^{[t]}$ for every $t \in \mathbb{F}_{\mathfrak{a}}$. Notice that if $a \neq b \in \mathfrak{a}$, then $D_{ab^{-1}} = \theta_a \circ \theta_{b^{-1}}(\mathsf{X}) = \emptyset$ and $D_{a^{-1}b} = \theta_{a^{-1}} \circ \theta_b(\mathsf{X}) = \emptyset$, so $S_{ab^{-1}} = S_{a^{-1}b} = 0$ in $\mathcal{O}_{\mathbf{X}}$. Thus $S_t \neq 0$ implies that $t = u_1 u_2 \cdots u_n$ where either $u_i \in \mathfrak{a}$ for all $i \in \{1, 2, \ldots, n\}$ or $u_i \in \mathfrak{a}^{-1}$ for all $i \in \{1, 2, \ldots, n\}$. Hence $S_t S_t^* \left(\sum_{a \in \mathfrak{a}} S_a\right)^{[t]} = S_t$ for every $t \in \mathbb{F}_{\mathfrak{a}}$, which shows that $\tilde{\phi} \circ \psi = \mathrm{Id}_{\mathcal{O}_{\mathbf{X}}}$.

Thus $\mathcal{O}_{\mathbf{X}}$ is isomorphic to $C(\mathsf{X}) \rtimes_{\sigma^\star} \mathbb{Z}$. □

Example 2 continued. Let (X^+, σ_+) be a one-sided shift space over the finite alphabet $\mathfrak{a}$ and let $\mathbf{X}^+$ be the discrete partial dynamical system defined in Example 2. Let $\mathfrak{a}^*$ denote the set of finite words with letters from $\mathfrak{a}$. For $\mu, \nu \in \mathfrak{a}^*$, we let

$$C(\mu, \nu) = \{\nu x \in \mathsf{X}^+ \mid \mu x \in \mathsf{X}^+\}.$$

We let $\mathfrak{B}(\mathsf{X}^+)$ be the abelian C^*-algebra of all bounded functions on X^+, and $\mathcal{D}_{\mathsf{X}^+}$ the C^*-subalgebra of $\mathfrak{B}(\mathsf{X}^+)$ generated by $\{1_{C(\mu,\nu)} \mid \mu, \nu \in \mathfrak{a}^*\}$.

The C^*-algebra $\mathcal{O}_{\mathsf{X}^+}$ is the universal C^*-algebra generated by partial isometries $\{S_a\}_{a \in \mathfrak{a}}$ satisfying that the map $1_{C(\mu,\nu)} \mapsto S_\nu S_\mu^* S_\mu S_\nu^*$ extends to a $*$-homomorphism from $\mathcal{D}_{\mathsf{X}^+}$ to $\mathcal{O}_{\mathsf{X}^+}$ (cf. [1, Remark 7.3]).

We will for each $a \in \mathfrak{a}$, by λ_a denote the map on $\mathcal{D}_{\mathsf{X}^+}$ given by

$$\lambda_a(f)(x) = \begin{cases} f(ax) & \text{if } ax \in \mathsf{X}^+, \\ 0 & \text{if } ax \notin \mathsf{X}^+, \end{cases}$$

and by ϕ_a the map on $\mathcal{D}_{\mathsf{X}^+}$ given by

$$\phi_a(f)(x) = \begin{cases} f(\sigma_+(x)) & \text{if } x \in D_a, \\ 0 & \text{if } x \notin D_a, \end{cases}$$

for $f \in \mathcal{D}_{\mathsf{X}^+}$ and $x \in \mathsf{X}^+$ (cf. [1, Proposition 4.3 and Lemma 8.2]).

Proposition 8. *$\mathcal{O}_{\mathbf{X}^+}$ is isomorphic to $\mathcal{O}_{\mathsf{X}^+}$.*

Proof. We claim that $\overline{\operatorname{span}}\{1_{D_t} \mid t \in \mathbb{F}_{\mathfrak{a}}\} = \mathcal{D}_{\mathsf{X}^+}$. To see this, first notice that if $\mu, \nu \in \mathfrak{a}^*$, then

$$C(\mu,\nu) = \theta_\nu(D_{\mu^{-1}}) = D_\nu \cap D_{\nu\mu^{-1}},$$

so $\mathcal{D}_{\mathsf{X}^+} \subseteq \overline{\operatorname{span}}\{1_{D_t} \mid t \in \mathbb{F}_{\mathfrak{a}}\}$.

If A is a subset of X^+ such that $1_A \in \mathcal{D}_{\mathsf{X}^+}$, then $1_{\theta_a(A)} = \lambda_a(1_A) \in \mathcal{D}_{\mathsf{X}^+}$ and $1_{\theta_{a^{-1}}(A)} = \phi_a(1_A) \in \mathcal{D}_{\mathsf{X}^+}$ for $a \in \mathfrak{a}$, so since

$$D_t = \theta_{u_1} \circ \theta_{u_2} \circ \cdots \circ \theta_{u_n}(\mathsf{X}^+),$$

where $t \in \mathbb{F}_{\mathfrak{a}}$ is written on reduced form $u_1 u_2 \cdots u_n$, we have that $\overline{\operatorname{span}}\{1_{D_t} \mid t \in \mathbb{F}_{\mathfrak{a}}\} \subseteq \mathcal{D}_{\mathsf{X}^+}$. Thus $\mathcal{D}_{\mathsf{X}^+} = \overline{\operatorname{span}}\{1_{D_t} \mid t \in \mathbb{F}_{\mathfrak{a}}\}$.

Since the map $D_t \mapsto S_t S_t^*$ extends to a Boolean homomorphism from $\mathcal{B}_X$ to the set of projections in $\mathcal{O}_{\mathbf{X}^+}$, the map

$$\sum_{t \in \mathbb{F}_{\mathfrak{a}}} \lambda_t 1_{\hat{D}_t} \mapsto \sum_{t \in \mathbb{F}_{\mathfrak{a}}} \lambda_t S_t S_t^*,\ \ \lambda_t \in \mathbb{C}$$

extends to a $*$-homomorphism ϕ from $\mathcal{D}_{\mathsf{X}^+}$ to $\mathcal{O}_{\mathbf{X}^+}$.

For $t \in \mathbb{F}_{\mathfrak{a}}$ and $a \in \mathfrak{a}$ are

$$\begin{aligned} \phi(1_{\theta_a(D_t)}) &= \phi(1_{D_a \cap D_{at}}) \\ &= S_a S_a^* S_{at} S_{at}^* \\ &= S_a S_a^* S_{at} S_{at}^* S_a S_a^* \\ &= S_a S_t S_t^* S_a^*, \end{aligned}$$

and

$$\begin{aligned} \phi(1_{\theta_{a^{-1}}(D_t)}) &= \phi(1_{D_{a^{-1}} \cap D_{a^{-1}t}}) \\ &= S_{a^{-1}} S_{a^{-1}}^* S_{a^{-1}t} S_{a^{-1}t}^* \\ &= S_{a^{-1}} S_{a^{-1}}^* S_{a^{-1}t} S_{a^{-1}t}^* S_{a^{-1}} S_{a^{-1}}^* \\ &= S_{a^{-1}} S_t S_t^* S_{a^{-1}}^*. \end{aligned}$$

So $\phi(\phi_a(f)) = S_a \phi(f) S_a^*$ and $\phi(\lambda_a(f)) = S_a^* \phi(f) S_a$ for $f \in \mathcal{D}_{\mathsf{X}^+}$ and $a \in \mathfrak{a}$. Thus

$$\begin{aligned} \phi(1_{C(\mu,\nu)}) &= \phi(1_{D_\nu(D_{\mu^{-1}})}) \\ &= \phi(\phi_{\nu_1} \circ \phi_{\nu_2} \cdots \circ \phi_{\mu_n} \circ \lambda_{\mu_m} \circ \cdots \circ \lambda_{\mu_1}(1)) \\ &= S_{\nu_1} S_{\nu_2} \cdots S_{\nu_n} S_{\mu_m}^* \cdots S_{\mu_1}^* S_{\mu_1} \cdots S_{\mu_m} S_{\nu_n}^* \cdots S_{\nu_1}^* \end{aligned}$$

for $\mu = \mu_1 \mu_2 \cdots \mu_m, \nu = \nu_1 \nu_2 \cdots \nu_n \in \mathfrak{a}^*$. Hence there is a $*$-homomorphism $\tilde{\phi}$ from $\mathcal{O}_{\mathsf{X}^+}$ to $\mathcal{O}_{\mathbf{X}^+}$ such that $\tilde{\phi}(S_a) = S_a$ for every $a \in \mathfrak{a}$.

Let us now turn towards $\mathcal{O}_{\mathsf{X}^+}$. Let $S_{a^{-1}} = S_a^*$ and let $S_e = 1$ and $S_t = S_{u_1} S_{u_2} \cdots S_{u_n}$, where $t = u_1 u_2 \cdots u_n \in \mathbb{F}_{\mathfrak{a}}$ is written in reduced form. Then (1), (2) and (3) of Theorem 4 hold.

The map $1_{C(\mu,\nu)} \mapsto S_\nu S_\mu^* S_\mu S_\nu^*$ extends to a $*$-homomorphism ψ from $\mathcal{D}_{\mathsf{X}^+}$ to $\mathcal{O}_{\mathsf{X}^+}$. It is easy to check that $\psi(\lambda_a(f)) = S_a^*\psi(f)S_a$ and $\psi(\phi_a(f)) = S_a\psi(f)S_{a^{-1}}$ for $a \in \mathfrak{a}$ and $f \in \mathcal{D}_{\mathsf{X}^+}$. Thus

$$\begin{aligned}\psi(1_{D_t}) &= \psi(1_{\theta_{u_1}\circ\theta_{u_2}\circ\cdots\circ\theta_{u_n}(\mathsf{X}^+)})\\ &= \psi(1\circ\theta_{u_n^{-1}}\circ\theta_{u_{n-1}^{-1}}\circ\cdots\circ\theta_{u_1^{-1}})\\ &= S_{u_1}S_{u_2}\cdots S_{u_n}S_{u_n}^*\cdots S_{u_1}^*\\ &= S_tS_t^*\end{aligned}$$

so also (4) of Theorem 4 holds. Hence there is a $*$-homomorphism ψ from $\mathcal{O}_{\mathbf{X}^+}$ to $\mathcal{O}_{\mathsf{X}^+}$ such that $\psi(S_t) = S_{u_1}S_{u_2}\cdots S_{u_n}$ for every $t \in \mathbb{F}_\mathfrak{a}$, where $t = u_1u_2\cdots u_n \in \mathbb{F}_\mathfrak{a}$ is written in reduced form.

We have that $\psi(\tilde\phi(S_a)) = \psi(S_a) = S_a$ for every $a \in \mathfrak{a}$, and since $\mathcal{O}_{\mathsf{X}^+}$ is generated by $\{S_a\}_{a\in\mathfrak{a}}$, this shows that $\psi\circ\tilde\phi = \mathrm{Id}_{\mathcal{O}_{\mathsf{X}^+}}$.

We also have that $\tilde\phi(\psi(S_t)) = \tilde\phi(S_{u_1}S_{u_2}\cdots S_{u_n}) = S_{u_1}S_{u_2}\cdots S_{u_n}$ for every $t \in \mathbb{F}_\mathfrak{a}$, where $t = u_1u_2\cdots u_n \in \mathbb{F}_\mathfrak{a}$ is written in reduced form.

If $u \in \mathfrak{a}\cup\mathfrak{a}^{-1}$ and $t \in \mathbb{F}_\mathfrak{a}$ written in reduced form does not begin with u, then $D_{ut} \subseteq D_u$, so

$$\begin{aligned}S_uS_t &= S_uS_u^*S_{ut}\\ &= S_uS_u^*S_{ut}S_{ut}^*S_{ut}\\ &= S_{ut}.\end{aligned}$$

This shows that $S_{u_1}S_{u_2}\cdots S_{u_n} = S_t$ for $t = u_1u_2\cdots u_n \in \mathbb{F}_\mathfrak{a}$ written in reduced form. Hence $\tilde\phi(\psi(S_t)) = S_t$ for every $t \in \mathbb{F}_\mathfrak{a}$, and since $\mathcal{O}_{\mathbf{X}^+}$ is generated by S_t, $t \in \mathbb{F}_\mathfrak{a}$, this shows that $\tilde\phi\circ\psi = \mathrm{Id}_{\mathcal{O}_{\mathbf{X}^+}}$. Thus $\mathcal{O}_{\mathbf{X}^+}$ and $\mathcal{O}_{\mathsf{X}^+}$ are isomorphic. □

Example 3 continued. Let $\mathcal{I}$ be an arbitrary index set and let $A = A(i,j)_{i,j\in\mathcal{I}}$ be a matrix with entries in $\{0,1\}$ and having no identically zero rows. Exel and Laca have in [4] defined a C^*-algebra $\mathcal{O}_A$ associated with A. It is the universal C^*-algebra generated by partial isometries $\{S_i\}_{i\in\mathcal{I}}$ satisfying:

(1) for each pair of finite subsets X and Y of $\mathcal{I}$ such that

$$A(X,Y,j) := \prod_{x\in X} A(x,j)\prod_{y\in Y}(1-A(y,j))$$

vanish for all but a finite number of j's,

$$\prod_{x\in X} S_x^*S_x \prod_{y\in Y}(1-S_y^*S_y) = \sum_{j\in\mathcal{I}} A(X,Y,j)S_jS_j^*,$$

(2) $S_i^*S_i$ and $S_j^*S_j$ commute, for all $i,j\in\mathcal{I}$,
(3) $S_i^*S_j = 0$, if $i \neq j \in \mathcal{I}$,
(4) $S_i^*S_iS_j = A(i,j)S_j$, for all $i,j\in\mathcal{I}$.

Let $\mathsf{X}_\mathbf{A}^+$ be the discrete partial dynamical system defined in Example 3. We then have the following result:

Proposition 9. *$\mathcal{O}_{\mathsf{X}_\mathbf{A}^+}$ is isomorphic to the unitization $\widetilde{O}_A$ of $\mathcal{O}_A$.*

Proof. It is easy to check that $\{S_i\}_{i\in\mathcal{I}} \subseteq \mathcal{O}_{\mathsf{X}^+_{\mathbf{A}}}$ satisfy condition (1), (2), (3) and (4) above. Thus there is a $*$-homomorphism ϕ from $\mathcal{O}_A$ to $\mathcal{O}_{\mathsf{X}^+_{\mathbf{A}}}$ such that $\phi(S_i) = S_i$ for all $i \in \mathcal{I}$.

It follows from Lemma 6 that there is an injective Boolean homomorphism from $\mathcal{B}_{\mathsf{X}^+_A}$ to the set of projections in $\mathcal{O}_{\mathsf{X}^+_{\mathbf{A}}}$ which maps D_t to $S_tS_t^*$. Thus there is an Boolean homomorphism from the Boolean algebra generated by $\{S_tS_t^*\}_{t\in\mathbb{F}_\mathcal{I}}$ to $\mathcal{B}_{\mathsf{X}^+_A}$, mapping $S_tS_t^*$ to D_t.

Let us now turn towards $\widetilde{O}_A$. We let for every $i \in \mathcal{I}$, $S_{a^{-1}} = S_a^*$ and we let $S_e = 1$ and $S_t = S_{u_1}S_{u_2}\cdots S_{u_n}$, where $t = u_1u_2\cdots u_n \in \mathbb{F}_\mathcal{I}$ is written in reduced form. Then (1), (2) and (3) of Theorem 4 hold.

The $*$-homomorphism ϕ induces a Boolean homomorphism from the set of projections in $\widetilde{O}_A$ to the set of projections in $\mathcal{O}_{\mathsf{X}^+_{\mathbf{A}}}$ which maps $S_tS_t^*$ to $S_tS_t^*$. Thus there is a Boolean homomorphism from the set of projections in $\mathcal{O}_A$ to $\mathcal{B}_{\mathsf{X}^+_A}$ which maps $S_tS_t^*$ to D_t. We claim that it is injective.

Let $\mathcal{I}^*$ be the set of finite words with letters from $\mathcal{I}$, and let for every $\mu \in \mathcal{I}^*$ and every pair (I, J) of finite subset of $\mathcal{I}$,

$$\begin{aligned} C(\mu, I, J) &= \theta_\mu\left(\left(\bigcap_{i\in I} D_{i^{-1}}\right)\bigcap\left(\bigcap_{j\in J}\neg D_{j^{-1}}\right)\right) \\ &= \{\mu x \in \mathsf{X}^+_A \mid \forall i \in I : ix \in \mathsf{X}^+_A, \forall j \in J : jx \notin \mathsf{X}^+_A\}, \end{aligned}$$

and let $\mathcal{B}'$ be the set of subsets of X^+_A which is a finite union of sets of the form

$$C(\mu, I, J) \cap \left(\bigcap_{k=1}^n \neg C(\mu_k, I_k, J_k)\right)$$

where $n \in \mathbb{N}_0$, $\mu, \mu_1, \dots, \mu_n \in \mathcal{I}^*$ and $I, I_1, \dots, I_n, J, J_1, \dots, J_n$ all are finite subsets of $\mathcal{I}$.

We claim that $\mathcal{B}' = \mathcal{B}_{\mathsf{X}^+_A}$. It is clear that $\mathcal{B}' \subseteq \mathcal{B}_{\mathsf{X}^+_A}$. We will show that $\mathcal{B}_{\mathsf{X}^+_A} \subseteq \mathcal{B}'$ by proving that $\mathcal{B}'$ is closed under taking union, intersection and complement and that $\theta_i(\mathcal{B}') \subseteq \mathcal{B}'$ and $\theta_{i^{-1}}(\mathcal{B}') \subseteq \mathcal{B}'$ for every $i \in \mathcal{I}$.

It is obvious that $\mathcal{B}'$ is closed under union. Suppose that $|\mu_1| \geq |\mu_2|$. Then

$$C(\mu_1, I_1, J_1) \cap C(\mu_2, I_2, J_2) = \begin{cases} C(\mu_1, I_1 \cup I_2, J_1 \cup J_2) & \text{if } \mu_1 = \mu_2, \\ & \text{if there is a } \nu \text{ such that } \mu_1 = \mu_2\nu, \\ C(\mu_1, I_1, J_1) & A(i, \nu_1) = 1 \text{ for all } i \in I \text{ and} \\ & A(j, \nu_1) = 0 \text{ for all } j \in J, \\ \emptyset & \text{else,} \end{cases}$$

so $\mathcal{B}'$ is closed under intersection and hence closed under complement.

We have for every $i \in \mathcal{I}$ that

$$\theta_i(C(\mu, I, J)) = C(i\mu, I, J),$$

so $\theta_i(\mathcal{B}') \subseteq \mathcal{B}'$. If $|\mu| > 0$, then

$$\theta_{i^{-1}}(C(\mu, I, J)) = \begin{cases} C(\nu, I, J) & \text{if } i\nu = \mu, \\ \emptyset & \text{else,} \end{cases}$$

and

$$\theta_{i^{-1}}(C(\emptyset, I, J)) = \begin{cases} C(\emptyset, \{i\}, \emptyset) & \text{if } A(i, \nu_1) = 1 \text{ for all } i \in I, \\ & \text{and } A(j, \nu_1) = 0 \text{ for all } j \in J, \\ \emptyset & \text{else,} \end{cases}$$

so $\theta_{i^{-1}}(\mathcal{B}') \subseteq \mathcal{B}'$.

Thus $\mathcal{B}' = \mathcal{B}_{\mathsf{X}_A^+}$, so in order to prove that the Boolean homomorphism is injective it is enough to show that if

$$C(\mu, I, J) \cap \left(\bigcap_{k=1}^{n} \neg C(\mu_k, I_k, J_k) \right) = \emptyset,$$

then

$$S_\mu \prod_{i \in I} S_i^* S_i \prod_{j \in J} (1 - S_j^* S_j) S_\mu^* \left(\prod_{k=1}^{n} \left(1 - S_{\mu_k} \prod_{i \in I_k} S_i^* S_i \prod_{j \in J_k} (1 - S_j^* S_j) S_{\mu_k}^* \right) \right) = 0.$$

Let $k \in \{1, 2, \ldots, n\}$. If $|\mu| > |\mu_k|$, then either $C(\mu, I, J) \cap \neg C(\mu_k, I_k, J_k) = C(\mu, I, J)$ or $\mu = \mu_k \alpha$ for some $\alpha \in \mathcal{I}^*$ which satisfies that $A(i, \alpha_1) = 1$ for all $i \in I_k$ and $A(j, \alpha_1) = 0$ for all $j \in J_k$. In the later case

$$\begin{aligned} S_\mu S_\mu^* S_{\mu_k} \prod_{i \in I_k} S_i^* S_i \prod_{j \in J_k} (1 - S_j^* S_j) S_{\mu_k}^* &= S_{\mu_k \alpha} S_{\mu_k \alpha}^* S_{\mu_k} \prod_{i \in I_k} S_i^* S_i \prod_{j \in J_k} (1 - S_j^* S_j) S_{\mu_k}^* \\ &= S_{\mu_k} S_\alpha S_\alpha^* S_{\mu_k}^* S_{\mu_k} \prod_{i \in I_k} S_i^* S_i \prod_{j \in J_k} (1 - S_j^* S_j) S_{\mu_k}^* \\ &= S_{\mu_k} S_\alpha S_\alpha^* \prod_{i \in I_k} S_i^* S_i \prod_{j \in J_k} (1 - S_j^* S_j) S_{\mu_k}^* \\ &= S_{\mu_k} S_\alpha S_\alpha^* S_{\mu_k}^* \\ &= S_\mu S_\mu^* \end{aligned}$$

and thus

$$S_\mu \prod_{i \in I} S_i^* S_i \prod_{j \in J} (1 - S_j^* S_j) S_\mu^* \left(\prod_{k=1}^{n} \left(1 - S_{\mu_k} \prod_{i \in I_k} S_i^* S_i \prod_{j \in J_k} (1 - S_j^* S_j) S_{\mu_k}^* \right) \right) = 0.$$

If $|\mu| = |\mu_k|$, then $C(\mu, I, J) \cap \neg C(\mu_k, I_k, J_k)$ is equal to either $C(\mu, I, J)$ or to

$$\left(\bigcup_{i \in I_k} C(\mu, I, J \cup \{i\}) \right) \cup \left(\bigcup_{j \in J_k} C(\mu, I \cup \{j\}, J) \right).$$

If $|\mu| < |\mu_k|$ and there is no $\alpha \in \mathcal{I}^*$ such that $\mu\alpha = \mu_k$, then $C(\mu, I, J) \cap \neg C(\mu_k, I_k, J_k) = C(\mu, I, J)$.

Thus we may assume that $\mu_k = \mu\alpha_k$ for every $k \in \{1, 2, \dots, n\}$. We then have that

$$C(\emptyset, I, J) \cap \left(\bigcap_{k=1}^{n} \neg C(\alpha_k, I_k, J_k) \right) = \theta_{\mu^{-1}} \left(C(\mu, I, J) \cap \left(\bigcap_{k=1}^{n} \neg C(\mu_k, I_k, J_k) \right) \right),$$

so if

$$C(\mu, I, J) \cap \left(\bigcap_{k=1}^{n} \neg C(\mu_k, I_k, J_k) \right) = \emptyset,$$

then

$$C(\emptyset, I, J) \cap \left(\bigcap_{k=1}^{n} \neg C(\alpha_k, I_k, J_k) \right) = \emptyset.$$

Assume that

$$C(\emptyset, I, J) \cap \left(\bigcap_{k=1}^{n} \neg C(\alpha_k, I_k, J_k) \right) = \emptyset.$$

Then

$$C(\emptyset, I, J) \subseteq \bigcup_{k=1}^{n} C(\alpha_k, I_k, J_k),$$

so $A(I, J, k) = 0$ for all but finitely many $k \in \mathcal{I}$. Thus

$$\prod_{i \in I} S_i^* S_i \prod_{j \in J} (1 - S_j^* S_j) = \sum_{l=1}^{m} S_{k_l} S_{k_l}^*$$

and

$$C(\emptyset, I, J) = \bigcup_{l=1}^{m} D_{k_l}$$

for some $k_1, k_2, \dots k_m \in \mathcal{I}$. Let $l \in \{1, 2, \dots, m\}$. Since

$$D_{k_l} \subseteq \bigcup_{k=1}^{n} C(\alpha_k, I_k, J_k),$$

either $k_l = \alpha_k$ for some $k \in \{1, 2, \dots, n\}$ or $A(\{k_l\}, \emptyset, r) = 0$ for all but finitely many $r \in \mathcal{I}$. In the later case, $S_{k_l} S_{k_l}^* = \sum_{t=1}^{h} S_{r_t} S_{r_t}^*$ for some $r_1, r_2, \dots r_h \in \mathcal{I}$, and thus

$$S_{k_l} S_{k_l}^* = S_{k_l} S_{k_l}^* S_{k_l} S_{k_l}^* = \sum_{t=1}^{h} S_{k_l} S_{r_t} S_{r_t}^* S_{k_l}^*.$$

Continuing in this way, we get that

$$\prod_{i \in I} S_i^* S_i \prod_{j \in J} (1 - S_j^* S_j) = \sum_{k \in F} S_{\alpha_k} S_{\alpha_k}^*$$

where F is a subset of $\{1, 2, \dots, n\}$. Thus

$$\prod_{i \in I} S_i^* S_i \prod_{j \in J} (1 - S_j^* S_j) \sum_{k=1}^{n} S_{\alpha_k} S_{\alpha_k}^* = \prod_{i \in I} S_i^* S_i \prod_{j \in J} (1 - S_j^* S_j),$$

so

$$S_\mu \prod_{i\in I} S_i^* S_i \prod_{j\in J}(1-S_j^*S_j)S_\mu^* \left(\prod_{k=1}^{n}\left(1-S_{\mu_k}\prod_{i\in I_k} S_i^*S_i \prod_{j\in J_k}(1-S_j^*S_j)S_{\mu_k}^*\right)\right)=0.$$

Hence there is a Boolean homomorphism from $\mathcal{B}_{\mathsf{X}_A^+}$ to the set of projections in $\widetilde{O}_A$, sending D_t to $S_tS_t^*$. Thus also (4) of Theorem 4 holds. So there is a unital $*$-homomorphism ψ from $\mathcal{O}_{\mathsf{X}_{\mathbf{A}}^+}$ to $\widetilde{O}_A$, such that $\psi(S_t) = S_{u_1}S_{u_2}\cdots S_{u_n}$, for $t = u_1u_2\cdots u_n \in \mathbb{F}_{\mathcal{I}}$ written in reduced form.

The $*$-homomorphism $\tilde{\phi}$ extends to a unital $*$-homomorphism. We then have that $\psi(\tilde{\phi}(S_i)) = \psi(S_i) = S_i$ for every $i \in \mathcal{I}$ and that $\psi(\tilde{\phi}(1)) = 1$, and since $\widetilde{O}_A$ is generated by $\{S_i\}_{i\in\mathcal{I}} \cup \{1\}$, this shows that $\psi \circ \tilde{\phi} = \mathrm{Id}_{\widetilde{\mathcal{O}}_A}$.

We also have that $\tilde{\phi}(\psi(S_t)) = \tilde{\phi}(S_{u_1}S_{u_2}\cdots S_{u_n}) = S_{u_1}S_{u_2}\cdots S_{u_n}$ for every $t \in \mathbb{F}_{\mathcal{I}}$, where $t = u_1u_2\cdots u_n \in \mathbb{F}_{\mathcal{I}}$ is written in reduced form.

If $u \in \mathcal{I}\cup\mathcal{I}^{-1}$ and $t \in \mathbb{F}_{\mathcal{I}}$ written in reduced form does not begin with u, then $D_{ut} \subseteq D_u$, so

$$\begin{aligned} S_uS_t &= S_uS_u^*S_{ut} \\ &= S_uS_u^*S_{ut}S_{ut}^*S_{ut} \\ &= S_{ut}. \end{aligned}$$

This shows that $S_{u_1}S_{u_2}\cdots S_{u_n} = S_t$ for $t = u_1u_2\cdots u_n \in \mathbb{F}_{\mathcal{I}}$ written in reduced form. Hence $\tilde{\phi}(\psi(S_t)) = S_t$ for every $t \in \mathbb{F}_{\mathcal{I}}$, and since $\mathcal{O}_{\mathsf{X}_{\mathbf{A}}^+}$ is generated by $S_t,\ t \in \mathbb{F}_{\mathcal{I}}$, this shows that $\tilde{\phi}\circ\psi = \mathrm{Id}_{\mathcal{O}_{\mathsf{X}_{\mathbf{A}}^+}}$. Thus $\mathcal{O}_{\mathsf{X}_{\mathbf{A}}^+}$ and $\widetilde{\mathcal{O}}_A$ are isomorphic. □

Example 4 continued. Let Λ be a k-graph which satisfies the standing hypothesis

$$0 < \#\Lambda^n(v) < \infty \text{ for every } v \in \Lambda^0 \text{ and } n \in \mathbb{N}^k$$

of [7]. Kumjian and Pask have in [7] defined a C^*-algebra $C^*(\Lambda)$ which is the universal C^*-algebra generated by a family $\{S_\lambda \mid \lambda \in \Lambda\}$ of partial isometries satisfying:

(1) $\{S_v \mid v \in \Lambda^0\}$ is a family of mutually orthogonal projections,
(2) $S_\lambda S_\mu = S_{\lambda\mu}$ for all $\lambda, \mu \in \Lambda$ such that $s(\lambda) = r(\mu)$,
(3) $S_\lambda^* S_\lambda = S_{s(\lambda)}$ for all $\lambda \in \Lambda$,
(4) for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$ we have $S_v = \sum_{\lambda\in\Lambda^n(v)} S_\lambda S_\lambda^*$.

Let $\mathbf{\Lambda}$ be the discrete partial dynamical system defined in Example 4. We then have the following result:

Proposition 10. *$\mathcal{O}_{\mathbf{\Lambda}}$ is isomorphic to the unitizaation $C^*(\widetilde{\Lambda})$ of $C^*(\Lambda)$..*

Proof. Let for every $v \in \Lambda^0$, $S_v = \sum_{\lambda\in\Lambda^p(v)} S_\lambda S_\lambda^* \in \mathcal{O}_{\mathbf{\Lambda}}$, where p is the element $(1,1,\dots,1) \in \mathbb{N}^k$. Consider the family $\{S_\lambda \mid \lambda \in \Lambda\}$ of partial isometries in $\mathcal{O}_{\mathbf{\Lambda}}$.

Since $\left\{\bigcup_{\lambda\in\Lambda^p(v)} D_\lambda \mid v \in \Lambda^0\right\}$ is a family of mutually disjoint subsets of Λ^∞, condition (1) above is satisfied. For $\lambda \in \Lambda$ is $D_{\lambda^{-1}} = \bigcup_{\mu\in\Lambda^p(s(\lambda))} D_\mu$, so condition (3) above is satisfied. If $v \in \Lambda^0$ and $n \in \mathbb{N}^k$, then

$$\bigcup_{\lambda \in \Lambda^n(v)} D_\lambda = \bigcup_{\lambda \in \Lambda^p(v)} D_\lambda,$$

so (4) above is satisfied. If $\lambda, \mu \in \Lambda \setminus \Lambda^0$ and $s(\lambda) = r(\mu)$, then $D_{\lambda\mu} \subseteq D_\lambda$, so

$$\begin{aligned} S_\lambda S_\mu &= S_\lambda S_\lambda^* S_{\lambda\mu} S_{\lambda\mu}^* S_{\lambda\mu} \\ &= S_{\lambda\mu}. \end{aligned}$$

If $\lambda \in \Lambda \setminus \Lambda^0$, $v \in \Lambda^0$ and $r(\lambda) = v$, then $D_\lambda \subseteq \bigcup_{\mu \in \Lambda^p(v)} D_\mu$, so

$$\begin{aligned} S_v S_\lambda &= \left(\sum_{\mu \in \Lambda^p(v)} S_\mu S_\mu^* \right) S_\lambda S_\lambda^* S_\lambda \\ &= S_\lambda \\ &= S_{v\lambda}. \end{aligned}$$

Finally, if $\lambda \in \Lambda \setminus \Lambda^0$, $v \in \Lambda^0$ and $s(\lambda) = v$, then $S_\lambda S_v = S_\lambda S_\lambda^* S_\lambda = S_\lambda = S_{\lambda v}$. Thus condition (2) above is satisfied. Hence there is a $*$-homomorphism ϕ from $C^*(\Lambda)$ to $\mathcal{O}_{\mathbf{\Lambda}}$ such that $\phi(S_\lambda) = S_\lambda$ for $\lambda \in \Lambda \setminus \Lambda^0$ and $\phi(S_v) = \sum_{\lambda \in \Lambda^p(v)} S_\lambda S_\lambda^* \in \mathcal{O}_{\mathbf{\Lambda}}$ for $v \in \Lambda^0$.

Let us now turn towards $C^*(\widetilde{\Lambda})$. We let for every $\lambda \in \Lambda \setminus \Lambda^0$, $S_{\lambda^{-1}} = S_\lambda^*$ and we let $S_e = 1$ and $S_t = S_{u_1} S_{u_2} \cdots S_{u_n}$, where $t = u_1 u_2 \cdots u_n \in G$ is written in reduced form. Then (1), (2) and (3) of Theorem 4 hold.

The $*$-homomorphism ϕ induces a Boolean homomorphism from the set of projections in $C^*(\widetilde{\Lambda})$ to the set of projections in $\mathcal{O}_{\mathbf{\Lambda}}$ which maps $S_t S_t^*$ to $S_t S_t^*$. Since there by Lemma 6 is an injective Boolean homomorphism from $\mathcal{B}_\Lambda$ to the set of projections in $\mathcal{O}_{\mathbf{\Lambda}}$ which maps D_t to $S_t S_t^*$, we get a Boolean homomorphism from the set of projections in $C^*(\Lambda)$ to $\mathcal{B}_\Lambda$ which maps $S_t S_t^*$ to D_t. We claim that it is injective.

Let for each $v \in \Lambda^0$, $D_v = \Lambda^\infty(v) = \bigcup_{\lambda \in \Lambda^p(v)} D_\lambda \in \mathcal{B}_\Lambda$, and let $\mathcal{B}'$ be the set of subsets of Λ^∞ which is a finite union of sets of the form

$$D_\lambda \cap \bigcap_{i=1}^{n} \neg D_{\lambda^i}$$

where $n \in \mathbb{N}_0$ and $\lambda, \lambda_1, \lambda_2, \ldots, \lambda_n \in \Lambda$.

We claim that $\mathcal{B}' = \mathcal{B}_\Lambda$. It is clear that $\mathcal{B}' \subseteq \mathcal{B}_\Lambda$. We will show that $\mathcal{B}_\Lambda \subseteq \mathcal{B}'$ by proving that $\mathcal{B}'$ is closed under taking union, intersection and complement and that $\theta_\lambda(\mathcal{B}') \subseteq \mathcal{B}'$ and $\theta_{\lambda^{-1}}(\mathcal{B}') \subseteq \mathcal{B}'$ for every $\lambda \in \Lambda \setminus \Lambda^0$.

It is clear that $\mathcal{B}'$ is closed under union. Let $\lambda, \mu \in \Lambda$ and let d be the least upper bound of $d(\lambda)$ and $d(\mu)$ in $\mathbb{N}_0^k$. If there is a $\nu \in \Lambda$ with $d(\nu) = d$ such that $\lambda\nu_1 = \nu$ and $\mu\nu_2 = \nu$ for some $\nu_1, \nu_2 \in \Lambda$, then $D_\lambda \cap D_\mu = D_\nu$. Otherwise $D_\lambda \cap D_\mu = \emptyset$. Hence $\mathcal{B}'$ is closed under intersection, and hence closed under complement.

Let $\lambda \in \Lambda \setminus \Lambda^0$ and $\mu \in \Lambda$. Then

$$\theta_\lambda(D_\mu) = \begin{cases} D_{\lambda\mu} & \text{if } s(\lambda) = r(\mu), \\ \emptyset & \text{else.} \end{cases}$$

Hence $\theta_\lambda(\mathcal{B}')\subseteq\mathcal{B}'$ for every $\lambda\in\Lambda\setminus\Lambda^0$.

Let $\lambda\in\Lambda\setminus\Lambda^0$ and $\mu\in\Lambda$ and let as before d be the least upper bound of $d(\lambda)$ and $d(\mu)$ in $\mathbb{N}_0^k$. If there are $\nu,\nu'\in\Lambda$ such that $d(\lambda)+d(\nu)=d(\mu)+d(\nu')=d$ and $\lambda\nu=\mu\nu'$, then $\theta_{\lambda^{-1}}(D_\mu)=D_\nu$. Otherwise $\theta_{\lambda^{-1}}(D_\mu)=\emptyset$. Hence $\theta_{\lambda^{-1}}(\mathcal{B}')\subseteq\mathcal{B}'$ for every $\lambda\in\Lambda\setminus\Lambda^0$.

Thus $\mathcal{B}'=\mathcal{B}_\Lambda$, so in order to prove that the Boolean homomorphism is injective it is enough to show that if

$$D_\lambda\cap\left(\bigcap_{k=1}^n\neg D_{\lambda_k}\right)=\emptyset,$$

then

$$S_\lambda S_\lambda^*\left(\prod_{k=1}^n 1-S_{\lambda_k}S_{\lambda_k}^*\right)=0.$$

Let $k\in\{1,2,\dots n\}$. If there does not exist a $\mu\in\Lambda$ such that $\lambda\mu=\lambda_k$ then $D_\lambda\cap\neg D_{\lambda_k}=D_\lambda$, so we may assume that for every $k\in\{1,2,\dots n\}$, $\lambda_k=\lambda\mu_k$ for some $\mu_k\in\Lambda$. We then have that

$$\theta_{\lambda^{-1}}\left(D_\lambda\cap\left(\bigcap_{k=1}^n\neg D_{\lambda_k}\right)\right)=D_{s(\lambda)}\cap\left(\bigcap_{k=1}^n\neg D_{\mu_k}\right),$$

so if

$$D_\lambda\cap\left(\bigcap_{k=1}^n\neg D_{\lambda_k}\right)=\emptyset,$$

then

$$D_{s(\lambda)}\cap\left(\bigcap_{k=1}^n\neg D_{\mu_k}\right)=\emptyset.$$

Choose $d\in\mathbb{N}^k$ such that $d\geq d(\mu_k)$ for every $k\in\{1,2,\dots,n\}$. Let $\mu\in\Lambda^d(s(\lambda))$. Since

$$D_\mu\subseteq D_{s(\lambda)}\subseteq\bigcup_{k=1}^n D_{\mu_k},$$

there is a $k\in\{1,2,\dots,n\}$ and a $\alpha\in\Lambda$ such that $\mu=\mu_k\alpha$. Then

$$S_\mu S_\mu^* S_{\mu_k}S_{\mu_k}^*=S_{\mu_k}S_\alpha S_\alpha^* S_{\mu_k}^* S_{\mu_k}S_{\mu_k}^*=S_{\mu_k}S_\alpha S_\alpha^* S_{\mu_k}^*=S_\mu S_\mu^*,$$

so

$$\begin{aligned}S_{s(\lambda)}S_{s(\lambda)}^*\sum_{k=1}^n S_{\mu_k}S_{\mu_k}^* &= \sum_{\mu\in\Lambda^d(s(\lambda))}S_\mu S_\mu^*\sum_{k=1}^n S_{\mu_k}S_{\mu_k}^*\\ &=\sum_{\mu\in\Lambda^d(s(\lambda))}S_\mu S_\mu^*\\ &=S_{s(\lambda)}S_{s(\lambda)}^*,\end{aligned}$$

and

$$\begin{aligned} S_\lambda S_\lambda^* \left(\prod_{k=1}^n 1 - S_{\lambda_k} S_{\lambda_k}^*\right) &= S_\lambda S_\lambda^* \left(\prod_{k=1}^n 1 - S_\lambda S_{\mu_k} S_{\mu_k}^* S_\lambda^*\right) \\ &= S_\lambda \left(\prod_{k=1}^n 1 - S_{s(\lambda)} S_{s(\lambda)}^* S_{\mu_k} S_{\mu_k}^*\right) S_\lambda^* \\ &= S_\lambda \left(\prod_{k=1}^n 1 - S_{s(\lambda)} S_{s(\lambda)}^*\right) S_\lambda^* \\ &= 0. \end{aligned}$$

Hence there is a Boolean homomorphism from $\mathcal{B}_\Lambda$ to the set of projections in $C^*(\widetilde{\Lambda})$, sending D_t to $S_t S_t^*$. Thus also (4) of Theorem 4 holds. So there is a unital $*$-homomorphism ψ from $\mathcal{O}_{\mathbf{\Lambda}}$ to $C^*(\widetilde{\Lambda})$, such that $\psi(S_t) = S_{u_1} S_{u_2} \cdots S_{u_n}$, for $t = u_1 u_2 \cdots u_n \in G$ written in reduced form.

The $*$-homomorphism $\phi : C^*(\Lambda) \to \mathcal{O}_{\mathbf{\Lambda}}$ extends to a unital $*$-homomorphism from $C^*(\widetilde{\Lambda})$ to $\mathcal{O}_{\mathbf{\Lambda}}$. We then have that $\psi(\tilde{\phi}(S_\lambda)) = \psi(S_\lambda) = S_\lambda$ for every $\lambda \in \Lambda$, and since $C^*(\widetilde{\Lambda})$ is generated by $\{S_\lambda\}_{\lambda \in \Lambda} \cup \{1\}$, this shows that $\psi \circ \tilde{\phi} = \mathrm{Id}_{C^*(\widetilde{\Lambda})}$.

We also have that $\phi(\psi(S_t)) = \phi(S_{u_1} S_{u_2} \cdots S_{u_n}) = S_{u_1} S_{u_2} \cdots S_{u_n}$ for every $t \in G$, where $t = u_1 u_2 \cdots u_n \in G$ is written in reduced form.

If $u \in \Lambda \setminus \Lambda^0 \cup \Lambda \setminus \Lambda^{0^{-1}}$ and $t \in G$ written in reduced form does not begin with u, then $D_{ut} \subseteq D_u$, so

$$\begin{aligned} S_u S_t &= S_u S_u^* S_{ut} \\ &= S_u S_u^* S_{ut} S_{ut}^* S_{ut} \\ &= S_{ut}. \end{aligned}$$

This shows that $S_{u_1} S_{u_2} \cdots S_{u_n} = S_t$ for $t = u_1 u_2 \cdots u_n \in G$ written in reduced form. Hence $\tilde{\phi}(\psi(S_t)) = S_t$ for every $t \in G$, and since $\mathcal{O}_{\mathbf{\Lambda}}$ is generated by $S_t,\ t \in G$, this shows that $\tilde{\phi} \circ \psi = \mathrm{Id}_{\mathcal{O}_{\mathbf{\Lambda}}}$. Thus $\mathcal{O}_{\mathbf{\Lambda}}$ and $C^*(\widetilde{\Lambda})$ are isomorphic. ☐

The hypothesis

$$0 < \#\Lambda^n(v) < \infty \text{ for every } v \in \Lambda^0 \text{ and } n \in \mathbb{N}^k$$

is not required for the construction of the partial dynamical system $\mathbf{\Lambda}$ and thus one can construct the C^*-algebra $\mathcal{O}_{\mathbf{\Lambda}}$ also in the case where Λ does not satisfy the hypothesis. It seems naturally also in this case to think of $\mathcal{O}_{\mathbf{\Lambda}}$ as the C^*-algebra associated to the k-graph Λ.

Example 5. Let $k \in \mathbb{N}$ and let $\mathcal{I}$ be an arbitrary discrete set. Endow $\mathcal{I}^{\mathbb{N}_0^k}$ with the product topology, and let for each $n \in \mathbb{N}_0^k$, $\sigma^n : \mathcal{I}^{\mathbb{N}_0^k} \to \mathcal{I}^{\mathbb{N}_0^k}$ be the shift mapping defined by $\sigma^n(x)(m) = x(m+n)$ for $x \in \mathcal{I}^{\mathbb{N}_0^k}$ and $m \in \mathbb{N}_0^k$.

Let $\mathcal{X}$ be a closed subset of $\mathcal{I}^{\mathbb{N}_0^k}$ such that $\sigma^n(\mathcal{X}) \subset \mathcal{X}$ for every $n \in \mathbb{N}_0^k$. Then $\mathcal{X}$ is a k-dimensional shift space over $\mathcal{I}$.

Notice that a k-dimensional shift is a generalization of one-sided shift spaces, k-graphs and the space X_A^+ associated to a matrix A.

Let for $i \in \{1, 2, \ldots, k\}$, $i(\mathbb{N}_0^k) = \{n \in \mathbb{N}_0^k \mid n_i = 0\}$ and let for each $x \in \mathcal{I}^{i(\mathbb{N}_0^k)}$,

$$D_x = \{y \in \mathcal{I}^{\mathbb{N}_0^k} \mid y_{|i(\mathbb{N}_0^k)} = x\}.$$

Let e_i be the element in $\mathbb{N}_0^k$ with every entries equal to 0 except the i'th which is equal to 1. Then $\sigma^{e_i}_{|D_x}$ is injective. Let $D_{x^{-1}} = \sigma^{e_i}_{|D_x}(D_x)$, $\theta_x = (\sigma^{e_i}_{|D_x})^{-1}$ and $\theta_{x^{-1}} = \sigma^{e_i}_{|D_x}$.

Let $\mathcal{J} = \bigcup_{i=1}^k i(\mathbb{N}_0^k)$, let $\mathbb{F}_\mathcal{J}$ be the free group generated by $\mathcal{J}$ and let for every $t \in \mathbb{F}_\mathcal{J}$ written on reduced form $u_1 u_2 \cdots u_n$, where $u_1, u_2, \ldots, u_n \in \mathcal{J} \cup \mathcal{J}^{-1}$,

$$D_t = \theta_{u_1} \circ \theta_{u_2} \circ \cdots \circ \theta_{u_n}(\mathcal{X})$$

and

$$\theta_t = \theta_{u_1} \circ \theta_{u_2} \circ \cdots \circ \theta_{u_n}.$$

Then $\mathcal{X} = (\mathcal{X}, \mathbb{F}_\mathcal{J}, \theta)$ is a discrete partial dynamical system which is a generalization of the discrete partial dynamical systems associated to one-sided shift spaces, k-graphs and $\{0,1\}$-matrices. Thus $\mathcal{O}_\mathcal{X}$ is a generalization of the C^*-algebra of one-sided shift spaces, k-graphs and $\{0,1\}$-matrices.

7. A short exact sequence

We will now show how an invariant ideal of $\mathcal{B}_X$ (i.e. a subset which is a union of sets from $\mathcal{B}_X$) gives raise to an ideal in $\mathcal{O}_\mathbf{X}$.

At the end of the section, we will apply this to two-sided shift spaces having a certain property and show that the crossed product of the shift space is a quotient of the C^*-algebra associated to the corresponding one-sided shift space.

Theorem 11. *Let $\mathbf{X} = (X, G, \theta)$ be a discrete partial dynamical system and let Y be an invariant (i.e. $\theta_t(Y) \subseteq Y$ for all $t \in G$) ideal of $\mathcal{B}_X$.*

Let $\mathcal{O}_\mathbf{Y}$ be the ideal of $\mathcal{O}_\mathbf{X}$ generated by elements of the form $S_{t_1} S_{t_2} \cdots S_{t_n}$ where $\theta_{t_1} \circ \theta_{t_2} \circ \cdots \circ \theta_{t_n}(X) \subseteq Y$. Then the map

$$S_t + \mathcal{O}_\mathbf{Y} \mapsto S_t$$

extends to a $*$-isomorphism from the quotient $\mathcal{O}_\mathbf{X}/\mathcal{O}_\mathbf{Y}$ to $\mathcal{O}_{\mathbf{X}/\mathbf{Y}}$, where $\mathbf{X}/\mathbf{Y}$ is the C^*-partial action $(\{D_t \cap (X \setminus Y)\}_{t \in G}, \{\theta_{t|X \setminus Y}\}_{t \in G})$ of G on $X \setminus Y$.

Proof. Let

$$\hat{Y} = \bigcup \{\hat{A} \mid A \in \mathcal{B}_X, A \subseteq Y\}.$$

Then $\hat{Y}$ is an open subset of $\hat{X}$, and for $t \in G$ are

$$\begin{aligned} \hat{\theta}_t(\hat{Y}) &= \bigcup \{\hat{\theta}_t(\hat{A}) \mid A \in \mathcal{B}_X, A \subseteq Y\} \\ &= \bigcup \{\widehat{\theta_t(A)} \mid A \in \mathcal{B}_X, A \subseteq Y\} \\ &\subseteq \hat{Y}, \end{aligned}$$

so $\hat{Y}$ is invariant.

