Indeterminate Moment Problems within the Askey-scheme

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Ph.D. thesis

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Preface

The great Russian mathematician Chebychev once asked the following question:

If for some positive function f,

$$\int_{-\infty}^{\infty} x^n f(x) dx = \int_{-\infty}^{\infty} x^n e^{-x^2} dx, \quad n = 0, 1, \ldots$$

can we then conclude that $f(x) = e^{-x^2}$?

With the terminology of today, this is the same as to ask if the normal density is uniquely determined by its moment sequence and it is well-known that the answer is *yes* in the sense that $f(x) = e^{-x^2}$ almost everywhere with respect to the Lebesgue measure on $\mathbb{R}$.

But what happens if the normal density is replaced with something else? Can we still count on the same answer?

In general, no. Suppose that X is a random variable which follows a normal distribution $N(0, \sigma^2)$. If the normal density is replaced by the density of $\exp(X)$ or $\sinh(X)$, then the answer to Chebychev's question is *no*.

Whereas the distribution of $\exp(X)$ – also called the lognormal distribution – is very common in statistics, the distribution of $\sinh(X)$ does not even have a name. But if we pass to the associated orthogonal polynomials, the picture changes. We enter the q-analogue of the Askey-scheme and meet the Stieltjes–Wigert polynomials and the continuous q^{-1}-Hermite polynomials.

The above remarks are meant as an appetizer for what to expect of the present thesis. What can be said when there is no longer uniqueness?

My scientific work as a Ph.D. student has resulted in 5 papers all of which are related to the indeterminate moment problem:

[1] Jacob S. Christiansen, *The moment problem associated with the q-Laguerre polynomials*, Constr. Approx. **19** (2003) 1–22.

[2] Jacob S. Christiansen, *The moment problem associated with the Stieltjes–Wigert polynomials*, J. Math. Anal. Appl. **277** (2003) 218–245.

[3] Jacob S. Christiansen and Mourad E. H. Ismail, *A moment problem and a family of integral evaluations*, to appear in Trans. Amer. Math. Soc.

[4] Jacob S. Christiansen, *Indeterminate moment problems related to birth and death processes with quartic rates*, to appear in J. Comp. Appl. Math.

[5] B. Malcolm Brown and Jacob S. Christiansen, *On the Krein and Friedrichs extension of a positive Jacobi operator*, to appear in Expo. Math.

The first two were written in continuation of my master thesis but they also contain results that I first discovered as a Ph.D. student.

The third one was started during my stay in Tampa, Florida visiting Professor Mourad Ismail and completed around half a year later. Among [1]-[5], it is the most extensive paper.

The fourth paper was written in connection with the 7th OPSFA which took place in Copenhagen, August 2003. It is based on work of Valent and others. The orthogonal polynomials in [4] do not have a hypergeometric or basic hypergeometric representation; they are closer related to the Jacobi elliptic functions. For that reason, it was decided not to include the paper [4] in the thesis.

The most recent paper [5], which is joint work with Professor Malcolm Brown from Cardiff, is not included in the thesis either. This paper is dealing with the moment problem from an operator point of view and differ in this way from the preceding four papers.

The present thesis is organized as follows. The first chapter, entitled "The moment problem", serves as an introduction. Starting from the fundamental work of Stieltjes and Hamburger, it gives an up-to-date picture of the theory of the moment problem on the real line with special focus on the indeterminate case.

Before we go into details with the indeterminate moment problems within the Askey-scheme, it is appropriate to give an introduction to basic hypergeometric series and Darboux's method. Partly because all the indeterminate cases appear within the q-analogue of the Askey-scheme and partly because Darboux's method is a very important tool in determining the entire functions from the Krein and Nevanlinna parametrizations.

After the preliminaries, a chapter with the same title as the thesis itself is to follow. The purpose of this principal chapter is twofold. First of all, it gives a setting for the three enclosed papers. But it also contains the authors complete knowledge – as of today – about the indeterminate moment problems within the Askey-scheme. It is not always the intention to go down to the last computational detail, especially not when everything is well explained in the literature. Then it is rather the idea to give a survey. On the other hand, we make sure to be careful in explaining new results, like for instance the Krein parametrization for the q-Meixner moment problem.

First of all, I would also like to thank my supervisor Professor Christian Berg. Not only for being a superb and inspiring adviser but also for sending me to summer schools and conferences around the world. It has been of great value to me.

I would also like to thank "Rejselegat for Matematikere" for giving me the opportunity to spend one year abroad. I especially express my thanks to Professor Mourad Ismail and Professor Richard Askey for teaching me a lot about orthogonal polynomials and special functions. I truly benefited from the year 2002 in the States.

Finally, I would like to thank my family for their faithful support throughout my studies.

Frederiksberg, 2004
Jacob Stordal Christiansen

Contents

The moment problem

Let $I \subseteq \mathbb{R}$ be an interval. For a positive measure μ on I the nth moment is defined as $\int_I x^n d\mu(x)$ – provided the integral exists. If we suppose that $(s_n)_{n\geq 0}$ is a sequence of real numbers, the moment problem on I consists of solving the following three problems:

(I) Does there exist a positive measure on I with moments $(s_n)_{n\geq 0}$?

In the affirmative,

(II) is this positive measure uniquely determined by the moments $(s_n)_{n\geq 0}$?

If this is not the case,

(III) how can one describe all positive measures on I with moments $(s_n)_{n\geq 0}$?

Without loss of generality we can always assume that $s_0 = 1$. This is just a question of normalizing the involved measures to be probability measures.

When μ is a positive measure with moments $(s_n)_{n\geq 0}$, we say that μ is a *solution* to the moment problem. If the solution to the moment problem is unique, the moment problem is called *determinate*. Otherwise the moment problem is said to be *indeterminate*.

In this section we shall give an introduction to the classical moment problem on the real line with special focus on the indeterminate case. For a more detailed discussion the reader is referred to Akhiezer [1], Berg [3] or Shohat and Tamarkin [23].

There are three essentially different types of (closed) intervals. Either two end-points are finite, one end-point is finite, or no end-points are finite. In the last case the interval is simply $\mathbb{R}$ and in the first two cases one can think of $[0, 1]$ and $[0, \infty)$. For historical reasons the moment problem on $[0, \infty)$ is called the *Stieltjes* moment problem and the moment problem on $\mathbb{R}$ is called the *Hamburger* moment problem. Moreover, the moment problem on $[0, 1]$ is referred to as the *Hausdorff* moment problem.

It is elementary linear algebra to verify that a positive measure with finite support is uniquely determined by its moments. Applying the approximation theorem of Weierstrass and the Riesz representation theorem, one can extend this result to hold for positive measures with compact support. The Hausdorff moment problem is therefore always determinate. As regards existence, Hausdorff [13] proved in 1923 that the moment problem has a solution on $[0, 1]$ if and only if the sequence $(s_n)_{n\geq 0}$ is completely monotonic.

Stieltjes introduced the moment problem on $[0, \infty)$ and solved the problems about existence and uniqueness in his famous memoir “Recherches sur les fractions continues” from 1894-95, see [24].

The memoir is devoted to the study of continued fractions of the form

$$\cfrac{1}{m_1 z + \cfrac{1}{l_1 + \cfrac{1}{m_2 z + \cfrac{1}{l_2 + \cdots}}}} \tag{1}$$

where $m_n, l_n > 0$ and $z \in \mathbb{C}$. We denote by $T_n(z)/U_n(z)$ the nth convergent (or nth approximant) and observe that $T_n(z)$ and $U_n(z)$ are polynomials in z. To be precise, $T_{2n}(z)$ and $T_{2n-1}(z)$ are polynomials of degree $n-1$ whereas $U_{2n}(z)$ and $U_{2n-1}(z)$ are polynomials of degree n. Moreover,

$$T_{2n}(0) = l_1 + \ldots + l_n, \quad U_{2n}(0) = T_{2n-1}(0) = 1 \quad \text{and} \quad U_{2n-1}(0) = 0.$$

The moment sequence $(s_n)_{n\geq 0}$ comes in via the asymptotic expansion

$$\frac{T_n(z)}{U_n(z)} = \frac{s_0}{z} - \frac{s_1}{z^2} + \frac{s_3}{z^3} - \ldots + (-1)^{n-1}\frac{s_{n-1}}{z^n} + \mathcal{O}\Big(\frac{1}{z^{n+1}}\Big), \quad |z| \to \infty.$$

In this way the nth convergent uniquely determines the real numbers $s_0, s_1, \ldots, s_{n-1}$. The condition $m_n, l_n > 0$ is equivalent to assuming that

$$\begin{vmatrix} s_0 & s_1 & \ldots & s_{n-1} \\ s_1 & s_2 & \ldots & s_n \\ \vdots & \vdots & & \vdots \\ s_{n-1} & s_n & \ldots & s_{2n-2} \end{vmatrix} > 0 \quad \text{and} \quad \begin{vmatrix} s_1 & s_2 & \ldots & s_n \\ s_2 & s_3 & \ldots & s_{n+1} \\ \vdots & \vdots & & \vdots \\ s_n & s_{n+1} & \ldots & s_{2n-1} \end{vmatrix} > 0,$$

which is necessary and sufficient for the moment problem to have a solution on $[0,\infty)$ with infinite support.

Stieltjes pointed out that one has to distinguish between two cases:

$$\sum_{n=1}^{\infty}\big(m_n + l_n\big) < \infty \quad \text{and} \quad \sum_{n=1}^{\infty}\big(m_n + l_n\big) = \infty.$$

In the first case – the *indeterminate* case – the continued fraction diverges for all $z \in \mathbb{C}$. However, the even convergents and the odd convergents each have a limit as $n \to \infty$ for $z \in \mathbb{C} \setminus (-\infty, 0]$. The limits are different and of the form

$$\lim_{n\to\infty} \frac{T_{2n}(z)}{U_{2n}(z)} = \int_0^\infty \frac{d\nu_1(t)}{z+t} \quad \text{and} \quad \lim_{n\to\infty} \frac{T_{2n-1}(z)}{U_{2n-1}(z)} = \int_0^\infty \frac{d\nu_2(t)}{z+t},$$

where ν_1 and ν_2 are different positive (and discrete) measures on $[0,\infty)$ with moments $(s_n)_{n\geq 0}$. In fact, the polynomials $T_{2n}(z)$, $U_{2n}(z)$, $T_{2n-1}(z)$, $U_{2n-1}(z)$ converge uniformly on compact subsets of $\mathbb{C}$ as $n \to \infty$:

$$\begin{aligned} &\lim_{n\to\infty} T_{2n}(z) = P(z), \qquad &&\lim_{n\to\infty} T_{2n-1}(z) = R(z), \\ &\lim_{n\to\infty} U_{2n}(z) = Q(z), \qquad &&\lim_{n\to\infty} U_{2n-1}(z) = S(z). \end{aligned} \tag{2}$$

The entire functions P, Q, R, S satisfy the relation

$$Q(z)R(z) - P(z)S(z) = 1, \quad z \in \mathbb{C},$$

and admit only simple zeros which are ≤ 0. As we shall see later on, these four functions play an important role in the description of the set of solutions to an indeterminate Stieltjes moment problem.

In the second case – the *determinate* case – the continued fraction converges uniformly on compact subsets of $\mathbb{C} \setminus (-\infty, 0]$ even though the polynomials $T_n(z)$ and $U_n(z)$ diverge as $n \to \infty$. The limit of the nth convergent has the form

$$\lim_{n\to\infty} \frac{T_n(z)}{U_n(z)} = \int_0^\infty \frac{d\nu(t)}{z+t},$$

where ν is a positive measure on $[0, \infty)$ with moments $(s_n)_{n\geq 0}$. In fact, ν is the only positive measure on $[0, \infty)$ with moments $(s_n)_{n\geq 0}$.

Hamburger continued the work of Stieltjes in the series of papers "Über eine Erweiterung des Stieltjesschen Momentenproblems" from 1920-21, see [12]. He was the first to treat the moment problem as a theory of its own and considered more general continued fractions than the one in (1). The role of $[0, \infty)$ in Stieltjes' work is taken over by the real line in Hamburger's work. A key result – often referred to as *Hamburger's theorem* – says that $(s_n)_{n\geq 0}$ is a moment sequence if and only if it is positive definite. But besides the question about existence, Hamburger was also interested in the question about uniqueness.

To avoid confusion at this point we emphasize that if $(s_n)_{n\geq 0}$ is a sequence of Stieltjes moments, then one has to distinguish between determinacy and indeterminacy in the sense of Stieltjes and in the sense of Hamburger. Obviously, an indeterminate Stieltjes moment problem is also indeterminate in the sense of Hamburger and if the solution to a determinate Hamburger moment problem is supported within $[0, \infty)$, the moment problem is also determinate in the sense of Stieltjes. But a determinate Stieltjes moment problem can just as well be determinate as indeterminate in the sense of Hamburger. In the following we let the words *determinate* and *indeterminate* refer to the Hamburger moment problem unless otherwise stated.

It is desirable to be able to tell whether the moment problem is determinate or indeterminate just by looking at the moment sequence $(s_n)_{n\geq 0}$. Hamburger came up with a solution to this problem by pointing out that the moment problem is determinate if and only if

$$\lim_{n\to\infty} \frac{\begin{vmatrix} s_0 & s_1 & \dots & s_{n-1} \\ s_1 & s_2 & \dots & s_n \\ \vdots & \vdots & & \vdots \\ s_{n-1} & s_n & \dots & s_{2n-2} \end{vmatrix}}{\begin{vmatrix} s_4 & s_5 & \dots & s_{n+1} \\ s_5 & s_6 & \dots & s_{n+2} \\ \vdots & \vdots & & \vdots \\ s_{n+1} & s_{n+2} & \dots & s_{2n-2} \end{vmatrix}} = 0.$$

More recently, Berg, Chen and Ismail [4] have proved that the moment problem is determinate if and only if the smallest eigenvalue of the Hankel matrix $\big((s_{i+j})_{0\leq i,j\leq n}\big)$ tends to 0 as $n \to \infty$. A simpler criterion, however, was given by Carleman in his treatise of quasi-analytic functions from 1926, see [8]. He proved that if

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{s_{2n}}} = \infty, \tag{3}$$

then the moment problem is determinate. Carleman's criterion has the disadvantage that it only gives a sufficient condition for the moment problem to be determinate. There are moment sequences $(s_n)_{n\geq 0}$ for which the series in (3) converges although the moment problem is determinate. But Carleman's criterion tells us that the moment problem is determinate unless the even moments tend to infinity quite rapidly. On the other hand, we cannot conclude that the moment problem is indeterminate just because the moment sequence increases very rapidly.

Given a positive measure μ with moments $(s_n)_{n\geq 0}$, the *orthonormal* polynomials (P_n) are characterized by $P_n(x)$ being a polynomial of degree n with positive leading coefficient such that

$$\int_{\mathbb{R}} P_n(x)P_m(x)\,d\mu(x) = \delta_{mn}, \quad n,m \geq 0.$$

The polynomials (P_n) only depend on the moment sequence $(s_n)_{n\geq 0}$ and they can be obtained from the formula

$$P_n(x) = \frac{1}{\sqrt{D_{n-1}D_n}} \begin{vmatrix} s_0 & s_1 & \dots & s_n \\ s_1 & s_2 & \dots & s_{n+1} \\ \vdots & \vdots & & \vdots \\ s_{n-1} & s_n & \dots & s_{2n-1} \\ 1 & x & \dots & x^n \end{vmatrix}, \tag{4}$$

where $D_n = \det\bigl((s_{i+j})_{0\leq i,j\leq n}\bigr)$ denotes the Hankel determinant. It is well-known that (P_n) satisfy a *three-term recurrence relation* of the form

$$xP_n(x) = b_nP_{n+1}(x) + a_nP_n(x) + b_{n-1}P_{n-1}(x), \quad n \geq 1, \tag{5}$$

where $a_n \in \mathbb{R}$ and $b_n > 0$. The initial conditions are $P_0(x) = 1$ and $P_1(x) = \frac{1}{b_0}(x - a_0)$. Vice versa, if (P_n) satisfy the above three-term recurrence relation (including the initial conditions) for some real sequences (a_n) and (b_n) with $b_n > 0$, then it follows by Favard's theorem that there exists a positive measure μ on $\mathbb{R}$ such that the polynomials (P_n) are orthonormal with respect to μ.

As can be read of from (5), the leading coefficient of $P_n(x)$ is given by $(b_0b_1\cdots b_{n-1})^{-1}$. The polynomials $p_n(x) := (b_0b_1\cdots b_{n-1})P_n(x)$ are therefore monic and they satisfy the three-term recurrence relation

$$xp_n(x) = p_{n+1}(x) + c_np_n(x) + \lambda_np_{n-1}(x), \quad n \geq 1, \tag{6}$$

where $c_n = a_n \in \mathbb{R}$ and $\lambda_n = b_{n-1}^2 > 0$.

The recurrence coefficients in (5) and (6) contain useful information about the moment problem. Carleman proved in 1922 that the moment problem is determinate if

$$\sum_{n=0}^{\infty} \frac{1}{b_n} = \infty. \tag{7}$$

This condition is clearly satisfied if the sequence (b_n) is bounded and if the sequence (a_n) is bounded too, the unique solution has compact support. Just like Carleman's condition (3), the condition (7) is only sufficient for the moment problem to be determinate. The moment problem may be determinate even though the series in (7) converges.

In the set-up of Stieltjes the recurrence coefficients from (5) are given by

$$a_n = \frac{1}{m_{n+1}}\Bigl(\frac{1}{l_n} + \frac{1}{l_{n+1}}\Bigr) \quad \text{and} \quad b_n = \frac{1}{l_{n+1}\sqrt{m_{n+1}m_{n+2}}}$$

with the convention that $a_0 = \frac{1}{m_1}\frac{1}{l_1}$. After a few computations we see that the moment problem is determinate in the sense of Stieltjes if (but *not* only if)

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{b_n}} = \infty.$$

Using the concept of chain sequences, Chihara proved the following result in [10]. On the assumption that

$$c_n \to \infty \quad \text{and} \quad \frac{\lambda_{n+1}}{c_n c_{n+1}} \to L < \frac{1}{4} \quad \text{as } n \to \infty,$$

the moment problem is determinate if

$$\liminf_{n\to\infty} c_n^{1/n} < \frac{1+\sqrt{1-4L}}{1-\sqrt{1-4L}}$$

and indeterminate if the opposite (strict) inequality holds. In particular, if c_n has the form

$$c_n = f_n q^{-n},$$

where $0 < q < 1$ and (f_n) is both bounded and bounded away from 0, then the moment problem is determinate if

$$L < \frac{q}{(1+q)^2}$$

and indeterminate if the opposite (strict) inequality holds.

Just like the orthonormal polynomials (P_n), the polynomials of the *second kind* (Q_n) are generated by the three-term recurrence relation (5) – but with initial conditions $Q_0(x) = 0$ and $Q_1(x) = 1/b_0$. Consequently, (P_n) and (Q_n) are linearly independent solutions to (5) and together they span the solution space. Notice that $Q_n(x)$ is a polynomial of degree $n-1$ and when μ is a positive measure with moments $(s_n)_{n\geq 0}$, we have

$$Q_n(x) = \int_{\mathbb{R}} \frac{P_n(x) - P_n(y)}{x-y}\, d\mu(y).$$

The orthonormal polynomials (P_n) and the polynomials of the second kind (Q_n) play a crucial role for the moment problem. Hamburger proved that the moment problem is indeterminate if and only if

$$\sum_{n=0}^{\infty} \big(P_n^2(0) + Q_n^2(0)\big) < \infty. \tag{8}$$

Actually, it is necessary and sufficient that there exists an $x \in \mathbb{R}$ such that (8) is fulfilled with x instead of 0. It is even necessary and sufficient that there exists a $z \in \mathbb{C}\setminus\mathbb{R}$ such that either $(P_n(z))$ or $(Q_n(z))$ belong to ℓ^2. In any case, when the moment problem is indeterminate the series

$$\sum_{n=0}^{\infty} |P_n(z)|^2 \quad \text{and} \quad \sum_{n=0}^{\infty} |Q_n(z)|^2$$

converge uniformly on compact subsets of $\mathbb{C}$.

Hamburger pointed out that in the set-up of Stieltjes the condition (8) is equivalent to

$$\sum_{n=1}^{\infty} m_{n+1}(l_1 + \ldots + l_n)^2 < \infty. \tag{9}$$

This simply follows from the fact that

$$P_n(z) = (-1)^n \sqrt{m_{n+1}/m_1}\, U_{2n}(-z)$$

and

$$Q_n(z) = (-1)^{n-1} \sqrt{m_{n+1} m_1}\, T_{2n}(-z).$$

The condition (9) enables us to determine whether a determinate Stieltjes moment problem is determinate or indeterminate in the sense of Hamburger.

Sometimes the natural starting point is not the orthogonal polynomials but a density $w(t)$ with moments $(s_n)_{n\geq 0}$. In this situation Krein [14] proved that the moment problem is indeterminate if

$$\frac{1}{\pi}\int_{\mathbb{R}} \frac{\log w(t)}{1+t^2}\, dt > -\infty. \tag{10}$$

Krein's condition (10) is only sufficient and not necessary for the moment problem to be indeterminate.

We shall now take a closer look at the set of solutions to an indeterminate Hamburger moment problem. Such a set – which we will denote by $\mathcal{V}_H$ – is clearly convex and therefore infinite. In fact, it is infinite dimensional. Equipped with the vague topology, $\mathcal{V}_H$ is a compact set in which the subsets of absolutely continuous, discrete and continuous singular solutions each are dense, see Berg and Christensen [5]. Moreover, Naimark [17] proved that μ is an extreme point in $\mathcal{V}_H$ if and only if the polynomials $\mathbb{C}[x]$ are dense in $L^1(\mathbb{R}, \mu)$.

The problem about describing $\mathcal{V}_H$ was solved by Nevanlinna in 1922 using complex function theory, see [18]. We call a function φ a Pick function if it is holomorphic in the upper half-plane $\operatorname{Im} z > 0$ and $\operatorname{Im} \varphi(z) \geq 0$ for $\operatorname{Im} z > 0$. By reflection in the real line any such function can be extended to a holomorphic function in $\mathbb{C} \setminus \mathbb{R}$. Nevanlinna proved that $\mathcal{V}_H$ can be parametrized by the space $\mathcal{P}$ of Pick functions augmented with the point ∞. The space $\mathcal{P}$ inherits the topology of the holomorphic functions on $\mathbb{C} \setminus \mathbb{R}$ and one can think of $\mathcal{P} \cup \{\infty\}$ as a one-point compactification of $\mathcal{P}$. The parametrization is established via the homeomorphism $\varphi \mapsto \mu_\varphi$ of $\mathcal{P} \cup \{\infty\}$ onto $\mathcal{V}_H$ given by

$$\int_{\mathbb{R}} \frac{d\mu_\varphi(t)}{t-z} = -\frac{A(z)\varphi(z) - C(z)}{B(z)\varphi(z) - D(z)}, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

where A, B, C, D are certain entire functions defined in terms of the orthonormal polynomials (P_n) and the polynomials of the second kind (Q_n). More precisely, A, B, C, D are the uniform limits (on compact subsets of $\mathbb{C}$) of the polynomials

$$\begin{aligned}
A_n(z) &= b_n\big(Q_n(0)Q_{n+1}(z) - Q_{n+1}(0)Q_n(z)\big),\\
B_n(z) &= b_n\big(Q_n(0)P_{n+1}(z) - Q_{n+1}(0)P_n(z)\big),\\
C_n(z) &= b_n\big(P_n(0)Q_{n+1}(z) - P_{n+1}(0)Q_n(z)\big),\\
D_n(z) &= b_n\big(P_n(0)P_{n+1}(z) - P_{n+1}(0)P_n(z)\big),
\end{aligned} \tag{11}$$

as $n \to \infty$. In a more compact form, we have

$$
\begin{aligned}
A(z) &= z\sum_{n=0}^{\infty} Q_k(0)Q_k(z), \quad C(z) = 1 + z\sum_{n=0}^{\infty} P_k(0)Q_k(z),\\
B(z) &= -1 + z\sum_{n=0}^{\infty} Q_k(0)P_k(z), \quad D(z) = z\sum_{n=0}^{\infty} P_k(0)P_k(z),
\end{aligned}
\tag{12}
$$

and the so-called *Nevanlinna matrix* $\left(\begin{smallmatrix} A & C \\ B & D \end{smallmatrix}\right)$ has determinant one for all $z \in \mathbb{C}$.

M. Riesz proved in 1923 that the entire functions A, B, C, D are of minimal exponential type, see [22]. In particular, their order is ≤ 1 (and if the order is 1, then the type is 0). Berg and Pedersen [6] have later proved that A, B, C, D have the same order, type and Phragmén–Lindelöf indicator function.

In some sense, to solve an indeterminate Hamburger moment problem means to find the Nevanlinna matrix. If one can express A, B, C, D – but in particular B and D – in terms of well-known functions, it may be possible to obtain solutions to the moment problem in a systematic way. With A, B, C, D at hand one can use the Stieltjes–Perron inversion formula to find the solution μ_φ corresponding to the Pick function φ. In particular, if

$$\varphi(z) = t, \quad \operatorname{Im} z \neq 0$$

for $t \in \mathbb{R} \cup \{\infty\}$, then μ_φ is a discrete measure of the form

$$\mu_t = \sum_{x\in\Lambda_t} \rho(x)\varepsilon_x, \tag{13}$$

where Λ_t denotes the set of zeros of $x \mapsto B(x)t - D(x)$ (or $x \mapsto B(x)$ if $t = \infty$) and $\rho : \mathbb{R} \to (0, 1)$ is given by

$$\frac{1}{\rho(x)} = \sum_{n=0}^{\infty} P_n^2(x) = B'(x)D(x) - B(x)D'(x), \quad x \in \mathbb{R}. \tag{14}$$

As usual, we denote by ε_x the unit mass at the point x. Moreover, if we set

$$\varphi(z) = \begin{cases} \beta + i\gamma, & \operatorname{Im} z > 0 \\ \beta - i\gamma, & \operatorname{Im} z < 0 \end{cases}$$

for $\beta \in \mathbb{R}$ and $\gamma > 0$, then μ_φ is absolutely continuous with density

$$\frac{d\mu_{\beta,\gamma}}{dx} = \frac{\gamma/\pi}{\big(\beta B(x) - D(x)\big)^2 + \big(\gamma B(x)\big)^2}, \quad x \in \mathbb{R}. \tag{15}$$

The solutions in (13) and (15) are interesting in different ways. The discrete measures in (13) are characterized by M. Riesz [21] to be the only solutions μ for which the polynomials $\mathbb{C}[x]$ are dense in $L^2(\mathbb{R}, \mu)$ or, equivalently, for which the polynomials (P_n) form an orthonormal basis for the Hilbert space $L^2(\mathbb{R}, \mu)$. They are called N-*extremal* solutions and are indeed extreme points in $\mathcal{V}_H$ – just not the only ones. As regards the densities in (15), the polynomials $\mathbb{C}[x]$ are not even dense in $L^1(\mathbb{R}, \mu_{\beta,\gamma})$. But among all the absolutely continuous measures in $\mathcal{V}_H$ with density, say $w(t)$, the solution $\mu_{0,1}$ is the one that maximizes the entropy integral in (10). More generally, Gabardo [11] proved that for fixed $\lambda = x + iy$ in the upper half-plane, the integral

$$\frac{1}{\pi}\int_{\mathbb{R}} \frac{y \log w(t)}{(x-t)^2 + y^2}\,dt$$

obtains its maximum value among all densities in $\mathcal{V}_H$ when

$$w(t) = \frac{d\mu_{\beta,\gamma}}{dt} \quad \text{and} \quad \frac{D(\lambda)}{B(\lambda)} = \beta - i\gamma.$$

Since $\mathcal{V}_H$ is convex, we notice that given $\varphi, \psi \in \mathcal{P} \cup \{\infty\}$ and $s \in [0,1]$ there exists a unique function $\chi \in \mathcal{P} \cup \{\infty\}$ such that

$$s\mu_\varphi + (1-s)\mu_\psi = \mu_\chi.$$

In fact, χ is given by

$$\chi = \frac{\varphi\psi B - \big(s\varphi + (1-s)\psi\big)D}{\big((1-s)\varphi + s\psi\big)B - D}$$

and this in particular means that

$$\tfrac{1}{2}(\mu_1 + \mu_{-1}) = \mu_{B/D} \quad \text{and} \quad \tfrac{1}{2}(\mu_0 + \mu_\infty) = \mu_{-D/B}.$$

Therefore, B/D and $-D/B$ are Pick functions.

The solutions in (13) are also called *canonical*. More generally, a solution μ_φ is called m-canonical or canonical of order m if the Pick function φ is a real rational function of degree m. Such solutions are discrete measures and if $\varphi = P/Q$ – assuming that P and Q are polynomials with real coefficients and no common zeros – then μ_φ is supported on the zeros of $x \mapsto B(x)P(x) - D(x)Q(x)$. For fixed m_0, the subset of canonical solutions of order $m \geq m_0$ is dense in $\mathcal{V}_H$. Moreover, if μ is canonical of order $m \geq 1$ then the polynomials $\mathbb{C}[x]$ are dense in $L^p(\mathbb{R},\mu)$ for $1 \leq p < 2$ but *not* for $p \geq 2$. In particular, the m-canonical solutions are extreme points in $\mathcal{V}_H$ and we see that $\mathcal{V}_H$ is one of those special convex sets in which the extreme points are dense.

Buchwalter and Cassier proved in [7] that a solution μ is m-canonical if and only if the closure of the polynomials $\mathbb{C}[x]$ has codimension m in $L^2(\mathbb{R},\mu)$. In fact, if μ is a discrete solution of the form

$$\mu = \sum_n m_n \varepsilon_{x_n},$$

then the codimension of the closure of $\mathbb{C}[x]$ in $L^2(\mathbb{R},\mu)$ can be computed as the sum of the series

$$\sum_n \Big(1 - \frac{m_n}{\rho(x_n)}\Big),$$

where ρ is defined in (14). See Bakan [2] for details. The above series converges if and only if μ is canonical of some order $m \geq 0$. At this point we stress that

$$\mu(\{x\}) \leq \rho(x), \quad x \in \mathbb{R}$$

for all $\mu \in \mathcal{V}_H$ and equality only holds when $\mu = \mu_t$ is N-extremal and $B(x)t - D(x) = 0$.

Suppose now that $(s_n)_{n\geq 0}$ is a sequence of Stieltjes moments such that the moment problem is indeterminate in the sense of Hamburger. In order to describe the set $\mathcal{V}_S$ of solutions to the Stieltjes moment problem, one can still use the Nevanlinna parametrization and just restrict oneself to consider only the Pick functions φ which have an analytic continuation to $\mathbb{C}\setminus[0,\infty)$ such that $\alpha \leq \varphi(x) \leq 0$ for $x<0$, see Pedersen [20]. The quantity $\alpha \leq 0$ is defined by

$$-\frac{1}{\alpha} = m_1 \sum_{n=1}^{\infty} l_n$$

or as the limit

$$\alpha = \lim_{n\to\infty} \frac{P_n(0)}{Q_n(0)},$$

and the moment problem is determinate in the sense of Stieltjes if and only if $\alpha = 0$.

For the indeterminate Stieltjes moment problem a slightly more elegant way to describe $\mathcal{V}_S$ is the *Krein parametrization*, see Krein [15] or Krein and Nudel'man [16, p. 199]. We denote by $\mathcal{S}$ the subspace of $\mathcal{P}$ consisting of those Pick functions σ which have an analytic continuation to $\mathbb{C}\setminus[0,\infty)$ such that $\sigma(x)\geq 0$ for $x<0$. In addition to this, $\mathcal{S}\cup\{\infty\}$ is a one-point compactification of $\mathcal{S}$ in the topology inherited from the holomorphic functions on $\mathbb{C}\setminus[0,\infty)$. The parametrization is established via the homeomorphism $\sigma \mapsto \nu_\sigma$ of $\mathcal{S}\cup\{\infty\}$ onto $\mathcal{V}_S$ given by

$$\int_0^\infty \frac{d\nu_\sigma(t)}{t-z} = \frac{P(-z)+\sigma(z)R(-z)}{Q(-z)+\sigma(z)S(-z)}, \quad z\in\mathbb{C}\setminus[0,\infty),$$

where P, Q, R, S are the entire functions from (2). In fact, $\left(\begin{smallmatrix} P & R \\ Q & S \end{smallmatrix}\right)$ is related to the Nevanlinna matrix by

$$\begin{aligned} P(z) &= A(-z) - \frac{1}{\alpha}C(-z), \quad & R(z) &= C(-z), \\ Q(z) &= -\Big(B(-z) - \frac{1}{\alpha}D(-z)\Big), \quad & S(z) &= -D(-z), \end{aligned} \tag{16}$$

and we see that $\nu_\sigma = \mu_\varphi$ exactly when

$$\sigma(z) = \frac{\varphi(z)-\alpha}{\alpha\,\varphi(z)}.$$

In particular, this means that

$$\nu_0 = \mu_\alpha, \quad \nu_\infty = \mu_0$$

and the only N-extremal solutions supported within $[0,\infty)$ are μ_t with $\alpha \leq t \leq 0$ or ν_s with $0 \leq s \leq \infty$.

We end this section by explaining the connection between Stieltjes moment problems and symmetric Hamburger moment problems. A moment problem is said to be *symmetric* if all moments of odd order are 0. In terms of the orthonormal polynomials (P_n) this is equivalent to

$$P_n(-x) = (-1)^n P_n(x) \text{ for all } n \geq 0$$

or equivalent to $a_n = 0$, where (a_n) is the sequence from the three-term recurrence relation (5). If we suppose that $(s_n)_{n\geq 0}$ is a sequence of Stieltjes moments, then the sequence $(s_0, 0, s_1, 0, s_2, \ldots)$ gives rise to a symmetric Hamburger moment problem which is indeterminate if and only if the original Stieltjes moment problem is indeterminate. Notice that Carleman's criterion (3) thus says that the Stieltjes moment problem is determinate if

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{s_n}} = \infty.$$

There is a one-to-one correspondence between solutions to the Stieltjes moment problem and symmetric solutions to the corresponding symmetric Hamburger moment problem, cf. [19, Prop. 4.1]. In fact, if the density $w(t)$, $t > 0$, has moments $(s_n)_{n\geq 0}$ then the density $|t|\, w(t^2)$, $t \in \mathbb{R}$, has moments $(s_0, 0, s_1, 0, s_2, \ldots)$. So the criterion (10) of Krein tells us that the Stieltjes moment problem is indeterminate if

$$\int_0^{\infty} \frac{\log w(t)}{\sqrt{t}(1+t)}\, dt > -\infty. \tag{17}$$

However, as we explain now, an indeterminate symmetric Hamburger moment problem also has non-symmetric solutions. The set of solutions to an indeterminate Hamburger moment problem is described via the Nevanlinna parametrization. When the moment problem is symmetric, Pedersen [19] proved that the solution μ_φ is symmetric if and only if the Pick function φ is odd (with the convention that ∞ is odd). Obviously, there are quite a few odd Pick functions but even more are certainly not odd. In particular, the only symmetric N-extremal solutions are μ_0 and μ_∞. Moreover, the absolutely continuous solutions in (15) are symmetric exactly when $\beta = 0$.

The Nevanlinna matrix $\left(\begin{smallmatrix} A & C \\ B & D \end{smallmatrix}\right)$ for the symmetric Hamburger moment problem can be obtained from the Nevanlinna matrix for the original Stieltjes moment problem, see Chihara [9]. But A, B, C, D are closer related to the entire functions P, Q, R, S of Stieltjes which appear in the Krein parametrization. In fact, we have

$$\begin{aligned} A(z) &= zP(-z^2), \quad & C(z) &= R(-z^2), \\ B(z) &= -Q(-z^2), \quad & D(z) &= -S(-z^2)/z, \end{aligned} \tag{18}$$

and the Stieltjes solution ν_σ thus corresponds to the symmetric solution μ_φ if and only if

$$\varphi(z) = -\frac{1}{z\sigma(z^2)}.$$

In particular, ν_0 corresponds to μ_∞ and ν_∞ corresponds to μ_0 whereas all other N-extremal Stieltjes solutions correspond to (symmetric) canonical solutions of order 1.

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Preliminaries

The object of this thesis is to study the indeterminate moment problems within the Askey-scheme. At this point we therefore include a short section on basic hypergeometric series. We also include a section about Darboux's method which is a powerful tool to determine the entire functions from the Krein and Nevanlinna parametrizations.

Basic hypergeometric series

In this section we give a brief introduction to the notation and basic results from the theory of q-series. For proofs and more details, the reader is referred to the monograph [G&R] of Gasper and Rahman.

Unless otherwise stated, we will always assume that q denotes a fixed number in the open interval $(0,1)$. That is,

$$0 < q < 1.$$

The *q-shifted factorials* are defined by

$$(a;q)_0 = 1, \qquad (a;q)_n = \prod_{k=1}^{n}(1-aq^{k-1}), \quad n \in \mathbb{N} \cup \{\infty\}$$

for $a \in \mathbb{C}$ and by

$$(a;q)_{-n} = \frac{1}{(aq^{-n};q)_n}, \quad n \in \mathbb{N}$$

for $a \in \mathbb{C} \setminus \{q^k \mid k \in \mathbb{N}\}$. When several q-shifted factorials occur, we shall use the compact notation

$$(a_1, a_2, \ldots, a_k; q)_n = (a_1;q)_n (a_2;q)_n \cdots (a_k;q)_n, \quad n \in \mathbb{Z} \cup \{\infty\}.$$

The *q-binomial coefficient* is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}.$$

In the Appendix I of [G&R] one can find a number of identities involving q-shifted factorials and q-binomial coefficients which we shall use unhesitatingly.