Let I be the ideal $\{f \in \bar{X} \mid f_{|\hat{X}\setminus\hat{Y}}\}$ of $\bar{X}$. It follows from lemma 3, the fact that Y is a union of finite intersection of sets from $\{D_t \mid t \in G\}$ and Stone-Weierstrass Theorem, that

$$I = \overline{\operatorname{span}}\{1_{\hat{D}_{t_1}\cap\hat{D}_{t_2}\cap\cdots\cap\hat{D}_{t_n}} \mid D_{t_1} \cap D_{t_2} \cap \cdots \cap D_{t_n} \subseteq Y\}.$$

Let $\mathbf{Y}$ be the C^*-partial action of G on I with ideals $\bar{D}_t \cap I$ and isomorphisms $\bar{\theta}_{t|I}$ (cf. [5, Proposition 3.1]). We then have that

$$\begin{aligned} I \rtimes_{\mathbf{Y}} G &= \overline{\operatorname{span}}\{fS_t \mid f \in \bar{D}_t \cap I, t \in G\} \\ &= \overline{\operatorname{span}}\{1_{\hat{D}_{t_1}\cap\hat{D}_{t_2}\cap\cdots\cap\hat{D}_{t_n}} S_{t_n} \mid D_{t_1} \cap D_{t_2} \cap \cdots \cap D_{t_n} \subseteq Y\}. \end{aligned}$$

We claim that $\mathcal{O}_{\mathbf{Y}} = I \rtimes_{\mathbf{Y}} G$. If $D_{t_1} \cap D_{t_2} \cap \cdots \cap D_{t_n} \subseteq Y$, then

$$\theta_{t_1} \circ \theta_{t_1^{-1}} \circ \theta_{t_2} \circ \theta_{t_2^{-1}} \circ \cdots \circ \theta_{t_n}(X) = D_{t_1} \cap D_{t_2} \cap \cdots \cap D_{t_n} \subseteq Y,$$

so

$$1_{\hat{D}_{t_1}\cap\hat{D}_{t_2}\cap\cdots\cap\hat{D}_{t_n}} S_{t_n} = S_{t_1} S_{t_1^{-1}} S_{t_2} S_{t_2^{-1}} \cdots S_{t_n} \in \mathcal{O}_{\mathbf{Y}}.$$

If $D_{t_1} \cap D_{t_2} \cap \cdots \cap D_{t_n} \subseteq Y$, then

$$\theta_{t_1} \circ \theta_{t_1 t_2} \circ \cdots \circ \theta_{t_1 t_2 \cdots t_n}(X) = D_{t_1} \cap D_{t_2} \cap \cdots \cap D_{t_n} \subseteq Y,$$

so

$$\begin{aligned} S_{t_1} S_{t_2} \cdots S_{t_n} &= S_{t_1} S_{t_1}^* S_{t_1} S_{t_2} S_{t_3} \cdots S_{t_n} \\ &= 1_{\hat{D}_{t_1}} S_{t_1 t_2} S_{t_3} \cdots S_{t_n} \\ &= 1_{\hat{D}_{t_1}} 1_{\hat{D}_{t_1 t_2}} \cdots 1_{\hat{D}_{t_1 t_2 \cdots t_n}} S_{t_n} \in I \rtimes_{\mathbf{Y}} G. \end{aligned}$$

Thus $\mathcal{O}_{\mathbf{Y}} = I \rtimes_{\mathbf{Y}} G$.

It follows from [5, Proposition 3.1] that $(A \rtimes_{\mathbf{X}} G)/(I \rtimes_{\mathbf{Y}} G)$ is isomorphic to $(A/I) \rtimes_{\dot{\mathbf{X}}} G$, where $\dot{\mathbf{X}}$ is the partial action $(\{\dot{D}_t\}_{t\in G}, \{\dot{\bar{\theta}}_t\}_{t\in G})$ of G on $\bar{X}/I$, where $\dot{D}_t = \{f \in \bar{X} \mid f_{|\hat{X}\setminus(\hat{X}\setminus\hat{Y}\cap\hat{D}_t)} = 0\}$ and

$$\dot{\bar{\theta}}_t(f)(\phi) = \begin{cases} f(\hat{\theta}_{t^{-1}}(\phi)) & \text{if } \phi \in \hat{D}_t \cap \hat{X} \setminus \hat{Y}, \\ 0 & \text{if } \phi \in \hat{X} \setminus (\hat{D}_t \cap \hat{X} \setminus \hat{Y}), \end{cases}$$

(cf. [5, Page 7]). We claim that $(A/I) \rtimes_{\dot{\mathbf{X}}} G$ is isomorphic to $\mathcal{O}_{\mathbf{X}/\mathbf{Y}}$. A/I is isomorphic to $C(\hat{X} \setminus \hat{Y})$, so it is enough to check that there is a Boolean isomorphism from the Boolean algebra on $X \setminus Y$ generated by $D_t \cap (X \setminus Y)$, $t \in G$ to the Boolean algebra on $\hat{X} \setminus \hat{Y}$ generated by $\hat{D}_t \cap (\hat{X} \setminus \hat{Y})$, $t \in G$ which intertwines θ_t^{-1} and $\hat{\theta}_t^{-1}$, and which maps $D_t \cap (X \setminus Y)$ to $\hat{D}_t \cap (\hat{X} \setminus \hat{Y})$ for every $t \in G$, and the existence of such an isomorphism follows from the fact that the map $A \mapsto \hat{A}$ is an Boolean isomorphism from $\mathcal{B}_X$ to the Boolean algebra on $\hat{X}$ generated by $\hat{D}_t$, $t \in G$. □

Let X be a two-sided shift space and let

$$\mathsf{X}^+ = \{(x_n)_{n\in\mathbb{N}_0} \mid (x_n)_{n\in\mathbb{Z}} \in \mathsf{X}\}$$

be the corresponding one-sided shift space. We denote the language of X by $\mathcal{L}(\mathsf{X})$. For every $x \in \mathsf{X}^+$ and every $k \in \mathbb{N}_0$ we set

$$\mathcal{P}_k(x) = \{\mu \in \mathcal{L}(\mathsf{X}) \mid \mu x \in \mathsf{X}^+, |\mu| = k\},$$

and define for every $l \in \mathbb{N}_0$ an equivalence relation $\sim_l$ on X^+ by

$$x \sim_l x' \Leftrightarrow \mathcal{P}_l(x) = \mathcal{P}_l(x').$$

We say (cf. [2]) that a shift space X has *property* $(*)$ if for every $\mu \in \mathcal{L}(\mathsf{X})$ there exists an $x \in \mathsf{X}^+$ such that $\mathcal{P}_{|\mu|}(x) = \{\mu\}$.

Let $\mathcal{I} = \{(k,l) \in \mathbb{N}_0^2 \mid k \leq l\}$. We define an order $\leq$ on $\mathcal{I}$ by

$$(k_1,l_1) \leq (k_2,l_2) \Leftrightarrow k_1 \leq k_2 \wedge l_1 - k_1 \leq l_2 - k_2.$$

For $(k,l) \in \mathcal{I}$ we define an equivalence relation ${}_k{\sim}_l$ on X^+ by

$$x \;{}_k{\sim}_l\; y \Leftrightarrow x_{[0,k[} = y_{[0,k[} \wedge \mathcal{P}_l(x_{[k,\infty[}) = \mathcal{P}_l(y_{[k,\infty[}).$$

Denote the inclusion of $\mathcal{D}_{\mathsf{X}^+}$ into $\mathcal{O}_{\mathsf{X}^+}$ by $\eta_{\mathcal{O}}$ and the inclusion of $C(\mathsf{X})$ into $C(\mathsf{X}) \rtimes_{\sigma^\star} \mathbb{Z}$ by $\eta_{\rtimes}$. We then have the follow result:

Proposition 12. *Let X be a two-sided shift space with the property $(*)$. Then there are surjective $*$-homomorphisms $\kappa : \mathcal{D}_{\mathsf{X}^+} \to C(\mathsf{X})$ and $\rho : \mathcal{O}_{\mathsf{X}^+} \to C(\mathsf{X}) \rtimes_{\sigma^\star} \mathbb{Z}$ making the diagram*

$$\begin{array}{ccc} \mathcal{D}_{\mathsf{X}^+} & \xrightarrow{\;\kappa\;} & C(\mathsf{X}) \\ \eta_{\mathcal{O}} \big\downarrow & & \big\downarrow \eta_{\rtimes} \\ \mathcal{O}_{\mathsf{X}^+} & \xrightarrow[\;\rho\;]{} & C(\mathsf{X}) \rtimes_{\sigma^\star} \mathbb{Z} \end{array}$$

commute. We furthermore have that

$$\kappa(1_{C(\mu,\nu)}) = 1_{\{x \in \mathsf{X} \mid x_{-m} = \mu_m, x_{-m+1} = \mu_{m-1}, \dots x_{-1} = \mu_1, x_0 = \nu_0, \dots, x_n = \nu_n\}}$$

for every $\mu = \mu_1 \cdots \mu_m, \nu = \nu_0, \dots, \nu_n \in \mathcal{L}(\mathsf{X})$.

Proof. Let for every $A \in \mathcal{B}_{\mathsf{X}^+}$,

$$\psi(A) = \{z \in \mathsf{X} \mid \forall (k,l) \in \mathcal{I} \exists x \in A : x_{[0,k[} = z_{[0,k[} \wedge \mathcal{P}_l(x_{[k,\infty[}) = \{z_{[k,\infty[}\}\}.$$

We claim that ψ is a surjective Boolean homomorphism from $\mathcal{B}_{\mathsf{X}^+}$ to $\mathcal{B}_{\mathsf{X}}$, mapping D_t to D_t for every $t \in \mathbb{F}_{\mathfrak{a}}$.

We will prove that by establishing a sequence of claims.

Claim 1.

$$\forall A \in \mathcal{B}_{\mathsf{X}^+} \exists (k,l) \in \mathcal{I} : x \;{}_k{\sim}_l\; x' \wedge x \in A \Rightarrow x' \in A.$$

Proof. Let

$$\mathcal{A} = \{A \in \mathcal{B}_{\mathsf{X}^+} \mid \exists (k,l) \in \mathcal{I} : x \;{}_k{\sim}_l\; x' \wedge x \in A \Rightarrow x' \in A\}.$$

We must then show that $\mathcal{A} = \mathcal{B}_{\mathsf{X}^+}$. Clearly $\mathsf{X}^+ \in \mathcal{A}$. Assume that $A, B \in \mathcal{A}$ and choose $(k_a,l_a),(k_b,l_b) \in \mathcal{I}$ such that

$$x \;{}_{k_a}{\sim}_{l_a}\; x' \wedge x \in A \Rightarrow x' \in A$$

and

$$x \;{}_{k_b}{\sim}_{l_b}\; x' \wedge x \in B \Rightarrow x' \in B.$$

Let $k = \max\{k_a,k_b\}$ and $l = \max\{l_a,l_b\}$. Then

$$x \;{}_k{\sim}_l\; x' \wedge x \in A \cap B \Rightarrow x' \in A \cap B,$$

so $A \cap B \in \mathcal{A}$. We also have that

$$x \;{}_{k_a}{\sim}_{l_a}\; x' \wedge x \in \neg A \Rightarrow x' \in \neg A,$$

$$x \,{}_{k_a+1}{\sim}_{l_a+1}\, x' \wedge x \in \theta_a(A) \Rightarrow x' \in \theta_a(A)$$

and

$$x \,{}_{k_a}{\sim}_{l_a+1}\, x' \wedge x \in \theta_{a^{-1}}(A) \Rightarrow x' \in \theta_{a^{-1}}(A),$$

so $\neg A, \theta_a(A), \theta_{a^{-1}}(A) \in \mathcal{A}$. Thus $\mathcal{A}$ is a Boolean algebra containing D_t for every $t \in \mathbb{F}_{\mathfrak{a}}$, which means that $\mathcal{A} = \mathcal{B}_{\mathsf{X}^+}$. □

Claim 2. *ψ is a Boolean homomorphism.*

Proof. Clearly $\psi(\emptyset) = \emptyset$. Let $A, B \in \mathcal{B}_{\mathsf{X}^+}$. It is obvious that $\psi(A \cap B) \subseteq \psi(A) \cap \psi(B)$. Assume that $z \in \psi(A) \cap \psi(B)$. Choose $(k_a, l_a), (k_b, l_b) \in \mathcal{I}$ such that

$$x \,{}_{k_a}{\sim}_{l_a}\, x' \wedge x \in A \Rightarrow x' \in A$$

and

$$x \,{}_{k_b}{\sim}_{l_b}\, x' \wedge x \in B \Rightarrow x' \in B.$$

Let $(k, l) \geq (k_a, l_a), (k_b, l_b)$. Choose $x^A \in A$ and $x^B \in B$ such that $z_{[0,k[} = x^A_{[0,k[} = x^B_{[0,k[}$ and $\mathcal{P}_l(x^A_{[k,\infty[}) = \mathcal{P}_l(x^B_{[k,\infty[}) = \{z_{[k-l,k[}\}$. Then $x^A \in A \cap B$. Thus $z \in \psi(A \cap B)$. So $\psi(A \cap B) = \psi(A) \cap \psi(B)$.

Assume now that $z \in \psi(\neg A)$. Let $(k, l) \geq (k_a, l_a)$. We then have that if $z_{[0,k[} = x_{[0,k[}$ and $\mathcal{P}_l(x_{[k,\infty[}) = \{z_{[k-l,k[}\}$, then $x \in \neg A$. Thus $z \in \neg\psi(A)$.

If $z \in \neg\psi(A)$, then there is $(k, l) \geq (k_a, l_a)$ such that $z_{[0,k[} \neq x_{[0,k[}$ or $\mathcal{P}_l(x_{[k,\infty[}) \neq \{z_{[k-l,k[}\}$ for every $x \in A$. Since X has property $(*)$, there is a $x \in \mathsf{X}^+$ such that $z_{[0,k[} = x_{[0,k[}$ and $\mathcal{P}_l(x_{[k,\infty[}) = \{z_{[k-l,k[}\}$. This x must belong to $\neg A$. Thus $z \in \psi(\neg A)$. So $\psi(\neg A) = \neg\psi(A)$. □

Claim 3. *$\psi(\theta_a(A)) = \theta_a(\psi(A))$ for every $A \in \mathcal{B}_{\mathsf{X}^+}$ and every $a \in \mathfrak{a}$.*

Proof. Let $z \in \psi(\theta_a(A))$ and let $(k, l) \in \mathcal{I}$. Then there is an $x \in \theta_a(A)$ such that $x_{[0,k+1[} = z_{[0,k+1[}$ and $\mathcal{P}_l(x_{[k+1,\infty[}) = \{z_{[k+1-l,k+1[}\}$. Then there is a $y \in A \cap D_{a^{-1}}$ such that $x = \theta_a(y)$. Hence $z_0 = x_0 = a$, $y_{[0,k[} = x_{[1,k+1[} = z_{[1,k+1[}$ and $\mathcal{P}_l(y_{[k,\infty[}) = \mathcal{P}_l(x_{[k+1,\infty[}) = \{z_{[k+1-l,k+1[}\}$. Thus $\sigma(z) \in \psi(A)$ and $z_0 = a$. Hence $z \in \theta_a(\psi(A))$.

Now let $z \in \theta_a(\psi(A))$. Then there is a $z' \in \psi(A)$ such that $z = \theta_a(z')$. For every $(k, l) \in \mathcal{I}$, there is a $x \in A$ such that $z'_{[0,k[} = x_{[0,k[}$ and $\mathcal{P}_l(x_{[k,\infty[}) = \{z'_{[k-l,k[}\}$. Then $z_{[0,k+1[} = ax_{[0,k[}$ and $\mathcal{P}_l(x_{[k,\infty[}) = \{z_{[k+1-l,k+1[}\}$. Thus $z \in \psi(\theta_a(A))$. □

Claim 4. *$\psi(\theta_{a^{-1}}(A)) = \theta_{a^{-1}}(\psi(A))$ for every $A \in \mathcal{B}_{\mathsf{X}^+}$ and every $a \in \mathfrak{a}$.*

Proof. Let $z \in \psi(\theta_{a^{-1}}(A))$ and let $(k, l) \in \mathcal{I}$. Then there is an $x \in \theta_{a^{-1}}(A)$ such that $x_{[0,k[} = z_{[0,k[}$ and $\mathcal{P}_{l+1}(x_{[k,\infty[}) = \{z_{[k-l+1,k[}\}$. Then there is a $y \in A \cap D_a$ such that $x = \theta_{a^{-1}}(y)$. Hence $z_{-1} = y_0 = a$, $y_{[0,k[} = x_{[-1,k-1[} = z_{[-1,k-1[}$ and $\mathcal{P}_l(y_{[k,\infty[} = \mathcal{P}_l(x_{[k-1,\infty[} = \{z_{[k-1-l,k-1[}\}$. Thus $\sigma^{-1}(z) \in \psi(A)$ and $z_{-1} = a$. Hence $z \in \theta_a(\psi(A))$.

Now let $z \in \theta_{a^{-1}}(\psi(A))$. Then there is a $z' \in \psi(A)$ such that $z = \theta_{a^{-1}}(z')$. For every $(k, l) \in \mathcal{I}$, there is a $x \in A$ such that $z'_{[0,k[} = x_{[0,k[}$ and $\mathcal{P}_l(x_{[k,\infty[}) = \{z'_{[k-l,k[}\}$. Then $z_{[0,k+1[} = ax_{[0,k[}$ and $\mathcal{P}_l(x_{[k,\infty[}) = \{z_{[k+1-l,k+1[}\}$. Thus $z \in \psi(\theta_a(A))$. □

It follows from Claim 3 and Claim 4 that

$$\begin{aligned}\psi(D_t) &= \psi(\theta_{u_1} \circ \theta_{u_2} \circ \cdots \circ \theta_{u_n}(\mathsf{X}^+)) \\ &= \theta_{u_1} \circ \theta_{u_2} \circ \cdots \circ \theta_{u_n} \circ \psi(\mathsf{X}^+) \\ &= \theta_{u_1} \circ \theta_{u_2} \circ \cdots \circ \theta_{u_n}(\mathsf{X}) \\ &= D_t\end{aligned}$$

for every $t = u_1 u_2 \cdots u_n \in \mathbb{F}_{\mathfrak{a}}$ written in reduced form. Thus ψ is a surjective Boolean homomorphism from $\mathcal{B}_{\mathsf{X}^+}$ to $\mathcal{B}_{\mathsf{X}}$.

Let Y be the kernel of ψ, i.e. $Y = \bigcup\{A \in \mathcal{B}_{\mathsf{X}^+} \mid \psi(A) = \emptyset\}$. Then Y is invariant and a ideal. Thus it follows from Theorem 11 that the map $S_t \mapsto S_t$ extends to a surjective $*$-homomorphism from $\mathcal{O}_{\mathbf{X}^+}$ to $\mathcal{O}_{\mathbf{X}^+/\mathbf{Y}}$.

Now, ψ induces a Boolean isomorphism from the Boolean algebra $\mathcal{B}_{\mathbf{X}^+/\mathbf{Y}} = \{A \cap (\mathsf{X}^+ \setminus Y) \mid A \in \mathcal{B}_{\mathbf{X}^+}\}$ to $\mathcal{B}_X$ which maps $D_t \cap (\mathsf{X}^+ \setminus Y)$ to D_t for every $t \in \mathbb{F}_{\mathfrak{a}}$. Thus it follows from Theorem 4 that there exists a $*$-isomorphism from $\mathcal{O}_{\mathbf{X}^+/\mathbf{Y}}$ to $\mathcal{O}_{\mathbf{X}}$ which maps S_t to S_t for every $t \in \mathbb{F}_{\mathfrak{a}}$. Thus there is a surjective $*$-homomorphism $\eta_{\mathcal{O}}$ from $\mathcal{O}_{\mathbf{X}^+}$ to $\mathcal{O}_{\mathbf{X}}$, which by Proposition 7 is isomorphic to $C(\mathsf{X}) \rtimes_{\sigma^\star} \mathbb{Z}$.

The image of $\mathcal{D}_{\mathsf{X}^+}$ inside $\mathcal{O}_{\mathbf{X}^+}$ is the C^*-subalgebra generated by $\{S_t S_t^* \mid t \in \mathbb{F}_{\mathfrak{a}}\}$, and the image of $C(\mathsf{X})$ inside $\mathcal{O}_{\mathbf{X}}$ is the C^*-subalgebra generated by $\{S_t S_t^* \mid t \in \mathbb{F}_{\mathfrak{a}}\}$. Since $\eta_{\mathcal{O}}$ maps S_t to S_t, the restriction of $\eta_{\mathcal{O}}$ to $\{S_t S_t^* \mid t \in \mathbb{F}_{\mathfrak{a}}\}$ is surjective onto $\{S_t S_t^* \mid t \in \mathbb{F}_{\mathfrak{a}}\}$. Thus there is a surjective $*$-homomorphism $\kappa : \mathcal{D}_{\mathsf{X}^+} \to C(\mathsf{X})$ which makes the diagram commute and which maps $1_{C(\mu,\nu)} = S_\nu S_\mu^* S_\mu S_\nu^*$ to

$$S_\nu S_\mu^* S_\mu S_\nu^* = 1_{\{x \in \mathsf{X} \mid x_{-m} = \mu_m, x_{-m+1} = \mu_{m-1}, \ldots x_{-1} = \mu_1, x_0 = \nu_0, \ldots, x_n = \nu_n\}}.$$

□

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3.9 Closing remarks

The reduced C^*-algebra $\mathcal{O}^{red}_{\mathsf{X}^+}$ associated with one-sided shift spaces has been studied by Matsumoto and the author in [3]. If the shift spaces satisfies a certain condition (I), then the reduced C^*-algebra $\mathcal{O}^{red}_{\mathsf{X}^+}$ is isomorphic to $\mathcal{O}^{red}_{\mathsf{X}^+}$, but it is know that this is not generally true.

A thorough discussing of the property $(*)$ can be found in the next paper.

I would be interesting to try look at the C^*-algebra associated to a higher dimensional shift on an infinite alphabet for some concrete examples and, for example, try to compute its K-theory.

Chapter 4

K_0 of the C^*-algebra associated to certain one-sided shift spaces

This chapter consists of the preprint *Matsumoto K-groups associated to certain shift spaces* which is written together with Søren Eilers.

The result from the previous chapter which connects the crossed product of a two-sided shift space and the C^*-algebra associated to the corresponding one-sided shift space, is the starting point for an examination of the K-theory of the C^*-algebra associated to the one-sided shift space corresponding to a two-sided shift space which has property $(*)$, and its relation with the K-theory of the crossed product of the two-sided shift space.

In this paper we start the examination by showing that the K_0-group of the crossed product of a two-sided shift space which has the property $(*)$ is a factor group of the K_0-group of the C^*-algebra associated to the corresponding one-sided shift space, and by giving a description of the K_0-group of the C^*-algebra associated to the one-sided shift space corresponding to a two-sided shift space having a certain property $(**)$, which is a strengthening of the property $(*)$, in terms of the K_0-group of the crossed product of the two-sided shift space and the *left special* elements of the two-sided shift space.

Note that in this paper the K_0-group of the crossed product of a two-sided shift space is called *the first cohomology group* and is denoted by $C(\underline{\mathsf{X}}, \mathbb{Z})/(\mathrm{Id} - (\sigma^{-1})^\star)(C(\underline{\mathsf{X}}, \mathbb{Z}))$ (cf. [1, Theorem 5.2]), and that the K_0-group of the C^*-algebra associated to the corresponding one-sided shift space is called *the Matsumoto's K-group* and is denoted by $K_0(\underline{\mathsf{X}})$ (cf. [11]).

Matsumoto K-groups associated to certain shift spaces

Toke Meier Carlsen and Søren Eilers

Abstract. In [24] Matsumoto associated to each shift space (also called a subshift) an abelian group which is now known as Matsumoto's K_0-group. It is defined as the cokernel of a certain map and resembles the first cohomology group of the dynamical system which has been studied in for example [2], [28], [13], [16] and [11] (where it is called the dimension group).

In this paper, we will for shift spaces having a certain property $(*)$, show that the first cohomology group is a factor group of Matsumoto's K_0-group. We will also for shift spaces having an additional property $(**)$, describe Matsumoto's K_0-group in terms of the first cohomology group and some extra information determined by the left special elements of the shift space.

We determine for a broad range of different classes of shift spaces if they have property $(*)$ and property $(**)$ and use this to show that Matsumoto's K_0-group and the first cohomology group are isomorphic for example for finite shift spaces and for Sturmian shift spaces.

Furthermore, the ground is laid for a description of the Matsumoto K_0-group as an *ordered* group in a forthcoming paper.

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Keywords and Phrases: Shift spaces, subshifts, symbolic dynamics, Matsumoto's K-groups, dimension groups, cohomology, special elements.

1 Introduction

Invariants for symbolic dynamical systems in the form of abelian groups have a fruitful history. Important examples are the dimension group defined by Krieger in [19] and [20], and the Bowen-Franks group defined in [1] by Bowen and Franks.

In [24] Matsumoto generalized the definition of dimension groups and Bowen-Franks groups to the whole class of shift spaces and introduced what is now known as Matsumoto's K-groups.
In another direction, Putnam [29], Herman, Putnam and Skau [16], Giordano, Putnam and Skau [15], Durand, Host and Skau [11] and Forrest [13] studied what they called the dimension group (it is not the same as Krieger's or Matsumoto's dimension group) for Cantor minimal systems. The same group has for a broader class of topological dynamical systems been studied in [2], [28] and [27] where it is shown that it is the first cohomology group of the standard suspension of the dynamical system in question.
It turns out that Matsumoto's K_0-group and the first cohomology group are closely related. We will for shift spaces having a certain property $(*)$, show that the first cohomology group is a factor group of Matsumoto's K_0-group, and we will also for shift spaces having an additional property $(**)$, describe Matsumoto's K_0-group in terms of the first cohomology group and some extra information determined by the left special elements of the shift space.
We will for a broad range of different classes of shift spaces, which includes shift of finite types, finite shift spaces, Sturmian shift spaces, substitution shift spaces and Toeplitz shift spaces, determine if they have property $(*)$ and property $(**)$. This will allow us to show that Matsumoto's K_0-group and the first cohomology group are isomorphic for example for finite shift spaces and for Sturmian shift spaces and to describe Matsumoto's K_0-group for substitution shift spaces in such a way that we in [8] can for every shift space associated with a aperiodic and primitive substitution present Matsumoto's K_0-group as a stationary inductive limit of a system associated to an integer matrix defined from combinatorial data which can be computed in an algorithmic way (cf. [6], [7]).
Since both Matsumoto's K_0-group and the first cohomology group are K_0-groups of certain C^*-algebras they come with a natural (pre)order structure. All the results presented in this paper hold not just in the category of abelian group, but also in the category of preordered groups. Since we do not know how to prove this without involving C^*-algebras we have decided to defer this to [9], where we also show that Matsumoto's K_0-group with order is a finer invariant than Matsumoto's K_0-group without order.
We wish to thank Yves Lacroix for helping us understand Toeplitz sequences.

2 PRELIMINARIES AND NOTATION

Throughout this paper Id will denote the identity map. For a map ϕ between two sets X and Y, we will by $\phi^\star$ denote the map which maps a function f on Y to the function $f \circ \phi$ on X.
Let $\mathfrak{a}$ be a finite set of symbols, and let $\mathfrak{a}^\sharp$ denote the set of finite, nonempty words with letters from $\mathfrak{a}$. Thus with ϵ denoting the *empty word*, $\epsilon \notin \mathfrak{a}^\sharp$. By $|\mu|$ we denote the *length* of a finite word μ (i.e. the number of letters in μ). The length of ϵ is 0.

2.1 Shift spaces

We equip
$$\mathfrak{a}^{\mathbb{Z}}, \mathfrak{a}^{\mathbb{N}_0}, \mathfrak{a}^{-\mathbb{N}}$$
with the product topology from the discrete topology on $\mathfrak{a}$. We will strive to denote elements of $\mathfrak{a}^{\mathbb{Z}}$ by z, elements of $\mathfrak{a}^{\mathbb{N}_0}$ by x and elements of $\mathfrak{a}^{-\mathbb{N}}$ by y. We define $\sigma : \mathfrak{a}^{\mathbb{Z}} \longrightarrow \mathfrak{a}^{\mathbb{Z}}$, $\sigma_+ : \mathfrak{a}^{\mathbb{N}_0} \longrightarrow \mathfrak{a}^{\mathbb{N}_0}$, and $\sigma_- : \mathfrak{a}^{-\mathbb{N}} \longrightarrow \mathfrak{a}^{-\mathbb{N}}$ by
$$(\sigma(z))_n = z_{n+1} \qquad (\sigma_+(x))_n = x_{n+1} \qquad (\sigma_-(y))_n = y_{n-1}.$$
Such maps we will refer to as *shift maps*.
A *shift space* is a closed subset of $\mathfrak{a}^{\mathbb{Z}}$ which is mapped into itself by σ. We shall refer to such spaces by "$\underline{\mathsf{X}}$".
With the obvious restriction maps
$$\pi_+ : \underline{\mathsf{X}} \longrightarrow \mathfrak{a}^{\mathbb{N}_0} \qquad \pi_- : \underline{\mathsf{X}} \longrightarrow \mathfrak{a}^{-\mathbb{N}}$$
we get
$$\sigma_+ \circ \pi_+ = \pi_+ \circ \sigma \qquad \sigma_- \circ \pi_- = \pi_- \circ \sigma^{-1}.$$
We denote $\pi_+(\underline{\mathsf{X}})$, respectively $\pi_-(\underline{\mathsf{X}})$, by $\underline{\mathsf{X}}^+$, respectively $\underline{\mathsf{X}}^-$, and notice that $\sigma_+(\underline{\mathsf{X}}^+) = \underline{\mathsf{X}}^+$ and $\sigma_-(\underline{\mathsf{X}}^-) = \underline{\mathsf{X}}^-$. For $z \in \mathfrak{a}^{\mathbb{Z}}$ and $n \in \mathbb{Z}$, we write
$$z_{[n,\infty[} = \pi_+(\sigma^n(z)).$$
The *language* of a shift space is the subset of $\mathfrak{a}^\sharp \cup \{\epsilon\}$ given by
$$\mathcal{L}(\underline{\mathsf{X}}) = \{z_{[n,m]} \mid z \in \underline{\mathsf{X}}, n \leq m \in \mathbb{Z}\}$$
where the interval subscript notation should be self-explanatory. A compactness argument shows that an element $z \in \mathfrak{a}^{\mathbb{Z}}$ (respectively $x \in \mathfrak{a}^{\mathbb{N}_0}$, $y \in \mathfrak{a}^{-\mathbb{N}}$) is in $\underline{\mathsf{X}}$ (respectively $\underline{\mathsf{X}}^+$, $\underline{\mathsf{X}}^-$) if and only if $z_{[n,m]} \in \mathcal{L}(\underline{\mathsf{X}})$ for all $n < m \in \mathbb{Z}$ (respectively $n < m \in \mathbb{N}_0$, $n < m \in -\mathbb{N}$) (cf. [21, Corollary 1.3.5 and Theorem 6.1.21]).
We say that shift spaces are *conjugate*, denoted by "$\simeq$", when they are homeomorphic via a map which intertwines the relevant shift maps. The concept of conjugacy also makes sense for the "one-sided" shift spaces $\underline{\mathsf{X}}^+$. If $\underline{\mathsf{X}}^+ \simeq \underline{\mathsf{Y}}^+$, then we say that $\underline{\mathsf{X}}$ and $\underline{\mathsf{Y}}$ are *one-sided conjugate*. It is not difficult to see that $\underline{\mathsf{X}}^+ \simeq \underline{\mathsf{Y}}^+ \Rightarrow \underline{\mathsf{X}} \simeq \underline{\mathsf{Y}}$ (cf. [21, §13.8]).
Finally we want to draw attention to a third kind of equivalence between shift spaces, called *flow equivalence*, which we denote by $\cong_f$. We will not define it here (see [26], [14], [2] or [21, §13.6] for the definition), but just notice that $\underline{\mathsf{X}} \simeq \underline{\mathsf{Y}} \Rightarrow \underline{\mathsf{X}} \cong_f \underline{\mathsf{Y}}$.
A *flow invariant* of a shift space $\underline{\mathsf{X}}$ is a mapping associating to each shift space another mathematical object, called the *invariant*, in such a way that flow equivalent shift spaces give isomorphic invariants. In the same way, a *conjugacy invariant* of $\underline{\mathsf{X}}$, respectively $\underline{\mathsf{X}}^+$, is a mapping associating to each

shift space an invariant in such a way that conjugate, respectively one-sided conjugate, shift spaces give isomorphic invariants.
Since $\underline{\mathsf{X}} \simeq \underline{\mathsf{Y}} \Rightarrow \underline{\mathsf{X}} \cong_f \underline{\mathsf{Y}}$, a flow invariant of $\underline{\mathsf{X}}$ is also a conjugacy invariant of $\underline{\mathsf{X}}$, and since $\underline{\mathsf{X}}^+ \simeq \underline{\mathsf{Y}}^+ \Rightarrow \underline{\mathsf{X}} \simeq \underline{\mathsf{Y}}$, a conjugacy invariant of $\underline{\mathsf{X}}$ is also a conjugacy invariant of $\underline{\mathsf{X}}^+$.

2.2 Special elements

We say (cf. [17]) that $z \in \underline{\mathsf{X}}$ is *left special* if there exists $z' \in \underline{\mathsf{X}}$ such that

$$z_{-1} \neq z'_{-1} \qquad \pi_+(z) = \pi_+(z').$$

It follows from [4, Proposition 2.4.1] (cf. [3, Theorem 3.9]) that a sufficient condition for a shift space $\underline{\mathsf{X}}$ to have a left special element is that $\underline{\mathsf{X}}$ is infinite. Conversely, the following proposition shows that this condition is necessary.

Proposition 2.1. *Let $\underline{\mathsf{X}}$ be a finite shift space. Then $\underline{\mathsf{X}}$ contains no left special element.*

Proof: Since $\underline{\mathsf{X}}$ is finite, every $z \in \underline{\mathsf{X}}$ is periodic. Hence if $\pi_+(z) = \pi_+(z')$, then $z = z'$. □

We say that the left special word z is *adjusted* if $\sigma^{-n}(z)$ is not left special for any $n \in \mathbb{N}$, and that z is *cofinal* if $\sigma^n(z)$ is not left special for any $n \in \mathbb{N}$. Thinking of left special words as those which are not deterministic from the right at index -1, the adjusted and cofinal left special words are those where this is the *leftmost* and *rightmost* occurrence of nondeterminacy, respectively.
Let $z, z' \in \underline{\mathsf{X}}$. If there exist an n and an M such that $z_m = z'_{n+m}$ for all $m > M$ then we say that z and z' are *right shift tail equivalent* and write $z \sim_r z'$. We will denote the right shift tail equivalence class of z by $\mathbf{z}$.

2.3 The first cohomology group

The first cohomology group (cf. [2]) of a shift space $\underline{\mathsf{X}}$ is the group

$$C(\underline{\mathsf{X}}, \mathbb{Z})/(\mathrm{Id} - (\sigma^{-1})^\star)(C(\underline{\mathsf{X}}, \mathbb{Z})).$$

Notice that usually σ is used instead of σ^{-1}, but for our purpose it is more naturally to use σ^{-1}, and we of course get the same group. $C(\underline{\mathsf{X}}, \mathbb{Z})/(\mathrm{Id} - (\sigma^{-1})^\star)(C(\underline{\mathsf{X}}, \mathbb{Z}))$ is the first Čech cohomology group of the standard suspension of $(\underline{\mathsf{X}}, \sigma)$ (cf. [27, IV.15. Theorem]). It is also isomorphic to the homotopy classes of continuous maps from the standard suspension of $(\underline{\mathsf{X}}, \sigma)$ into the circle (cf. [27, page 60]).
It is proved in [2, Theorem 1.5] that $C(\underline{\mathsf{X}}, \mathbb{Z})/(\mathrm{Id} - (\sigma^{-1})^\star)(C(\underline{\mathsf{X}}, \mathbb{Z}))$ is a flow invariant of $\underline{\mathsf{X}}$ and thus also a conjugacy invariant of $\underline{\mathsf{X}}$ and $\underline{\mathsf{X}}^+$.

2.4 Past equivalence and Matsumoto's K-theory

Let $\underline{\mathsf{X}}$ be a shift space. For every $x \in \underline{\mathsf{X}}^+$ and every $k \in \mathbb{N}$ we set

$$\mathcal{P}_k(x) = \{\mu \in \mathcal{L}(\underline{\mathsf{X}}) \mid \mu x \in \underline{\mathsf{X}}^+), |\mu| = k\},$$

and define for every $l \in \mathbb{N}$ an equivalence relation $\sim_l$ on $\underline{\mathsf{X}}^+$ by

$$x \sim_l x' \Leftrightarrow \mathcal{P}_l(x) = \mathcal{P}_l(x').$$

Likewise we let for every $x \in \underline{\mathsf{X}}^+$

$$\mathcal{P}_\infty(x) = \{y \in \underline{\mathsf{X}}^- \mid yx \in \underline{\mathsf{X}})\},$$

and define an equivalence relation $\sim_\infty$ on $\underline{\mathsf{X}}^+$ by

$$x \sim_\infty x' \Leftrightarrow \mathcal{P}_\infty(x) = \mathcal{P}_\infty(x').$$

The set

$$\mathcal{ND}_\infty(\underline{\mathsf{X}}^+) = \{x \in \underline{\mathsf{X}}^+ \mid \exists k \in \mathbb{N} : |\mathcal{P}_k(x)| > 1\}$$

then consists exactly of all words on the form $z_{[n,\infty[}$ where z is left special and $n \in \mathbb{N}_0$.

Following Matsumoto ([23]), we denote by $[x]_l$ the equivalence class of x and refer to the relation as *l-past equivalence.*

Obviously the set of equivalence classes of the l-past equivalence relation $\sim_l$ is finite. We will denote the number of such classes $m(l)$ and enumerate them $\mathcal{E}^l_s$ with $s \in \{1, \dots, m(l)\}$. For each $l \in \mathbb{N}$, we define an $m(l+1) \times m(l)$-matrix I^l by

$$(\mathsf{I}^l)_{rs} = \begin{cases} 1 & \text{if } \mathcal{E}^{l+1}_r \subseteq \mathcal{E}^l_s \\ 0 & \text{otherwise,} \end{cases}$$

and note that I^l induces a group homomorphism from $\mathbb{Z}^{m(l)}$ to $\mathbb{Z}^{m(l+1)}$. We denote by $\mathbb{Z}_{\underline{\mathsf{X}}}$ the group given by the inductive limit

$$\varinjlim(\mathbb{Z}^{m(l)}, \mathsf{I}^l).$$

For a subset $\mathcal{E}$ of $\underline{\mathsf{X}}^+$ and a finite word μ we let $\mu\mathcal{E} = \{\mu x \in \underline{\mathsf{X}}^+ \mid x \in \mathcal{E}\}$. For each $l \in \mathbb{N}$ and $a \in \mathfrak{a}$ we define an $m(l+1) \times m(l)$-matrix

$$(\mathsf{L}^l_a)_{rs} = \begin{cases} 1 & \text{if } \emptyset \neq a\mathcal{E}^{l+1}_r \subseteq \mathcal{E}^l_s \\ 0 & \text{otherwise,} \end{cases}$$

and letting $\mathsf{L}^l = \sum_{a \in \mathfrak{a}} \mathsf{L}^l_a$ we get a matrix inducing a group homeomorphism from $\mathbb{Z}^{m(l)}$ to $\mathbb{Z}^{m(l+1)}$. Since one can prove that $\mathsf{L}^{l+1} \circ \mathsf{I}^l = \mathsf{I}^{l+1} \circ \mathsf{L}^l$, a group endomorphism λ on $\mathbb{Z}_{\underline{\mathsf{X}}}$ is induced.

THEOREM 2.2 (CF. [24], [25, THEOREM]). *Let $\underline{\mathsf{X}}$ be a shift space. The group*

$$K_0(\underline{\mathsf{X}}) = \mathbb{Z}_{\underline{\mathsf{X}}}/(\mathrm{Id} - \lambda)\mathbb{Z}_{\underline{\mathsf{X}}},$$

is a conjugacy invariant of $\underline{\mathsf{X}}$ and $\underline{\mathsf{X}}^+$, and a flow invariant of $\underline{\mathsf{X}}$.

2.5 THE SPACE $\Omega_{\underline{\mathsf{X}}}$

We will now give an alternative description of $K_0(\underline{\mathsf{X}})$. $K_0(\underline{\mathsf{X}})$ is defined by taking a inductive limits of $\mathbb{Z}^{m(l)}$, where $\mathbb{Z}^{m(l)}$ could be thought of as $C(\underline{\mathsf{X}}^+/\sim_l, \mathbb{Z})$.
We will now do things in different order. First we will take the projective limit of $\underline{\mathsf{X}}^+/\sim_l$ and then look at the continuous functions from the projective limit to $\mathbb{Z}$.
Since $\sim_l$ is coarser than $\sim_{l+1}$, there is a projection π_l of $\underline{\mathsf{X}}^+/\sim_{l+1}$ on $\underline{\mathsf{X}}^+/\sim_l$.

DEFINITION 2.3 (CF. [23, PAGE 682]). *Let $\underline{\mathsf{X}}$ be a shift space. $\Omega_{\underline{\mathsf{X}}}$ is the compact topological space given by the projective limit $\varprojlim(\underline{\mathsf{X}}^+/\sim_l, \pi_l)$.*

We will identify $\Omega_{\underline{\mathsf{X}}}$ with the closed subspace

$$\{([x_n]_n)_{n\in\mathbb{N}_0} \mid \forall n \in \mathbb{N}_0 : x_{n+1} \sim_n x_n\}$$

of $\prod_{l=0}^{\infty} \underline{\mathsf{X}}^+/\sim_l$, where $\prod_{l=0}^{\infty} \underline{\mathsf{X}}^+/\sim_l$ is endowed with the product of the discrete topologies.
Notice that if we identify $C(\underline{\mathsf{X}}^+/\sim_l, \mathbb{Z})$ with $\mathbb{Z}^{m(l)}$, then $\mathbf{I}^l$ is the map induced by π_l, so $C(\Omega_{\underline{\mathsf{X}}}, \mathbb{Z})$ can be identified with $\mathbb{Z}_{\underline{\mathsf{X}}}$.
If $([x_n]_n)_{n\in\mathbb{N}_0} \in \Omega_{\underline{\mathsf{X}}}$, then

$$\{([x'_n]_n)_{n\in\mathbb{N}_0} \in \Omega_{\underline{\mathsf{X}}} \mid x'_1 \sim_1 x_1\}$$

is an clopen subset of $\Omega_{\underline{\mathsf{X}}}$, and if $a \in \mathcal{P}_1(x_1)$, then $([ax'_n]_n)_{n\in\mathbb{N}_0} \in \Omega_{\underline{\mathsf{X}}}$ for every $([x'_n]_n)_{n\in\mathbb{N}_0} \in \Omega_{\underline{\mathsf{X}}}$ with $x'_1 \sim_1 x_1$, and the map

$$([x'_n]_n)_{n\in\mathbb{N}_0} \mapsto ([ax'_n]_n)_{n\in\mathbb{N}_0}$$

is a continuous map on $\{([x'_n]_n)_{n\in\mathbb{N}_0} \in \Omega_{\underline{\mathsf{X}}} \mid x'_1 \sim_1 x_1\}$. This allows us to define a map $\lambda_{\underline{\mathsf{X}}} : C(\Omega_{\underline{\mathsf{X}}}, \mathbb{Z}) \to C(\Omega_{\underline{\mathsf{X}}}, \mathbb{Z})$ in the following way:

DEFINITION 2.4. *Let $\underline{\mathsf{X}}$ be a shift space, $h \in C(\Omega_{\underline{\mathsf{X}}}, \mathbb{Z})$ and $([x_n]_n)_{n\in\mathbb{N}_0} \in \Omega_{\underline{\mathsf{X}}}$. Then we let*

$$\lambda_{\underline{\mathsf{X}}}(h)(([x_n]_n)_{n\in\mathbb{N}_0}) = \sum_{a\in\mathcal{P}_1(x_1)} h([ax_n]_{n\in\mathbb{N}_0}).$$

Under the identification of $C(\Omega_{\underline{\mathsf{X}}}, \mathbb{Z})$ and $\mathbb{Z}_{\underline{\mathsf{X}}}$, $\lambda_{\underline{\mathsf{X}}}$ is equal to λ, thus we have the following proposition:

PROPOSITION 2.5. *Let $\underline{\mathsf{X}}$ be a shift space. Then $K_0(\underline{\mathsf{X}})$ and*

$$C(\Omega_{\underline{\mathsf{X}}}, \mathbb{Z})/(\mathrm{Id} - \lambda_{\underline{\mathsf{X}}})(C(\Omega_{\underline{\mathsf{X}}}, \mathbb{Z}))$$

are isomorphic as groups.

3 Property (*) and (**)

We will introduce the properties $(*)$ and $(**)$ and show that they are invariant under flow equivalence and thus under conjugacy. At the end of the section, we will for various examples of shift spaces determine if they have property $(*)$ and $(**)$.

Definition 3.1. *We say that a shift space* $\underline{\mathsf{X}}$ *has* property $(*)$ *if for every* $\mu \in \mathcal{L}(\underline{\mathsf{X}})$ *there exists an* $x \in \underline{\mathsf{X}}^+$ *such that* $\mathcal{P}_{|\mu|}(x) = \{\mu\}$.

Definition 3.2. *We say that a shift space* $\underline{\mathsf{X}}$ *has* property $(**)$ *if it has property* $(*)$ *and if the number of left special words of* $\underline{\mathsf{X}}$ *is finite, and no such left special word is periodic.*

Since flow equivalence is generated by conjugacy and symbolic expansion (cf. [25, Lemma 2.1] and [26]), it is, in order to prove the following proposition, enough to check that $(*)$ and $(**)$ are invariant under symbolic expansion and conjugacy. Although this is easy, it is also tedious, so we will omit it.

Proposition 3.3. *The properties* $(*)$ *and* $(**)$ *are invariant under flow equivalence.*

Example 3.4. It follows from Proposition 2.1 that if a shift space $\underline{\mathsf{X}}$ is finite, then it contains no left special element, and thus has property $(**)$.

Example 3.5. An infinite shift of finite type does not have property $(*)$.

Proof: Let $\underline{\mathsf{X}}$ be a shift of finite type. This means (cf. [21, Chapter 2]) that there is a $k \in \mathbb{N}_0$ such that

$$\underline{\mathsf{X}} = \{z \in \mathfrak{a}^{\mathbb{Z}} \mid \forall n \in \mathbb{Z} : z_{[n,n+k]} \in \mathcal{L}(\underline{\mathsf{X}})\}.$$

Suppose that $\underline{\mathsf{X}}$ has property $(*)$. Let $\mathcal{L}(\underline{\mathsf{X}})_k = \{\mu \in \mathcal{L}(\underline{\mathsf{X}}) \mid |\mu| = k\}$, and notice that if $\mu, \nu, \omega \in \mathcal{L}(\underline{\mathsf{X}})_k$ and $\mu\nu, \nu\omega \in \mathcal{L}(\underline{\mathsf{X}})$, then $\mu\nu\omega \in \mathcal{L}(\underline{\mathsf{X}})$.
Let $\mu \in \mathcal{L}(\underline{\mathsf{X}})_k$. Then there is a $x \in \underline{\mathsf{X}}^+$ such that $\mathcal{P}_{|\mu|}(x) = \{\mu\}$. Let $\mu' = x_{[0,k[}$. Suppose that $\nu \in \mathcal{L}(\underline{\mathsf{X}})_k$ and $\nu\mu' \in \mathcal{L}(\underline{\mathsf{X}})$. Then $\nu x \in \underline{\mathsf{X}}^+$, so ν must be equal to μ. Thus there is for every $\mu \in \mathcal{L}(\underline{\mathsf{X}})_k$ a $\mu' \in \mathcal{L}(\underline{\mathsf{X}})_k$ such that

$$\nu \in \mathcal{L}(\underline{\mathsf{X}})_k \wedge \nu\mu' \in \mathcal{L}(\underline{\mathsf{X}}) \Rightarrow \nu = \mu.$$

Since $\mathcal{L}(\underline{\mathsf{X}})_k$ is finite and the map $\mu \mapsto \mu'$ is injective, there is for every $\nu \in \mathcal{L}(\underline{\mathsf{X}})_k$ a $\mu \in \mathcal{L}(\underline{\mathsf{X}})_k$ such that $\nu = \mu'$. Hence there is for every $\mu \in \mathcal{L}(\underline{\mathsf{X}})_k$ a unique $\mu' \in \mathcal{L}(\underline{\mathsf{X}})_k$ such that $\mu\mu' \in \mathcal{L}(\underline{\mathsf{X}})$ and a unique $\mu'' \in \mathcal{L}(\underline{\mathsf{X}})_k$ such that $\mu''\mu \in \mathcal{L}(\underline{\mathsf{X}})$. Thus every $z \in \underline{\mathsf{X}}$ is determined by $z_{[0,k[}$, but since $\mathcal{L}(\underline{\mathsf{X}})_k$ is finite, this implies that $\underline{\mathsf{X}}$ is finite. □

Example 3.6. An infinite minimal shift space $\underline{\mathsf{X}}$ has property $(**)$ precisely when the number of left special words of $\underline{\mathsf{X}}$ is finite.

Proof: Since no elements in such a shift space is periodic, we only need to prove that property (∗) follows from finiteness of the number of left special elements. Let $\mu \in \mathcal{L}(\underline{\mathsf{X}})$ and pick any $x \in \underline{\mathsf{X}}^+$. Since $\underline{\mathsf{X}}^+$ is infinite and minimal, x is not periodic, and since the set of left special words is finite there exists $N \in \mathbb{N}$ such that $\sigma^n(x)$ is not left special for any $n \geq N$. Since $\underline{\mathsf{X}}^+$ is minimal there exists a $k \geq N$ such that $x_{[k+1,k+|\mu|]} = \mu$. Hence $\mathcal{P}_{|\mu|}(\sigma^{k+|\mu|+1}(x)) = \{\mu\}$. □

EXAMPLE 3.7. If z is a non-regular Toeplitz sequence, then the shift space

$$\overline{\mathcal{O}(z)} = \overline{\{\sigma^n(z) \mid n \in \mathbb{Z}\}}$$

has property (∗).

Proof: Let $\mu \in \mathcal{L}(\overline{\mathcal{O}(z)})$. Since $\overline{\mathcal{O}(z)}$ is minimal (cf. [32, page 97]), there is an $m \in \mathbb{N}$ such that $z_{[-m-|\mu|,-m[} = \mu$. We claim that $\mathcal{P}_{|\mu|}(z_{[-m,\infty[}) = \{\mu\}$. Assume that $z' \in \overline{\mathcal{O}(z)}$ and $z'_{[-m,\infty[} = z_{[-m,\infty[}$. Then $\pi(z') = \pi(z)$, where π is the factor map of $\overline{\mathcal{O}(z)}$ onto its maximal equicontinuous factor $(G, \hat{1})$ (cf. [32, Theorem 2.2]), because since $z'_{[-m,\infty[} = z_{[-m,\infty[}$, the distance between $\sigma^n(z')$ and $\sigma^n(z)$, and thus the distance between $\hat{1}^n(\pi(z'))$ and $\hat{1}^n(\pi(z))$, goes to 0 as n goes to infinity, but since $\hat{1}$ is equicontinuous, this implies that $\pi(z') = \pi(z)$. Since π is one-to-one on the set of Toeplitz sequences (cf. [32, Corollary 2.4]), $z' = z$. Thus $\mathcal{P}_{|\mu|}(z_{[-m,\infty[}) = \{\mu\}$. □

The following example shows that property (∗∗) does not follow from property (∗).

EXAMPLE 3.8. We will construct a non-regular Toeplitz sequence $z \in \{0,1\}^{\mathbb{Z}}$ such that the shift space

$$\overline{\mathcal{O}(z)} = \overline{\{\sigma^n(z) \mid n \in \mathbb{Z}\}}$$

has infinitely many left special elements and thus does not have property (∗∗). We will construct z by using the technique introduced by Susan Williams in [32, Section 4]. We will use the same notation as in [32, Section 4]. We let $Y = \{0,1\}^{\mathbb{Z}}$ and defined $(p_i)_{i\in\mathbb{N}}$ recursively by setting $p_1 = 3$ and $p_{i+1} = 3^{r_i+i}p_i$ for $i \in \mathbb{N}$, where r_i is as defined in [32, Section 4]. We then have that

$$\frac{p_i\beta_{r_i}}{p_{i+1}} = \frac{2^{r_i}}{3^{r_i+i}} < 3^{-i}$$

so

$$\sum_{i=1}^{\infty} \frac{p_i\beta_{r_i}}{p_{i+1}}$$

converges, so z is non-regular by [32, Proposition 4.1].

CLAIM. *The shift space $\overline{\mathcal{O}(z)}$ has infinitely many left special elements.*

Proof: Let D be as defined on [32, page 103]. If

$$g \in \pi(\{z' \in D \mid -1 \in \mathrm{Aper}(z')\}),$$

$y, y' \in Y$, $y_{[0,\infty[} = y'_{[0,\infty[}$ and $y_{-1} \neq y'_{-1}$, then $\phi(g,y)_{[0,\infty[} = \phi(g,y')_{[0,\infty[}$ and $\phi(g,y)_{-1} \neq \phi(g,y')_{-1}$, where ϕ is the map define on [32, page 103]. Thus $\phi(g,y)$ and $\phi(g,y')$ are left special elements, and since

$$\pi(\{z' \in D \mid -1 \in \mathrm{Aper}(z')\}) \times \{y \in Y \mid y \text{ is left special}\}$$

is infinite and contained in $\pi(D) \times Y$, where ϕ is $1-1$ on, $\overline{\mathcal{O}(z)}$ has infinitely many left special elements. □

4 The first cohomology group is a factor of $K_0(\underline{\mathsf{X}})$

We will now show that if a shift space $\underline{\mathsf{X}}$ has property $(*)$, then the first cohomology group is a factor group of $K_0(\underline{\mathsf{X}})$.

Suppose that a shift space $\underline{\mathsf{X}}$ has property $(*)$. We can then define a map $\iota_{\underline{\mathsf{X}}}$ from $\underline{\mathsf{X}}^-$ into $\Omega_{\underline{\mathsf{X}}}$ in the following way: For each $y \in \underline{\mathsf{X}}^-$ and each $n \in \mathbb{N}_0$ we choose an $x_n \in \underline{\mathsf{X}}^+$ such that $\mathcal{P}_n(x_n) = \{y_{[-n,-1]}\}$. Then $([x_n]_n)_{n\in\mathbb{N}_0} \in \Omega_{\underline{\mathsf{X}}}$, and we denote this element by $\iota_{\underline{\mathsf{X}}}(y)$. $\iota_{\underline{\mathsf{X}}}$ is obviously injective and continuous. We denote the map

$$(\iota_{\underline{\mathsf{X}}} \circ \pi_-)^\star : C(\Omega_{\underline{\mathsf{X}}}, \mathbb{Z}) \to C(\underline{\mathsf{X}}, \mathbb{Z})$$

by κ.

PROPOSITION 4.1. *Let $\underline{\mathsf{X}}$ be a shift space which has property $(*)$. Then there is a surjective group homomorphism $\bar{\kappa}$ from $C(\Omega_{\underline{\mathsf{X}}}, \mathbb{Z})/(\mathrm{Id} - \lambda_{\underline{\mathsf{X}}})(C(\Omega_{\underline{\mathsf{X}}}, \mathbb{Z}))$ to $C(\underline{\mathsf{X}}, \mathbb{Z})/(\mathrm{Id} - (\sigma^{-1})^\star)(C(\underline{\mathsf{X}}, \mathbb{Z}))$ which makes the following diagram commute:*

$$\begin{array}{ccc} C(\Omega_{\underline{\mathsf{X}}}, \mathbb{Z}) & \xrightarrow{\kappa} & C(\underline{\mathsf{X}}, \mathbb{Z}) \\ \downarrow & & \downarrow \\ C(\Omega_{\underline{\mathsf{X}}}, \mathbb{Z})/(\mathrm{Id} - \lambda_{\underline{\mathsf{X}}})(C(\Omega_{\underline{\mathsf{X}}}, \mathbb{Z})) & \xrightarrow{\bar{\kappa}} & C(\underline{\mathsf{X}}, \mathbb{Z})/(\mathrm{Id} - (\sigma^{-1})^\star)(C(\underline{\mathsf{X}}, \mathbb{Z})) \end{array}$$

Proof: Let q be the quotient map from $C(\underline{\mathsf{X}}, \mathbb{Z})$ to

$$C(\underline{\mathsf{X}}, \mathbb{Z})/(\mathrm{Id} - (\sigma^{-1})^\star)(C(\underline{\mathsf{X}}, \mathbb{Z})).$$

We will show that $q \circ \kappa$ is surjective and that $(\mathrm{Id} - \lambda_{\underline{\mathsf{X}}})(C(\Omega_{\underline{\mathsf{X}}}, \mathbb{Z})) \subseteq \ker(q \circ \kappa)$. This will prove the existence of $\bar{\kappa}$.

$q \circ \kappa$ is surjective: Given $f \in C(\underline{\mathsf{X}}, \mathbb{Z})$. Our goal is to find a function $g \in C(\Omega_{\underline{\mathsf{X}}}, \mathbb{Z})$ which is mapped to $q(f)$ by $q \circ \kappa$.

Since f is continuous, there are $k,m\in\mathbb{N}$ such that $z_{[-k,m]}=z'_{[-k,m]}\Rightarrow f(z)=f(z')$. Thus

$$z_{[-k-m-1,-1]}=z'_{[-k-m-1,-1]}\Rightarrow f\circ\sigma^{-(m+1)}(z)=f\circ\sigma^{-(m+1)}(z').$$

Define a function g from $\Omega_{\underline{\mathsf{X}}}$ to $\mathbb{Z}$ by

$$g(([x_n]_n)_{n\in\mathbb{N}_0})=\begin{cases} f\circ\sigma^{-(m+1)}(z) & \text{if } \mathcal{P}_{k+m+1}(x_{k+m+1})=\{z_{[-k-m-1,-1]}\},\\ 0 & \text{if } \#\mathcal{P}_{k+m+1}(x_{k+m+1})>1.\end{cases}$$

Then $g\in C(\Omega_{\underline{\mathsf{X}}},\mathbb{Z})$, and $g\circ\iota_{\underline{\mathsf{X}}}\circ\pi_-=f\circ\sigma^{-(m+1)}$. Thus $q\circ\kappa(g)=q(f)$.

$(\mathrm{Id}-\lambda_{\underline{\mathsf{X}}})(C(\Omega_{\underline{\mathsf{X}}},\mathbb{Z}))\subseteq\ker(q\circ\kappa)$*:* Let $g\in C(\Omega_{\underline{\mathsf{X}}},\mathbb{Z})$ and $y\in\underline{\mathsf{X}}^-$. Then $\lambda_{\underline{\mathsf{X}}}(g)(\iota_{\underline{\mathsf{X}}}(y))=g(\iota_{\underline{\mathsf{X}}}(\sigma_-(y)))$, so

$$\kappa(\lambda_{\underline{\mathsf{X}}}(g))=g\circ\iota_{\underline{\mathsf{X}}}\circ\pi_-\circ\sigma^{-1},$$

which shows that $(\mathrm{Id}-\lambda_{\underline{\mathsf{X}}})(g)\in\ker(q\circ\kappa)$. □

The following corollary now follows from Theorem 2.5:

Corollary 4.2. *Let $\underline{\mathsf{X}}$ be a shift space which has property $(*)$. Then $C(\underline{\mathsf{X}},\mathbb{Z})/(\mathrm{Id}-(\sigma^{-1})^\star)(C(\underline{\mathsf{X}},\mathbb{Z}))$ is a factor group of $K_0(\underline{\mathsf{X}})$.*

5 K_0 of shift spaces having property $(**)$

We saw in the last section that if a shift space $\underline{\mathsf{X}}$ has property $(*)$, then the first cohomology group is a factor group of $K_0(\underline{\mathsf{X}})$. This stems from the fact that property $(*)$ causes an inclusion of $\underline{\mathsf{X}}^-$ into $\Omega_{\underline{\mathsf{X}}}$, and thus a surjection of $C(\Omega_{\underline{\mathsf{X}}},\mathbb{Z})$ onto $C(\underline{\mathsf{X}}^-,\mathbb{Z})$. We will now for shift spaces having property $(**)$ describe K_0 in terms of the first cohomology group and some extra information determined by the left special elements of the shift space.

We will first define the group $\mathcal{G}_{\underline{\mathsf{X}}}$ which is a subgroup of the external direct product of $C(\underline{\mathsf{X}}^-,\mathbb{Z})$ and an infinite product of copies of $\mathbb{Z}$, and isomorphic to $C(\Omega_{\underline{\mathsf{X}}},\mathbb{Z})$. Next, we will define the group $G_{\underline{\mathsf{X}}}$ which is a the external direct product of $C(\underline{\mathsf{X}},\mathbb{Z})$ and an infinite sum of copies of $\mathbb{Z}$, and has a factor group which is isomorphic to $K_0(\underline{\mathsf{X}})$. We will round off by relating this with the fact that the first cohomology group is a factor group of $K_0(\underline{\mathsf{X}})$ and look at some examples.

Lemma 5.1. *Let $\underline{\mathsf{X}}$ be a shift space which has property $(*)$. Then*

$$\iota_{\underline{\mathsf{X}}}(\underline{\mathsf{X}}^-)=\{([x_n]_n)_{n\in\mathbb{N}_0}\in\Omega_{\underline{\mathsf{X}}}\mid\forall n\in\mathbb{N}_0:\#\mathcal{P}_n(x_n)=1\}.$$

Proof: Clearly

$$\iota_{\underline{\mathsf{X}}}(\underline{\mathsf{X}}^-)\subseteq\{([x_n]_n)_{n\in\mathbb{N}_0}\in\Omega_{\underline{\mathsf{X}}}\mid\forall n\in\mathbb{N}_0:\#\mathcal{P}_n(x_n)=1\}.$$

Suppose $([x_n]_n)_{n\in\mathbb{N}_0} \in \Omega_{\underline{X}}$ and $\mathcal{P}_n(x_n) = \{\mu_n\}$ for every $n \in \mathbb{N}_0$. Let for every $n \in \mathbb{N}$, y_{-n} be the first letter of μ_n. Since $y_{[-n,-1]} = \mu_n$ for every $n \in \mathbb{N}$, $y \in \underline{X}^-$, and clearly $\iota_{\underline{X}}(y) = ([x_n]_n)_{n\in\mathbb{N}_0}$. $\square$

Denote by $\mathcal{I}_{\underline{X}}$ the set $\mathcal{ND}_\infty(\underline{X}^+)/\sim_\infty$. We will now define a map $\phi_{\underline{X}}$ from $\mathcal{I}_{\underline{X}}$ to $\Omega_{\underline{X}}$. We see that for $x \in \mathcal{ND}_\infty(\underline{X}^+)$, $([x]_n)_{n\in\mathbb{N}_0} \in \Omega_{\underline{X}}$, and we notice that $x \sim_\infty \tilde{x}$, if and only if $([x]_n)_{n\in\mathbb{N}_0} = ([\tilde{x}]_n)_{n\in\mathbb{N}_0}$. So if we let $\phi_{\underline{X}}([x]_\infty) = ([x]_n)_{n\in\mathbb{N}_0}$, then $\phi_{\underline{X}}$ is well-defined and injective.

LEMMA 5.2. *Let $\underline{X}$ be a shift space which has property $(*)$. Then $\iota_{\underline{X}}(\underline{X}^-) \cap \phi_{\underline{X}}(\mathcal{I}_{\underline{X}}) = \emptyset$, and if $\underline{X}$ has property $(**)$, then $\iota_{\underline{X}}(\underline{X}^-) \cup \phi_{\underline{X}}(\mathcal{I}_{\underline{X}}) = \Omega_{\underline{X}}$.*

Proof: If $([x_n]_n)_{n\in\mathbb{N}_0} \in \iota_{\underline{X}}(\underline{X}^-)$, then according to Lemma 5.1, $\#\mathcal{P}_n(x_n) = 1$ for every $n \in \mathbb{N}_0$, and if $([x_n]_n)_{n\in\mathbb{N}_0} \in \phi_{\underline{X}}(\mathcal{I}_{\underline{X}})$, then $\#\mathcal{P}_n(x_n) > 1$ for some $n \in \mathbb{N}_0$. Hence $\iota_{\underline{X}}(\underline{X}^-) \cap \phi_{\underline{X}}(\mathcal{I}_{\underline{X}}) = \emptyset$.
Suppose that $\underline{X}$ has property $(**)$. If $([x_n]_n)_{n\in\mathbb{N}_0} \in \Omega_{\underline{X}} \setminus \iota_{\underline{X}}(\underline{X}^-)$, then according to Lemma 5.1, there is an $n \in \mathbb{N}_0$ such that $\#\mathcal{P}_n(x_n) > 1$, and since there only are finitely many left special words, $[x_n]_n$ is finite. Since $[x_k]_k \neq \emptyset$ and $[x_{k+1}]_{k+1} \subseteq [x_k]_k$ for every $k \in \mathbb{N}_0$, this implies that $\bigcap_{k\in\mathbb{N}_0}[x_k]_k$ is not empty. Let $x \in \bigcap_{k\in\mathbb{N}_0}[x_k]_k$. Since $\#\mathcal{P}_n(x) = \#\mathcal{P}_n(x_n) > 1$, $x \in \mathcal{ND}_\infty(\underline{X}^+)$, and since $([x_n]_n)_{n\in\mathbb{N}_0} = \phi_{\underline{X}}([x]_\infty)$, $([x_n]_n)_{n\in\mathbb{N}_0} \in \phi_{\underline{X}}(\mathcal{I}_{\underline{X}})$. $\square$

5.1 THE GROUP $\mathcal{G}_{\underline{X}}$

We will from now on assume that $\underline{X}$ has property $(**)$. Let for every function $h : \Omega_{\underline{X}} \to \mathbb{Z}$,

$$\gamma_{\underline{X}}(h) = (h \circ \iota_{\underline{X}}, (h(\phi_{\underline{X}}(\mathsf{i})))_{\mathsf{i}\in\mathcal{I}_{\underline{X}}}).$$

It follows from Lemma 5.2 that $\gamma_{\underline{X}}$ is a bijective correspondence between functions from $\Omega_{\underline{X}}$ to $\mathbb{Z}$ and pairs $(g, (\alpha_{\mathsf{i}})_{\mathsf{i}\in\mathcal{I}_{\underline{X}}})$, where g is a function from $\underline{X}^-$ to $\mathbb{Z}$ and each α_{i} is an integer.

LEMMA 5.3. *$(g, (\alpha_{\mathsf{i}})_{\mathsf{i}\in\mathcal{I}_{\underline{X}}}) \in \gamma_{\underline{X}}(C(\Omega_{\underline{X}}, \mathbb{Z}))$ if and only if there is an $n_0 \in \mathbb{N}_0$ such that*

1. $\forall y, y' \in \underline{X}^- : y_{[-n_0,-1]} = y'_{[-n_0,-1]} \Rightarrow g(y) = g(y')$,
2. $\forall x, x' \in \mathcal{ND}_\infty(\underline{X}^+) : [x]_{n_0} = [x']_{n_0} \Rightarrow \alpha_{[x]_\infty} = \alpha_{[x']_\infty}$,
3. $\forall x \in \mathcal{ND}_\infty(\underline{X}^+), y \in \underline{X}^- : \mathcal{P}_{n_0}(x) = \{y_{[-n,-1]}\} \Rightarrow \alpha_{[x]_\infty} = g(y)$.

Proof: A function from $\Omega_{\underline{X}}$ to $\mathbb{Z}$ is continuous if and only if there is an $n_0 \in \mathbb{N}_0$ such that

$$[x_{n_0}]_{n_0} = [x'_{n_0}]_{n_0} \Rightarrow h(([x_n]_n)_{n\in\mathbb{N}_0}) = h(([x'_n]_n)_{n\in\mathbb{N}_0})$$

for $([x_n]_n)_{n\in\mathbb{N}_0}, ([x'_n]_n)_{n\in\mathbb{N}_0} \in \Omega_{\underline{X}}$, and since

$$[x_{n_0}]_{n_0} = [x'_{n_0}]_{n_0} \Leftrightarrow y_{[-n_0,-1]} = y'_{[-n_0,-1]}$$

for $y, y' \in \underline{\mathsf{X}}^-$ if $([x_n]_n)_{n\in\mathbb{N}_0} = \iota_{\underline{\mathsf{X}}}(y)$ and $([x'_n]_n)_{n\in\mathbb{N}_0} = \iota_{\underline{\mathsf{X}}}(y')$, and

$$[x]_{n_0} = [x'_{n_0}]_{n_0} \Leftrightarrow \mathcal{P}_{n_0}(x) = \{y_{[-n_0,-1]}\}$$

for $x \in \mathcal{ND}_\infty(\underline{\mathsf{X}}^+)$ and $y \in \underline{\mathsf{X}}^-$ if $([x_n]_n)_{n\in\mathbb{N}_0} = \iota_{\underline{\mathsf{X}}}(y)$, the conclusion follows. □

DEFINITION 5.4. *Let $\underline{\mathsf{X}}$ be a shift space which has property $(**)$. We denote $\gamma_{\underline{\mathsf{X}}}(C(\Omega_{\underline{\mathsf{X}}},\mathbb{Z}))$ by $\mathcal{G}_{\underline{\mathsf{X}}}$, and we let for every function $g : \underline{\mathsf{X}}^- \to \mathbb{Z}$ and $(\alpha_{\mathsf{i}})_{\mathsf{i}\in\mathcal{I}_{\underline{\mathsf{X}}}} \in \mathbb{Z}^{\mathcal{I}}_{\underline{\mathsf{X}}}$,*

$$\mathcal{A}_{\underline{\mathsf{X}}}((g,(\alpha_{\mathsf{i}})_{\mathsf{i}\in\mathcal{I}_{\underline{\mathsf{X}}}})) = (g\circ\sigma_-,(\tilde{\alpha}_{\mathsf{i}})_{\mathsf{i}\in\mathcal{I}_{\underline{\mathsf{X}}}}),$$

where

$$\tilde{\alpha}_{[x]_\infty} = \sum_{\substack{x'\in\mathcal{ND}_\infty(\underline{\mathsf{X}}^+)\\ \sigma_+(x')=x}} \alpha_{[x']_\infty} + \sum_{\substack{z\in\underline{\mathsf{X}}\\ z_{[0,\infty[}\notin\mathcal{ND}_\infty(\underline{\mathsf{X}}^+)\\ z_{[1,\infty[}=x}} g(\pi_-(z)).$$

LEMMA 5.5. *$\mathcal{A}_{\underline{\mathsf{X}}}$ maps $\mathcal{G}_{\underline{\mathsf{X}}}$ into $\mathcal{G}_{\underline{\mathsf{X}}}$, and the following diagram commutes:*

$$\begin{array}{ccc} C(\Omega_{\underline{\mathsf{X}}},\mathbb{Z}) & \xrightarrow{\gamma_{\underline{\mathsf{X}}}} & \mathcal{G}_{\underline{\mathsf{X}}} \\ \downarrow{\scriptstyle\lambda_{\underline{\mathsf{X}}}} & & \downarrow{\scriptstyle\mathcal{A}_{\underline{\mathsf{X}}}} \\ C(\Omega_{\underline{\mathsf{X}}},\mathbb{Z}) & \xrightarrow{\gamma_{\underline{\mathsf{X}}}} & \mathcal{G}_{\underline{\mathsf{X}}}. \end{array}$$

Proof: Let $h \in C(\Omega_{\underline{\mathsf{X}}},\mathbb{Z})$ and $([x_n]_n)_{n\in\mathbb{N}_0} \in \Omega_{\underline{\mathsf{X}}}$. Then

$$\lambda_{\underline{\mathsf{X}}}(h)(([x_n]_n)_{n\in\mathbb{N}_0}) = \sum_{a\in\mathcal{P}_1(x_1)} h([ax_n]_{n\in\mathbb{N}_0}).$$

If $([x_n]_n)_{n\in\mathbb{N}_0} \in \iota_{\underline{\mathsf{X}}}(\underline{\mathsf{X}}^-)$, then $\#\mathcal{P}_1(x_1) = 1$ and $\iota_{\underline{\mathsf{X}}}(\sigma_-(\iota_{\underline{\mathsf{X}}}^{-1}(([x_n]_n)_{n\in\mathbb{N}_0}))) = [ax_n]_{n\in\mathbb{N}_0}$, where $a \in \mathcal{P}_1(x_1)$. Thus

$$\lambda_{\underline{\mathsf{X}}}(h)(([x_n]_n)_{n\in\mathbb{N}_0}) = h(([ax_n]_n)_{n\in\mathbb{N}_0}) = \gamma_{\underline{\mathsf{X}}}^{-1}\circ\mathcal{A}_{\underline{\mathsf{X}}}\circ\gamma_{\underline{\mathsf{X}}}(h)(([x_n]_n)_{n\in\mathbb{N}_0}).$$

Now assume that $([x_n]_n)_{n\in\mathbb{N}_0} \in \phi_{\underline{\mathsf{X}}}(\mathcal{I}_{\underline{\mathsf{X}}})$ and choose $x \in \mathcal{ND}_\infty(\underline{\mathsf{X}}^+)$ such that $\phi_{\underline{\mathsf{X}}}([x]_\infty) = ([x_n]_n)_{n\in\mathbb{N}_0}$. We claim that

$$\sum_{a\in\mathcal{P}_1(x_1)} h([ax_n]_{n\in\mathbb{N}_0}) = \sum_{\substack{x'\in\mathcal{ND}_\infty(\underline{\mathsf{X}}^+)\\ \sigma_+(x')=x}} h(\phi_{\underline{\mathsf{X}}}([x']_\infty)) + \sum_{\substack{z\in\underline{\mathsf{X}}\\ z_{[0,\infty[}\notin\mathcal{ND}_\infty(\underline{\mathsf{X}}^+)\\ z_{[1,\infty[}=x}} h(\iota_{\underline{\mathsf{X}}}(z_{[-\infty,-1]})). \quad (1)$$

To see this, let $a \in \mathcal{P}_1(x_1)$.
Assume first that $([ax_n]_n)_{n\in\mathbb{N}_0} \in \iota_{\underline{\mathsf{X}}}(\underline{\mathsf{X}}^-)$, and let z be the element of $\mathfrak{a}^{\mathbb{Z}}$ satisfying $z_{]-\infty,0[} = \iota_{\underline{\mathsf{X}}}^{-1}(([ax_n]_n)_{n\in\mathbb{N}_0})$, $z_0 = a$, and $z_{[1,\infty[} = x$. Then $z \in \underline{\mathsf{X}}$,

$z_{[0,\infty[} \notin \mathcal{ND}_\infty(\underline{\mathsf{X}}^+)$, $z_{[1,\infty[} = x$, and $\iota_{\underline{\mathsf{X}}}(z_{]-\infty,-1]}) = [ax_n]_{n\in\mathbb{N}_0}$. If on the other hand $\tilde{z}$ is an element of $\underline{\mathsf{X}}$ which satisfies $\tilde{z}_{[0,\infty[} \notin \mathcal{ND}_\infty(\underline{\mathsf{X}}^+)$, and $\tilde{z}_{[1,\infty[} = x$, then $\tilde{z}_0 \in \mathcal{P}_1(x_1)$, and $\iota_{\underline{\mathsf{X}}}(\tilde{z}_{]-\infty,-1]}) = ([\tilde{z}_0 x_n]_n)_{n\in\mathbb{N}_0}$
Then assume that $([ax_n]_n)_{n\in\mathbb{N}_0} \in \phi_{\underline{\mathsf{X}}}(\mathcal{I}_{\underline{\mathsf{X}}})$. Then $ax \in \mathcal{ND}_\infty(\underline{\mathsf{X}}^+)$, $\sigma_+(ax) = x$, and $\phi_{\underline{\mathsf{X}}}([ax]_\infty) = [ax_n]_{n\in\mathbb{N}_0}$. On the other hand, if $x' \in \mathcal{ND}_\infty(\underline{\mathsf{X}}^+)$ and $\sigma_+(x') = x$, then $x'_0 \in \mathcal{P}_1(x_1)$, and $\phi_{\underline{\mathsf{X}}}([x']_\infty) = [x'_0 x_n]_{n\in\mathbb{N}_0}$.
Thus (1) holds, and

$$\begin{aligned}\lambda_{\underline{\mathsf{X}}}(h)(([x_n]_n)_{n\in\mathbb{N}_0}) &= \sum_{a\in\mathcal{P}_1(x_1)} h([ax_n]_{n\in\mathbb{N}_0}) \\ &= \sum_{\substack{x'\in\mathcal{ND}_\infty(\underline{\mathsf{X}}^+)\\ \sigma_+(x')=x}} h(\phi_{\underline{\mathsf{X}}}([x']_\infty)) + \sum_{\substack{z\in\underline{\mathsf{X}}\\ z_{[0,\infty[}\notin\mathcal{ND}_\infty(\underline{\mathsf{X}}^+)\\ z_{[1,\infty[}=x}} h(\iota_{\underline{\mathsf{X}}}(z_{[-\infty,-1]})) \\ &= \gamma_{\underline{\mathsf{X}}}^{-1}\circ\mathcal{A}_{\underline{\mathsf{X}}}\circ\gamma_{\underline{\mathsf{X}}}(h)(([x_n]_n)_{n\in\mathbb{N}_0}).\end{aligned}$$

This shows that $\mathcal{A}_{\underline{\mathsf{X}}} = \gamma_{\underline{\mathsf{X}}}\circ\lambda_{\underline{\mathsf{X}}}\circ\gamma_{\underline{\mathsf{X}}}^{-1}$, so $\mathcal{A}_{\underline{\mathsf{X}}}$ maps $\mathcal{G}_{\underline{\mathsf{X}}}$ into $\mathcal{G}_{\underline{\mathsf{X}}}$, and the diagram commutes. ☐

COROLLARY 5.6. *Let $\underline{\mathsf{X}}$ be a shift space which has property* $(**)$. *Then $K_0(\underline{\mathsf{X}})$ and*

$$\mathcal{G}_{\underline{\mathsf{X}}}/(\mathrm{Id}-\mathcal{A}_{\underline{\mathsf{X}}})\mathcal{G}_{\underline{\mathsf{X}}}$$

are isomorphic as groups.

5.2 THE SPACE $\mathcal{I}_{\underline{\mathsf{X}}}$

In order to get a better understanding of the group $\mathcal{G}_{\underline{\mathsf{X}}}$ and the map $\mathcal{A}_{\underline{\mathsf{X}}}$, we will now try to describe $\mathcal{I}_{\underline{\mathsf{X}}}$ in the case where $\underline{\mathsf{X}}$ has properties $(**)$. For that we will need the concept of right shift tail equivalence.
Denote the set of those right shift tail equivalence classes of $\underline{\mathsf{X}}$ which contains a left special element by $\mathcal{J}_{\underline{\mathsf{X}}}$. Notice that it is finite. Let for every $\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}$, $M_{\mathbf{j}}$ be the set of adjusted left special elements belonging to $\mathbf{j}$. Notice that there only is a finite - but positive - number of elements in $M_{\mathbf{j}}$.
Let us take a closer look at $\pi_+(\mathbf{j})$. It is clear that

$$\pi_+(\mathbf{j}) = \{z_{[n,\infty[} \mid z \in M_{\mathbf{j}}, n \in \mathbb{Z}\},$$

and it follows from the definition of adjusted left special elements that $z_{[n,\infty[} \in \mathcal{ND}_\infty(\underline{\mathsf{X}}^+)$ if and only if $n \geq 0$. It is easy to see that if $z, z' \in M_{\mathbf{j}}$ and $n, n' < 0$, then

$$z_{[n,\infty[} = z'_{[n',\infty[} \Leftrightarrow z = z' \wedge n = n'.$$

Contrary to this, it might happen that $z_{[n,\infty[} = z'_{[n',\infty[}$ for $z \neq z'$ if $n, n' \geq 0$. In fact, it turns out that $\mathbf{j}$ has a "common tail".

DEFINITION 5.7. *Let $\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}$. An $x \in \underline{\mathsf{X}}^+$ such that there for every $z \in \mathbf{j}$ is an $n \in \mathbb{Z}$ such that $z_{[n,\infty[} = x$ is called a common tail of* $\mathbf{j}$.

LEMMA 5.8. *Let z be a left special element. Then $z_{[0,\infty[}$ is a common tail of* $\mathbf{z}$ *if and only if z is cofinal.*

Proof: Assume that z is cofinal and let $z' \in \mathbf{z}$. Then there are $n, n' \in \mathbb{Z}$ such that $z_{[n,\infty[} = z'_{[n',\infty[}$, and since z is cofinal, $z_{[0,n[} = z'_{[n'-n,n'[}$ if $n > 0$. If $n \leq 0$, then obviously $z_{[0,\infty[} = z'_{[n'-n,\infty[}$. Thus $z_{[0,\infty[}$ is a common tail of $\mathbf{z}$.
Assume now that z is not cofinal. Then there is a $z' \in \mathbf{z}$ and an $n \in \mathbb{N}$ such that $z_{[n,\infty[} = z'_{[n,\infty[}$, but $z_{n-1} \neq z'_{n-1}$. If $z_{[0,\infty[}$ is a common tail of $\mathbf{z}$, then there is a $n' \in \mathbb{Z}$ such that $z'_{[n',\infty[} = z_{[0,\infty[}$, and since $z_{n-1} \neq z'_{n-1}$, $n' \neq 0$. But we then have for $k \geq n$ that $z'_{k+n'} = z_k = z'_k$, which cannot be true, since there are no periodic left special words. □

DEFINITION 5.9. *An $x \in \underline{\mathsf{X}}^+$ is called isolated if there is a $k \in \mathbb{N}_0$ such that $[x]_k = \{x\}$.*

LEMMA 5.10. *Every $\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}$ has an isolated common tail.*

Proof: Let z be the cofinal left special element of $\mathbf{j}$. Then $z_{[0,\infty[}$, and thus $z_{[n,\infty[}$ for every $n \in \mathbb{N}_0$, is a common tail by Lemma 5.8. Since there only are finitely many left special words, $[z_{[0,\infty[}]_1$ is finite. Hence there is an $n \in \mathbb{N}$ such that

$$x \in [z_{[0,\infty[}]_1 \wedge x_{[0,n]} = z_{[0,n]} \Rightarrow x = z_{[0,\infty[}.$$

Thus $[z_{[n,\infty[}]_{n+1} = \{z_{[n,\infty[}\}$ and therefore $z_{[n,\infty[}$ is an isolated common tail. □

REMARK 5.11. In [22] Matsumoto introduced the condition (I) for shift spaces, which is a generalization of the condition (I) for topological Markov shifts in the sense of Cuntz and Krieger (cf. [10]).
A shift space $\underline{\mathsf{X}}$ satisfies condition (I) if and only if $\underline{\mathsf{X}}^+$ has no isolated elements (cf. [22, Lemma 5.1]). Thus, it follows from Lemma 5.10 that a shift space which has property (∗∗) does not satisfy condition (I).

Choose once and for all, for each $\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}$ an isolated common tail $x^{\mathbf{j}}$ and a $z^{\mathbf{j}}$ such that $z^{\mathbf{j}}_{[0,\infty[} = x^{\mathbf{j}}$.

REMARK 5.12. Notice that $z^{\mathbf{j}}_{[n,\infty[}$ is isolated for every $\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}$ and every $n \in \mathbb{N}_0$, because if $[z^{\mathbf{j}}_{[0,\infty[}]_k = \{z^{\mathbf{j}}_{[0,\infty[}\}$, then $[z^{\mathbf{j}}_{[n,\infty[}]_{k+n} = \{z^{\mathbf{j}}_{[n,\infty[}\}$.

Let $\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}$. Since $x^{\mathbf{j}}$ is a common tail of $\mathbf{j}$, there is for every $z \in M_{\mathbf{j}}$ an $n_z \in \mathbb{N}_0$ such that $z_{[n_z,\infty[} = x^{\mathbf{j}}$. Let

$$K_{\mathbf{j}} = \{[z_{[n,\infty[}]_\infty \mid z \in M_{\mathbf{j}}, 0 \leq n \leq n_z\}.$$

LEMMA 5.13.

$$\bigcup_{\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}} \left(K_{\mathbf{j}} \cup \{[z^{\mathbf{j}}_{[n,\infty[}]_\infty \mid n \in \mathbb{N}_0\} \right) = \mathcal{I}_{\underline{\mathsf{X}}}$$

and

$$K_{\mathbf{j}} \cap \{[z^{\mathbf{j}}_{[n,\infty[}]_\infty \mid n \in \mathbb{N}_0\} = \{z^{\mathbf{j}}_{[0,\infty[}\}$$

for each $\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}$.

Proof: Let $x \in \mathcal{ND}_\infty(\underline{\mathsf{X}}^+)$. Then there are an adjusted left special word z and an $n \in \mathbb{N}_0$ such that $x = z_{[n,\infty[}$. If $n > n_z$, then

$$x = z_{[n,\infty[} = z^{\mathbf{z}}_{[n-n_z,\infty[}.$$

If $n \leq n_z$, then $[x]_\infty \in K_{\mathbf{z}}$. Thus

$$\bigcup_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}} \left(K_{\mathbf{j}} \cup \{[z^{\mathbf{j}}_{[n,\infty[}]_\infty \mid n \in \mathbb{N}\}\right) = \mathcal{I}_{\underline{\mathsf{X}}}.$$

Assume that $n > 0$ and $[z^{\mathbf{j}}_{[n,\infty[}]_\infty \in K_{\mathbf{j}}$. Since $z^{\mathbf{j}}_{[n,\infty[}$ is isolated, this implies that $z^{\mathbf{j}}_{[n,\infty[} = z_{[m,\infty[}$ for some $z \in M_{\mathbf{j}}$ and $0 \leq m \leq n_z$. But then

$$z_{[m,\infty[} = z^{\mathbf{j}}_{[n,\infty[} = z_{[n_z+n,\infty[}$$

which cannot be true since there are no periodic left special words.
Thus

$$K_{\mathbf{j}} \cap \{[z^{\mathbf{j}}_{[n,\infty[}]_\infty \mid \mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}, n \in \mathbb{N}_0\} = \{z^{\mathbf{j}}_{[0,\infty[}\}$$

for each $\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}$. □

LEMMA 5.14. *The map*

$$(\mathbf{j}, n) \mapsto [z^{\mathbf{j}}_{[n,\infty[}]_\infty,$$

from $\mathcal{J}_{\underline{\mathsf{X}}} \times \mathbb{N}_0$ *to* $\mathcal{I}_{\underline{\mathsf{X}}}$ *is injective.*

Proof: Assume that $[z^{\mathbf{j}_1}_{[n_1,\infty[}]_\infty = [z^{\mathbf{j}_2}_{[n_2,\infty[}]_\infty$. Since $z^{\mathbf{j}_1}_{[n_1,\infty[}$ is isolated, $z^{\mathbf{j}_1}_{[n_1,\infty[}$ must be equal to $z^{\mathbf{j}_2}_{[n_2,\infty[}$. This implies that $z^{\mathbf{j}_1}$ and $z^{\mathbf{j}_2}$ are right shift tail equivalent, so $\mathbf{j}_1 = \mathbf{j}_2$, and since there are no periodic left special words, n_1 and n_2 must be equal. □

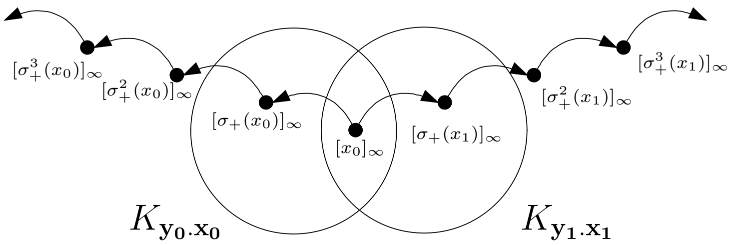
We will now look at $\mathcal{I}_{\underline{\mathsf{X}}}$ for three examples. First let $\underline{\mathsf{X}}$ be the shift space associated with the Morse substitution

$$0 \mapsto 01, \quad 1 \mapsto 10.$$

The shift space $\underline{\mathsf{X}}$ is minimal and has 4 left special elements:

$$y_0.x_0 \quad y_0.x_1 \quad y_1.x_0 \quad y_1.x_1$$

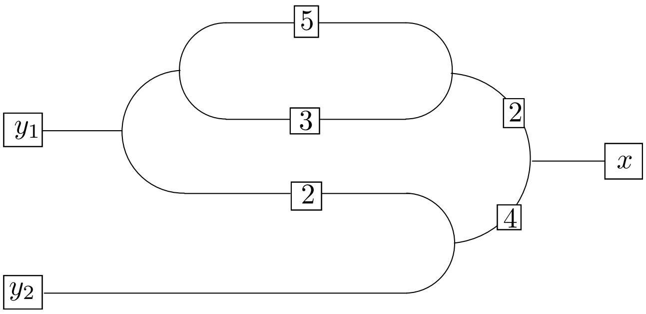
where y_0, y_1 are the fixpoints in $\underline{\mathsf{X}}^-$ of the substitution ending with 0 respectively 1, and x_0, x_1 are the fixpoints in $\underline{\mathsf{X}}^+$ of the substitution beginning with 0 respectively 1. Thus it follows from Example 3.6 that $\underline{\mathsf{X}}$ has property $(**)$. We see that $\mathcal{J}_{\underline{\mathsf{X}}}$ consists of 2 elements: $\mathbf{y_0.x_0}$ and $\mathbf{y_1.x_1}$. Notice that although all of the 4 left special elements are cofinal (and adjusted) neither x_0 nor x_1 are isolated, but $\sigma_+(x_0)$ and $\sigma_+(x_1)$ are, so we can choose $\sigma(y_0.x_0)$ and $\sigma(y_1.x_1)$ as $z^{\mathbf{y_0.x_0}}$ and $z^{\mathbf{y_1.x_1}}$ respectively. We then have that $\mathcal{I}_{\underline{\mathsf{X}}}$ looks like this:



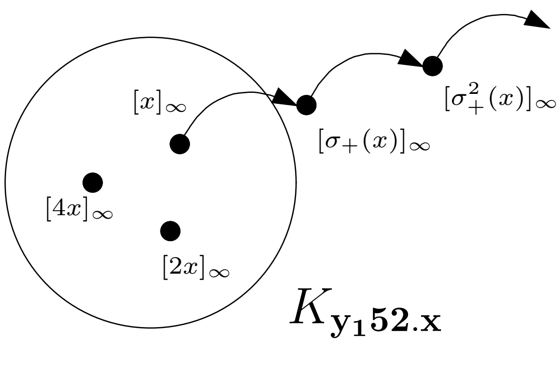
where an arrow from a to b means that in Definition 5.4 $\tilde{\alpha}_b = \alpha_a$. We notice further that $\tilde{\alpha}_{[x_0]_\infty} = g(\sigma_-(y_0)) + g(\sigma_-(y_1))$.
Our second example is the shift space associated to the substitution

$$1 \mapsto 123514, \quad 2 \mapsto 124, \quad 3 \mapsto 13214, \quad 4 \mapsto 14124, \quad 5 \mapsto 15214.$$

The shift space $\underline{\mathsf{X}}$ is minimal and has 8 left special elements (4 adjusted and 4 cofinal) as illustrated on this figure:



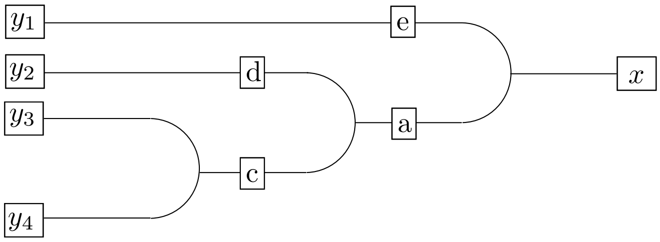
where $x \in \underline{\mathsf{X}}^+$ and $y_1, y_2 \in \underline{\mathsf{X}}^-$. Thus it follows from 3.6 that $\underline{\mathsf{X}}$ has property (∗∗). The set $\mathcal{J}_{\underline{\mathsf{X}}}$ consists of one element $\mathbf{y_1 52.x}$, and since x is isolated, we can choose $y_1 52.x$ as $z^{\mathbf{y_1 52.x}}$. We then have that $\mathcal{I}_{\underline{\mathsf{X}}}$ looks like this:



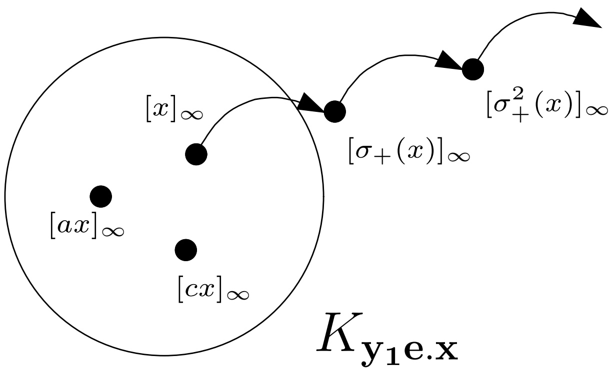
where an arrow from a to b means that in Definition 5.4 $\tilde{\alpha}_b = \alpha_a$. We notice further that $\tilde{\alpha}_{[x]_\infty} = \alpha_{[2x]_\infty} + \alpha_{[4x]_\infty}$, $\tilde{\alpha}_{[2x]_\infty} = 2g(y_1)$ and $\tilde{\alpha}_{[4x]_\infty} = g(y_1) + g(\sigma_-(y_2))$.
The third example is the shift space associated to the substitution

$$a \mapsto adbac, \quad b \mapsto aedbbc, \quad c \mapsto ac, \quad d \mapsto adac, \quad e \mapsto aecadbac.$$

The shift space $\underline{\mathsf{X}}$ is minimal and has 9 left special elements (3 adjusted, 4 cofinal and 2 which are neither adjusted nor cofinal) as illustrated on this figure:



where $x \in \underline{\mathsf{X}}^+$ and $y_1, y_2, y_3, y_4 \in \underline{\mathsf{X}}^-$. Thus it follows from 3.6 that $\underline{\mathsf{X}}$ has property $(**)$. The set $\mathcal{J}_{\underline{\mathsf{X}}}$ consists of one element $\mathbf{y_1 e.x}$, and since x is isolated, we can choose $y_1 e.x$ as $z^{\mathbf{y_1 e.x}}$. We then have that $\mathcal{I}_{\underline{\mathsf{X}}}$ looks like this:



where an arrow from a to b means that in Definition 5.4 $\tilde{\alpha}_b = \alpha_a$. We notice further that $\tilde{\alpha}_{[x]_\infty} = \alpha_{[ax]_\infty} + g(y_1)$, $\tilde{\alpha}_{[ax]_\infty} = \alpha_{[cx]_\infty} + g(y_2)$ and $\tilde{\alpha}_{[cx]_\infty} = g(\sigma_-(y_3)) + g(\sigma_+(y_4))$.

5.3 $K_0(\underline{\mathsf{X}})$ IS A FACTOR OF $G_{\underline{\mathsf{X}}}$

We are now ready to define the group $G_{\underline{\mathsf{X}}}$ which has a factor which is isomorphic to $\mathcal{G}_{\underline{\mathsf{X}}}/(\mathrm{Id} - \mathcal{A}_{\underline{\mathsf{X}}})(\mathcal{G}_{\underline{\mathsf{X}}})$.

Loosely speaking, the idea is to simplify $\mathcal{G}_{\underline{\mathsf{X}}}$ in three ways. First we replace $\underline{\mathsf{X}}^-$ by $\underline{\mathsf{X}}$, secondly we collapse for each $\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}$, $K_{\mathbf{j}}$ to one point, which makes it possible to replace $\mathcal{I}_{\underline{\mathsf{X}}}$ by $\mathcal{J}_{\underline{\mathsf{X}}} \times \mathbb{N}_0$, and thirdly we replace the continuity condition of Lemma 5.3 by the condition that the sequence is eventually 0. By doing this, we of course do not get a group which is isomorphic to $\mathcal{G}_{\underline{\mathsf{X}}}$, but it turns out that we still get isomorphic cokernels.

DEFINITION 5.15. *Let $\underline{\mathsf{X}}$ be a shift space which has property $(**)$. Denote by $G_{\underline{\mathsf{X}}}$ the group $C(\underline{\mathsf{X}},\mathbb{Z}) \oplus \sum_{n\in\mathbb{N}_0} \mathbb{Z}^{\mathcal{J}_{\underline{\mathsf{X}}}}$, let $A_{\underline{\mathsf{X}}}$ be the map from $G_{\underline{\mathsf{X}}}$ to itself defined by*

$$(f, (a_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}, n\in\mathbb{N}_0}) \mapsto (f\circ\sigma^{-1}, (\tilde{a}_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}, n\in\mathbb{N}_0}),$$

where $\tilde{a}_0^{\mathbf{j}} = \sum_{z\in M_{\mathsf{j}}} f(\sigma^{-1}(z)) - f(\sigma^{-1}(z^{\mathbf{j}}))$, and $\tilde{a}_n^{\mathbf{j}} = a_{n-1}^{\mathbf{j}}$ for $n>0$, and let ψ be the map from $\mathcal{G}_{\underline{\mathsf{X}}}$ to $G_{\underline{\mathsf{X}}}$ defined by

$$(g, (\alpha_{\mathsf{i}})_{\mathsf{i}\in\mathcal{I}_{\underline{\mathsf{X}}}}) \mapsto (g\circ\pi_-, (a_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}, n\in\mathbb{N}_0}),$$

where $a_0^{\mathbf{j}} = \sum_{\mathsf{i}\in K_{\mathbf{j}}} \alpha_{\mathsf{i}} - g(\pi_-(z^{\mathbf{j}}))$ and $a_n^{\mathbf{j}} = \alpha_{[z^{\mathbf{j}}_{[n,\infty[}]_\infty} - g(z^{\mathbf{j}}_{]-\infty,n[})$ for $n>0$.