The *basic hypergeometric series* ${}_r\phi_s$ is defined by

$$ {}_r\phi_s\left(\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix}\,\middle|\, q;z\right) = \sum_{n=0}^{\infty} \frac{(a_1,\dots,a_r;q)_n}{(b_1,\dots,b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}\frac{z^n}{(q;q)_n}. $$

It is assumed that none of the denominator factors vanish. If one of the numerator parameters equals q^{-k} for some $k \geq 0$, then the series terminates and represents a polynomial in z. Otherwise the radius of convergence is ∞ if $r < s+1$, or 1 if $r = s+1$ and 0 if $r > s+1$. The special case $r = s+1$ is particularly important and reads

$$ {}_{s+1}\phi_s\left(\begin{matrix} a_1,\dots,a_{s+1} \\ b_1,\dots,b_s \end{matrix}\,\middle|\, q;z\right) = \sum_{n=0}^{\infty} \frac{(a_1,\dots,a_{s+1};q)_n}{(b_1,\dots,b_s;q)_n}\frac{z^n}{(q;q)_n}, \quad |z|<1. $$

The *bilateral basic hypergeometric series* ${}_r\psi_r$ is defined by

$$ {}_r\psi_r\left(\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_r \end{matrix}\,\middle|\, q;z\right) = \sum_{n=-\infty}^{\infty} \frac{(a_1,\dots,a_r;q)_n}{(b_1,\dots,b_r;q)_n} z^n. $$

It is assumed that z and the parameters are such that each term of the series is well-defined. Since the series can be written as

$$ \sum_{n=0}^{\infty} \frac{(a_1,\dots,a_r;q)_n}{(b_1,\dots,b_r;q)_n} z^n + \sum_{n=1}^{\infty} \frac{(q/b_1,\dots,q/b_r;q)_n}{(q/a_1,\dots,q/a_r;q)_n}\left(\frac{b_1\cdots b_r}{a_1\cdots a_r z}\right)^n, $$

the region of convergence is the annulus

$$ \left|\frac{b_1\cdots b_r}{a_1\cdots a_r}\right| < |z| < 1. $$

The most important summation formula for q-series is the *q-binomial theorem*,

$$ {}_1\phi_0\left(\begin{matrix} a \\ - \end{matrix}\,\middle|\, q;z\right) = \frac{(az;q)_\infty}{(z;q)_\infty} \qquad (a\in\mathbb{C},\ |z|<1), \tag{i} $$

which is due to Cauchy and Heine. The two special cases,

$$ \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty} \qquad (|z|<1), \tag{ii} $$

$$ \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{(q;q)_n} z^n = (-z;q)_\infty \qquad (z\in\mathbb{C}), \tag{iii} $$

go back to Euler. Notice that $e_q(z) = 1/(z;q)_\infty$ as well as $E_q(z) = (-z;q)_\infty$ can be thought of as q-analogues of the exponential function. A finite version of the q-binomial theorem is

$$ \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}} z^k = (z;q)_n \qquad (z\in\mathbb{C}). \tag{iv} $$

Jacobi's *triple product identity*,

$$ \sum_{n=-\infty}^{\infty} (-1)^n q^{\binom{n}{2}} z^n = (z, q/z, q;q)_\infty \qquad (z\neq 0), \tag{v} $$

is a special case of *Ramanujan's sum*,

$$ {}_1\psi_1\left(\begin{matrix} a \\ b \end{matrix}\,\middle|\, q; z\right) = \frac{(q, b/a, az, q/az; q)_\infty}{(b, q/a, z, b/az; q)_\infty} \qquad (|b/a| < |z| < 1), \tag{vi} $$

which also contains the q-binomial theorem. Another generalization of the q-binomial theorem is *Heine's transformation formula*,

$$ {}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}\,\middle|\, q; z\right) = \frac{(b, az; q)_\infty}{(c, z; q)_\infty}\, {}_2\phi_1\left(\begin{matrix} c/b, z \\ az \end{matrix}\,\middle|\, q; b\right) \qquad (a, c \in \mathbb{C},\ |b| < 1,\ |z| < 1) \tag{vii} $$

$$ = \frac{(c/b, bz; q)_\infty}{(c, z; q)_\infty}\, {}_2\phi_1\left(\begin{matrix} abz/c, b \\ bz \end{matrix}\,\middle|\, q; \frac{c}{b}\right) \qquad (a \in \mathbb{C},\ |c/b| < 1,\ |z| < 1) \tag{viii} $$

$$ = \frac{(abz/c; q)_\infty}{(z; q)_\infty}\, {}_2\phi_1\left(\begin{matrix} c/a, c/b \\ c \end{matrix}\,\middle|\, q; \frac{abz}{c}\right) \qquad (|abz/c| < 1,\ |z| < 1), \tag{ix} $$

which leads to the *q-Gauss sum*,

$$ {}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}\,\middle|\, q; \frac{c}{ab}\right) = \frac{(c/a, c/b; q)_\infty}{(c, c/ab; q)_\infty} \qquad (|c/ab| < 1), \tag{x} $$

and the *${}_1\phi_1$–transformation*,

$$ {}_1\phi_1\left(\begin{matrix} a \\ c \end{matrix}\,\middle|\, q; b\right) = \frac{(b; q)_\infty}{(c; q)_\infty}\, {}_1\phi_1\left(\begin{matrix} ab/c \\ b \end{matrix}\,\middle|\, q; c\right) \qquad (a, b, c \in \mathbb{C}). \tag{xi} $$

A special case of the q-Gauss sum is

$$ {}_1\phi_1\left(\begin{matrix} b \\ c \end{matrix}\,\middle|\, q; \frac{c}{b}\right) = \frac{(c/b; q)_\infty}{(c; q)_\infty} \qquad (b, c \in \mathbb{C}) \tag{xii} $$

and a finite version is the *q-Chu–Vandermonde formula*,

$$ \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(b; q)_k}{(c; q)_k} (-1)^k q^{\binom{k}{2}} \left(\frac{c}{b}\right)^k = \frac{(c/b; q)_n}{(c; q)_n} \qquad (b, c \in \mathbb{C}). \tag{xiii} $$

The summation formula

$$ {}_1\phi_1\left(\begin{matrix} a \\ 0 \end{matrix}\,\middle|\, q; -q\right) = \frac{(aq; q^2)_\infty}{(q; q^2)_\infty} \qquad (a \in \mathbb{C}) \tag{xiv} $$

is a special case of the q-Kummer sum,

$$ {}_2\phi_1\left(\begin{matrix} a, b \\ aq/b \end{matrix}\,\middle|\, q; -\frac{q}{b}\right) = \frac{(aq, aq^2/b^2; q^2)_\infty}{(q; q^2)_\infty (-q/b, aq/b; q)_\infty} \qquad (a \in \mathbb{C},\ |b| > q). \tag{xv} $$

A more complete list of summation formulas can be found in Appendix II of [G&R] and Appendix III contains a number of transformation formulas.

Finally, we mention the *Ramanujan q-beta integral*,

$$ \int_0^\infty x^{c-1} \frac{(-ax; q)_\infty}{(-x; q)_\infty}\, dx = \frac{(a, q^{1-c}; q)_\infty}{(q, aq^{-c}; q)_\infty} \frac{\pi}{\sin \pi c} \qquad (c > 0,\ |a| < q^c), \tag{xvi} $$

and the more general *Askey–Roy q-beta integral*,

$$\int_0^\infty x^{c-1}\frac{(-ax,-bq/x;q)_\infty}{(-x,-q/x;q)_\infty}dx = \frac{(ab,q^c,q^{1-c};q)_\infty}{(q,aq^{-c},bq^c;q)_\infty}\frac{\pi}{\sin\pi c} \qquad (c>0,\ |a|<q^c,\ |b|<q^{-c}). \quad \text{(xvii)}$$

For details on these two integrals, [A&R] and [Ask] are better references than [G&R].

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Darboux's method

The aim of this section is to shortly present and explain Darboux's method. For additional information the reader is referred to Olver [Olv].

Suppose that $f(z)=\sum_{n=0}^\infty a_n z^n$ is holomorphic in a neighbourhood of 0. Sometimes it is useful to know the behaviour of the coefficients a_n when $n\to\infty$. In the chapter to follow, for instance, we are dealing with generating functions for orthogonal polynomials and are interested in the behaviour of the n'th polynomial as $n\to\infty$. If the radius of convergence for the above power series is $<\infty$, then the asymptotic behaviour of a_n when $n\to\infty$ is determined by the singularities closest to 0. Indeed, we have the following result due to Darboux in 1878.

Theorem (Darboux's method).
Suppose that $f(z)=\sum_{n=0}^\infty a_n z^n$ is holomorphic in $|z|<r$ for some $r\in(0,\infty)$ and has a finite number of singularities on $|z|=r$. If $g(z)=\sum_{n=0}^\infty b_n z^n$ is also holomorphic in $|z|<r$ and $f-g$ is continuous on $|z|=r$, then $a_n-b_n=o(r^{-n})$ for $n\to\infty$.

Proof. Since $f-g$ is holomorphic in $|z|<r$ and continuous on $|z|=r$, Cauchy's integral formula tells us that

$$a_n-b_n=\frac{1}{2\pi i}\int_{|z|=r}\frac{f(z)-g(z)}{z^{n+1}}dz=\frac{1}{2\pi r^n}\int_0^{2\pi}\left(f(re^{i\theta})-g(re^{i\theta})\right)e^{-in\theta}d\theta.$$

According to Riemann–Lebesgue's lemma,

$$\frac{1}{2\pi}\int_0^{2\pi}\left(f(re^{i\theta})-g(re^{i\theta})\right)e^{-in\theta}d\theta \longrightarrow 0 \quad \text{for } n\to\infty$$

and this proves the theorem. □

The point of Darboux's method is to find an appropriate comparison function $g(z)=\sum_{n=0}^\infty b_n z^n$, where the asymptotic behaviour of the coefficients b_n is known. In concrete examples one can often improve the estimate of the error term. In particular, if $f-g$ is m times continuously differentiable on $|z|=r$ then integration by parts leads to

$$a_n - b_n = o(n^{-m}r^{-n}) \quad \text{as } n \to \infty.$$

The simplest situation is the case where all the singularities of f on the circle of convergence are poles. Then one can let g be the sum of the principal parts at the poles. Suppose for simplicity that f is holomorphic in $|z|<r$ and only has one simple pole, say z_1, on $|z|=r$ with $a=\operatorname{Res}(f,z_1)$. If P_1 denotes the principal part of f at z_1, that is,

$$P_1(z) = \frac{a}{z-z_1} = -\sum_{n=0}^{\infty} \frac{a}{z_1^{n+1}} z^n, \quad |z|<r,$$

then $h_1 = f - P_1$ is holomorphic in $|z|<r_1$ for some $r_1>r$ and has a power series expansion, say $h_1(z) = \sum_{n=0}^{\infty} b_n z^n$ in $|z|<r_1$. For each $\rho<r_1$, the series

$$h_1(\rho e^{i\theta}) = \sum_{n=0}^{\infty} b_n \rho^n e^{in\theta}, \quad \theta \in \mathbb{R}$$

is the Fourier series of the C^∞-function $\theta \mapsto h_1(\rho e^{i\theta})$ and therefore $b_n\rho^n \to 0$ for $n\to\infty$, that is,

$$b_n = o(\rho^{-n}) \quad \text{for } n\to\infty.$$

All in all we conclude that

$$a_n = -\frac{a}{z_1^{n+1}} + o(\rho^{-n}) \quad \text{as } n\to\infty$$

and when $\rho>r$, the estimate for the error term is better than $o(r^{-n})$.

In the situation where f again only has one simple pole, say z_2, on $|z|=r_1$ with $b=\operatorname{Res}(f,z_2)$, we can give an even better estimate for the error term. Let P_2 denote the principal part of f at z_2. Then $h_2 = f-P_1-P_2$ is holomorphic in $|z|<r_2$ for some $r_2>r_1$ and with $h_2(z)=\sum_{n=0}^{\infty} c_n z^n$ in $|z|<r_2$, we have for each $\rho<r_2$ that

$$c_n = o(\rho^{-n}) \quad \text{for } n\to\infty.$$

Realizing that when $\rho>r_1$,

$$-\frac{b}{z_2^{n+1}} + o(\rho^{-n}) = \mathcal{O}(r_1^{-n}) \quad \text{as } n\to\infty,$$

it follows that

$$a_n = -\frac{a}{z_1^{n+1}} + \mathcal{O}(r_1^{-n}) \quad \text{as } n\to\infty.$$

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Indeterminate moment problems within the Askey-scheme

The very classical orthogonal polynomials of Hermite, Laguerre and Jacobi are characterized by a number of common properties. For instance they satisfy a second order linear differential equation of the Sturm–Liouville type, they possess a Rodrigues type formula and they have derivatives which again are orthogonal polynomials.

Over the years, several attempts have been made to characterize orthogonal polynomials in such a way that all important families fit into the same scheme. In this chapter the starting point will be the so-called *Askey-scheme* which consists of all orthogonal polynomials that can be obtained from the Askey–Wilson polynomials as limit or special cases. Just as for the very classical orthogonal polynomials, the polynomials in the Askey-scheme share a number of common properties. But the underlying operator need no longer be usual differentiation. Difference operators and more or less complicated q-difference operators also come into play.

A standard reference to the Askey-scheme is the report [26] of Koekoek and Swarttouw. We shall for short refer to this report as K&S.

In the present chapter we start by classifying the moment problems within the Askey-scheme. The aim is to single out the indeterminate cases in preparation for a more extensive study. The main part of the chapter is devoted to the Krein and Nevanlinna parametrizations. In the literature only the Nevanlinna parametrization seems to be commonly used but we insist on using the Krein parametrization whenever the moment problem is indeterminate in the sense of Stieltjes.

Once the entire functions from the Krein or Nevanlinna parametrization are computed, we seek to relate them to already known q-special functions. But more importantly, we use the entire functions to obtain solutions to the moment problems in a systematic way. In particular, we try to find the N-extremal solutions or at least to say as much as possible about the zeros of Q and S (or B and D). When the moment problem is only indeterminate in the sense of Hamburger, we also try to find absolutely continuous solutions of the form (15).

In the last part of the chapter we give a more elementary approach to the indeterminate moment problems within the Askey-scheme. The moment sequences are brought into focus and we explain the role of the q-Pearson equation which appear in connection with the second order q-difference equation.

Classification

As is well-known, there are no indeterminate moment problems within the classical Askey-scheme. Within the q-analogue of the Askey-scheme, on the other hand, there is a mixture of determinate and indeterminate moment problems. The aim of the present section is to pick out the indeterminate ones. This is effectively done by considering the three-term recurrence relations.

Though it is explicitly stated in K&S that $0 < q < 1$, we shall for some time just think of q as an arbitrary positive number. The special case $q = 1$ leads to the classical Askey-scheme and if $0 < q < 1$, the situation is described in K&S. If $q > 1$, it is common to replace q by $1/q$ and then again think of q as a number between 0 and 1.

The first observation is the fact that *q-Racah*, *q-Hahn*, *dual q-Hahn*, *quantum q-Krawtchouk*, *q-Krawtchouk*, *affine q-Krawtchouk* and *dual q-Krawtchouk* all satisfy a finite orthogonality relation. It is therefore trivial that the associated moment problems are determinate.

For *Askey–Wilson*, *continuous q-Hahn*, *big q-Jacobi*, *continuous q-Jacobi*, *little q-Jacobi* and *alternative q-Charlier* the recurrence coefficients are rational functions in q^n. One easily checks that the coefficients in question are bounded in n for all $q > 0$ and therefore the associated moment problems are determinate. Moreover, the unique solutions have bounded support. As a matter of form we mention that Askey–Wilson, continuous q-Hahn, big q-Jacobi, continuous q-Jacobi and little q-Jacobi are invariant under the interchange $q \leftrightarrow 1/q$. That is, the parameters change and one may have to rescale the variable but we stay inside the same family of polynomials.

For *continuous dual q-Hahn*, *Al-Salam–Chihara*, *q-Meixner–Pollaczek*, *big q-Laguerre*, *continuous big q-Hermite*, *continuous q-Laguerre*, *little q-Laguerre*, *Al-Salam–Carlitz I*, *continuous q-Hermite* and *discrete q-Hermite I* the recurrence coefficients are polynomials in q^n. The associated moment problems can therefore only be indeterminate if $q > 1$. Similarly, for *q-Meixner*, *q-Laguerre*, *q-Charlier*, *Al-Salam–Carlitz II*, *Stieltjes–Wigert* and *discrete q-Hermite II* the recurrence coefficients are polynomials in q^{-n}. Hence, the associated moment problems are determinate unless $0 < q < 1$.

Some of the polynomials in the q-analogue of the Askey-scheme are related to other polynomials within the same scheme via the interchange $q \leftrightarrow 1/q$. For instance big q-Laguerre turns into q-Meixner when q is replaced by $1/q$. The exact connection is

$$P_n(x; a, b; 1/q) = \frac{1}{(q/b; q)_n} M_n(xq/a; 1/a, -b; q).$$

Furthermore, little q-Laguerre is related to q-Laguerre via $q \leftrightarrow 1/q$ and the same holds for Al-Salam–Carlitz I and II as well as discrete q-Hermite I and II.

As a consequence we only need to consider *continuous dual q^{-1}-Hahn*, *Al-Salam–Chihara II*, *q^{-1}-Meixner–Pollaczek*, *q-Meixner*, *continuous big q^{-1}-Hermite*, *continuous q^{-1}-Laguerre*, *q-Laguerre*, *q-Charlier*, *Al-Salam–Carlitz II*, *continuous q^{-1}-Hermite*, *Stieltjes–Wigert* and *discrete q-Hermite II* in order to single out the indeterminate cases. By Al-Salam–Chihara II we mean Al-Salam–Chihara when $q > 1$ and q is replaced by $1/q$.

The continuous dual q-Hahn polynomials $p_n(x|q) := p_n(x; a, b, c|q)$ are generated by the three-term recurrence relation

$$\begin{aligned} 2xp_n(x|q) = p_{n+1}(x|q) &+ \big[(a+b+c)q^n + abcq^{n-1}(1 - q^n - q^{n+1})\big]p_n(x|q) \\ &+ (1 - q^n)(1 - abq^{n-1})(1 - acq^{n-1})(1 - bcq^{n-1})p_{n-1}(x|q), \quad n \geq 0 \end{aligned}$$

with initial conditions $p_{-1}(x|q) = 0$ and $p_0(x|q) = 1$. We define the continuous dual q^{-1}-Hahn polynomials by

$$P_n(x|q) := P_n(x; a, b, c|q) = (-i)^n p_n(ix; ia, ib, ic|1/q)$$

and they satisfy the three-term recurrence relation

$$\begin{aligned} 2xP_n(x|q) = P_{n+1}(x|q) + q^{-2n}\big[(a+b+c)q^n + abc(1+q-q^{n+1})\big]P_n(x|q) \\ + q^{-4n+3}(1-q^n)(ab+q^{n-1})(ac+q^{n-1})(bc+q^{n-1})P_{n-1}(x|q), \quad n \geq 0. \end{aligned} \tag{1.1}$$

Notice that a, b, c appear in a symmetric way and the polynomials are orthogonal on the real line if

$$a+b+c \in \mathbb{R} \quad \text{and} \quad ab, ac, bc \geq 0.$$

By the criterion of Chihara mentioned in the introduction, it is easy to see that the associated moment problem is indeterminate when $abc \neq 0$. The recurrence coefficients for the monic orthogonal polynomials are

$$c_n = \tfrac{1}{2}q^{-2n}\big[(a+b+c)q^n + abc(1+q-q^{n+1})\big]$$

and

$$\lambda_n = \tfrac{1}{4}q^{-4n+3}(1-q^n)(ab+q^{n-1})(ac+q^{n-1})(bc+q^{n-1}).$$

In particular, c_n has the form

$$c_n = f_n q^{-2n},$$

where (f_n) is both bounded and bounded away from 0 (when $abc \neq 0$). It is immediately established that

$$\lim_{n\to\infty} \frac{\lambda_{n+1}}{c_n c_{n+1}} = \frac{q}{(1+q)^2} < \frac{1}{4}$$

and the result follows.

The special case $c = 0$ leads to the Al-Salam–Chihara polynomials of type II, denoted by $Q_n(x|q) := Q_n(x; a, b|q)$. The three-term recurrence relation (1.1) reduces to

$$\begin{aligned} 2xQ_n(x|q) = Q_{n+1}(x|q) + q^{-n}(a+b)Q_n(x|q) \\ + q^{-2n+1}(1-q^n)(ab+q^{n-1})Q_{n-1}(x|q), \quad n \geq 0 \end{aligned} \tag{1.2}$$

and the polynomials are orthogonal on the real line exactly if

$$a+b \in \mathbb{R} \quad \text{and} \quad ab \geq 0.$$

Since $Q_n(-x; -a, -b|q) = (-1)^n Q_n(x; a, b|q)$, we can assume that $\mathrm{Re}(a), \mathrm{Re}(b) \geq 0$. Askey and Ismail [4] proved that the associated moment problem is indeterminate if and only if

$$a^2 + b^2 < (q + 1/q)\, ab,$$

which means that either $\bar{a} = b$ or $q < a/b < 1/q$. When $\bar{a} = b$, it is common to replace $a+b$ and ab by respectively $2c\cos\theta$ and c^2 for $c \geq 0$ and $0 \leq \theta \leq \pi/2$. In this way we obtain the q^{-1}-Meixner–Pollaczek polynomials which are studied by Chihara and Ismail [12]. The special case $\theta = \pi/2$ corresponds to $a = -b$ and leads to the symmetric Al-Salam–Chihara polynomials of type II, see [15]. When $a = q^{\frac{\alpha+1}{2}}$ and $b = q^{\frac{\alpha}{2}}$ for $\alpha \in \mathbb{R}$, we are dealing with the continuous q^{-1}-Laguerre polynomials. Since $a/b = \sqrt{q}$, the associated moment problem is indeterminate.

The continuous big q^{-1}-Hermite polynomials are obtained by setting $b = 0$ in (1.2). They are orthogonal on the real line for all $a \in \mathbb{R}$ but the associated moment problem is only indeterminate when $a = 0$. We thus end up with the continuous q^{-1}-Hermite polynomials studied by Ismail and Masson [22]. They satisfy the three-term recurrence relation

$$2xh_n(x|q) = h_{n+1}(x|q) + q^{-n}(1-q^n)h_{n-1}(x|q), \quad n \geq 0 \tag{1.3}$$

and can also be obtained from the continuous q^{-1}-Laguerre polynomials by letting $\alpha \to \infty$.

We now return to the continuous dual q^{-1}-Hahn polynomials. When $abc \neq 0$, it makes sense to replace x by $(abc)x$ and then $P_n\big((abc)x|q\big)/(abc)^n$ by $R_n(x|q)$. The resulting three-term recurrence relation is

$$\begin{aligned}2xR_n(x|q) = R_{n+1}(x|q) &+ q^{-2n}\big[(1/ab + 1/ac + 1/bc)q^n + 1 + q - q^{n+1}\big]R_n(x|q)\\ &+ q^{-4n+3}(1-q^n)(1+q^{n-1}/ab)(1+q^{n-1}/ac)(1+q^{n-1}/bc)R_{n-1}(x|q), \quad n \geq 0\end{aligned}$$

and we replace $1/ab$, $1/ac$, $1/bc$ with a, b, c to get

$$\begin{aligned}2xR_n(x|q) = R_{n+1}(x|q) &+ q^{-2n}\big[(a+b+c)q^n + 1 + q - q^{n+1}\big]R_n(x|q)\\ &+ q^{-4n+3}(1-q^n)(1+aq^{n-1})(1+bq^{n-1})(1+cq^{n-1})R_{n-1}(x|q), \quad n \geq 0.\end{aligned} \tag{1.4}$$

The polynomials generated by (1.4) are orthogonal on the real line if $a, b, c > -1$. Moreover, if two of the parameters a, b, c are complex conjugates (and *not* of the form $-q^{-k}$ for some $k \geq 0$) and the third parameter is > -1, then the polynomials are also orthogonal on $\mathbb{R}$.

Setting $a = 0$, replacing b by $-bq$, c by q/c, x by xq/c and then $R_n(xq/c|q)$ by $(q/c)^n P_n(x;q)$, we obtain the q-Meixner polynomials satisfying the three-term recurrence relation

$$\begin{aligned}2xP_n(x;q) = P_{n+1}(x;q) &+ cq^{-2n}\big[(1/c - b - 1)q^n + 1 + 1/q\big]P_n(x;q)\\ &+ cq^{-4n+1}(1-q^n)(1-bq^n)(c+q^n)P_{n-1}(x;q), \quad n \geq 0.\end{aligned}$$

They are orthogonal on the real line when $b < 1/q$ and $c > 0$. To follow K&S we consider the polynomials

$$M_n(x;q) := M_n(x;b,c;q) = \frac{(-1)^n q^{n^2}}{c^n (bq;q)_n} P_n(x/2;q),$$

which satisfy the three-term recurrence relation

$$\begin{aligned}q^{2n+1}(1-x)M_n(x;q) = c(1-bq^{n+1})M_{n+1}(x;q) &+ q(1-q^n)(c+q^n)M_{n-1}(x;q)\\ &- \big[c(1-bq^{n+1}) + q(1-q^n)(c+q^n)\big]M_n(x;q), \quad n \geq 0.\end{aligned}$$

Notice that the polynomials $F_n(x) = M_n(x+1;q)$ are birth and death polynomials, i.e., they satisfy a three-term recurrence relation of the form

$$(\lambda_n + \mu_n - x)F_n(x) = \lambda_n F_{n+1}(x) + \mu_n F_{n-1}(x), \quad n \geq 0,$$

where $\lambda_n > 0$ for $n \geq 0$ and $\mu_0 \geq 0$, $\mu_n > 0$ for $n \geq 1$. Such polynomials are known to be orthogonal on $[0, \infty)$, see e.g. Karlin and McGregor [25]. As a matter of fact, the birth and death rates are related to the coefficients in the continued fraction (1) via

$$m_n = \pi_{n-1} \quad \text{and} \quad l_n = \frac{1}{\pi_n \mu_n},$$

where

$$\pi_n = \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n}.$$

Without loss of generality we can assume that $m_1 = \pi_0 = 1$ and if $\mu_0 = 0$, the associated moment problem is therefore indeterminate in the sense of Stieltjes if and only if

$$\sum_{n=1}^{\infty} \left(\pi_n + \frac{1}{\pi_n \mu_n} \right) < \infty.$$

Moreover, the moment problem is indeterminate in the sense of Hamburger exactly if

$$\sum_{n=1}^{\infty} \pi_n \left(\sum_{k=0}^{n} \frac{1}{\pi_k \mu_k} \right)^2 < \infty.$$

It is now an easy matter to verify that the Stieltjes moment problem associated with the q-Meixner polynomials is indeterminate for $b < 1/q$ and $c > 0$.

The special case $b = 0$ leads to the q-Charlier polynomials. Usually c is also replaced by a and the associated Stieltjes moment problem is thus indeterminate for $a > 0$. If we set $b = q^\alpha$, replace x by $cq^\alpha x$ and let $c \to \infty$, we obtain the q-Laguerre polynomials studied by Moak [32]. See also [23] and [13]. They are orthogonal on $[0, \infty)$ for $\alpha > -1$ and the associated Stieltjes moment problem is always indeterminate. To obtain the Al-Salam–Carlitz polynomials of type II, set $b = -a/c$ and let $c \to 0$. The three-term recurrence relation is

$$xV_n^{(a)}(x;q) = V_{n+1}^{(a)}(x;q) + q^{-n}(a+1)V_n^{(a)}(x;q) + aq^{-2n+1}(1-q^n)V_{n-1}^{(a)}(x;q), \quad n \geq 0 \tag{1.5}$$

and it is curious that (1.5) almost can be obtained from (1.2). One just needs to replace the factor $(ab + q^{n-1})$ with $(ab + cq^{n-1})$ and then set $c = 0$ and $b = 1$. Berg and Valent [8] considered the birth and death polynomials

$$F_n(x) = \frac{(-1)^n q^{\binom{n}{2}}}{a^n} V_n^{(a)}(x+1;q),$$

which satisfy the three-term recurrence relation

$$-q^n x F_n(x) = aF_{n+1}(x) - (a+1-q^n)F_n(x) + (1-q^n)F_{n-1}(x), \quad n \geq 0. \tag{1.6}$$

They are orthogonal on $[0, \infty)$ for $a > 0$ and the associated Stieltjes moment problem is indeterminate if and only if $1 < a < 1/q$. Furthermore, the moment problem is indeterminate in the sense of Hamburger when $q < a < 1/q$.

The Stieltjes–Wigert polynomials are obtained from the q-Laguerre polynomials by replacing x with $q^{-\alpha}x$ and letting $\alpha \to \infty$. They can also be obtained from the q-Charlier polynomials by replacing x by ax and letting $a \to \infty$. The associated Stieltjes moment problem is indeterminate and studied in [11], [14]. The special case $a = 1$ of the q-Charlier polynomials seems to be interesting too. We shall refer to these polynomials as the special q-Charlier polynomials. They satisfy the three-term recurrence relation

$$-q^{2n+1} x c_n(x;q) = c_{n+1}(x;q) - (1+q)c_n(x;q) + q(1-q^{2n})c_{n-1}(x;q), \quad n \geq 0$$

and can also be obtained from the Al-Salam–Carlitz polynomials of type II by setting $a = 1/q$ after replacing q with q^2.

In general, the Al-Salam–Carlitz polynomials of type II are not orthogonal for $a \leq 0$. But when $a = -1$, they are orthogonal on the imaginary axis. The polynomials

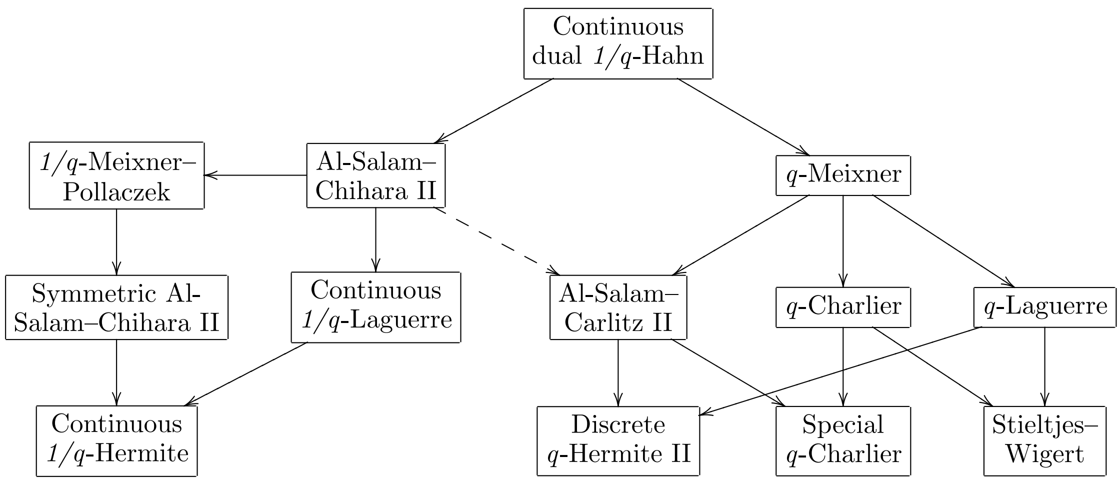
$$\tilde{h}_n(x;q) = (-i)^n V_n^{(-1)}(ix;q)$$

satisfy the three-term recurrence relation

$$x\tilde{h}_n(x;q) = \tilde{h}_{n+1}(x;q) + q^{-2n+1}(1-q^n)\tilde{h}_{n-1}(x;q), \quad n \geq 0$$

and are known as the discrete q-Hermite polynomials of type II. The associated moment problem is indeterminate in the sense of Hamburger. One can namely think of it as the symmetric moment problem corresponding to the q-Laguerre moment problem when $\alpha = -1/2$ and q is replaced by q^2.

We end this section with a table of the indeterminate moment problems within the Askey-Scheme.



As will be explained later on, the above table originates from simple transformations of the normal distribution. To be more precise, if the random variable X follows a normal distribution $N(0,\sigma^2)$ then the distribution of $\exp(X)$ is associated with the Stieltjes–Wigert polynomials whereas the distribution of $\sinh(X)$ is closely related to the continuous q^{-1}-Hermite polynomials.

The Krein and Nevanlinna parametrizations

In some sense, to solve an indeterminate moment problem on the real line $\mathbb{R}$ or on the half-line $[0,\infty)$ means to find the four entire functions from the Nevanlinna or Krein parametrization. A standard technique to compute these entire functions is to use generating functions.

Definition 1.1. By a generating function $G(x,t)$ for a sequence $\{P_n(x)\}$ of orthogonal polynomials we mean a series of the form

$$G(x,t) = \sum_n c_n P_n(x) t^n, \tag{1.7}$$

where (c_n) is a sequence of real (or complex) numbers and t can be thought of as a power series variable.

The series in (1.7) will only be of interest to us if the radius of convergence (in t) is neither 0 nor ∞. In this situation one can use Darboux's method to obtain the large n behaviour of $P_n(x)$. Darboux's method is well explained in Olver's book [33] on asymptotics and special functions but for the sake of completeness we have included a short description of the method in the preliminaries.

At the very top of the hierarchy presented in the previous section we find the continuous dual q^{-1}-Hahn polynomials generated by the three-term recurrence relation (1.1). Using the same procedure as in the proof of Lemma 1.2 below, one can establish the generating function

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)}}{(ab)^n(-1/ab,q;q)_n} P_n(x|q)t^n = \frac{(-t/a;q)_\infty}{(ct;q)_\infty}\,{}_2\phi_2\left(\begin{matrix} e^y/a, -e^{-y}/a \\ -1/ab, -t/a \end{matrix}\,\middle|\, q; -t/b\right), \quad |ct|<1.$$

Darboux's method then leads to

$$\frac{(-1)^n q^{n(n-1)}}{(abc)^n(-1/ab,q;q)_n} P_n(x|q) = \frac{(-1/ac;q)_\infty}{(q;q)_\infty}\,{}_2\phi_2\left(\begin{matrix} e^y/a, -e^{-y}/a \\ -1/ab, -1/ac \end{matrix}\,\middle|\, q; -1/bc\right) + \mathcal{O}(q^n)$$

but the above asymptotics is not suitable for finding an expression for the entire function D from the Nevanlinna parametrization. As of today, the continuous dual q^{-1}-Hahn moment problem still seems to be unsolved in the sense of finding the Nevanlinna matrix. A one-parameter family of solutions to the moment problem is obtained by Koelink and Stokman in [27].

Below continuous dual q^{-1}-Hahn the hierarchy breaks into two parts: The q-Meixner tableau and the Al-Salam–Chihara II tableau. In the next two sections we shall throw light on either case.

The q-Meixner tableau

The main new result of this section is Theorem 1.3 in which the entire functions from the Krein parametrization for the q-Meixner moment problem are computed. A number of corollaries is then to follow. It is briefly explained in the section on classification how to come from the q-Meixner polynomials to the special or limit cases which are important enough to have a name. We obtain the entire functions from the Krein or Nevanlinna parametrizations for the associated moment problems as corollaries.

We denote by $\{M_n(x;q)\}$ the solution to the three-term recurrence relation

$$\begin{aligned} q^{2n+1}(1-x)M_n(x;q) =& c(1-bq^{n+1})M_{n+1}(x;q) + q(1-q^n)(c+q^n)M_{n-1}(x;q) \\ &- \left[c(1-bq^{n+1}) + q(1-q^n)(c+q^n)\right]M_n(x;q) \end{aligned} \tag{1.8}$$

with initial conditions

$$M_{-1}(x;q) = 0, \quad M_0(x;q) = 1$$

and by $\{M_n^*(x;q)\}$ the solution with initial conditions

$$M_0^*(x;q) = 0, \quad M_1^*(x;q) = -\frac{q}{c(1-bq)}.$$

The first part of the following lemma is also stated in K&S.

Lemma 1.2. *Suppose that $b < 1/q$ and $c > 0$. For $|t| < 1$, we have*

$$\sum_{n=0}^{\infty} \frac{(bq;q)_n}{(-q/c,q;q)_n} M_n(x;q) t^n = \frac{1}{(t;q)_\infty} {}_1\phi_1 \left(\begin{matrix} -x/bc \\ -q/c \end{matrix} \middle| q; btq \right) \tag{1.9}$$

and

$$\sum_{n=1}^{\infty} \frac{(bq;q)_n}{(-q/c,q;q)_n} M_n^*(x;q) t^n = \frac{1}{(t;q)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^n q^{\binom{n+1}{2}} t^n}{(c+q^n)(q;q)_n} \sum_{k=0}^{n-1} \frac{(-bcq^{1-n}/x;q)_k}{(-cq^{1-n};q)_k} x^k. \tag{1.10}$$

Proof. Denote by $M(x,t)$ the formal power series

$$M(x,t) := \sum_{n=0}^{\infty} \frac{(bq;q)_n}{(-q/c,q;q)_n} M_n(x;q) t^n$$

and notice that $M(x,0) = 1$. Multiply with $(bq;q)_n t^{n+1}/(-q/c,q;q)_n$ in (1.8) and sum from $n = 0$ to ∞ to obtain

$$\begin{aligned}(1-x)tqM(x,tq^2) =& cM(x,t) + (1-c)M(x,tq) - M(x,tq^2) - ctM(x,t) \\ &+ bctqM(x,tq) - ctqM(x,t) - (1-c)tqM(x,tq) \\ &+ tqM(x,tq^2) + ct^2qM(x,t) - bct^2q^2M(x,tq)\end{aligned}$$

or simply the homogeneous equation

$$c(1-t)(1-tq)M(x,t) + (1-c+bctq)(1-tq)M(x,tq) - (1-xtq)M(x,tq^2) = 0. \tag{1.11}$$

Hence, the formal expression $N(x,t) := (t;q)_\infty M(x,t)$ satisfies

$$cN(x,t) + (1-c+bctq)N(x,tq) - (1-xtq)N(x,tq^2) = 0$$

and if we suppose that $N(x,t)$ has the form $\sum_{n=0}^{\infty} a_n t^n$, we get $a_0 = 1$ and

$$ca_n + (1-c)q^n a_n + bcq^n a_{n-1} - q^{2n} a_n + xq^{2n-1} a_{n-1} = 0, \quad n > 0,$$

that is,

$$(1+q^n/c)(1-q^n)a_n + bq^n(1+xq^{n-1}/bc)a_{n-1} = 0, \quad n > 0.$$

It follows recursively that

$$a_n = (-1)^n q^{\binom{n+1}{2}} \frac{(-x/bc;q)_n}{(-q/c,q;q)_n} b^n$$

and we have formally established (1.9). To make the proof rigorous notice that the right-hand side in (1.9), say $f(t)$, is holomorphic in $|t| < 1$. Therefore, $f(t)$ has a power series expansion around 0, say

$$f(t) = \sum_{n=0}^{\infty} f_n t^n, \quad |t| < 1.$$

Using the command qsumdiffeq in Maple as described by Koepf in [29, Chap. 10], it is easily obtained that

$$c(1-t)(1-tq)f(t) + (1-c+bctq)(1-tq)f(tq) - (1-xtq)f(tq^2) = 0.$$

Of course, this equation can also be verified by hand. In any case, the coefficients (f_n) are given by $f_0 = f(0) = 1$ and

$$\begin{aligned} q(1-x) &= (c+q)(1-q)f_1 - c(1-bq), \\ q^{2n-1}(1-x)f_{n-1} &= (c+q^n)(1-q^n)f_n + cq(1-bq^{n-1})f_{n-2} \\ &\quad - \big[c(1-bq^n) + q(1-q^{n-1})(c+q^{n-1})\big]f_{n-1}, \quad n \geq 2. \end{aligned}$$

Since the polynomials

$$M_n(x) := \frac{(bq;q)_n}{(-q/c,q;q)_n} M_n(x;q)$$

satisfy the same three-term recurrence relation and the initial conditions coincide, we conclude that f_n has the desired form.