PROPOSITION 5.16. *Let $\underline{\mathsf{X}}$ be a shift space which has property $(**)$. Then there is an isomorphism*

$$\bar{\psi} : \mathcal{G}_{\underline{\mathsf{X}}}/(\mathrm{Id}-\mathcal{A}_{\underline{\mathsf{X}}})(\mathcal{G}_{\underline{\mathsf{X}}}) \to G_{\underline{\mathsf{X}}}/(\mathrm{Id}-A_{\underline{\mathsf{X}}})\big(G_{\underline{\mathsf{X}}}\big)$$

which makes the following diagram commute:

$$\begin{array}{ccc}
\mathcal{G}_{\underline{\mathsf{X}}} & \xrightarrow{\psi} & G_{\underline{\mathsf{X}}} \\
\downarrow & & \downarrow \\
\mathcal{G}_{\underline{\mathsf{X}}}/(\mathrm{Id}-\mathcal{A}_{\underline{\mathsf{X}}})(\mathcal{G}_{\underline{\mathsf{X}}}) & \xrightarrow{\bar{\psi}} & G_{\underline{\mathsf{X}}}/(\mathrm{Id}-A_{\underline{\mathsf{X}}})\big(G_{\underline{\mathsf{X}}}\big)
\end{array}$$

Proof: Let ρ be the projection from $G_{\underline{\mathsf{X}}}$ to $G_{\underline{\mathsf{X}}}/(\mathrm{Id}-A_{\underline{\mathsf{X}}})\big(G_{\underline{\mathsf{X}}}\big)$. We will prove the existence of $\bar{\psi}$ by showing the following 3 things about $\rho\circ\psi$: a) that it is surjective, b) that $\ker(\rho\circ\psi) \subseteq (\mathrm{Id}-\mathcal{A}_{\underline{\mathsf{X}}})(\mathcal{G}_{\underline{\mathsf{X}}})$, and c) that $(\mathrm{Id}-\mathcal{A}_{\underline{\mathsf{X}}})(\mathcal{G}_{\underline{\mathsf{X}}}) \subseteq \ker(\rho\circ\psi)$.

a) $\rho\circ\psi$ is surjective: Let $(f, (a_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}, n\in\mathbb{N}_0}) \in G_{\underline{\mathsf{X}}}$. Our goal is to find an element of $\mathcal{G}_{\underline{\mathsf{X}}}$ which is mapped to $\rho((f, (a_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}, n\in\mathbb{N}_0}))$ by $\rho\circ\psi$.
Since f is continuous, there are $n,m\in\mathbb{N}$ such that $z_{[-n,m]} = z'_{[-n,m]} \Rightarrow f(z) = f(z')$. Thus

$$z_{[-n-m-1,-1]} = z'_{[-n-m-1,-1]} \Rightarrow f\circ\sigma^{-(m+1)}(z) = f\circ\sigma^{-(m+1)}(z').$$

Hence there is an $g\in C(\underline{\mathsf{X}}^-,\mathbb{Z})$ such that $g\circ\pi_- = f\circ\sigma^{-(m+1)}$.
Choose for each $\mathsf{i}\in\bigcup_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}} K_{\mathbf{j}}$ an $l_{\mathsf{i}}\in\mathbb{N}_0$ such that $\mathcal{P}_{l_{\mathsf{i}}}(x) > 1$ for every $x\in\mathsf{i}$, and let $N = n+m+1+\max\{l_{\mathsf{i}} \mid \mathsf{i}\in\bigcup_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}} K_{\mathbf{j}}\}$. Let $\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}$. Then there is

an $m_{\mathbf{j}} \in \mathbb{N}_0$ such that $\mathcal{P}_N(z^{\mathbf{j}}_{[n,\infty[}) > 1$ for $0 \le n < m_{\mathbf{j}}$ and $\mathcal{P}_N(z^{\mathbf{j}}_{[n,\infty[}) = 1$ for $n \ge m_{\mathbf{j}}$.
Set for each $\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}$ and $n \ge m_{\mathbf{j}}$,

$$\alpha_{[z^{\mathbf{j}}_{[n,\infty[}]_\infty} = g(z^{\mathbf{j}}_{]-\infty,n[}),$$

and let $\alpha_{\mathsf{i}} = 0$ for

$$\mathsf{i} \in \mathcal{I}_{\underline{\mathsf{X}}} \setminus \{[z^{\mathbf{j}}_{[n,\infty[}]_\infty \mid \mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}, n \ge m_{\mathbf{j}}\} = \bigcup_{\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}} \Big(K_{\mathbf{j}} \cup \{[z^{\mathbf{j}}_{[n,\infty[}]_\infty \mid 0 \le n \le m_{\mathbf{j}}\} \Big).$$

We then have that $y_{[-N,-1]} = y'_{[-N,-1]} \Rightarrow g(y) = g(y')$, $\mathcal{P}_N(x) > 1 \Rightarrow \alpha_{[x]_\infty} = 0$, and $\mathcal{P}_N(x) = \{y_{[-N,-1]}\} \Rightarrow \alpha_{[x]_\infty} = g(y)$ for $y, y' \in \underline{\mathsf{X}}^-$ and $x \in \underline{\mathsf{X}}^+$. Hence $(g, (\alpha_{\mathsf{i}})_{\mathsf{i} \in \mathcal{I}_{\underline{\mathsf{X}}}}) \in \mathcal{G}_{\underline{\mathsf{X}}}$. We also have that

$$\psi((g, (\alpha_{\mathsf{i}})_{\mathsf{i} \in \mathcal{I}_{\underline{\mathsf{X}}}})) = A^{m+1}_{\underline{\mathsf{X}}}(f, (a^{\mathbf{j}}_n)_{\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}, n \in \mathbb{N}_0}) + (0, (\tilde{a}^{\mathbf{j}}_n)_{\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}, n \in \mathbb{N}_0})$$

for some $(\tilde{a}^{\mathbf{j}}_n)_{\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}, n \in \mathbb{N}_0} \in \sum_{n \in \mathbb{N}_0} \mathbb{Z}^{\mathcal{J}_{\underline{\mathsf{X}}}}$. Since

$$\rho(f, (a^{\mathbf{j}}_n)_{\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}, n \in \mathbb{N}_0}) = \rho(A^{m+1}_{\underline{\mathsf{X}}}(f, (a^{\mathbf{j}}_n)_{\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}, n \in \mathbb{N}_0})),$$

it is enough to find $(0, (\tilde{\alpha}_{\mathsf{i}})_{\mathsf{i} \in \mathcal{I}_{\underline{\mathsf{X}}}}) \in \mathcal{G}_{\underline{\mathsf{X}}}$ such that $\psi(0, (\tilde{\alpha}_{\mathsf{i}})_{\mathsf{i} \in \mathcal{I}_{\underline{\mathsf{X}}}}) = (0, (\tilde{a}^{\mathbf{j}}_n)_{\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}, n \in \mathbb{N}_0})$, because then

$$\rho \circ \psi((g, (\alpha_{\mathsf{i}})_{\mathsf{i} \in \mathcal{I}_{\underline{\mathsf{X}}}}) - (0, (\tilde{\alpha}_{\mathsf{i}})_{\mathsf{i} \in \mathcal{I}_{\underline{\mathsf{X}}}})) = \rho(f, (a^{\mathbf{j}}_n)_{\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}, n \in \mathbb{N}_0}).$$

Let $\mathsf{i} \in \mathcal{I}_{\underline{\mathsf{X}}}$. If $\mathsf{i} \in \bigcup_{\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}} K_{\mathbf{j}} \setminus \{[x^{\mathbf{j}}]_\infty\}$, then we let $\tilde{\alpha}_{\mathsf{i}} = 0$, and if $\mathsf{i} = [z^{\mathbf{j}}_{[n,\infty[}]_\infty$, $\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}$, $n \in \mathbb{N}_0$, then we let $\tilde{\alpha}_{\mathsf{i}} = a^{\mathbf{j}}_n$. We claim that $(0, (\tilde{\alpha}_{\mathsf{i}})_{\mathsf{i} \in \mathcal{I}_{\underline{\mathsf{X}}}}) \in \mathcal{G}_{\underline{\mathsf{X}}}$ and $\psi(0, (\tilde{\alpha}_{\mathsf{i}})_{\mathsf{i} \in \mathcal{I}_{\underline{\mathsf{X}}}}) = (0, (\tilde{a}^{\mathbf{j}}_n)_{\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}, n \in \mathbb{N}_0})$.
Choose an $N \in \mathbb{N}_0$ such that $\tilde{a}^{\mathbf{j}}_n = 0$ for every $\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}$ and $n \ge N$. Let $\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}$. Since $z^{\mathbf{j}}_{[n,\infty[}$ is isolated for every $\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}$ and every $n \in \mathbb{N}_0$ (cf. Remark 5.12), there is for each $0 \le n < N$ a $k^{\mathbf{j}}_n \in \mathbb{N}_0$ such that $[z^{\mathbf{j}}_{[n,\infty[}]_{k^{\mathbf{j}}_n} = \{z^{\mathbf{j}}_{[n,\infty[}\}$ and by increasing $k^{\mathbf{j}}_n$ if necessary, we may (and will) assume that $\#\mathcal{P}_{k^{\mathbf{j}}_n}(z^{\mathbf{j}}_{[n,\infty[}) > 1$. Let

$$M = \max\{k^{\mathbf{j}}_n \mid \mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}, 0 \le n < N\}.$$

We then have that $[z^{\mathbf{j}}_{[n,\infty[}]_M = \{z^{\mathbf{j}}_{[n,\infty[}\}$ and $\#\mathcal{P}_M(z^{\mathbf{j}}_{[n,\infty[}) > 1$ for $\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}$ and $0 \le n < N$. Thus if $\tilde{\alpha}_{[x]_\infty} \ne 0$, then $[x]_M = \{x\}$ and $\#\mathcal{P}_M(x) > 1$. So we have that $[x]_M = [x']_M \Rightarrow \tilde{\alpha}_{[x]_\infty} = \tilde{\alpha}_{[x']_\infty}$, and $\mathcal{P}_M(x) = \{y_{[-M,-1]}\} \Rightarrow \tilde{\alpha}_{[x]_\infty} = 0$. Hence $(0, (\tilde{\alpha}_i)_{\mathsf{i} \in \mathcal{I}_{\underline{\mathsf{X}}}}) \in \mathcal{G}_{\underline{\mathsf{X}}}$.
Let $\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}$. Then we have that $\tilde{\alpha}_{\mathsf{i}} = 0$ for $\mathsf{i} \in K_{\mathbf{j}} \setminus \{x^{\mathbf{j}}\}$ and $\tilde{\alpha}_{[z^{\mathbf{j}}_{[n,\infty[}]_\infty} = \tilde{a}^{\mathbf{j}}_n$ for $n \in \mathbb{N}_0$. Hence $\psi(0, (\tilde{\alpha}_i)_{\mathsf{i} \in \mathcal{I}_{\underline{\mathsf{X}}}}) = (0, (\tilde{a}^{\mathbf{j}}_n)_{\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}, n \in \mathbb{N}_0})$.

b) $\ker(\rho\circ\psi)\subseteq(\mathrm{Id}-\mathcal{A}_{\underline{X}})(\mathcal{G}_{\underline{X}})$*:* Assume that $(g,(\alpha_{\mathsf{i}})_{\mathsf{i}\in\mathcal{I}_{\underline{X}}})\in\ker(\rho\circ\psi)$. We must then find $(\tilde{g},(\tilde{\alpha}_{\mathsf{i}})_{\mathsf{i}\in\mathcal{I}_{\underline{X}}})\in\mathcal{G}_{\underline{X}}$ such that $(\mathrm{Id}-\mathcal{A}_{\underline{X}})(\tilde{g},(\tilde{\alpha}_{\mathsf{i}})_{\mathsf{i}\in\mathcal{I}_{\underline{X}}})=(g,(\alpha_{\mathsf{i}})_{\mathsf{i}\in\mathcal{I}_{\underline{X}}})$.
Since $\rho\circ\psi((g,(\alpha_{\mathsf{i}})_{\mathsf{i}\in\mathcal{I}_{\underline{X}}}))=0$, there is $(f,(a_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{X}},n\in\mathbb{N}_0})\in G_{\underline{X}}$ such that $(\mathrm{Id}-A_{\underline{X}})(f,(a_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{X}},n\in\mathbb{N}_0})=\psi(g,(\alpha_{\mathsf{i}})_{\mathsf{i}\in\mathcal{I}_{\underline{X}}})$. Hence $f-f\circ\sigma^{-1}=g\circ\pi_-$. The function f is continuous, so there are $n,m\in\mathbb{N}$ such that $z_{[-n,m]}=z'_{[-n,m]}\Rightarrow f(z)=f(z')$. We claim that

$$z_{[-\infty,-1]}=z'_{[-\infty,-1]}\Rightarrow f(z)=f(z').$$

Assume that there are z,z' such that $z_{[-\infty,-1]}=z'_{[-\infty,-1]}$ and $f(z)\neq f(z')$. Since $f=f\circ\sigma^{-1}+g\circ\pi_-$, $f(\sigma^{-1}(z))\neq f(\sigma^{-1}(z'))$. Similarly $f\circ\sigma^{-1}=f\circ\sigma^{-2}+g\circ\pi_-\circ\sigma^{-1}$, so $f(\sigma^{-2}(z))\neq f(\sigma^{-2}(z'))$. Continuing in this way we get that $f(\sigma^{-m}(z))\neq f(\sigma^{-m}(z'))$, but this can not be true since $\sigma^{-m}(z)_{[-n,m]}=\sigma^{-m}(z')_{[-n,m]}$.
Thus there is a $\tilde{g}\in C(\underline{X}^-,\mathbb{Z})$ such that $\tilde{g}\circ\pi_-=f$ and hence $\tilde{g}-\tilde{g}\circ\sigma_-=g$. Set for every $\mathbf{j}\in\mathcal{J}_{\underline{X}}$ and $n\in\mathbb{N}$,

$$\tilde{\alpha}_{[z^{\mathbf{j}}_{[n,\infty[}]_\infty}=a_n^{\mathbf{j}}+\tilde{g}(z^{\mathbf{j}}_{]-\infty,n[}).$$

Let $\mathsf{i}\in\bigcup_{\mathbf{j}\in\mathcal{J}_{\underline{X}}}K_{\mathbf{j}}$. Choose $x_{\mathsf{i}}\in\mathcal{ND}_\infty(\underline{X}^+)$ such that $\mathsf{i}=[x_{\mathsf{i}}]_\infty$. There is for each $z\in\underline{X}$ which satisfies $\pi_+(z)=x_{\mathsf{i}}$, a unique $m_z\in\mathbb{N}_0$ such that $\sigma^{-m_z}(z)$ is an adjusted left special word. Let

$$L_{\mathsf{i}}=\{[z_{[-m,\infty[}]_\infty\mid\pi_+(z)=x_{\mathsf{i}},0\leq m\leq m_z\}\subseteq\mathcal{I}_{\underline{X}},$$

$$B_{\mathsf{i}}=\{\sigma^{-m_z}(z)\mid\pi_+(z)=x_{\mathsf{i}}\}\subseteq\underline{X},$$

and

$$\tilde{\alpha}_{\mathsf{i}}=\sum_{\mathsf{i}'\in L_{\mathsf{i}}}\alpha_{\mathsf{i}'}+\sum_{z\in B_{\mathsf{i}}}f(\sigma^{-1}(z)).$$

Notice that even though B_{i} depends on the choice of x_{i}, $\tilde{\alpha}_{\mathsf{i}}$ does not, because $f=\tilde{g}\circ\pi_-$.
Since $(a_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{X}},n\in\mathbb{N}_0}\in\sum_{n\in\mathbb{N}_0}\mathbb{Z}^{\mathcal{J}_{\underline{X}}}$, there is an N_1 such that $a_n^{\mathbf{j}}=0$ for $\mathbf{j}\in\mathcal{J}_{\underline{X}}$ and $n\geq N_1$, and since $\tilde{g}$ is continuous, there is an $N_2\in\mathbb{N}$ such that

$$y_{[-N_2,-1]}=y'_{[-N_2,-1]}\Rightarrow\tilde{g}(y)=\tilde{g}(y')$$

for $y,y'\in\underline{X}^-$. Let $\mathbf{j}\in\mathcal{J}_{\underline{X}}$. Since $z^{\mathbf{j}}_{[n,\infty[}$ is isolated for every $\mathbf{j}\in\mathcal{J}_{\underline{X}}$ and every $n\in\mathbb{N}_0$ (cf. Remark 5.12), there is for each $0\leq n<\max\{N_1,N_2\}$ a $k_n^{\mathbf{j}}\in\mathbb{N}$ such that $[z^{\mathbf{j}}_{[n,\infty[}]_{k_n^{\mathbf{j}}}=\{z^{\mathbf{j}}_{[n,\infty[}\}$, and by increasing $k_n^{\mathbf{j}}$ if necessary, we may (and will) assume that $\#\mathcal{P}_{k_n^{\mathbf{j}}}(z^{\mathbf{j}}_{[n,\infty[})>1$. Let

$$N_3=\max\{k_n^{\mathbf{j}}\mid\mathbf{j}\in\mathcal{J}_{\underline{X}},0\leq n<\max\{N_1,N_2\}\}.$$

Since $\bigcup_{\mathbf{j}\in\mathcal{J}_{\underline{X}}}K_{\mathbf{j}}$ is finite, there is an N_4 such that $[x]_{N_4}=[x']_{N_4}\Rightarrow[x]_\infty=[x']_\infty$ and $\mathcal{P}_{N_4}(x)>1$ for $[x]_\infty,[x']_\infty\in\bigcup_{\mathbf{j}\in\mathcal{J}_{\underline{X}}}K_{\mathbf{j}}$. Let $M=\max\{N_1,N_2,N_3,N_4\}$. We claim that

1. $\forall y, y' \in \underline{\mathsf{X}}^- : y_{[-M,-1]} = y'_{[-M,-1]} \Rightarrow \tilde{g}(y) = \tilde{g}(y')$,

2. $\forall x, x' \in \mathcal{ND}_\infty(\underline{\mathsf{X}}^+) : [x]_M = [x']_M \Rightarrow \tilde{\alpha}_{[x]_\infty} = \tilde{\alpha}_{[x']_\infty}$,

3. $\forall x \in \mathcal{ND}_\infty(\underline{\mathsf{X}}^+), y \in \underline{\mathsf{X}}^- : \mathcal{P}_M(x) = \{y_{[-M,-1]}\} \Rightarrow \tilde{\alpha}_{[x]_\infty} = \tilde{g}(y)$,

which implies that $(\tilde{g}, (\tilde{\alpha}_\mathsf{i})_{\mathsf{i}\in\mathcal{I}_{\underline{\mathsf{X}}}}) \in \mathcal{G}_{\underline{\mathsf{X}}}$. 1. follows from the fact that $M \geq N_2$. Notice that if

$$[x]_\infty \in \bigcup_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}} \left(K_\mathbf{j} \cup \{[z^\mathbf{j}_{[n,\infty[}]_\infty \mid 0 < n < \max\{N_1, N_2\}\}\right),$$

then $x' \in [x]_M \Rightarrow x \sim_\infty x'$. This takes care of 2. in the case where $[x]_\infty \in \bigcup_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}} \left(K_\mathbf{j} \cup \{[z^\mathbf{j}_{[n,\infty[}]_\infty \mid 0 < n < \max\{N_1, N_2\}\}\right)$. For $n \geq \max\{N_1, N_2\}$, $\alpha_{[z^\mathbf{j}_{[n,\infty[}]_\infty} = \tilde{g}(z^\mathbf{j}_{]-\infty,n[})$, and since

$$\begin{aligned}[z^\mathbf{j}_{[n,\infty[}]_M = [z^{\mathbf{j}'}_{[n',\infty[}]_M \Rightarrow & z^\mathbf{j}_{[n-N_2,n[} = z^{\mathbf{j}'}_{[n'-N_2,n'[} \\ \Rightarrow & \tilde{g}(z^\mathbf{j}_{]-\infty,n[}) = \tilde{g}(z^{\mathbf{j}'}_{]-\infty,n'[})\end{aligned}$$

2. and 3. hold.
Let $(\tilde{\tilde{g}}, (\tilde{\tilde{\alpha}}_\mathsf{i})_{\mathsf{i}\in\mathcal{I}_{\underline{\mathsf{X}}}}) = \mathcal{A}_{\underline{\mathsf{X}}}(\tilde{g}, (\tilde{\alpha}_\mathsf{i})_{\mathsf{i}\in\mathcal{I}_{\underline{\mathsf{X}}}})$. Then $\tilde{\tilde{g}} = \tilde{g} \circ \sigma_-$, and for $\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}$ and $n \in \mathbb{N}$

$$\tilde{\tilde{\alpha}}_{[z^\mathbf{j}_{[n+1,\infty[}]_\infty} = \tilde{\alpha}_{[z^\mathbf{j}_{[n,\infty[}]_\infty} = a^\mathbf{j}_n + \tilde{g}(z^\mathbf{j}_{]-\infty,n[}).$$

Let $\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}$. Then $L_{[x^\mathbf{j}]_\infty} = K_\mathbf{j}$ and $B_{[x^\mathbf{j}]_\infty} = M_\mathbf{j}$, so

$$\begin{aligned}\tilde{\tilde{\alpha}}_{[z^\mathbf{j}_{[1,\infty[}]_\infty} &= \tilde{\alpha}_{[x^\mathbf{j}]_\infty} \\ &= \sum_{\mathsf{i}\in K_\mathbf{j}} \alpha_\mathsf{i} + \sum_{z\in M_\mathbf{j}} f(\sigma^{-1}(z)) \\ &= a^\mathbf{j}_0 + g(\pi_-(z^\mathbf{j})) + f(\sigma^{-1}(z^\mathbf{j})) \\ &= a^\mathbf{j}_0 + \tilde{g}(z^\mathbf{j}_{]-\infty,0[}),\end{aligned}$$

where the third equality sign follows from the fact that

$$(\mathrm{Id} - A_{\underline{\mathsf{X}}})(f, (a^\mathbf{j}_n)_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}, n\in\mathbb{N}_0}) = \psi(g, (\alpha_\mathsf{i})_{\mathsf{i}\in\mathcal{I}_{\underline{\mathsf{X}}}}),$$

and the fourth follows from the facts that $\tilde{g} \circ \pi_- = f$ and $\tilde{g} - \tilde{g} \circ \sigma_- = g$.
If $[x]_\infty \in K_\mathbf{j}$, then $L_{[x]_\infty}$ is the disjoint union of $L_{[x']_\infty}$, where $[x']_\infty \in \mathcal{I}_{\underline{\mathsf{X}}}$ and $\sigma_+(x') = x$, and $\{[x]_\infty\}$, and $B_{[x]_\infty}$ is the disjoint union of $B_{[x']_\infty}$, where $[x']_\infty \in \mathcal{I}_{\underline{\mathsf{X}}}$ and $\sigma_+(x') = x$, and $\{\sigma(z) \mid z \in \underline{\mathsf{X}}, z_{[0,\infty[} \notin \mathcal{ND}_\infty(\underline{\mathsf{X}}^+), z_{[1,\infty[} = x\}$.

Hence

$$\begin{aligned}\tilde{\tilde{\alpha}}_{[x]_\infty} &= \sum_{\substack{[x']_\infty \in \mathcal{I}_{\underline{X}} \\ \sigma_+(x')=x}} \tilde{\alpha}_{[x']_\infty} + \sum_{\substack{z\in\underline{X} \\ z_{[0,\infty[}\notin\mathcal{ND}_\infty(\underline{X}^+) \\ z_{[1,\infty[}=x}} \tilde{g}(z_{]-\infty,-1]}) \\ &= \sum_{\mathsf{i}\in L_{[x]_\infty}} \alpha_{\mathsf{i}} - \alpha_{[x]_\infty} + \sum_{z\in B_{[x]_\infty}} f(\sigma^{-1}(z)) \\ &= \tilde{\alpha}_{[x]_\infty} - \alpha_{[x]_\infty}.\end{aligned}$$

So $\tilde{g} - \tilde{\tilde{g}} = g$, $\tilde{\alpha}_{\mathsf{i}} - \tilde{\tilde{\alpha}}_{\mathsf{i}} = \tilde{\alpha}_{\mathsf{i}} - \tilde{\alpha}_{\mathsf{i}} + \alpha_{\mathsf{i}} = \alpha_{\mathsf{i}}$ for $\mathsf{i} \in K_{\mathbf{j}}$, and

$$\tilde{\alpha}_{[z^{\mathbf{j}}_{[n,\infty[}]_\infty} - \tilde{\tilde{\alpha}}_{[z^{\mathbf{j}}_{[n,\infty[}]_\infty} = a^{\mathbf{j}}_n + \tilde{g}(z^{\mathbf{j}}_{]-\infty,n[}) - a^{\mathbf{j}}_{n-1} - \tilde{g}(z^{\mathbf{j}}_{]-\infty,n-1[}) = \alpha_{[z^{\mathbf{j}}_{[n,\infty[}]_\infty}.$$

for $\mathbf{j} \in \mathcal{J}_{\underline{X}}$ and $n \in \mathbb{N}$. Thus $(\mathrm{Id} - \mathcal{A}_{\underline{X}})(\tilde{g}, (\tilde{\alpha}_{\mathsf{i}})_{\mathsf{i}\in\mathcal{I}_{\underline{X}}}) = (\tilde{g}, (\tilde{\alpha}_{\mathsf{i}})_{\mathsf{i}\in\mathcal{I}_{\underline{X}}}) - (\tilde{\tilde{g}}, (\tilde{\tilde{\alpha}}_{\mathsf{i}})_{\mathsf{i}\in\mathcal{I}_{\underline{X}}}) = (g, (\alpha_{\mathsf{i}})_{\mathsf{i}\in\mathcal{I}_{\underline{X}}})$.

c) $(\mathrm{Id} - \mathcal{A}_{\underline{X}})(\mathcal{G}_{\underline{X}}) \subseteq \ker(\rho\circ\psi)$*:* Let $(g, (\alpha_{\mathsf{i}})_{\mathsf{i}\in\mathcal{I}_{\underline{X}}}) \in \mathcal{G}_{\underline{X}}$. Set $f = g\circ\pi_-$ and $a^{\mathbf{j}}_n = \alpha_{[z^{\mathbf{j}}_{[n,\infty[}]_\infty} - g(z^{\mathbf{j}}_{]-\infty,n[})$ for $\mathbf{j} \in \mathcal{J}_{\underline{X}}$ and $n \in \mathbb{N}_0$. Then $(f, (a^{\mathbf{j}}_n)_{\mathbf{j}\in\mathcal{J}_{\underline{X}}, n\in\mathbb{N}_0}) \in G_{\underline{X}}$, and since $f - f\circ\sigma^{-1} = (g - g\circ\sigma_-)\circ\pi_-$,

$$a^{\mathbf{j}}_n - a^{\mathbf{j}}_{n-1} = \alpha_{[z^{\mathbf{j}}_{[n,\infty[}]_\infty} - \alpha_{[\sigma_+^{n-1}(x^{\mathbf{j}})]} - (g - g\circ\sigma_-)(z^{\mathbf{j}}_{]-\infty,n[})$$

for $\mathbf{j} \in \mathcal{J}_{\underline{X}}$ and $n \in \mathbb{N}$, and

$$a^{\mathbf{j}}_0 - \sum_{z\in M_{\mathbf{j}}} f(\sigma^{-1}(z)) = \alpha_{[x^{\mathbf{j}}]_\infty} - \sum_{\substack{z\in\underline{X} \\ z_{[0,\infty[}\notin\mathcal{ND}_\infty(\underline{X}^+) \\ z_{[1,\infty[}\in\mathbf{j}}} g(z_{]-\infty,-1]}) - g(\pi_-(z^{\mathbf{j}}))$$

for $\mathbf{j} \in \mathcal{J}_{\underline{X}}$, we have that

$$(\mathrm{Id} - A_{\underline{X}})(f, (a^{\mathbf{j}}_n)_{\mathbf{j}\in\mathcal{J}_{\underline{X}}, n\in\mathbb{N}_0}) = \psi((\mathrm{Id} - \mathcal{A}_{\underline{X}})(g, (\alpha_{\mathsf{i}})_{\mathsf{i}\in\mathcal{I}_{\underline{X}}})).$$

Thus $(\mathrm{Id} - \mathcal{A}_{\underline{X}})(\mathcal{G}_{\underline{X}}) \subseteq \ker(\rho\circ\psi)$. □

The next theorem now immediately follows from Corollary 5.6:

THEOREM 5.17. *Let* $\underline{X}$ *be a shift space which has property* $(**)$*. Then* $K_0(\underline{X})$ *and*

$$G_{\underline{X}}/(\mathrm{Id} - A_{\underline{X}})\left(G_{\underline{X}}\right)$$

are isomorphic as groups.

5.4 EXAMPLES

EXAMPLE 5.18. Let $\underline{\mathsf{X}}$ be a finite shift space. Then $K_0(\underline{\mathsf{X}})$ and

$$C(\underline{\mathsf{X}},\mathbb{Z})/(\mathrm{Id}-(\sigma^{-1})^\star)(C(\underline{\mathsf{X}},\mathbb{Z}))$$

are isomorphic as groups.

Proof: We saw in Example 3.4, that a finite shift space has property $(**)$ and has no left special elements. Thus $\mathcal{J}_{\underline{\mathsf{X}}}=\emptyset$, so $G_{\underline{\mathsf{X}}}=C(\underline{\mathsf{X}},\mathbb{Z})$ and $A_{\underline{\mathsf{X}}}=(\sigma^{-1})^\star$ and it follows from 5.17, that $K_0(\underline{\mathsf{X}})$ and

$$C(\underline{\mathsf{X}},\mathbb{Z})/(\mathrm{Id}-(\sigma^{-1})^\star)(C(\underline{\mathsf{X}},\mathbb{Z}))$$

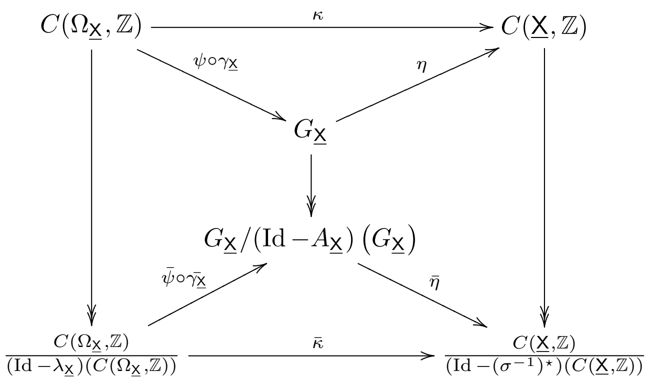
are isomorphic as groups. □

Let η be the canonical projection from $G_{\underline{\mathsf{X}}}$ to $C(\underline{\mathsf{X}},\mathbb{Z})$. We tie things up with the following proposition:

PROPOSITION 5.19. *Let $\underline{\mathsf{X}}$ be a shift space which has property $(**)$. Then there is a surjective group homomorphism*

$$\bar{\eta}: G_{\underline{\mathsf{X}}}/(\mathrm{Id}-A_{\underline{\mathsf{X}}})\left(G_{\underline{\mathsf{X}}}\right)\to C(\underline{\mathsf{X}},\mathbb{Z})/(\mathrm{Id}-(\sigma^{-1})^\star)(C(\underline{\mathsf{X}},\mathbb{Z}))$$

which makes the following diagram commute:



where $\bar{\gamma}_{\underline{\mathsf{X}}}$ is the map from $C(\Omega_{\underline{\mathsf{X}}},\mathbb{Z})/(\mathrm{Id}-\lambda_{\underline{\mathsf{X}}})(C(\Omega_{\underline{\mathsf{X}}},\mathbb{Z}))$ to $\mathcal{G}_{\underline{\mathsf{X}}}/(\mathrm{Id}-\mathcal{A}_{\underline{\mathsf{X}}})\mathcal{G}_{\underline{\mathsf{X}}}$ induced by $\gamma_{\underline{\mathsf{X}}}$.

Proof: Since $\eta\circ A_{\underline{\mathsf{X}}}=(\sigma^{-1})^\star\circ\eta$, η induces a map from $G_{\underline{\mathsf{X}}}/(\mathrm{Id}-A_{\underline{\mathsf{X}}})\left(G_{\underline{\mathsf{X}}}\right)$ to $C(\underline{\mathsf{X}},\mathbb{Z})/(\mathrm{Id}-(\sigma^{-1})^\star)(C(\underline{\mathsf{X}},\mathbb{Z}))$. It is easy to check that this map makes the diagram commute. □

COROLLARY 5.20. *Let $\underline{\mathsf{X}}$ be a shift space which has property $(**)$ and only has two left special words. Then $\bar{\eta}$ is an isomorphism from $G_{\underline{\mathsf{X}}}/(\mathrm{Id}-A_{\underline{\mathsf{X}}})\left(G_{\underline{\mathsf{X}}}\right)$ to $C(\underline{\mathsf{X}},\mathbb{Z})/(\mathrm{Id}-(\sigma^{-1})^\star)(C(\underline{\mathsf{X}},\mathbb{Z}))$. Thus $K_0(\underline{\mathsf{X}})$ and*

$$C(\underline{\mathsf{X}},\mathbb{Z})/(\mathrm{Id}-(\sigma^{-1})^\star)(C(\underline{\mathsf{X}},\mathbb{Z}))$$

are isomorphic as groups.

Proof: If $\underline{\mathsf{X}}$ only has two left special words, z_1 and z_2, then they must necessarily be right shift tail equivalent, so $\mathcal{J}_{\underline{\mathsf{X}}} = \{\mathbf{j}\}$, where $\mathbf{j} = \mathbf{z_1} = \mathbf{z_2}$. We also have that $z_{1[0,\infty[} = z_{2[0,\infty[}$ is an isolated common tail of $\mathbf{j}$, so we can choose z_2 to be $z^{\mathbf{j}}$. The set $M_{\mathbf{j}}$ is equal to $\{z_1, z_2\}$, so

$$A_{\underline{\mathsf{X}}}((f,(a_n^{\mathbf{j}})_{n\in\mathbb{N}_0})) = (f\circ\sigma^{-1},(\tilde{a}_n^{\mathbf{j}})_{n\in\mathbb{N}_0}),$$

where $\tilde{a}_0^{\mathbf{j}} = f(\sigma^{-1}(z_1))$, and $\tilde{a}_n^{\mathbf{j}} = a_{n-1}^{\mathbf{j}}$ for $n > 0$.
Suppose that $\eta((f,(a_n^{\mathbf{j}})_{n\in\mathbb{N}_0})) \in (\mathrm{Id} - (\sigma^{-1})^\star)(C(\underline{\mathsf{X}},\mathbb{Z}))$. Then there is a $g \in C(\underline{\mathsf{X}},\mathbb{Z})$ such that $f = g - g\circ\sigma^{-1}$. Since $(a_n^{\mathbf{j}})_{n\in\mathbb{N}_0} \in \sum_{n\in\mathbb{N}_0}\mathbb{Z}$, there is an $N \in \mathbb{N}_0$ such that $a_n^{\mathbf{j}} =$ for $n > N$. Let $c = -g(\sigma^{-1}(z_1)) - \sum_{n=0}^{N} a_n^{\mathbf{j}}$ and $h \in C(\underline{\mathsf{X}},\mathbb{Z})$ the function f plus the constant c, and let $b_n^{\mathbf{j}} = \sum_{i=0}^{n} a_i^{\mathbf{j}} + g(\sigma^{-1}(z_1)) + c$ for $n \in \mathbb{N}_0$. Then $b_n^{\mathbf{j}} =$ for $n > N$, so $(h,(b_n^{\mathbf{j}})_{n\in\mathbb{N}_0}) \in G_{\underline{\mathsf{X}}}$, and

$$(\mathrm{Id} - A_{\underline{\mathsf{X}}})((h,(b_n^{\mathbf{j}})_{n\in\mathbb{N}_0})) = (f,(a_n^{\mathbf{j}})_{n\in\mathbb{N}_0}),$$

which prove that $\bar{\eta}$ is injective and thus an isomorphism. □

EXAMPLE 5.21. As noted in [12], a Sturmian shift space $\underline{\mathsf{X}}_\alpha$, $\alpha \in [0,1]\backslash\mathbb{Q}$ is minimal and has two special words. Thus it follows from Example 3.6 and Corollary 5.20 that $K_0(\underline{\mathsf{X}}_\alpha)$ and

$$C(\underline{\mathsf{X}}_\alpha,\mathbb{Z})/(\mathrm{Id} - (\sigma^{-1})^\star)(C(\underline{\mathsf{X}}_\alpha,\mathbb{Z}))$$

are isomorphic as groups.
In [31] it is shown that

$$C(\underline{\mathsf{X}}_\alpha,\mathbb{Z})/(\mathrm{Id} - (\sigma^{-1})^\star)(C(\underline{\mathsf{X}}_\alpha,\mathbb{Z}))$$

is isomorphic to $\mathbb{Z}+\mathbb{Z}\alpha$ as an *ordered* group. Thus it follows that $K_0(\underline{\mathsf{X}}_\alpha)$ and $\mathbb{Z}+\mathbb{Z}\alpha$ are isomorphic as groups.
In [9, Corollary 5.2] we prove that $K_0(\underline{\mathsf{X}}_\alpha)$ with the order structure mentioned in the Introduction is isomorphic to $\mathbb{Z}+\mathbb{Z}\alpha$.

EXAMPLE 5.22. It is proved in [30, pp. 90 and 107] that if τ is an aperiodic and primitive substitution, then the associated shift space X_τ is minimal and only has a finite number of left special words. Thus by Example 3.6, X_τ has property $(**)$. It follows from [6, Proposition 3.5] that if τ furthermore is proper and elementary, then $\pi_+(z)$ is isolated for every left special word z. Thus $K_0(\mathsf{X}_\tau)$ is isomorphic to the cokernel of the map

$$A_\tau(f,[(a_0^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\mathsf{X}_\tau}},(a_1^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\mathsf{X}_\tau}},\dots]) =$$
$$\left(f\circ\sigma^{-1},\left[\left(\left(\sum_{z\in M_{\mathbf{j}}} f(\sigma^{-1}(z))\right) - f(\sigma^{-1}(z^{\mathbf{j}}))\right)_{\mathbf{j}\in\mathcal{J}_{\mathsf{X}_\tau}},(a_0^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\mathsf{X}_\tau}},(a_1^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\mathsf{X}_\tau}},\dots\right]\right)$$

defined on

$$G_\tau = C(\underline{\mathsf{X}}_\tau, \mathbb{Z}) \oplus \sum_{i=0}^{\infty} \mathbb{Z}^{\mathcal{J}_{\mathsf{X}_\tau}},$$

where $\mathcal{J}_{\mathsf{X}_\tau}$ and $M_{\mathbf{j}}$ are as defined in section 5.2, and $z^{\mathbf{j}}$ is a cofinal special element belonging to the right shift tail equivalence class $\mathbf{j}$.
In the notation of [8], $\mathcal{J}_{\mathsf{X}_\tau} = \{\widetilde{\mathsf{y}}^1, \widetilde{\mathsf{y}}^2, \dots, \widetilde{\mathsf{y}}^{\mathsf{n}_\tau}\}$, $M_{\widetilde{\mathsf{y}}^{\mathsf{j}}} = \{\mathsf{y}^j_1, \mathsf{y}^j_2, \dots, \mathsf{y}^j_{p_j+1}\}$ and $z^{\widetilde{\mathsf{y}}^{\mathsf{j}}} = \widetilde{\mathsf{y}}^j$. In [8], this is used for every aperiodic and primitive (but not necessarily proper or elementary) substitution τ, to present $K_0(\mathsf{X}_\tau)$ as a stationary inductive limit of a system associated to an integer matrix defined from combinatorial data which can be computed in an algorithmic way (cf. [6] and [7]).

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4.7 Closing remarks

One can show that if $\underline{\mathsf{X}}$ is shift space which has property $(**)$, then the kernel of the quotient map $\rho : \mathcal{O}_{\underline{\mathsf{X}}^+} \to C(\underline{\mathsf{X}}) \rtimes_{\sigma^\star} \mathbb{Z}$ is a finite number of copies of the compact operators. In fact the number of copies is the same as the number of elements in $\mathcal{J}_{\underline{\mathsf{X}}}$.

The description of K_0 giving in the paper is the starting point for an thorough investigation and description of the K_0-group of the C^*-algebra associated to the one-sided shift space of a substitutional dynamical systems as a stationary inductive limit of finite abelian preordered groups which is carried out in the last three papers.

Chapter 5

Special elements in substitutional dynamical systems

This chapter consists of the preprint *A graph approach to computing non-determinacy in substitutional dynamical systems* which is written together with Søren Eilers.

In order to give the previously mentioned description of K_0 for substitutional dynamical system, we need an understanding of the structure of *special elements* in substitutional dynamical systems. This paper describes an algorithm for finding them.

A graph approach to computing nondeterminacy in substitutional dynamical systems

Toke M. Carlsen and Søren Eilers

Revised version, February 2004

Abstract

We present an algorithm which for any aperiodic and primitive substitution outputs a finite representation of each special word in the shift space associated to that substitution, and determines when such representations are equivalent under orbit and shift tail equivalence. The algorithm has been implemented and applied in the study of certain new invariants for flow equivalence of substitutional dynamical systems.

1 Preliminaries

1.1 Introduction

Most elements in substitutional dynamical systems, given as doubly infinite sequences, have unique pasts and futures in the sense that one one-sided infinite subsequence determines the other. The importance of those elements which do not have this property, the *special elements*, is well understood in the theory of substitutions and the dynamical systems associated to them.

Determining K-groups of certain C^*-algebras we found, as described in [4], an invariant of flow equivalence (cf. [16]) — akin and related to the dimension groups considered in [8] — of substitutional systems based on combinatorial and textual properties of the special elements. For each primitive and aperiodic substitution τ on an alphabet $\mathfrak{a}$ this invariant is an ordered group defined as a stationary inductive limit of group endomorphisms on $\mathbb{Z}^{|\mathfrak{a}|} \oplus \mathbb{Z}^{\mathsf{n}_\tau}$ induced from a $(|\mathfrak{a}| + \mathsf{n}_\tau) \times (|\mathfrak{a}| + \mathsf{n}_\tau)$ block matrix

$$\begin{bmatrix} \mathbf{A}_\tau & 0 \\ \mathbf{B}_\tau & \mathbf{Id} \end{bmatrix}$$

where the block $\mathbf{A}_\tau$ is the abelianization matrix of the substitution in question. To compute the integer n_τ and the matrix $\mathbf{B}_\tau$ one needs a coherent finite representation of all the special words of the substitution, and to determine which among the special words are equivalent under the natural relations of *orbit* and *right shift tail* equivalence (see Definition 1.4). In a recent paper [7] we prove by example that the resulting invariant contains information not accessible by any other flow invariant known to us, such as the dimension groups in [8], the configuration graph (see page 6 below), or the numerical index used in conjuction with the notion of weak equivalence in [1]. The example is an explicit substitution τ on $\{a, b, c, d\}$ such that the matrices associated to τ and its opposite τ^{-1}, respectively, become

$$\begin{bmatrix} 6 & 9 & 3 & 9 & 0 & 0 \\ 12 & 18 & 6 & 18 & 0 & 0 \\ 6 & 9 & 3 & 9 & 0 & 0 \\ 36 & 54 & 18 & 54 & 0 & 0 \\ 10 & 13 & 4 & 12 & 1 & 0 \\ 6 & 8 & 2 & 8 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 6 & 9 & 3 & 9 & 0 & 0 \\ 12 & 18 & 6 & 18 & 0 & 0 \\ 6 & 9 & 3 & 9 & 0 & 0 \\ 36 & 54 & 18 & 54 & 0 & 0 \\ 2 & 7 & 2 & 7 & 1 & 0 \\ 2 & 7 & 2 & 7 & 0 & 1 \end{bmatrix}$$

Needless to say, having computer based tools to compute these components of our invariant is very useful in the study of it. The project described above thus naturally lead us to concern ourselves with computability of the aforementioned words and quantities associated to the class of special elements associated to a given substitution, and failing to find algorithms meeting our needs in the literature, we developed the approach presented in the present note. Our algorithm outputs a finite representation of each special word, and determines when such representations are equivalent under *shift tail equivalence*, a naturally occurring relation of importance in our invariant.

We wish to acknowledge [2], to which our work is closely related. Although the ends and ambitions of the present note and [2] do not overlap, the means seem to do. Our method was developed independently, but we recieved [2] before writing up this note. Although we have not attempted to do so, the results in our Section 3 could most likely be developed using the methods in [2], and vice versa. The series of reductions based on replacing the substitutions in [2] is, however, computationally inconvenient for our purposes. Our Section 4 has no analogue there.

Our paper is organized as follows. In Section 2, after having singled out the class of elementary substitutions and explained how to reduce the problem to this case, we associate certain graphs to such substitutions and explain how they give rise to a class of adjusted left special words. We also define a class of adjusted left special words arising from τ-periodic one-sided words. In Section 3 we then proceed to prove that each adjusted left special word is on the list generated in the previous section, and prove a separation result of importance in our forthcoming paper [6]. The main technical tool is the one-sided substitute for injectivity of τ considered as a map on its two-sided shift space, cf. Lemma 1.2 below, which we shall be able to derive from the work of Mossé ([14]). In Section 4 we describe an algorithm for determining shift tail and orbit equivalence of the output of the algorithm described and proved in the previous section. The paper ends with a summary of the algorithm and a few remarks of relevance to related work.

1.2 Substitutions

We refer to [9], [8], and [17] for a thorough introduction to this subject and shall here only lay out notation. Letting $\mathfrak{a}$ denote a finite set or alphabet, we denote by $\mathfrak{a}^\sharp$ the set of nonempty finite words in $\mathfrak{a}$. For $w \in \mathfrak{a}^\sharp$, we let $|w|$ denote the number of letters and index the letters of w from 0 to $|w|-1$. A *substitution* is simply a map $\tau : \mathfrak{a} \longrightarrow \mathfrak{a}^\sharp$. We can extend τ to $\mathfrak{a}^\sharp$ or to

$$\mathfrak{a}^{\mathbb{Z}}, \mathfrak{a}^{\mathbb{N}_0}, \mathfrak{a}^{-\mathbb{N}} \tag{1}$$

(with $\mathbb{N}_0 = \{0, 1, \dots\}, -\mathbb{N} = \mathbb{Z}\backslash\mathbb{N}_0$) in the obvious way, and define powers of τ recursively. To define the action of τ on $\mathfrak{a}^{\mathbb{Z}}$ we need to specify that the word resulting from the substitution of the letter at index 0 of a doubly infinite sequence x will be placed starting at index 0 in $\tau(x)$. We thus have

$$\tau(y.x) = \tau(y).\tau(x)$$

where, as we will do in the following, we have used a dot to indicate the position separating $-\mathbb{N}$ and $\mathbb{N}_0$. We denote by τ^{-1} the *opposite substitution* defined by reversing each word $\tau(\cdot)$. Finally, an *abelianization matrix* A_τ is associated to τ as the $|\mathfrak{a}| \times |\mathfrak{a}|$-matrix counting at row b and column a the number of occurrences of b in $\tau(a)$.

We equip the sequence spaces mentioned in (1) with the product topology from the discrete topology on $\mathfrak{a}$, and define $\sigma : \mathfrak{a}^{\mathbb{Z}} \longrightarrow \mathfrak{a}^{\mathbb{Z}}$ by $(\sigma(x))_n = x_{n+1}$. Maps of this type we will refer to as *shift maps*. A *two-sided shift space* is a closed subset of $\mathfrak{a}^{\mathbb{Z}}$ which is mapped onto itself by σ. We shall refer to such spaces by "$\underline{\mathsf{X}}$" with possible subscripts. Generally speaking, a *one-sided shift space* is a closed subset of $\mathfrak{a}^{\mathbb{N}_0}$ or $\mathfrak{a}^{-\mathbb{N}}$ which is mapped into itself by the unique shift map. We are only interested in those one-sided shift spaces which can be produced from two-sided shift spaces by projection, and denote these spaces by X^+ and X^-, respectively. There is a rich theory of shift spaces; we refer to [12] and [11].

For $-\infty \leq i < j \leq \infty$ we use interval notation $x_{[i,j]}$ to denote the (possibly infinite) subword of x corresponding to the indices between i and j. We write $x_{[i,j[} = x_{[i,j-1]}$ when it makes sense and is convenient. Unless specified otherwise, we index finite words by nonnegative indices starting with 0, and right or left infinite words by $\mathbb{N}_0$ or $-\mathbb{N}$.

The *language* of a two-sided shift space is the subset of $\mathfrak{a}^\sharp \cup \{\epsilon\}$, where ϵ denotes the empty word, given by

$$\mathcal{L}(\underline{\mathsf{X}}) = \{x_{[i,j]} \mid x \in \underline{\mathsf{X}}, i \leq j \in \mathbb{Z}\}.$$

Conversely, a subset $\mathcal{G} \subseteq \mathfrak{a}^\sharp \cup \{\epsilon\}$ defines a shift space; the smallest shift space $\underline{\mathsf{X}}_{\mathcal{G}}$ such that $\mathcal{G} \subseteq \mathcal{L}(\underline{\mathsf{X}}_{\mathcal{G}})$. With $\mathcal{G} = \{\tau^n(a) \mid n \in \mathbb{N}, a \in \mathfrak{a}\}$ we arrive at the *substitutional dynamical systems* denoted $\underline{\mathsf{X}}_\tau$ which will be our main concern in the present paper.

We single out two important properties of substitutions below. The notation "$\mathbf{A} > 0$" indicates that the matrix $\mathbf{A}$ has only positive entries.

Definition 1.1 A substitution τ is *primitive* if $|\mathfrak{a}| > 1$ and

$$\exists n \in \mathbb{N} : \mathbf{A}_\tau^n > 0.$$

A substitution τ is *aperiodic* if $|\underline{\mathsf{X}}_\tau| = \infty$.

It is decidable when a given substitution has these properties, cf. [15] and [19]. Primitive and aperiodic substitutions yield *minimal* shift spaces: all orbits $\{\sigma^n(x) \mid n \in \mathbb{Z}\}$ are dense, cf. [17]. Consequently, there are no (ultimately) *σ-periodic words* in $\underline{\mathsf{X}}_\tau, \mathsf{X}_\tau^+$ or X_τ^- for such τ: if $x_{k+n} = x_{m+n}$ for all n in the various index sets, then $k = m$. Further, we have

Lemma 1.2 *[[14], cf. [8, Corollary 10]] The map induced by τ on $\underline{\mathsf{X}}_\tau$ is injective, when τ is primitive and aperiodic.*

Example 1.3 *The following substitutions are all primitive and aperiodic:*

$$\begin{aligned}
\tau_1 &: 1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 123;\\
\tau_2 &: 0 \mapsto 003210, 1 \mapsto 00, 2 \mapsto 00, 3 \mapsto 00220;\\
\tau_3 &: a \mapsto aba, b \mapsto baab;\\
\tau_4 &: 0 \mapsto 10, 1 \mapsto 0;\\
\tau_5 &: a \mapsto accdadbb, b \mapsto acdcbadb, c \mapsto aacdcdbb, d \mapsto accbdadb;\\
\tau_6 &: a \mapsto accbbadd, b \mapsto accdbabd, c \mapsto aacbbcdd, d \mapsto acbcdabd.
\end{aligned}$$

The following notation is convenient. When $w_0, \ldots, w_{n-1}$ is a finite list of words in $\mathcal{L}(\underline{\mathsf{X}}_\tau)$, we define

$$\begin{aligned}
[w_0, \ldots, w_{n-1}]^+ &= w_0\tau(w_1)\cdots\tau^{n-1}(w_{n-1})\tau^n(w_0)\tau^{n+1}(w_1)\cdots \in \mathfrak{a}^{\mathbb{N}_0},\\
[w_{n-1}, \ldots, w_0]^- &= \cdots\tau^{n+1}(w_1)\tau^n(w_0)\tau^{n-1}(w_{n-1})\cdots\tau(w_1)w_0 \in \mathfrak{a}^{-\mathbb{N}}.
\end{aligned}$$

1.3 Orbit classes and special elements

Definition 1.4 Let $\underline{\mathsf{X}}$ be a two-sided shift space. We define three equivalence relations on $x, y \in \underline{\mathsf{X}}$ in the following way:

(i) If there exists an n such that $x_m = y_{n+m}$ for all $m \in \mathbb{Z}$ then we say that x and y are *orbit equivalent* and write $x \sim_o y$.

(ii) If there exist an n and an N such that $x_m = y_{n+m}$ for all $m > N$ then we say that x and y are *right shift tail equivalent* and write $x \sim_r y$.

(iii) If there exist an n and an N such that $x_m = y_{n+m}$ for all $m < N$ then we say that x and y are *left shift tail equivalent* and write $x \sim_l y$.

Notice that $x \sim_o y$ implies that $x \sim_r y, x \sim_l y$, so $\sim_r$ and $\sim_l$ induce equivalence relations on $\underline{\mathsf{X}}/\sim_o$ which we also will denote by $\sim_r$ and $\sim_l$. We call an orbit class $[x]$ in $\underline{\mathsf{X}}/\sim_o$ *left special* ([10, ¶5]) if there exists $[y] \in \underline{\mathsf{X}}/\sim_o$

such that $[x] \neq [y]$, but $[x] \sim_r [y]$. A *left special word* $x \in \underline{\mathsf{X}}$ is a representative of such an orbit class with the property that $y \in \underline{\mathsf{X}}$ exists with

$$x_{-1} \neq y_{-1} \qquad x_{[0,\infty[} = y_{[0,\infty[}.$$

We say that the left special word x is *adjusted* if $\sigma^{-n}(x)$ is not left special for any $n \in \mathbb{N}$.

The symmetric definition defines a class of (adjusted) right special words. Classical results ([17, p. 107], [3, Theorem 3.9]) give:

Theorem 1.5 *When τ is aperiodic and primitive, then the number of (left or right) special orbit classes is finite, but nonzero.*

Note that as a consequence of this, there is always an adjusted special word representing each special orbit class. Clearly this word is unique.

A nice way of describing the structure of special words using the equivalence relations $\sim_r$ and $\sim_l$ on $\underline{\mathsf{X}}/\sim_o$, suggested to us by an electronic exchange with Charles Holton, is by means of a bipartite graph defined as follows. The vertex set of the graph will be contained in the disjoint union of $\underline{\mathsf{X}}/\sim_r$ and of $\underline{\mathsf{X}}/\sim_l$, and for each orbit class $[x]_o$ with x a special element, we let an edge connect $A \in \underline{\mathsf{X}}/\sim_r$ with $B \in \underline{\mathsf{X}}/\sim_l$ if $[x]_o \in A$ and $[x]_o \in B$, and we label that edge $[x]_o$. Removing all vertices with no edges, we arrive at a bipartite graph which we shall denote as the *configuration graph* of $\underline{\mathsf{X}}$. The theorem above shows that this is a finite graph when the shift space arises from a substitution. Examples are given in 4.7 below.

Lemma 1.6 *The configuration graph is an invariant of conjugacy and flow equivalence ([16]) of the substitutional dynamical systems.*

Proof: Since a conjugacy is a sliding block code (cf. [12, 1.5]), one easily sees that it must preserve special words and all the relevant equivalence relations. Similarly, any *expansion map* induced by

$$a_0 \mapsto a_0 b \qquad a_i \mapsto a_i, i > 0$$

sending biinfinite sequences on the alphabet $\mathfrak{a} = \{a_0, a_1, \ldots, a_n\}$ to biinfinite sequences on $\mathfrak{a} \cup \{b\}$ will take special words to special words in a manner preserving all the relations, and since the same can be said about the inverse of this map which deletes all occurrences of b, we get that expansion maps preserve configuration graphs. This proves the second claim since flow equivalence on shift spaces is generated by conjugacy and expansion according to [13, Lemma 2.1]. □

Note that the lemma implies that also the number of orbit or shift tail classes of special elements is a flow invariant.

2 Collecting special elements

2.1 Elementary and simplifiable substitutions

We recall from [18, p. 17] that a substitution τ on the alphabet $\mathfrak{a}$ is *simplifiable* if it can be factored $\tau = f \circ g$ for maps

$$f : \mathfrak{b} \longrightarrow \mathfrak{a}^\sharp \qquad g : \mathfrak{a} \longrightarrow \mathfrak{b}^\sharp$$

where $|\mathfrak{b}| < |\mathfrak{a}|$. We say that the substitution $\upsilon = g \circ f$ is a simplification of τ in this case. In case τ is not simplifiable, we call it *elementary.*

It is decidable whether a substitution is simplifiable or elementary, cf. [18, p. 17], and a succession of simplifications, ending with an elementary substitution, can be computed in the simplifiable case. Composing the $2n$ maps involved in a simplification in n steps to the elementary substitution υ, we get f, g with the property

$$\tau^n = f \circ g \qquad \upsilon^n = g \circ f. \tag{2}$$

This was used in [15] to provide an algorithm for deciding aperodicity by reducing to the elementary case. We shall use a similar strategy to compute the set of special elements for a given substitution, based on Proposition 2.2 below.

First, however, we need to concern ourselves with establishing our key substitution properties for simplifications. Simplifications preserve aperiodicity – this is a key observation in [15] – but a simplification of a primitive substitution may fail to be primitive. However, the following holds:

Lemma 2.1 *If a primitive and aperiodic substitution τ is simplified to an elementary substitution υ, then υ is primitive and aperiodic.*

Proof: It follows easily that A_τ and A_υ are strongly shift equivalent, cf. [12]. Note further that A_υ must be essential, as otherwise a letter could be deleted from the alphabet. Applying [12, Proposition 4.5.10], we get the desired result. □

Proposition 2.2 *Let τ be a primitive and aperiodic simplifiable substitution and let υ be an elementary simplification with maps f, g satisfying* (2) *above. The map from $\underline{\mathsf{X}}_\upsilon$ to $\underline{\mathsf{X}}_\tau$ induced by f preserves orbit and shift tail equivalences, and maps the (left, right) special orbits of $\underline{\mathsf{X}}_\upsilon$ bijectively onto the set of (left, right) special orbits of $\underline{\mathsf{X}}_\tau$.*

Proof: Clearly the maps induced by f and g preserve all three kinds of equivalence. Note also that they are injective because of Lemma 1.2; in the case of f because υ (and υ^n) is primitive by Lemma 2.1.

Clearly, then, both maps send special elements to special elements. Let $x_1, \ldots, x_n$ be a choice of orbit inequivalent special words of $\underline{\mathsf{X}}_\tau$, representing all such orbit classes. We have that $\tau(x_1), \ldots, \tau(x_n)$ are orbit inequivalent special words of $\underline{\mathsf{X}}_\tau$, since τ is injective and $\underline{\mathsf{X}}_\tau$ is aperiodic. Hence each orbit class of special elements is realized by a representative of the form $f(g(x_i))$, where $g(x_i)$ is special. □

Example 2.3 *The substitutions $\tau_1, \tau_3, \tau_4, \tau_5$ and τ_6 are elementary, but τ_2 is simplifiable to*

$$p \mapsto ppqp, q \mapsto pprrppppp, r \mapsto pp$$

using f given by $p \mapsto 0, q \mapsto 321, r \mapsto 2$ and g given by $0 \mapsto ppqp, 1 \mapsto pp, 2 \mapsto pp, 3 \mapsto pprrp$.

2.2 NS-covers and their graphs

In the following, we assume that the alphabet $\mathfrak{a}$ is equipped with some well-ordering ">"; in the examples, we just use alphabetical or numerical order.

Let $\mathcal{W}$ be a finite set of nonempty words. By $\mathcal{W}\widehat{\times}\mathcal{W}$ we denote the set

$$\{(v, w) \mid v, w \in \mathcal{W}, v_{|v|-1} > w_{|w|-1}\}$$

consisting of pairs of words from $\mathcal{W}$ which end in different letters, arranged so that the word ending in the first letter according to ">" is first among the two.

Definition 2.4 Let τ be a primitive and aperiodic substitution. We say that the finite family $\mathcal{W} \subseteq \mathcal{L}(\underline{\mathsf{X}}_\tau)$ is an NS-cover of τ (a nonsuffix cover) if

$$\mathsf{Cyl}^-(w) = \{x \in \underline{\mathsf{X}}_\tau \mid x_{[-|w|,-1]} = w\}, \qquad w \in \mathcal{W}$$

forms a disjoint partition of $\underline{\mathsf{X}}_\tau$, and if for every pair $(v, w) \in \mathcal{W}\widehat{\times}\mathcal{W}$ one can write

$$\tau(v) = tv'z \qquad \tau(w) = uw'z \tag{3}$$

where $t, u, z \in \mathcal{L}(\underline{\mathsf{X}}_\tau)$ with $t, u \neq \epsilon$, and where either

$$(v', w') \in \mathcal{W}\widehat{\times}\mathcal{W} \tag{+}$$

or

$$(w', v') \in \mathcal{W}\widehat{\times}\mathcal{W}. \tag{−}$$

Not every primitive and aperiodic substitution posseses an NS-cover – our example τ_2 provides an example of this behavior as seen in Example 5.2 below. However, the following shall suffice for our purposes:

Proposition 2.5 *If a primitive and aperiodic substitution is elementary, it posseses an NS-cover. Indeed, there is a computable integer n such that the set*

$$\{w \in \mathcal{L}(\underline{\mathsf{X}}_\tau) \mid |w| = n\}$$

is an NS-cover.

Proof: By [18, Theorem 1.6, p. 126], the integer

$$p = \sum_{a\in\mathfrak{a}}(|\tau(a)| - 1) + \max_{a\in\mathfrak{a}} |\tau(a)|$$

has the property that if words $v, w \in \mathcal{L}(\underline{\mathsf{X}}_\tau)$ end in different letters, and both $\tau(v)$ and $\tau(w)$ have the suffix z, then $|z| \leq p$.

By primitivity, one letter $a \in \mathfrak{a}$ has the property $|\tau(a)| \geq 2$, and we may find m such that a occurs in $\tau^m(b)$ for every $b \in \mathfrak{a}$, and with

$$n = 2(p + 1) \max\{|\tau^m(b)| \mid b \in \mathfrak{a}\}$$

we thus have that a occurs $p + 1$ times in v if $|v| = n$. Thus

$$|\tau(v)| \geq p + 1 + |v| = p + n + 1$$

for each such v. In (3), this leaves n letters to read off v', w' ending in different letters, and at least one more letter to read off t, u. □

Implementation remark 2.6 *In practice one finds that the value of* n *determined above is often much larger than needed. It is hence recommendable to simply try* $n = 1$, $n = 2$, *etc. until one reaches a sufficiently large length. We do not know whether the bound given is reachable.*

Example 2.7 *For the substitutions considered in Example 1.3, the smallest number* n *such that the set of all words in the associated language is an* NS*-cover is*

τ_1	τ_1^{-1}	υ_2	υ_2^{-1}	$\tau_3 = \tau_3^{-1}$	τ_4	τ_4^{-1}
2	3	3	4	1	4	2

Let now τ be a primitive and aperiodic substitution with an NS-cover $\mathcal{W}$. We define a multiply labeled graph $\mathcal{G}_{\tau,\mathcal{W}}$ of τ and $\mathcal{W}$ as follows. Choose as vertex set $\mathcal{V}_{\tau,\mathcal{W}} = \mathcal{W}\widehat{\times}\mathcal{W}$ and define for each $(v,w) \in \mathcal{W}\widehat{\times}\mathcal{W}$ a threefold labeled edge

$$(v', w') \xrightarrow{z,t,u} (v, w),$$

where v', w' and z, t, u are the (obviously unique) elements satisfying (3). Let $\mathcal{E}_\tau$ denote the set of all such edges with (v,w) ranging over $\mathcal{W}\widehat{\times}\mathcal{W}$ and define labelings

$$\mathfrak{L} : \mathcal{E} \longrightarrow \mathcal{L}(\underline{\mathsf{X}}_\tau), \mathfrak{L}_+ : \mathcal{E} \longrightarrow \mathcal{L}(\underline{\mathsf{X}}_\tau), \mathfrak{L}_- : \mathcal{E} \longrightarrow \mathcal{L}(\underline{\mathsf{X}}_\tau)$$

accordingly, associating z, t, u, respectively, to the edge in question. Finally, we label any edge of $\mathcal{G}_\tau$ by

$$\mathfrak{L}' : \mathcal{E} \longrightarrow \mathfrak{s} = \{+, -\},$$

according to which of the alternatives in Definition 2.4 is met. We will need to consider $\{+,-\}$ as the group $\mathbb{Z}_2$, but find this notation more suggestive.

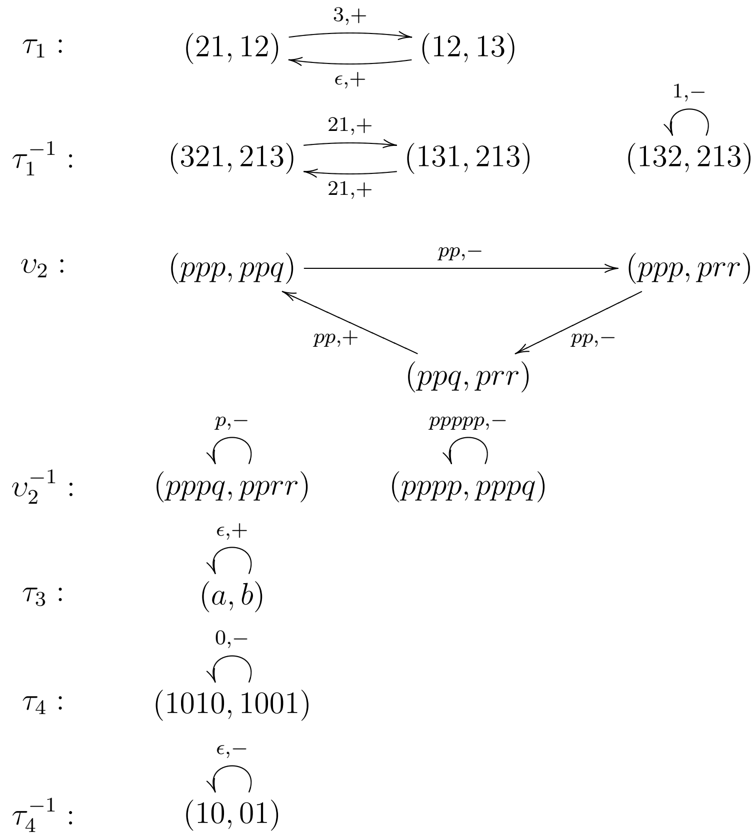
In the following definition, the *essential part* of a given graph is the subgraph defined by deleting all vertices which do not have both incoming and outgoing edges.

Definition 2.8 The graph $\mathcal{G}_{\tau,\mathcal{W}}$ is the essential part of $(\mathcal{V}_{\tau,\mathcal{W}}, \mathcal{E}_{\tau,\mathcal{W}})$ labeled by the restrictions of the labelings.

Corollary 2.9 *For every primitive and aperiodic substitution* τ *with an* NS*-cover* $\mathcal{W}$, $\mathcal{G}_{\tau,\mathcal{W}}$ *is a nonempty forest of cycles.*

Proof: By construction, each vertex of $\mathcal{G}_\tau$ has only one incoming edge. Since $\mathcal{G}_\tau$ is essential, each vertex has at least one, and thus exactly one, outgoing edge. We conclude that $\mathcal{G}_\tau$ is a forest of cycles. Since $(\mathcal{V}_\tau, \mathcal{E}_\tau)$ defined above has at least one cycle, so does $\mathcal{G}_\tau$. $\square$

Example 2.10 *For each substitution in Example 1.3 we state $\mathcal{G}_\tau$ associated to the NS-covers consisting of all words of a certain length, as found in Example 2.7. The graphs are decorated with $\mathfrak{L}, \mathfrak{L}'$.*



Lemma 2.11 *Let τ be a primitive and aperiodic substitution. When $\mathcal{W}$ is an NS-cover for τ, then it is also an NS-cover for τ^N for $N \geq 1$. The underlying graph of $\mathcal{G}_{\tau^N,\mathcal{W}}$ is identical to the higher power graph $(\mathcal{G}_{\tau,\mathcal{W}})^N$,*

with edges representing paths on $\mathcal{G}_{\tau\mathcal{W}}$ of length N. It is labeled by

$$\mathfrak{L}(e_0,\ldots,e_{N-1}) = \mathfrak{L}(e_0)\tau(\mathfrak{L}(e_1))\cdots\tau^{N-1}(\mathfrak{L}(e_{n-1})),$$
$$\mathfrak{L}'(e_0,\ldots,e_{N-1}) = \prod_{i=0}^{N-1}\mathfrak{L}'(e_i),$$

and

$$\mathfrak{L}_\pm(e_0,\ldots,e_{N-1}) = \tau^{N-1}(\mathfrak{L}_{s_{N-1}}(e_{N-1}))\cdots\tau(\mathfrak{L}_{s_1}(e_1))\mathfrak{L}_{s_0}(e_0),$$

where $s_i \in \{+,-\}$ is defined recursively by $s_0 = \pm$ and $s_{i+1} = \mathfrak{L}'(e_i)s_i$.

Proof: To see the first claim, note that when $(v,w) \in \mathcal{W}\widehat{\times}\mathcal{W}$

$$\tau^2(v) = \tau(tv'z) = \tau(t)\tau(v')\tau(z) = \tau(t)t'v''z'\tau(z),$$
$$\tau^2(w) = \tau(tw'z) = \tau(u)\tau(w')\tau(z) = \tau(u)u'w''z'\tau(z),$$

where (v',w') and (v'',w'') are elements of $\mathcal{W}\widehat{\times}\mathcal{W}$. This forms the basis of an induction argument proving that $\mathcal{W}$ is an NS-cover for τ^N. The remaining claims are straightforward. □

Proposition 2.12 *Let τ be a primitive and aperiodic substitution. There is an N such that $\mathcal{G}_{\tau^N,\mathcal{W}}$ is a forest of loops all labeled $+$ by $\mathfrak{L}'$.*

Proof: As seen in the proof of Corollary 2.9, $\mathcal{G}_{\tau,\mathcal{W}}$ is a forest of cycles. The power m defined as the least common multiple of all the lengths of cycles will lead to a graph with loops only. Then $N = 2m$ will suffice. □

We shall say that N is a $\mathcal{W}$*-basic power* of the primitive and aperiodic substitution τ (relative to the NS-cover $\mathcal{W}$) if $\mathcal{G}_{\tau^N,\mathcal{W}}$ meets the conditions of Proposition 2.12. Our result above proves that every primitive and aperiodic substitution with an NS-cover $\mathcal{W}$ has an $\mathcal{W}$-basic power. Suppose further that τ^{-1} has an NS-cover $\mathcal{W}'$. We say that N is an $\mathcal{W},\mathcal{W}'$-bibasic power if N is $\mathcal{W}$-basic for τ and $\mathcal{W}'$-basic for τ^{-1}.

A class of left special words on bracket form can be read of the graph thus associated to a $\mathcal{W}$-basic power of a substitution. Indeed, whenever (v,w) is a vertex in the graph, e is the loop at that vertex, and whenever $\mathfrak{L}(e) \neq \epsilon$ we have that

$$\tau^{mN}(v) = \tau^{N(m-1)}(\mathfrak{L}_+(e))\cdots\tau^N(\mathfrak{L}_+(e))\mathfrak{L}_+(e)v\mathfrak{L}(e)\tau^N(\mathfrak{L}(e))\cdots\tau^{N(m-1)}(\mathfrak{L}(e))$$
$$\tau^{mN}(w) = \tau^{N(m-1)}(\mathfrak{L}_-(e))\cdots\tau^N(\mathfrak{L}_-(e))\mathfrak{L}_-(e)w\mathfrak{L}(e)\tau^N(\mathfrak{L}(e))\cdots\tau^{N(m-1)}(\mathfrak{L}(e))$$

are words growing to infinity in both directions, as $\mathfrak{L}_+(e) \neq \epsilon, \mathfrak{L}_-(e) \neq \epsilon$ by definition of NS-covers. Thus both

$$[\overbrace{\epsilon, \ldots, \epsilon}^{N-1}, \mathfrak{L}_+(e)]^- v.[\mathfrak{L}(e), \overbrace{\epsilon, \ldots, \epsilon}^{N-1}]^+$$

and

$$[\overbrace{\epsilon, \ldots, \epsilon}^{N-1}, \mathfrak{L}_-(e)]^- w.[\mathfrak{L}(e), \overbrace{\epsilon, \ldots, \epsilon}^{N-1}]^+$$

are elements of $\underline{\mathsf{X}}_\tau$. Since v and w end in different letters, these elements are left special. We shall denote the class of such left special words by $\mathcal{S}_\mathcal{W}$ (lat. *sinister*). By considering opposite substitutions and reverting the output of the procedure described above, we get a set of right special elements which we denote by $\mathcal{D}_\mathcal{W}$ (lat. *dexter*).

Implementation remark 2.13 *It is a theoretical convenience to work with special words read off a graph associated to basic powers, but as a consequence of our construction the words may also be read off graphs associated to smaller powers, notably $N = 1$.*

When the graph is no longer a forest of $+$-labeled loops, one proceeds as follows. For each vertex (v, w) in $\mathcal{G}_{\tau,\mathcal{W}}$, one follows outgoing edges

$$e_0, \ldots, e_n$$

until e_n ends at (v, w). One defines $s_i \in \{+, -\}$ recursively by

$$s_0 = +, \qquad s_{i+1} = \mathfrak{L}'(e_i) s_i.$$

If $s_n = +$ one records

$$[\mathfrak{L}_{s_n}(e_n), \ldots, \mathfrak{L}_{s_0}(e_0)]^- v.[\mathfrak{L}(e_0), \ldots, \mathfrak{L}(e_n)]^+,$$
$$[\mathfrak{L}_{-s_n}(e_n), \ldots, \mathfrak{L}_{-s_0}(e_0)]^- w.[\mathfrak{L}(e_0), \ldots, \mathfrak{L}(e_n)]^+.$$

If $s_n = -$ one needs to consider

$$[\mathfrak{L}_{-s_n}(e_n), \ldots, \mathfrak{L}_{-s_0}(e_0), \mathfrak{L}_{s_n}(e_n), \ldots, \mathfrak{L}_{s_0}(e_0)]^- v.[\mathfrak{L}(e_0), \ldots, \mathfrak{L}(e_n)]^+,$$
$$[\mathfrak{L}_{s_n}(e_n), \ldots, \mathfrak{L}_{s_0}(e_0), \mathfrak{L}_{-s_n}(e_n), \ldots, \mathfrak{L}_{-s_0}(e_0)]^- w.[\mathfrak{L}(e_0), \ldots, \mathfrak{L}(e_n)]^+.$$

Obviously, we just get different – shorter – bracket representations of the elements of $\mathcal{S}_\mathcal{W}$ this way.

Example 2.14 *Reading off elements on the graphs found in Example 2.10 we get*

$$\begin{aligned}
\tau_1 :& \mathcal{S}_{\mathcal{W}_2} = \{[1,13]^-12.[\epsilon,3]^+, [12,12]^-13.[\epsilon,3]^+, [13,1]^-21.[3,\epsilon]^+, [12,12]^-12.[3,\epsilon]^+\};\\
& \mathcal{D}_{\mathcal{W}_3} = \{[12,12]^-.123[12,23]^+, [12,12]^-.312[13,13]^+,\\
&\qquad [12,12]^-.131[23,12]^+, [1,1]^-.231[213,312]^+,\\
&\qquad [1,1]^-.312[312,213]^+\};\\
\upsilon_2 :& \mathcal{S}_{\mathcal{W}_3} = \{[pqpppqp, pqpppqpppppp, pqp]^-ppp.[pp,pp,pp]^+,\\
&\qquad [pqp, pqpppqp, pqpppqpppppp]^-prr.[pp,pp,pp]^+,\\
&\qquad [pqpppqpppppp, pqp, pqpppqp]^-ppq.[pp,pp,pp]^+\};\\
& \mathcal{D}_{\mathcal{W}_4} = \{[ppppp, ppppp]^-.qppp[qpp, pqpppqpppqpp]^+,\\
&\qquad [ppppp, ppppp]^-.rrpp[pqpppqpppqpp, qpp]^+,\\
&\qquad [p,p]^-.pppp[rrpppqpppqpppqpp, qpppqpppqpp]^+,\\
&\qquad [p,p]^-.qppp[qpppqpppqpp, rrpppqpppqpppqpp]^+\};\\
\tau_3 :& \mathcal{S}_{\mathcal{W}_1} = \emptyset;\\
& \mathcal{D}_{\mathcal{W}_1} = \emptyset.\\
\tau_4 :& \mathcal{S}_{\mathcal{W}_4} = \{[0,0]^-1010.[0,0]^+, [0,0]^-1001.[0,0]^+\}\\
& \mathcal{D}_{\mathcal{W}_4} = \emptyset.
\end{aligned}$$

2.3 τ-periodic points

We call elements $y \in \mathsf{X}^+$, respectively $x \in \mathsf{X}^-$, *τ-periodic* when $\tau^n(y) = y$, respectively $\tau^n(x) = x$, for some $n \geq 1$. Let a be the last letter of a τ-periodic x and b the first letter of a τ-periodic y. If $ab \in \mathcal{L}(\underline{\mathsf{X}}_\tau)$, then because every finite subword of $x.y$ is contained in $\tau^{nk}(ab)$ for some k, we have that $x.y \in \underline{\mathsf{X}}_\tau$. And if another τ-periodic word $x' \in \mathsf{X}^-$ ends in $a' \neq a$ for which $a'b \in \mathcal{L}(\underline{\mathsf{X}}_\tau)$, then $x.y$ and $x'.y$ are left special elements.

The class $\mathcal{S}_p$ of left special elements obtained this way is computable. For a letter $a \in \mathfrak{a}$ gives rise to a τ-periodic word precisely when $\tau^n(a)$ begins or ends in a, and there is a computable smallest integer N such that all possible first and last letters are attained at some power $n \leq N$. Furthermore, the set of two-letter words of $\mathcal{L}(\underline{\mathsf{X}}_\tau)$ is computable.

Definition 2.15 For any NS-cover $\mathcal{W}$ we write

$$\mathcal{S}_{p\mathcal{W}} = \mathcal{S}_{\mathcal{W}} \cup \mathcal{S}_p.$$

The symmetric notation is applied to $\mathcal{D}$ as well.

We also note that such a left special element can be written on bracket form. Indeed,

$$x.y = [\overbrace{\epsilon, \ldots, \epsilon}^{N-1}, v]^{-} a.b[w, \overbrace{\epsilon, \ldots, \epsilon}^{N-1}]^{+}$$

where $\tau^N(a) = va$ and $\tau^N(b) = bw$. Note that $v, w \neq \epsilon$ by primitivity.

Example 2.16 *$\mathcal{S}_p$ and $\mathcal{D}_p$ are empty for $\tau_1, \tau_2, \tau_5, \tau_6$, but*

$$\begin{aligned} \tau_3 :& \mathcal{S}_p = \{[ab]^{-}a.a[ba]^{+}, [baa]^{-}b.a[ba]^{+}\}; \\ & \mathcal{D}_p = \{[ab]^{-}a.a[ba]^{+}, [ab]^{-}a.b[aab]^{+}\}; \\ \tau_4 :& \mathcal{S}_p = \emptyset \\ & \mathcal{D}_p = \{[1]^{-}0.0[10, \epsilon]^{+}, [1]^{-}0.1[0, \epsilon]^{+}\}. \end{aligned}$$

3 The structure of special words

In the previous section we defined two classes of left special words which we denoted by $\mathcal{S}_{\mathcal{W}}$ and $\mathcal{S}_p$, respectively, and let $\mathcal{S}_{p\mathcal{W}}$ denote their union. In the present section we are going to prove that $\mathcal{S}_{p\mathcal{W}}$ coincides with the set of left special words.

3.1 Auxiliary results

The following is a one-sided substitute for Lemma 1.2. It is proved using techniques from [14].

Lemma 3.1 *For a primitive and aperiodic substitution τ, let $x, y \in \underline{\mathsf{X}}_\tau$. If $\tau(x) \sim_r \tau(y)$, then $x \sim_r y$.*

Proof: We first note, as in [14], that there exists $M \in \mathbb{N}$ such that

$$\tau^M(a) = \tau^M(b) \Longleftrightarrow \tau^{M-1}(a) = \tau^{M-1}(b)$$

for any $a, b \in \mathfrak{a}$. Since τ is primitive, we may choose M so large that an element $u \in \underline{\mathsf{X}}_\tau$ has the property $\tau^M(u) = u$. As a consequence of [14], cf. [8, Corollary 12], we may choose $x', y' \in \underline{\mathsf{X}}_\tau$ such that

$$x \sim_o \tau^{M-1}(x') \qquad y \sim_o \tau^{M-1}(y').$$

By assumption, $\tau^M(x') \sim_r \tau^M(y')$. We fix $i, j \in \mathbb{Z}$ such that $\tau^M(x')_{i+n} = \tau^M(y')_{j+n}$ for all $n \in \mathbb{N}_0$.

We define for each $k \in \mathbb{N}_0$

$$a_k = |\tau^M(x'_{[0,k[})| - i \qquad b_k = |\tau^M(y'_{[0,k[})| - j$$

and let $A = \{a_0, a_1, \dots\}$, $B = \{b_0, b_1, \dots\}$. Our first goal is to prove that $A \cap [L, \infty[= B \cap [L, \infty[$ for an $L \in \mathbb{N}$ chosen according to Mossé's two-sided reckognizability property for τ^M. This property states that with $e_h^M = |\tau^M(u_{[0,h[})|$ and

$$u_{[e_h^M - L, e_h^M + L]} = u_{[l-L, l+L]}$$

then $l \in \{e_k^M \mid k \in \mathbb{N}_0\}$, cf. [14, Définition 1.2]. Hence let $k \in A, k \geq L$ and choose, using minimality, integers r, s such that

$$u_{[r, r+k+L+i]} = x'_{[0,k+L+i]} \qquad u_{[s,s+k+L+j]} = y'_{[0,k+L+j]}.$$

Then

$$\{h + e_r^M + i \mid h \in \{a_0, \dots, a_{k+i+L}\}\} = \{e_h^M \mid r \leq h \leq r + k + i + L\}, \quad (4)$$

and

$$\{h + e_s^M + j \mid h \in \{b_0, \dots, b_{k+j+L}\}\} = \{e_h^M \mid s \leq h \leq s + k + j + L\}. \quad (5)$$

Choose by (4) $h \in \{r, \dots, r + k + i + L\}$ such that $e_h^M = k + e_r^M + i$. Since

$$\begin{aligned} u_{[e_h^M - L, e_h^M + L]} &= u_{[e_r^M + k + i - L, e_r^M + k + i + L]} \\ &= \tau^M(x')_{[k+i-L, k+i+L]} \\ &= \tau^M(y')_{[k+j-L, k+j+L]} \\ &= u_{[e_s^M + k + j - L, e_s^M + k + j + L]} \end{aligned}$$

it follows by [14, Théorème 3.1 bis] that $e_s^M + k + j = e_h^M$ for some h, where obviously $s \leq h \leq s + k + j$. Using (5) we get that $k \in B$, as required to prove $A \cap [L, \infty[\subseteq B \cap [L, \infty[$. The symmetric argument proves the other inclusion.

Let

$$n = \min\{h \in \mathbb{N}_0 \mid a_h \geq L\} \qquad m = \min\{h \in \mathbb{N}_0 \mid b_h \geq L\}.$$

By what we have already proved, $\{a_n, a_{n+1}, \dots\} = \{b_m, b_{m+1}, \dots\}$, so

$$\begin{aligned} \tau^M(x'_{n+d}) &= \tau^M(x')_{[a_{n+d}+i, a_{n+d+1}+i[} \\ &= \tau^M(y')_{[a_{n+d}+j, a_{n+d+1}+j[} \\ &= \tau^M(y')_{[b_{m+d}+j, b_{m+d+1}+j[} \\ &= \tau^M(y'_{m+d}) \end{aligned}$$

for every $d \in \mathbb{N}_0$. By our initial assumption on M, also $\tau^{M-1}(x'_{n+d}) = \tau^{M-1}(y'_{m+d})$. Consequently, $x \sim_r y$, as desired. □

3.2 The main theorem

Lemma 3.2 *Let τ be a primitive and aperiodic substitution with an NS-cover $\mathcal{W}$, and assume that N is a $\mathcal{W}$-basic power for τ. Suppose that $x_1, x_2 \in \underline{\mathsf{X}}_\tau$ are elements of the form*

$$x_i = \widetilde{x_i} v_i . \tilde{x}$$

where $(v_1, v_2) \in \mathcal{W} \widehat{\times} \mathcal{W}$. Then there exist $y_1, y_2 \in \underline{\mathsf{X}}_\tau$ of the form

$$y_i = \widetilde{y_i} v_i . \tilde{y}$$

such that $x_i = \sigma^{-|\mathfrak{L}(e)|}(\tau^N(y_i))$, where e is the unique loop at $(v_1, v_2) \in \mathcal{G}_{\tau^N, \mathcal{W}}$.

Proof: Let x_i be of the described form. As a consequence of [14], cf. [8, Corollary 12], there exist $w_i \in \underline{\mathsf{X}}_\tau$, and integers m_i with $0 \leq m_i < |\tau^N((w_i)_0)|$ such that

$$x_i = \sigma^{m_i}(\tau^N(w_i)).$$

We get that

$$(\tau^N(w_1))_{[m_1, \infty[} = (x_1)_{[0,\infty[} = (x_2)_{[0,\infty[} = (\tau^N(w_2))_{[m_2, \infty[}$$

and since τ^N is also aperiodic and primitive, Lemma 3.1 applies to yield $n_i \in \mathbb{Z}$ such that

$$(w_1)_{[n_1, \infty[} = (w_2)_{[n_2, \infty[}. \tag{6}$$

We may and shall assume that among the pairs $(n_1, n_2) \in \mathbb{Z}^2$ satisfying (6), the sum n_1+n_2 is minimal. For if (n_1^j, n_2^j) satisfied (6) with $n_1^j+n_2^j \longrightarrow -\infty$, we could use the fact that there are no σ-periodic points in X_τ^+ to prove that $n_1^j - n_2^j$ is constant and then conclude that $w_1 \sim_o w_2$. This would lead to the contradiction $x_1 \sim_o x_2$.

Now by minimality

$$(w_1)_{n_1-1} \neq (w_2)_{n_2-1}.$$

Choose $\ell_i \in \mathbb{Z}$ and $u_1, u_2 \in \mathcal{W}$ such that

$$(w_i)_{[\ell_i,\infty[} = u_i\tilde{y}$$

where $\tilde{y} = (w_1)_{[n_1,\infty[} = (w_2)_{[n_2,\infty[}$. Since we are working with a $\mathcal{W}$-basic power N for τ, we have

$$\tau^N(u_i\tilde{y}) = t_i u_i z \tau^N(\tilde{y}),$$

and since this is a segment of x_i, we get that z begins at index 0. Consequently, $(u_1, u_2) = (v_1, v_2)$, and since $z = \mathfrak{L}(e)$, the result is established. □

Theorem 3.3 *Let τ be a primitive and aperiodic substitution with an NS-cover $\mathcal{W}$. If $x \in \underline{\mathsf{X}}_\tau$ is a left special word, then $x \in \mathcal{S}_{p\mathcal{W}}$.*

Proof: Choose a $\mathcal{W}$-basic power N for τ. Let $x_1, x_2 \in \underline{\mathsf{X}}_\tau$ be given with $(x_1)_{[0,\infty[} = (x_2)_{[0,\infty[}$, but $(x_1)_{-1} \neq (x_2)_{-1}$. By definition of NS-covers, $x_i \in \mathsf{Cyl}^-(v_i)$ for some unique $v_i \in \mathcal{W}$. Note that after interchanging x_1 and x_2 if necessary, we may assume that $(v_1, v_2) \in \mathcal{W}\widehat{\times}\mathcal{W}$. Apply Lemma 3.2 to get elements y_i and an edge e with the properties stated there, and note that these elements are also left special and satisfy that $y_i \in \mathsf{Cyl}^-(v_i)$.

When $|\mathfrak{L}(e)| = 0$, Lemma 3.2 can be iterated to get that

$$x_i \in \bigcap_{i=1}^{\infty} \tau^{iN}(\underline{\mathsf{X}}_\tau),$$

and this in turn implies that the x_i are τ-periodic. When $|\mathfrak{L}(e)| > 0$, we let $z = \mathfrak{L}(e)$ and note that Lemma 3.2 may be iterated to prove that

$$x_i = [\overbrace{\epsilon, \ldots, \epsilon}^{N-1}, t_i]^- u_i.[z, \overbrace{\epsilon, \ldots, \epsilon}^{N-1}]^+.$$

We note that this word lies in $\mathcal{S}_\mathcal{W}$ by construction. □

Remark 3.4 Note that when $|\mathfrak{L}(e)| > 0$ for all edges of $\mathcal{G}_{\tau^N,\mathcal{W}}$ the first case does not occur and every left special word can be found in $\mathcal{S}_{\mathcal{W}}$. This will always be the case when τ is *proper* in the sense that there exists $M \in \mathbb{N}$ and letters $r, l \in \mathfrak{a}$ such that every word $\tau^M(a), a \in \mathfrak{a}$ begins in l and ends in r. This leads to the following separation result of importance in our forthcoming paper [6].

Proposition 3.5 *Let τ be a proper, primitive and aperiodic substitution with an NS-cover $\mathcal{W}$. For every left special word x of X_τ^+ two words $v_1, v_2 \in \mathcal{W}$ may be chosen with the property*

$$\forall y \in \mathsf{X}_\tau^+ : v_1 y, v_2 y \in \mathsf{X}_\tau^+ \Longrightarrow x = y.$$

Proof: As seen in Remark 3.4, $x \in \mathcal{S}_{\mathcal{W}}$. Let (v_1, v_2) be a node of $\mathcal{G}_{\tau^N,\mathcal{W}}$, where N is a $\mathcal{W}$-basic power, from which we may initiate an infinite walk on $\mathcal{G}_{\tau^N,\mathcal{W}}$ with labels yielding x. Using Lemma 3.2 as in the proof of Theorem 3.3 we get that when $v_i y \in \mathsf{X}_\tau^+$, then y will coincide with the labels read off this infinite walk, and hence $y = x$. □

4 Deciding shift tail and orbit equivalence

We have managed to generate all special words of an aperiodic and primitive substitution. We have not yet, however, given an algorithm to decide which special words are adjusted. Similarly, since our method may output two or more elements which are orbit equivalent, or even identical, we have not yet explained how to count the number of orbit classes of special words or to compute the configuration graph. We solve these problems in the present section.

4.1 A decidable relation

In this section we consider the following auxiliary relation, prove that it is decidable on words on the form $[v]^- u.[w]^+$, and that it is closely related to right shift tail equivalence.

Definition 4.1 Let τ be aperiodic and primitive, and consider $x, y \in \underline{\mathsf{X}}_\tau$. We write $x \hookrightarrow y$ when there exists $n \in \mathbb{N}_0$ such that $x_{[0,\infty[} = y_{[n,\infty[}$.

We first see in Lemma 4.2 that it is decidable when $[v]^+ = ([w]^+)_{[k,\infty[}$ for k up to a certain predefined integer, and then pass to general k in Lemma 4.3.

Lemma 4.2 *Let τ be aperiodic and primitive, and let $v, w \in \mathcal{L}(\underline{\mathsf{X}}_\tau)\backslash\{\epsilon\}$ and $0 \leq k \leq |w|$ be given. We have*

$$[v]^+ = ([w]^+)_{[k,\infty[}$$

if and only if

$$v = w_{[k,|w|[}\tau(w_{[0,k[}).$$

Proof: If the equality of words holds, we get

$$\begin{aligned} [v]^+ &= [w_{[k,|w|[}\tau(w_{[0,k[})]^+ \\ &= w_{[k,|w|[}\tau(w_{[0,k[})\tau(w_{[k,|w|[}\tau(w_{[0,k[}))\tau^2(w_{[k,|w|[}\tau(w_{[0,k[}))\cdots \\ &= w_{[k,|w|[}\tau(w)\tau^2(w)\cdots. \end{aligned}$$

In the other direction, first note that if $[v]^+ = [w]^+_{[k,\infty[}$ and $0 \leq k \leq |w|$,

$$\begin{aligned} ([v]^+)_{[|v|+k,\infty[} &= \tau([v]^+)_{[k,\infty[} \\ &= \tau([w]^+_{[k,\infty[})_{[k,\infty[} \\ &= \tau([w]^+)_{[k+|\tau(w_{[0,k[})|,\infty[} \\ &= [w]^+_{[k+|\tau(w_{[0,k[})|+|w|,\infty[} \\ &= ([v]^+)_{[\tau(w_{[0,k[})|+|w|,\infty[} \end{aligned}$$

using the assumption on k in the third step. Since otherwise a subword of $[v]^+$ would be σ-periodic, the two segments must agree, so the length of v must coincide with the length of $w_{[k,|w|[}\tau(w_{[0,k[})$. Reading off letters from the left we get equality of the words themselves. □

Lemma 4.3 *Let τ be aperiodic and primitive, and consider $v, w \in \mathcal{L}(\underline{\mathsf{X}}_\tau)\backslash\{\epsilon\}$. If $[v]^+ \hookrightarrow [w]^+$ then*

$$v = (\tau^i(w)\tau^{i+1}(w))_{[j,j+|v|[}$$

where $i \in \mathbb{N}_0$ satisfies $|\tau^i(w)| \leq |v| \leq |\tau^{i+1}(w)|$ and $0 \leq j \leq |\tau^i(w)|$.

Proof: Assume that $[v]^+ = ([w]^+)_{[n,\infty[}$ and write

$$n = \sum_{k=0}^{i-1} |\tau^k(w)| + j$$

with $j \in \{0, \dots, |\tau^i(w)| - 1\}$. Then $[v]^+ = ([w]^+)_{[n,\infty[} = ([\tau^i(w)]^+)_{[j,\infty[}$, and by Lemma 4.2 applied with $\tau^i(w)$ in place of w we get that v is a subword of $\tau^i(w)\tau^{i+1}(w)$, and that

$$|v| = |\tau^i(w)| - j + |\tau(\tau^i(w)_{[0,j[})|.$$

Since $|\tau(\tau^i(w)_{[0,j[})| \geq j$ we have $|v| \geq |\tau^i(w)|$, and since we have, for any word u and any $\ell \leq |u|$,

$$\ell + |\tau(u_{[\ell,|u|[})| \geq |u|$$

we can apply this to $u = \tau^i(w)$ and get that

$$\begin{aligned} |\tau^{i+1}(w)| &= |\tau(\tau^i(w)_{[0,j[})| + |\tau(\tau^i(w)_{[j,|\tau^i(w)|[})| \\ &= |v| - |\tau^i(w)| + j + |\tau(\tau^i(w)_{[j,|\tau^i(w)|[})| \\ &\geq |v|, \end{aligned}$$

as desired. □

Note that when w and v are given in the lemma above, there is only a finite number of i satisfying

$$|\tau^i(w)| \leq |v| \leq |\tau^{i+1}(w)|.$$

Thus "$\hookrightarrow$" becomes decidable for elements of $\underline{\mathsf{X}}_\tau$ given on the form $[v]^-u.[w]^+$. To tie this in with right shift tail equivalence, we note:

Proposition 4.4 *Let τ be aperiodic and primitive, and consider a finite set $\mathcal{B} \subseteq \underline{\mathsf{X}}_\tau$ which contains all left special words of $\underline{\mathsf{X}}_\tau$. Then the equivalence relation induced by $\hookrightarrow$ on $\mathcal{B}$ coincides with right shift tail equivalence.*

Proof: Since the other implication is obvious, let $x, x' \in \mathcal{B}$ and assume that $x \sim_r x'$ to find a series of elements in $\mathcal{B}$, related by "$\hookrightarrow$", passing between x and x'. More precisely, assume that $x_{[m,\infty[} = x'_{[m',\infty[}$, where we may and shall assume that the pair (m, m') is chosen such that among pairs of nonnegative integers with this property, $m + m'$ is least possible. If $m = 0$

or $m' = 0$ we have $x \hookrightarrow x'$ or $x' \hookrightarrow x$. If $m, m' > 0$, we get by the minimality assumption that $x_{[m-1,\infty[} \neq x'_{[m'-1,\infty[}$ holds, whence $\sigma^m(x)$ is left special, and thus $x_{[m,\infty[} = x'_{[m',\infty[} = x''_{[0,\infty[}$ with $x'' \in \mathcal{B}$. Consequently, $x'' \hookrightarrow x$ and $x'' \hookrightarrow x'$, wherefrom it follows that x and x' are related by any symmetric and transitive relation extending "$\hookrightarrow$". □

Example 4.5 *For τ_4 our algorithm has produced a set*

$$\{[0,0]^-1010.[0,0]^+, [0,0]^-1001.[0,0]^+, [1]^-0.0[10,\epsilon]^+, [1]^-0.1[0,\epsilon]^+\}$$

of special words which we enumerate $x_1, \ldots, x_4$. Applying the results of the present section to τ_4^2 we get that $x_1 \hookrightarrow x_2, x_2 \hookrightarrow x_1, x_3 \hookrightarrow x_1, x_4 \hookrightarrow x_1$, but $x_3 \not\hookrightarrow x_4$ and $x_4 \not\hookrightarrow x_3$. This demonstrates that the symmetrized relation induced by "$\hookrightarrow$" is not an equivalence relation, and hence not the same as right shift tail equivalence.