To establish (1.10) one can repeat the above procedure with only a few modifications. Let $M^*(x,t)$ denote the formal power series

$$M^*(x,t) := \sum_{n=1}^{\infty} \frac{(bq;q)_n}{(-q/c,q;q)_n} M_n^*(x;q)t^n$$

and notice that $M^*(x,0) = 0$. Multiply with $(bq;q)_n t^{n+1}/(-q/c,q;q)_n$ in (1.8) and sum from $n = 1$ to ∞ to obtain the inhomogeneous equation

$$c(1-t)(1-tq)M^*(x,t) + (1-c+bctq)(1-tq)M^*(x,tq) - (1-xtq)M^*(x,tq^2) = -tq. \quad (1.12)$$

Accordingly, the formal expression $N^*(x,t) := (t;q)_\infty M^*(x,t)$ satisfies

$$cN^*(x,t) + (1-c+bctq)N^*(x,tq) - (1-xtq)N^*(x,tq^2) = (tq;q)_\infty - (tq^2;q)_\infty$$

and by (iii), we have

$$(tq;q)_\infty - (tq^2;q)_\infty = \sum_{n=1}^{\infty} (-1)^n \frac{q^{\binom{n+1}{2}}}{(q;q)_{n-1}} t^n.$$

So if $N^*(x,t)$ has the form $\sum_{n=0}^{\infty} a_n^* t^n$, it follows that $a_0^* = 0$ and

$$(c+q^n)(1-q^n)a_n^* + bcq^n(1+xq^{n-1}/bc)a_{n-1}^* = (-1)^n \frac{q^{\binom{n+1}{2}}}{(q;q)_{n-1}}, \quad n > 0.$$

By recursion we finally get

$$a_n^* = \frac{(-1)^n q^{\binom{n+1}{2}}}{(c+q^n)(q;q)_n} \sum_{k=0}^{n-1} \frac{(-bcq^{1-n}/x;q)_k}{(-cq^{1-n};q)_k} x^k$$

and the right-hand side of (1.10) is obtained. We skip the details in making the last part of the proof rigorous. □

The above lemma prepares the way for the following result.

Theorem 1.3. *Suppose that $b < 1/q$, $c > 0$ and consider the indeterminate Stieltjes moment problem associated with the q-Meixner polynomials $M_n(z+1;b,c;q)$. The entire functions from the Krein parametrization are given by*

$$\begin{aligned}
P(z) &= -\frac{(-q/c;q)_\infty}{(bq;q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^n q^{\binom{n+1}{2}}}{(c+q^n)(q;q)_n}\sum_{k=0}^{n-1}\frac{(-\frac{bcq^{1-n}}{1-z};q)_k}{(-cq^{1-n};q)_k}(1-z)^k,\\
Q(z) &= {}_1\phi_1\left(\begin{matrix}1-z\\ bq\end{matrix}\,\middle|\, q;-q/c\right),\\
R(z) &= 1-\frac{z}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^n q^{\binom{n+1}{2}+n}}{(c+q^n)(q;q)_n}\sum_{k=0}^{n-1}\frac{(-\frac{bcq^{1-n}}{1-z};q)_k}{(-cq^{1-n};q)_k}(1-z)^k,\\
S(z) &= \frac{z\,(bq^2;q)_\infty}{(-q/c,q;q)_\infty}\,{}_1\phi_1\left(\begin{matrix}q(1-z)\\ bq^2\end{matrix}\,\middle|\, q;-q/c\right).
\end{aligned}$$

Proof. Observe first of all that the orthonormal polynomials are given by

$$P_n(z) := (-1)^n\sqrt{\frac{(bq;q)_n q^n}{(-q/c,q;q)_n}}M_n(z+1;q).$$

They satisfy namely the three-term recurrence relation (5) with

$$a_n = \frac{c(1-bq^{n+1})+q(1-q^n)(c+q^n)}{q^{2n+1}} \quad\text{and}\quad b_n = \frac{c\sqrt{(1+q^{n+1}/c)(1-bq^{n+1})(1-q^{n+1})}}{q^{2n+3/2}}.$$

The polynomials of the second kind are given by

$$Q_n(z) := (-1)^n\sqrt{\frac{(bq;q)_n q^n}{(-q/c,q;q)_n}}M_n^*(z+1;q).$$

It follows from the three-term recurrence relation (1.8) that

$$c(1-bq^{n+1})\big(M_{n+1}(1;q)-M_n(1;q)\big) = q(1-q^n)(c+q^n)\big(M_n(1;q)-M_{n-1}(1;q)\big).$$

Since $M_1(1;q)=M_0(1;q)=1$, we have $M_n(1;q)=1$ for all $n\geq 0$ and therefore

$$P_n(0) = (-1)^n\sqrt{\frac{(bq;q)_n q^n}{(-q/c,q;q)_n}}.$$

Now set $t=q$ in (1.9) as well as (1.10) and combine (12) with (16) to obtain expressions for S and R. The ${}_1\phi_1$-transformation (xi) leads to the final expression for S.

An application of Darboux's method tells us that, as $n\to\infty$,

$$\frac{(bq;q)_n}{(-q/c,q;q)_n}M_n(z;q) = \frac{1}{(q;q)_\infty}\,{}_1\phi_1\left(\begin{matrix}-z/bc\\ -q/c\end{matrix}\,\middle|\, q;bq\right)+\mathcal{O}(q^n)$$

and

$$\frac{(bq;q)_n}{(-q/c,q;q)_n}M_n^*(z;q) = \frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^n q^{\binom{n+1}{2}}}{(c+q^n)(q;q)_n}\sum_{k=0}^{n-1}\frac{(-bcq^{1-n}/z;q)_k}{(-cq^{1-n};q)_k}z^k+\mathcal{O}(q^n).$$

We thus conclude that

$$Q(z) = \lim_{n\to\infty} M_n(1-z;q) = \frac{(-q/c;q)_\infty}{(bq;q)_\infty} {}_1\phi_1\left(\begin{matrix} -\frac{1-z}{bc} \\ -q/c \end{matrix} \,\middle|\, q; bq\right)$$

and in a similar way P can be obtained as the limit

$$P(z) = -\lim_{n\to\infty} M_n^*(1-z;q).$$

Again, the ${}_1\phi_1$-transformation (xi) leads to the desired expression for Q. □

The q-Charlier polynomials are just a special case of the q-Meixner polynomials. An easy consequence of Theorem 1.3 is thus the following result.

Corollary 1.4. *Suppose that $a > 0$ and consider the indeterminate Stieltjes moment problem associated with the q-Charlier polynomials $C_n(z+1;a;q)$. The entire functions from the Krein parametrization are given by*

$$\begin{aligned}
P(z) &= -(-q/a;q)_\infty \sum_{n=1}^\infty \frac{(-1)^n q^{\binom{n+1}{2}}}{(a+q^n)(q;q)_n} \sum_{k=0}^{n-1} \frac{(1-z)^k}{(-aq^{1-n};q)_k}, \\
Q(z) &= {}_1\phi_1\left(\begin{matrix} 1-z \\ 0 \end{matrix} \,\middle|\, q; -q/a\right), \\
R(z) &= 1 - \frac{z}{(q;q)_\infty} \sum_{n=1}^\infty \frac{(-1)^n q^{\binom{n+1}{2}+n}}{(a+q^n)(q;q)_n} \sum_{k=0}^{n-1} \frac{(1-z)^k}{(-aq^{1-n};q)_k}, \\
S(z) &= \frac{z}{(-q/a,q;q)_\infty} {}_1\phi_1\left(\begin{matrix} q(1-z) \\ 0 \end{matrix} \,\middle|\, q; -q/a\right).
\end{aligned}$$

Proof. Set $b = 0$ in Theorem 1.3 and replace c with a. □

In general, the ${}_1\phi_1$'s in Theorem 1.3 and Corollary 1.4 cannot be summed and it seems very hard to find a simple closed form for their zeros. However, in the special case $a = 1$ a remarkable simplification occurs. Using the special case (xiv) of the q-Kummer sum, we find that

$$Q(z) = \frac{(q(1-z);q^2)_\infty}{(q;q^2)_\infty}, \quad S(z) = \frac{(1-z;q^2)_\infty}{(q;q)_\infty} \tag{1.13}$$

and as a direct consequence, two of the N-extremal solutions are given by

$$\nu_0 = (q;q^2)_\infty \sum_{n=0}^\infty \frac{q^{\binom{2n+1}{2}}}{(q;q)_{2n+1}} \varepsilon_{q^{-(2n+1)}-1} \tag{1.14}$$

and

$$\nu_\infty = (q;q^2)_\infty \sum_{n=0}^\infty \frac{q^{\binom{2n}{2}}}{(q;q)_{2n}} \varepsilon_{q^{-2n}-1}. \tag{1.15}$$

We skip the computational details at this point since a more general result (Prop. 1.7) is just around the corner.

It is interesting to notice that the zeros of Q and S in (1.13) both are of the form

$$1 - Aq^{-2n}, \quad n \geq 0$$

for some constant $A > 0$. One should not expect the zeros of Q and S from Theorem 1.3 or Corollary 1.4 to be just as simple but their asymptotic behaviour turns out to be the same. A general theorem of Bergweiler and Hayman [9] leads to the following result.

Proposition 1.5. *Consider the entire function Q (or S) from Theorem 1.3 and let $0 \geq x_1 > \ldots > x_n > \ldots$ denote its zeros. There exists a constant $A > 0$ such that $1 - x_n \sim Aq^{-2n}$ as $n \to \infty$. More precisely,*

$$1 - x_n = Aq^{-2n}\big(1 + \mathcal{O}(q^n)\big) \quad \text{as } n \to \infty.$$

Proof. We learn from Maple (using qsumdiffeq) that the function $M_\infty(z) = Q(1-z)$ satisfies the equation

$$cq(z-b)M_\infty(z) + \big[q(1-c-bc)z + bc(1+q)\big]M_\infty(zq) - (1-zq)(bc+zq)M_\infty(zq^2) = 0.$$

Clearly, this is a functional equation of the form

$$\sum_{j=0}^{2} a_j(z)M_\infty(zq^j) = 0,$$

where the a_j's are polynomials. Following the notation of [9], we have $p_0 = p_1 = 1$, $p_2 = 2$ and $d_0 = d_1 = 0$, $d_2 = 1$. Thus, the Newton–Puiseux diagram only has two vertices, namely $(0,0)$ and $(2,1)$, and the hypothesis of [9, Thm. 2] is satisfied. In our set-up, c is replaced by q, $M = 1$, $N = 2$ and $\rho = \sqrt{q}$. The proposed statement follows easily and a similar result for the zeros of S can be obtained by replacing z with zq and b with bq. □

The Al-Salam–Carlitz polynomials of type II are a certain limit case of the q-Meixner polynomials. The associated moment problem is studied by Berg and Valent in [8] and on the following pages we go through some of their important results.

Corollary 1.6. *Suppose that $1 < a < 1/q$ and consider the indeterminate Stieltjes moment problem associated with the Al-Salam–Carlitz II polynomials $V_n^{(a)}(z+1;q)$. The entire functions from the Krein parametrization are given by*

$$P(z) = \frac{(q;q)_\infty}{a-1}\, {}_2\phi_1\left(\begin{matrix}\frac{1-z}{a}, 0\\ q/a\end{matrix}\,\middle|\, q;q\right), \quad R(z) = {}_2\phi_1\left(\begin{matrix}1-z, 0\\ aq\end{matrix}\,\middle|\, q;q\right),$$

$$Q(z) = \frac{(\frac{1-z}{a};q)_\infty}{(1/a;q)_\infty}, \qquad S(z) = \frac{(1-z;q)_\infty}{(aq,q;q)_\infty}.$$

Proof. The proof is based on Lemma 1.2 rather than Theorem 1.3. When we set $b = -a/c$ and let $c \to 0$, the second order q-difference equations (1.11) and (1.12) reduces to the first order equations

$$(1-t)(1-at)V(x,t) - (1-xt)V(x,tq) = 0, \quad V(x,0) = 1$$

and

$$(1-t)(1-at)V^*(x,t) - (1-xt)V^*(x,tq) = -t, \quad V^*(x,0) = 0.$$

Such a simplification occurs because the recurrence coefficients in (1.6) are linear in q^n. When $|t|\,,|at|<1$, iteration leads to

$$V(x,t)=\frac{(xt;q)_\infty}{(t,at;q)_\infty},\quad V^*(x,t)=-t\sum_{n=0}^\infty\frac{(xt;q)_n}{(t,at;q)_{n+1}}q^n$$

and the expressions for P, Q, R, S can now be obtained in the same way as in the proof of Theorem 1.3. □

Notice that the zeros of Q and S from Corollary 1.6 are respectively

$$x_n=1-aq^{-n},\ n\geq 0\quad\text{and}\quad y_n=1-q^{-n},\ n\geq 0.$$

It is a matter of form to prove the following result.

Proposition 1.7. *In the set-up of Corollary 1.6, the N-extremal solutions ν_0 and ν_∞ are given by*

$$\nu_0=(q/a;q)_\infty\sum_{n=0}^\infty\frac{a^{-n}q^{n^2}}{(q/a,q;q)_n}\varepsilon_{aq^{-n}-1}. \tag{1.16}$$

and

$$\nu_\infty=(aq;q)_\infty\sum_{n=0}^\infty\frac{a^{n}q^{n^2}}{(aq,q;q)_n}\varepsilon_{q^{-n}-1}. \tag{1.17}$$

Proof. We just have to evaluate the function ρ defined by

$$1/\rho(x)=Q(x)S'(x)-Q'(x)S(x)$$

at the points x_n and y_n. Since the derivative of the function $x\mapsto(x;q)_\infty$ is given by

$$-(x;q)_\infty\sum_{k=0}^\infty\frac{q^k}{1-xq^k},$$

we have

$$\rho(x)=\frac{(1/a,aq,q;q)_\infty}{(1-x,\frac{1-x}{a};q)_\infty}\Big(\sum_{k=0}^\infty\frac{q^k}{1-(1-x)q^k}-\sum_{k=0}^\infty\frac{q^k}{a-(1-x)q^k}\Big)^{-1}.$$

Using the fact that

$$(t;q)_\infty\sum_{k=0}^\infty\frac{q^k}{1-tq^k}\Big|_{t=q^{-n}}=(q^{-n};q)_n(q;q)_\infty q^n,$$

we thus get

$$\rho(x_n)=(q/a;q)_\infty\frac{a^{-n}q^{n^2}}{(q/a,q;q)_n},\quad\rho(y_n)=(aq;q)_\infty\frac{a^nq^{n^2}}{(aq,q;q)_n}$$

and the proof is completed. □

Remark 1.8. As mentioned in the end of the section on classification, the special q-Charlier polynomials can be obtained from the Al-Salam–Carlitz polynomials of type II by replacing q with q^2 and setting $a=1/q$. In this way Proposition 1.7 leads to the solutions (1.14) and (1.15).

When $q < a \le 1$, the Al-Salam–Carlitz II moment problem is no longer indeterminate in the sense of Stieltjes but still indeterminate in the sense of Hamburger. The following result is more general than Corollary 1.6.

Theorem 1.9. *Suppose that $q < a < 1/q$ and consider the indeterminate Hamburger moment problem associated with the Al-Salam–Carlitz II polynomials $V_n^{(a)}(z+1;q)$. If $\xi(a)$ denotes the constant*

$$\xi(a) = \frac{(q;q)_\infty}{a-1} {}_2\phi_1\left(\begin{matrix} 1/a, 0 \\ q/a \end{matrix} \,\middle|\, q;q\right),$$

then the entire functions from the Nevanlinna parametrization are given by

$$A(z) = \frac{(q;q)_\infty}{a-1} {}_2\phi_1\left(\begin{matrix} \frac{1+z}{a}, 0 \\ q/a \end{matrix} \,\middle|\, q;q\right) - \xi(a) {}_2\phi_1\left(\begin{matrix} 1+z, 0 \\ aq \end{matrix} \,\middle|\, q;q\right), \quad C(z) = {}_2\phi_1\left(\begin{matrix} 1+z, 0 \\ aq \end{matrix} \,\middle|\, q;q\right),$$

$$B(z) = -\frac{(\frac{1+z}{a};q)_\infty}{(1/a;q)_\infty} + \xi(a)\frac{(1+z;q)_\infty}{(aq,q;q)_\infty}, \qquad D(z) = -\frac{(1+z;q)_\infty}{(aq,q;q)_\infty},$$

where the expressions for A and B have to be interpreted as the limits

$$A^{(1)}(z) = (q;q)_\infty \sum_{n=0}^\infty \frac{(1+z;q)_n}{(q;q)_n^2} q^n \left(\sum_{k=0}^{n-1} \frac{(1+z)q^k}{1-(1+z)q^k} - 2\sum_{k=1}^n \frac{q^k}{1-q^k} - \sum_{k=1}^\infty \frac{q^k}{(q;q)_k(1-q^k)} \right)$$

and

$$B^{(1)}(z) = -\frac{(1+z;q)_\infty}{(q;q)_\infty} \left(\sum_{k=0}^\infty \frac{(1+z)q^k}{1-(1+z)q^k} + 1 - 2\sum_{k=1}^\infty \frac{q^k}{1-q^k} - \sum_{k=1}^\infty \frac{q^k}{(q;q)_k(1-q^k)} \right)$$

when $a = 1$.

We shall not give a detailed proof of the above theorem but merely comment on the special case $a = 1$. Berg and Valent proved in [8] that the entire functions A, B, C, D from Theorem 1.9 depend continuously on a for $q < a < 1/q$. Since the recurrence coefficients in (1.6) are continuous in a (and q), this result can be obtained by proving that

$$\sum_{n=0}^\infty P_n^2(0) \quad \text{and} \quad \sum_{n=0}^\infty Q_n^2(0)$$

converge uniformly in compact subsets of $\{(a,q) \,|\, q < a < 1/q\}$, cf. [8, Prop. 2.4.1 & Rem. 2.4.2]. As usual, (P_n) denote the orthonormal polynomials and (Q_n) the polynomials of the second kind. Recalling that the derivative of $(a;q)_n$ is given by

$$-(a;q)_n \sum_{k=0}^{n-1} \frac{q^k}{1-aq^k}$$

for $n \in \mathbb{N} \cup \{\infty\}$, the limits of $A(z)$ and $B(z)$ as $a \to 1$ can be found using l'Hospital's rule on the first terms in the expressions

$$A(z) = \frac{(q;q)_\infty}{a-1} \sum_{n=0}^\infty \frac{q^n}{(q;q)_n} \left(\frac{(\frac{1+z}{a};q)_n}{(q/a;q)_n} - \frac{(1+z;q)_n}{(aq;q)_n} \right)$$
$$- (q;q)_\infty \sum_{k=1}^\infty \frac{q^k}{(q;q)_k(a-q^k)} \sum_{n=0}^\infty \frac{(1+z;q)_n}{(aq,q;q)_n} q^n$$

and

$$B(z)=\frac{1}{a-1}\left(\frac{(1+z;q)_\infty}{(aq;q)_\infty}-a\frac{(\frac{1+z}{a};q)_\infty}{(q/a;q)_\infty}\right)+\sum_{k=1}^{\infty}\frac{q^k}{(q;q)_k(a-q^k)}\frac{(1+z;q)_\infty}{(aq;q)_\infty}.$$

The discrete measures in Proposition 1.7 are also N-extremal solutions when $q<a\le 1$. For $a=1$ they actually coincide and represent the unique solution on $[0,\infty)$. In the case $q<a<1$, the measure (1.17) is still the unique solution on $[0,\infty)$ whereas the measure (1.16) has exactly one negative mass point, namely $a-1$.

The solution (1.17) was discovered by Al-Salam and Carlitz [1]. Only some years later, Chihara [10] pointed out that it is N-extremal and the first example of this kind.

We conclude with the following result also due to Berg and Valent [8]. The counterpart corresponding to $a=1$ is stated in [8, Prop. 4.7.2].

Proposition 1.10. *Suppose that $a\neq 1$. In the set-up of Theorem 1.9, we have for each $c>0$ an absolutely continuous solution with density*

$$v_c(x)=\frac{|a-1|}{\pi a/c}\frac{(q/a,aq,q;q)_\infty}{(\frac{x+1}{a};q)_\infty^2+c^2(x+1;q)_\infty^2},\quad x\in\mathbb{R}. \tag{1.18}$$

Proof. Suppose that β lies strictly between 0 and $-1/\xi(a)$ and is related to γ by the equation

$$\gamma^2=-\beta\big(\beta+1/\xi(a)\big).$$

A few computations then lead to

$$\big(\beta B(x)-D(x)\big)^2+\gamma^2B^2(x)=-\frac{\beta}{\xi(a)}\big(B(x)+\xi(a)D(x)\big)^2+(\beta\xi(a)+1)D^2(x)$$

and with $b=-\xi(a)\gamma/\beta$, the density (15) has the form

$$\frac{d\mu_{\beta,\gamma}}{dx}=\frac{b/\pi}{\big(B(x)+\xi(a)D(x)\big)^2+\big(bD(x)\big)^2},\quad x\in\mathbb{R}.$$

Observe that b can take any positive value when β varies between 0 and $-1/\xi(a)$ and replace b with

$$\frac{ca}{|a-1|}\frac{(aq,q;q)_\infty}{(q/a;q)_\infty}$$

to obtain the density in (1.18). $\square$

The q-Laguerre polynomials are yet another limit case of the q-Meixner polynomials. They were studied by Moak [32] and more recently by Ismail and Rahman [23]. See also [13].

Corollary 1.11. *Suppose that $\alpha>-1$ and consider the indeterminate Stieltjes moment problem associated with the q-Laguerre polynomials $L_n^{(\alpha)}(z;q)$. The entire functions from the Krein* *parametrization are given by*

$$\begin{aligned}
P(z) &= -\frac{q^{\alpha}}{(q^{\alpha+1};q)_\infty}\sum_{n=1}^{\infty}(-1)^n\frac{q^{\binom{n+1}{2}}}{(q;q)_n}\sum_{k=0}^{n-1}(zq^{n-k};q)_k q^{\alpha k},\\
Q(z) &= \sum_{n=0}^{\infty}\frac{q^{n(n+\alpha)}}{(q^{\alpha+1},q;q)_n}z^n,\\
R(z) &= 1-\frac{zq^{\alpha}}{(q;q)_\infty}\sum_{n=1}^{\infty}(-1)^n\frac{q^{\binom{n+1}{2}+n}}{(q;q)_n}\sum_{k=0}^{n-1}(zq^{n-k};q)_k q^{\alpha k},\\
S(z) &= \frac{z(q^{\alpha+2};q)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\frac{q^{n(n+\alpha+1)}}{(q^{\alpha+2},q;q)_n}z^n.
\end{aligned}$$

Proof. Set $b = q^{\alpha}$ and let $c \to \infty$ in the expressions for

$$cq^{\alpha}P(cq^{\alpha}z+1), \quad Q(cq^{\alpha}z+1), \quad R(cq^{\alpha}z+1), \quad S(cq^{\alpha}z+1)/cq^{\alpha}$$

from Theorem 1.3. □

The expressions for Q and S are due to Moak [32], but can also be found in [23] and [13]. Moak pointed out that Q and S are closely related to the second q-Bessel function defined by

$$J_{\nu}^{(2)}(z;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\left(\frac{z}{2}\right)^{\nu}\sum_{n=0}^{\infty}(-1)^n\frac{q^{n(n+\nu)}}{(q^{\nu+1},q;q)_n}\left(\frac{z}{2}\right)^{2n}.$$

To be precise, we have

$$Q(-z) = \frac{(q;q)_\infty}{(q^{\alpha+1};q)_\infty}z^{-\frac{\alpha}{2}}J_{\alpha}^{(2)}(2\sqrt{z};q) \quad \text{and} \quad S(-z) = -z^{\frac{1-\alpha}{2}}J_{\alpha+1}^{(2)}(2\sqrt{z};q).$$

Moreover, Moak proved that the zeros of Q and S are very well separated. That is, if $0 \geq x_1 > \ldots > x_n > \ldots$ denote the zeros of Q or S, then

$$\frac{x_{n+1}}{x_n} > q^{-2} \quad \text{for } n \geq 1.$$

The zeros of the second q-Bessel function were first studied by Ismail in [20]. Recently, the results on their asymptotic behaviour have been improved considerably by Hayman. To begin with, notice that Q satisfies the functional equation

$$Q(z) - (1-q^{\alpha})Q(zq) + q^{\alpha}(1-zq)Q(zq^2) = 0.$$

According to [9, Thm. 2] this means that the zeros of Q behave like

$$x_n = Aq^{-2n}\big(1+\mathcal{O}(q^n)\big) \quad \text{as } n \to \infty,$$

where $A < 0$ is some constant. But in [19] Hayman establishes the more precise result saying that there exists a sequence (b_k) of real numbers such that for each $N \in \mathbb{N}$ we have the asymptotic expansion

$$x_n = -q^{1-2n}\left(1+\sum_{k=1}^{N}b_k q^{nk} + \mathcal{O}\big(q^{(N+1)n}\big)\right) \quad \text{as } n \to \infty.$$

We shall not go into details on how to determine the real numbers b_k but only mention that the first one is given by $b_1 = -(1+q^\alpha)(q,q^3;q^2)_\infty/(q^2;q^2)_\infty^2$.

The expressions for P and R in Corollary 1.11 differ a little from the ones in [23] and [13]. To make it clear that we are dealing with different expressions for the same functions, we point out that the generating function (1.10) reduces to the generating function established in [23, p. 161–162] when we take the appropriate limit. Moreover, it is verified in [13] that A, B, C, D from [23] and A, B, C, D from [13] coincide.

The Stieltjes–Wigert polynomials are a limit case of both the q-Laguerre polynomials and the q-Charlier polynomials.

Corollary 1.12. *Consider the indeterminate Stieltjes moment problem associated with the Stieltjes–Wigert polynomials $S_n(z;q)$. The entire functions from the Krein parametrization are given by*

$$\begin{aligned}
P(z) &= \sum_{n=1}^{\infty}(-1)^n \frac{q^{\binom{n}{2}}}{(q;q)_n}\sum_{k=1}^{n}(-1)^k q^{nk-\binom{k}{2}} z^{k-1},\\
Q(z) &= \sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n} z^n,\\
R(z) &= 1+\frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty}(-1)^n \frac{q^{\binom{n+1}{2}}}{(q;q)_n}\sum_{k=1}^{n}(-1)^k q^{nk-\binom{k}{2}} z^{k},\\
S(z) &= \frac{z}{(q;q)_\infty}\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(q;q)_n} z^n.
\end{aligned}$$

Proof. As explained in [13], the result follows by letting $\alpha \to \infty$ in the expressions for

$$q^{-\alpha}P(q^{-\alpha}z), \quad Q(q^{-\alpha}z), \quad R(q^{-\alpha}z), \quad q^{\alpha}S(q^{-\alpha}z)$$

from Corollary 1.11. The result can also be obtained from Corollary 1.4 in a similar way. □

The above expressions coincide with the ones in [14, Thm. 3.5]. As regards P and R, one just has to interchange the order of summation. Notice that the entire functions Q and S can be written in terms of the function

$$\Phi(z) = \sum_{n=0}^{\infty}\frac{q^{n^2}z^n}{(q;q)_n},$$

also called the entire Rogers–Ramanujan function. This function and its zeros were studied in [14]. It was proved that the zeros, say $0 > x_1 > \ldots > x_n > \ldots$, are very well separated and obtained as a corollary that $x_{n+1}/x_n \to q^{-2}$ as $n \to \infty$. On the way the author used the curious identity

$$(-zq;q)_\infty \sum_{n=0}^{\infty}\frac{q^{n(n+1)}z^n}{(-zq,q^2;q^2)_n} = \Phi(z) = (-zq^2;q^2)_\infty \sum_{n=0}^{\infty}\frac{q^{n^2}z^n}{(-zq^2,q^2;q^2)_n}$$

due to Rogers [35], and the fact that Φ satisfies the functional equation

$$\Phi(z) - \Phi(zq) - zq\Phi(zq^2) = 0.$$

We mention here that this equation in itself implies the zeros to behave like

$$x_n = Aq^{-2n}\big(1+\mathcal{O}(q^n)\big) \quad \text{as } n\to\infty$$

for some constant $A<0$. It was also pointed out in [14] that $\Phi(-1)$ and $\Phi(-q)$ appear in the celebrated Rogers–Ramanujan identities

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q,q^4;q^5)_\infty}$$

and

$$\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(q;q)_n} = \frac{1}{(q^2,q^3;q^5)_\infty}.$$

Recently, Andrews and Berndt have discovered that Φ appears on p. 57 of Ramanujan's lost notebook [34] in the remarkable identity

$$\sum_{n=0}^{\infty}\frac{q^{n^2}z^n}{(q;q)_n} = \prod_{n=1}^{\infty}\left(1+\frac{zq^{2n-1}}{1-q^ny_1-q^{2n}y_2-q^{3n}y_3-\dots}\right), \tag{1.19}$$

where

$$y_1 = \frac{1}{(1-q)\psi^2(q)},\quad y_2=0,\quad y_4=y_1y_3,$$
$$y_3 = \frac{q+q^3}{(1-q)(1-q^2)(1-q^3)\psi^2(q)} - \frac{\sum_{n=0}^{\infty}\frac{(2n+1)q^{2n+1}}{1-q^{2n+1}}}{(1-q)^3\psi^6(q)},$$

and

$$\psi(q) = \sum_{n=0}^{\infty} q^{\binom{n+1}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}.$$

As usual, Ramanujan just states this formula without giving a proof. The identity (1.19) is presumably true for $|q|<1$ but Andrews [2] only succeeded in proving it when $0<q<1/4$. The idea of his proof is first to show that the zeros of Φ are given exactly by the convergent series

$$x_n = -q^{1-2n}\sum_{j=0}^{\infty}a_{n,j}q^j,$$

where the $a_{n,j}$'s are uniquely determined from

$$0=\sum_{k=0}^{\infty}\frac{q^{k(k+1)-2nk}}{(q;q)_k}\Big(-\sum_{j=0}^{\infty}a_{n,j}q^j\Big)^k \quad\text{or}\quad 0=\sum_{k=-\infty}^{\infty}q^{k(k+1)}(q^{n+k+1};q)_\infty\Big(-\sum_{j=0}^{\infty}a_{n,j}q^j\Big)^k.$$

The next step is then to establish the identity

$$\sum_{j=0}^{\infty}a_{n,j}q^j = 1-\sum_{i=1}^{\infty}y_iq^{ni},$$

from which (1.19) follows. For the record, we mention as Andrews that

$$
\begin{aligned}
x_1 &= -\,q^{-2}(1 - q + q^2 - 2q^3 + 4q^4 - 8q^5 + 16q^6 - 33q^7 + \ldots) \\
x_2 &= -\,q^{-4}(1 - q^2 + q^3 - 2q^4 + 4q^5 - 7q^6 + 11q^7 - 18q^8 + 33q^9 - \ldots) \\
x_3 &= -\,q^{-6}(1 - q^3 + q^4 - 2q^5 + 4q^6 - 7q^7 + 11q^8 - 17q^9 + 27q^{10} \\
&\qquad\qquad - 43q^{11} + 68q^{12} - 112q^{13} + \ldots) \\
x_4 &= -\,q^{-8}(1 - q^4 + q^5 - 2q^6 + 4q^7 - 7q^8 + 11q^9 - 17q^{10} + 27q^{11} \\
&\qquad\qquad - 42q^{12} + 62q^{13} - 91q^{14} + 138q^{15} - 213q^{16} + \ldots) \\
x_5 &= -\,q^{-10}(1 - q^5 + q^6 - 2q^7 + 4q^8 - 7q^9 + 11q^{10} - 17q^{11} + 27q^{12} \\
&\qquad\qquad - 42q^{13} + 62q^{14} - 90q^{15} + 132q^{16} - 192q^{17} + 275q^{18} - 398q^{19} + \ldots)
\end{aligned}
$$

and in the search for patterns we notice that $a_{n,j} = 0$ for $j = 1, \ldots, n-1$ and $a_{n+1,j+1} = a_{n,j}$ for $j = 1, \ldots, 2n+2$ at least when n is small.

The discrete q-Hermite polynomials of type II can be obtained from the q-Laguerre polynomials in the same way as one can obtain the Hermite polynomials from the Laguerre polynomials. The relationship is given by

$$\tilde{h}_{2n}(x;q) = (-1)^n \frac{(q^2;q^2)_n}{q^{n(2n-1)}} L_n^{(-1/2)}(x^2;q^2)$$

and

$$\tilde{h}_{2n+1}(x;q) = (-1)^n \frac{(q^2;q^2)_n}{q^{n(2n-1)}} x L_n^{(1/2)}(x^2;q^2)$$

and we arrive at the following result.

Corollary 1.13. *Consider the indeterminate Hamburger moment problem associated with the discrete q-Hermite II polynomials $\tilde{h}_n(z;q)$. The entire functions from the Nevanlinna parametrization are given by*

$$
\begin{aligned}
A(z) &= -\frac{z}{(q;q^2)_\infty} \sum_{n=1}^{\infty} (-1)^n \frac{q^{n(n+1)}}{(q^2;q^2)_n} \sum_{k=0}^{n-1} \frac{(-z^2 q^{2(n-k)};q^2)_k}{q^{k+1}}, \\
B(z) &= -\sum_{n=0}^{\infty} (-1)^n \frac{q^{\binom{2n}{2}}}{(q;q)_{2n}} z^{2n}, \\
C(z) &= 1 + \frac{z^2}{(q^2;q^2)_\infty} \sum_{n=1}^{\infty} (-1)^n \frac{q^{n(n+3)}}{(q^2;q^2)_n} \sum_{k=0}^{n-1} \frac{(-z^2 q^{2(n-k)};q^2)_k}{q^{k+1}}, \\
D(z) &= \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} (-1)^n \frac{q^{\binom{2n+1}{2}}}{(q;q)_{2n+1}} z^{2n+1}.
\end{aligned}
$$

Proof. The result follows from (18) after setting $\alpha = -1/2$ and replacing q by q^2 in Corollary 1.11. □

The entire functions B and D are essentially the basic trigonometric functions Cos_q, Sin_q defined by

$$\mathrm{Cos}_q(z) = \tfrac{1}{2}\big(E_q(iz) + E_q(-iz)\big) \quad \text{and} \quad \mathrm{Sin}_q(z) = \tfrac{1}{2i}\big(E_q(iz) - E_q(-iz)\big),$$

where E_q is one of the q-analogues of the exponential function, cf. (iii). To be precise, we have

$$B(z) = -\operatorname{Cos}_q(z) \quad \text{and} \quad D(z) = \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty} \operatorname{Sin}_q(z).$$

By applying the identity

$$\operatorname{Cos}_q^2(z) + \operatorname{Sin}_q^2(z) = E_q(iz)E_q(-iz) = (-z^2;q^2)_\infty,$$

Ismail and Rahman [23] observed that when $\beta = 0$ and $\gamma = (q;q^2)_\infty/(q^2;q^2)_\infty$, the density (15) has the form

$$\frac{d\mu_{0,\gamma}}{dx} = \frac{1}{\pi} \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \frac{1}{(-x^2;q^2)_\infty}, \quad x \in \mathbb{R}. \tag{1.20}$$

This solution was also found by Berg [6], but in a different way. Unlike the zeros of the classical trigonometric functions, the zeros of Cos_q and Sin_q are rather complicated. If $0 < x_1 < \ldots < x_n < \ldots$ denote the positive zeros of Cos_q and $0 = y_0 < y_1 < \ldots < y_n < \ldots$ the positive zeros of Sin_q, Suslov proved in [37] that

$$x_n = q^{3/2-2n} - c(q) + o(1)$$

and

$$y_n = q^{1/2-2n} - c(q) + o(1)$$

as $n \to \infty$, where $c(q)$ is the constant given by

$$c(q) = \frac{\sqrt{q}}{2(1-q)} \frac{(q^2;q^4)_\infty}{(q^4;q^4)_\infty}.$$

We end by noticing that the expressions for B and D lead to a simple expression for the function ρ defined in (14). It turns out that

$$\rho(x) = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \frac{1}{(-x^2;q^2)_\infty} \left(\sum_{n=0}^{\infty} \frac{q^n}{1 + x^2 q^{2n}} \right)^{-1}, \quad x \in \mathbb{R}.$$

The Al-Salam–Chihara II tableau

In this section we take a closer look at the Al-Salam–Chihara II moment problem. Following the approach of Askey and Ismail in [4], we start by proving that the moment problem is indeterminate if and only if $\bar{a} = b$ or $q < a/b < 1/q$. The most general result is then Theorem 1.15 due to Chihara and Ismail [12]. In this theorem the entire functions from the Nevanlinna parametrization are computed when $\bar{a} = b$ and $a \neq b$. As corollaries we present the Nevanlinna matrices for the symmetric Al-Salam–Chihara II moment problem and the continuous q^{-1}-Hermite moment problem.

We denote by $\{Q_n(x|q)\}$ the solution to the three-term recurrence relation

$$2xQ_n(x|q) = Q_{n+1}(x|q) + q^{-n}(a+b)Q_n(x|q) + q^{-2n+1}(1-q^n)(ab+q^{n-1})Q_{n-1}(x|q)$$

with initial conditions

$$Q_{-1}(x|q) = 0, \quad Q_0(x|q) = 1$$

and by $\{Q_n^*(x|q)\}$ the solution with initial conditions

$$Q_0^*(x|q) = 0, \quad Q_1^*(x|q) = 2.$$

For reasons to be explained later on it is appropriate to write

$$x = \sinh y$$

and throughout the section we shall assume that x and y are related in this way.