4.2 Deciding equivalences

Theorem 4.6 *Let τ be a primitive and aperiodic substitution with an NS-cover $\mathcal{W}$, and with an NS-cover $\mathcal{W}'$ given for τ^{-1}. On a finite set $\mathcal{B}$ with*

$$\mathcal{S}_{p\mathcal{W}} \cup \mathcal{D}_{p\mathcal{W}'} \subseteq \mathcal{B} \subseteq \underline{\mathsf{X}}_\tau$$

of elements finitely presented on the form $[v]^-u.[w]^+$, right and left shift tail equivalence, as well as orbit equivalence, is decidable. Furthermore, it is decidable which special words are adjusted.

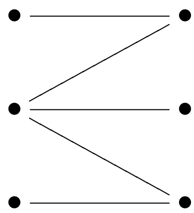
Proof: Let $x = [v]^-u.[w]^+$ and $x' = [v']^-u'.[w']^+$ denote generic elements of $\mathcal{B}$. We have seen in Proposition 4.4 that right shift tail equivalence is generated by a relation which is decidable by Lemma 4.3. By symmetry, the same is true for left shift tail equivalence. To decide orbit equivalence, we first decide whether the shift tail equivalences hold, noting that the algorithms described above provide us with integers satisfying

$$([v]^-)_{]-\infty,l]} = ([w']^-)_{]-\infty,l']} \qquad ([w]^+)_{[m,\infty[} = ([w']^+)_{[m',\infty[}.$$

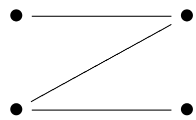
Checking orbit equivalence is hence reduced to comparing finite segments containing u and u'. The adjusted left special words are then those left special words x with the property that $y \hookrightarrow x$ for each other left special $y \in \mathcal{B}$ in the same orbit class. □

Example 4.7 *Our algorithm leads to the following configuration data graphs:*

τ_1 :

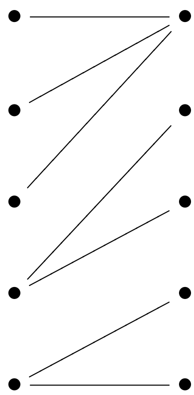


τ_3 :

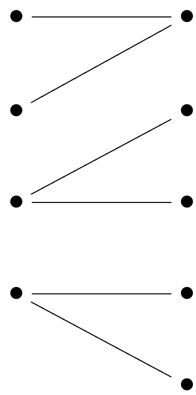


τ_4 :

τ_2, υ_2 :



τ_5, τ_6 :



5 Conclusion

In conclusion, our algorithm is laid out as follows. We have implemented it as a Java applet, see [5].

(a) Check that τ is aperiodic and primitive ([15],[19]).

(b) Decide whether τ is simplifiable or elementary ([18]).

(c) If τ is elementary, let $\upsilon = \tau$. If τ is simplifiable, compute a simplification υ of τ ([18]).

(d) Compute an integer such that $\mathcal{W}_n$ is an NS-cover for υ (2.5).

(e) Compute graphs $\mathcal{G}_{\upsilon,\mathcal{W}_n}$ and $\mathcal{G}_{\upsilon^{-1},\mathcal{W}_n}$ and read off sets $\mathcal{S}_{\mathcal{W}}$ and $\mathcal{D}_{\mathcal{W}'}$ (2.8, 2.13).

(f) Compute sets $\mathcal{S}_p$ and $\mathcal{D}_p$ (2.15).

(g) Determine shift tail and orbit equivalence among elements in $\mathcal{S}_{p\mathcal{W}} \cup \mathcal{D}_{p\mathcal{W}'}$ (4.6).

(h) If $\tau \neq \upsilon$, transfer special elements back to the alphabet of τ (2.2).

Remark 5.1 Obviously steps (c) and (h) are redundant when τ is already elementary. And as noted in Remark 3.4, we may skip step (f) and work directly with $\mathcal{S}_{\mathcal{W}}$ and $\mathcal{D}_{\mathcal{W}'}$ when $\mathfrak{L}(e) \neq \epsilon$ for all edges of $\mathcal{G}_{\tau^N,\mathcal{W}}$ and $\mathcal{G}_{(\tau^{-1})^N,\mathcal{W}'}$, when N is an $\mathcal{W},\mathcal{W}'$-bibasic power. This will always be the case when τ is *proper* in the sense defined there.

Example 5.2 *Applying step (h) of the algorithm one gets that two of the right special elements of* τ_2 *are* $[00000]^-.123000[12300, 012300012300012300]^+$ *and* $[00000]^-.2200[012300012300012300, 12300]^+$. *This proves that* τ_2 *has no* NS*-cover, for since* $[00000]^-1$ *and* $[00000]^-2$ *are mapped to the same sequence under* τ_2*, there are words of any length with this problematic behavior.*

Remark 5.3 Our paper [4] shows how finer data associated to special words may be used to distinguish the flow classes of τ_5 and τ_6 even though their configuration graphs coincide.

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Chapter 6

K_0 of the C^*-algebra associated to substitutional dynamical systems

This chapter consists of the paper *Augmenting dimension group invariants for substitution dynamics* in which the promised description of the K_0-group of the C^*-algebra associated to the one-sided shift space of a substitutional dynamical systems as a stationary inductive limit of finite abelian groups is given.

Notice that as in Chapter 4, the K_0-group of the C^*-algebra associated to the corresponding one-sided shift space is denoted by $K_0(\underline{\mathsf{X}})$, but that the K_0-group of the crossed product of a two-sided shift space is called *the dimension group* and is denoted by $DG(\underline{\mathsf{X}})$ (cf. [4]).

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Augmenting dimension group invariants for substitution dynamics

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Abstract. We present new invariants for substitutional dynamical systems. Our main contribution is a flow invariant which is strictly finer than, but related and akin to, the dimension groups of Herman, Putnam and Skau. We present this group as a stationary inductive limit of a system associated to an integer matrix defined from combinatorial data based on the class of *special words* of the dynamical system.

1. *Introduction*

The topics of *topological dynamics* and *operator algebras* are tied together in a way allowing fruitful bidirectional (although asymmetrical) transport of ideas from one area of research to another. The main benefit for operator algebras from this transport of ideas seems to be the definition of several important classes of C^*-algebras associated to dynamical systems. The main benefit for topological dynamics seems to be the discovery of conjugacy invariants, especially ordered groups arising from K-theory for operator algebras.

The contribution in the present paper is of the latter kind, based on a contribution by Matsumoto of the former. Indeed, computing the K-groups of C^*-algebras associated to certain shift spaces, we shall arrive at a flow (and hence conjugacy) invariant for these. This invariant is closely related to, but finer than, the dimension groups for substitutional shift spaces defined by Herman, Putnam and Skau in [**18**], and studied in this particular setting by Durand, Host and Skau in [**14**].

The ground-breaking work of Cuntz and Krieger [**13**] showed how to associate, in a natural and conjugacy invariant way, a C^*-algebra to a shift space of finite type.

There has been a large amount of attention in the operator algebra community to endeavors to generalize this construction, as the Cuntz-Krieger class holds a pivotal position in the theory of *purely infinite* C^*-algebras. Some work has taken the graph picture of such a shift space as a starting point of generalization to a non-finite setting. Other work, notably that of K. Matsumoto, has looked towards the full class of shift spaces on a finite alphabet. Indeed, in a series of papers [**24**]–[**28**] Matsumoto has managed to associate a certain C^*-algebra to *any* such shift space, and to gather much information about the algebras.

At the core of the interplay between operator algebras and dynamics lies an idea originating with Krieger to study the K-groups of the operator algebras in question, employing the fact that these will be invariants of the shift spaces when the C^*-algebras are. This idea allowed the realization of the dimension groups originating in Elliott's work [**15**] on AF algebras as the conjugacy invariant now well known.

Such a strategy has been successfully pursued in work of Matsumoto ([**24**]-[**29**]) (and of Krieger and Matsumoto ([**21**])) leading to a complete description of these K-groups which does not involve C^*-algebras, and to new insight in several important classes of shift spaces. Taking the vastness of the class covered by Matsumoto's work into account, it is no surprise that the best general description of such algebras — given in terms of "past equivalence" — is not readily computable. However, for the class of *substitutional dynamical spaces* which is the focus of the present paper, a very concrete description of this group can be given taking into account the ordered group arising as the K-theory of a completely different C^*-algebraic construction.

Indeed, such shift spaces will give rise to *minimal* topological dynamical systems, and as shown in work by Putnam [**33**] and Giordano, Putnam and Skau [**17**], the canonical *crossed product* associated hereto falls in a well studied class of C^*-algebras. This work was the starting point for work by Durand, Host and Skau [**14**] and by Forrest [**16**] leading to new and readily computable conjugacy invariants for such systems.

In the present paper, we shall compute the Matsumoto K-groups for any primitive and aperiodic substitution shift space in terms of an integer matrix giving rise to a dimension group through a standard inductive limit construction. The starting point of our work is the intermediate presentation of the K-groups given in [**10**] which then, in the present paper, leads to a complete description of the Matsumoto K-group as an inductive limit of a stationary system just like the K-groups considered in [**14**]. Indeed, this part of our computation is a (not completely trivial) adaptation of methods from that paper.

1.1. *A recurring example* Throughout the paper we shall use the substitutions

$$\tau(a) = accdadbb \qquad \tau(b) = acdcbadb$$
$$\tau(c) = aacdcdbb \qquad \tau(d) = accbdadb$$

and

$$\upsilon(a) = accbbadd \qquad \upsilon(b) = accdbabd$$
$$\upsilon(c) = aacbbcdd \qquad \upsilon(d) = acbcdabd$$

to illustrate the nature of our invariant, and to demonstrate how it is computed. The corresponding shift spaces are strong orbit equivalent and hence indistinguishable by the invariant of Durand, Host and Skau. This pair of examples is also resistant to the method of comparing the configuration of the special elements or asymptotic orbits (cf. [**1**]), suggested to us by Charles Holton. Indeed, the "configuration data" of all right or left tail equivalence classes of special elements (see [**10**]) are identical. From [**1**, 3.10] τ and υ are flow equivalent precisely when the derived substitutions τ^* and υ^* defined there are weakly equivalent. However, since computer experiments indicate that both τ^* and υ^* allow squares ww but no triples www in their respective languages, the method given in [**1**] for establishing flow inequivalence does not seem to work here.

Nevertheless, we can use our invariant to prove that the shift spaces associated to these two substitutions are not flow equivalent. We will return to this example in 2.9, 3.4, 3.6, 3.8 and 5.17 below.

1.2. *Acknowledgments* This work is the result of a long process starting when we were both visiting The Mathematical Sciences Research Institute, Berkeley, CA, in the fall of 2000. We wish to thank the Danish Science Research Counsil and Herborgs Fond for making this visit possible. We are also grateful to Klaus Thomsen for directing our attention to the class of shift spaces considered in the paper, and to Ian Putnam for hospitality and suggestions during a visit by the first author to University of Victoria. We are also grateful to Charles Holton for a productive email exchange during the process.

2. *Preliminaries and notation*

Let $\mathfrak{a}$ be a finite set of symbols, and let $\mathfrak{a}^\sharp$ denote the set of finite, nonempty words with letters from $\mathfrak{a}$. Thus with ϵ denoting the empty word, $\epsilon \notin \mathfrak{a}^\sharp$.

2.1. *Substitutions and shifts* We refer to [**14**] and [**34**] for an introduction to this subject. A *substitution* is simply a map

$$\tau : \mathfrak{a} \longrightarrow \mathfrak{a}^\sharp.$$

We can extend τ to $\mathfrak{a}^\sharp$ in the obvious way, and thereby define powers of τ recursively by

$$\tau^n(a) = \tau(\tau^{n-1}(a)).$$

We find the following notation convenient:

Definition 2.1. Let $v, w \in \mathfrak{a}^\sharp$. We say that v *occurs in* w and write

$$v \dashv w$$

when $w = w'vw''$ for suitable $w', w'' \in \mathfrak{a}^\sharp \cup \{\epsilon\}$.

For $w \in \mathfrak{a}^\sharp$, we call the number of letters in w the *length* and denote it $|w|$.

We set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $-\mathbb{N} = \mathbb{Z}\backslash\mathbb{N}_0$, and equip $\mathfrak{a}^{\mathbb{Z}}$ and $\mathfrak{a}^{\mathbb{N}_0}$ with the product topology from the discrete topology on $\mathfrak{a}$, and define $\sigma : \mathfrak{a}^{\mathbb{Z}} \longrightarrow \mathfrak{a}^{\mathbb{Z}}$ and $\sigma_+ : \mathfrak{a}^{\mathbb{N}_0} \longrightarrow \mathfrak{a}^{\mathbb{N}_0}$ by

$$(\sigma(x))_n = x_{n+1}, \qquad (\sigma_+(x))_n = x_{n+1}.$$

Such maps we will refer to as *shift maps.* A *two-sided shift space* is a closed subset of $\mathfrak{a}^{\mathbb{Z}}$ which is mapped onto itself by σ. We shall refer to such spaces by "$\underline{\mathsf{X}}$" with possible subscripts; note that $\sigma(\underline{\mathsf{X}}) = \underline{\mathsf{X}}$. A *one-sided shift space* is a closed subset of $\mathfrak{a}^{\mathbb{N}_0}$ which is mapped into itself by σ_+. We refer to such spaces by X^+, and remark that $\sigma_+(\mathsf{X}^+) \neq \mathsf{X}^+$ is possible. There is a rich theory of shift spaces; we refer to [**23**] and [**20**] and shall not give details here, but just establish notation. However, the method for describing such spaces by way of *languages* and *forbidden words* deserves explicit mentioning here.

We can further extend τ to $\mathfrak{a}^{\mathbb{N}_0}, \mathfrak{a}^{-\mathbb{N}}$ and $\mathfrak{a}^{\mathbb{Z}}$ in the obvious way. It is necessary in the last case, however, to specify that the word resulting from the substitution of the letter at index 0 of a doubly infinite sequence x will be placed starting at index 0 in $\tau(x)$. Using a dot to indicate the position separating $-\mathbb{N}$ and $\mathbb{N}_0$, as we will do in the following, we thus have

$$\tau(y.x) = \tau(y).\tau(x)$$

The *language* of a shift space is the subset of $\mathfrak{a}^\sharp \cup \{\epsilon\}$ given by

$$\mathcal{L}(\underline{\mathsf{X}}) = \{w \in \mathfrak{a}^\sharp \cup \{\epsilon\} \mid \exists x \in \underline{\mathsf{X}} : w \dashv x\}$$

(extending the notation "$\dashv$" in the obvious way). With the obvious restriction maps

$$\pi_+ : \underline{\mathsf{X}} \longrightarrow \mathfrak{a}^{\mathbb{N}_0},$$

we get

$$\sigma_+ \circ \pi_+ = \pi_+ \circ \sigma$$

and immediately note that $\pi_+(\underline{\mathsf{X}})$ is a one-sided shift space. Sometimes it is more suggestive to write

$$x_{[n,\infty[} = \pi_+(\sigma^n(x))$$

for $n \in \mathbb{Z}$.

Whenever $\mathcal{F} \subseteq \mathfrak{a}^\sharp$ is given, we define a two-sided shift space by

$$\underline{\mathsf{X}}_{\mathcal{F}} = \{(x_i) \in \mathfrak{a}^{\mathbb{Z}} \mid \forall i < j \in \mathbb{Z} : x_i \cdots x_j \notin \mathcal{F}\}.$$

One can prove that every two-sided shift space has such a description.

We say that shift spaces are *conjugate*, denoted by "$\simeq$", when they are homeomorphic via a map which intertwines the relevant shift maps. A *conjugacy invariant* is a mapping associating to a class of shift spaces another mathematical object, called the *invariant*, in such a way that conjugate shift spaces give isomorphic invariants.

The weaker notion of *flow equivalence* among two-sided shift spaces is also of importance here. This notion is defined using the *suspension flow space* of $(\underline{\mathsf{X}}, \sigma)$ defined as $S\underline{\mathsf{X}} = (\underline{\mathsf{X}} \times \mathbb{R})/\sim$ where the equivalence relation $\sim$ is generated by requiring that $(x, t+1) \sim (\sigma(x), t)$. Equipped with the quotient topology, we get a compact space with a *continuous flow* consisting of a family of maps (ϕ_t) defined by $\phi_t([x, s]) = [x, s+t]$. We say that two shift spaces $\underline{\mathsf{X}}$ and $\underline{\mathsf{X}}'$ are *flow equivalent* and write $\underline{\mathsf{X}} \cong_f \underline{\mathsf{X}}'$ if a homeomorphism $F : S\underline{\mathsf{X}} \longrightarrow S\underline{\mathsf{X}}'$ exists with the property that for every $x \in S\underline{\mathsf{X}}$ there is a monotonically increasing map $f_x : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$F(\phi_t(x)) = \phi'_{f_x(t)}(F(x)).$$

In words, F takes flow orbits to flow orbits in an orientation-preserving way. It is not hard to see that conjugacy implies flow equivalence.

We derive shift spaces from substitutions as follows:

Definition 2.2. With τ a substitution, we set

$$\mathcal{F}_\tau = \mathfrak{a}^\sharp \backslash \{w \in \mathfrak{a}^\sharp \mid \exists n \in \mathbb{N}, a \in \mathfrak{a} : w \dashv \tau^n(a)\}.$$

We abbreviate $\underline{\mathsf{X}}_{\mathcal{F}_\tau} = \underline{\mathsf{X}}_\tau$.

Clearly the maps derived from τ above sends $\underline{\mathsf{X}}_\tau$ back in itself.

2.2. *Classes of substitutions* In this section, we single out and discuss some important properties of substitutions:

Definition 2.3. A substitution τ is *primitive* if $|\mathfrak{a}| > 1$ and

$$\exists n \in \mathbb{N} \forall a, b \in \mathfrak{a} : b \dashv \tau^n(a).$$

Note that $\mathcal{F}_\tau = \mathcal{F}_{\tau^n}$ and (hence $\underline{\mathsf{X}}_\tau = \underline{\mathsf{X}}_{\tau^n}$) irrespective of $n \in \mathbb{N}$, when τ is primitive. Furthermore, in the primitive case, $\underline{\mathsf{X}}_\tau$ is *minimal* in the sense that every orbit $\{\sigma^m(x) \mid m \in \mathbb{Z}\}$ is dense, see [**34**, p. 90].

We are not interested in the case where $\underline{\mathsf{X}}_\tau$ is finite, and hence consider only the following class.

Definition 2.4. A substitution τ is *aperiodic* if $\underline{\mathsf{X}}_\tau$ is infinite.

The following concepts are useful in determining whether or not a substitution is aperiodic.

Definition 2.5. A substitution τ on the alphabet $\mathfrak{a}$ is *intertwined* with a substitution υ on the alphabet $\mathfrak{b}$ if $\tau = g \circ f$ and $\upsilon = f \circ g$ for some maps

$$f : \mathfrak{a} \longrightarrow \mathfrak{b}^\sharp \qquad g : \mathfrak{b} \longrightarrow \mathfrak{a}^\sharp.$$

We say that υ is a *simplification* of τ if $|\mathfrak{b}| < |\mathfrak{a}|$. In case τ has no simplification, we call it *elementary.*

It is computable whether a substitution is elementary or not, and there is an algorithmic way to produce a sequence of simplifications ending with an elementary substitution in the latter case, cf. [**35**, p. 17]. Since simplification preserves aperiodicity, this reduces the problem of deciding aperiodicity to the elementary case, in which it is readily decidable, cf. [**32**].

The final property we shall consider is perhaps less natural than the others:

DEFINITION 2.6. A substitution τ is *proper* if for some $\tau' : \mathfrak{a} \longrightarrow \mathfrak{a}^\sharp \cup \{\epsilon\}$,

$$\exists n \in \mathbb{N} \exists l, r \in \mathfrak{a} \forall a \in \mathfrak{a} : \tau^n(a) = l\tau'(a)r.$$

In [**14**, Proposition 20, Lemma 21] an algorithmic way is given for passing from a primitive and aperiodic substitution τ' to a primitive, aperiodic and proper substitution τ such that $\underline{\mathsf{X}}_{\tau'} \simeq \underline{\mathsf{X}}_\tau$. There is hence no restriction, when the goal is to discuss conjugacy or flow equivalence of aperiodic and primitive substitution shift spaces, in working with the proper ones among them.

Furthermore, when a proper, primitive and aperiodic substitution τ' is simplified to an elementary substitution τ'', the resulting substitution will also be proper, primitive and aperiodic. That properness is preserved after incrementing the power n in Definition 2.6 is obvious, and the other two claims are proved in [**8**, Lemma 2.1] and [**32**]. Observe that $\underline{\mathsf{X}}_{\tau'}$ and $\underline{\mathsf{X}}_{\tau''}$ may fail to be conjugate, but we do have

PROPOSITION 2.7. *If τ and υ are intertwined primitive substitutions, then $\underline{\mathsf{X}}_\tau \cong_f \underline{\mathsf{X}}_\upsilon$.*

Proof: Assume that $g \circ f = \tau$ and $f \circ g = \upsilon$ with notation as in Definition 2.5. We prove the claim by defining

$$F : S\underline{\mathsf{X}}_\tau \longrightarrow S\underline{\mathsf{X}}_\upsilon$$

by $F([x, s]) = [f(x), s|f(x_0)|]$ when $s \in [0, 1[$ and $x \in \underline{\mathsf{X}}_\tau$. Checking that F is defined and continuous is straightforward; we shall give details for injectivity and surjectivity of F.

Suppose first that $[f(x), s|f(x_0)|] = [f(y), t|f(y_0)|]$ for some $s, t \in [0, 1[$ and $x, y \in \underline{\mathsf{X}}_\tau$. By definition, there is an $n \in \mathbb{Z}$ with the property

$$\sigma^n(f(x)) = f(y) \qquad s|f(x_0)| = t|f(y_0)| + n.$$

Reversing the roles of x and y if necessary, we may assume that $n \geq 0$. Choose $m \in \mathbb{N}_0$ maximal with the property that

$$|\tau(x_{[0,m[})| \leq |g(f(x)_{[0,n[})|$$

and set $i = |g(f(x)_{[0,n[})| - |\tau(x_{[0,m[})|$. Since $0 \leq i < |\tau(x_m)|$ and

$$\begin{aligned} \sigma^i(\tau(\sigma^m(x))) &= \sigma^{|g(f(x)_{[0,n[})|}(\tau(x)) \\ &= \sigma^{|g(f(x)_{[0,n[})|}(g(f(x))) \\ &= g(\sigma^n(f(x))) \\ &= g(f(y)) \\ &= \tau(y), \end{aligned}$$

we get by Mossé recognizability ([**30**], cf. [**14**, Corollary 12]) that $i = 0$, $\sigma^m(x) = y$ and $|\tau(x_{[0,m[})| = |g(f(x)_{[0,n[})|$. Hence $|f(x_{[0,m[})| = n$. We conclude that $m = n = 0$ because if $m > 0$, then

$$s|f(x_0)| < |f(x_0)| \leq |f(x_{[0,m[})| = n,$$

which would make it impossible for $s|f(x_0)|$ to equal $t|f(y_0)| + n$. Hence $x = y$, and $s|f(x_0)| = t|f(y_0)|$ so that $s = t$ as desired.

To see that F is surjective, let $x \in \underline{\mathsf{X}}_\upsilon$ and $s \in [0, 1[$ be given. Choose $y \in \underline{\mathsf{X}}_\upsilon$ and $k \in [0, |\upsilon(y_0)|[$ such that $x = \sigma^k(\upsilon(y))$. Let

$$n = \max\{n' \in \mathbb{N}_0 \mid |f(g(y)_{[0,n'[})| \leq s + k\}$$

and

$$r = \frac{s + k - |f(g(y)_{[0,n[})|}{|f(g(y)_n)|}.$$

Then $r \in [0, 1[$, and

$$\begin{aligned} F([\sigma^n(g(y)), r]) &= [f(\sigma^n(g(y))), r|f(g(y)_n)|] \\ &= [\sigma^{|f(g(y)_{[0,n[}))|}(f(g(y))), r|f(g(y)_n)|] \\ &= [\sigma^{|f(g(y)_{[0,n[}))|}(\upsilon(y)), r|f(g(y)_n)|] \\ &= [\sigma^k(\upsilon(y)), s] = [x, s] \end{aligned}$$

because $k + s = |f(g(y)_{[0,n[}))| + r|f(g(y)_n)|$.

We have established that F is a homeomorphism. Since it obviously maps orbits to orbits in an orientation-preserving way, $\underline{\mathsf{X}}_\tau$ and $\underline{\mathsf{X}}_\upsilon$ are flow equivalent. □

COROLLARY 2.8. *If τ is simplified to τ', then $\underline{\mathsf{X}}_\tau \cong_f \underline{\mathsf{X}}_{\tau'}$.*

EXAMPLE 2.9. *The substitutions τ and υ are aperiodic, elementary, primitive, and proper on $\{a, b, c, d\}$.*

3. *Components of the invariant*

3.1. *Basic quantities* Fix a primitive and aperiodic substitution τ. In this section, we shall associate a collection of combinatoric data to τ which we shall employ in our theoretical work, as well as in our invariants, below.

As we are navigating mathematical waters close to known undecidable quantities, the reader might worry about computability of these data. Fortunately, we have been able to find efficient algorithms for computing all the data described below. The algorithms are suffiently simple that we have found ourselves capable of implementing them in a Java applet ([**9**]), and although we have not studied the complexity of these algorithms, we have found that they can compute invariants for substitutions such as τ and υ in a matter of seconds. A presentation of our algorithmic results, proved by methods partially related to [**2**], will appear elsewhere, in [**8**].

We say (cf. [**19**]) that $y \in \underline{\mathsf{X}}_\tau$ is *left special* if there exists $y' \in \underline{\mathsf{X}}_\tau$ such that

$$y_{-1} \neq y'_{-1} \qquad \pi_+(y) = \pi_+(y').$$

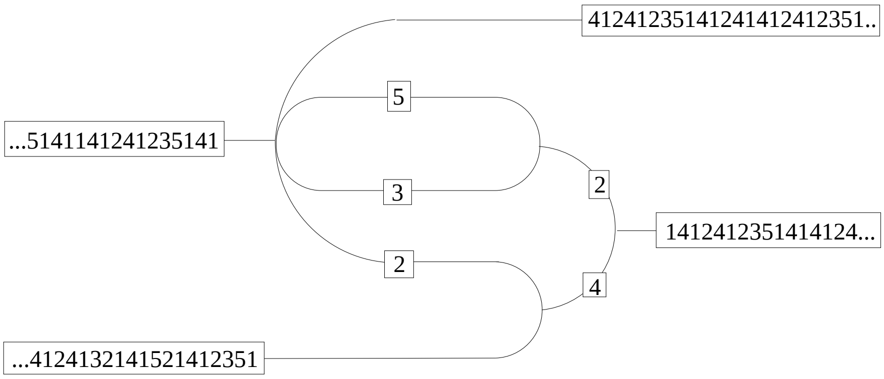

FIGURE 1. Special words for ω

By [**34**, p. 107] and [**3**, Theorem 3.9], there is a finite, but nonzero, number of left special words.

We say that the left special word y is *adjusted* if $\sigma^{-n}(y)$ is not left special for any $n \in \mathbb{N}$, and that y is *cofinal* if $\sigma^n(y)$ is not left special for any $n \in \mathbb{N}$. Thinking of left special words as those which are not deterministic from the right at index -1, the adjusted and cofinal left special words are those where this is the *leftmost* and *rightmost* occurrence of nondeterminacy, respectively.

Let $x, y \in \underline{\mathsf{X}}_\tau$. If there exist an n and an M such that $x_m = y_{n+m}$ for all $m > M$ then we say that x and y are *right shift tail equivalent* and write $x \sim_r y$. One defines right special elements using

$$y_0 \neq y'_0 \qquad \pi_-(y) = \pi_-(y'),$$

and left shift tail equivalence and $\sim_l$ in the obvious way. When a left special word y is cofinal, every word in its right shift tail equivalence class will end in $\pi_+(y)$.

REMARK 3.1. Quite often, all the special words of a substitution are simultaneously adjusted and cofinal. There are exceptions, though, as illustrated by Figure 1 which indicates the relations among all the special words of the aperiodic and primitive substitution ω on $\{1, 2, 3, 4, 5\}$ given by $\omega(1) = 123514, \omega(2) = 124, \omega(3) = 13214, \omega(4) = 14124, \omega(5) = 15214$. The element

$$...5141141241235141 52.1412412351414124...$$

is an example of a cofinal left special element which is not adjusted left special. Shifting it to

$$...51411412412351415.21412412351414124...$$

one achieves an element which is adjusted left special, but not cofinal. Shifting once more, one gets an element which is simultaneously adjusted and cofinal *right* special.

DEFINITION 3.2. When τ is an aperiodic and primitive substitution, we denote the number of right shift tail equivalence classes of left special elements of $\underline{\mathsf{X}}_\tau$ by n_τ.

It is not hard to see directly that this number is a flow invariant for substitutional systems, but in fact it will follow from our main result as noted in Theorem 6.2.

As described in Section 2.2 there is an algorithmic way of passing from a given aperiodic and primitive substitution τ' to an aperiodic, elementary, primitive and proper substitution τ in the same flow equivalence class. Now as concluded in [**8**, Remark 5.1] (cf. also [**1**], [**2**]), there is an integer n such that each left special word y can be represented (nonuniquely) as

$$\cdots \tau^{3n}(v)\tau^{2n}(v)\tau^{n}(v)vu.w\tau^{n}(w)\tau^{2n}(w)\tau^{3n}(w)\ldots$$

for some $n \in \mathbb{N}$ and some $v, u, w \in \mathcal{L}(\underline{\mathsf{X}}_\tau)\backslash\{\epsilon\}$ with the property that $\tau^n(u) = vuw$. Since $\underline{\mathsf{X}}_\tau = \underline{\mathsf{X}}_{\tau^n}$ we shall in the following, with no loss of generality, pass to τ^n and assume that $n = 1$. Our definition below becomes more complicated by the fact that elementarity does not always pass to powers.

DEFINITION 3.3. We say that a substitution τ is *basic* if it has the form $(\tau')^n$ for some aperodic, elementary, primitive and proper substitution τ', and if all its left special words have the form

$$\cdots \tau^{3}(v)\tau^{2}(v)\tau(v)vu.w\tau(w)\tau^{2}(w)\tau^{3}(w)\ldots \tag{3.1}$$

for suitable $u, v, w \in \mathcal{L}(\underline{\mathsf{X}}_\tau)\backslash\{\epsilon\}$ such that

$$\tau(u) = vuw. \tag{3.2}$$

As outlined above, there is an algorithm yielding for every aperiodic and primitive substitution υ a basic substitution $\tau = (\tau')^n$ with $\underline{\mathsf{X}}_\upsilon \cong_f \underline{\mathsf{X}}_\tau$. We may hence work only with basic substituions as long as we are interested in invariants of flow equivalence.

Our paper [**8**] provides algorithms for computing and representing each left special element in $\underline{\mathsf{X}}_\tau$ as in (3.1) and (3.2), to determine which of these elements are adjusted or cofinal, and which among them are right shift tail equivalent. Thus we may, in what follows, use the convenient notation

$$[w]^- = \cdots \tau^{n+1}(w)\tau^{n}(w)\tau^{n-1}(w)\cdots\tau(w)w \in \mathfrak{a}^{-\mathbb{N}}$$
$$[w]^+ = w\tau(w)\cdots\tau^{n-1}(w)\tau^{n}(w)\tau^{n+1}(w)\cdots \in \mathfrak{a}^{\mathbb{N}_0}$$

to describe all the (adjusted, cofinal) left special words.

EXAMPLE 3.4. *Both τ and υ are basic substitutions. The left special elements of $\underline{\mathsf{X}}_\tau$ are*

$$[accd]^-a.[dbb]^+, [aacd]^-c.[dbb]^+, [acdc]^-b.[adb]^+, [accb]^-d.[adb]^+.$$

which are all simultaneously adjusted and cofinal. The left special elements of $\underline{\mathsf{X}}_\upsilon$ are

$$[accbb]^-a.[dd]^+, [aacbb]^-c.[dd]^+, [acbc]^-d.[abd]^+, [accd]^-b.[abd]^+,$$

also all adjusted and cofinal. Since $[dbb]^+ \not\sim_r [adb]^+$ and $[dbb]^+ \not\sim_r [adb]^+$, $\mathsf{n}_\tau = \mathsf{n}_\upsilon = 2$.

DEFINITION 3.5. When τ is a basic substitution, equipped with some ordering of the right shift tail equivalence classes containing left special elements, we define $\mathsf{p}_\tau \in \mathbb{N}^{\mathsf{n}_\tau}$ by

$$\mathsf{p}_\tau = (p_1, \dots, p_{\mathsf{n}_\tau}),$$

where $p_i + 1$ is the number of adjusted left special words in each such class.

Note that by the definition of right special words, $p_i \geq 1$ for all i.

Enumerating the output of our algorithm we organize all the adjusted left special words as

$$\begin{gathered} \mathsf{y}_1^1, \mathsf{y}_2^1, \dots, \mathsf{y}_{p_1+1}^1 \\ \mathsf{y}_1^2, \mathsf{y}_2^2, \dots, \mathsf{y}_{p_2+1}^2 \\ \vdots \\ \mathsf{y}_1^{\mathsf{n}_\tau}, \mathsf{y}_2^{\mathsf{n}_\tau}, \dots, \mathsf{y}_{p_{\mathsf{n}_\tau}+1}^{\mathsf{n}_\tau} \end{gathered}$$

where

$$\mathsf{y}_k^j = [\mathsf{v}_k^j]^- \mathsf{u}_k^j.[\mathsf{w}_k^j]^+, \qquad \tau(\mathsf{u}_k^j) = \mathsf{v}_k^j \mathsf{u}_k^j \mathsf{w}_k^j.$$

Finally, we denote the last letter of each word u_k^j by a_k^j.

We further choose *one* cofinal left special element in each right tail equivalence class, and denote it $\widetilde{\mathsf{y}}^j$. As above, we write

$$\widetilde{\mathsf{y}}^j = [\widetilde{\mathsf{v}}^j]^- \widetilde{\mathsf{u}}^j.[\widetilde{\mathsf{w}}^j]^+, \qquad \tau(\widetilde{\mathsf{u}}^j) = \widetilde{\mathsf{v}}^j \widetilde{\mathsf{u}}^j \widetilde{\mathsf{w}}^j.$$

and denote by $\widetilde{\mathsf{a}}^j$ the last letter of $\widetilde{\mathsf{u}}^j$.

3.2. *Matrices* The number $\#[a, w]$ counts the number of occurrences of the letter a in the word w. As usual (cf. [**14**]) one associates to any substitution τ the *abelianization matrix* which is the $|\mathfrak{a}| \times |\mathfrak{a}|$-matrix $\mathbf{A}_\tau$ given by

$$(\mathbf{A}_\tau)_{a,b} = \#[b, \tau(a)].$$

EXAMPLE 3.6.

$$\mathbf{A}_\tau = \mathbf{A}_\upsilon = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{bmatrix}.$$

We shall define a rectangular matrix based on different data of the same nature. The reader may share our initial surprise that this definition will eventually lead to an invariant of conjugacy and flow equivalence.

DEFINITION 3.7. To a basic substitution τ one associates the $\mathsf{n}_\tau \times |\mathfrak{a}|$-matrix $\mathbf{E}_\tau$ given by

$$(\mathbf{E}_\tau)_{j,b} = \left(\sum_{k=1}^{p_j+1} e_{\tau, \mathsf{a}_k^j, \mathsf{w}_k^j}(b) \right) - e_{\tau, \widetilde{\mathsf{a}}^j, \widetilde{\mathsf{w}}^j}(b)$$

with

$$e_{\tau,a,w}(b) = \max(0, \#[b, \tau(a)] - \#[b, aw])$$

and with $\mathsf{a}_k^j, \widetilde{\mathsf{a}}^j$ and $\mathsf{w}_k^j, \widetilde{\mathsf{w}}^j$ given as in Section 3.1.

In all the applications of $e_{\tau,a,w}$ we either have that $\tau(a)$ is a proper subword of aw, in which case the term vanishes, or that $\tau(a)$ ends in aw, in which case the contribution of the term is a count of the remaining letters in $\tau(a)$

EXAMPLE 3.8. *Enumerating the elements given Example 3.6 in the order* $\mathsf{y}_1^1, \mathsf{y}_2^1 = \widetilde{\mathsf{y}}^1, \mathsf{y}_1^2, \mathsf{y}_2^2 = \widetilde{\mathsf{y}}^2$, *we would have*

$$\mathsf{E}_\tau = \begin{bmatrix} \#[\bullet, accd] + \#[\bullet, aacd] - \#[\bullet, aacd] \\ \#[\bullet, acdc] + \#[\bullet, accb] - \#[\bullet, accb] \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 1 & 0 & 2 & 1 \end{bmatrix}$$

and similarly

$$\mathsf{E}_\upsilon = \begin{bmatrix} \#[\bullet, accbb] \\ \#[\bullet, acbc] \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 & 0 \\ 1 & 1 & 2 & 0 \end{bmatrix}.$$

DEFINITION 3.9. To a basic substitution τ one associates the $(|\mathfrak{a}| + \mathsf{n}_\tau) \times (|\mathfrak{a}| + \mathsf{n}_\tau)$-matrix

$$\widetilde{\mathsf{A}}_\tau = \begin{bmatrix} \mathsf{A}_\tau & 0 \\ \mathsf{E}_\tau & \mathsf{Id} \end{bmatrix}.$$

4. *Matsumoto K-groups*

The *Matsumoto K-groups* with which we are concerned in the present paper can be efficiently defined directly, using the concept of past equivalence. They were, however, discovered as the (ordered) K-groups associated to certain classes of C^*-algebras. The results in the present paper do not depend directly or indirectly on an analysis of C^*-algebras, so we shall employ the most fundamental definition and repeat it for the benefit of the reader in section 4.1 below.

However, since our results were developed in this category and subsequently translated to a more basic setting, and since we do have further results (see Section 6) which we do not know how to get without this machinery, we find that a brief outline of how our work is positioned in an operator algebraic setting may be in order. We do this in section 4.3 below, which may be skipped by any reader not operator algebraically inclined.

4.1. *Past equivalence*

Let $\underline{\mathsf{X}}$ be a two-sided shift space. For every $x \in \pi_+(\underline{\mathsf{X}})$ and every $k \in \mathbb{N}$ we set

$$\mathcal{P}_k(x) = \{\mu \in \mathcal{L}(\underline{\mathsf{X}}) \mid \mu x \in \pi_+(\underline{\mathsf{X}}), |\mu| = k\},$$

and define for every $l \in \mathbb{N}$ an equivalence relation $\sim_l$ on $\pi_+(\underline{\mathsf{X}})$ by

$$x \sim_l y \Leftrightarrow \mathcal{P}_l(x) = \mathcal{P}_l(y).$$

Following Matsumoto ([**25**], [**27**]), we denote by $[x]_l$ the equivalence class of x and refer to the relation as *l-past equivalence.*

Obviously the set of equivalence classes of the l-past equivalence relation $\sim_l$ is finite. We will denote the number of such classes $m(l)$ and enumerate them $\mathcal{E}_s^l$ with $s \in \{1, \dots, m(l)\}$. For each $l \in \mathbb{N}$, we define an $m(l+1) \times m(l)$-matrix I^l by

$$(\mathsf{I}^l)_{rs} = \begin{cases} 1 & \text{if } \mathcal{E}_r^{l+1} \subseteq \mathcal{E}_s^l \\ 0 & \text{otherwise,} \end{cases}$$

and note that I^l induces a group homomorphism from $\mathbb{Z}^{m(l)}$ to $\mathbb{Z}^{m(l+1)}$. We denote by $\mathbb{Z}_{\underline{\mathsf{X}}}$ the group given by the inductive limit

$$\varinjlim(\mathbb{Z}^{m(l)}, \mathsf{I}^l).$$

For a subset $\mathcal{E}$ of $\pi_+(\underline{\mathsf{X}})$ and a finite word μ we let $\mu\mathcal{E} = \{\mu x \in \pi_+(\underline{\mathsf{X}}) \mid x \in \mathcal{E}\}$.

For each $l \in \mathbb{N}$ and $a \in \mathfrak{a}$ we define an $m(l+1) \times m(l)$-matrix

$$(\mathsf{L}_a^l)_{rs} = \begin{cases} 1 & \text{if } \emptyset \neq a\mathcal{E}_r^{l+1} \subseteq \mathcal{E}_s^l \\ 0 & \text{otherwise,} \end{cases}$$

and letting $\mathsf{L}^l = \sum_{a \in \mathfrak{a}} \mathsf{L}_a^l$ we get a matrix inducing a group homeomorphism from $\mathbb{Z}^{m(l)}$ to $\mathbb{Z}^{m(l+1)}$. Since one can prove that $\mathsf{L}^{l+1} \circ \mathsf{I}^l = \mathsf{I}^{l+1} \circ \mathsf{L}^l$, a group endomorphism λ on $\mathbb{Z}_{\underline{\mathsf{X}}}$ is induced.

Definition 4.1. [Cf. [**25**, Theorem 4.9], [**27**], [**29**, Theorem], [**5**, pp. 67-68]] Let $\underline{\mathsf{X}}$ be a two-sided shift space. The group

$$K_0(\underline{\mathsf{X}}) = \mathbb{Z}_{\underline{\mathsf{X}}} / (\mathrm{Id} - \lambda)\mathbb{Z}_{\underline{\mathsf{X}}},$$

is a conjugacy invariant of $\underline{\mathsf{X}}$ and $\pi_+(\underline{\mathsf{X}})$, and a flow invariant of $\underline{\mathsf{X}}$.

4.2. *An intermediate description* The dimension group $DG(X, \sigma)$ of a Cantor minimal system (X, σ) — a dynamical system where X is a Cantor set in which every σ-orbit is dense — is the cokernel of the map

$$\mathrm{Id} - (\sigma^{-1})^\sharp : C(X, \mathbb{Z}) \longrightarrow C(X, \mathbb{Z}),$$

equipped with the quotient order induced from $C(X, \mathbb{N}_0)$.

When τ is an aperiodic and primitive substitution, $(\underline{\mathsf{X}}_\tau, \sigma)$ is a Cantor minimal system. The technical basis of our results is a similar description of the Matsumoto K_0-group of a basic substitution as the cokernel of a certain map, based on a set of choices made as described in Section 3.1. Such a description can, for the special kind of shift spaces considered here, be inferred from a theoretically straightforward, but rather technical, analysis of l-past equivalence relation (or the lambda-graph) of the substitutional dynamics, noting that it may be correlated with the structure of the left special words. It is not the same cokernal description as the – much more general – basic tool in [**27**]. We defer the proofs of this to our paper [**10**], and shall here just present the results, laying out notation along the way.

Fix a basic substitution τ. We shall work extensively with elements of $\mathbb{Z}^{\mathsf{n}_\tau}$, and fix here notation for such. We shall prefer the index $j \in \{1, \dots, \mathsf{n}_\tau\}$ and write

$$\underline{x} = \left(x^j\right)_{j=1}^{\mathsf{n}_\tau}, \underline{y} = \left(y^j\right)_{j=1}^{\mathsf{n}_\tau}, \underline{x}_i = \left(x_i^j\right)_{j=1}^{\mathsf{n}_\tau},$$

etc., for such vectors. The vector $\underline{\delta}_{j_0}$ has zero entries except at index j_0, where the entry is 1.

We now define a group

$$G_\tau = C(\underline{\mathsf{X}}_\tau, \mathbb{Z}) \oplus \sum_{i=0}^{\infty} \mathbb{Z}^{\mathsf{n}_\tau}$$

and, based on a set of choices made as described in Section 3.1, a map $A_\tau : G_\tau \longrightarrow G_\tau$ given hereon by

$$A_\tau(f, [\underline{x}_0, \underline{x}_1, \dots]) = \left(f \circ \sigma^{-1}, \left[\left(\left(\sum_{k=1}^{p_j+1} f(\sigma^{-1}(\mathsf{y}_k^j)) \right) - f(\sigma^{-1}(\widetilde{\mathsf{y}}^j)) \right)_{j=1}^{\mathsf{n}_\tau}, \underline{x}_0, \underline{x}_1, \dots \right] \right),$$

with y_k^j and $\widetilde{\mathsf{y}}^j$ defined as in Section 3.1 above. The following result of [**10**] forms the basis of our alternative characterization of $K_0(\underline{\mathsf{X}}_\tau)$ involving G_τ and A_τ.

PROPOSITION 4.2. *When τ is a basic substitution, then $K_0(\underline{\mathsf{X}}_\tau)$ is isomorphic to the cokernel of the map A_τ.*

4.3. *Related C^*-algebras* There is a universal construction associating to most dynamical systems a C^*-algebra called the *crossed product*. In the seminal case of a $\mathbb{Z}$-action given by a homeomorphism ϕ of a compact Hausdorff space X, one first passes to the C^*-dynamical system of the transpose $\phi^\sharp$ acting on $C(X)$ by composition, and constructs therefrom a C^*-algebra denoted $C(X) \rtimes_{\phi^\sharp} \mathbb{Z}$ which captures the dynamics in non-commutative structure. A crossed product algebra $C(\underline{\mathsf{X}}) \rtimes_{\sigma^\sharp} \mathbb{Z}$ can hence be associated to each two-sided shift space, and the universality of the construction proves that such an associated C^*-algebra is a conjugacy invariant. But in this special case, another invariant C^*-algebra is available to us via the one-sided shift $\pi_+(\underline{\mathsf{X}})$.

The C^*-algebras first considered by Matsumoto can be constructed from such a one-sided shift space in several equivalent ways – by a universal construction based on generators and relations, or by invoking standard constructions in C^*-algebras based on either groupoids ([**7**]) or Hilbert C^*-bimodules ([**5**]). The original approach in [**24**] based on a Fock space construction may in some cases lead to a different algebra, see [**12**]. Each of these approaches have independent virtues and add to the accumulated value of this concept. When a one-sided shift space X^+ is given, we denote this C^*-algebra by $\mathcal{O}_{\mathsf{X}^+}$. Such C^*-algebras can be used to provide conjugacy invariants up to either one-sided or two-sided conjugacy as follows. Here and below, $\mathbb{K}$ denotes the C^*-algebra of compact operators on a separable Hilbert spaces.

THEOREM 4.3. *[[**5**, Theorem 4.1.4], [**6**]] Let X^+ and Y^+ be one-sided shift spaces. We have*

$$\mathsf{X}^+ \simeq \mathsf{Y}^+ \Longrightarrow \mathcal{O}_{\mathsf{X}^+} \simeq \mathcal{O}_{\mathsf{Y}^+}.$$

Furthermore, when $\underline{\mathsf{X}}$ and $\underline{\mathsf{Y}}$ are two-sided shift spaces, we have

$$\underline{\mathsf{X}} \simeq \underline{\mathsf{Y}} \Longrightarrow \underline{\mathsf{X}} \cong_f \underline{\mathsf{Y}} \Longrightarrow \mathcal{O}_{\pi_+(\underline{\mathsf{X}})} \otimes \mathbb{K} \simeq \mathcal{O}_{\pi_+(\underline{\mathsf{Y}})} \otimes \mathbb{K}.$$

This observation goes back to Matsumoto for a large family of shift spaces with the so-called *property (I)*, see [**24**, Proposition 5.8], [**28**, Corollary 6.2] and [**26**, Lemma 4.5]. However, the one-sided shift spaces associated to the two-sided shifts under investigation, those of the form $\pi_+(\underline{\mathsf{X}}_\tau)$, rarely have this property.

Definition 4.4. $\mathcal{O}_\tau = \mathcal{O}_{\pi_+(\underline{\mathsf{X}}_\tau)}$.

These C^*-algebras bear relevance for the groups considered in the present paper through a K-functor. In fact,

$$\begin{aligned} DG(\underline{\mathsf{X}}_\tau) &= K_0(C(\underline{\mathsf{X}}_\tau) \rtimes_{\sigma^\sharp} \mathbb{Z}) \\ K_0(\underline{\mathsf{X}}_\tau) &= K_0(\mathcal{O}_\tau). \end{aligned}$$

In general, the C^*-algebras $C(\underline{\mathsf{X}}) \rtimes_{\sigma^\sharp} \mathbb{Z}$ and $\mathcal{O}_{\pi_+(\underline{\mathsf{X}})}$ will be very different. For instance, when $\underline{\mathsf{X}} = \underline{\mathsf{X}}_{\mathcal{F}}$ where $\mathcal{F}$ is a finite set (a so-called shift of finite type), the crossed product will always have a very rich ideal structure, whereas the algebra considered by Matsumoto becomes the *Cuntz-Krieger* algebra associated to $\mathcal{F}$, which is a simple C^*-algebra under modest assumptions. When $\mathcal{F} = \mathcal{F}_\tau$, as proved in [**7**], there is an extension of C^*-algebras

$$0 \longrightarrow \mathbb{K}^{\mathsf{n}_\tau} \longrightarrow \mathcal{O}_\tau \longrightarrow C(\underline{\mathsf{X}}_\tau) \rtimes_{\sigma^\sharp} \mathbb{Z} \longrightarrow 0 \tag{4.3}$$

showing that $\mathcal{O}_\tau$ is non-simple, with the crossed product as a quotient.

Since $C(\underline{\mathsf{X}}_\tau) \rtimes_{\sigma^\sharp} \mathbb{Z}$ is simple because the underlying dynamical system is minimal, this gives a complete description of the ideal structure of $\mathcal{O}_\tau$. However, a reader unfamiliar with the extension theory of C^*-algebras should probably be explicitly warned that such a description offers very little concrete information about the algebra in general. In many cases, the theorem of Brown-Douglas-Fillmore in conjunction with the Universal coefficient theorem in Kasparov's theory proved by Rosenberg and Schochet shows that there are uncountably many nonisomorphic algebras having such a decomposition.

5. *Inductive limit descriptions*

A main accomplishment in [**14**] is the description of $DG(\underline{\mathsf{X}}_\tau)$ (see Section 4.2) as a stationary inducetive limit with matrices for the connecting maps read off directly from the substitution. A main result is the following.

Theorem 5.1. *[[**14**], Theorem 22(i)] There is an order isomorphism*

$$DG(\underline{\mathsf{X}}_\tau) \simeq \varinjlim(\mathbb{Z}^{|\mathfrak{a}|}, \mathbf{A}_\tau)$$

where each $\mathbb{Z}^{|\mathfrak{a}|}$ *is ordered by*

$$(x_a) \geq 0 \Longleftrightarrow \forall a \in \mathfrak{a} : x_a \geq 0.$$

We have found analogous results for the ordered group $K_0(\underline{\mathsf{X}}_\tau)$, but will in the present paper restrain ourselves to give, in Theorem 5.8 below, an inductive limit description of $K_0(\underline{\mathsf{X}}_\tau)$ as a group.

Computing the order structure requires a deeper analysis of the interrelations among certain C^*-algebras, employing the fact that $DG(\underline{\mathsf{X}}_\tau) = K_0(C(\underline{\mathsf{X}}_\tau) \rtimes_{\sigma^\sharp} \mathbb{Z})$ and $K_0(\underline{\mathsf{X}}_\tau) = K_0(\mathcal{O}_\tau)$, cf. Section 4.3. We defer this to [**11**], but the interested reader is referred to Section 6 for a brief overview of our results.

5.1. *Kakutani-Rohlin partitions* Theorem 5.1 is achieved from the cokernal description of the dimension group (see Section 4.2 above) using a sequence of Kakutani-Rohlin partitions of $\underline{\mathsf{X}}_\tau$ and direct computations of the actions hereupon by τ. We are going to follow the lead of [**14**], adapting crucial techniques to our somewhat more complicated setting. As in that paper, we abbreviate

$$[a] = \{x \in \underline{\mathsf{X}}_\tau \mid x_0 = a\},$$

and note that by [**14**, Corollary 13] — a consequence of the work by Mossé [**31**], [**30**] — the family of sets

$$\sigma^{-i}\tau^m[a], \qquad a \in \mathfrak{a} \quad, i \in \{0, \dots, |\tau(a)^m| - 1\}, \tag{5.4}$$

forms a (clopen) disjoint partition of $\underline{\mathsf{X}}_\tau$ for each $m \in \mathbb{N}$, when τ is any aperiodic substitution.

To set up notation and motivate our adaptation, we will sketch how the Kakutani-Rohlin partitions are used in [**14**] to prove Theorem 5.1 in the case of proper substitutions. We do this to allow references to parts of this proof in our proof of Theorem 5.8 below.

For any fixed $m \in \mathbb{N}$, we use the notation $\Xi = (\xi_{i,a})$ to denote a collection of integers where $a \in \mathfrak{a}, i \in \{0, \dots, |\tau^m(a)| - 1\}$. For each such collection, we define a function on $\underline{\mathsf{X}}_\tau$ by

$$f_\Xi = \sum_{a \in \mathfrak{a}} \sum_{i=0}^{|\tau^m(a)|-1} \xi_{i,a} 1_{\sigma^{-i}\tau^m[a]}.$$

Definition 5.2. Fix $m \in \mathbb{N}$. We define $\mathcal{CE}_\tau[m]$ as the set all integer collections defined above, and let

$$\mathrm{rk}_\tau[m] = \{f_\Xi \in C(\underline{\mathsf{X}}_\tau, \mathbb{Z}) \mid \Xi \in \mathcal{CE}_\tau[m]\}.$$

The subset of $\mathcal{CE}_\tau[m]$ with the further property that

$$\xi_{0,a} = \xi_{0,b} \qquad \forall a, b \in \mathfrak{a}$$

we denote by $\mathcal{CE}^c_\tau[m]$, and let $\mathrm{rk}^c_\tau[m]$ be the corresponding subspace of $\mathrm{rk}_\tau[m]$.

Our properness assumption enters our proof as follows, cf. [**14**, Proposition 14(iv)]:

Proposition 5.3. *If τ is a proper, primitive and aperiodic substitution, then*

$$\bigcup_{m=1}^{\infty} \mathrm{rk}_\tau[m] = \bigcup_{m=1}^{\infty} \mathrm{rk}^c_\tau[m] = C(\underline{\mathsf{X}}_\tau, \mathbb{Z}).$$

As the family generates $C(\underline{\mathsf{X}}_\tau, \mathbb{Z})$, the proof of Theorem 5.1 may be reduced to check that $\psi_{m+1} = \mathsf{A}_\tau \circ \psi_m$ where

$$\psi_m : \mathrm{rk}_\tau[m] \longrightarrow \mathbb{Z}^{|\mathfrak{a}|}, \qquad \psi_m\left(\sum_{a\in\mathfrak{a}} \sum_{i=0}^{|\tau^m(a)|-1} \alpha_{i,a} 1_{\sigma^{-i}\tau^m[a]}\right) = \left(\sum_{i=0}^{|\tau^m(a)|-1} \alpha_{i,\bullet}\right)_{\bullet\in\mathfrak{a}}$$

so that a map

$$\psi_\infty : C(\underline{\mathsf{X}}_\tau, \mathbb{Z}) \longrightarrow \varinjlim(\mathbb{Z}^{|\mathfrak{a}|}, \mathsf{A}_\tau)$$

is induced, and to check that this map is surjective and has the property that $\ker(\psi_\infty) = \mathrm{Im}(\mathrm{Id} - (\sigma^{-1})^\sharp)$. An isomorphism

$$\overline{\psi}_\infty : C(\underline{\mathsf{X}}_\tau, \mathbb{Z}) / \mathrm{Im}(\mathrm{Id} - (\sigma^{-1})^\sharp) \longrightarrow \varinjlim(\mathbb{Z}^{|\mathfrak{a}|}, \mathsf{A}_\tau)$$

is then induced.

We need to consider the interrelations between sets of the form $\sigma^{-n}\tau^{m+1}[a]$ and $\sigma^{-n'}\tau^m[a']$. Doing so is eased by the following perhaps somewhat counterintuitive notation, which we shall use for the remainder of Section 5.

Notation 5.4. *Let $w \in \mathfrak{a}^\sharp$. By $w^{[h]}$ we denote the letter at position h in w* from right to left, *starting with index* 0 *at the rightmost letter. By $w^{]h,0]}$ we denote the subword of w consisting of the h rightmost letters.*

It is straightforward (but tedious) to check that

$$\sigma^{-(|\tau^m(\tau(a)^{]h,0]})|+k)}\tau^{m+1}[a] \subseteq \sigma^{-k}\tau^m[\tau(a)^{[h]}] \tag{5.5}$$

for any $a \in \mathfrak{a}$, $m \in \mathbb{N}_0$, $h \in \{0, \ldots, |\tau(a)| - 1\}$ and $k \in \mathbb{N}_0$. Letting $k \in \{0, \ldots, |\tau^m(\tau(a)^{[h]})|\}$ one covers the sets in the $(m+1)$st level of the Rohlin-Kakutani partition exactly once. Consequently, $\mathrm{rk}_\tau[m] \subseteq \mathrm{rk}_\tau[m+1]$ and $\mathrm{rk}^c_\tau[m] \subseteq \mathrm{rk}^c_\tau[m+1]$.

We end this section by defining a numerical quantity associated to the kind of words used to describe right special elements and observing two basic properties of it:

Definition 5.5. For $w \in \mathcal{L}(\underline{\mathsf{X}}_\tau)$, and $m \in \mathbb{N}$ we set $\ell(m, w) = \sum_{i=0}^{m-1} |\tau^i(w)|$. We also let $\ell(0, w) = 0$.

Observation 5.6. When $[v]^-u.[w]^+ \in \underline{\mathsf{X}}$ for $u, v, w \in \mathcal{L}(\underline{\mathsf{X}}_\tau)\backslash\{\epsilon\}$ with $\tau(u) = vuw$, and u ends in $a \in \mathfrak{a}$,

$$[v]^-u.[w]^+ \in \sigma^{-\ell(m,w)}\tau^m[a]$$

for every $m \in \mathbb{N}_0$.

Proof: An inductive argument based on

$$\tau([v]^-u.[w]^+) = [\tau(v)]^-vuw.[\tau(w)]^+ = \sigma^{|w|}([v]^-u.[w]^+).$$

□

OBSERVATION 5.7. Let $u, v, w \in \mathcal{L}(\underline{\mathsf{X}}_\tau)\backslash\{\epsilon\}$ with $\tau(u) = vuw$, and assume that u ends in $a \in \mathfrak{a}$. For any $m \in \mathbb{N}_0$ and $h \leq |\tau(a)|$, we have

$$|\tau^m(\tau(a)^{]h,0]})| \geq \ell(m+1, w) + 1$$

if and only if $h \geq |w| + 1$.

Proof: Suppose first that $h \leq |w|$. Then $\tau(a)^{]h,0]} \dashv w$ so that

$$|\tau^m(\tau(a)^{]h,0]})| \leq |\tau^m(w)| \leq \ell(m+1, w).$$

Induction after m is required to prove the other implication, so assume that $h \geq |w| + 1$ and note that this assumption is equivalent with the case $m = 0$. For $m > 0$ we further note that $aw \dashv \tau(a)^{]h,0]}$ because of the way that u, a and w are interrelated. Thus

$$\begin{aligned} |\tau^m(\tau(a)^{]h,0]})| &\geq |\tau^m(a)| + |\tau^m(w)| \\ &\geq |\tau^{m-1}(\tau(a)^{]h,0]})| + |\tau^m(w)| \\ &\geq \ell(m, w) + 1 + |\tau^m(w)| \\ &= \ell(m+1, w) + 1 \end{aligned}$$

using the induction hypothesis at the third inequality sign. □

5.2. *A stationary inductive system* The main result of our paper is the following:

THEOREM 5.8. *Let τ be a basic substitution. There is a group isomorphism*

$$K_0(\underline{\mathsf{X}}_\tau) \simeq \varinjlim(\mathbb{Z}^{|\mathfrak{a}|} \oplus \mathbb{Z}^{\mathsf{n}_\tau}/\mathsf{p}_\tau\mathbb{Z}, \widetilde{\mathbf{A}}_\tau).$$

We recall that there is an algorithmic way of passing from any aperiodic and primitive substitution to one which is basic, staying in the same flow equivalence class. Since K_0 is an invariant of flow equivalence, the result above can be used to compute the Matsumoto K_0-group of any aperiodic and primitive substitution.

We note right away that the group $K_0(\underline{\mathsf{X}}_\tau)$ has the group $DG(\underline{\mathsf{X}}_\tau)$ computed in [**14**], as a quotient. The corresponding kernel is simply $\mathbb{Z}^{\mathsf{n}_\tau}/\mathsf{p}_\tau\mathbb{Z}$. But as we shall see, this extension is not split in general, making room for storage of additional information in the non-vanishing cross-term $\mathbf{E}_\tau$.

COROLLARY 5.9. *Let τ be a basic substitution. The short exact sequence*

$$0 \longrightarrow \mathbb{Z}^{\mathsf{n}_\tau}/\mathsf{p}_\tau\mathbb{Z} \xrightarrow{P} \mathbb{Z}^{|\mathfrak{a}|} \oplus \mathbb{Z}^{\mathsf{n}_\tau}/\mathsf{p}_\tau\mathbb{Z} \xrightarrow{R} \mathbb{Z}^{|\mathfrak{a}|} \longrightarrow 0$$

induces a short exact sequence

$$0 \longrightarrow \mathbb{Z}^{\mathsf{n}_\tau}/\mathsf{p}_\tau\mathbb{Z} \xrightarrow{P_\infty} K_0(\underline{\mathsf{X}}_\tau) \xrightarrow{R_\infty} DG(\underline{\mathsf{X}}_\tau) \longrightarrow 0\ .$$

Proof: Observe that $P \circ \mathrm{Id}_{\mathbb{Z}^{\mathsf{n}_\tau}/\mathsf{p}_\tau\mathbb{Z}} = \widetilde{\mathbf{A}}_\tau \circ P$ and $R \circ \widetilde{\mathbf{A}}_\tau = \mathbf{A}_\tau \circ R$. □

Definition 5.10. When $\Xi \in \mathcal{CE}_\tau[m]$, we define $\widetilde{\Xi}, \widehat{\Xi} \in \mathcal{CE}_\tau[m]$ by

$$\widetilde{\xi}_{i,a} = \begin{cases} \xi_{i+1,a} & 0 \le i < |\tau^m(a)| - 1 \\ \xi_{0,a} & i = |\tau^m(a)| - 1, \end{cases}$$
$$\widehat{\xi}_{i,a} = \sum_{k=i}^{|\tau^m(a)|-1} \xi_{k,a}.$$

Lemma 5.11. *If* $\Xi \in \mathcal{CE}^c_\tau[m]$ *then* $f_{\widetilde{\Xi}} = f_\Xi \circ \sigma^{-1}$.

Proof: Let c denote the mutual value at the lower level of Ξ. First note that if $x \in \sigma^{-i}\tau^m[a]$ is given with $i < |\tau^m(a)| - 1$, $\sigma^{-1}(x) \in \sigma^{-(i+1)}\tau^m[a]$. Further, if $x \in \sigma^{-(|\tau^m(a)|-1)}\tau^m[a]$, say with $x = \sigma^{-(|\tau^m(a)|-1)}(y)$ where $y \in \tau^m([a])$, we can write

$$\sigma^{-1}(x) = \sigma^{-|\tau^m(a)|}\tau^m(y) = \tau^m(\sigma^{-1}(y))$$

such that $\sigma^{-1}(x) \in \sigma^{-0}\tau^m[b]$ for b chosen as the second letter of y. Thus for any $x \in \underline{\mathsf{X}}_\tau$, we have

$$f_{\widetilde{\Xi}}(x) = \begin{cases} \xi_{i+1,a} & x \in \sigma^{-i}\tau^m[a], i < |\tau^m(a)| - 1 \\ c & x \in \sigma^{-i}\tau^m[a], i = |\tau^m(a)| - 1 \end{cases} = f_\Xi(\sigma^{-1}(x)).$$

□

Lemma 5.12. *If* $\Xi \in \mathcal{CE}_\tau[m]$ *and satisfies*

$$\sum_{i=0}^{|\tau^m(a)|-1} \xi_{i,a} = 0 \qquad \forall a \in \mathfrak{a}, \tag{5.6}$$

then $\widehat{\Xi} \in \mathcal{CE}^c_\tau[m]$*, and*

$$(\mathrm{Id} - A_\tau)(f_{\widehat{\Xi}}, [\underline{0}, \underline{0}, \ldots]) =$$
$$\left(f_\Xi, \left[\left(\sum_{i=\ell(m,\widetilde{\mathsf{w}}^j)+1}^{|\tau^m(\widetilde{\mathsf{a}}^j)|-1} \xi_{i,\widetilde{\mathsf{a}}^j} - \sum_{k=1}^{p_j+1} \sum_{i=\ell(m,\mathsf{w}^j_k)+1}^{|\tau^m(\mathsf{a}^j_k)|-1} \xi_{i,\mathsf{a}^j_k} \right)_{j=1}^{\mathsf{n}_\tau}, \underline{0}, \underline{0}, \ldots \right] \right).$$

Proof: By (5.6), $\widehat{\Xi} \in \mathcal{CE}^c_\tau[m]$. So we get by Lemma 5.11 that $f_{\widehat{\Xi}} \circ \sigma^{-1} = f_{\widetilde{\widehat{\Xi}}}$. By (5.6) again,

$$\widehat{\Xi} - \widetilde{\widehat{\Xi}} = \Xi.$$

Finally according to Observation 5.6,

$$f_{\widehat{\Xi}}(\sigma^{-1}(\mathsf{y}^j_k)) = \widehat{\xi}_{\ell(m,\mathsf{w}^j_k)+1,\mathsf{a}^j_k} = \sum_{i=\ell(m,\mathsf{w}^j_k)+1}^{|\tau^m(\mathsf{a}^j_k)|-1} \xi_{i,\mathsf{a}^j_k}$$

and similarly for $\widetilde{\mathsf{y}}^j$. □

We are now ready to define the family of maps which shall give the desired identification between $G_\tau/\operatorname{Im}(\mathrm{Id} - A_\tau)$ and a stationary inductive system.

Definition 5.13. The maps

$$\Psi_m : \mathrm{rk}_\tau[m] \oplus \sum_{i=0}^{m} \mathbb{Z}^{\mathsf{n}_\tau} \longrightarrow \mathbb{Z}^{|\mathfrak{a}|} \oplus \mathbb{Z}^{\mathsf{n}_\tau}/\mathsf{p}_\tau\mathbb{Z}$$

are given by

$$\Psi_m(f_\Xi, [\underline{x}_0, \underline{x}_1, \dots, \underline{x}_m]) =$$
$$\left(\sum_{i=0}^{|\tau^m(\mathsf{a}_k^j)|-1} \xi_{i,a}, \left(\sum_{i=0}^{m} x_i^j + \sum_{k=1}^{p_j+1} \sum_{i=\ell(m,\mathsf{w}_k^j)+1}^{|\tau^m(\mathsf{a}_k^j)|-1} \xi_{i,\mathsf{a}_k^j} - \sum_{i=\ell(m,\widetilde{\mathsf{w}}^j)+1}^{|\tau^m(\widetilde{\mathsf{a}}^j)|-1} \xi_{i,\widetilde{\mathsf{a}}^j} \right)_{j=1}^{\mathsf{n}_\tau} + \mathsf{p}_\tau\mathbb{Z} \right).$$

Note that Ψ_m is well-defined because $\sigma^{-i}\tau^m[a] \neq \emptyset$.

We have seen in Lemma 5.11 that $(\sigma^{-1})^\sharp$ maps $\mathrm{rk}_\tau^c[m]$ to $\mathrm{rk}_\tau[m]$. Therefore, A_τ restricts to a map

$$\mathrm{rk}_\tau^c[m] \oplus \sum_{i=0}^{m-1} \mathbb{Z}^{\mathsf{n}_\tau} \longrightarrow \mathrm{rk}_\tau[m] \oplus \sum_{i=0}^{m} \mathbb{Z}^{\mathsf{n}_\tau}$$

which we shall also denote by A_τ.

Proposition 5.14. *The sequence*

$$\mathrm{rk}_\tau^c[m] \oplus \sum_{i=0}^{m-1} \mathbb{Z}^{\mathsf{n}_\tau} \xrightarrow{\mathrm{Id}-A_\tau} \mathrm{rk}_\tau[m] \oplus \sum_{i=0}^{m} \mathbb{Z}^{\mathsf{n}_\tau} \xrightarrow{\Psi_m} \mathbb{Z}^{|\mathfrak{a}|} \oplus \mathbb{Z}^{\mathsf{n}_\tau}/\mathsf{p}_\tau\mathbb{Z}$$

is exact.

Proof: Direct computations, using among other things that there are p_j+1 positive ξ-terms and one negative ξ-term in the j entry of the second coordinate of the image of Ψ_m, show that $\Psi_m \circ (\mathrm{Id}-A_\tau) = 0$. And if

$$(f_\Xi, [\underline{x}_0, \dots, \underline{x}_m]) \in \ker \Psi_m,$$

then the conditions of Lemma 5.12 are met for Ξ, and $\widehat{\Xi} \in \mathcal{CE}_\tau^c[m]$. Note also that for suitable $c \in \mathbb{Z}$,

$$\sum_{i=0}^{m} x_i^j + \left(\sum_{k=1}^{p_j+1} \sum_{i=\ell(m,\mathsf{w}_k^j)+1}^{|\tau^m(\mathsf{a}_k^j)|-1} \xi_{i,\mathsf{a}_k^j} \right) - \sum_{i=\ell(m,\widetilde{\mathsf{w}}^j)+1}^{|\tau^m(\widetilde{\mathsf{a}}^j)|-1} \xi_{i,\widetilde{\mathsf{a}}^j} = p_j c$$

for each $j \in \{1, \ldots, \mathsf{n}_\tau\}$. With $C \in \mathcal{CE}^c_\tau[m]$ a constant scheme which each entry set to c, we have that $\widehat{\widehat{\Xi}} + C$ induces a function $g \in \mathrm{rk}^c_\tau[m]$ for which

$$\begin{aligned}
&(\mathrm{Id} - A_\tau)\left(g, \left[-\sum_{i=1}^{m} \underline{x}_i, -\sum_{i=2}^{m} \underline{x}_i, \ldots, -\sum_{i=m-1}^{m} \underline{x}_i, -\sum_{i=m}^{m} \underline{x}_i\right]\right) \\
&= \left(f_\Xi, \left[-\sum_{i=1}^{m} \underline{x}_i - \left(\sum_{k=1}^{p_j+1} \sum_{i=\ell(m,\mathsf{w}^j_k)+1}^{|\tau^m(\mathsf{a}^j_k)|-1} \xi_{i,\mathsf{a}^j_k} - \right.\right.\right. \\
&\left.\left.\left. \sum_{i=\ell(m,\widetilde{\mathsf{w}}^j)+1}^{|\tau^m(\widetilde{\mathsf{a}}^j)|-1} \xi_{i,\widetilde{\mathsf{a}}^j} + c\mathsf{p}_\tau\right), \underline{x}_1, \ldots, \underline{x}_m\right]\right) \\
&= (f_\Xi, [\underline{x}_0, \underline{x}_1, \ldots, \underline{x}_m]).
\end{aligned}$$

□

We shall work with the following basic elements of $\mathrm{rk}_\tau[m] \oplus \sum_{i=0}^{m} \mathbb{Z}^{\mathsf{n}_\tau}$. For each $\bullet \in \mathfrak{a}$, we let

$$e^m_\bullet = (1_{\tau^m\lceil\bullet\rceil}, [\underline{0}, \ldots, \underline{0}])$$
$$f^m_i = (0, [\underline{\delta}_i, \underline{0}, \ldots, \underline{0}])$$

with $\underline{\delta}_i$ referring to Kronecker delta. Further, we define a vector $\Delta_\bullet \in \mathbb{Z}^{|\mathfrak{a}|}$ using Kronecker delta again.

Lemma 5.15. *For each m,*

$$\Psi_m(e^m_\bullet) = (\Delta_\bullet, 0) \qquad \Psi_m(f^m_j) = (0, \delta_j + \mathsf{p}_\tau\mathbb{Z}),$$

and under the imbedding $\mathrm{rk}_\tau[m] \hookrightarrow \mathrm{rk}_\tau[m+1]$

$$\Psi_{m+1}(e^m_\bullet) = (\mathsf{A}\Delta_\bullet, \mathsf{E}\Delta_\bullet + \mathsf{p}_\tau\mathbb{Z}) \qquad \Psi_{m+1}(f^m_j) = (0, \delta_j + \mathsf{p}_\tau\mathbb{Z}).$$

Proof: The set of claims concerning f^m_j are straightforward; the second coordinate of $\Psi_m(e^m_\bullet)$ vanishes as described because evaluation begins at a nonzero index. To compute $\Psi_{m+1}(e^m_\bullet)$, we note that as a consequence of (5.5)

$$\tau^m\lceil\bullet\rceil = \bigcup_{\substack{a \in \mathfrak{a} \\ j \in \{1, \ldots, |\tau(a)|\} \\ \tau(a)^{[j]} = \bullet}} \sigma^{-|\tau^m(\tau(a)^{]j,0]})|}\tau^{m+1}[a].$$

This means that the element in $\mathcal{CE}_\tau[m+1]$ inducing the function of $e^m_\bullet$ is given by

$$\xi_{i,a} = \begin{cases} 1 & \exists h : \tau(a)^{[h]} = \bullet, i = |\tau^m(\tau(a)^{]h,0]})| \\ 0 & \text{otherwise.} \end{cases}$$

Now

$$\begin{aligned}
\sum_{i=0}^{|\tau^{m+1}(a)|-1} \xi_{i,a} &= \sum_{i=0}^{|\tau^{m+1}(a)|-1} \#\{h \mid \tau(a)^{[h]} = \bullet, i = |\tau^m(\tau(a)^{]h,0]})|\} \\
&= \#\{h \mid \tau(a)^{[h]} = \bullet\} \\
&= \#[\bullet, \tau(a)] = (\mathbf{A})_{a,\bullet}
\end{aligned}$$

and similarly

$$\begin{aligned}
\sum_{i=\ell(m+1,\mathsf{w}_k^j)+1}^{|\tau^{m+1}(\mathsf{a}_k^j)|-1} \xi_{i,\mathsf{a}_k^j} &= \sum_{i=\ell(m+1,\mathsf{w}_k^j)+1}^{|\tau^{m+1}(\mathsf{a}_k^j)|-1} \#\{h \mid \tau(\mathsf{a}_k^j)^{[h]} = \bullet, i = |\tau^m(\tau(a)^{]h,0]})|\} \\
&= \#\{h \mid \tau(\mathsf{a}_k^j)^{[h]} = \bullet, |\tau^m(\tau(\mathsf{a}_k^j)^{]h,0]})| \geq \ell(m+1,\mathsf{w}_k^j)+1\} \\
&= \#\{h \mid \tau(\mathsf{a}_k^j)^{[h]} = \bullet, h \geq |\mathsf{w}_k^j| + 1\}
\end{aligned}$$

according to Lemma 5.7. If $|\tau(\mathsf{a}_k^j)| \leq |\mathsf{w}_k^j| + 1$ this sum evaluates to 0, otherwise we get a count of the letter $\bullet$ in what is to the left of $\mathsf{a}_k^j\mathsf{w}_k^j$ in $\tau(\mathsf{a}_k^j)$, corresponding to our Definition 3.7. The same argument applies to $\widetilde{\mathsf{a}}^j$ and $\widetilde{\mathsf{w}}^j$. Thus

$$\left(\sum_{k=1}^{p_j+1} \sum_{i=\ell(m+1,\mathsf{w}_k^j)+1}^{|\tau^{m+1}(\mathsf{a}_k^j)|-1} \xi_{i,\mathsf{a}_k^j}\right) - \sum_{i=\ell(m,\widetilde{\mathsf{w}}^j)+1}^{|\tau^m(\widetilde{\mathsf{a}}^j)|-1} \xi_{i,\widetilde{\mathsf{a}}^j} = (\mathbf{E}_\tau)_{j,\bullet}$$

as desired. □

PROPOSITION 5.16. *The diagram*

$$\begin{array}{ccc}
\mathrm{rk}_\tau[m] \oplus \sum_{i=0}^{m} \mathbb{Z}^{\mathsf{n}_\tau} & \xrightarrow{\Psi_m} & \mathbb{Z}^{|\mathfrak{a}|} \oplus \mathbb{Z}^{\mathsf{n}_\tau}/\mathsf{p}_\tau\mathbb{Z} \\
\big\downarrow & & \big\downarrow \widetilde{\mathbf{A}} \\
\mathrm{rk}_\tau[m+1] \oplus \sum_{i=0}^{m+1} \mathbb{Z}^{\mathsf{n}_\tau} & \xrightarrow[\Psi_{m+1}]{} & \mathbb{Z}^{|\mathfrak{a}|} \oplus \mathbb{Z}^{\mathsf{n}_\tau}/\mathsf{p}_\tau\mathbb{Z}
\end{array}$$

commutes.

Proof: By Lemma 5.15 and the definition of $\widetilde{\mathbf{A}}_\tau$, the diagram commutes on $e_\bullet^m$ and f_j^m, and on the subgroup that they generate. For a general $(f, [\underline{x}_0, \ldots, \underline{x}_m]) \in \mathrm{rk}_\tau[m] \oplus \sum_{i=0}^m \mathbb{Z}^{\mathsf{n}_\tau}$ we note that Proposition 5.14 proves the claim as the images of $e_\bullet^m$ and f_j^m generate $\mathbb{Z}^{|\mathfrak{a}|} \oplus \mathbb{Z}^{\mathsf{n}_\tau}/\mathsf{p}_\tau\mathbb{Z}$. □

Proof of 5.8: By Proposition 5.14 and Proposition 5.16, the diagram

$$\begin{array}{ccccc}
\mathrm{rk}_\tau^c[m] \oplus \sum_{i=0}^{m-1} \mathbb{Z}^{\mathsf{n}_\tau} & \xrightarrow{\mathrm{Id}-A_\tau} & \mathrm{rk}_\tau[m] \oplus \sum_{i=0}^{m} \mathbb{Z}^{\mathsf{n}_\tau} & \xrightarrow{\Psi_m} & \mathbb{Z}^{|\mathfrak{a}|} \oplus \mathbb{Z}^{\mathsf{n}_\tau}/\mathsf{p}_\tau\mathbb{Z} \\
\big\downarrow & & \big\downarrow & & \big\downarrow \widetilde{\mathbf{A}} \\
\mathrm{rk}_\tau^c[m+1] \oplus \sum_{i=0}^{m} \mathbb{Z}^{\mathsf{n}_\tau} & \xrightarrow{\mathrm{Id}-A_\tau} & \mathrm{rk}_\tau[m+1] \oplus \sum_{i=0}^{m+1} \mathbb{Z}^{\mathsf{n}_\tau} & \xrightarrow[\Psi_{m+1}]{} & \mathbb{Z}^{|\mathfrak{a}|} \oplus \mathbb{Z}^{\mathsf{n}_\tau}/\mathsf{p}_\tau\mathbb{Z}
\end{array}$$

is commutative and exact for each m. Furthermore, since Lemma 5.15 shows that Ψ_m is surjective for each m, the rightmost horizontal maps in the diagram are surjections. Since taking inductive limits is an exact functor, we get that

$$G_\tau \xrightarrow{\mathrm{Id}-A_\tau} G_\tau \xrightarrow{\Psi_\infty} \varinjlim(\mathbb{Z}^{|\mathsf{a}|} \oplus \mathbb{Z}^{\mathsf{n}_\tau}/\mathsf{p}_\tau\mathbb{Z}, \widetilde{\mathbf{A}}_\tau) \longrightarrow 0$$

is exact, where we have used Lemma 5.3 to identify

$$\bigcup_{m\in\mathbb{N}} \left(\mathrm{rk}^c_\tau[m] \oplus \sum_{i=0}^{m-1} \mathbb{Z}^{\mathsf{n}_\tau} \right) = \bigcup_{m\in\mathbb{N}} \left(\mathrm{rk}_\tau[m] \oplus \sum_{i=0}^{m-1} \mathbb{Z}^{\mathsf{n}_\tau} \right) = G_\tau.$$

□

EXAMPLE 5.17. *The matrices*

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \qquad \begin{bmatrix} 2 & 0 \\ 2 & 0 \\ 2 & 0 \\ 2 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

induce maps $\chi : \mathbb{Z}^4 \oplus \mathbb{Z}^2/(1,1)\mathbb{Z} \longrightarrow \mathbb{Z}^2$ *and* $\eta : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^4 \oplus \mathbb{Z}^2/(1,1)\mathbb{Z}$ *with the property that* $\chi \circ \eta = \left[\begin{smallmatrix} 8 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ *and* $\eta \circ \chi = \widetilde{\mathbf{A}}_\tau$*; the latter since*

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 1 & 1 & 0 \\ 1 & 0 & 2 & 1 & 0 & 1 \end{bmatrix}$$

as a map from $\mathbb{Z}^4 \oplus \mathbb{Z}^2/(1,1)\mathbb{Z}$ *to* $\mathbb{Z}^2/(1,1)\mathbb{Z}$*. Similarly, we may reduce our description of* $K_0(\underline{\mathsf{X}}_\upsilon)$ *to a stationary system with* $\left[\begin{smallmatrix} 8 & 0 \\ 2 & 1 \end{smallmatrix}\right]$*.*

One now easily finds that

$$K_0(\underline{\mathsf{X}}_\tau) \simeq \mathbb{Z}[\frac{1}{2}] \oplus \mathbb{Z}$$

and, by $\left[\begin{smallmatrix} 8^{-k} & 0 \\ -2 & 7 \end{smallmatrix}\right] \left[\begin{smallmatrix} 8 & 0 \\ 2 & 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 8^{1-k} & 0 \\ -2 & 7 \end{smallmatrix}\right]$*, that*

$$K_0(\underline{\mathsf{X}}_\upsilon) = \{(8^{-k}x, 7y - 2x) \in \mathbb{Q}^2 \mid k \in \mathbb{N}, x, y \in \mathbb{Z}\}.$$

One sees that $K_0(\underline{\mathsf{X}}_\tau) \not\simeq K_0(\underline{\mathsf{X}}_\upsilon)$ *– and hence that* $\underline{\mathsf{X}}_\tau \not\cong_f \underline{\mathsf{X}}_\upsilon$ *– by proving that any element in* $K_0(\underline{\mathsf{X}}_\upsilon)$ *which is divisibe by any power of two is also divisible by seven. This is not the case for* $K_0(\underline{\mathsf{X}}_\tau)$*.*

6. *Finer invariants*

6.1. *Symmetrized invariants* Our focus on left special elements makes our invariant non-symmetric. It is easy to find examples of pairs of substitutions τ, υ which cannot be distinguished by our invariant, but such that their opposites τ^{-1}, υ^{-1} – the same substitutions, but read from right to left – can.

Thus a strictly finer flow invariant may be achieved by considering $K_0(\underline{\mathsf{X}}_\tau) \oplus K_0(\underline{\mathsf{X}}_{\tau^{-1}})$.

6.2. *Pointed groups* The K_0-group associated to a unital C^*-algebra posseses a distinguished element [1] corresponding to the unit of the C^*-algebra. This element is an invariant of isomorphism of such algebras, so according to Theorem 4.3 we have that $(K_0(\underline{\mathsf{X}}_\tau), [1])$ is an invariant of one-sided conjugacy of $\pi_+(\underline{\mathsf{X}}_\tau)$.

When τ is basic, [1] is the image of $(1, \dots, 1) \oplus 0$ from the first copy of $\mathbb{Z}^{|\mathfrak{a}|} \oplus \mathbb{Z}^{\mathsf{n}_\tau}/\mathsf{p}_\tau\mathbb{Z}$ in our description of $K_0(\underline{\mathsf{X}}_\tau)$. Since this distinguished element is not an invariant of flow equivalence we do not at present know how to compute it when τ is simplifiable.

6.3. *Ordered groups* The K_0-group associated to any C^*-algebra posseses a canonical order structure stemming from the fact that it is given as a Grothendieck group of a semigroup of equivalence classes of self-adjoint projections. The order structure may be degenerate in the sense that elements can be simultaneously positive and negative, but often holds important and natural information on the algebras in question.

In the case of crossed products associated to Cantor minimal systems, for instance, the order on the K_0-group is part of the complete invariant for (strong) orbit equivalence given in [**17**]. Similarly, since it can be given as the K_0-groups of a C^*-algebra, $K_0(\underline{\mathsf{X}}_\tau)$ has an order structure which is a flow invariant for the underlying substitutional dynamics.

In our paper [**11**] we give examples showing that this ordered group carries more information than the group itself, by proving that $K_0(\underline{\mathsf{X}}_\tau)$ may fail to be order isomorphic to $K_0(\underline{\mathsf{X}}_{\tau^{-1}})$, even though the K-groups are isomorphic as groups. We also give the following complete description of this ordered group:

Theorem 6.1. *Let τ be a basic substitution. There is an order isomorphism*

$$K_0(\underline{\mathsf{X}}_\tau) \simeq \varinjlim(\mathbb{Z}^{|\mathfrak{a}|} \oplus \mathbb{Z}^{\mathsf{n}_\tau}/\mathsf{p}_\tau\mathbb{Z}, \widetilde{\mathbf{A}}_\tau)$$

where each $\mathbb{Z}^{|\mathfrak{a}|} \oplus \mathbb{Z}^{\mathsf{n}_\tau}/\mathsf{p}_\tau\mathbb{Z}$ is ordered by

$$[(x_a), (y_i)] \geq 0 \iff \forall a \in \mathfrak{a} : x_a \geq 0.$$

This result shows in particular that the order on $K_0(\underline{\mathsf{X}}_\tau)$ is the quotient order induced by the order on $DG(\underline{\mathsf{X}}_\tau)$ via the map R_∞ considered in the proof of Theorem 5.9. As will be explained in [**11**], this phenomenon extends beyond substitutional shift spaces.

Let us quote another result from [**11**], stating the potentially finest invariant conceivable to us from our work above. Such an invariant can be extracted from the six term exacty sequence associated to the extension (4.3), which becomes

$$\begin{array}{ccccc} \mathbb{Z}^{\mathsf{n}_\tau} & \longrightarrow & K_0(\underline{\mathsf{X}}_\tau) & \longrightarrow & DG(\underline{\mathsf{X}}_\tau) \\ \uparrow & & & & \downarrow \\ \mathbb{Z} & \longleftarrow & 0 & \longleftarrow & 0. \end{array}$$

To describe the maps, apart from p_τ and R_∞, we use $Q : \mathbb{Z}^{\mathsf{n}_\tau} \longrightarrow \mathbb{Z}^{|\mathfrak{a}|} \oplus \mathbb{Z}^{\mathsf{n}_\tau}/\mathsf{p}_\tau\mathbb{Z}$ defined by

$$Q(\underline{x}) = (0, \underline{x} + \mathsf{p}_\tau\mathbb{Z}),$$

and its composition Q_1 with the canonical mapping from the first instance of $\mathbb{Z}^{|\mathfrak{a}|} \oplus \mathbb{Z}^{\mathsf{n}_\tau}/\mathsf{p}_\tau\mathbb{Z}$ in the inductive system to the inductive limit in our description of $K_0(\underline{\mathsf{X}}_\tau)$:

Corollary 6.2. *Let τ be a basic substitution. The exact complex*

$$\mathbf{K}_\tau : \qquad 0 \longrightarrow \mathbb{Z} \xrightarrow{\mathsf{p}_\tau} \mathbb{Z}^{\mathsf{n}_\tau} \xrightarrow{Q_1} K_0(\underline{\mathsf{X}}_\tau) \xrightarrow{R_\infty} DG(\underline{\mathsf{X}}_\tau) \longrightarrow 0,$$

where $\mathbb{Z}, \mathbb{Z}^{\mathsf{n}_\tau}, K_0(\underline{\mathsf{X}}_\tau)$ and $DG(\underline{\mathsf{X}}_\tau)$ should be considered as ordered groups and $\mathsf{p}_\tau, Q_1, R_\infty$ as positive homomorphisms, is a flow invariant of $\underline{\mathsf{X}}_\tau$.

6.4. *Open questions* It would be most interesting to know exactly which relation on the substitution shift spaces $\underline{\mathsf{X}}_\tau$ is induced by isomorphism of the stabilized algebra $\mathcal{O}_\tau \otimes \mathbb{K}$, or by isomorphism of the invariants mentioned above. Our examples above show that this relation is stronger than strong orbit equivalence, cf. [**17**]. There are classification results, notably those of Lin and Su ([**22**]), which could apply to the class of C^*-algebras in question, but we have not yet attempted to pursue this question.

As mentioned above, and documented in [**8**], the constituents of our invariants are effectively computable. However, this does not in itself lead to the conclusion that isomorphism of our invariants is decidable. Related work by Bratteli *et al* ([**4**]) proves decidability of the invariant which is complete for strong orbit equivalence – it would seem reasonable to expect that the result can be extended.

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6.8 Closing remarks

In the next chapter we show that the description of K_0 in fact also holds if we regard it as a preordered group.

It would be interesting to conduct a similar description of K_0 for other classes of shift spaces. Toeplitz flows seems as a natural candidate. They have property ($*$), but not necessarily property ($**$) (cf. Example 3.7 and 3.8 of Chapter 4), but there seems to be a connection between the l-past equivalence structure used to describe K_0 and the periodic structure of a Toeplitz flow (cf. [16]).

Chapter 7

Ordered K_0 of the C^*-algebra associated to certain one-sided shift spaces

We round off with the preprint *Ordered K-groups associated to substitutional dynamics* which is written together with Søren Eilers.

It contains a description of the order on K_0 of the C^*-algebra associated to the one-sided shift space corresponding to a two-sided shift space which has property $(*)$. It also proofs that the description of the K_0-group of the C^*-algebra associated to the one-sided shift space of a substitutional dynamical systems given in the previous chapter also holds in the category of preordered groups and it shows that the ordered K_0-group of the C^*-algebra associated to the one-sided shift space of a substitutional dynamical systems is a finer invariant than the K_0-group without order.

Noticed that in this paper the C^*-algebra associated to the one-sided shift space $\underline{\mathsf{X}}^+$ corresponding to a two-sided shift space $\underline{\mathsf{X}}$ is denoted $\mathcal{O}_{\underline{\mathsf{X}}}$.

Ordered K-groups associated to substitutional dynamics

Toke M. Carlsen and Søren Eilers

Final version, March 2004

1 Introduction

The Matsumoto K_0-group ([25]) is an invariant of flow equivalence (cf. [29]) associated to any shift space in a fashion closely related to the way the groups of Bowen and Franks are associated to shifts of finite type ([3]). This group is not always easily computable, but in a previous paper ([9]), we have given a concrete inductive limit description of the Matsumoto K_0-groups associated to substitutional shift spaces, proved that they contain the dimension groups of the system (cf. [15]), and demonstrated by examples that the Matsumoto K_0-group often carries more information than the dimension group.

Using the fact that the Matsumoto K_0-group can be defined as the K_0-group associated to a certain stabilized C^*-algebra ([23], [22], [13]) which is itself a flow invariant of the shift space, one sees that it carries an ordered structure which is also a flow invariant. It is well known in the theory of C^*-algebras associated to shifts of finite type, the so called Cuntz-Krieger algebras ([14]), that in this case the order structure is degenerate and redundant. The main goal of the present paper is to perform a further analysis of our previous description of the Matsumoto K_0-group of substitutional shift spaces, leading to a complete description of the order it carries as well (Theorem 6.1). In contrast to the case for shifts of finite type, we shall give an example proving that we hence arrive at a finer flow invariant than what the group, in itself, offers.

Many of the methods employed in this paper apply to a much wider class of shift spaces. However, we can not at present point to other classes of shift spaces where this ordered group is manifestly new and important. In some classes, like the class of Sturmian shift spaces ([17]), the order structure is already present in the well studied dimension group (Corollary 5.2). In other classes, like the class of Toeplitz flows ([34]) we conjecture that the order structure carries interesting information, but the group it sits in is too complex for us to understand it well enough to prove that this is in fact the case. Nevertheless, to pave the way for future applications, we shall state our main results (Theorems 4.3, 5.1) in the most general framework known to us, making only the requirements of properties $(*)$ and $(**)$ defined below. Outside of the class of shifts of finite type, which only meet these properties when they are finite, these properties can often be established. The properties are automatic for primitive substitutional systems.

Since we know of no way to define a flow invariant order structure on the Matsumoto K_0-groups without referring to its operator algebraic origin, we in an essential fashion need to work with C^*-algebras in the present paper. This is in contrast to our description of the Matsumoto K_0-group as a group ([9]), which could employ a more direct description of these groups using the past equivalence structure (or the closely related lambda-graph) of the shift spaces in question.

We wish to thank Ken Goodearl for suggesting to us where to look for the counterexample given in Section 6.

2 General preliminaries

Let $\mathbb{N}_0 = \{0, 1, \dots\}$ and $-\mathbb{N} = \mathbb{Z}\backslash\mathbb{N}_0$. We equip

$$\mathfrak{a}^{\mathbb{Z}}, \mathfrak{a}^{\mathbb{N}_0}, \mathfrak{a}^{-\mathbb{N}}$$

with the product topology from the discrete topology on $\mathfrak{a}$, and define $\sigma : \mathfrak{a}^{\mathbb{Z}} \longrightarrow \mathfrak{a}^{\mathbb{Z}}$, $\sigma_+ : \mathfrak{a}^{\mathbb{N}_0} \longrightarrow \mathfrak{a}^{\mathbb{N}_0}$, and $\sigma_- : \mathfrak{a}^{-\mathbb{N}} \longrightarrow \mathfrak{a}^{-\mathbb{N}}$ by

$$(\sigma(x))_n = x_{n+1} \qquad (\sigma_+(x))_n = x_{n+1} \qquad (\sigma_-(x))_n = x_{n-1}.$$

Such maps we will refer to as *shift maps.*

A *two-sided shift space* is a closed subset of $\mathfrak{a}^{\mathbb{Z}}$ which is mapped onto itself by σ. We shall refer to such spaces by "$\underline{\mathsf{X}}$" with possible subscripts; note that $\sigma(\underline{\mathsf{X}}) = \underline{\mathsf{X}}$. A *one-sided shift space* is either a closed subset of $\mathfrak{a}^{\mathbb{N}_0}$ which is mapped into itself by σ_+, or a closed subset of $\mathfrak{a}^{-\mathbb{N}}$ which is mapped into itself by σ_-. We refer to such spaces by X^+ and X^-, respectively.

With the obvious restriction maps

$$\pi_+ : \underline{\mathsf{X}} \longrightarrow \mathfrak{a}^{\mathbb{N}_0} \qquad \pi_- : \underline{\mathsf{X}} \longrightarrow \mathfrak{a}^{-\mathbb{N}}$$

we get

$$\sigma_+ \circ \pi_+ = \pi_+ \circ \sigma \qquad \sigma_- \circ \pi_- = \pi_- \circ \sigma^{-1}, \tag{1}$$

and immediately note that $\underline{\mathsf{X}}^+ = \pi_+(\underline{\mathsf{X}})$ and $\underline{\mathsf{X}}^- = \pi_-(\underline{\mathsf{X}})$ are one-sided shift spaces. In general, $\sigma_+(\mathsf{X}^+) \neq \mathsf{X}^+$ (or $\sigma_-(\mathsf{X}^-) \neq \mathsf{X}^-$) is possible, but as a consequence of (1) we always have $\sigma_+(\underline{\mathsf{X}}^+) = \underline{\mathsf{X}}^+$.

Let $\mathfrak{a}^\sharp$ be the set of finite nonempty words with letters from $\mathfrak{a}$, equipped with the length map $|\cdot| : \mathfrak{a}^\sharp \longrightarrow \mathbb{N}$. The *language* of a shift space is the subset of $\mathfrak{a}^\sharp \cup \{\epsilon\}$ given by

$$\mathcal{L}(\underline{\mathsf{X}}) = \{x_{[n,m]} \mid x \in \underline{\mathsf{X}}, n \leq m \in \mathbb{Z}\}$$

where the interval subscript notation should be self-explanatory. Each μ in the language gives rise to a non-empty *cylinder set*

$$\mathsf{Cyl}^+(\mu) = \{x \in \mathsf{X}^+ \mid x_{[0,|\mu|-1]} = \mu\}$$

Clearly these sets form a base for the topology of X^+. Similar bases can be given for the topologies of X^- and $\underline{\mathsf{X}}$, but we will not need this here.

We call $x, y \in \underline{\mathsf{X}}$ *right shift tail equivalent* and write $x \sim_r y$ when there exist $n, M \in \mathbb{Z}$ with

$$x_m = y_{n+m}, \qquad m \geq M.$$

We say that a pair of (one- or two-sided) shift spaces are *conjugate* when there is a homeomorphism between them which intertwines the relevant shift maps. We indicate conjugacy by the symbol "$\simeq$" and write "$\underline{\mathsf{X}} \cong_f \underline{\mathsf{Y}}$ when two-sided shift spaces are *flow equivalent* in the sense considered, e.g., in [29], [18], [4] or [21, §13.6]. As noted in Theorem 3.2 below there is a hierarchy among one-sided conjugacy, two-sided conjugacy and flow equivalence.

Let X^+ be a one-sided shift space. As in [24], for every $x \in \mathsf{X}^+$ and every $k \in \mathbb{N}$ we set

$$\mathcal{P}_k(x) = \{\mu \in \mathfrak{a}^k \mid \mu x \in \mathsf{X}^+\},$$

and write $x \sim_k y$ or $[x]_k = [y]_k$ if $\mathcal{P}_k(x) = \mathcal{P}_k(y)$. We say (cf. [20]) that $y \in \underline{\mathsf{X}}$ is *left special* if there exists $y' \in \underline{\mathsf{X}}$ such that

$$y_{-1} \neq y'_{-1} \qquad \pi_+(y) = \pi_+(y').$$

Definition 2.1 [12] We say that a shift space $\underline{\mathsf{X}}$ has *property* $(*)$ if for every $\mu \in \mathcal{L}(\underline{\mathsf{X}})$ there exists an $x \in \underline{\mathsf{X}}^+$ such that $\mathcal{P}_{|\mu|}(x) = \{\mu\}$.