Askey and Ismail [4] considered the polynomials

$$v_n(x) := \frac{(-1)^n q^{\binom{n}{2}}}{(q;q)_n} Q_n(x|q),$$

which are generated by the three-term recurrence relation

$$-2xq^n v_n(x) = (1 - q^{n+1}) v_{n+1}(x) - (a+b) v_n(x) + (ab + q^{n-1}) v_{n-1}(x).$$

Since the recurrence coefficients are linear in q^n, it is an easy matter to establish the following generating functions.

Lemma 1.14. *Suppose that $a + b \in \mathbb{R}$ and $ab \geq 0$. For $|at|, |bt| < 1$, we have*

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}}}{(q;q)_n} Q_n(x|q) t^n = \frac{(e^y t, -e^{-y} t; q)_\infty}{(at, bt; q)_\infty} \tag{1.21}$$

and

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}}}{(q;q)_n} Q_n^*(x|q) t^n = -2t \sum_{n=0}^{\infty} \frac{(e^y t, -e^{-y} t; q)_n}{(at, bt; q)_{n+1}} q^n. \tag{1.22}$$

In order to determine the large n behaviour of $Q_n(x|q)$ we have to consider a few different cases. Notice first that the conditions $a + b \in \mathbb{R}$ and $ab \geq 0$ imply that $\bar{a} = b$ when $|a| = |b|$ and $a, b \in \mathbb{R}$ when $|a| \neq |b|$.

When $a = b$, the singularity of (1.21) (or (1.22)) closest to zero is a double pole at $t = 1/a$. So Darboux's method leads to

$$\frac{(-1)^n q^{\binom{n}{2}}}{a^n (q;q)_n} Q_n(x|q) = (n+1) \frac{(e^y/a, -e^{-y}/a; q)_\infty}{(q;q)_\infty^2} + \mathcal{O}(q^n)$$

and

$$\frac{(-1)^n q^{\binom{n}{2}}}{a^n (q;q)_n} Q_n^*(x|q) = -\frac{2(n+1)}{a} {}_2\phi_1 \left(\left. \begin{matrix} e^y/a, -e^{-y}/a \\ q \end{matrix} \right| q; q \right) + \mathcal{O}(q^n).$$

When $\bar{a} = b$ and $a \neq b$, we see that (1.21) (or (1.22)) has two simple poles at $t = 1/a$ and $t = 1/\bar{a}$ as the singularities closest to zero. Darboux's method therefore tells us that

$$\frac{(-1)^n q^{\binom{n}{2}}}{(q;q)_n} Q_n(x|q) = a^n \frac{(e^y/a, -e^{-y}/a; q)_\infty}{(\bar{a}/a, q; q)_\infty} + \bar{a}^n \frac{(e^y/\bar{a}, -e^{-y}/\bar{a}; q)_\infty}{(a/\bar{a}, q; q)_\infty} + \mathcal{O}(|a|^n q^n)$$

and

$$\frac{(-1)^n q^{\binom{n}{2}}}{(q;q)_n} Q_n^*(x|q) = -\frac{2a^{n-1}}{1-\bar{a}/a}\,{}_2\phi_1\left(\begin{matrix} e^y/a, -e^{-y}/a \\ \bar{a}q/a \end{matrix}\,\middle|\, q;q\right) - \frac{2\bar{a}^{n-1}}{1-a/\bar{a}}\,{}_2\phi_1\left(\begin{matrix} e^y/\bar{a}, -e^{-y}/\bar{a} \\ aq/\bar{a} \end{matrix}\,\middle|\, q;q\right) + \mathcal{O}(|a|^n\, q^n).$$

Finally, when $|a| \neq |b|$ the singularity of (1.21) (or (1.22)) closest to zero is a simple pole at either $t = 1/a$ or $t = 1/b$. Assuming that $a > b$, Darboux's method gives

$$\frac{(-1)^n q^{\binom{n}{2}}}{(q;q)_n} Q_n(x|q) = a^n \frac{(e^y/a, -e^{-y}/a; q)_\infty}{(b/a, q; q)_\infty} + \mathcal{O}(p^n)$$

and

$$\frac{(-1)^n q^{\binom{n}{2}}}{(q;q)_n} Q_n^*(x|q) = -\frac{2a^{n-1}}{1-b/a}\,{}_2\phi_1\left(\begin{matrix} e^y/a, -e^{-y}/a \\ bq/a \end{matrix}\,\middle|\, q;q\right) + \mathcal{O}(p^n),$$

where p denotes the maximum value of $|b|$ and $|aq|$.

With the asymptotic behaviour of $Q_n(x|q)$ at hand, Askey and Ismail [4] determined the values of a and b for which the Al-Salam–Chihara II moment problem is indeterminate. Notice that the orthonormal polynomials are given by

$$P_n(x) := \frac{q^{n^2/2}}{\sqrt{(ab)^n(-1/ab, q; q)_n}} Q_n(x|q)$$

and satisfy the three-term recurrence relation (5) with

$$a_n = \frac{a+b}{2q^n} \quad \text{and} \quad b_n = \frac{\sqrt{(ab+q^n)(1-q^{n+1})}}{2q^{n+1/2}}.$$

Similarly, the polynomials of the second kind are given by

$$Q_n(x) := \frac{q^{n^2/2}}{\sqrt{(ab)^n(-1/ab, q; q)_n}} Q_n^*(x|q).$$

When $a = b$, we have

$$|P_n(x)|^2 \sim \frac{(n+1)^2 q^n}{(-1/a^2; q)_\infty (q;q)_\infty^3} \left|(e^y/a, -e^{-y}/a; q)_\infty\right|^2$$

and when $\bar{a} = b$ and $a \neq b$, we have

$$\begin{aligned} |P_n(x)|^2 &\sim \frac{q^n/|a|^{2n}}{(-1/a\bar{a}, q; q)_\infty} \left| a^n \frac{(e^y/a, -e^{-y}/a; q)_\infty}{(\bar{a}/a; q)_\infty} + \bar{a}^n \frac{(e^y/\bar{a}, -e^{-y}/\bar{a}; q)_\infty}{(a/\bar{a}; q)_\infty} \right|^2 \\ &\leq \frac{q^n}{(-1/a\bar{a}, q; q)_\infty} \frac{\left(|(e^y/a, -e^{-y}/a; q)_\infty| + |(e^y/\bar{a}, -e^{-y}/\bar{a}; q)_\infty|\right)^2}{|(\bar{a}/a; q)_\infty|^2}. \end{aligned}$$

Consequently, the moment problem is indeterminate if $\bar{a} = b$. When $a > b$, we have

$$|P_n(x)|^2 \sim \frac{(aq/b)^n}{(-1/ab, q; q)_\infty (b/a; q)_\infty^2} \left|(e^y/a, -e^{-y}/a; q)_\infty\right|^2$$

and since the infinite product $(e^y/a, -e^{-y}/a; q)_\infty$ never vanishes for $y \in \mathbb{C}\backslash\mathbb{R}$, the moment problem is only indeterminate if $a/b < 1/q$. Similarly, when $a < b$ the moment problem is only indeterminate if $a/b > q$. All in all we therefore conclude that the Al-Salam–Chihara II moment problem is indeterminate if and only if $\bar{a} = b$ or $q < a/b < 1/q$.

We now consider the case $\bar{a} = b$ in every detail. Recall that the Al-Salam–Chihara polynomials of type II reduce to the q^{-1}-Meixner–Pollaczek polynomials in this situation. It is convenient to write $a = ce^{i\theta}$ for $c \geq 0$ and $0 \leq \theta \leq \pi/2$. Moreover, for $x \in \mathbb{R}$ we define

$$\tfrac{1}{2}R(x)e^{i\zeta(x)} := \frac{(e^{y-i\theta}/c, -e^{-y-i\theta}/c; q)_\infty}{(e^{-2i\theta}, q; q)_\infty} \tag{1.23}$$

and

$$\tfrac{1}{2}S(x)e^{i\eta(x)} := {}_2\phi_1\left(\begin{matrix} e^{y-i\theta}/c, -e^{-y-i\theta}/c \\ qe^{-2i\theta}\end{matrix} \,\middle|\, q; q\right), \tag{1.24}$$

where $R(x), S(x) > 0$ and $\zeta(x), \eta(x) \in \mathbb{R}$. The above definitions only make sense when $a \neq b$ (or $c > 0$ and $0 < \theta \leq \pi/2$) in case of which

$$\frac{(-1)^n q^{\binom{n}{2}}}{c^n (q;q)_n} Q_n(x|q) = R(x)\cos\big(n\theta + \zeta(x)\big) + \mathcal{O}(q^n)$$

and

$$\frac{(-1)^n q^{\binom{n}{2}}}{c^n (q;q)_n} Q_n^*(x|q) = -S(x)\frac{\sin\big(n\theta + \eta(x)\big)}{c\sin\theta} + \mathcal{O}(q^n).$$

The following result is due to Chihara and Ismail [12].

Theorem 1.15. *Suppose that $c > 0$, $0 < \theta \leq \pi/2$ and consider the indeterminate Hamburger moment problem associated with the q^{-1}-Meixner–Pollaczek polynomials $P_n(x; c, \theta|q)$. For $x \in \mathbb{R}$, the entire functions from the Nevanlinna parametrization are given by*

$$\begin{aligned}
A(x) &= \frac{(q;q)_\infty}{2(-1/c^2;q)_\infty}\frac{1}{c\sin\theta}S(0)S(x)\sin\big(\eta(x) - \eta(0)\big),\\
B(x) &= \frac{-(q;q)_\infty}{2(-1/c^2;q)_\infty}S(0)R(x)\cos\big(\zeta(x) - \eta(0)\big),\\
C(x) &= \frac{(q;q)_\infty}{2(-1/c^2;q)_\infty}R(0)S(x)\cos\big(\eta(x) - \zeta(0)\big),\\
D(x) &= \frac{(q;q)_\infty}{2(-1/c^2;q)_\infty}c\sin\theta R(0)R(x)\sin\big(\zeta(x) - \zeta(0)\big),
\end{aligned}$$

where $R(x)$, $\zeta(x)$ resp. $S(x)$, $\eta(x)$ are defined in (1.23) *and* (1.24).

Proof. The result follows from (11) after some computations involving trigonometric functions. For instance we have

$$\begin{aligned}
C(x) =& \frac{R(0)S(x)}{2\sin\theta}\lim_{n\to\infty}\frac{(q;q)_{n+1}}{(-1/c^2;q)_n}\\
&\times\Big(\cos\big(n\theta + \zeta(0)\big)\sin\big((n+1)\theta + \eta(x)\big) - \cos\big((n+1)\theta + \zeta(0)\big)\sin\big(n\theta + \eta(x)\big)\Big)
\end{aligned}$$

and the factor in brackets reduces to

$$\tfrac{1}{2}\sin\bigl(\theta+\eta(x)-\zeta(0)\bigr)+\tfrac{1}{2}\sin\bigl(\theta-\eta(x)+\zeta(x)\bigr)=\sin\theta\cos\bigl(\eta(x)-\zeta(0)\bigr),$$

which is independent of n. $\square$

Remark 1.16. When $q<a/b<1/q$, the large n behaviour of $Q_n(x|q)$ presented on the previous pages is not suitable for finding the entire functions from the Nevanlinna parametrization.

The Al-Salam–Chihara polynomials of type II are symmetric exactly when $a=-b$. The following result is derived in [15].

Corollary 1.17. *Suppose that $c>0$ and consider the indeterminate Hamburger moment problem associated with the symmetric Al-Salam–Chihara II polynomials $Q_n(x;c|q)$. For $x\in\mathbb{R}$, the entire functions from the Nevanlinna parametrization are given by*

$$A(x)=\frac{2(q^2;q^2)_\infty}{c(-1/c^2;q^2)_\infty}S(x)\sin\eta(x),\qquad C(x)=\frac{(q;q^2)_\infty}{(-q/c^2;q^2)_\infty}S(x)\cos\eta(x),$$
$$B(x)=-\frac{(q^2;q^2)_\infty}{(-1/c^2;q^2)_\infty}R(x)\cos\zeta(x),\qquad D(x)=\frac{c(q;q^2)_\infty}{2(-q/c^2;q^2)_\infty}R(x)\sin\zeta(x),$$

where $R(x),S(x)>0$ and $\zeta(x),\eta(x)\in\mathbb{R}$ are defined via

$$R(x)e^{i\zeta(x)}=\frac{(-ie^y/c,ie^{-y}/c;q)_\infty}{(q^2;q^2)_\infty},\qquad S(x)e^{i\eta(x)}={}_2\phi_1\left(\begin{matrix}-ie^y/c,ie^{-y}/c\\-q\end{matrix}\,\middle|\,q;q\right).$$

Proof. Set $\theta=\pi/2$ in Theorem 1.15 and notice that

$$R(0)e^{i\zeta(0)}=\frac{(-1/c^2;q^2)_\infty}{(q^2;q^2)_\infty},\quad S(0)e^{i\eta(0)}={}_1\phi_0\left(\begin{matrix}-1/c^2\\-\end{matrix}\,\middle|\,q^2;q\right)=\frac{(-q/c^2;q^2)_\infty}{(q;q^2)_\infty}.$$

In particular, $\zeta(0)=\eta(0)=0$ and the result follows. $\square$

A simple computation leads to

$$R^2(x)\cos^2\zeta(x)+R^2(x)\sin^2\zeta(x)=\frac{(-e^{2y}/c^2,-e^{-2y}/c^2;q^2)_\infty}{(q^2;q^2)_\infty^2}$$

so that with $\beta=0$ and

$$\gamma=\frac{c}{2}\frac{(-1/c^2,q;q^2)_\infty}{(-q/c^2,q^2;q^2)_\infty},$$

the solution (15) has the form

$$\frac{d\mu_{0,\gamma}}{dx}=\frac{2}{c\pi}\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\frac{(-1/c^2;q)_\infty}{(-e^{2y}/c^2,-e^{-2y}/c^2;q^2)_\infty},\quad x=\sinh y\in\mathbb{R}. \tag{1.25}$$

This observation was the starting point of [15] and the solution (1.25) plays a central role throughout the paper. It is used to construct a family of discrete solutions (in §6) and it also appears in the first version of the q-Sturm–Liouville equation (in §9).

The continuous q^{-1}-Hermite polynomials are the special case $a=b=0$ of the Al-Salam–Chihara polynomials of type II. But as mentioned in [15], they can also be obtained from the symmetric Al-Salam–Chihara II polynomials by setting $c^2=1/q$ and replacing q^2 with q. In this way we arrive at the following result.

Corollary 1.18. *Consider the indeterminate Hamburger moment problem associated with the continuous q^{-1}-Hermite polynomials. The entire functions from the Nevanlinna parametrization are given by*

$$A(x) = \frac{4qx}{1-q}\,{}_2\phi_1\left(\begin{matrix} q^2e^{2y}, q^2e^{-2y} \\ q^3 \end{matrix}\,\middle|\, q^2; q\right), \quad C(x) = {}_2\phi_1\left(\begin{matrix} e^{2y}, e^{-2y} \\ q \end{matrix}\,\middle|\, q^2; q^2\right),$$
$$B(x) = -\frac{(qe^{2y}, qe^{-2y}; q^2)_\infty}{(q;q^2)_\infty^2}, \qquad D(x) = x\frac{(q^2e^{2y}, q^2e^{-2y}; q^2)_\infty}{(q;q)_\infty}.$$

Proof. The proof is based on four identities derived in [22, (5.37), (5.38) & (5.46), (5.47)]. Two of them have the form

$$(-i\sqrt{q}e^y, i\sqrt{q}e^{-y}; q)_\infty + (i\sqrt{q}e^y, -i\sqrt{q}e^{-y}; q)_\infty = 2\frac{(q^2e^{2y}, q^2e^{-2y}; q^4)_\infty}{(q;q^2)_\infty (q^2;q^4)_\infty}$$

and

$$(-i\sqrt{q}e^y, i\sqrt{q}e^{-y}; q)_\infty - (i\sqrt{q}e^y, -i\sqrt{q}e^{-y}; q)_\infty = 4i\sqrt{q}x\frac{(q^4e^{2y}, q^4e^{-2y}; q^4)_\infty}{(q;q^2)_\infty (q^2;q^4)_\infty}$$

while the other two can be written as

$$\begin{aligned} {}_2\phi_1\left(\begin{matrix} -i\sqrt{q}e^y, i\sqrt{q}e^{-y} \\ -q \end{matrix}\,\middle|\, q; q\right) &+ {}_2\phi_1\left(\begin{matrix} i\sqrt{q}e^y, -i\sqrt{q}e^{-y} \\ -q \end{matrix}\,\middle|\, q; q\right) \\ &= \frac{2(q^4;q^4)_\infty}{(q;q)_\infty}\,{}_2\phi_1\left(\begin{matrix} e^{2y}, e^{-2y} \\ q^2 \end{matrix}\,\middle|\, q^4; q^4\right) \end{aligned}$$

and

$$\begin{aligned} {}_2\phi_1\left(\begin{matrix} -i\sqrt{q}e^y, i\sqrt{q}e^{-y} \\ -q \end{matrix}\,\middle|\, q; q\right) &- {}_2\phi_1\left(\begin{matrix} i\sqrt{q}e^y, -i\sqrt{q}e^{-y} \\ -q \end{matrix}\,\middle|\, q; q\right) \\ &= \frac{4iq\sqrt{q}x(q^6;q^4)_\infty}{(q;q)_\infty}\,{}_2\phi_1\left(\begin{matrix} q^4e^{2y}, q^4e^{-2y} \\ q^6 \end{matrix}\,\middle|\, q^4; q^2\right) \end{aligned}$$

when Heine's transformation formula (ix) is applied to the right-hand sides. To obtain the desired expression for, say D, set $c = 1/\sqrt{q}$ in Corollary 1.17 and realize that

$$\begin{aligned} \frac{(q;q^2)_\infty}{4i\sqrt{q}(-q^2;q^2)_\infty}&\left(\frac{(-i\sqrt{q}e^y, i\sqrt{q}e^{-y}; q)_\infty}{(q^2;q^2)_\infty} - \frac{(i\sqrt{q}e^y, -i\sqrt{q}e^{-y}; q)_\infty}{(q^2;q^2)_\infty}\right) \\ &= x\frac{(q;q^2)_\infty}{(q^4;q^4)_\infty}\frac{(q^4e^{2y}, q^4e^{-2y}; q^4)}{(q;q^2)_\infty(q^2;q^4)_\infty} = x\frac{(q^4e^{2y}, q^4e^{-2y}; q^4)}{(q^2;q^2)_\infty}. \end{aligned}$$

Finally, replace q^2 with q. □

The above result is due to Ismail and Masson [22, §5]. We can apply Heine's transformation formula (ix) to the expressions for A and C in [22] to obtain the somewhat simpler expressions presented here.

Ismail and Masson pointed out that B and D are closely related to two of the four theta functions given by

$$\vartheta_1(x) = 2q^{1/4}\sin x\,(q^2e^{2ix}, q^2e^{-2ix}, q^2; q^2)_\infty, \quad \vartheta_3(x) = (-qe^{2ix}, -qe^{-2ix}, q^2; q^2)_\infty,$$
$$\vartheta_2(x) = 2q^{1/4}\cos x\,(-q^2e^{2ix}, -q^2e^{-2ix}, q^2; q^2)_\infty, \quad \vartheta_4(x) = (qe^{2ix}, qe^{-2ix}, q^2; q^2)_\infty.$$

To be precise, we have

$$B(x) = -\frac{\vartheta_4(iy)}{(q;q)_\infty(q;q^2)_\infty} \quad \text{and} \quad D(x) = \frac{\vartheta_1(iy)}{2iq^{1/4}(q;q)_\infty(q^2;q^2)_\infty}.$$

It is remarkable that one can find all of the N-extremal solutions, see [22, §6]. The first step is to show that

$$\rho(x) = \frac{1}{(-qe^{2y}, -qe^{-2y}, q; q)_\infty}, \quad x = \sinh y \in \mathbb{R}. \tag{1.26}$$

Since the orthonormal polynomials are given by

$$P_n(x) = \frac{q^{n(n+1)/4}}{\sqrt{(q;q)_n}} h_n(x|q),$$

this follows from the q-Mehler formula

$$\sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{(q;q)_n} h_n(\sinh y|q)h_n(\sinh v|q)t^n = \frac{(-te^{y+v}, -te^{-y-v}, te^{y-v}, te^{-y+v}; q)_\infty}{(t^2/q;q)_\infty}, \quad |t| < \sqrt{q}$$

established by Ismail and Masson in [22, Thm. 2.1]. The next step is to find the zeros of $x \mapsto B(x)t - D(x)$ for fixed $t \in \mathbb{R} \cup \{\infty\}$. It is not hard to see that the zeros of B and D are given by

$$\tfrac{1}{2}(q^{-n-1/2} - q^{n+1/2}) \quad \text{resp.} \quad \tfrac{1}{2}(q^{-n} - q^n) \quad \text{for } n \in \mathbb{Z}.$$

But this only solves the problem for the particular values $t = 0$ and $t = \infty$. The trick is to introduce a reparametrization of $t \in \mathbb{R} \cup \{\infty\}$. Since $\rho(x) > 0$, the function D/B is strictly decreasing from 0 to $-\infty$ on the interval

$$\big[0, \tfrac{1}{2}(q^{-1/2} - q^{1/2})\big)$$

and strictly decreasing from ∞ to 0 on the interval

$$\big(\tfrac{1}{2}(q^{-1/2} - q^{1/2}), \tfrac{1}{2}(1/q - q)\big].$$

For given $t \in \mathbb{R} \cup \{\infty\}$, let $w = \sinh v$ denote the unique number in $\big[0, \frac{1}{2}(1/q - q)\big)$ such that $t = D(w)/B(w)$ (assuming that $1/0 = \infty$). The problem is now reduced to finding the zeros of $x \mapsto B(x)D(w) - D(x)B(w)$ for fixed $w \in \big[0, \frac{1}{2}(1/q - q)\big)$. Using the addition formulas

$$\vartheta_1(y \pm v)\vartheta_4(y \mp v)\vartheta_2\vartheta_3 = \vartheta_1(y)\vartheta_4(y)\vartheta_2(v)\vartheta_3(v) \pm \vartheta_2(y)\vartheta_3(y)\vartheta_1(v)\vartheta_4(v),$$

where

$$\vartheta_2 := \vartheta_2(0) = 2q^{1/4}(-q^2;q^2)^2_\infty(q^2;q^2)_\infty, \quad \vartheta_3 := \vartheta_3(0) = (-q;q^2)^2_\infty(q^2;q^2)_\infty, \tag{1.27}$$

we see that

$$\begin{aligned}B(x)D(w)-D(x)B(w)&=\frac{\vartheta_1(iy)\vartheta_4(iv)-\vartheta_1(iv)\vartheta_4(iy)}{2iq^{1/4}(q;q)_\infty^3}\\&=\frac{\vartheta_2\big(i(y+v)/2\big)\vartheta_3\big(i(y+v)/2\big)\vartheta_1\big(i(y-v)/2\big)\vartheta_4\big(i(y-v)/2\big)}{2i\sqrt{q}(q;q)_\infty(q^2;q^2)_\infty^4}\end{aligned}$$

and inserting the infinite product representations for the theta functions, we end up with

$$D(x)B(w)-B(x)D(w)=\frac{e^v}{2(q;q)_\infty}(-qe^{y+v},-e^{-y-v},qe^{-y+v},e^{y-v};q)_\infty \tag{1.28}$$

or

$$D(x)B(w)-B(x)D(w)=\frac{(ae^y,-ae^{-y},qe^{-y}/a,-qe^y/a;q)_\infty}{2a(q;q)_\infty}$$

if we set $a=e^{-v}$. For fixed $t\in\mathbb{R}\cup\{\infty\}$, the zeros of $x\mapsto B(x)t-D(x)$ are therefore given by

$$\tfrac12(q^{-n}/a-q^na),\quad n\in\mathbb{Z}$$

exactly when

$$t=\frac{D\big((1/a-a)/2\big)}{B\big((1/a-a)/2\big)}=\frac{a-1/a}{2}\frac{(q^2a^2,q^2/a^2,q;q^2)_\infty}{(qa^2,q/a^2,q^2;q^2)_\infty}\quad\text{and}\quad q<a\le 1.$$

We have the following result.

Proposition 1.19. *In the set-up of Corollary 1.18, the N-extremal solutions μ_t, $t\in\mathbb{R}\cup\{\infty\}$ are given by*

$$\mu_{t(a)}=\frac{1}{(-a^2,-q/a^2,q;q)_\infty}\sum_{n=-\infty}^{\infty}a^{4n}q^{\binom{2n}{2}}(1+a^2q^{2n})\,\varepsilon_{\frac12(q^{-n}/a-q^na)},$$

where

$$t(a)=\frac{a-1/a}{2}\frac{(q^2a^2,q^2/a^2,q;q^2)_\infty}{(qa^2,q/a^2,q^2;q^2)_\infty}\quad\textit{for } q<a\le 1.$$

Proof. It is only left to evaluate the function ρ from (1.26) at the zeros of $B(x)t-D(x)$. We get

$$\begin{aligned}\rho\big(\tfrac12(q^{-n}/a-q^na)\big)&=\frac{1}{(-q^{1-2n}/a^2,-q^{1+2n}a^2,q;q)_\infty}\\&=\frac{(-qa^2;q)_{2n}}{(-q^{1-2n}/a^2;q)_{2n}}\frac{1}{(-q/a^2,-qa^2,q;q)_\infty}\\&=\frac{1+a^2q^{2n}}{1+a^2}\frac{a^{4n}q^{\binom{2n}{2}}}{(-q/a^2,-qa^2,q;q)_\infty}\end{aligned}$$

and the desired result follows. □

Besides the N-extremal solutions we can also find solutions of the form (15). For given $\beta\in\mathbb{R}$ and $\gamma>0$ the problem is to find an expression for

$$|B(x)(\beta+i\gamma)-D(x)|^2=\big(\beta B(x)-D(x)\big)^2+\big(\gamma B(x)\big)^2,\quad x\in\mathbb{R}.$$

Using the relation

$$-\vartheta_3^2\vartheta_1^2(x)+\vartheta_2^2\vartheta_4^2(x)=\vartheta_4^2\vartheta_2^2(x),$$

where

$$\vartheta_4 := \vartheta_4(0)=(q;q^2)_\infty^2(q^2;q^2)_\infty$$

and ϑ_2,ϑ_3 are defined in (1.27), we see that

$$D^2(x)+\gamma^2B^2(x)=\frac{(q,-q^2;q^2)_\infty^2(-e^{2y},-e^{-2y},-q^2e^{2y},-q^2e^{-2y};q^2)_\infty}{4(-q;q)_\infty^2(-q,q^2;q^2)_\infty^2}$$

exactly when

$$\gamma=\frac{(q;q^2)_\infty(-q^2;q^2)_\infty^2}{(q^2;q^2)_\infty(-q;q^2)_\infty^2}.$$

In this way we obtain the solution

$$\frac{d\mu_{0,\gamma}}{dx}=\frac{4}{\pi}\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\frac{(-q;q)_\infty^2}{(-e^{2y},-e^{-2y},-q^2e^{2y},-q^2e^{-2y};q^2)_\infty},\quad x=\sinh y\in\mathbb{R}.$$

But many more solutions of the form (15) are derived in [22, §7]. Since $-D/B$ is a Pick function, we can write

$$-\frac{D(w)}{B(w)}=\beta+i\gamma,\quad \beta\in\mathbb{R}\text{ and }\gamma>0$$

for every $w=\sinh v$ in the upper half-plane $\operatorname{Im} w>0$. Notice that γ is given by

$$\gamma=-\frac{1}{2i}\left(\frac{D(w)}{B(w)}-\frac{D(\bar{w})}{B(\bar{w})}\right)$$

and the solution (15) therefore takes the form

$$\frac{d\mu_{\beta,\gamma}}{dx}=\frac{1}{2\pi i}\frac{B(w)D(\bar{w})-D(w)B(\bar{w})}{|B(x)D(w)+D(x)B(w)|^2},\quad x\in\mathbb{R}.$$

Recalling that B is even and D is odd, we get

$$B(x)D(w)+D(x)B(w)=D(x)B(-w)-B(x)D(-w)$$

and (1.28) now leads to

$$\frac{d\mu_{\beta,\gamma}}{dx}=\frac{c}{\pi i}\frac{(-c\bar{c},-q/c\bar{c},\bar{c}/c,qc/\bar{c};q)_\infty}{|(ce^y,-ce^{-y},-qe^y/c,qe^{-y}/c;q)_\infty|^2},\quad x=\sinh y\in\mathbb{R}$$

if we set $c=e^v$.

Proposition 1.20. *In the set-up of Corollary 1.18, we have for each $c=re^{i\theta}$ with $q<r\le 1$ and $0<\theta\le\pi/2$ an absolutely continuous solution with density*

$$v_c(x)=\frac{c}{\pi i}\frac{(-c\bar{c},-q/c\bar{c},\bar{c}/c,qc/\bar{c};q)_\infty}{|(ce^y,-ce^{-y},-qe^y/c,qe^{-y}/c;q)_\infty|^2},\quad x=\sinh y\in\mathbb{R}.$$

Proof. The mapping $c = e^v \mapsto \sinh v = w$ is a one-to-one mapping of the set

$$K = \{re^{i\theta} \mid r > 0,\, 0 < \theta < \pi/2\} \cup \{re^{i\theta} \mid 0 < r \leq 1,\, \theta = \pi/2\}$$

onto the upper half-plane $\operatorname{Im} w > 0$. Realizing that $v_{cq}(x) = v_c(x)$, it suffices to consider $c \in K$ with $q < |c| \leq 1$. $\square$

Remark 1.21. When $\theta = \pi/2$, the solutions in Proposition 1.20 can be obtained from (1.25), see [15, §5] for details.

A number of other solutions are also derived in [22, §7]. But for odd reasons the simplest solution seems to be missing. We here refer to the density

$$w(x) = \frac{2q^{1/8}}{\sqrt{2\pi \log q^{-1}}} e^{2\frac{\left(\log(x+\sqrt{x^2+1})\right)^2}{\log q}}, \quad x \in \mathbb{R} \tag{1.29}$$

discovered by Atakishiyev, Frank and Wolf in [5].

As the present section illustrates, the Krein and Nevanlinna parametrizations are powerful tools when dealing with indeterminate moment problems. But sometimes other approaches can be fruitful as well and this is what the next section is about. For instance, we have not at all considered the moment sequences and only come across the orthogonal polynomials through their three-term recurrence relation.

A second approach

In the previous section we started at the highest level of indeterminate moment problems within the Askey-scheme. The advantage in doing so is the fact that results on a higher level immediately lead to results on lower levels by taking limits or considering special cases. But one has to be careful not to overlook things only of interest at the lower levels.

In this section we start at the very bottom of the Askey-scheme and try to work our way up through the scheme. The main tools will now be the moment sequence $(s_n)_{n\geq 0}$ and the explicit form of the orthogonal polynomials. Furthermore, the second order q-difference equation or q-Sturm–Liouville equation will come into play.

As is well-known, the Hermite polynomials are orthogonal with respect to the normal distribution. If the random variable X follows a normal distribution $N(0, \sigma^2)$, then $\exp(X)$ follows the so-called lognormal distribution. The density of the lognormal distribution is given by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \frac{1}{x} e^{-\frac{(\log x)^2}{2\sigma^2}}, \quad x > 0$$

and the moments are found to be $s_n = e^{\frac{1}{2}n^2\sigma^2}$ for $n \in \mathbb{Z}$. The lognormal distribution is widely used in statistics, see e.g. [17].

As regards the lognormal moment problem, the criterion (17) of Krein tells us that it is indeterminate in the sense of Stieltjes. The associated orthogonal polynomials were computed by Wigert in

[38] directly from the formula (4). They are known as the Stieltjes–Wigert polynomials. To follow the notation of K&S we set $q = e^{-\sigma^2}$ and consider the density

$$v(x) = \frac{q^{1/8}}{\sqrt{2\pi \log q^{-1}}} \frac{1}{\sqrt{x}} e^{\frac{1}{2} \frac{(\log x)^2}{\log q}}, \quad x > 0 \tag{1.30}$$

which has the moments $s_n = q^{-n(n+1)/2}$ for $n \in \mathbb{Z}$. Notice that $v(x) = \sqrt{q} f(x\sqrt{q})$ and $\sigma^2 > 0$ corresponds to $0 < q < 1$.

The Stieltjes–Wigert moment problem is studied in [11] and [14]. The observation that v satisfies the functional equation

$$xv(x) = v(xq), \quad x > 0 \tag{1.31}$$

gets things moving. The first result to prove is the fact that any (absolutely continuous) probability density which satisfies (1.31) has the moments $q^{-n(n+1)/2}$ for $n \in \mathbb{Z}$. Using the Askey–Roy q-beta integral (xvii), one can thus show that for any $c > 0$ the density

$$v_c(x) = \frac{\sin \pi c}{\pi} \frac{(q;q)_\infty}{(q^c, q^{1-c}; q)_\infty} \frac{q^{c(1-c)} x^{c-1}}{(-q^{1-c}x, -q^c/x; q)_\infty}, \quad x > 0 \tag{1.32}$$

is a solution to the Stieltjes–Wigert moment problem. Moreover, it follows that new solutions can be obtained by multiplying $v(x)$ from (1.30) with positive q-periodic functions, i.e., functions having the same value at x and xq for $x > 0$. In fact, Stieltjes [36] pointed out that for $\lambda \in [-1, 1]$ the densities

$$\tilde{v}_\lambda(x) = v(x)\Big(1 + \lambda \sin\big(2\pi \tfrac{\log x}{\log q}\big)\Big), \quad x > 0 \tag{1.33}$$

all have the same moments.

The next step is to generalize the functional equation (1.31) to an equation for positive measures. One can prove that any probability measure μ on $[0, \infty)$ satisfying the equation

$$\tau_{q^{-1}}(\mu) = qx d\mu(x) \tag{1.34}$$

has the moments $q^{-n(n+1)/2}$ for $n \in \mathbb{Z}$. Here, $\tau_a(\mu)$ denotes the image measure of μ under $\tau_a : x \mapsto ax$ and $g(x)d\mu(x)$ denotes the measure with density g with respect to μ. Notice that if μ is a discrete measure, then (1.34) means that $c > 0$ is a mass point of μ if and only if cq likewise is a mass point of μ and $\mu(\{cq\}) = qc\mu(\{c\})$. Using the triple product identity (v), we thus see that for each $c > 0$ the discrete measure

$$\lambda_c = \frac{1}{(-cq, -1/c, q; q)_\infty} \sum_{n=-\infty}^{\infty} c^n q^{\binom{n+1}{2}} \varepsilon_{cq^n} \tag{1.35}$$

is a solution to the Stieltjes–Wigert moment problem.

One step further is to consider the transformation T defined by

$$\mu \xrightarrow{T} \tau_q\big(qx d\mu(x)\big).$$

Whenever μ is a solution to the Stieltjes–Wigert moment problem, it is easily seen that $T(\mu)$ is again a solution. Notice that a positive measure μ is a fixed point of T exactly when μ satisfies (1.34). The solutions presented so far are therefore all invariant under T. However, the m-canonical solutions are not fixed points of T. A major part of [14] is devoted to studying the action of T on solutions to the moment problem. In Thm. 3.6 the action of T is described at the level of Pick functions and in Thm. 3.7 it is proved that starting from the N-extremal solution μ_0, we have

$$T^{(2n+1)}(\mu_0) = \mu_{R_n} \quad \text{and} \quad T^{(2n+2)}(\mu_0) = \mu_{\tilde{R}_n}$$

for explicitly known real rational functions R_n, $\tilde{R}_n$ of order $\le n$. In addition to this, it is proved that

$$T^{(2n+1)}(\mu_0) \to \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} q^{\binom{2n+2}{2}} \varepsilon_{q^{2n+1}} \tag{1.36}$$

and

$$T^{(2n+2)}(\mu_0) \to \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} q^{\binom{2n+1}{2}} \varepsilon_{q^{2n}} \tag{1.37}$$

as $n \to \infty$. The solutions in (1.36) and (1.37) were constructed by Berg in [7] to illustrate that infinitely many solutions to the Stieltjes–Wigert moment problem are supported on the geometric progression $\{q^n \mid n \in \mathbb{Z}\}$. The highlight of [14] is without doubt Thm. 3.9.

There is a functional equation similar to the one in (1.31) for the q-Laguerre moment problem. As stated in [13], it reads

$$q^\alpha(1+x)v(x) = v(xq), \quad x > 0. \tag{1.38}$$

One easily checks that the function $x^\alpha/(-x;q)_\infty$ satisfies (1.38) and using the Ramanujan q-beta integral (xvi), we see that the density

$$v^{(\alpha)}(x) = -\frac{\sin \pi\alpha}{\pi} \frac{(q;q)_\infty}{(q^{-\alpha};q)_\infty} \frac{x^\alpha}{(-x;q)_\infty}, \quad x > 0 \tag{1.39}$$

has the q-Laguerre moments, namely

$$s_n = q^{-\alpha n - \binom{n+1}{2}} (q^{\alpha+1};q)_n.$$

In the next place, it follows from Ramanujan's sum (vi) that for every $c > 0$ the discrete measure

$$\lambda_c^{(\alpha)} = \frac{(-q/c, q^{\alpha+1};q)_\infty}{(-cq^{\alpha+1}, -1/cq^\alpha, q;q)_\infty} \sum_{n=-\infty}^{\infty} (-c;q)_n q^{n(\alpha+1)} \varepsilon_{cq^n} \tag{1.40}$$

is a solution to the q-Laguerre moment problem. The solutions in (1.39) and (1.40) were found by Moak [32] and it is explained in [13] how they lead to the Stieltjes–Wigert solutions (1.32) and (1.35) by letting $\alpha \to \infty$.

The discrete solutions in (1.40) (and (1.35)) are not m-canonical and so the codimension of $\mathbb{C}[x]$ in the Hilbert space $L^2(\lambda_c^{(\alpha)})$ is $+\infty$. Nevertheless, Ciccoli, Koelink and Koornwinder [16] were able to find a sequence $(M_p^{(\alpha;c)})_{p\in\mathbb{Z}}$ of functions (expressed as ${}_1\phi_1$' s) such that these functions together with the q-Laguerre polynomials form an orthogonal basis for $L^2(\lambda_c^{(\alpha)})$.