Definition 2.2 [12] We say that a shift space $\underline{\mathsf{X}}$ has *property* $(**)$ if it has property $(*)$ and if the number of left special words of $\underline{\mathsf{X}}$ is finite, and no such left special word is periodic.

It is proved in [12, Example 3.6] that an infinite minimal shift space $\underline{\mathsf{X}}$ has property $(**)$ precisely when the number of left special words of $\underline{\mathsf{X}}$ is finite. Further, we see

Type of shift space	$(*)$	$(**)$	Reference in [12]
Finite shift	Yes	Yes	3.4
Infinite shift of finite type	No	No	3.5
Sturmian shift	Yes	Yes	5.21
Primitive substitutional shift	Yes	Yes	5.22 or 3.4
Non-regular Toeplitz flow	Yes	Not always	3.7, 3.8

3 C^*-algebras associated to shift spaces

In this section we shall introduce certain C^*-algebras and describe their K-theory. We recommend [28],[30] as general sources for the theory of C^*-algebras and [2],[32] for general sources of the K-theory of C^*-algebras.

One of the easiest K-theory computations in C^*-algebra theory is that of $C(X)$, where X is a zero-dimensional compact Hausdorff space. Indeed, one gets that $K_0(C(X)) = C(X,\mathbb{Z})$ in a way which associates to any class $[p] \in K_0(C(X))$ with p a *projection* in $C(X)$ the *function* p with values in $\{0,1\}$. The map thus induced is an order isomorphism, so we shall identify $K_0(C(X))$ and $C(X,\mathbb{Z})$ is such cases considered below.

Whenever a homeomorphism T of a compact Hausdorff space X is given, the adjoint action $T^\star$ on $C(X)$ gives rise to a C^*-algebraic crossed product $C(X) \rtimes_{T^\star} \mathbb{Z}$. There is a canonical $*$-homomorphism

$$\eta_\rtimes : C(X) \longrightarrow C(X) \rtimes_{T^\star} \mathbb{Z}$$

which we shall denote as indicated. When (X,T) is a two-sided shift space $(\underline{\mathsf{X}},\sigma)$, the ordered K_0-groups of such systems are completely described by the following result:

Theorem 3.1 *[[4, Theorem 5.2], cf. [19]] Let (X,T) be a dynamical system with X a zero-dimensional, metrizable and compact space. There exists an order isomorphism $\chi_{\rtimes}$ making the diagram*

$$\begin{array}{ccc} K_0(C(X)) & \xlongequal{\quad\quad\quad} & C(X,\mathbb{Z}) \\ {\scriptstyle (\eta_{\rtimes})_*}\downarrow & & \downarrow \\ K_0(C(X)\rtimes_{T^\star}\mathbb{Z}) & \xrightarrow[\chi_{\rtimes}]{} & C(X,\mathbb{Z})/\operatorname{Im}(\operatorname{Id}-(T^{-1})^\star) \end{array}$$

commute, where the rightmost map is the canonical quotient map.

Here and below, we use the notation $(\cdot)_*$ to indicate the group homomorphism functorially associated to a $*$-homomorphism. We shall use repeatedly that such induced maps are always positive.

Another C^*-algebra, introduced in the work by K. Matsumoto in [22] (cf. [13] and [7]), is available in the special case where (X,T) is a shift space $(\underline{\mathsf{X}},\sigma)$. This algebra can be defined in several equivalent ways – we shall briefly outline the construction based on Hilbert C^*-bimodules, see [6] for a detailed exposition.

Starting with a two-sided shift space $\underline{\mathsf{X}}$, we define a C^*-algebra

$$\mathcal{B}=\{f:\underline{\mathsf{X}}^+\longrightarrow\mathbb{C}\mid f\text{ is bounded}\}.$$

Inside this algebra, we have elements

$$g_\mu \quad=\quad 1_{\sigma_+^{|\mu|}(\mathsf{Cyl}^+(\mu))},\qquad \mu\in\mathcal{L}(\underline{\mathsf{X}})$$

and we will define a C^*-subalgebra $\mathcal{A}_{\underline{\mathsf{X}}}$ of $\mathcal{B}$ by

$$\mathcal{A}_{\underline{\mathsf{X}}}=C^*(g_\mu\mid\mu\in\mathcal{L}(\underline{\mathsf{X}}))$$

We also let $\mathcal{A}_a$ be the ideal of $\mathcal{A}_{\underline{\mathsf{X}}}$ generated, for a fixed $a\in\mathfrak{a}$, by g_a. We may equip $\mathcal{H}_{\underline{\mathsf{X}}}=\bigoplus_{a\in\mathfrak{a}}\mathcal{A}_a$ as a Hilbert C^*-bimodule over $\mathcal{A}_{\underline{\mathsf{X}}}$ by letting $\mathcal{A}_{\underline{\mathsf{X}}}$ act by multiplication on the right and through the map $\phi:\mathcal{A}_{\underline{\mathsf{X}}}\longrightarrow\mathcal{L}(\mathcal{H}_{\underline{\mathsf{X}}})$ defined by

$$\phi(f)((f_a)_{a\in\mathfrak{a}})=(\lambda_a(f)f_a)_{a\in\mathfrak{a}}$$

where

$$\lambda_a(f)(x)=\begin{cases} f(ax) & ax\in\underline{\mathsf{X}}^+ \\ 0 & ax\notin\underline{\mathsf{X}}^+ \end{cases}.$$

A construction of great and growing importance in C^*-algebra theory, the (augmented!) Cuntz-Pimsner algebra defined in [31], now leads to a C^*-algebra

$$\mathcal{O}_{\underline{\mathsf{X}}}$$

for which we have:

Theorem 3.2 *[[22, Proposition 5.8], [26, Corollary 6.2], [27, Proposition 9.2]] Let* $\underline{\mathsf{X}}$ *and* $\underline{\mathsf{Y}}$ *be two-sided shift spaces. We have*

$$\begin{array}{ccccc}
\underline{\mathsf{X}}^+ \simeq \underline{\mathsf{Y}}^+ & \Longrightarrow & \underline{\mathsf{X}} \simeq \underline{\mathsf{Y}} & \Longrightarrow & \underline{\mathsf{X}} \cong_f \underline{\mathsf{Y}} \\
\Downarrow & & & & \Downarrow \\
\mathcal{O}_{\underline{\mathsf{X}}} \simeq \mathcal{O}_{\underline{\mathsf{Y}}} & & & & \mathcal{O}_{\underline{\mathsf{X}}} \otimes \mathbb{K} \simeq \mathcal{O}_{\underline{\mathsf{Y}}} \otimes \mathbb{K} \\
& \searrow & & \swarrow & \\
& & [K_0(\mathcal{O}_{\underline{\mathsf{X}}}), K_0(\mathcal{O}_{\underline{\mathsf{X}}})_+] \simeq [K_0(\mathcal{O}_{\underline{\mathsf{Y}}}), K_0(\mathcal{O}_{\underline{\mathsf{Y}}})_+] & &
\end{array}$$

Parallel to the situtation for crossed products, the construction above leaves us with a canonical $*$-homomorphism

$$\eta_{\mathcal{O}} : \mathcal{A}_{\underline{\mathsf{X}}} \longrightarrow \mathcal{O}_{\underline{\mathsf{X}}}.$$

By construction, $\mathcal{A}_{\underline{\mathsf{X}}}$ is a unital commutative AF algebra ([5]). It may hence be written $C(\Omega_{\underline{\mathsf{X}}})$ where the spectrum $\Omega_{\underline{\mathsf{X}}}$ is a totally disconnected compact Hausdorff space (cf. [24, Corollary 4.7]). As explained in [12, §2.4] we can give a concrete description of $\Omega_{\underline{\mathsf{X}}}$ as the space

$$\{([x_n]_n)_{n\in\mathbb{N}_0} \mid \forall n \in \mathbb{N}_0 : x_{n+1} \sim_n x_n\}.$$

On $C(\Omega_{\underline{\mathsf{X}}})$, we consider $\lambda_{\underline{\mathsf{X}}}$ given by

$$\lambda_{\underline{\mathsf{X}}}(h)(([x_n]_n)_{n\in\mathbb{N}_0}) = \sum_{a\in\mathcal{P}_1(x_1)} h([ax_n]_n)_{n\in\mathbb{N}_0}$$

Further we see that property $(*)$ allows the definition of a continuous and injective map $\iota_{\underline{\mathsf{X}}} : \underline{\mathsf{X}}^- \longrightarrow \Omega_{\underline{\mathsf{X}}}$ by

$$\iota_{\underline{\mathsf{X}}}(y) = ([x_n]_n)_{n\in\mathbb{N}_0}$$

where x_n is chosen with $\mathcal{P}_n(x_n) = \{y_{[-n,0[}\}$. We set

$$\kappa = (\iota \circ \pi_-)^\star : C(\Omega_{\underline{\mathsf{X}}}) \longrightarrow C(\underline{\mathsf{X}})$$

and note that this map may also be considered as a map from $C(\Omega_{\underline{\mathsf{X}}}, \mathbb{Z})$ to $C(\underline{\mathsf{X}}, \mathbb{Z})$. Using this structure, the first author proves the following result, which is the key to our results in this paper:

Theorem 3.3 *[[8]] Let $\underline{\mathsf{X}}$ be a two-sided shift space with property $(*)$. There is a surjective $*$-homomorphism*

$$\rho : \mathcal{O}_{\underline{\mathsf{X}}} \longrightarrow C(\underline{\mathsf{X}}) \rtimes_{\sigma^\star} \mathbb{Z},$$

making the diagram

$$\begin{array}{ccccc}
\mathcal{A}_{\underline{\mathsf{X}}} & = & C(\Omega_{\underline{\mathsf{X}}}) & \xrightarrow{\ \kappa\ } & C(\underline{\mathsf{X}}) \\
\downarrow{\scriptstyle \eta_{\mathcal{O}}} & & & & \downarrow{\scriptstyle \eta_{\rtimes}} \\
\mathcal{O}_{\underline{\mathsf{X}}} & & \xrightarrow{\qquad \rho \qquad} & & C(\underline{\mathsf{X}}) \rtimes_{\sigma^\star} \mathbb{Z}
\end{array}$$

commute.

4 Quotient order

In [23, Theorem 4.9] Matsumoto establishes a group isomorphism

$$\chi_{\mathcal{O}} : K_0(\mathcal{O}_{\underline{\mathsf{X}}}) \longrightarrow \frac{C(\Omega_{\underline{\mathsf{X}}}, \mathbb{Z})}{\operatorname{Im}(\operatorname{Id} - \lambda_{\underline{\mathsf{X}}})}$$

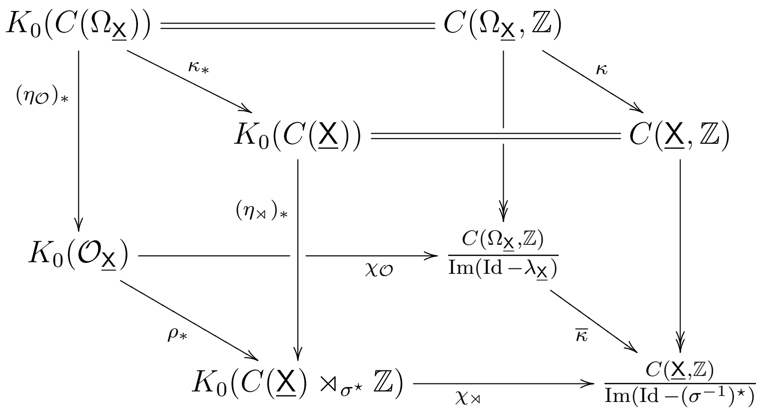
with the property

$$\chi_{\mathcal{O}} \circ (\eta_{\mathcal{O}})_*([p]) = p + \operatorname{Im}(\operatorname{Id} - \lambda_{\underline{\mathsf{X}}}) \tag{2}$$

for any projection in $\mathcal{A}_{\underline{\mathsf{X}}}$, considered as a $\{0,1\}$-valued continuous function on $\Omega_{\underline{\mathsf{X}}}$. Since the group on the right hand side may be described directly in terms of the shift space, this characterization allows a more direct analysis of the group $K_0(\mathcal{O}_{\underline{\mathsf{X}}})$ which in this context is often denoted simply $K_0(\underline{\mathsf{X}})$. A wide ranging analysis along these lines is carried out in Matumoto's work, and under extra assumptions such as property $(*)$, $(**)$ or the shift space being substitutional, we have contributed in [9] and [12].

Returning to the origin of $K_0(\mathcal{O}_{\underline{\mathsf{X}}})$ as an *ordered* group – and under the assumption of property $(*)$ – we shall prove that $\chi_{\mathcal{O}}$ is in fact an order isomorphism. Our starting point is the following diagram:

Proposition 4.1 *Let $\underline{\mathsf{X}}$ be a shift space with property $(*)$. Then the diagram*



is commutative, where all double-headed arrows indicate canonical quotient maps.

Proof: Commutativity of the back face follows from (2) since elements of the form $[p]$ generate $K_0(C(\Omega_{\underline{\mathsf{X}}}))$. The left face of the diagram commutes because of Theorem 3.3, and commutativity of the right face is noted in [12, Proposition 4.1]. Further, the front face commutes according to Theorem 3.1.

Since commutativity of the top square is obvious, the bottom face may now be seen to commute by a diagram chase since, as seen on the back face of the diagram, $(\eta_{\mathcal{O}})_*$ is onto as a consequence of the fact that $\chi_{\mathcal{O}}$ is an isomorphism. □

Proposition 4.2 *Let $\underline{\mathsf{X}}$ be a shift space with property $(*)$. The canonical order on $C(\Omega_{\underline{\mathsf{X}}}, \mathbb{Z}) / \operatorname{Im}(\operatorname{Id} - \lambda_{\underline{\mathsf{X}}})$ is the order induced via $\overline{\kappa}$ by the canonical order on $C(\underline{\mathsf{X}}, \mathbb{Z}) / \operatorname{Im}(\operatorname{Id} - (\sigma^{-1})^\star)$ in the sense that*

$$f + \operatorname{Im}(\operatorname{Id} - \lambda_{\underline{\mathsf{X}}}) \geq 0 \Longleftrightarrow \overline{\kappa}(f + \operatorname{Im}(\operatorname{Id} - \lambda_{\underline{\mathsf{X}}})) \geq 0.$$

Proof: Since $\overline{\kappa}$ is induced by the positive map κ the forward implication is obvious. Thus assume that $\overline{\kappa}(f + \operatorname{Im}(\operatorname{Id} - \lambda_{\underline{\mathsf{X}}})) \geq 0$ and note that this means that there exists $g \in C(\underline{\mathsf{X}}, \mathbb{Z})$ with the property that

$$f \circ \iota_{\underline{\mathsf{X}}} \circ \pi_- + (g - g \circ \sigma^{-1}) \geq 0.$$

We shall find $h_1, h_2, h_3 \in C(\Omega_{\underline{\mathsf{X}}}, \mathbb{Z})$ and $N_1, N_2, N_3 \in \mathbb{N}$ such that

$$f + \sum_{i=1}^{3} (\mathrm{Id} - \lambda_{\underline{\mathsf{X}}}^{N_i})(h_i) \geq 0. \tag{3}$$

This proves positivity since we then have

$$f + (\mathrm{Id} - \lambda_{\underline{\mathsf{X}}}) \left[\sum_{i=1}^{3} \sum_{n=0}^{N_i - 1} \lambda_{\underline{\mathsf{X}}}^{n}(h_i) \right] \geq 0.$$

By continuity of g, there exists an N_1 such that $g \circ \sigma^{-N_1}(x)$ depends only on $\pi_-(x)$. Thus as seen in the proof of [12, Proposition 4.1] there is $\widetilde{g} \in C(\Omega_{\underline{\mathsf{X}}}, \mathbb{Z})$ with the property that $\widetilde{g} \circ \iota_{\underline{\mathsf{X}}} \circ \pi_- = g \circ \sigma^{-N_1}$. We have that

$$f \circ \iota_{\underline{\mathsf{X}}} \circ \pi_- \circ \sigma^{-N_1} + g \circ \sigma^{-N_1} - g \circ \sigma^{-N_1 - 1} \geq 0,$$

so since π_- is surjective and we have proved in [12, Proposition 4.1] that

$$(\iota_{\underline{\mathsf{X}}})^\star \circ \lambda_{\underline{\mathsf{X}}} = (\iota_{\underline{\mathsf{X}}} \circ \sigma_-)^\star = (\sigma_-)^\star \circ (\iota_{\underline{\mathsf{X}}})^\star.$$

we get

$$\begin{aligned}
0 &\leq f \circ \iota_{\underline{\mathsf{X}}} \circ \sigma_-^{N_1} + \widetilde{g} \circ \iota_{\underline{\mathsf{X}}} - \widetilde{g} \circ \iota_{\underline{\mathsf{X}}} \circ \sigma_- \\
&= \left((\sigma_-)^{\star N_1} \circ (\iota_{\underline{\mathsf{X}}})^\star \right)(f) + (\iota_{\underline{\mathsf{X}}})^\star(\widetilde{g}) - \left((\sigma_-)^\star \circ (\iota_{\underline{\mathsf{X}}})^\star \right)(\widetilde{g}) \\
&= (\iota_{\underline{\mathsf{X}}})^\star \circ \lambda_{\underline{\mathsf{X}}}^{N_1}(f) + (\iota_{\underline{\mathsf{X}}})^\star(\widetilde{g}) - (\iota_{\underline{\mathsf{X}}})^\star \circ \lambda_{\underline{\mathsf{X}}}(\widetilde{g}) \\
&= (\iota_{\underline{\mathsf{X}}})^\star \left(f - (\mathrm{Id} - \lambda_{\underline{\mathsf{X}}}^{N_1})(f) + (\mathrm{Id} - \lambda_{\underline{\mathsf{X}}})(\widetilde{g}) \right).
\end{aligned}$$

Let

$$\widetilde{f} = f - (\mathrm{Id} - \lambda_{\underline{\mathsf{X}}}^{N_1})(f) + (\mathrm{Id} - \lambda_{\underline{\mathsf{X}}})(\widetilde{g}).$$

We have seen that $\widetilde{f} \circ \iota_{\underline{\mathsf{X}}}$ is a nonnegative function.

By continuity of $\widetilde{f}$ we get an N_3 such that

$$x_{N_3} \sim_{N_3} x'_{N_3} \Longrightarrow \widetilde{f}(([x_n])_{n \in \mathbb{N}_0}) = \widetilde{f}(([x'_n])_{n \in \mathbb{N}_0}).$$

Thus, if $([x_n])_{n \in \mathbb{N}_0}$ is given with $\#\mathcal{P}_{N_3}(x_{N_3}) = 1$ we get that

$$\widetilde{f}(([x_n])_{n \in \mathbb{N}_0}) = \widetilde{f}(\iota_{\underline{\mathsf{X}}}(y)) \geq 0,$$

where $y \in \mathsf{X}^-$ is chosen with $\mathcal{P}_{N_3}(x_{N_3}) = \{y_{]-N_3,0[}\}$. Note that in every point $([x_n])_{n\in\mathbb{N}_0}$ where we do not know that $\widetilde{f}$ is nonnegative, we have $\#\mathcal{P}_{N_3}(x_{N_3}) > 1$.

Define, for any $c \in \mathbb{Z}$, $k_c \in C(\Omega_{\underline{\mathsf{X}}}, \mathbb{Z})$ as the constant function with value c. One sees by induction that

$$(\lambda_{\underline{\mathsf{X}}})^{N_3}(k_c)(([x_n]_n)_{n\in\mathbb{N}_0}) = c\#\mathcal{P}_{N_3}(x_{N_3}),$$

so with $c = \min(0, \min_{\Omega_{\underline{\mathsf{X}}}} \widetilde{f})$ we get

$$\widetilde{f} + (\mathrm{Id} - (\lambda_{\underline{\mathsf{X}}})^{N_3})(k_c) \geq 0.$$

This proves (3) with $N_2 = 1, h_1 = -f, h_2 = \widetilde{g}$, and $h_3 = k_c$. □

Theorem 4.3 *Let $\underline{\mathsf{X}}$ be a shift space with property (*). Then $\chi_{\mathcal{O}}$ is an order isomorphism*

$$\chi_{\mathcal{O}} : K_0(\mathcal{O}_{\underline{\mathsf{X}}}) \longrightarrow \frac{C(\Omega_{\underline{\mathsf{X}}}, \mathbb{Z})}{\mathrm{Im}(\mathrm{Id} - \lambda_{\underline{\mathsf{X}}})}.$$

when the codomain is equipped with quotient order.

Proof: If $x \in K_0(\mathcal{O}_{\underline{\mathsf{X}}})_+$ we get that

$$\overline{\kappa} \circ \chi_{\mathcal{O}}(x) = \chi_{\rtimes} \circ \rho_*(x) \geq 0$$

which proves positivity of $\chi_{\mathcal{O}}(x)$ by Proposition 4.2.

In the other direction, lift $\chi_{\mathcal{O}}(x)$ to $f \in C(\Omega_{\underline{\mathsf{X}}}, \mathbb{Z})_+$ and note that

$$0 \leq (\eta_{\mathcal{O}})_*(f) = x.$$

□

As noted in [12, Proposition 2.5] the map $\chi_{\mathcal{O}}$ is a group isomorphism regardless of property $(*)$. One may check that it is also an order isomorphism for certain shifts of finite type, but we do not know if it could fail to be an order isomorphism in general. A completely different approach would seem to be needed outside of the case with property $(*)$.

Several useful corollaries can be deduced from this result.

Corollary 4.4 *Let $\underline{\mathsf{X}}$ be an shift spaces with property $(*)$. The order of $K_0(\mathcal{O}_{\underline{\mathsf{X}}})$ is the quotient order induced by ρ_* in the sense that*

$$z \in K_0(\mathcal{O}_{\underline{\mathsf{X}}})_+ \iff \rho_*(z) \in K_0(C(\underline{\mathsf{X}}) \rtimes_{\sigma^*} \mathbb{Z})_+.$$

Proof: Since we now know that both maps $\chi_{\rtimes}$ and $\chi_{\mathcal{O}}$ are order isomorphisms, we can establish the claim by the diagram

$$\begin{array}{ccc} K_0(\mathcal{O}_{\underline{\mathsf{X}}}) & \xrightarrow{\rho_*} & K_0(C(\underline{\mathsf{X}}) \rtimes_{\sigma^\star} \mathbb{Z}) \\ {\scriptstyle \chi_{\mathcal{O}}}\big\downarrow & & \big\downarrow{\scriptstyle \chi_{\rtimes}} \\ \frac{C(\Omega_{\underline{\mathsf{X}}},\mathbb{Z})}{\mathrm{Im}(\mathrm{Id}-\lambda_{\underline{\mathsf{X}}})} & \xrightarrow[\overline{\kappa}]{} & \frac{C(\underline{\mathsf{X}},\mathbb{Z})}{\mathrm{Im}(\mathrm{Id}-(\sigma^{-1})^\star)} \end{array}$$

by appealing to Proposition 4.2. □

The previous result describes $K_0(\mathcal{O}_{\underline{\mathsf{X}}})$ as an ordered group by proving that the order it carries is given as the quotient order induced by ρ_* and the order on $K_0(C(\underline{\mathsf{X}}) \rtimes_{\sigma^\star} \mathbb{Z})$. Note that this means that as soon as ρ_* is not injective, then $[K_0(\mathcal{O}_{\underline{\mathsf{X}}}), K_0(\mathcal{O}_{\underline{\mathsf{X}}})_+]$ will be degenerate in the sense that elements besides 0 are simultaneously positive and negative. It is, in particular, not a dimension group in the strict sense of [16].

This does not, however, mean that the order on $K_0(\mathcal{O}_{\underline{\mathsf{X}}})$ is redundant or completely determined by the order on $K_0(C(\underline{\mathsf{X}}) \rtimes_{\sigma^\star} \mathbb{Z})$. Indeed, the example given in Section 6 shows that for substitutional systems $\underline{\mathsf{X}}$ and $\underline{\mathsf{Y}}$ we may have

$$K_0(\mathcal{O}_{\underline{\mathsf{X}}}) \simeq K_0(\mathcal{O}_{\underline{\mathsf{Y}}})$$

as well as

$$[K_0(C(\underline{\mathsf{X}}) \rtimes_{\sigma^\star} \mathbb{Z}), K_0(C(\underline{\mathsf{X}}) \rtimes_{\sigma^\star} \mathbb{Z})_+] \simeq [K_0(C(\underline{\mathsf{Y}}) \rtimes_{\sigma^\star} \mathbb{Z}), K_0(C(\underline{\mathsf{Y}}) \rtimes_{\sigma^\star} \mathbb{Z})_+],$$

yet

$$[K_0(\mathcal{O}_{\underline{\mathsf{X}}}), K_0(\mathcal{O}_{\underline{\mathsf{X}}})_+] \not\simeq [K_0(\mathcal{O}_{\underline{\mathsf{Y}}}), K_0(\mathcal{O}_{\underline{\mathsf{Y}}})_+].$$

The following result gives an algebraic reformulation of the extra information captured by the order on $K_0(\mathcal{O}_{\underline{\mathsf{X}}})$.

Corollary 4.5 *Let $\underline{\mathsf{X}}$ and $\underline{\mathsf{Y}}$ both be minimal shift spaces with finitely many left special words. The following are equivalent*

(i) $[K_0(\mathcal{O}_{\underline{\mathsf{X}}}), K_0(\mathcal{O}_{\underline{\mathsf{X}}})_+] \simeq [K_0(\mathcal{O}_{\underline{\mathsf{Y}}}), K_0(\mathcal{O}_{\underline{\mathsf{Y}}})_+]$

(ii) There exist group isomorphisms ψ, ψ', ψ'' with ψ' an order isomorphism such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & k_{\underline{\mathsf{X}}} & \hookrightarrow & K_0(\mathcal{O}_{\underline{\mathsf{X}}}) & \xrightarrow{\rho_*} & K_0(C(\underline{\mathsf{X}}) \rtimes_{\sigma^\star} \mathbb{Z}) & \longrightarrow & 0 \\ & & {\scriptstyle \psi''}\big\downarrow & & {\scriptstyle \psi}\big\downarrow & & {\scriptstyle \psi'}\big\downarrow & & \\ 0 & \longrightarrow & k_{\underline{\mathsf{Y}}} & \hookrightarrow & K_0(\mathcal{O}_{\underline{\mathsf{Y}}}) & \xrightarrow[\rho_*]{} & K_0(C(\underline{\mathsf{Y}}) \rtimes_{\sigma^\star} \mathbb{Z}) & \longrightarrow & 0 \end{array}$$

commutes, where $k_{\underline{\mathsf{X}}}$ and $k_{\underline{\mathsf{Y}}}$ denote the kernels of the respective ρ_*-maps.

Proof: The shift spaces have property $(*)$ as seen in [12, Example 3.6]. Suppose the K_0-groups are order isomorphic via ψ. Further note that, as seen in [19], $K_0(C(\underline{\mathsf{X}}) \rtimes_{\sigma^\star} \mathbb{Z})$ is a dimension group and hence in particular has no nonzero element which is simultaneously positive and negative.

This means, by Corollary 4.4, that

$$k_{\underline{\mathsf{X}}} = \{x \in K_0(\mathcal{O}_{\underline{\mathsf{X}}}) \mid 0 \leq x \leq 0\}.$$

Since the same charancterization may be given for $\underline{\mathsf{Y}}$, we get that $\psi(k_{\underline{\mathsf{X}}}) = k_{\underline{\mathsf{Y}}}$. Thus isomorphisms ψ'' and ψ' are induced, and clearly ψ' will be an order isomorphism.

In the other direction, we see that ψ will be an order isomorphism by

$$\psi(z) \geq 0 \Longleftrightarrow \rho_*(\psi(z)) \geq 0 \Longleftrightarrow \psi'(\rho_*(z)) \geq 0 \Longleftrightarrow \rho_*(z) \geq 0 \Longleftrightarrow z \geq 0.$$

□

Note that the condition in (ii) above is not the one defining equivalence in $\mathrm{Ext}(K_0(C(\underline{\mathsf{X}}) \rtimes_{\sigma^\star} \mathbb{Z}), k_{\underline{\mathsf{X}}})$. Hence, even though there would seem to be a relation between the condition and the theory of extensions of C^*-algebras, the order description remains more useful for our purposes than one involving $KK^1(C(\underline{\mathsf{X}}) \rtimes_{\sigma^\star} \mathbb{Z}, \ker \rho)$.

5 Finer descriptions

We will now concentrate on the case when the shift space $\underline{\mathsf{X}}$ has property $(**)$. In this case, as seen in [12, Section 5.3], we can give a description of $K_0(\mathcal{O}_{\underline{\mathsf{X}}})$ as a cokernel of a map on

$$G_{\underline{\mathsf{X}}} = C(\underline{\mathsf{X}}, \mathbb{Z}) \oplus \sum_{n \in \mathbb{N}_0} \mathbb{Z}^{\mathcal{J}_{\underline{\mathsf{X}}}},$$

where the index set $\mathcal{J}_{\underline{\mathsf{X}}}$ is the set of those right shift tail equivalence classes of $\underline{\mathsf{X}}$ which contains a left special element. Notice that it is finite.

We say that a left special words is *adjusted* when $\sigma^{-n}(y)$ is not left special for any $n \in \mathbb{N}$. As a consequence of property $(**)$ each right shift tail class

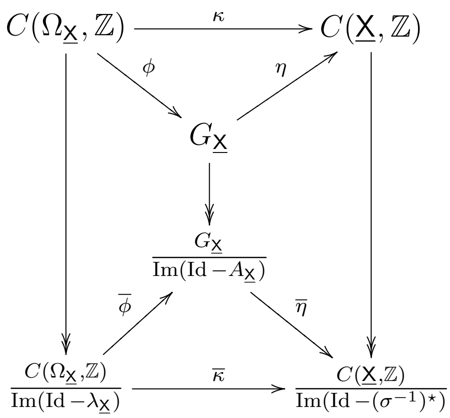
$\mathbf{j}$ of a left special word contains at least one, and at most finitely many, adjusted left special word. We call that set $M_{\mathbf{j}}$.

We let $A_{\underline{\mathsf{X}}}$ be the map from $G_{\underline{\mathsf{X}}}$ to itself defined by

$$(f, (a_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}, n\in\mathbb{N}_0}) \mapsto (f \circ \sigma^{-1}, (\tilde{a}_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}, n\in\mathbb{N}_0}),$$

where $\tilde{a}_0^{\mathbf{j}} = \sum_{z\in M_{\mathbf{j}}} f(\sigma^{-1}(z)) - f(\sigma^{-1}(z^{\mathbf{j}}))$, and $\tilde{a}_n^{\mathbf{j}} = a_{n-1}^{\mathbf{j}}$ for $n > 0$.

We have proved in [12, Proposition 5.16] that $K_0(\mathcal{O}_{\underline{\mathsf{X}}})$ is isomorphic to $G_{\underline{\mathsf{X}}}/\operatorname{Im}(\operatorname{Id} - A_{\underline{\mathsf{X}}})$. The advantage of this description over the one available for property $(*)$ is that $G_{\underline{\mathsf{X}}}$ manifestly contains $C(\underline{\mathsf{X}}, \mathbb{Z})$ via the canonical restriction map $\eta : G_{\underline{\mathsf{X}}} \longrightarrow C(\underline{\mathsf{X}}, \mathbb{Z})$. Indeed, we prove in [12, Proposition 5.19] that there is a group homomorphism $\phi : C(\Omega_{\underline{\mathsf{X}}}, \mathbb{Z}) \longrightarrow G_{\underline{\mathsf{X}}}$ such that



commutes and $\overline{\phi}$ is an isomorphism. We conclude:

Theorem 5.1 *Let $\underline{\mathsf{X}}$ be a shift space with property $(**)$. When*

$$\frac{G_{\underline{\mathsf{X}}}}{\operatorname{Im}(\operatorname{Id} - A_{\underline{\mathsf{X}}})}$$

is equipped with quotient order from the (degenerate) order

$$(f, (a_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}, n\in\mathbb{N}_0}) \geq 0 \Longleftrightarrow f \geq 0,$$

the map $\widetilde{\chi}_{\mathcal{O}} = \overline{\phi} \circ \chi_{\mathcal{O}}$ becomes an order isomorphism.

Proof: According to Theorem 4.3 it suffices to prove that $\overline{\phi}$ is an order isomorphism. Because of our choice of order on $G_{\underline{\mathsf{X}}}$ we clearly have

$$f \geq 0 \Longrightarrow \kappa(f) \geq 0 \Longrightarrow \eta(\phi(f)) \geq 0 \Longrightarrow \phi(f) \geq 0$$

so that $\overline{\phi}$ is positive.

In the other direction, since η is positive, we get

$$y \geq 0 \Longrightarrow \overline{\eta}(y) \geq 0 \Longrightarrow \overline{\kappa}(\overline{\phi}^{-1}(y)) \geq 0 \Longrightarrow \overline{\phi}^{-1}(y)$$

according to Proposition 4.2. □

From [12, Corollary 5.20] we immediately get the following:

Corollary 5.2 *When $\underline{\mathsf{X}}$ is a shift space which has property $(**)$ and only has two left special words $K_0(\mathcal{O}_{\underline{\mathsf{X}}})$ and $K_0(C(\underline{\mathsf{X}}) \rtimes_{\sigma\star} \mathbb{Z})$ are isomorphic as ordered groups.*

Example 5.3 *For a Sturmian shift space with parameter α (see [17, §6]),we get*

$$K_0(\mathcal{O}_{\underline{\mathsf{X}}_\alpha}) \cong \mathbb{Z} + \alpha\mathbb{Z},$$

cf. [33].

6 Substitutional shift spaces

We now consider shift spaces $\underline{\mathsf{X}}_\tau$ associated to aperiodic and primitive substitutions τ via

$$\mathcal{L}(\underline{\mathsf{X}}_\tau) = \{\tau^N(a)_{[n,m]} \mid a \in \mathfrak{a}, N \in \mathbb{N}, 1 \leq n \leq m \leq |\tau^N(a)|\}.$$

In [12] we found a representation of $K_0(\mathcal{O}_{\underline{\mathsf{X}}_\tau})$ as a stationary inductive limit of finitely generated groups. We may now prove that this description also captures the order structure. We refer the reader to [9, 3.3] for the definition of the class of basic substitutions, for a discussion of why working with this class is no restriction, and for the definitions of the combinatorial data A_τ, n_τ, p_τ and $\widetilde{\mathsf{A}}_\tau$ associated with the substitution τ on the alphabet $\mathfrak{a}$.

Theorem 6.1 *Let τ be a basic substitution. There is an order isomorphism*

$$K_0(\mathcal{O}_{\underline{\mathsf{X}}_\tau}) \simeq \varinjlim(\mathbb{Z}^{|\mathfrak{a}|} \oplus \mathbb{Z}^{\mathsf{n}_\tau}/\mathsf{p}_\tau\mathbb{Z}, \widetilde{\mathsf{A}}_\tau)$$

where each $\mathbb{Z}^{|\mathfrak{a}|} \oplus \mathbb{Z}^{\mathsf{n}_\tau}/\mathsf{p}_\tau\mathbb{Z}$ is ordered by

$$((x_a), (y_i) + \mathsf{p}_\tau\mathbb{Z}) \geq 0 \Longleftrightarrow \forall a \in \mathfrak{a} : x_a \geq 0.$$

Proof: The identification in [15] and [9], respectively, of $K_0(C(\underline{\mathsf{X}}_\tau) \rtimes_\sigma \mathbb{Z})$ and $K_0(\mathcal{O}_{\underline{\mathsf{X}}_\tau})$ as limits of stationary inductive systems of finitely generated groups, was found using a Kakutani-Rohlin partition

$$\bigcup_{m=1}^{\infty} \mathrm{rk}_\tau[m] = C(\underline{\mathsf{X}}_\tau, \mathbb{Z})$$

and maps

$$\psi_m : \mathrm{rk}_\tau[m] \longrightarrow \mathbb{Z}^{|\mathsf{a}|} \qquad \Psi_m : \mathrm{rk}_\tau[m] \longrightarrow \mathbb{Z}^{|\mathsf{a}|} \oplus \mathbb{Z}^{\mathsf{n}_\tau}/\mathsf{p}_\tau\mathbb{Z}$$

defined in such a way that $\mathsf{A}_\tau \circ \psi_m = \psi_{m+1}$ and $\widetilde{\mathsf{A}}_\tau \circ \Psi_m = \Psi_{m+1}$. Thus maps

$$\psi_\infty : C(\underline{\mathsf{X}}_\tau, \mathbb{Z}) \longrightarrow \varinjlim(\mathbb{Z}^{|\mathsf{a}|}, \mathsf{A}_\tau) \qquad \Psi_\infty : G_{\underline{\mathsf{X}}_\tau} \longrightarrow \varinjlim(\mathbb{Z}^{|\mathsf{a}|} \oplus \mathbb{Z}^{\mathsf{n}_\tau}/\mathsf{p}_\tau\mathbb{Z}, \widetilde{\mathsf{A}}_\tau)$$

are induced, and since we know by the proofs of [15, Theorem 22] and [9, Theorem 5.8] (cf. [9, Proposition 5.14]) that

$$\ker \psi_\infty = \mathrm{Im}(\mathrm{Id} - (\sigma^{-1})^\star) \qquad \ker \Psi_\infty = \mathrm{Im}(\mathrm{Id} - A_{\underline{\mathsf{X}}}) \tag{4}$$

this establishes the stated isomorphisms.

Let $R : \mathbb{Z}^{|\mathsf{a}|} \oplus \mathbb{Z}^{\mathsf{n}_\tau}/\mathsf{p}_\tau\mathbb{Z} \longrightarrow \mathbb{Z}^{|\mathsf{a}|}$ be the projection map. Since we have directly by definition that

$$\begin{array}{ccc} \mathrm{rk}_\tau[m] \oplus \displaystyle\sum_{i=0}^{m} \mathbb{Z}^{\mathsf{n}_\tau} & \xrightarrow{\ \eta\ } & \mathrm{rk}_\tau[m] \\ \Big\downarrow{\scriptstyle \Psi_m} & & \Big\downarrow{\scriptstyle \psi_m} \\ \mathbb{Z}^{|\mathsf{a}|} \oplus \mathbb{Z}^{\mathsf{n}_\tau}/\mathsf{p}_\tau\mathbb{Z} & \xrightarrow[\ R\ ]{} & \mathbb{Z}^{|\mathsf{a}|} \end{array}$$

we may pass to limits and get that

$$\begin{array}{ccc} G_{\underline{\mathsf{X}}} & \xrightarrow{\ \eta\ } & C(\underline{\mathsf{X}}, \mathbb{Z}) \\ \Big\downarrow{\scriptstyle \Psi_\infty} & & \Big\downarrow{\scriptstyle \psi_\infty} \\ \varinjlim(\mathbb{Z}^{|\mathsf{a}|} \oplus \mathbb{Z}^{\mathsf{n}_\tau}/\mathsf{p}_\tau\mathbb{Z}, \widetilde{\mathsf{A}}_\tau) & \xrightarrow[\ R_\infty\ ]{} & \varinjlim(\mathbb{Z}^{|\mathsf{a}|}, \mathsf{A}_\tau) \end{array}$$

is commutative. Comparing (4) and Theorem 5.1 we get a diagram

$$\begin{array}{ccccc}
K_0(\mathcal{O}_{\underline{\mathsf{X}}}) & \xrightarrow{\widetilde{\chi}_{\mathcal{O}}} & \frac{G_{\underline{\mathsf{X}}}}{\mathrm{Im}(\mathrm{Id}-A_{\underline{\mathsf{X}}})} & \xrightarrow{\overline{\eta}} & \frac{C(\underline{\mathsf{X}},\mathbb{Z})}{\mathrm{Im}(\mathrm{Id}-(\sigma^{-1})^{\star})} \\
 & & \downarrow \overline{\Psi_\infty} & & \downarrow \overline{\psi_\infty} \\
 & & \varinjlim(\mathbb{Z}^{|\mathfrak{a}|} \oplus \mathbb{Z}^{\mathsf{n}_\tau}/\mathsf{p}_\tau\mathbb{Z}, \widetilde{\mathbf{A}}_\tau) & \xrightarrow[R_\infty]{} & \varinjlim(\mathbb{Z}^{|\mathfrak{a}|}, \mathbf{A}_\tau)
\end{array}$$

where from [15, Theorem 22], $\overline{\psi_\infty}$ is an order isomorphism. Thus $\overline{\Psi_\infty} \circ \widetilde{\chi}_{\mathcal{O}}$ becomes an order isomorphism when $\varinjlim(\mathbb{Z}^{|\mathfrak{a}|} \oplus \mathbb{Z}^{\mathsf{n}_\tau}/\mathsf{p}_\tau\mathbb{Z}, \widetilde{\mathbf{A}}_\tau)$ is equipped with the order induced from $\varinjlim(\mathbb{Z}^{|\mathfrak{a}|}, \mathbf{A}_\tau)$ via R_∞. This is the same as equipping each $\mathbb{Z}^{|\mathfrak{a}|} \oplus \mathbb{Z}^{\mathsf{n}_\tau}/\mathsf{p}_\tau\mathbb{Z}$ in the inductive system with the order induced from $\mathbb{Z}^{|\mathfrak{a}|}$ via R, as required. □

The remainder of the paper is devoted to proving by example that the ordered groups thus obtained carry information which is not available by any other flow equivalent means known to us. For this end, we consider a substitution τ given on the alphabet $\mathfrak{a} = \{a, b, c, d\}$ by

$$\begin{aligned}
\tau(a) &= dbdbaaaaadddddddbbbbbbcccabd \\
\tau(b) &= d^{11}b^{10}cccca^{10}bdbbbddddbbbbccaadd \\
\tau(c) &= dbdbaaaacdddddddbbbbbbcaacbd \\
\tau(d) &= d^{11}b^{10}cccca^{10}dddddbd^{31}b^{39}c^{12}a^{24}dddddbbbbccaadd
\end{aligned}$$

where "$\bullet^i$" is just an abbreviation of the concatenation of i instances of "$\bullet$". Surely shorter examples could be found – the repeated letters are only used to get computationally convenient invariants.

Computations using our program [11], cf. [10], show that this substitution is aperiodic, elementary and basic with $\mathsf{n}_\tau = 2$ and $\mathsf{p}_\tau = (1, 1)$. Using the notation

$$[w]^+ = w\tau(w)\cdots\tau^n(w)\cdots \in \mathfrak{a}^{\mathbb{N}_0}$$
$$[w]^- = \cdots\tau^n(w)\cdots\tau(w)w \in \mathfrak{a}^{-\mathbb{N}}$$

as in [10], we may choose cofinal representatives (cf. [9, §3.1])

$$[d^{11}b^{10}cccca^{10}dddddbd^{31}b^{39}c^{12}a^{24}]^- d.[ddddbbbbccaadd]^+$$
$$[dbdbaaaaadddddddbbbbbbccc]^- a.[bd]^+$$

for the two orbit classes of special words, and hence arrive at the augmented matrix

$$\widetilde{\mathsf{A}}_\tau = \begin{bmatrix} 6 & 9 & 3 & 9 & 0 & 0 \\ 12 & 18 & 6 & 18 & 0 & 0 \\ 6 & 9 & 3 & 9 & 0 & 0 \\ 36 & 54 & 18 & 54 & 0 & 0 \\ 10 & 13 & 4 & 12 & 1 & 0 \\ 6 & 8 & 2 & 8 & 0 & 1 \end{bmatrix}.$$

Now consider

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 4 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad S = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

noting that

$$RS = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 4 & 6 & 2 & 6 & 0 & 0 \\ 1 & 2 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

has the property that $(RS)^2$ induces the same map as $\widetilde{\mathsf{A}}_\tau$ on $\mathbb{Z}^4 \oplus \mathbb{Z}^2/(1,1)\mathbb{Z}$. Since

$$SR = \begin{bmatrix} 6 & 3 & 0 \\ 6 & 3 & 0 \\ -3 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}^2$$

we have shown that $K_0(\mathcal{O}_{\underline{\mathsf{X}}_\tau})$ is isomorphic, as an ordered group, to the stationary inductive limit of $\mathbb{Z}^3$ equipped with the order

$$(x_1, x_2, x_3) \geq 0 \iff x_1 \geq 0 \wedge x_2 \geq 0$$

and the matrix

$$B = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

We define

$$X_n = \begin{bmatrix} 0 & 1 & 2 \\ 3^{-n} & (1-3^{-n})/2 & 1 \end{bmatrix}$$

and note that $X_{n+1}B = X_n$. Hence a map

$$X_\infty : K_0(\mathcal{O}_{\underline{\mathsf{X}}_\tau}) \longrightarrow \mathbb{Q}^2$$

is defined. One easily sees that X_∞ is an isomorphism onto $\mathbb{Z} \oplus \mathbb{Z}[1/3]$, and that

$$X_\infty \circ P_\infty = X_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

whence we get that the short exact sequence associated to τ becomes

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\left[\begin{smallmatrix}2\\1\end{smallmatrix}\right]} \mathbb{Z}\oplus\mathbb{Z}[\tfrac{1}{3}] \longrightarrow \mathbb{Z}[\tfrac{1}{3}] \longrightarrow 0$$

This short exact sequence does not split.

Now consider the oppostite substitution τ^{-1} with $\tau^{-1}(\bullet)$ equalling $\tau(\bullet)$ read from right to left. Again the substitution is aperiodic, elementary and basic, with $\mathsf{n}_\tau = 2$ and $\mathsf{p}_\tau = (1,1)$, and it has augmented matrix

$$\widetilde{\mathsf{A}}_{\tau^{-1}} = \begin{bmatrix} 6 & 9 & 3 & 9 & 0 & 0\\ 12 & 18 & 6 & 18 & 0 & 0\\ 6 & 9 & 3 & 9 & 0 & 0\\ 36 & 54 & 18 & 54 & 0 & 0\\ 2 & 7 & 2 & 7 & 1 & 0\\ 2 & 7 & 2 & 7 & 0 & 1 \end{bmatrix}$$

if one chooses

$$[ddaaccbbbbddddda^{24}c^{12}b^{39}d^{31}bdddd]^{-}d.[a^{10}ccccb^{10}dddddddddd]^{+}$$
$$[dbacccbbbbbbdddddd]^{-}a.[aaaabdbd]^{+}$$

Similar computations with

$$R_{op} = \left[\begin{smallmatrix}1&0&0\\1&1&0\\1&0&0\\4&2&0\\0&0&1\\0&0&0\end{smallmatrix}\right] \qquad S_{op} = \left[\begin{smallmatrix}1&1&0&1&0&0\\0&1&1&1&0&0\\0&0&0&0&1&-1\end{smallmatrix}\right]$$
$$B_{op} = \left[\begin{smallmatrix}2&1&0\\2&1&0\\0&0&1\end{smallmatrix}\right] \qquad X_{op,n} = \left[\begin{smallmatrix}0&0&1\\3^{-n}&3^{-n}/2&0\end{smallmatrix}\right]$$

show that we get

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} \mathbb{Z}\oplus\mathbb{Z}[\tfrac{1}{3}] \longrightarrow \mathbb{Z}[\tfrac{1}{3}] \longrightarrow 0$$

whence the short exact sequence splits and the two ordered groups $K_0(\mathcal{O}_{\underline{\mathsf{X}}_\tau})$ and $K_0(\mathcal{O}_{\underline{\mathsf{X}}_{\tau^{-1}}})$ are nonisomorphic by Corollary 4.5 above, even though they are identical as groups.

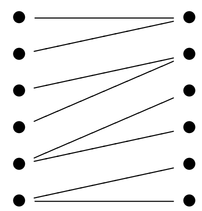
We have chosen the example so that no other flow invariant known to us can tell the flow equivalence classes of $\underline{\mathsf{X}}_\tau$ and $\underline{\mathsf{X}}_{\tau^{-1}}$ apart. Indeed, since the abelianization matrices of τ and τ^{-1} are identical, the invariant of [15] cannot detect any difference. Similarly we get in the notation of [1, Theorem 3.10] that

$$(\tau^{-1})^* = (\tau^*)^{-1}$$

so that

$$\sup\{n \in \mathbb{N} \mid \exists w : w^n \in \mathcal{L}(\underline{\mathsf{X}}_{\tau^*})\} = \sup\{n \in \mathbb{N} \mid \exists w : w^n \in \mathcal{L}(\underline{\mathsf{X}}_{(\tau^{-1})^*})\},$$

rendering the method of [1] inapplicable here. And finally, the configuration data graph (cf. [10]) of τ is symmetric; indeed it is given by



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7.8 Closing remarks

Since the statement of Theorem 4.3 does not depend on property $(*)$, it is natural to ask if the results is true in generale and not just for shift spaces having property $(*)$.

Off other interesting questions in this area is what equivalence relation does the C^*-algebra $\mathcal{O}_{\mathsf{X}^+}$ induce on shift spaces? Said in another way: Is there a symbolic dynamical condition for when to shift spaces have isomorphic (or Morita equivalent) C^*-algebras $\mathcal{O}_{\mathsf{X}}$?

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