A simple computation shows that if $v(x)$ satisfies (1.31), then $x^\alpha(-q/x;q)_\infty v(x)$ satisfies (1.38). Using an integral of Hardy [18, p. 20-21], we thus see that the density

$$\tilde{v}^{(\alpha)}(x) = \frac{q^{\binom{\alpha+1}{2}+1/8}}{\sqrt{2\pi \log q^{-1}}} (q^{\alpha+1}, -q/x;q)_\infty x^{\alpha-1/2} e^{\frac{1}{2}\frac{(\log x)^2}{\log q}}, \quad x > 0 \tag{1.41}$$

is a solution to the q-Laguerre moment problem. Similarly, the Askey–Roy q-beta integral (xvii) tells us that for each $c > 0$ the density

$$v_c^{(\alpha)}(x) = q^{c(\alpha+1-c)} \frac{\sin \pi c}{\pi} \frac{(q, q^{\alpha+1}; q)_\infty}{(q^c, q^{1-c}; q)_\infty} \frac{(-q/x; q)_\infty x^{c-1}}{(-q^{\alpha+1-c}x, -q^{-\alpha+c}/x; q)_\infty}, \quad x > 0 \tag{1.42}$$

is a solutions to the q-Laguerre moment problem.

It is now natural to ask where the functional equations (1.31) and (1.38) come from. The answer is to be found by considering the q-Sturm–Liouville equation since we are dealing with the q-Pearson equations for the Stieltjes–Wigert and q-Laguerre polynomials, see e.g. Marcellán and Medem [30].

The next thing to do is then to consider the q-Pearson equations for the other polynomials in the q-Meixner tableau. Fortunately, it is easy to find these equations since all the preliminary work has been done by Medem, Álvarez-Nodarse and Marcellán in [31]. Using their results, we see that the q-Pearson equation for the q-Meixner polynomials is given by

$$(1-x)(1+x/bc)v(x) = (1-x/b)v(xq), \quad x > 0 \tag{1.43}$$

and the q-Pearson equation for the Al-Salam–Carlitz polynomials of type II is given by

$$(1-x)(1-x/a)v(x) = v(xq), \quad x > 0. \tag{1.44}$$

Assuming that v is continuous at $x = 0$ with $v(0) = 1$, iteration of (1.43) and (1.44) leads to

$$v(x) = \frac{(x/b; q)_\infty}{(x, -x/bc; q)_\infty} \quad \text{resp.} \quad v(x) = \frac{1}{(x, x/a; q)_\infty}.$$

Because of the factor $(x; q)_\infty$ in the denominator, the above functions are never positive for all $x > 0$ (except if $a = 1$ or $b = 1$). Apparently, the q-Pearson equation does not lead to solutions to the q-Meixner and Al-Salam–Carlitz II moment problems.

The q-Pearson equation for the q-Charlier polynomials reads

$$(1-x)v(x) = -av(xq), \quad x > 0. \tag{1.45}$$

Setting $a = q^\beta$ for $\beta \in \mathbb{R}$, we observe that if $v(x)$ satisfies (1.31) then $(q/x; q)_\infty v(x)/x^\beta$ satisfies (1.45). Moreover, we have the following result.

Proposition 1.22. *Suppose that $f(x)$ satisfies (1.45) and $\int_0^\infty f(x)dx = 1$. Then*

$$\int_0^\infty x^n f(x)dx = a^n q^{-\binom{n+1}{2}} (-q/a; q)_n, \quad n \geq 0.$$

Proof. The proof is by induction. If

$$\int_0^\infty x^n f(x)dx = a^n q^{-\binom{n+1}{2}} (-q/a; q)_n \quad \text{for some } n \geq 0,$$

then

$$\begin{aligned} \int_0^\infty x^{n+1} f(x)dx &= \int_0^\infty x^n \big(f(x) + af(xq)\big)dx \\ &= \int_0^\infty x^n f(x)dx + aq^{-(n+1)} \int_0^\infty x^n f(x)dx = a^{n+1} q^{-\binom{n+2}{2}} (-q/a; q)_{n+1} \end{aligned}$$

and the result follows. □

The q-Charlier moments are indeed given by

$$s_n = a^n q^{-\binom{n+1}{2}}(-q/a;q)_n.$$

However, we have to take into account that any function ($\not\equiv 0$) satisfying (1.45) changes sign infinitely often on the interval $(0,1)$ — and so does the factor $(q/x;q)_\infty$. Therefore, it seems as if the q-Pearson equation does not lead to solutions to the q-Charlier moment problem either.

In any case, we can generalize (1.45) to an equation for signed measures on $[0,\infty)$ and get

$$-a\tau_{q^{-1}}(\mu) = q(1-x)d\mu(x). \tag{1.46}$$

For a discrete measure μ this equation means that $-a\mu(\{cq\}) = q(1-c)\mu(\{c\})$ for all $c \geq 0$. Suppose now that $\mu^{(a)}$ satisfies (1.46) and

$$\mu^{(a)}(\{1\}) > 0, \quad \mu^{(a)}(\{c\}) = 0 \quad \text{for } q < c < 1.$$

Then

$$\mu^{(a)}(\{q^{-n}\}) = \frac{a^n q^{\binom{n}{2}}}{(q;q)_n}\mu^{(a)}(\{1\}), \quad n \geq 0$$

and $\mu^{(a)}(\{c\}) = 0$ for all other values of $c \geq 0$. So $\mu^{(a)}$ is a positive measure on $[0,\infty)$ and with $\mu^{(a)}(\{1\}) = 1/(-a;q)_\infty$, it becomes a probability measure of the form

$$\mu^{(a)} = \frac{1}{(-a;q)_\infty}\sum_{n=0}^{\infty}\frac{a^n q^{\binom{n}{2}}}{(q;q)_n}\varepsilon_{q^{-n}}. \tag{1.47}$$

The above solution to the q-Charlier moment problem is contained in K&S and is derived in a different way by Koelink [28]. Besides the solution (1.47), Koelink also gives the relation

$$\sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}}}{(q;q)_k}C_n(q^{-k};a;q)C_m(q^{-k};1/a;q) = 0 \quad \text{for all } n,m$$

in [28, Cor. 4.2.]. So when $a=1$,

$$\sum_{k=0}^{\infty}\frac{q^{\binom{k}{2}}}{(q;q)_k}C_n(q^{-k})C_m(q^{-k}) = \sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}}}{(q;q)_k}C_n(q^{-k})C_m(q^{-k}) = 0 \quad \text{for } n \neq m$$

or

$$\sum_{k=0}^{\infty}\frac{q^{\binom{2k}{2}}}{(q;q)_{2k}}C_n(q^{-2k})C_m(q^{-2k}) = \sum_{k=0}^{\infty}\frac{q^{\binom{2k+1}{2}}}{(q;q)_{2k+1}}C_n(q^{-(2k+1)})C_m(q^{-(2k+1)}) = 0 \quad \text{for } n \neq m$$

and we are led to the solutions (1.14) and (1.15). In fact, $\mu^{(1)} = \frac{1}{2}(\nu_0 + \nu_\infty)$ when the mass points of ν_1, ν_∞ are shifted to the right by 1. This shift of mass points just corresponds to considering the polynomials $C_n(x;a;q)$ instead of $C_n(x+1;a;q)$.

We now derive the solution to the q-Meixner moment problem which is contained in K&S.

Proposition 1.23. *When $0 \le b < 1/q$ and $c > 0$, the discrete measure*

$$\mu^{(b,c)} = \frac{(-bcq;q)_\infty}{(-c;q)_\infty} \sum_{n=0}^{\infty} \frac{(bq;q)_n}{(-bcq,q;q)_n} c^n q^{\binom{n}{2}} \varepsilon_{q^{-n}}$$

is a solution to the q-Meixner moment problem.

Proof. We start by recalling that the q-Meixner polynomials can be written as

$$M_n(x;q) = {}_2\phi_1\left(\begin{matrix} q^{-n}, x \\ bq \end{matrix} \,\middle|\, q; -q^{n+1}/c \right), \tag{1.48}$$

see for example K&S. The identity (xii) tells us that $\mu^{(b,c)}$ is a probability measure and therefore it suffices to prove that

$$\int_0^\infty M_n(x;q) x^m d\mu^{(b,c)}(x) = 0, \quad m < n$$

or, equivalently,

$$\int_0^\infty M_n(x;q)(-x/bc;q)_m d\mu^{(b,c)}(x) = 0, \quad m < n.$$

Inserting the expression for $M_n(x;q)$ from (1.48), we get

$$\begin{aligned}\int_0^\infty M_n(x;q)(-x/bc;q)_m d\mu^{(b,c)}(x) =& \sum_{k=0}^{n} \frac{(q^{-n};q)_k}{(bq,q;q)_k} (-q^{n+1}/c)^k \int_0^\infty (x;q)_k (-x/bc;q)_m d\mu^{(b,c)}(x) \\ =& \frac{(-bcq;q)_\infty}{(-c;q)_\infty} \sum_{k=0}^{n} \frac{(q^{-n};q)_k}{(bq,q;q)_k} (-q^{n+1}/c)^k \\ & \times \sum_{j=0}^{\infty} \frac{(q^{-j};q)_k (-q^{-j}/bc;q)_m (bq;q)_j}{(-bcq,q;q)_j} c^j q^{\binom{j}{2}}\end{aligned}$$

and since $(q^{-j};q)_k = 0$ for $j < k$, the inner sum reduces to

$$\begin{aligned}&(-1)^k q^{\binom{k}{2}} (-1/bc;q)_m \sum_{j=k}^{\infty} \frac{(bq;q)_j}{(q;q)_{j-k}(-bcq^{1-m};q)_j} c^j q^{\binom{j}{2} - j(k+m)} \\ &\qquad = (-1)^k \frac{(bq;q)_k}{(-bcq^{1-m};q)_k} c^k q^{-k(m+1)} (-1/bc;q)_m \sum_{j=0}^{\infty} \frac{(bq^{k+1};q)_j}{(-bcq^{k+1-m},q;q)_j} c^j q^{\binom{j}{2} - jm}.\end{aligned}$$

Once more, we apply (xii) to get

$$\sum_{j=0}^{\infty} \frac{(bq^{k+1};q)_j}{(-bcq^{k+1-m},q;q)_j} c^j q^{\binom{j}{2} - jm} = \frac{(-cq^{-m};q)_\infty}{(-bcq^{k+1-m};q)_\infty}$$

and the original integral thus simplifies to

$$\int_0^\infty M_n(x;q)(-x/bc;q)_m d\mu^{(b,c)}(x) = \frac{(-q/c;q)_m}{(bq)^m} \sum_{k=0}^{n} \frac{(q^{-n};q)_k}{(q;q)_k} q^{(n-m)k}.$$

By the q-binomial theorem, the finite sum on the right-hand side reduces to $(q^{-m};q)_n$ and the desired result follows. $\square$

The above proof is not very constructive but it illustrates how one can use brute force to verify that that a given positive measure is solution to the moment problem.

We still have left to consider the discrete q-Hermite polynomials of type II. The q-Pearson equation is given by

$$(1+x^2)w(x) = w(xq), \quad x \in \mathbb{R}$$

and assuming that w is continuous at $x=0$ with $w(0)=1$, iteration leads to $w(x) = 1/(-x^2;q^2)_\infty$. In other words, we obtain the solution in (1.20).

The easiest way to obtain other solutions is to recall that the discrete q-Hermite II moment problem is the symmetrized version of the q-Laguerre moment problem when $\alpha = -1/2$ and q is replaced by q^2. In particular, if $d\mu^{(\alpha)}(x) = v^{(\alpha)}(x;q)dx$ is a q-Laguerre solution then

$$d\mu(x) = |x|\, v^{(-1/2)}(x^2;q^2)dx$$

is a discrete q-Hermite II solution. In this way we get the density

$$\tilde{w}(x) = \frac{(q;q^2)_\infty}{2\sqrt{\pi \log q^{-1}}} \frac{(-q^2/x^2;q^2)_\infty}{|x|} e^{\frac{(\log|x|)^2}{\log q}}, \quad x \neq 0 \tag{1.49}$$

and also the densities

$$w_c(x) = \frac{\sin \pi c}{\pi} \frac{(q;q)_\infty}{(q^{2c}, q^{2(1-c)};q^2)_\infty} \frac{q^{c(1-2c)}(-q^2/x^2;q^2)_\infty}{(-q^{1-2c}x^2, -q^{1+2c}/x^2;q^2)_\infty} \frac{x^{2c}}{|x|}, \quad x \neq 0. \tag{1.50}$$

Since $w_{c+1}(x) = w_c(x)$, it suffices to consider the densities in (1.50) for $c \in (0,1]$. The same remark applies to the solutions in (1.32) and (1.42). Notice that $w_{1/2}(x)$ coincide with (1.20) and

$$w_1(x) = \frac{1}{2\log q^{-1}} \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty} \frac{(-q^2/x^2;q^2)_\infty}{(-qx^2, -q/x^2;q^2)_\infty} \frac{1}{|x|}, \quad x \neq 0.$$

It is interesting that the densities in (1.49) and (1.50) all satisfy the q-Pearson equation but only $w_{1/2}(x)$ is continuous at $x=0$.

As regards discrete solutions, we mention that if $\mu^{(\alpha)} = \sum_n m^{(\alpha)}(x_n;q)\varepsilon_{x_n}$ is a q-Laguerre solution then

$$\mu = \tfrac{1}{2} \sum_n m^{(-1/2)}(x_n;q^2)\big(\varepsilon_{-\sqrt{x_n}} + \varepsilon_{\sqrt{x_n}}\big)$$

is a discrete q-Hermite II solution. Hence, we get the solutions

$$\kappa_c = \frac{(-q^2/c^2, q;q^2)_\infty}{2(-c^2q, -q/c^2, q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} (-c^2;q^2)_n q^n \big(\varepsilon_{-cq^n} + \varepsilon_{cq^n}\big). \tag{1.51}$$

and just as for the solutions in (1.35) and (1.40), it suffices to consider $c \in (q,1]$ because of the periodicity $\kappa_{cq} = \kappa_c$.

All discrete q-Hermite II solutions that can be obtained from solutions to the q-Laguerre moment problem are by nature symmetric. Among the many non-symmetric discrete q-Hermite II solutions, no one seems to be explicitly known. For the continuous q^{-1}-Hermite moment problem, on the other hand, we have seen examples of symmetric as well as non-symmetric solutions in the previous section.

Suppose again that the random variable X follows a normal distribution $N(0, \sigma^2)$. We now consider the distribution of $\sinh(X)$ given by the density

$$f(x) = \frac{2}{\sqrt{2\pi\sigma^2}} \frac{1}{\sqrt{x^2+1}} e^{-2\frac{\left(\log(x+\sqrt{x^2+1})\right)^2}{\sigma^2}}, \quad x \in \mathbb{R}. \tag{1.52}$$

This distribution is mentioned in [24, §4.3] but seems to have very little say in statistics and does not even have a name. We will refer to it as the alternative lognormal distribution. It only becomes interesting when we realize how closely related it is to the continuous q^{-1}-Hermite polynomials.

Askey was the first to give an orthogonality relation for the continuous q^{-1}-Hermite polynomials. Using the Askey–Roy q-beta integral (xvii), he proved in [3] that

$$\int_{\mathbb{R}} \frac{h_n(\sinh y|q) h_m(\sinh y|q)}{(-qe^{2y}, -qe^{-2y}, q; q)_\infty} dy = \log q^{-1} (q;q)_n q^{-\binom{n+1}{2}} \delta_{n,m}.$$

By the same integral we can compute the continuous q^{-1}-Hermite moments and get

$$s_{2n} = \frac{(-1)^n}{2^{2n}} \sum_{k=-n}^{n} \binom{2n}{n+k} (-1)^k q^{-\binom{k+1}{2}}, \quad s_{2n+1} = 0 \quad \text{for } n \geq 0$$

The odd moments vanish since the moment problem is symmetric.

We now modify the density (1.52) by removing the factor $\sqrt{x^2+1}$ in the denominator and set $q = e^{-\sigma^2}$. The new density turns out to be (1.29) and a straightforward computation of the integrals

$$\int_{\mathbb{R}} x^n w(x) dx, \quad n \geq 0$$

verifies that (1.29) is a solution to the continuous q^{-1}-Hermite moment problem. Whereas the Stieltjes–Wigert polynomials are orthogonal with respect to the lognormal distribution, the continuous q^{-1}-Hermite polynomials are orthogonal with respect to a slight variation of the alternative lognormal distribution. This explains why it is appropriate to set $x = \sinh y$.

Inspired by the Stieltjes–Wigert moment problem, we notice that for $\lambda \in [-1, 1]$ the densities

$$\tilde{w}_\lambda(x) = w(x)\Big(1 + \lambda \sin\big(4\pi \tfrac{\log(x+\sqrt{x^2+1})}{\log q}\big)\Big), \quad x \in \mathbb{R} \tag{1.53}$$

all have the same moments. As the following result shows, this is no coincidence.

Proposition 1.24. *Suppose that μ is a positive measure on $[0, \infty)$ such that*

$$\int_0^\infty x^n d\mu(x) = q^{-\binom{n+1}{2}}, \quad n \in \mathbb{Z}.$$

If μ is absolutely continuous, say $d\mu(x) = f(x)dx$, then

$$\int_{\mathbb{R}} x^{2n}\big(x + \sqrt{x^2+1}\big) f\big(1 + 2x(x + \sqrt{x^2+1})\big) dx = \frac{(-1)^n}{2^{2n+1}} \sum_{k=-n}^{n} \binom{2n}{n+k} (-1)^k q^{-\binom{k+1}{2}}, \quad n \geq 0.$$

Proof. Denote by ν the image measure of μ under the map $x \mapsto \frac{1}{2}(\sqrt{x} - 1/\sqrt{x})$. A tedious computation shows that

$$\begin{aligned}\int_{\mathbb{R}} x^{2n} d\nu(x) &= \frac{1}{2^{2n}} \int_0^\infty \left(\sqrt{x} - 1/\sqrt{x}\right)^{2n} d\mu(x) \\ &= \frac{1}{2^{2n}} \sum_{k=0}^{2n} \binom{2n}{k} (-1)^k \int_0^\infty x^{k-n} d\mu(x) \\ &= \frac{(-1)^n}{2^{2n}} \sum_{k=-n}^{n} \binom{2n}{n+k} (-1)^k q^{-\binom{k+1}{2}}.\end{aligned}$$

If μ has the form $d\mu(x) = f(x)dx$, then

$$d\nu(x) = 2\big(x + \sqrt{x^2+1}\big) f\big(1 + 2x(x + \sqrt{x^2+1})\big)\big(1 + x/\sqrt{x^2+1}\big)dx$$

and we therefore see that

$$\int_{\mathbb{R}} x^{2n} d\nu(x) = 2 \int_{\mathbb{R}} x^{2n} \big(x + \sqrt{x^2+1}\big) f\big(1 + 2x(x + \sqrt{x^2+1})\big)dx.$$

This completes the proof. $\square$

In particular, the above result tells us that if the density

$$g(x) := 2\big(x + \sqrt{x^2+1}\big) f\big(1 + 2x(x + \sqrt{x^2+1})\big), \quad x \in \mathbb{R}$$

is an even function, then the absolutely continuous measure $d\nu(x) = g(x)dx$ is a solution to the continuous q^{-1}-Hermite moment problem. In this way we are led to the solutions in (1.53) and we are also led to the densities

$$w_c(x) = \frac{\sin \pi c}{\pi} \frac{(q;q)_\infty}{(q^c, q^{1-c}; q)_\infty} \frac{2q^{c(1-c)} e^{y(2c-1)}}{(-q^{1-c}e^{2y}, -q^c e^{-2y}; q)_\infty}, \quad x = \sinh y \in \mathbb{R}. \tag{1.54}$$

Clearly, $w_c(x)$ is only an even function when $c = 1/2$ or $c = 1$. To verify that

$$\int_{\mathbb{R}} x^{2n+1} w_c(x) dx = 0, \quad n \geq 0$$

for all $c \in (0, 1]$, one can use the Askey–Roy q-beta integral (xvii) and the fact that

$$\sum_{k=0}^{2n+1} \binom{2n+1}{k} (-1)^k \left(q^{-\binom{k-n+1/2}{2}} + q^{-\binom{k-n-1/2}{2}}\right) = 0.$$

But we rather refer to [15, §5], where the solutions in (1.54) are derived in a different way. In short, the method in [15] is based on the fact that eigenfunctions corresponding to distinct eigenvalues of a symmetric operator are mutually orthogonal. Following the strategy of [21], it is proved that the densities in (1.54) satisfy a q-Sturm–Liouville equation for the continuous q^{-1}-Hermite polynomials. Integration by parts then leads to the orthogonality and it is only left to check that $\int_{\mathbb{R}} w_c(x)dx = 1$.

There is a counterpart of Proposition 1.24 for discrete measures. Using this result, we see that the solutions in (1.36) and (1.37) lead to the two symmetric N-extremal solutions μ_∞ and μ_0.

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CONSTRUCTIVE APPROXIMATION

The Moment Problem Associated with the q-Laguerre Polynomials

Jacob Stordal Christiansen

Abstract. We consider the indeterminate Stieltjes moment problem associated with the q-Laguerre polynomials. A transformation of the set of solutions, which has all the classical solutions as fixed points, is established and we present a method to construct, for instance, continuous singular solutions. The connection with the moment problem associated with the Stieltjes–Wigert polynomials is studied; we show how to come from q-Laguerre solutions to Stieltjes–Wigert solutions by letting the parameter $\alpha \to \infty$, and we explain how to lift a Stieltjes–Wigert solution to a q-Laguerre solution at the level of Pick functions. Based on two generating functions, expressions for the four entire functions from the Nevanlinna parametrization are obtained.

1. Introduction

In this paper we follow the notation of Gasper and Rahman [14] for basic hypergeometric series. We will always assume that $0 < q < 1$.

Recall from the general theory of the moment problem (see, e.g., Akhiezer [1]) that the Nevanlinna parametrization gives a one-to-one correspondence between the set of Pick functions (including ∞) and the set of solutions to an indeterminate Hamburger moment problem. If μ_φ is the solution corresponding to the Pick function φ, then the Stieltjes transform of μ_φ is given by

$$\int_{\mathbf{R}} \frac{1}{t-x}\, d\mu_\varphi(t) = -\frac{A(x)\varphi(x) - C(x)}{B(x)\varphi(x) - D(x)}, \qquad x \in \mathbf{C}\setminus\mathbf{R},$$

where A, B, C, and D are certain entire functions defined in terms of the orthonormal polynomials (P_n) and (Q_n) by

$$A(x) = x\sum_{n=0}^{\infty} Q_n(0)Q_n(x),$$

$$B(x) = -1 + x\sum_{n=0}^{\infty} Q_n(0)P_n(x),$$

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$$C(x) = 1 + x \sum_{n=0}^{\infty} P_n(0) Q_n(x),$$

$$D(x) = x \sum_{n=0}^{\infty} P_n(0) P_n(x),$$

and according to the Stieltjes–Perron inversion formula the measure μ_φ is uniquely determined by its Stieltjes transform.

The q-Laguerre polynomials belong to the Askey-scheme of basic hypergeometric orthogonal polynomials and, according to Koekoek and Swarttouw [16], they are defined by

$$L_n^{(\alpha)}(x;q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n} {}_1\varphi_1\left(\begin{matrix} q^{-n} \\ q^{\alpha+1} \end{matrix}; q, -q^{n+\alpha+1}x\right). \tag{1.0.1}$$

It is easily seen that $L_n^{(\alpha)}((1-q)x;q) \to L_n^{(\alpha)}(x)$ for $q \to 1^-$, where $L_n^{(\alpha)}(x)$ denotes the usual Laguerre polynomials, and this explains the name. The q-Laguerre polynomials (with x replaced by $(1-q)x$) are studied by Moak in [17]. Using Ramanujan's q-beta integral [2, p. 513]:

$$\int_0^\infty x^{c-1} \frac{(-ax;q)_\infty}{(-x;q)_\infty}\, dx = \frac{(a, q^{1-c};q)_\infty}{(q, aq^{-c};q)_\infty} \frac{\pi}{\sin \pi c} \qquad (c > 0, |a| < q^c), \tag{1.0.2}$$

and the q-Chu–Vandermonde formula of Heine [14, p. 11]:

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k \frac{(a;q)_k}{(c;q)_k} q^{\binom{k}{2}} \left(\frac{c}{a}\right)^k = \frac{(c/a;q)_n}{(c;q)_n}, \tag{1.0.3}$$

Moak proved that the q-Laguerre polynomials are orthogonal on $(0,\infty)$ with respect to the weight function $x^\alpha/(-x;q)_\infty$. The function $E_q(x) = (-x;q)_\infty$ is a q-analogue of the exponential function, so the q-Laguerre polynomials are orthogonal with respect to a q-analogue of the gamma distribution $x^\alpha e^{-x}$. Since this orthogonality relation only holds when $\alpha > -1$, we will always assume that $\alpha > -1$. Using Ramanujan's ${}_1\psi_1$ formula [2, p. 502]:

$$\sum_{n=-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n} x^n = \frac{(q, b/a, ax, q/ax;q)_\infty}{(b, q/a, x, b/ax;q)_\infty} \qquad (|b/a| < |x| < 1), \tag{1.0.4}$$

Moak also proved that the q-Laguerre polynomials are orthogonal with respect to the discrete measures

$$\sum_{n=-\infty}^{\infty} \frac{q^{n(\alpha+1)}}{(-cq^n;q)_\infty} \varepsilon_{cq^n} \qquad (c > 0),$$

where ε_x denotes the Dirac measure at the point x. This in particular means that the moment problem associated with the q-Laguerre polynomials is indeterminate as a Stieltjes moment problem. See also Askey [4] for the connection between (1.0.2) and (1.0.4).

The weight function $x^\alpha/(-x;q)_\infty$ satisfies the functional equation

$$f(qx) = q^\alpha (1+x) f(x) \qquad (x > 0), \tag{1.0.5}$$

as was pointed out by Chihara in [12]. In Section 2 we will take this functional equation as a starting point and we prove that any normalized solution of (1.0.5) has the same moments as $x^\alpha/(-x;q)_\infty$. The same equation leads to a transformation of the set of solutions to the moment problem. It turns out that the classical solutions are fixed points of this transformation and by analyzing the condition for a solution to be a fixed point, we even find a method to construct continuous singular solutions to the moment problem.

As Askey pointed out in [3] the q-Laguerre polynomials converge to the Stieltjes–Wigert polynomials for $\alpha \to \infty$, to be precise

$$L_n^{(\alpha)}(q^{-\alpha}x;q) \to S_n(x;q) \qquad \text{for} \quad \alpha \to \infty,$$

where the Stieltjes–Wigert polynomials, according to Koekoek and Swarttouw [16], are defined by

$$S_n(x;q) = \frac{1}{(q;q)_n}\,{}_1\varphi_1\left(\begin{matrix} q^{-n} \\ 0 \end{matrix};q,-q^{n+1}x\right). \tag{1.0.6}$$

In Szegő [19] and Chihara [10] the Stieltjes–Wigert polynomials are normalized in a slightly different way. In Section 3 we consider the limit $\alpha \to \infty$ at several levels. First we show how to obtain solutions to the moment problem associated with the Stieltjes–Wigert polynomials by letting $\alpha \to \infty$ in various orthogonality measures for the q-Laguerre polynomials. The solution corresponding to the weight function $x^\alpha/(-x;q)_\infty$ does not have a unique limit for $\alpha \to \infty$, but it turns out to have a family $(\omega_c)_{0<c\le 1}$ of accumulation points. For each ω_c we give a solution $\omega_c^{(\alpha)}$ to the moment problem associated with the q-Laguerre polynomials such that $\omega_c^{(\alpha)}$ converges to ω_c for $\alpha \to \infty$. All the solutions being considered can also be found in Berg [7], [8], but here we focus on the connection between the two moment problems. In the next place we establish the connection between the four entire functions from the Nevanlinna parametrizations for the moment problems, see (3.0.15) below, and with this result available it is possible to explain the limit $\alpha \to \infty$ at the level of Pick functions. In Theorem 3.1 we show how to lift a solution μ to the moment problem associated with the Stieltjes–Wigert polynomials to a solution $\mu^{(\alpha)}$ to the moment problem associated with the q-Laguerre polynomials (at least for $\alpha > 0$), and it is made precise how $\mu^{(\alpha)}$ converges to μ for $\alpha \to \infty$.

In some sense to solve an indeterminate moment problem means to find the four entire functions from the Nevanlinna parametrization. For the moment problem associated with the q-Laguerre polynomials this was partly done by Moak in [17], where the entire functions $B^{(\alpha)}$ and $D^{(\alpha)}$ are expressed in terms of q-Bessel functions. The work was completed by Ismail and Rahman in [15]. Using Darboux's method on a certain generating function they established an asymptotic relation for the q-Laguerre polynomials $\tilde{L}_n^{(\alpha)}(x;q)$ of the second kind, and this made it possible to give expressions for $A^{(\alpha)}$ and $C^{(\alpha)}$. In Section 4 we will present two generating functions, one for the q-Laguerre polynomials and one for the q-Laguerre polynomials of the second kind. The latter seems to be new. From these generating functions we derive expressions for $A^{(\alpha)}$, $B^{(\alpha)}$, $C^{(\alpha)}$, and $D^{(\alpha)}$, and we end by clarifying that our expressions coincide with the expressions of Ismail and Rahman.

2. A Transformation of the Set of Solutions

As previously mentioned the q-Laguerre polynomials are orthogonal on $(0,\infty)$ with respect to the weight function $x^\alpha/(-x;q)_\infty$, to be precise

$$\int_0^\infty L_m^{(\alpha)}(x;q)L_n^{(\alpha)}(x;q)\frac{x^\alpha}{(-x;q)_\infty}\,dx = -\frac{(q^{\alpha+1};q)_n}{q^n(q;q)_n}\frac{(q^{-\alpha};q)_\infty}{(q;q)_\infty}\frac{\pi}{\sin\pi\alpha}\delta_{mn}, \tag{2.0.7}$$

where the right-hand side has to be interpreted as the limit

$$\log q^{-1} q^{-\binom{k+1}{2}-n}(q^{n+1};q)_k\delta_{mn}$$

when $\alpha = k = 0, 1, \ldots$. Therefore, the absolutely continuous measure

$$\nu^{(\alpha)} = -\frac{\sin\pi\alpha}{\pi}\frac{(q;q)_\infty}{(q^{-\alpha};q)_\infty}\frac{x^\alpha}{(-x;q)_\infty}\,dx \tag{2.0.8}$$

is a probability measure on $(0,\infty)$ and by (1.0.2) the moments of $\nu^{(\alpha)}$ are

$$\int_0^\infty x^n\,d\nu^{(\alpha)}(x) = q^{-\alpha n-\binom{n+1}{2}}(q^{\alpha+1};q)_n. \tag{2.0.9}$$

So we consider the moment problem associated with the moment sequence (2.0.9). The functional equation (1.0.5) becomes useful because of the following result.

Proposition 2.1. *Let f be a positive and measurable function defined on the interval $(0,\infty)$. If f satisfies the functional equation $f(qx) = q^\alpha(1+x)f(x)$ and $\int_0^\infty f(x)\,dx = c \in (0,\infty)$, then the absolutely continuous measure with density $(1/c)f$ has the moments* (2.0.9).

Proof. Without loss of generality we can assume that $\int_0^\infty f(x)\,dx = 1$. If $\int_0^\infty x^n f(x)\,dx = q^{-\alpha n-\binom{n+1}{2}}(q^{\alpha+1};q)_n$ for some $n \geq 0$, then

$$\begin{aligned}
\int_0^\infty x^{n+1}f(x)\,dx &= \int_0^\infty x^n(q^{-\alpha}f(qx) - f(x))\,dx \\
&= q^{-\alpha-n}\int_0^\infty (qx)^n f(qx)\,dx - \int_0^\infty x^n f(x)\,dx \\
&= q^{-\alpha-n-1}\int_0^\infty y^n f(y)\,dy - \int_0^\infty x^n f(x)\,dx \\
&= (q^{-\alpha-n-1}-1)q^{-\alpha n-\binom{n+1}{2}}(q^{\alpha+1};q)_n \\
&= q^{-\alpha(n+1)-\binom{n+2}{2}}(q^{\alpha+1};q)_{n+1},
\end{aligned}$$

and the assertion follows by induction. ∎

As a rule, it is easier to verify that the density of an absolutely continuous measure satisfies the functional equation (1.0.5) than to calculate all the moments, and this is the advantage of Proposition 2.1. However, the conditions from Proposition 2.1 are only sufficient and not necessary.

Suppose that f_1 and f_2 are two functions which satisfy the functional equation (1.0.5). If f_2 is strictly positive, then the quotient f_1/f_2 is a well-defined q-periodic function, i.e., a function on $(0,\infty)$ satisfying the very simple functional equation $g(qx) = g(x)$. So we conclude that every solution of (1.0.5) has the form $g(x)x^\alpha/(-x;q)_\infty$ for some q-periodic function g, and if g additionally is positive, measurable, and $\int_0^\infty g(x)x^\alpha/(-x;q)_\infty\,dx = c \in (0,\infty)$, then the absolutely continuous measure with density $(1/c)g(x)x^\alpha/(-x;q)_\infty$ has the moments (2.0.9), see Proposition 2.1.

There is a natural way to construct discrete measures with the moments (2.0.9). These measures can also be found in [17], but here we emphasize the method. Suppose that f is a strictly positive function which satisfies the functional equation (1.0.5)—such as $x^\alpha/(-x;q)_\infty$—and consider for $c > 0$ the measure $\kappa_c^{(\alpha)}$ given by

$$\kappa_c^{(\alpha)} = \frac{1}{cf(c)K^{(\alpha)}(c)} \sum_{n=-\infty}^{\infty} cq^n f(cq^n)\varepsilon_{cq^n},$$

where the constant $K^{(\alpha)}(c)$ is chosen such that $\kappa_c^{(\alpha)}$ becomes a probability measure. It is easily seen by induction that f satisfies the functional equation

$$f(xq^n) = (-x;q)_n q^{\alpha n} f(x), \qquad x > 0,$$

for each $n \in \mathbf{Z}$, so the measure $\kappa_c^{(\alpha)}$ does not depend on the specific f and is given by

$$\kappa_c^{(\alpha)} = \frac{1}{K^{(\alpha)}(c)} \sum_{n=-\infty}^{\infty} (-c;q)_n q^{n(\alpha+1)} \varepsilon_{cq^n}. \tag{2.1.1}$$

By (1.0.4) we have

$$K^{(\alpha)}(c) = \frac{(q, -cq^{\alpha+1}, -1/cq^\alpha; q)_\infty}{(-q/c, q^{\alpha+1}; q)_\infty},$$

and the moments of $\kappa_c^{(\alpha)}$ are (2.0.9). Actually $\kappa_{qc}^{(\alpha)} = \kappa_c^{(\alpha)}$, so it suffices to consider $\kappa_c^{(\alpha)}$ for $q < c \le 1$.

For any number $a > 0$ we let τ_a denote the map given by $\tau_a(x) = ax$. For a measure μ on $\mathbf{R}$ the image measure $\tau_a(\mu)$ under τ_a is defined by

$$\tau_a(\mu)(B) = \mu(a^{-1}B)$$

for any Borel set $B \subset \mathbf{R}$. The following result gives rise to a transformation of the set of solutions to the indeterminate Stieltjes moment problem.

Proposition 2.2. *Suppose that $\mu^{(\alpha)}$ is a measure on* $[0,\infty)$ *with the moments* (2.0.9). *Then the support of $\upsilon^{(\alpha)} = \tau_q(q^{\alpha+1}(1+x)\,d\mu^{(\alpha)}(x))$ is contained in* $[0,\infty)$ *and $\upsilon^{(\alpha)}$ has the moments* (2.0.9).

Proof. The proof is straightforward. The support of $\upsilon^{(\alpha)}$ is certainly contained in $[0,\infty)$ and

$$\begin{aligned}\int_0^\infty x^n\,d\upsilon^{(\alpha)}(x) &= \int_0^\infty (qx)^n q^{\alpha+1}(1+x)\,d\mu^{(\alpha)}(x)\\ &= q^{n+\alpha+1}\left(\int_0^\infty x^n\,d\mu^{(\alpha)}(x)+\int_0^\infty x^{n+1}\,d\mu^{(\alpha)}(x)\right)\\ &= q^{n+\alpha+1}\left(q^{-\alpha n-\binom{n+1}{2}}(q^{\alpha+1};q)_n+q^{-\alpha(n+1)-\binom{n+2}{2}}(q^{\alpha+1};q)_{n+1}\right)\\ &= q^{-\alpha n-\binom{n+1}{2}}(q^{\alpha+1};q)_n. \qquad \blacksquare\end{aligned}$$

We will denote the transformation by $T^{(\alpha)}$, that is, $T^{(\alpha)}(\mu^{(\alpha)})=\tau_q(q^{\alpha+1}(1+x)\,d\mu^{(\alpha)}(x))$. A probability measure $\mu^{(\alpha)}$ is a fixed point of $T^{(\alpha)}$ if and only if $\mu^{(\alpha)}$ satisfies the equation

$$\tau_{q^{-1}}(\mu^{(\alpha)})=q^{\alpha+1}(1+x)\,d\mu^{(\alpha)}(x). \tag{2.2.1}$$

When $\mu^{(\alpha)}=f(x)\,dx$ this equation corresponds exactly to the functional equation (1.0.5), and when $\mu^{(\alpha)}$ is a discrete measure this equation tells us that $c>0$ is a mass point of $\mu^{(\alpha)}$ exactly when qc likewise is a mass point of $\mu^{(\alpha)}$ and $\mu^{(\alpha)}(\{qc\})=q^{\alpha+1}(1+c)\mu^{(\alpha)}(\{c\})$. The latter property is certainly satisfied by $\kappa_c^{(\alpha)}$, so the measures $\nu^{(\alpha)}$ and $\kappa_c^{(\alpha)}$, see (2.0.8) and (2.1.1), are fixed points of $T^{(\alpha)}$.

As a matter of fact we can classify all the absolutely continuous and all the discrete fixed points of $T^{(\alpha)}$. Whenever g is a positive, measurable, and q-periodic function on $(0,\infty)$ such that $\int_0^\infty g(x)x^\alpha/(-x;q)_\infty\,dx=1$, the measure $\mu^{(\alpha)}=g(x)x^\alpha/(-x;q)_\infty\,dx$ is a fixed point of $T^{(\alpha)}$, and there is no other way to find absolutely continuous fixed points of $T^{(\alpha)}$. The discrete fixed points of $T^{(\alpha)}$ are precisely the countable convex combinations of the measures $\kappa_c^{(\alpha)}$ for varying $c>0$.

It is worthwhile dwelling somewhat on (2.2.1). Suppose that $\mu^{(\alpha)}$ is a finite measure on $(0,\infty)$, which satisfies this equation. Just as in the proof of Proposition 2.1, it follows that $\mu^{(\alpha)}$ is a solution to the moment problem provided that $\mu^{(\alpha)}$ is a probability measure. By induction it is easy to see that

$$\tau_{q^{-n}}(\mu^{(\alpha)})=q^{n(\alpha+1)}(-x;q)_n\,d\mu^{(\alpha)}(x)$$

for all $n\in\mathbf{Z}$, and this means that $\mu^{(\alpha)}$ is uniquely determined by its restriction $\mu^{(\alpha)}|_{(q,1]}$ to the interval $(q,1]$ or any other interval of the form $(q^{n+1},q^n]$, $n\in\mathbf{Z}$. For if $\mu^{(\alpha)}|_{(q,1]}=\rho^{(\alpha)}$, then

$$\mu^{(\alpha)}|_{(q^{n+1},q^n]}=\tau_{q^n}(q^{n(\alpha+1)}(-x;q)_n\,d\rho^{(\alpha)}(x)) \qquad \text{for} \quad n\in\mathbf{Z},$$

and $\bigcup_{n=-\infty}^\infty(q^{n+1},q^n]=(0,\infty)$. On the other hand, we have the following result.

Proposition 2.3. *Suppose that $\rho^{(\alpha)}$ is a finite measure on $(q,1]$. Then there is exactly one way to extend $\rho^{(\alpha)}$ to a finite measure $\mu^{(\alpha)}$ on $(0,\infty)$ such that $\mu^{(\alpha)}$ satisfies (2.2.1).*

Proof. Simply define

$$\mu^{(\alpha)}|_{(q^{n+1},q^n]} = \tau_{q^n}(q^{n(\alpha+1)}(-x;q)_n\, d\rho^{(\alpha)}(x)) \qquad \text{for} \quad n \in \mathbf{Z},$$

that is,

$$\mu^{(\alpha)}(q^n B) = q^{n(\alpha+1)} \int_B (-x;q)_n\, d\rho^{(\alpha)}(x)$$

for any Borel set $B \subset (q,1]$. In this way

$$\begin{aligned} \tau_{q^{-1}}(\mu^{(\alpha)}|_{(q^{n+1},q^n]}) &= \tau_{q^{n-1}}(q^{n(\alpha+1)}(-x;q)_n\, d\rho^{(\alpha)}(x)) \\ &= q^{\alpha+1}(1+x)\, d\tau_{q^{n-1}}(q^{(n-1)(\alpha+1)}(-x;q)_{n-1}\, d\rho^{(\alpha)}(x))(x) \\ &= q^{\alpha+1}(1+x)\, d\mu^{(\alpha)}|_{(q^n,q^{n-1}]}(x), \end{aligned}$$

so the measure $\mu^{(\alpha)}$ satisfies the desired equation. Note that $\mu^{(\alpha)}$ is a finite measure since

$$\begin{aligned} \mu^{(\alpha)}((0,\infty)) &= \sum_{n=-\infty}^{\infty} q^{n(\alpha+1)} \int_{(q,1]} (-x;q)_n\, d\rho^{(\alpha)}(x) \\ &\le \rho^{(\alpha)}((q,1]) \left(\sum_{n=0}^{\infty} (-1;q)_n q^{n(\alpha+1)} + \sum_{n=1}^{\infty} \frac{q^{\binom{n}{2}-n(\alpha+1)}}{(-q;q)_n} \right) < \infty. \quad \blacksquare \end{aligned}$$

Starting from a finite measure $\rho^{(\alpha)}$ on the interval $(q,1]$ we can thus construct a solution to the moment problem by, if necessary, normalizing the extension $\mu^{(\alpha)}$ from Proposition 2.3. The solution obtained from $\rho^{(\alpha)}$ is of the same type as $\rho^{(\alpha)}$, so if $\rho^{(\alpha)}$ is a continuous singular measure we will end up with a continuous singular solution to the moment problem.

3. The Limit $\alpha \to \infty$ and the Connection with the Moment Problem Associated with the Stieltjes–Wigert Polynomials

The Stieltjes–Wigert polynomials (1.0.6) are known to be orthogonal polynomials associated with the moment sequence $(q^{-\binom{n+1}{2}})$, see [13].

Suppose that $\mu^{(\alpha)}$ is a measure on $(0,\infty)$ with the moments (2.0.9). Then $\tau_{q^\alpha}(\mu^{(\alpha)})$ has the moments

$$\int_0^\infty x^n\, d\tau_{q^\alpha}(\mu^{(\alpha)})(x) = \int_0^\infty (q^\alpha x)^n\, d\mu^{(\alpha)}(x) = q^{-\binom{n+1}{2}}(q^{\alpha+1};q)_n \tag{3.0.1}$$

and since $(q^{\alpha+1};q)_n \to 1$ for $\alpha \to \infty$, it follows that any vague accumulation point of $(\tau_{q^\alpha}(\mu^{(\alpha)}))$ for $\alpha \to \infty$ has the moments $q^{-\binom{n+1}{2}}$. Furthermore, such accumulation points always exist (see, e.g., [1, pp. 30–32]).

Let us take a look at the situation when $\mu^{(\alpha)} = \kappa^{(\alpha)}_{cq^{-\alpha}}$ and $\mu^{(\alpha)} = \nu^{(\alpha)}$, respectively.

The discrete measure $\tau_{q^\alpha}(\kappa^{(\alpha)}_{cq^{-\alpha}})$ is given by

$$\tau_{q^\alpha}(\kappa^{(\alpha)}_{cq^{-\alpha}}) = \frac{1}{K^{(\alpha)}(cq^{-\alpha})} \sum_{n=-\infty}^{\infty} (-cq^{-\alpha};q)_n (q^{\alpha+1})^n \varepsilon_{cq^n},$$

so independent of α this measure is supported on the set $\{cq^n \mid n \in \mathbf{Z}\}$. Therefore the question is how the masses behave when $\alpha \to \infty$. Since

$$\frac{(-cq^{-\alpha};q)_n (q^{\alpha+1})^n}{K^{(\alpha)}(cq^{-\alpha})} = c^n q^{\binom{n+1}{2}} (-q^{\alpha+1-n}/c;q)_n \frac{(-q^{\alpha+1}/c, q^{\alpha+1};q)_\infty}{(q,-cq,-1/c;q)_\infty},$$

we get, for each n, that

$$\frac{(-cq^{-\alpha};q)_n (q^{\alpha+1})^n}{K^{(\alpha)}(cq^{-\alpha})} \to \frac{c^n q^{\binom{n+1}{2}}}{(q,-cq,-1/c;q)_\infty} \qquad \text{for} \quad \alpha \to \infty.$$

This means that $\tau_{q^\alpha}(\kappa^{(\alpha)}_{cq^{-\alpha}})$ converges to

$$\lambda_c = \frac{1}{L(c)} \sum_{n=-\infty}^{\infty} c^n q^{\binom{n+1}{2}} \varepsilon_{cq^n} \tag{3.0.2}$$

for $\alpha \to \infty$, where $L(c) = (-cq, -1/c, q; q)_\infty$. The measures $(\lambda_c)_{c>0}$—found by Chihara in [11]—are well-known discrete solutions to the moment problem associated with the Stieltjes–Wigert polynomials. Since $\lambda_{c/q} = \lambda_c$, it suffices to consider λ_c for $q < c \leq 1$.

The absolutely continuous measure $\tau_{q^\alpha}(\nu^{(\alpha)})$ is given by the density

$$v^{(\alpha)}(x) = -\frac{\sin \pi\alpha}{\pi} \frac{(q;q)_\infty}{(q^{-\alpha};q)_\infty} q^{-\alpha(\alpha+1)} \frac{x^\alpha}{(-q^{-\alpha}x;q)_\infty}, \qquad x > 0.$$

We cannot, as before, simply let $\alpha \to \infty$ because the limit does not exist. To begin with, suppose that $\alpha = k$ with $k = 0, 1, \ldots$. Then the density $v^{(\alpha)}$ has the form

$$v^{(k)}(x) = \frac{1}{\log q^{-1}} \frac{q^{-\binom{k+1}{2}}}{(q;q)_k} \frac{x^k}{(-q^{-k}x;q)_\infty}, \qquad x > 0,$$

and since

$$\frac{q^{-\binom{k+1}{2}}}{(q;q)_k} \frac{x^k}{(-q^{-k}x;q)_\infty} = \frac{1}{(q,-q/x;q)_k} \frac{1}{(-x;q)_\infty},$$

it follows that

$$v^{(k)}(x) \to \frac{1}{\log q^{-1}} \frac{1}{(q,-q/x,-x;q)_\infty} \qquad \text{for} \quad k \to \infty.$$

So the measure

$$\omega = \frac{1}{\log q^{-1}} \frac{1}{(-x,-q/x,q;q)_\infty}\, dx \tag{3.0.3}$$

is a certain accumulation point of $(\tau_{q^\alpha}(\nu^{(\alpha)}))$ and this was mentioned by Askey in [3]. But there is more to be said. Suppose that $\alpha = k + c$ with $k = 0, 1, \ldots$ and $0 < c < 1$. Then the density $v^{(\alpha)}$ can be written as

$$\begin{aligned} v^{(k+c)}(x) &= -\frac{\sin \pi(k+c)}{\pi} \frac{(q;q)_\infty}{(q^{-k-c};q)_\infty} q^{-(k+c)(k+c+1)} \frac{x^{k+c}}{(-q^{-k-c}x;q)_\infty} \\ &= \frac{\sin \pi c}{\pi} \frac{(q;q)_\infty}{(q^c;q)_{k+1}(q^{1-c};q)_\infty} q^{c(1-c)} \frac{x^{c-1}}{(-q^c/x;q)_{k+1}(-q^{1-c}x;q)_\infty}, \end{aligned}$$

and hence for fixed c it follows that

$$v^{(k+c)}(x) \to q^{c(1-c)} \frac{\sin \pi c}{\pi} \frac{(q;q)_\infty}{(q^c, q^{1-c}; q)_\infty} \frac{x^{c-1}}{(-q^c/x, -q^{1-c}x; q)_\infty} \qquad \text{for} \quad k \to \infty.$$

In this way we have obtained the absolutely continuous measures

$$(3.0.4) \quad \omega_c = q^{c(1-c)} \frac{\sin \pi c}{\pi} \frac{(q;q)_\infty}{(q^c, q^{1-c}; q)_\infty} \frac{x^{c-1}}{(-q^c/x, -q^{1-c}x; q)_\infty} dx \qquad (0 < c < 1),$$

as accumulation points of $(\tau_{q^\alpha}(\nu^{(\alpha)}))$. Note that $\omega = \omega_1$ and, due to the fact that $\omega_{c+1} = \omega_c$, it suffices to consider ω_c for $0 < c \leq 1$.

The connection between $(\nu^{(\alpha)})$ and (ω_c) inspires us to find a family of absolutely continuous measures with the moments (2.0.9). The moments of ω_c can be found by using the Askey–Roy q-beta integral [2, p. 514], [5], [6]:

$$(3.0.5) \qquad \int_0^\infty x^{c-1} \frac{(-ax, -bq/x; q)_\infty}{(-x, -q/x; q)_\infty} dx = \frac{(ab, q^c, q^{1-c}; q)_\infty}{(q, aq^{-c}, bq^c; q)_\infty} \frac{\pi}{\sin \pi c}$$
$$(c > 0, |a| < q^c, |b| < q^{-c}),$$

and the idea is that this integral also leads to solutions to the moment problem associated with the q-Laguerre polynomials.

Let $\omega_c^{(\alpha)}$ denote the absolutely continuous measure with density

$$(3.0.6) \qquad w_c^{(\alpha)}(x) \;=\; q^{c(\alpha+1-c)} \frac{\sin \pi c}{\pi} \frac{(q, q^{\alpha+1}, -q/x; q)_\infty}{(q^c, q^{1-c}, -q^{\alpha+1-c}x, -q^{-\alpha+c}/x; q)_\infty} x^{c-1},$$
$$(x > 0).$$

It is straightforward to see that $\tau_{q^\alpha}(\omega_c^{(\alpha)})$ converges to ω_c for $\alpha \to \infty$ and by (3.0.5) we have

$$\int_0^\infty x^{c-1} \frac{(-q/x; q)_\infty}{(-q^{\alpha+1-c}x, -q^{-\alpha+c}/x; q)_\infty} dx = q^{-c(\alpha+1-c)} \int_0^\infty y^{c-1} \frac{(-q^{\alpha+2-c}/y; q)_\infty}{(-y, -q/y; q)_\infty} dy$$
$$= q^{-c(\alpha+1-c)} \frac{(q^c, q^{1-c}; q)_\infty}{(q, q^{\alpha+1}; q)_\infty} \frac{\pi}{\sin \pi c}.$$

So $\omega_c^{(\alpha)}$ is a probability measure on $(0, \infty)$ and since

$$\frac{(-1/x; q)_\infty}{(-q^{\alpha+2-c}x, -q^{-\alpha-1+c}/x; q)_\infty} (qx)^{c-1} = q^\alpha (1+x) \frac{(-q/x; q)_\infty}{(-q^{\alpha+1-c}x, -q^{-\alpha+c}/x; q)_\infty} x^{c-1},$$

it follows from Proposition 2.1 that $\omega_c^{(\alpha)}$ has the moments (2.0.9).

Note that $\nu^{(\alpha)} = \omega_{\alpha+1}^{(\alpha)}$ and the measure $\omega^{(\alpha)} = \omega_1^{(\alpha)}$ has the form

$$(3.0.7) \qquad \omega^{(\alpha)} = \frac{q^\alpha}{\log q^{-1}} \frac{(q^{\alpha+1}, -q/x; q)_\infty}{(q, -q^\alpha x, -q^{-\alpha+1}/x; q)_\infty} dx.$$

After having studied the situation when $\alpha \to \infty$ at the level of weight functions, we will now consider the situation at the level of Pick functions. Of course not every Pick function φ gives rise to a measure μ_φ supported on $[0, \infty)$. In this connection the quantity $\beta \leq 0$ given by

$$\beta = \lim_{n\to\infty} \frac{P_n(0)}{Q_n(0)}$$

plays an important part. As Pedersen proved in [18, Théorème 1], the measure μ_φ corresponding to the Pick function φ is supported on $[0, \infty)$ precisely if φ has an analytic continuation to $\mathbf{C}\backslash[0, \infty)$ and $\beta \leq \varphi(x) \leq 0$ for $x < 0$.

The q-Laguerre polynomials of the second kind are defined by

$$\tilde{L}_n^{(\alpha)}(x; q) = \int_0^\infty \frac{L_n^{(\alpha)}(x; q) - L_n^{(\alpha)}(y; q)}{x - y}\, d\mu^{(\alpha)}(y),$$

where $\mu^{(\alpha)}$ is any measure on $[0, \infty)$ with the moments (2.0.9). Explicitly, we have

$$L_n^{(\alpha)}(x; q) = \frac{(q^{\alpha+1}; q)_n}{(q; q)_n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k \frac{q^{\alpha k + k^2}}{(q^{\alpha+1}; q)_k} x^k \tag{3.0.8}$$

and

$$\begin{aligned} \tilde{L}_n^{(\alpha)}(x; q) &= \frac{(q^{\alpha+1}; q)_n}{(q; q)_n} \\ &\quad \times \sum_{m=0}^{n-1} \left(\sum_{k=m+1}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k \frac{q^{\alpha k + k^2}}{(q^{\alpha+1}; q)_k} q^{-\alpha(k-m-1) - \binom{k-m}{2}} \right. \\ &\qquad \left. \times\, (q^{\alpha+1}; q)_{k-m-1} \right) x^m \\ &= \frac{(q^{\alpha+1}; q)_n}{(q; q)_n} \sum_{m=0}^{n-1} \frac{q^{\alpha(m+1) - \binom{m+1}{2}}}{(q^{\alpha-m}; q)_{m+1}} \\ &\quad \times \left(\sum_{k=m+1}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k \frac{(q^{\alpha-m}; q)_k}{(q^{\alpha+1}; q)_k} q^{\binom{k}{2} + (m+1)k} \right) x^m. \end{aligned} \tag{3.0.9}$$

The inner sum $\sum_{k=m+1}^{n} \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q (-1)^k [(q^{\alpha-m}; q)_k/(q^{\alpha+1}; q)_k] q^{\binom{k}{2}+(m+1)k}$ is the tail in the q-Chu–Vandermonde formula (1.0.3) and can also be written as

$$\frac{(q^{m+1}; q)_n}{(q^{\alpha+1}; q)_n} - \sum_{k=0}^{m} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k \frac{(q^{\alpha-m}; q)_k}{(q^{\alpha+1}; q)_k} q^{\binom{k}{2}+(m+1)k}.$$

From (2.0.7) we see that the orthonormal polynomials $(P_n^{(\alpha)})$ and $(Q_n^{(\alpha)})$ associated with the moment sequence (2.0.9) have the form

$$P_n^{(\alpha)}(x) = \sqrt{q^n \frac{(q; q)_n}{(q^{\alpha+1}; q)_n}}\, L_n^{(\alpha)}(x; q) \tag{3.0.10}$$

and

$$Q_n^{(\alpha)}(x) = \sqrt{q^n \frac{(q;q)_n}{(q^{\alpha+1};q)_n}}\, \tilde{L}_n^{(\alpha)}(x;q).$$

In particular, we have

$$P_n^{(\alpha)}(0) = \sqrt{q^n \frac{(q^{\alpha+1};q)_n}{(q;q)_n}} \tag{3.0.11}$$

and

$$Q_n^{(\alpha)}(0) = \frac{q^\alpha}{1-q^\alpha}\left(\frac{(q;q)_n}{(q^{\alpha+1};q)_n} - 1\right) P_n^{(\alpha)}(0) \qquad \text{for} \quad \alpha \neq 0.$$

The expression for $Q_n^{(\alpha)}(0)$ has a singularity at $\alpha = 0$. Seeing that the function $(xq;q)_n$ is differentiable with derivative $-(xq;q)_n \sum_{k=1}^n q^k/(1-xq^k)$, it follows that

$$\begin{aligned} \frac{q^\alpha}{1-q^\alpha}\left(\frac{(q;q)_n}{(q^{\alpha+1};q)_n} - 1\right) &= \frac{q^\alpha}{(q^{\alpha+1};q)_n} \frac{(q;q)_n - (q^{\alpha+1};q)_n}{1-q^\alpha} \\ &\to -\sum_{k=1}^n \frac{q^k}{1-q^k} \qquad \text{for} \quad \alpha \to 0. \end{aligned}$$

So the singularity at $\alpha = 0$ is removable and $Q_n^{(0)}(0) = -\sum_{k=1}^n [q^k/(1-q^k)] P_n^{(0)}(0)$. Since we are interested in the situation when $\alpha \to \infty$, usually we will not be interested in the special case $\alpha = 0$. Whenever $Q_n^{(\alpha)}(0)$ plays a part, we will assume that $\alpha \neq 0$.

The quantity $\beta^{(\alpha)} = \lim_{n\to\infty} P_n^{(\alpha)}(0)/Q_n^{(\alpha)}(0)$ is given by

$$\begin{aligned} \beta^{(\alpha)} &= \lim_{n\to\infty} \frac{1-q^\alpha}{q^\alpha}\left(\frac{(q;q)_n}{(q^{\alpha+1};q)_n} - 1\right)^{-1} \\ &= \frac{1-q^\alpha}{q^\alpha}\left(\frac{(q;q)_\infty}{(q^{\alpha+1};q)_\infty} - 1\right)^{-1}, \end{aligned} \tag{3.0.12}$$

and it is noteworthy that

$$\left(\frac{(q;q)_\infty}{(q^{\alpha+1};q)_\infty} - 1\right)^{-1} \leq \frac{1}{(q;q)_\infty - 1} \qquad \text{for} \quad \alpha > 0. \tag{3.0.13}$$

For if (P_n) and (Q_n) are the orthonormal polynomials associated with the moment sequence $(q^{-\binom{n+1}{2}})$, then (see, e.g., [13]):

$$\lim_{n\to\infty} \frac{P_n(0)}{Q_n(0)} = \frac{1}{(q;q)_\infty - 1}. \tag{3.0.14}$$

There is a close connection between the moment problems associated with the moment sequences (2.0.9) and (3.0.1). The orthonormal polynomials $(\tilde{P}_n^{(\alpha)})$ associated with the moment sequence (3.0.1) are given by $\tilde{P}_n^{(\alpha)}(x) = P_n^{(\alpha)}(q^{-\alpha}x)$, and the orthonormal polynomials $(\tilde{Q}_n^{(\alpha)})$ of the second kind are given by $\tilde{Q}_n^{(\alpha)}(x) = q^{-\alpha} Q_n^{(\alpha)}(q^{-\alpha}x)$. Therefore the four entire functions $\tilde{A}^{(\alpha)}$, $\tilde{B}^{(\alpha)}$, $\tilde{C}^{(\alpha)}$, and $\tilde{D}^{(\alpha)}$ from the Nevanlinna parametrization can be written as

$$\begin{aligned}
\tilde{A}^{(\alpha)}(x) &= x\sum_{n=0}^{\infty}\tilde{Q}_n^{(\alpha)}(0)\tilde{Q}_n^{(\alpha)}(x) = q^{-2\alpha}x\sum_{n=0}^{\infty}Q_n^{(\alpha)}(0)Q_n^{(\alpha)}(q^{-\alpha}x) = q^{-\alpha}A^{(\alpha)}(q^{-\alpha}x),\\
\tilde{B}^{(\alpha)}(x) &= -1+x\sum_{n=0}^{\infty}\tilde{Q}_n^{(\alpha)}(0)\tilde{P}_n^{(\alpha)}(x) = -1+q^{-\alpha}x\sum_{n=0}^{\infty}Q_n^{(\alpha)}(0)P_n^{(\alpha)}(q^{-\alpha}x)\\
&= B^{(\alpha)}(q^{-\alpha}x),\\
\tilde{C}^{(\alpha)}(x) &= 1+x\sum_{n=0}^{\infty}\tilde{P}_n^{(\alpha)}(0)\tilde{Q}_n^{(\alpha)}(x) = 1+q^{-\alpha}x\sum_{n=0}^{\infty}P_n^{(\alpha)}(0)Q_n^{(\alpha)}(q^{-\alpha}x)\\
&= C^{(\alpha)}(q^{-\alpha}x),\\
\tilde{D}^{(\alpha)}(x) &= x\sum_{n=0}^{\infty}\tilde{P}_n^{(\alpha)}(0)\tilde{P}_n^{(\alpha)}(x) = x\sum_{n=0}^{\infty}P_n^{(\alpha)}(0)P_n^{(\alpha)}(q^{-\alpha}x) = q^{\alpha}D^{(\alpha)}(q^{-\alpha}x).
\end{aligned}$$

Let $(-1,\infty]$ denote the set $(-1,\infty)\cup\{\infty\}$. We can think of $(-1,\infty]$ as a (metrizable) topological space by defining a basis of neighborhoods at ∞ as $(r,\infty)\cup\{\infty\}$. In this way $[r,\infty]$ clearly becomes a compact set for each $r>-1$, and a function f defined on the interval $(-1,\infty)$ has a unique continuous extension to $(-1,\infty]$ provided that $\lim_{x\to\infty}f(x)$ exists. Simply define $f(\infty)$ as $\lim_{x\to\infty}f(x)$.

In this setup, since $(q^{\alpha+1};q)_n\to 1$ for $\alpha\to\infty$, the moments (3.0.1) are continuous as a function of α for $\alpha\in(-1,\infty]$. Recall that

$$\begin{aligned}
\tilde{P}_n^{(\alpha)}(0) &= P_n^{(\alpha)}(0) = \sqrt{q^n\frac{(q^{\alpha+1};q)_n}{(q;q)_n}},\\
\tilde{Q}_n^{(\alpha)}(0) &= q^{-\alpha}Q_n^{(\alpha)}(0) = \frac{1}{1-q^{\alpha}}\left(\frac{(q;q)_n}{(q^{\alpha+1};q)_n}-1\right)P_n^{(\alpha)}(0)
\end{aligned}$$

and note that

$$\tilde{P}_n^{(\alpha)}(0)\underset{\alpha\to\infty}{\longrightarrow}\sqrt{\frac{q^n}{(q;q)_n}}$$

and

$$\tilde{Q}_n^{(\alpha)}(0)\underset{\alpha\to\infty}{\longrightarrow}((q;q)_n-1)\sqrt{\frac{q^n}{(q;q)_n}}.$$

Hence the series

$$\sum_{n=0}^{\infty}\tilde{P}_n^{(\alpha)}(0)^2 \qquad\text{and}\qquad \sum_{n=0}^{\infty}\tilde{Q}_n^{(\alpha)}(0)^2$$

are uniformly convergent for $\alpha\in[1,\infty]$, since they are both dominated by some constant times the convergent series $\sum_{n=0}^{\infty}q^n/(q;q)_n$ for $\alpha\in[1,\infty)$.

In [9, Proposition 2.4.1] Berg and Valent consider the situation where the moment problem depends on a parameter in a metric space. In our case the metric space is $[1,\infty]$

and it follows that $\tilde{A}^{(\alpha)}$, $\tilde{B}^{(\alpha)}$, $\tilde{C}^{(\alpha)}$, and $\tilde{D}^{(\alpha)}$ are continuous for $(x, \alpha) \in \mathbf{C} \times [1, \infty]$. Since $q^{-\binom{n+1}{2}}(q^{\alpha+1}; q)_n \to q^{-\binom{n+1}{2}}$ for $\alpha \to \infty$, this in particular means that

(3.0.15)
$$\begin{aligned} q^{-\alpha} A^{(\alpha)}(q^{-\alpha}x) &= \tilde{A}^{(\alpha)}(x) \to A(x), \\ B^{(\alpha)}(q^{-\alpha}x) &= \tilde{B}^{(\alpha)}(x) \to B(x), \\ C^{(\alpha)}(q^{-\alpha}x) &= \tilde{C}^{(\alpha)}(x) \to C(x), \\ q^{\alpha} D^{(\alpha)}(q^{-\alpha}x) &= \tilde{D}^{(\alpha)}(x) \to D(x), \end{aligned}$$

for $\alpha \to \infty$, where A, B, C, and D are the four entire functions from the Nevanlinna parametrization for the moment problem associated with the Stieltjes–Wigert polynomials. Due to the fact that a continuous function on a compact set is uniformly continuous, the above convergence is uniform on compact subsets of $\mathbf{C}$.

We are now ready to present the connection between the moment problems associated with the moment sequences $(q^{-\binom{n+1}{2}})$ and (2.0.9) at the level of Pick functions.

Theorem 3.1. *Suppose that μ_φ is a measure on $[0, \infty)$ with the moments $q^{-\binom{n+1}{2}}$. Here φ denotes the corresponding Pick function. For each $\alpha > 0$ the function $\psi^{(\alpha)}$ given by*

$$\psi^{(\alpha)}(x) = \frac{1 - q^{\alpha}}{q^{\alpha}} \varphi(q^{\alpha} x)$$

is a Pick function corresponding to a measure $\mu^{(\alpha)}_{\psi^{(\alpha)}}$ on $[0, \infty)$ with the moments (2.0.9) *and*

$$\tau_{q^\alpha}(\mu^{(\alpha)}_{\psi^{(\alpha)}}) \to \mu_\varphi \qquad \text{for} \quad \alpha \to \infty.$$

Proof. Clearly $\psi^{(\alpha)}$ is a Pick function for $\alpha > 0$. Recall that φ has an analytic continuation to $\mathbf{C}\backslash[0, \infty)$ such that

$$\frac{1}{(q; q)_\infty - 1} \le \varphi(x) \le 0 \qquad \text{for} \quad x < 0,$$

see (3.0.14). This means that $\psi^{(\alpha)}$ has an analytic continuation to $\mathbf{C}\backslash[0, \infty)$ too, and in order to prove that $\psi^{(\alpha)}$ corresponds to a measure on $[0, \infty)$ with the prescribed moments, we have to verify that

$$\beta^{(\alpha)} \le \psi^{(\alpha)}(x) \le 0 \qquad \text{for} \quad x < 0,$$

where

$$\beta^{(\alpha)} = \frac{1 - q^{\alpha}}{q^{\alpha}} \left(\frac{(q; q)_\infty}{(q^{\alpha+1}; q)_\infty} - 1 \right)^{-1}.$$

This is an immediate consequence of (3.0.13). The Stieltjes transform of $\tau_{q^\alpha}(\mu^{(\alpha)}_{\psi^{(\alpha)}})$ is given by

$$\int_0^\infty \frac{1}{t - x}\, d\tau_{q^\alpha}(\mu^{(\alpha)}_{\psi^{(\alpha)}})(t) \;=\; \int_0^\infty \frac{1}{q^\alpha t - x}\, d\mu^{(\alpha)}_{\psi^{(\alpha)}}(t)$$

$$
\begin{aligned}
&= \frac{1}{q^\alpha}\int_0^\infty \frac{1}{t-q^{-\alpha}x}\,d\mu^{(\alpha)}_{\psi^{(\alpha)}}(t)\\
&= -\frac{1}{q^\alpha}\frac{A^{(\alpha)}(q^{-\alpha}x)\psi^{(\alpha)}(q^{-\alpha}x)-C^{(\alpha)}(q^{-\alpha}x)}{B^{(\alpha)}(q^{-\alpha}x)\psi^{(\alpha)}(q^{-\alpha}x)-D^{(\alpha)}(q^{-\alpha}x)}\\
&= -\frac{A^{(\alpha)}(q^{-\alpha}x)[(1-q^\alpha)/q^\alpha]\varphi(x)-C^{(\alpha)}(q^{-\alpha}x)}{B^{(\alpha)}(q^{-\alpha}x)(1-q^\alpha)\varphi(x)-q^\alpha D^{(\alpha)}(q^{-\alpha}x)},
\end{aligned}
$$

so it follows from (3.0.15) that

$$
\begin{aligned}
\int_0^\infty \frac{1}{t-x}\,d\tau_{q^\alpha}(\mu^{(\alpha)}_{\psi^{(\alpha)}})(t) \;\to\; & -\frac{A(x)\varphi(x)-C(x)}{B(x)\varphi(x)-D(x)}\\
&= \int_0^\infty \frac{1}{t-x}\,d\mu_\varphi(t) \qquad \text{for} \quad \alpha\to\infty.
\end{aligned}
$$

Since a finite measure is uniquely determined by its Stieltjes transform, we have thus proved that

$$
\tau_{q^\alpha}(\mu^{(\alpha)}_{\psi^{(\alpha)}}) \to \mu_\varphi \qquad \text{for} \quad \alpha\to\infty. \qquad \blacksquare
$$

4. The Nevanlinna Parametrization

In this section we will present some explicit expressions for the four entire functions $A^{(\alpha)}$, $B^{(\alpha)}$, $C^{(\alpha)}$, and $D^{(\alpha)}$ from the Nevanlinna parametrization. This was done by Moak in [17], but only for $D^{(\alpha)}$ and $B^{(\alpha)}$, and by Ismail and Rahman in [15]. However, the expressions for $C^{(\alpha)}$ and $A^{(\alpha)}$ given by Ismail and Rahman are rather complicated, and it is not easy to explain what happens when $\alpha\to\infty$. Furthermore, the present method only depends on two generating functions and appears to be more direct. The first generating function can also be found in Koekoek and Swarttouw [16].

Proposition 4.1. *For $\gamma\in\mathbf{C}$ and $|t|<1$, we have*

$$
\sum_{n=0}^\infty \frac{(\gamma;q)_n}{(q^{\alpha+1};q)_n}t^n L_n^{(\alpha)}(x;q) = \frac{(\gamma t;q)_\infty}{(t;q)_\infty}\sum_{n=0}^\infty \frac{(\gamma;q)_n}{(\gamma t,q^{\alpha+1},q;q)_n}q^{\alpha n+n^2}(-tx)^n,
$$

$$
\begin{aligned}
\sum_{n=0}^\infty \frac{(\gamma;q)_n}{(q^{\alpha+1};q)_n}t^n \tilde{L}_n^{(\alpha)}(x;q) &= \frac{(\gamma t;q)_\infty}{(t;q)_\infty}\sum_{n=0}^\infty \frac{q^{\alpha(n+1)-\binom{n+1}{2}}}{(q^{\alpha-n};q)_{n+1}}\\
&\quad\times\left(\sum_{k=n+1}^\infty \frac{(\gamma,q^{\alpha-n};q)_k}{(\gamma t,q^{\alpha+1},q;q)_k}q^{\binom{k}{2}+(n+1)k}(-t)^k\right)x^n.
\end{aligned}
$$

In particular, with $\gamma=q^{\alpha+1}$ and $t=q$, we have

$$
\sum_{n=0}^\infty q^n L_n^{(\alpha)}(x;q) = \frac{(q^{\alpha+2};q)_\infty}{(q;q)_\infty}\sum_{n=0}^\infty(-1)^n\frac{q^{\alpha n+n(n+1)}}{(q^{\alpha+2},q;q)_n}x^n,
$$

$$\sum_{n=0}^{\infty} q^n \tilde{L}_n^{(\alpha)}(x;q) = \frac{(q^{\alpha+2};q)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{\alpha(n+1)-\binom{n+1}{2}}}{(q^{\alpha-n};q)_{n+1}} \times \left(\sum_{k=n+1}^{\infty} (-1)^k \frac{(q^{\alpha-n};q)_k}{(q^{\alpha+2},q;q)_k} q^{\binom{k}{2}+(n+2)k} \right) x^n,$$

and with $\gamma = t = q$, we have

$$\sum_{n=0}^{\infty} \frac{(q;q)_n}{(q^{\alpha+1};q)_n} q^n L_n^{(\alpha)}(x;q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{\alpha n + n(n+1)}}{(q^{\alpha+1};q)_n (q;q)_{n+1}} x^n,$$

$$\sum_{n=0}^{\infty} \frac{(q;q)_n}{(q^{\alpha+1};q)_n} q^n \tilde{L}_n^{(\alpha)}(x;q) = \sum_{n=0}^{\infty} \frac{q^{\alpha(n+1)-\binom{n+1}{2}}}{(q^{\alpha-n};q)_{n+1}} \times \left(\sum_{k=n+1}^{\infty} (-1)^k \frac{(q^{\alpha-n};q)_k}{(q^{\alpha+1};q)_k (q;q)_{k+1}} q^{\binom{k}{2}+(n+2)k} \right) x^n.$$

Proof. The point of this proof is to interchange the order of summation. We are certainly allowed to do so because of absolute convergence. Hence

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(\gamma;q)_n}{(q^{\alpha+1};q)_n} t^n L_n^{(\alpha)}(x;q) &= \sum_{n=0}^{\infty} (\gamma;q)_n t^n \sum_{k=0}^{n} (-1)^k \frac{q^{\alpha k + k^2}}{(q^{\alpha+1},q;q)_k (q;q)_{n-k}} x^k \\
&= \sum_{k=0}^{\infty} (-1)^k \frac{q^{\alpha k + k^2}}{(q^{\alpha+1},q;q)_k} x^k \sum_{n=k}^{\infty} \frac{(\gamma;q)_n}{(q;q)_{n-k}} t^n \\
&= \sum_{k=0}^{\infty} (-1)^k \frac{(\gamma;q)_k}{(q^{\alpha+1},q;q)_k} q^{\alpha k + k^2} t^k x^k \sum_{n=0}^{\infty} \frac{(\gamma q^k;q)_n}{(q;q)_n} t^n
\end{aligned}$$

and, similarly,

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(\gamma;q)_n}{(q^{\alpha+1};q)_n} t^n \tilde{L}_n^{(\alpha)}(x;q) \\
&\quad = \sum_{n=0}^{\infty} (\gamma;q)_n t^n \sum_{m=0}^{n-1} \frac{q^{\alpha(m+1)-\binom{m+1}{2}}}{(q^{\alpha-m};q)_{m+1}} \\
&\qquad \times \left(\sum_{k=m+1}^{n} (-1)^k \frac{(q^{\alpha-m};q)_k}{(q^{\alpha+1},q;q)_k (q;q)_{n-k}} q^{\binom{k}{2}+(m+1)k} \right) x^m \\
&\quad = \sum_{m=0}^{\infty} \frac{q^{\alpha(m+1)-\binom{m+1}{2}}}{(q^{\alpha-m};q)_{m+1}} \\
&\qquad \times \left(\sum_{n=m+1}^{\infty} (\gamma;q)_n t^n \sum_{k=m+1}^{n} (-1)^k \frac{(q^{\alpha-m};q)_k}{(q^{\alpha+1},q;q)_k (q;q)_{n-k}} q^{\binom{k}{2}+(m+1)k} \right) x^m
\end{aligned}$$

$$= \sum_{m=0}^{\infty} \frac{q^{\alpha(m+1)-\binom{m+1}{2}}}{(q^{\alpha-m};q)_{m+1}} \left(\sum_{k=m+1}^{\infty} (-1)^k \frac{(q^{\alpha-m};q)_k}{(q^{\alpha+1},q;q)_k} q^{\binom{k}{2}+(m+1)k} \sum_{n=k}^{\infty} \frac{(\gamma;q)_n}{(q;q)_{n-k}} t^n \right) x^m$$

$$= \sum_{m=0}^{\infty} \frac{q^{\alpha(m+1)-\binom{m+1}{2}}}{(q^{\alpha-m};q)_{m+1}}$$

$$\times \left(\sum_{k=m+1}^{\infty} (-1)^k \frac{(\gamma, q^{\alpha-m};q)_k}{(q^{\alpha+1},q;q)_k} q^{\binom{k}{2}+(m+1)k} t^k \sum_{n=0}^{\infty} \frac{(\gamma q^k;q)_n}{(q;q)_n} t^n \right) x^m.$$

By the q-binomial theorem, we have

$$\sum_{n=0}^{\infty} \frac{(\gamma q^k;q)_n}{(q;q)_n} t^n = \frac{(\gamma t q^k;q)_\infty}{(t;q)_\infty},$$

so, consequently,

$$\sum_{n=0}^{\infty} \frac{(\gamma;q)_n}{(q^{\alpha+1};q)_n} t^n L_n^{(\alpha)}(x;q) = \frac{(\gamma t;q)_\infty}{(t;q)_\infty} \sum_{k=0}^{\infty} \frac{(\gamma;q)_k}{(\gamma t, q^{\alpha+1}, q;q)_k} q^{\alpha k + k^2} (-tx)^k$$

and

$$\sum_{n=0}^{\infty} \frac{(\gamma;q)_n}{(q^{\alpha+1};q)_n} t^n \tilde{L}_n^{(\alpha)}(x;q) = \frac{(\gamma t;q)_\infty}{(t;q)_\infty} \sum_{m=0}^{\infty} \frac{q^{\alpha(m+1)-\binom{m+1}{2}}}{(q^{\alpha-m};q)_{m+1}}$$

$$\times \left(\sum_{k=m+1}^{\infty} \frac{(\gamma, q^{\alpha-m};q)_k}{(\gamma t, q^{\alpha+1}, q;q)_k} q^{\binom{k}{2}+(m+1)k} (-t)^k \right) x^m. \qquad \blacksquare$$

Remark 4.2. The inner sum $\sum_{k=n+1}^{\infty}(-1)^k[(q^{\alpha-n};q)_k/(q^{\alpha+2},q;q)_k]q^{\binom{k}{2}+(n+2)k}$ is the tail in the following special version of the q-Gauss sum

$$\sum_{n=0}^{\infty} (-1)^n \frac{(a;q)_n}{(c,q;q)_n} q^{\binom{n}{2}} \left(\frac{c}{a}\right)^n = \frac{(c/a;q)_\infty}{(c;q)_\infty}. \tag{4.2.1}$$

Thus we may write this inner sum as

$$\frac{(q^{n+2};q)_\infty}{(q^{\alpha+2};q)_\infty} - \sum_{k=0}^{n} (-1)^k \frac{(q^{\alpha-n};q)_k}{(q^{\alpha+2},q;q)_k} q^{\binom{k}{2}+(n+2)k}.$$

Almost the same can be said about the inner sum $\sum_{k=n+1}^{\infty}(-1)^k[(q^{\alpha-n};q)_k/(q^{\alpha+1};q)_k (q;q)_{k+1}]q^{\binom{k}{2}+(n+2)k}$.

Theorem 4.3. *The four entire functions from the Nevanlinna parametrization are given by*

$$A^{(\alpha)}(x) = -\sum_{n=0}^{\infty} \frac{q^{\alpha(n+1)-\binom{n+1}{2}}}{(q^{\alpha-n};q)_{n+1}} \left(\sum_{k=n+1}^{\infty} (-1)^k \frac{(q^{\alpha-n};q)_k}{(q^{\alpha},q;q)_k} q^{\binom{k}{2}+nk} \right) x^n$$

$$- \frac{q^\alpha}{1-q^\alpha} \frac{(q^{\alpha+2};q)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{\alpha n - \binom{n}{2}}}{(q^{\alpha-n+1};q)_n} \times \left(\sum_{k=n}^{\infty} (-1)^k \frac{(q^{\alpha-n+1};q)_k}{(q^{\alpha+2},q;q)_k} q^{\binom{k}{2}+(n+1)k} \right) x^n,$$

$$B^{(\alpha)}(x) = -\sum_{n=0}^{\infty} (-1)^n \frac{q^{\alpha n+n(n-1)}}{(q^\alpha,q;q)_n} x^n - \frac{q^\alpha}{1-q^\alpha} x \frac{(q^{\alpha+2};q)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} (-1)^n \frac{q^{\alpha n+n(n+1)}}{(q^{\alpha+2},q;q)_n} x^n,$$

$$C^{(\alpha)}(x) = \frac{(q^{\alpha+2};q)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{\alpha n - \binom{n}{2}}}{(q^{\alpha-n+1};q)_n} \left(\sum_{k=n}^{\infty} (-1)^k \frac{(q^{\alpha-n+1};q)_k}{(q^{\alpha+2},q;q)_k} q^{\binom{k}{2}+(n+1)k} \right) x^n,$$

$$D^{(\alpha)}(x) = x \frac{(q^{\alpha+2};q)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} (-1)^n \frac{q^{\alpha n+n(n+1)}}{(q^{\alpha+2},q;q)_n} x^n.$$

Proof. Start by recalling from (3.0.10) that

$$P_n^{(\alpha)}(x) = \sqrt{q^n \frac{(q;q)_n}{(q^{\alpha+1};q)_n}} L_n^{(\alpha)}(x;q) \quad \text{and} \quad Q_n^{(\alpha)}(x) = \sqrt{q^n \frac{(q;q)_n}{(q^{\alpha+1};q)_n}} \tilde{L}_n^{(\alpha)}(x;q),$$

and recall from (3.0.11) that

$$P_n^{(\alpha)}(0) = \sqrt{q^n \frac{(q^{\alpha+1};q)_n}{(q;q)_n}} \quad \text{and} \quad Q_n^{(\alpha)}(0) = \frac{q^\alpha}{1-q^\alpha} \left(\frac{(q;q)_n}{(q^{\alpha+1};q)_n} - 1 \right) P_n^{(\alpha)}(0).$$

According to Proposition 4.1 we thus have

$$\begin{aligned} D^{(\alpha)}(x) &= x \sum_{n=0}^{\infty} P_n^{(\alpha)}(0) P_n^{(\alpha)}(x) \\ &= x \sum_{n=0}^{\infty} q^n L_n^{(\alpha)}(x;q) \\ &= x \frac{(q^{\alpha+2};q)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} (-1)^n \frac{q^{\alpha n+n(n+1)}}{(q^{\alpha+2},q;q)_n} x^n, \end{aligned}$$

$$\begin{aligned} B^{(\alpha)}(x) &= -1 + x \sum_{n=0}^{\infty} Q_n^{(\alpha)}(0) P_n^{(\alpha)}(x) \\ &= -1 + x \frac{q^\alpha}{1-q^\alpha} \sum_{n=0}^{\infty} \left(\frac{(q;q)_n}{(q^{\alpha+1};q)_n} - 1 \right) q^n L_n^{(\alpha)}(x;q) \\ &= -1 + \sum_{n=0}^{\infty} (-1)^n \frac{q^{\alpha(n+1)+n(n+1)}}{(q^\alpha,q;q)_{n+1}} x^{n+1} - \frac{q^\alpha}{1-q^\alpha} D^{(\alpha)}(x) \\ &= -\sum_{n=0}^{\infty} (-1)^n \frac{q^{\alpha n+n(n-1)}}{(q^\alpha,q;q)_n} x^n - \frac{q^\alpha}{1-q^\alpha} D^{(\alpha)}(x), \end{aligned}$$

$$
\begin{aligned}
C^{(\alpha)}(x) &= 1 + x\sum_{n=0}^{\infty} P_n^{(\alpha)}(0)Q_n^{(\alpha)}(x) \\
&= 1 + x\sum_{n=0}^{\infty} q^n \tilde{L}_n^{(\alpha)}(x;q) \\
&= 1 + \frac{(q^{\alpha+2};q)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\frac{q^{\alpha(n+1)-\binom{n+1}{2}}}{(q^{\alpha-n};q)_{n+1}} \\
&\quad \times \left(\sum_{k=n+1}^{\infty}(-1)^k\frac{(q^{\alpha-n};q)_k}{(q^{\alpha+2},q;q)_k}q^{\binom{k}{2}+(n+2)k}\right)x^{n+1} \\
&= \frac{(q^{\alpha+2};q)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\frac{q^{\alpha n-\binom{n}{2}}}{(q^{\alpha-n+1};q)_n}\left(\sum_{k=n}^{\infty}(-1)^k\frac{(q^{\alpha-n+1};q)_k}{(q^{\alpha+2},q;q)_k}q^{\binom{k}{2}+(n+1)k}\right)x^n, \\
A^{(\alpha)}(x) &= x\sum_{n=0}^{\infty} Q_n^{(\alpha)}(0)Q_n^{(\alpha)}(x) \\
&= x\frac{q^\alpha}{1-q^\alpha}\sum_{n=0}^{\infty}\left(\frac{(q;q)_n}{(q^{\alpha+1};q)_n}-1\right)q^n\tilde{L}_n^{(\alpha)}(x;q) \\
&= \frac{q^\alpha}{1-q^\alpha} + \sum_{n=0}^{\infty}\frac{q^{\alpha(n+2)-\binom{n+1}{2}}}{(q^{\alpha-n};q)_{n+1}}\left(\sum_{k=n+1}^{\infty}(-1)^k\frac{(q^{\alpha-n};q)_k}{(q^\alpha,q;q)_{k+1}}q^{\binom{k}{2}+(n+2)k}\right)x^{n+1} \\
&\quad - \frac{q^\alpha}{1-q^\alpha}C^{(\alpha)}(x) \\
&= -\sum_{n=0}^{\infty}\frac{q^{\alpha(n+1)-\binom{n+1}{2}}}{(q^{\alpha-n};q)_{n+1}}\left(\sum_{k=n+1}^{\infty}(-1)^k\frac{(q^{\alpha-n};q)_k}{(q^\alpha,q;q)_k}q^{\binom{k}{2}+nk}\right)x^n \\
&\quad - \frac{q^\alpha}{1-q^\alpha}C^{(\alpha)}(x).
\end{aligned}
$$

In the computations of $C^{(\alpha)}$ and $A^{(\alpha)}$ we have used (4.2.1) in the last steps. ■

Since the expressions for $A^{(\alpha)}$ and $B^{(\alpha)}$ look complicated, we point out that

$$
\begin{aligned}
B^{(\alpha)}(x) + \frac{q^\alpha}{1-q^\alpha}D^{(\alpha)}(x) &= -\sum_{n=0}^{\infty}(-1)^n\frac{q^{\alpha n+n(n-1)}}{(q^\alpha,q;q)_n}x^n, \\
A^{(\alpha)}(x) + \frac{q^\alpha}{1-q^\alpha}C^{(\alpha)}(x) &= -\sum_{n=0}^{\infty}\frac{q^{\alpha(n+1)-\binom{n+1}{2}}}{(q^{\alpha-n};q)_{n+1}} \\
&\quad \times\left(\sum_{k=n+1}^{\infty}(-1)^k\frac{(q^{\alpha-n};q)_k}{(q^\alpha,q;q)_k}q^{\binom{k}{2}+nk}\right)x^n.
\end{aligned}
$$

Theorem 4.3 gives the power series expansion of the four entire functions $A^{(\alpha)} + [q^\alpha/(1-q^\alpha)]C^{(\alpha)}$, $B^{(\alpha)} + [q^\alpha/(1-q^\alpha)]D^{(\alpha)}$, $C^{(\alpha)}$, and $D^{(\alpha)}$.

It is easy to see that the expressions for $D^{(\alpha)}$ and $B^{(\alpha)}$ from Theorem 4.3 agree with the expressions found by Moak in [17] and by Ismail and Rahman in [15]. But it is not obvious that the present expressions for $C^{(\alpha)}$ and $A^{(\alpha)}$ coincide with the expressions from [15]. Here Ismail and Rahman also find a generating function for the q-Laguerre polynomials of the second kind. With $c = -q^{\alpha+1}$ in [15, (2.14)] we have $p_n(x) = \tilde{L}_n^{(\alpha)}(x; q)$ and [15, (2.15)] becomes

$$P(x,t) = \sum_{n=0}^{\infty} t^n \tilde{L}_n^{(\alpha)}(x; q).$$

In the Appendix in [15, (2.24)] it is proved that $(t; q)_\infty P(x,t)$ can be written as

$$\tag{4.3.1} -\sum_{n=0}^{\infty} \frac{(tq^{n+1}; q)_\infty (-x)^n}{(q^{-\alpha}; q)_{n+1}} + (tq^{\alpha+1}, -x; q)_\infty \times \sum_{n=0}^{\infty} \frac{(-x)^n}{(q^{-\alpha}; q)_{n+1}} \sum_{m=0}^{\infty} \frac{(-x)^m}{(tq^{\alpha+1}, q; q)_m}$$

and with this result available, Ismail and Rahman establish an asymptotic relation for $\tilde{L}_n^{(\alpha)}(x; q)$ by using Darboux's method. Subsequently, it is possible to give expressions for $C^{(\alpha)}$ and $A^{(\alpha)}$.

We will end this paper by showing that (4.3.1) can be obtained from Proposition 4.1. The first step is to rewrite the inner sum using the q-Gauss sum. Next we apply Cauchy multiplication and compare the coefficients. It turns out that the situation reduces to a certain version of the q-Chu–Vandermonde formula.

With $\gamma = q^{\alpha+1}$ in Proposition 4.1 we find that $(t; q)_\infty \sum_{n=0}^{\infty} t^n \tilde{L}_n^{(\alpha)}(x; q)$ has the form

$$\tag{4.3.2} (tq^{\alpha+1}; q)_\infty \sum_{n=0}^{\infty} \frac{q^{\alpha(n+1)-\binom{n+1}{2}}}{(q^{\alpha-n}; q)_{n+1}} \left(\sum_{k=n+1}^{\infty} \frac{(q^{\alpha-n}; q)_k}{(tq^{\alpha+1}, q; q)_k} q^{\binom{k}{2}+(n+1)k} (-t)^k \right) x^n.$$

According to (4.2.1) the inner sum can be written as

$$\frac{(tq^{n+1}; q)_\infty}{(tq^{\alpha+1}; q)_\infty} - \sum_{k=0}^{n} \frac{(q^{\alpha-n}; q)_k}{(tq^{\alpha+1}, q; q)_k} q^{\binom{k}{2}+(n+1)k} (-t)^k$$

and since

$$\frac{q^{\alpha(n+1)-\binom{n+1}{2}}}{(q^{\alpha-n}; q)_{n+1}} = -\frac{(-1)^n}{(q^{-\alpha}; q)_{n+1}},$$

the expression in (4.3.2) is equal to

$$-\sum_{n=0}^{\infty} \frac{(tq^{n+1}; q)_\infty (-x)^n}{(q^{-\alpha}; q)_{n+1}} + (tq^{\alpha+1}; q)_\infty \times \sum_{n=0}^{\infty} \frac{(-x)^n}{(q^{-\alpha}; q)_{n+1}} \sum_{k=0}^{n} \frac{(q^{\alpha-n}; q)_k}{(tq^{\alpha+1}, q; q)_k} q^{\binom{k}{2}+(n+1)k} (-t)^k.$$

So we have to realize that

$$(-x;q)_\infty \sum_{n=0}^{\infty} \frac{(-x)^n}{(q^{-\alpha};q)_{n+1}} \sum_{m=0}^{\infty} \frac{(-x)^m}{(tq^{\alpha+1},q;q)_m} = \sum_{n=0}^{\infty} \frac{(-x)^n}{(q^{-\alpha};q)_{n+1}} \sum_{k=0}^{n} \frac{(q^{\alpha-n};q)_k}{(tq^{\alpha+1},q;q)_k} q^{\binom{k}{2}+(n+1)k}(-t)^k$$

or, equivalently,

$$\sum_{n=0}^{\infty} \frac{(-x)^n}{(q^{1-\alpha};q)_n} \sum_{m=0}^{\infty} \frac{(-x)^m}{(tq^{\alpha+1},q;q)_m} = \frac{1}{(-x;q)_\infty} \sum_{n=0}^{\infty} \frac{(-x)^n}{(q^{1-\alpha};q)_n} \times \sum_{k=0}^{n} \frac{(q^{\alpha-n};q)_k}{(tq^{\alpha+1},q;q)_k} q^{\binom{k}{2}+(n+1)k}(-t)^k. \tag{4.3.3}$$

Using Euler's power series expansion of $1/(x;q)_\infty$ we get, by Cauchy multiplication, that (4.3.3) has the form

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{(-1)^k}{(q^{1-\alpha};q)_k} \frac{(-1)^{n-k}}{(tq^{\alpha+1},q;q)_{n-k}} \right) x^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{(-1)^k}{(q^{1-\alpha};q)_k} \left(\sum_{m=0}^{k} \frac{(q^{\alpha-k};q)_m}{(tq^{\alpha+1},q;q)_m} q^{\binom{m}{2}+(k+1)m}(-t)^m \right) \frac{(-1)^{n-k}}{(q;q)_{n-k}} \right) x^n$$

and therefore it suffices to prove that

$$\sum_{k=0}^{n} \frac{1}{(q^{1-\alpha};q)_k (tq^{\alpha+1},q;q)_{n-k}} = \sum_{k=0}^{n} \frac{1}{(q^{1-\alpha};q)_k (q;q)_{n-k}} \times \sum_{m=0}^{k} \frac{(q^{\alpha-k};q)_m}{(tq^{\alpha+1},q;q)_m} q^{\binom{m}{2}+(k+1)m}(-t)^m. \tag{4.3.4}$$

Unfortunately we do not have

$$\frac{1}{(tq^{\alpha+1};q)_{n-k}} = \sum_{m=0}^{k} \frac{(q^{\alpha-k};q)_m}{(tq^{\alpha+1},q;q)_m} q^{\binom{m}{2}+(k+1)m}(-t)^m.$$

Since (4.3.4) is an identity between rational functions in t, we multiply with $(tq^{\alpha+1};q)_n$ in order to get an identity between polynomials in t. Using that

$$q^{m(k-\alpha)} \frac{(q^{\alpha-k};q)_m}{(q^{1-\alpha};q)_k} = (-1)^m \frac{q^{\binom{m}{2}}}{(q^{1-\alpha};q)_{k-m}}$$

we obtain

$$\sum_{k=0}^{n} \frac{(tq^{\alpha+1+n-k};q)_k}{(q^{1-\alpha};q)_k (q;q)_{n-k}} = \sum_{k=0}^{n} \frac{1}{(q;q)_{n-k}} \times \sum_{m=0}^{k} \frac{(tq^{\alpha+1+m};q)_{n-m}}{(q^{1-\alpha};q)_{k-m}(q;q)_m} q^{m^2+m\alpha} t^m. \tag{4.3.5}$$

By the finite version of the q-binomial theorem, the left-hand side in (4.3.5) is

$$\sum_{k=0}^{n}\frac{1}{(q^{1-\alpha};q)_k(q;q)_{n-k}}\sum_{l=0}^{k}\begin{bmatrix}k\\l\end{bmatrix}_q(-1)^l q^{\binom{l}{2}}(tq^{\alpha+1+n-k})^l$$
$$=\sum_{l=0}^{n}(-1)^l q^{\binom{l}{2}+l(\alpha+1)}\left(\sum_{k=l}^{n}\begin{bmatrix}k\\l\end{bmatrix}_q\frac{q^{l(n-k)}}{(q^{1-\alpha};q)_k(q;q)_{n-k}}\right)t^l$$

and the right-hand side in (4.3.5) is

$$\sum_{k=0}^{n}\frac{1}{(q;q)_{n-k}}\sum_{m=0}^{k}\frac{q^{m^2+m\alpha}}{(q^{1-\alpha};q)_{k-m}(q;q)_m}t^m\sum_{l=0}^{n-m}\begin{bmatrix}n-m\\l\end{bmatrix}_q(-1)^l q^{\binom{l}{2}}(tq^{\alpha+1+m})^l$$
$$=\sum_{m=0}^{n}\frac{q^{m^2+m\alpha}}{(q;q)_m}\sum_{k=m}^{n}\frac{1}{(q^{1-\alpha};q)_{k-m}(q;q)_{n-k}}\sum_{l=m}^{n}\begin{bmatrix}n-m\\l-m\end{bmatrix}_q(-1)^{l-m}q^{\binom{l-m}{2}+(l-m)(\alpha+1+m)}t^l$$
$$=\sum_{l=0}^{n}(-1)^l q^{\binom{l}{2}+l(\alpha+1)}\left(\sum_{m=0}^{l}\begin{bmatrix}n-m\\l-m\end{bmatrix}_q(-1)^m\frac{q^{\binom{m}{2}}}{(q;q)_m}\sum_{k=m}^{n}\frac{1}{(q^{1-\alpha};q)_{k-m}(q;q)_{n-k}}\right)t^l.$$

Thus we have to prove that

$$\sum_{k=l}^{n}\begin{bmatrix}k\\l\end{bmatrix}_q\frac{q^{l(n-k)}}{(q^{1-\alpha};q)_k(q;q)_{n-k}}=\sum_{m=0}^{l}\begin{bmatrix}n-m\\l-m\end{bmatrix}_q(-1)^m\frac{q^{\binom{m}{2}}}{(q;q)_m}\sum_{k=m}^{n}\frac{1}{(q^{1-\alpha};q)_{k-m}(q;q)_{n-k}}$$

or

$$\sum_{k=0}^{n}\begin{bmatrix}k\\l\end{bmatrix}_q\frac{q^{l(n-k)}}{(q^{1-\alpha};q)_k(q;q)_{n-k}}=\sum_{m=0}^{n}\begin{bmatrix}n-m\\l-m\end{bmatrix}_q(-1)^m\frac{q^{\binom{m}{2}}}{(q;q)_m}\sum_{k=m}^{n}\frac{1}{(q^{1-\alpha};q)_{k-m}(q;q)_{n-k}}$$

since $\begin{bmatrix}k\\l\end{bmatrix}_q=0$ for $k<l$ and $\begin{bmatrix}n-m\\l-m\end{bmatrix}_q=0$ for $m>l$. Here the right-hand side is

$$\sum_{m=0}^{n}\begin{bmatrix}n-m\\l-m\end{bmatrix}_q(-1)^m\frac{q^{\binom{m}{2}}}{(q;q)_m}\sum_{k=0}^{n-m}\frac{1}{(q^{1-\alpha};q)_k(q;q)_{n-k-m}}$$
$$=\sum_{k=0}^{n}\frac{1}{(q^{1-\alpha};q)_k(q;q)_{n-k}}\sum_{m=0}^{n-k}\begin{bmatrix}n-m\\l-m\end{bmatrix}_q\begin{bmatrix}n-k\\m\end{bmatrix}_q(-1)^m q^{\binom{m}{2}}$$

and since

$$\begin{bmatrix}n-m\\l-m\end{bmatrix}_q=\begin{bmatrix}n\\l\end{bmatrix}_q\frac{(q^{-l};q)_m}{(q^{-n};q)_m}q^{m(l-n)},$$

it follows from (1.0.3) that

$$\sum_{m=0}^{n-k}\begin{bmatrix}n-m\\l-m\end{bmatrix}_q\begin{bmatrix}n-k\\m\end{bmatrix}_q(-1)^m q^{\binom{m}{2}}=\begin{bmatrix}n\\l\end{bmatrix}_q\frac{(q^{l-n};q)_{n-k}}{(q^{-n};q)_{n-k}}=\begin{bmatrix}k\\l\end{bmatrix}_q q^{l(n-k)}.$$

This establishes the desired identity.

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The moment problem associated with the Stieltjes–Wigert polynomials

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Abstract

We consider the indeterminate Stieltjes moment problem associated with the Stieltjes–Wigert polynomials. After a presentation of the well-known solutions, we study a transformation T of the set of solutions. All the classical solutions turn out to be fixed under this transformation but this is not the case for the so-called canonical solutions. Based on generating functions for the Stieltjes–Wigert polynomials, expressions for the entire functions A, B, C, and D from the Nevanlinna parametrization are obtained. We describe $T^{(n)}(\mu)$ for $n \in \mathbb{N}$ when $\mu = \mu_0$ is a particular N-extremal solution and explain in detail what happens when $n \to \infty$.

Keywords: Indeterminate moment problems; Stieltjes–Wigert polynomials; Nevanlinna parametrization

1. Introduction

T.J. Stieltjes was the first to give examples of indeterminate moment problems. In [18] he pointed out that if f is an odd function satisfying $f(u+1/2) = \pm f(u)$, then

$$\int_0^\infty u^n u^{-\log u} f(\log u)\, du = 0$$

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for all $n \in \mathbb{Z}$. In particular,

$$\int_0^\infty u^n u^{-\log u} \sin(2\pi \log u)\, du = 0, \quad n \in \mathbb{Z},$$

so independent of λ we have

$$\int_0^\infty \frac{1}{\sqrt{\pi}} u^n u^{-\log u}\bigl(1 + \lambda \sin(2\pi \log u)\bigr)\, du = \frac{1}{\sqrt{\pi}} \int_0^\infty u^n u^{-\log u}\, du = e^{(n+1)^2/4}.$$

In other words, for $\lambda \in [-1, 1]$ the densities

$$w_\lambda(u) = \frac{1}{\sqrt{\pi}} u^{-\log u}\bigl(1 + \lambda \sin(2\pi \log u)\bigr), \quad u > 0,$$

have the same moments.

More generally, one could consider the weight function[1]

$$w(x) = \frac{1}{\sqrt{\pi}} k x^{-k^2 \log x}, \quad x > 0, \tag{1.1}$$

which has the moments

$$s_n = \int_0^\infty x^n w(x)\, dx = e^{(n+1)^2/4k^2}. \tag{1.2}$$

Here $k > 0$ is a constant (and $k = 1$ corresponds to Stieltjes' example). This was done by Wigert in [20]. He succeeded in finding the orthonormal polynomials (P_n) corresponding to $w(x)$ using the general formula

$$P_n(x) = \frac{1}{\sqrt{D_{n-1} D_n}} \begin{vmatrix} s_0 & s_1 & \dots & s_n \\ s_1 & s_2 & \dots & s_{n+1} \\ \vdots & \vdots & & \vdots \\ s_{n-1} & s_n & \dots & s_{2n-1} \\ 1 & x & \dots & x^n \end{vmatrix}, \quad n \geqslant 1, \tag{1.3}$$

where (s_n) denotes the moment sequence and $D_n = \det((s_{i+j})_{0 \leqslant i,j \leqslant n})$ denotes the Hankel determinant. If we set $q = e^{-1/2k^2}$, the moment sequence (1.2) has the form $s_n = q^{-(n+1)^2/2}$ and it is readily seen that all the determinants in (1.3) are of the Vandermonde type. Following the notation of Gasper and Rahman [13] for basic hypergeometric series, Wigert's expressions are

$$P_n(x) = (-1)^n \frac{q^{n/2+1/4}}{\sqrt{(q;q)_n}} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{k^2+k/2} x^k, \tag{1.4}$$

[1] Note that $w(x)/x$ is the density of the log-normal distribution with parameter $\sigma^2 = 1/2k^2 > 0$.

cf. Szegö [19] and Chihara [9], where these polynomials are called the Stieltjes–Wigert polynomials. Wigert also considered the behaviour of $P_n(x)$ when $n \to \infty$ and proved that

$$(-1)^n q^{-n/2} P_n(x) \to \frac{q^{1/4}}{\sqrt{(q;q)_\infty}} \sum_{k=0}^{\infty} (-1)^k \frac{q^{k^2+k/2}}{(q;q)_k} x^k \quad \text{for } n \to \infty. \tag{1.5}$$

The convergence is uniform on compact subsets of $\mathbb{C}$.

Later, Chihara [10] pointed out that the weight function $w(x)$ satisfies the functional equation

$$w(xq) = \sqrt{q}\, x\, w(x), \quad x > 0, \tag{1.6}$$

and this observation led to the discovery of a family of discrete measures with the same moments as $w(x)$. The discrete version of the functional equation (1.6) is the following. Suppose that μ is a discrete measure. Then $c > 0$ is a mass point of μ exactly if cq likewise is a mass point of μ and $\mu(\{cq\}) = cq\sqrt{q}\,\mu(\{c\})$. This property is certainly satisfied by the measures

$$\mu_c = \frac{1}{\sqrt{q}M(c)} \sum_{n=-\infty}^{\infty} c^n q^{n+n^2/2} \varepsilon_{cq^n}, \quad c > 0, \tag{1.7}$$

where $M(c)$ is some constant depending on c and ε_x denotes the Dirac measure at the point x. Setting $M(c) = (-cq\sqrt{q}, -1/c\sqrt{q}, q; q)_\infty$, it follows by the Jacobi triple product identity [2, p. 497]

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{\binom{n}{2}} x^n = (x, q/x, q; q)_\infty, \quad x \neq 0, \tag{1.8}$$

and the translation invariance of $\sum_{-\infty}^{\infty}$ that each μ_c has the moments $q^{-(n+1)^2/2}$.

In [5] Askey and Roy presented a symmetric q-analogue of the usual beta integral. With a and b instead of q^{a+c} and q^{b-c}, their formula reads

$$\int_0^\infty t^{c-1} \frac{(-at, -bq/t; q)_\infty}{(-t, -q/t; q)_\infty}\, dt = \frac{(ab, q^c, q^{1-c}; q)_\infty}{(q, aq^{-c}, bq^c; q)_\infty} \frac{\pi}{\sin \pi c},$$
$$c > 0,\ |a| < q^c,\ |b| < q^{-c}. \tag{1.9}$$

When $a = b = 0$, (1.9) simplifies to

$$\int_0^\infty \frac{t^{c-1}}{(-t, -q/t; q)_\infty}\, dt = \frac{(q^c, q^{1-c}; q)_\infty}{(q;q)_\infty} \frac{\pi}{\sin \pi c}, \quad c > 0,$$

and we have

$$\int_0^\infty t^n \frac{t^{c-1}}{(-t, -q/t; q)_\infty}\, dt = q^{-cn-\binom{n}{2}} \frac{(q^c, q^{1-c}; q)_\infty}{(q;q)_\infty} \frac{\pi}{\sin \pi c}, \quad c > 0. \tag{1.10}$$

Setting $c = 3/2$, the right-hand side in (1.10) becomes

$$q^{-n-n^2/2}\frac{(q^{3/2}, q^{-1/2}; q)_\infty}{(q; q)_\infty}(-\pi) = q^{-(n+1)^2/2}\frac{\pi(\sqrt{q}; q)_\infty^2}{(q; q)_\infty}$$

so the weight function

$$\widetilde{w}(x) = \frac{(q; q)_\infty}{\pi(\sqrt{q}; q)_\infty^2}\frac{\sqrt{x}}{(-x, -q/x; q)_\infty}, \quad x > 0, \tag{1.11}$$

has the moments $q^{-(n+1)^2/2}$. This observation was made by Askey in [4] and introduces a new weight function for the polynomials (1.4).

As a basic knowledge of the theory of the moment problem we shall refer to Akhiezer [1]. Recall that the Nevanlinna parametrization gives a one-to-one correspondence between the set of Pick functions (including ∞) and the set of solutions to an indeterminate Hamburger moment problem. If μ_φ is the solution corresponding to the Pick function φ, then the Stieltjes transform of μ_φ is given by

$$\int_{\mathbb{R}} \frac{1}{t - x}\, d\mu_\varphi(t) = -\frac{A(x)\varphi(x) - C(x)}{B(x)\varphi(x) - D(x)}, \quad x \in \mathbb{C} \setminus \mathbb{R}, \tag{1.12}$$

where A, B, C, and D are certain entire functions defined in terms of the orthonormal polynomials (P_n) and (Q_n) by

$$A(x) = x\sum_{n=0}^{\infty} Q_n(0)Q_n(x), \qquad C(x) = 1 + x\sum_{n=0}^{\infty} P_n(0)Q_n(x),$$
$$B(x) = -1 + x\sum_{n=0}^{\infty} Q_n(0)P_n(x), \qquad D(x) = x\sum_{n=0}^{\infty} P_n(0)P_n(x).$$

According to the Stieltjes–Perron inversion formula, the measure μ_φ is uniquely determined by its Stieltjes transform.

The solutions corresponding to the Pick function being a real constant (or ∞) are called N-extremal and the solutions corresponding to the Pick function being a real rational function are called canonical. To be precise, the solutions are called n-canonical or canonical of order n if the Pick function is a real rational function of degree n. Thus, canonical of order 0 is the same as N-extremal. It is well-known that canonical solutions are discrete. If $\varphi = P/Q$ (assuming that P and Q are polynomials with real coefficients and no common zeros), then μ_φ is supported on the zeros of the entire function $B(x)P(x) - D(x)Q(x)$. In particular, the N-extremal solution μ_t is supported on the zeros of $B(x)t - D(x)$ (or $B(x)$ when $t = \infty$).

Considering a Stieltjes moment problem, of course not every Pick function gives rise to a Stieltjes solution. In this connection the quantity $\alpha \leqslant 0$ defined by

$$\alpha = \lim_{n\to\infty}\frac{P_n(0)}{Q_n(0)} \tag{1.13}$$

plays an important part. As Pedersen proved in [17], the measure μ_φ corresponding to the Pick function φ is supported within $[0, \infty)$ precisely if φ has an analytic continuation to

$\mathbb{C} \setminus [0, \infty)$ such that $\alpha \leqslant \varphi(x) \leqslant 0$ for $x < 0$. In particular, the only N-extremal Stieltjes solutions are μ_t with $\alpha \leqslant t \leqslant 0$. Furthermore, it is well-known that the moment problem is determinate in the sense of Stieltjes exactly if $\alpha = 0$.

This paper is organized as follows. In Section 2 we start by adjusting the normalization in order to follow the normalization in Koekoek and Swarttouw [14]. Then we present the well-known solutions to the moment problem and explain how to obtain them. These solutions can also be found in Berg [6,7]. The functional equation $f(xq) = xf(x)$ is of great importance both in connection with absolutely continuous and discrete solutions. A transformation T of the set of solutions is established and we classify the absolutely continuous and discrete fixed points. These include all the well-known absolutely continuous solutions and a wide class of the well-known discrete solutions. However, some of the well-known discrete solutions are only fixed under $T^{(2)}$. A method to construct continuous singular solutions to the moment problem concludes the section. In Section 3 we introduce the Stieltjes–Wigert polynomials. These polynomials are proportional to the orthonormal polynomials and converge uniformly on compact subsets of $\mathbb{C}$ when $n \to \infty$. We show that the zeros of the Stieltjes–Wigert polynomials are very well separated, that is, the ratio between two consecutive zeros is strictly greater than q^{-2}. Based on generating functions for the Stieltjes–Wigert polynomials, expressions for the four entire functions from the Nevanlinna parametrization are obtained in terms of their power series expansions. Concerning the canonical solutions to the moment problem an entire function Φ becomes important. The zeros of Φ turn out to be closely related to the supports of certain N-extremal and canonical solutions. However, the zeros of Φ cannot be found explicitly but since Φ is proportional to the limit of the Stieltjes–Wigert polynomials when $n \to \infty$, these zeros are very well separated. Moreover, in the end of the section we get as a corollary that the ratio between two consecutive zeros of Φ actually converges to q^{-2}. The canonical solutions are not fixed points of the transformation T defined in Section 2. We describe T at the level of Pick functions and show that T maps a canonical solution into another canonical solution. For the particular N-extremal solution μ_0 we are able to describe $T^{(n)}(\mu_0)$ for each $n \in \mathbb{N}$. There is a difference between n odd and n even. We show that the limits of $T^{(2n+1)}(\mu_0)$ and $T^{(2n+2)}(\mu_0)$ exist when $n \to \infty$ and coincide with already known solutions to the moment problem.

2. The classical solutions

Let us start by adjusting the normalization in order to follow the standard reference, Koekoek and Swarttouw [14]. So instead of $w(x)$ we consider the weight function

$$v(x) = \frac{w(x\sqrt{q}\,)}{x}, \quad x > 0,$$

that is, explicitly we have

$$v(x) = \frac{q^{1/8}}{\sqrt{2\pi \log q^{-1}}} \frac{1}{\sqrt{x}} e^{\frac{1}{2}\frac{(\log x)^2}{\log q}}, \quad x > 0. \tag{2.1}$$

Note that v satisfies the functional equation

$$v(xq) = xv(x), \quad x > 0 \tag{2.2}$$

and is the density of a probability measure v on $(0, \infty)$ with the moments

$$\int_0^\infty x^n v(x)\,dx = q^{-\binom{n+1}{2}}. \tag{2.3}$$

Using the same procedure as Wigert in [20], we find that the orthonormal polynomials (P_n) associated with the moment sequence (2.3) are given by

$$P_n(x) = (-1)^n \sqrt{\frac{q^n}{(q;q)_n}} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{k^2} x^k, \quad n \geqslant 0. \tag{2.4}$$

We stress that

$$P_n(x) = (-1)^n \sqrt{q^n (q;q)_n}\, S_n(x;q),$$

where $S_n(x;q)$ denotes the Stieltjes–Wigert polynomials given by

$$S_n(x;q) = \frac{1}{(q;q)_n} {}_1\varphi_1\left(\begin{matrix} q^{-n} \\ 0 \end{matrix}; q, -q^{n+1}x \right), \quad n \geqslant 0,$$

see Koekoek and Swarttouw [14].

The functional equation (2.2) is important due to the following observation which is also contained in Chihara's paper [11].

Proposition 2.1. *Let f be a positive measurable function defined on the interval $(0, \infty)$. If f satisfies the functional equation $f(xq) = xf(x)$ and*

$$\int_0^\infty f(x)\,dx = c \in (0, \infty),$$

then the absolutely continuous measure with density $\frac{1}{c} f$ has the moments $q^{-\binom{n+1}{2}}$.

Remark 2.2. The conditions in Proposition 2.1 are sufficient but not necessary.

Proof. Without loss of generality we can assume that $\int_0^\infty f(x)\,dx = 1$. For if this is not the case, one can simply replace f by $\frac{1}{c} f$. If f satisfies the functional equation $xf(x) = f(xq)$, it is seen by induction that f satisfies the functional equation

$$q^{\binom{n}{2}} x^n f(x) = f\left(xq^n\right) \tag{2.5}$$

for each $n \in \mathbb{Z}$ and, consequently,

$$\int_0^\infty x^n f(x)\,dx = q^{-\binom{n}{2}} \int_0^\infty f\left(xq^n\right)dx = q^{-\binom{n}{2}} q^{-n} \int_0^\infty f(x)\,dx = q^{-\binom{n+1}{2}}.$$

□

So the question is whether we know of any positive and integrable functions on $(0,\infty)$, which satisfy the functional equation (2.2)—besides v of course. At this point the functions f_c given by

$$f_c(x)=\frac{x^{c-1}}{(-q^{1-c}x,-q^c/x;q)_\infty},\quad x>0,$$

become relevant. They certainly satisfy the functional equation (2.2) and by the Askey–Roy q-beta integral (1.9), we have

$$\int_0^\infty f_c(x)\,dx=q^{c(c-1)}\frac{(q^c,q^{1-c};q)_\infty}{(q;q)_\infty}\frac{\pi}{\sin\pi c}.$$

Therefore, by Proposition 2.1 the absolutely continuous measures v_c with densities

$$v_c(x)=q^{c(1-c)}\frac{\sin\pi c}{\pi}\frac{(q;q)_\infty}{(q^c,q^{1-c};q)_\infty}\frac{x^{c-1}}{(-q^{1-c}x,-q^c/x;q)_\infty},\quad x>0, \tag{2.6}$$

have the moments (2.3). Since $v_{c+1}=v_c$, it suffices to consider v_c for $c\in(0,1]$.

As Askey stated in [3] (but only for $c=1$), the densities $v_c(x)$ appear to be certain (normalized) accumulation points of the weight function

$$v^{(\alpha)}(x)=\frac{x^\alpha}{(-x;q)_\infty},\quad x>0,$$

for the q-Laguerre polynomials when $\alpha\to\infty$. It is well known, see [14], that the q-Laguerre polynomials given by

$$L_n^{(\alpha)}(x;q)=\frac{(q^{\alpha+1};q)_n}{(q;q)_n}\,{}_1\varphi_1\left(\begin{matrix}q^{-n}\\ q^{\alpha+1}\end{matrix};q,-q^{n+\alpha+1}x\right),\quad n\geqslant 0,$$

in a suitable way converge to the Stieltjes–Wigert polynomials when $\alpha\to\infty$ and results on convergence at the level of orthogonality measures can be worked out as well. For the precise statements and computations, the reader is referred to [12].

If one should be tempted to look at the graphs of the densities v and v_c for some fixed value of q, say $q=1/2$, the variation turns out to be surprisingly small. For a minute one might be afraid that the measures are not different at all. However, the measures cannot coincide because v_c can be considered as a meromorphic function in $\mathbb{C}\setminus\{i\beta\mid\beta\geqslant 0\}$ with simple poles at $-q^{c+n}$ for $n\in\mathbb{Z}$, whereas v can be considered as a holomorphic function in $\mathbb{C}\setminus\{i\beta\mid\beta\geqslant 0\}$.

Let us now return to the functional equation (2.2) and suppose that f_1 and f_2 are two functions satisfying this equation. If f_2 is strictly positive, then the quotient $g=f_1/f_2$ is well defined and it satisfies the simple functional equation

$$g(x)=g(xq),\quad x>0.$$

So the two functions differ at the most by a factor which in a certain sense is periodic—what we shall call q-periodic. In other words, if we know one strictly positive solution to the functional equation (2.2), we can get all the others by multiplying with q-periodic functions. Therefore, whenever g is a positive, measurable and q-periodic function such that

$$\int_0^\infty v(x)g(x)\,dx = c \in (0,\infty),$$

the absolutely continuous measure with density $\frac{1}{c}v(x)g(x)$, $x > 0$, has the moments (2.3). This is exactly Stieltjes' observation in full generality—he only considered the case $q = 1/2$. Since the sine function is periodic with period 2π, it can be made q-periodic by changing the argument to $2\pi \log x/\log q$. In order to get a positive function, just add the constant 1 and obviously the function remains positive and q-periodic if the sine term is multiplied by any constant λ between -1 and 1. It is easily verified that

$$\int_0^\infty v(x)\sin\left(2\pi\frac{\log x}{\log q}\right)dx = 0$$

so for $\lambda \in [-1,1]$, the densities

$$\tilde{v}_\lambda(x) = v(x)\left(1 + \lambda\sin\left(2\pi\frac{\log x}{\log q}\right)\right), \quad x > 0, \tag{2.7}$$

have the same moments. Note that each $\tilde{v}_\lambda(x)$ is a convex combination of the end points $\tilde{v}_{-1}(x)$ and $\tilde{v}_1(x)$, to be precise

$$\tilde{v}_\lambda(x) = \frac{1-\lambda}{2}\tilde{v}_{-1}(x) + \frac{1+\lambda}{2}\tilde{v}_1(x).$$

After this, let us turn the attention to discrete solutions to the moment problem. Suppose that f is a strictly positive function satisfying the functional equation (2.2) and consider for $c > 0$ the discrete measure λ_c supported on $\{cq^n \mid n \in \mathbb{Z}\}$ and given by

$$\lambda_c\big(\{cq^n\}\big) = \frac{1}{f(c)L(c)}q^n f\big(cq^n\big), \quad n \in \mathbb{Z}.$$

Here $L(c)$ is a constant which ensures that λ_c is a probability measure. Recall from (2.5) that

$$f\big(cq^n\big) = q^{\binom{n}{2}}c^n f(c), \quad n \in \mathbb{Z},$$

so independent of f, the measure λ_c is given by

$$\lambda_c = \frac{1}{L(c)}\sum_{n=-\infty}^{\infty}(cq)^n q^{\binom{n}{2}}\varepsilon_{cq^n}. \tag{2.8}$$

According to the Jacobi triple product identity (1.8), we have $L(c) = (-cq, -1/c, q; q)_\infty$ and using the translation invariance of $\sum_{-\infty}^{\infty}$, we see that

$$\int_0^\infty x^n\,d\lambda_c(x) = q^{-\binom{n+1}{2}}.$$

Since $\lambda_{c/q} = \lambda_c$, it suffices to consider λ_c for $c \in (q, 1]$ and this perfectly agrees with the fact that a function satisfying the functional equation (2.2) is uniquely determined by its values on the interval $(q, 1]$.

The particular solution λ_1 is supported on the geometric progression $\{q^n \mid n \in \mathbb{Z}\}$ and one could ask if this is the only solution supported within this special set. The answer is in the negative, see [6], where Berg pointed out that for $s \in [-1, 1]$, the measures

$$\kappa_s = \frac{1}{L(1)} \sum_{n=-\infty}^{\infty} q^{\binom{n+1}{2}} \bigl(1 + s(-1)^n\bigr) \varepsilon_{q^n} \tag{2.9}$$

have the same moments. To justify this, one has to realize that

$$\sum_{n=-\infty}^{\infty} \bigl(q^n\bigr)^k q^{\binom{n+1}{2}} (-1)^n = 0$$

which is a consequence of the Jacobi triple product identity (1.8). The end points κ_{-1} and κ_1 are supported on $\{q^{2n+1} \mid n \in \mathbb{Z}\}$ and $\{q^{2n} \mid n \in \mathbb{Z}\}$, respectively, and we stress that each κ_s can be thought of as a convex combination of κ_{-1} and κ_1, to be precise

$$\kappa_s = \frac{1-s}{2} \kappa_{-1} + \frac{1+s}{2} \kappa_1.$$

On the previous pages we have given a survey of the well-known solutions to the moment problem. To learn even more about the structure of these solutions and to obtain further insight, we shall now introduce a transformation of the set V of solutions. But first some notation. For $a > 0$, let τ_a denote the map given by $\tau_a(x) = ax$ and recall that the image measure $\tau_a(\mu)$ of a measure μ on $[0, \infty)$ under τ_a is defined by

$$\tau_a(\mu)(B) = \mu\bigl(a^{-1}B\bigr)$$

for all Borel sets $B \subset [0, \infty)$.

Proposition 2.3. *Suppose that μ is a measure on $[0, \infty)$ with moments $q^{-\binom{n+1}{2}}$. Then the support of $\nu = \tau_q(qx\, d\mu(x))$ is contained in $[0, \infty)$ and ν has the moments $q^{-\binom{n+1}{2}}$.*

Proof. The proof is straightforward. The support of ν is certainly contained in $[0, \infty)$ and

$$\int_0^\infty x^n\, d\nu(x) = \int_0^\infty (qx)^n qx\, d\mu(x) = q^{n+1} \int_0^\infty x^{n+1}\, d\mu(x) = q^{-\binom{n+1}{2}}.$$

□

The above proposition gives rise to the following definition.

Definition 2.4. We denote by $T : V \mapsto V$ the map given by $T(\mu) = \tau_q(qx\, d\mu(x))$.

A probability measure μ is a fixed point of T if and only if it satisfies the equation

$$\tau_{q^{-1}}(\mu) = qx\, d\mu(x). \tag{2.10}$$

When μ is absolutely continuous with density, say $f(x)$, this equation exactly corresponds to the functional equation $f(xq) = xf(x)$, $x > 0$ and when μ is a discrete measure, the equation tells us that $c > 0$ is a mass point of μ exactly when cq likewise is a mass point of μ and $\mu(\{cq\}) = qc\mu(\{c\})$. The latter property is satisfied by the measures λ_c in (2.8).

As a matter of fact, we can classify all the absolutely continuous and all the discrete fixed points of T. Whenever g is a positive, measurable and q-periodic function on $(0,\infty)$ such that

$$\int_0^\infty v(x)g(x)\,dx = 1,$$

the absolutely continuous measure with density $v(x)g(x)$, $x > 0$ is a fixed point of T and every absolutely continuous fixed point of T has this form (for some g). The discrete fixed points of T are precisely the countable convex combinations of the measures λ_c.

So nearly all the solutions presented till now are fixed points of T. The only exception is the measures κ_s in (2.9) when $s \neq 0$. For $-1 < s < 1$, the support of κ_s is the geometric progression $\{q^n \mid n \in \mathbb{Z}\}$ and T has at most one fixed point with this support. However, we know that $\kappa_0 = \lambda_1$ is a fixed point of T. In general, it turns out that $T(\kappa_s) = \kappa_{-s}$ so all the measures κ_s are fixed points of $T^{(2)}$.

It is worth while dwelling somewhat on Eq. (2.10) since this is the full generalization of the functional equation (2.2). Suppose that μ is a finite measure on $(0,\infty)$ which satisfies this equation or, equivalently,

$$\mu(qB) = q\int_B x\,d\mu(x)$$

for all Borel sets $B \subset (0,\infty)$. By induction, we have

$$\tau_{q^{-n}}(\mu) = q^{\binom{n+1}{2}} x^n\,d\mu(x), \quad n \in \mathbb{Z},$$

and this means that

$$\int_0^\infty x^n\,d\mu(x) = q^{-\binom{n+1}{2}} \int_0^\infty d\tau_{q^{-n}}(\mu)(x).$$

So if μ is a probability measure, it surely has the moments (2.3). But furthermore, we see that μ is uniquely determined by its restriction $\mu|_{(q,1]}$ to the interval $(q,1]$ or any other interval of the form $(q^{n+1},q^n]$ for some $n \in \mathbb{Z}$. For if $\mu|_{(q,1]} = \nu$, then

$$\mu|_{(q^{n+1},q^n]} = \tau_{q^n}\big(q^{\binom{n+1}{2}} x^n\,d\nu(x)\big)$$

for each $n \in \mathbb{Z}$ and $\bigcup_{n=-\infty}^{\infty}(q^{n+1},q^n] = (0,\infty)$.

On the other hand, suppose that ν is any finite measure on $(q,1]$. Then there is exactly one way to extend ν to a finite measure μ on $(0,\infty)$ such that μ satisfies Eq. (2.10). Simply define

$$\mu|_{(q^{n+1},q^n]} = \tau_{q^n}\big(q^{\binom{n+1}{2}} x^n\,d\nu(x)\big), \quad n \in \mathbb{Z},$$

that is,

$$\mu\big(q^n B\big) = q^{\binom{n+1}{2}} \int\limits_B x^n \, d\nu(x)$$

for all Borel sets $B \subset (q, 1]$. In this way,

$$\tau_{q^{-1}}(\mu|_{(q^{n+1},q^n]}) = qx \, d\mu|_{(q^n,q^{n-1}]}(x), \quad n \in \mathbb{Z},$$

so the measure μ satisfies the desired equation and it is a finite measure since

$$\mu\big((0,\infty)\big) = \sum_{n=-\infty}^{\infty} q^{\binom{n+1}{2}} \int\limits_{(q,1]} x^n \, d\nu(x)$$
$$\leqslant \nu\big((q,1]\big)\left(1/q \sum_{n=0}^{\infty} q^{\binom{n}{2}} + \sum_{n=0}^{\infty} q^{\binom{n+1}{2}}\right) < \infty.$$

Starting from a finite measure ν on the interval $(q, 1]$, we can thus construct a solution to the moment problem by, if necessary, normalizing the extension μ. The solution obtained from ν is of the same type as ν. So if ν is a continuous singular measure, we end up with a continuous singular solution to the moment problem.

Similar observations was made by Pakes in [15]. Using a slightly different notation, he proved that a measure μ is solution to (2.10) if and only if μ has the form

$$\mu = K \sum_{n=-\infty}^{\infty} \tau_{q^n}\big(q^{\binom{n+1}{2}} x^n \, d\nu(x)\big),$$

where K is some constant and ν is a finite measure supported within the interval $(q, 1]$.

3. The N-extremal solutions and canonical solutions

The orthonormal polynomials (P_n) associated with the moment sequence (2.3) are given explicitly in (2.4). Recall that the polynomials (Q_n) of the second kind are defined by

$$Q_n(x) = \int \frac{P_n(x) - P_n(y)}{x - y} \, d\mu(y), \quad n \geqslant 0,$$

where μ is any measure with the moments s_n $(= q^{-\binom{n+1}{2}}$ in our case). Obviously, $Q_0(x) = 0$ and when $P_n(x) = \sum_{k=0}^{n} c_k x^k$, we have

$$Q_n(x) = \sum_{m=0}^{n-1} \left(\sum_{k=m+1}^{n} c_k s_{k-m-1} \right) x^m, \quad n \geqslant 1.$$

Consequently, the polynomials (Q_n) of the second kind associated with the moment sequence (2.3) are given by

$$Q_n(x) = (-1)^n \sqrt{\frac{q^n}{(q;q)_n}} \sum_{m=0}^{n-1} q^{-\binom{m+1}{2}} \left(\sum_{k=m+1}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}+(m+1)k} \right) x^m,$$
$$n \geqslant 1. \tag{3.1}$$

Remark 3.1. The inner sum $\sum_{k=m+1}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}+(m+1)k}$ is the tail in the finite version of the q-binomial theorem [2, p. 490]

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}} x^k = (x;q)_n. \tag{3.2}$$

Therefore, we could also write this sum as

$$\left(q^{m+1};q\right)_n - \sum_{k=0}^{m} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}+(m+1)k}.$$

From time to time we shall be dealing with the Stieltjes–Wigert polynomials of the first and second kind given by

$$S_n(x;q) = \frac{1}{(q;q)_n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{k^2} x^k, \quad n \geqslant 0,$$

and

$$\widetilde{S}_n(x;q) = \frac{1}{(q;q)_n} \sum_{m=0}^{n-1} q^{-\binom{m+1}{2}} \left(\sum_{k=m+1}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}+(m+1)k} \right) x^m, \quad n \geqslant 1,$$

that is, $P_n(x) = (-1)^n \sqrt{q^n (q;q)_n} S_n(x;q)$ and $Q_n(x) = (-1)^n \sqrt{q^n (q;q)_n}\, \widetilde{S}_n(x;q)$.

It is essential that $S_n(x;q)$ and $\widetilde{S}_n(x;q)$ converge uniformly on compact subsets of $\mathbb{C}$ when $n \to \infty$. In fact, $S_n(x;q) \to \Phi(x)/(q;q)_\infty$ and $\widetilde{S}_n(x;q) \to \Psi(x)/(q;q)_\infty$ for $n \to \infty$, where Φ and Ψ denote the entire functions

$$\Phi(x) = \sum_{k=0}^{\infty} (-1)^k \frac{q^{k^2}}{(q;q)_k} x^k \tag{3.3}$$

and

$$\Psi(x) = \sum_{m=0}^{\infty} q^{-\binom{m+1}{2}} \left(\sum_{k=m+1}^{\infty} (-1)^k \frac{q^{\binom{k}{2}+(m+1)k}}{(q;q)_k} \right) x^m. \tag{3.4}$$

From the general theory of orthogonal polynomials it is well known that $S_n(x;q)$ has n simple positive zeros and that the polynomials $S_{n-1}(x;q)$ and $S_n(x;q)$ have no common zeros. Moreover, the zeros of $S_{n-1}(x;q)$ and $S_n(x;q)$ interlace, that is, $S_{n-1}(x;q)$ has exactly one zero between two consecutive zeros of $S_n(x;q)$.

Since the Stieltjes–Wigert polynomials are orthogonal with respect to the discrete measures λ_c in (2.8), it follows that $S_n(x;q)$ has at most one zero in the open interval (cq, c) for each $c > 0$. In other words, the n zeros of $S_n(x;q)$, say $0 < x_{n,1} < \cdots < x_{n,n}$, are separated and this was mentioned by Chihara in [10]. Using the identity

$$S_{n-1}(x;q) = \left(1 - q^n\right) S_n(x;q) + x q^n S_{n-1}(xq;q), \tag{3.5}$$

which can be verified by direct computations, Chihara proved in [11] that

$$x_{n,m} < x_{n-1,m} < q x_{n,m+1}.$$

So in a sense, the mth zero of $S_{n-1}(x;q)$ lies in the first part of the interval from the mth to the $(m+1)$th zero of $S_n(x;q)$ and we have

$$\frac{x_{n,m+1}}{x_{n,m}} > q^{-1}. \tag{3.6}$$

Referring to (3.6), we say that the zeros of $S_n(x;q)$ are well separated. Using the identity

$$S_n(x;q) = \bigl(1 + xq^{n+1}\bigr)S_n(xq;q) - qxS_n\bigl(xq^2;q\bigr), \tag{3.7}$$

which can also be verified by direct computations, we shall give a refinement of the separation property (3.6). Assume that $S_n(x;q) > 0$ for $x_{n,m} < x < x_{n,m+1}$. The case $S_n(x;q) < 0$ can be handled in a completely similar way. Since $x_{n,m} < qx_{n,m+1} < x_{n,m+1}$, this in particular means that $S_n(qx_{n,m+1};q) > 0$. The open interval $(qx_{n,m}, x_{n,m})$ contains no zero of $S_n(x;q)$ and, consequently, $S_n(x;q) < 0$ for $qx_{n,m} < x < x_{n,m}$. Suppose now that $q^2x_{n,m+1} \leqslant x_{n,m}$. Since $qx_{n,m} < q^2x_{n,m+1}$, this results in $S_n(q^2x_{n,m+1};q) \leqslant 0$ which clearly contradicts the identity (3.7). Therefore, we have $q^2x_{n,m+1} > x_{n,m}$ or, equivalently,

$$\frac{x_{n,m+1}}{x_{n,m}} > q^{-2} \tag{3.8}$$

and we say that the zeros of $S_n(x;q)$ are very well separated.

Remark 3.2. One should not expect to find a stronger separation property than (3.8) after looking at the zeros of $S_2(x;q)$. For instance, $x_{2,2}/x_{2,1} < q^{-3}$ when $q = 1/2$.

In some sense, to solve an indeterminate moment problem means to find the four entire functions A, B, C, and D from the Nevanlinna parametrization. Based on generating functions for the Stieltjes–Wigert polynomials, we shall give expressions for these functions. The generating function for $S_n(x;q)$ is also stated in Koekoek and Swarttouw [14].

Proposition 3.3. *For $\gamma \in \mathbb{C}$ and $|t| < 1$, we have*

$$\sum_{n=0}^{\infty}(\gamma;q)_n t^n S_n(x;q) = \frac{(\gamma t;q)_\infty}{(t;q)_\infty}\sum_{n=0}^{\infty}(-1)^n\frac{(\gamma;q)_n}{(\gamma t,q;q)_n}q^{n^2}(xt)^n,$$

$$\sum_{n=0}^{\infty}(\gamma;q)_n t^n \widetilde{S}_n(x;q) = \frac{(\gamma t;q)_\infty}{(t;q)_\infty}\sum_{n=0}^{\infty}q^{-\binom{n+1}{2}} \times\left(\sum_{k=n+1}^{\infty}(-1)^k\frac{(\gamma;q)_k}{(\gamma t,q;q)_k}q^{\binom{k}{2}+(n+1)k}t^k\right)x^n.$$

In particular, with $\gamma = 0$ and $t = q$ we have

$$\sum_{n=0}^{\infty}q^n S_n(x;q) = \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n\frac{q^{n(n+1)}}{(q;q)_n}x^n,$$

$$\sum_{n=0}^{\infty}q^n \widetilde{S}_n(x;q) = \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}q^{-\binom{n+1}{2}}\left(\sum_{k=n+1}^{\infty}(-1)^k\frac{q^{\binom{k}{2}+(n+2)k}}{(q;q)_k}\right)x^n,$$

and with $\gamma = t = q$ we have

$$\sum_{n=0}^{\infty}(q;q)_n q^n S_n(x;q) = \sum_{n=0}^{\infty}(-1)^n \frac{q^{n(n+1)}}{(q;q)_{n+1}} x^n,$$
$$\sum_{n=0}^{\infty}(q;q)_n q^n \widetilde{S}_n(x;q) = \sum_{n=0}^{\infty} q^{-\binom{n+1}{2}} \left(\sum_{k=n+1}^{\infty} (-1)^k \frac{q^{\binom{k}{2}+(n+2)k}}{(q;q)_{k+1}} \right) x^n.$$

Remark 3.4. The inner sum $\sum_{k=n+1}^{\infty}(-1)^k q^{\binom{k}{2}+(n+2)k}/(q;q)_k$ is the tail in Euler's formula [2, p. 490]

$$\sum_{n=0}^{\infty}(-1)^n \frac{q^{\binom{n}{2}}}{(q;q)_n} x^n = (x;q)_\infty. \tag{3.9}$$

So this sum can also be written as

$$\left(q^{n+2};q\right)_\infty - \sum_{k=0}^{n}(-1)^k \frac{q^{\binom{k}{2}+(n+2)k}}{(q;q)_k}.$$

Concerning the inner sum $\sum_{k=n+1}^{\infty}(-1)^k q^{\binom{k}{2}+(n+2)k}/(q;q)_{k+1}$, we can say almost the same.

Proof. The point of the proof is to interchange the order of summation and use the q-binomial theorem [2, p. 488]

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} x^n = \frac{(ax;q)_\infty}{(x;q)_\infty}, \quad |x| < 1. \tag{3.10}$$

Absolute convergence assures that we can change the summation. Hence

$$\begin{aligned}\sum_{n=0}^{\infty}(\gamma;q)_n t^n S_n(x;q) &= \sum_{n=0}^{\infty}(\gamma;q)_n t^n \sum_{k=0}^{n}(-1)^k \frac{q^{k^2}}{(q;q)_k (q;q)_{n-k}} x^k \\ &= \sum_{k=0}^{\infty}(-1)^k \frac{q^{k^2}}{(q;q)_k} x^k \sum_{n=k}^{\infty} \frac{(\gamma;q)_n}{(q;q)_{n-k}} t^n \\ &= \sum_{k=0}^{\infty}(-1)^k \frac{(\gamma;q)_k}{(q;q)_k} q^{k^2} t^k x^k \sum_{n=0}^{\infty} \frac{(\gamma q^k;q)_n}{(q;q)_n} t^n\end{aligned}$$

and similarly

$$\begin{aligned}&\sum_{n=0}^{\infty}(\gamma;q)_n t^n \widetilde{S}_n(x;q) \\ &\quad = \sum_{n=0}^{\infty}(\gamma;q)_n t^n \sum_{m=0}^{n-1} q^{-\binom{m+1}{2}} \left(\sum_{k=m+1}^{n} (-1)^k \frac{q^{\binom{k}{2}+(m+1)k}}{(q;q)_k (q;q)_{n-k}} \right) x^m\end{aligned}$$

$$= \sum_{m=0}^{\infty} q^{-\binom{m+1}{2}} \left(\sum_{n=m+1}^{\infty} (\gamma;q)_n t^n \sum_{k=m+1}^{n} (-1)^k \frac{q^{\binom{k}{2}+(m+1)k}}{(q;q)_k (q;q)_{n-k}} \right) x^m$$

$$= \sum_{m=0}^{\infty} q^{-\binom{m+1}{2}} \left(\sum_{k=m+1}^{\infty} (-1)^k \frac{q^{\binom{k}{2}+(m+1)k}}{(q;q)_k} \sum_{n=k}^{\infty} \frac{(\gamma;q)_n}{(q;q)_{n-k}} t^n \right) x^m$$

$$= \sum_{m=0}^{\infty} q^{-\binom{m+1}{2}} \left(\sum_{k=m+1}^{\infty} (-1)^k \frac{(\gamma;q)_k}{(q;q)_k} q^{\binom{k}{2}+(m+1)k} t^k \sum_{n=0}^{\infty} \frac{(\gamma q^k;q)_n}{(q;q)_n} t^n \right) x^m.$$

By the q-binomial theorem (3.10), we have

$$\sum_{n=0}^{\infty} \frac{(\gamma q^k;q)_n}{(q;q)_n} t^n = \frac{(\gamma t q^k;q)_\infty}{(t;q)_\infty}$$

so it follows that

$$\sum_{n=0}^{\infty} (\gamma;q)_n t^n S_n(x;q) = \frac{(\gamma t;q)_\infty}{(t;q)_\infty} \sum_{k=0}^{\infty} (-1)^k \frac{(\gamma;q)_k}{(\gamma t, q;q)_k} q^{k^2} (xt)^k$$

and

$$\sum_{n=0}^{\infty} (\gamma;q)_n t^n \widetilde{S}_n(x;q) = \frac{(\gamma t;q)_\infty}{(t;q)_\infty} \sum_{m=0}^{\infty} q^{-\binom{m+1}{2}} \times \left(\sum_{k=m+1}^{\infty} (-1)^k \frac{(\gamma;q)_k}{(\gamma t, q;q)_k} q^{\binom{k}{2}+(m+1)k} t^k \right) x^m.$$

□

The special cases from Proposition 3.3 leads to the following result.

Theorem 3.5. *The four entire functions A, B, C, and D from the Nevanlinna parametrization are given by*

$$A(x) = -\sum_{n=0}^{\infty} q^{-\binom{n+1}{2}} \left(\sum_{k=n+1}^{\infty} (-1)^k \frac{q^{\binom{k}{2}+nk}}{(q;q)_k} \right) x^n - \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} q^{-\binom{n}{2}} \left(\sum_{k=n}^{\infty} (-1)^k \frac{q^{\binom{k}{2}+(n+1)k}}{(q;q)_k} \right) x^n,$$

$$B(x) = -\sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n-1)}}{(q;q)_n} x^n - \frac{x}{(q;q)_\infty} \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)}}{(q;q)_n} x^n,$$

$$C(x) = \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} q^{-\binom{n}{2}} \left(\sum_{k=n}^{\infty} (-1)^k \frac{q^{\binom{k}{2}+(n+1)k}}{(q;q)_k} \right) x^n,$$

$$D(x) = \frac{x}{(q;q)_\infty} \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)}}{(q;q)_n} x^n.$$

Proof. From (2.4) we see that

$$P_n(0) = (-1)^n \sqrt{\frac{q^n}{(q;q)_n}} \tag{3.11}$$

and using the finite version of the q-binomial theorem (3.2), we get from (3.1) that

$$Q_n(0) = \big((q;q)_n - 1\big) P_n(0). \tag{3.12}$$

Recalling that $P_n(x) = (-1)^n \sqrt{q^n (q;q)_n} S_n(x;q)$ and $Q_n(x) = (-1)^n \sqrt{q^n (q;q)_n} \times \widetilde{S}_n(x;q)$, we thus obtain

$$\begin{aligned}
D(x) &= x \sum_{n=0}^{\infty} P_n(0) P_n(x) = x \sum_{n=0}^{\infty} q^n S_n(x;q) \\
&= \frac{x}{(q;q)_\infty} \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)}}{(q;q)_n} x^n, \\
B(x) &= -1 + x \sum_{n=0}^{\infty} Q_n(0) P_n(x) = -1 + x \sum_{n=0}^{\infty} \big((q;q)_n - 1\big) q^n S_n(x;q) \\
&= -1 - \sum_{n=0}^{\infty} (-1)^{n+1} \frac{q^{n(n+1)}}{(q;q)_{n+1}} x^{n+1} - D(x) \\
&= -\sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n-1)}}{(q;q)_n} x^n - D(x), \\
C(x) &= 1 + x \sum_{n=0}^{\infty} P_n(0) Q_n(x) = 1 + x \sum_{n=0}^{\infty} q^n \widetilde{S}_n(x;q) \\
&= 1 + \frac{x}{(q;q)_\infty} \sum_{n=0}^{\infty} q^{-\binom{n+1}{2}} \left(\sum_{k=n+1}^{\infty} (-1)^k \frac{q^{\binom{k}{2}+(n+2)k}}{(q;q)_k} \right) x^n \\
&= \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} q^{-\binom{n}{2}} \left(\sum_{k=n}^{\infty} (-1)^k \frac{q^{\binom{k}{2}+(n+1)k}}{(q;q)_k} \right) x^n, \\
A(x) &= x \sum_{n=0}^{\infty} Q_n(0) Q_n(x) = x \sum_{n=0}^{\infty} \big((q;q)_n - 1\big) q^n \widetilde{S}_n(x;q) \\
&= 1 + \sum_{n=0}^{\infty} q^{-\binom{n+1}{2}} \left(\sum_{k=n+1}^{\infty} (-1)^k \frac{q^{\binom{k}{2}+(n+2)k}}{(q;q)_{k+1}} \right) x^{n+1} - C(x) \\
&= -\sum_{n=0}^{\infty} q^{-\binom{n+1}{2}} \left(\sum_{k=n+1}^{\infty} (-1)^k \frac{q^{\binom{k}{2}+nk}}{(q;q)_k} \right) x^n - C(x).
\end{aligned}$$

In the computations of C and A, we have used Euler's formula (3.9) in the last steps. □

The expressions for A and B are more complicated than the expressions for C and D. However, we obviously have

$$B(x)+D(x)=-\sum_{n=0}^{\infty}(-1)^n\frac{q^{n(n-1)}}{(q;q)_n}x^n$$

and

$$A(x)+C(x)=-\sum_{n=0}^{\infty}q^{-\binom{n+1}{2}}\left(\sum_{k=n+1}^{\infty}(-1)^k\frac{q^{\binom{k}{2}+nk}}{(q;q)_k}\right)x^n.$$

The quantity α in (1.13) is explicitly given by

$$\alpha=\lim_{n\to\infty}\frac{1}{(q;q)_n-1}=\frac{1}{(q;q)_\infty-1} \tag{3.13}$$

since $Q_n(0)=((q;q)_n-1)P_n(0)$, see (3.12). Due to the fact that $0<(q;q)_\infty<1$, this in particular means that $\alpha<-1$. Realizing that $-1/\alpha=1-(q;q)_\infty$, simple computations give that

$$B(x)-\frac{1}{\alpha}D(x)=-\sum_{n=0}^{\infty}(-1)^n\frac{q^{n^2}}{(q;q)_n}x^n$$

and

$$A(x)-\frac{1}{\alpha}C(x)=-\sum_{n=0}^{\infty}q^{-\binom{n+1}{2}}\left(\sum_{k=n+1}^{\infty}(-1)^k\frac{q^{\binom{k}{2}+(n+1)k}}{(q;q)_k}\right)x^n.$$

In the light of Theorem 3.5, we have thus established the power series expansions of the entire functions C, D, $A+C$, $B+D$, $A-\frac{1}{\alpha}C$, and $B-\frac{1}{\alpha}D$. One should note that

$$D(x)=\frac{x}{(q;q)_\infty}\Phi(xq),\qquad B(x)+D(x)=-\Phi(x/q)\quad\text{and}$$
$$B(x)-\frac{1}{\alpha}D(x)=-\Phi(x),$$

whereas

$$A(x)-\frac{1}{\alpha}C(x)=-\Psi(x),$$

cf. (3.3) and (3.4). In particular, we have

$$\lim_{n\to\infty}\frac{\widetilde{S}_n(x;q)}{S_n(x;q)}=\frac{\Psi(x)}{\Phi(x)}=\frac{A(x)-\frac{1}{\alpha}C(x)}{B(x)-\frac{1}{\alpha}D(x)}=\frac{A(x)\alpha-C(x)}{B(x)\alpha-D(x)}$$
$$\text{for } x\in\mathbb{C}\setminus[0,\infty). \tag{3.14}$$

We will now focus on the canonical solutions to the moment problem and especially on the N-extremal solutions. Since a canonical solution is discrete and supported on the zeros of an entire function, these solutions cannot be convex combinations of the measures λ_c in (2.8). For 0 is an accumulation point of the set $\{cq^n \mid n\in\mathbb{Z}\}$ and the zeros of an entire function cannot have an accumulation point without the function being identically zero. Compare with [10], where Chihara made it clear that the measures λ_c are not N-extremal. Consequently, the canonical solutions are not fixed points of the transformation T in Definition 2.4.

Recall that the only N-extremal solutions supported within $[0,\infty)$ are μ_t when $\alpha \leqslant t \leqslant 0$. In our case, three of these solutions are leaping to the eye, namely μ_t when $t \in \{0,-1,\alpha\}$. In order to find these solutions explicitly, one needs to know the zeros of Φ since μ_0 is supported on the zeros of $\Phi(xq)$ (plus 0), μ_α is supported on the zeros of $\Phi(x)$ and μ_{-1} is supported on the zeros of $\Phi(x/q)$. However, the zeros of Φ cannot be found explicitly.

Since the zeros of $S_n(x;q)$ in a certain sense converge to the zeros of Φ, we are able to show that the zeros of Φ are very well separated. For each $m \in \mathbb{N}$, the sequence $(x_{n,m})$ is decreasing in n and thus convergent, say $x_{n,m} \to x_m$ for $n \to \infty$. Since $S_n(x;q)$ converge uniformly to $\Phi(x)/(q;q)_\infty$ on compact subsets of $\mathbb{C}$, the limit points x_m are zeros of Φ and since $\Phi(0)=1$, we have $x_1 > 0$. Recalling that the zeros of $S_n(x;q)$ are very well separated, the points x_m are surely well separated, at the worst $x_{m+1}/x_m \geqslant q^{-2}$. According to Rouché's theorem, the points x_m are the only zeros of Φ. For if $x_m < y < x_{m+1}$, then the closed ball with center at y and radius $r < \min(y-x_m, x_{m+1}-y)$ contains no zero of $S_n(x;q)$ for n sufficiently large. Due to the uniform convergence, this is also the case for Φ and, in particular, y is not a zero of Φ. It is easy to see from (3.7) by letting $n \to \infty$ that

$$\Phi(x) = \Phi(xq) - qx\Phi\bigl(xq^2\bigr) \tag{3.15}$$

and with a similar argumentation as for $S_n(x;q)$, it therefore follows that the zeros of Φ are very well separated, that is, $x_{m+1}/x_m > q^{-2}$.

It is straightforward to see that Φ is a q-analogue of the exponential function and an entire function of order 0. The latter implies that A, B, C, and D from Theorem 3.5 also are entire functions of order 0 since these functions are known to have the same order, see [8].

To underline the fact that Φ is a very interesting and complicated function, we point out that

$$\Phi(-1) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \prod_{n=0}^{\infty} \bigl(1-q^{5n+1}\bigr)^{-1}\bigl(1-q^{5n+4}\bigr)^{-1}$$

and

$$\Phi(-q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n} = \prod_{n=0}^{\infty} \bigl(1-q^{5n+2}\bigr)^{-1}\bigl(1-q^{5n+3}\bigr)^{-1}.$$

These are the famous Rogers–Ramanujan identities, cf. [2, p. 565].

We shall now make the preparations for describing the transformation T at the level of Pick functions. If μ is a measure on $[0,\infty)$ with moments (2.3), then the moments of $\tilde{\mu} = qx\,d\mu(x)$ are

$$\int_0^\infty x^n\,d\tilde{\mu}(x) = q\int_0^\infty x^{n+1}\,d\mu(x) = q^{-\binom{n+1}{2}-n}. \tag{3.16}$$

The key is to look at the connection between the moment problems associated with the moment sequences (2.3) and (3.16). Suppose that μ is a probability measure on $(0,\infty)$ satisfying Eq. (2.10). Since μ has the moments (2.3), we know that

$$\int_0^\infty P_m(x)P_n(x)\,d\mu(x)=\delta_{mn}$$

and, equivalently,

$$\int_0^\infty P_m(xq)P_n(xq)\,d\tau_{q^{-1}}(\mu)(x)=\delta_{mn}.$$

This means that the orthonormal polynomials $(\widetilde{P}_n)$ associated with the moment sequence (3.16) are given by $\widetilde{P}_n(x)=P_n(xq)$. Moreover, the polynomials $(\widetilde{Q}_n)$ of the second kind are given by $\widetilde{Q}_n(x)=qQ_n(xq)$ since

$$\begin{aligned}\int_0^\infty \frac{\widetilde{P}_n(x)-\widetilde{P}_n(y)}{x-y}\,d\tau_{q^{-1}}(\mu)(y)&=\int_0^\infty \frac{P_n(xq)-P_n(y)}{x-y/q}\,d\mu(y)\\&=q\int_0^\infty \frac{P_n(xq)-P_n(y)}{xq-y}\,d\mu(y).\end{aligned}$$

In this way, we see that the entire functions from the Nevanlinna parametrization for the two moment problems are related by

$$\begin{aligned}\widetilde{A}(x)&=x\sum_{n=0}^\infty \widetilde{Q}_n(0)\widetilde{Q}_n(x)=q^2x\sum_{n=0}^\infty Q_n(0)Q_n(xq)=qA(xq),\\ \widetilde{B}(x)&=-1+x\sum_{n=0}^\infty \widetilde{Q}_n(0)\widetilde{P}_n(x)=-1+qx\sum_{n=0}^\infty Q_n(0)P_n(xq)=B(xq),\\ \widetilde{C}(x)&=1+x\sum_{n=0}^\infty \widetilde{P}_n(0)\widetilde{Q}_n(x)=1+qx\sum_{n=0}^\infty P_n(0)Q_n(xq)=C(xq),\\ \widetilde{D}(x)&=x\sum_{n=0}^\infty \widetilde{P}_n(0)\widetilde{P}_n(x)=x\sum_{n=0}^\infty P_n(0)P_n(xq)=D(xq)/q.\end{aligned}$$

On the other hand, a general result given by Pedersen in [16, Proposition 6.3] tells us that

$$\begin{pmatrix}\widetilde{A}(x)\\ \widetilde{B}(x)\\ \widetilde{C}(x)\\ \widetilde{D}(x)\end{pmatrix}=M(x)\begin{pmatrix}A(x)\\ B(x)\\ C(x)\\ D(x)\end{pmatrix},$$

where $M(x)$ denotes the matrix

$$\begin{pmatrix} qx(1-D'(0)) & -q(1-D'(0)) & -\frac{qx}{\alpha}(1-D'(0))-q & \frac{q}{\alpha}(1-D'(0))+\frac{q}{x} \\ 0 & 1-D'(0) & 0 & -\frac{1}{\alpha}(1-D'(0))-\frac{1}{x} \\ xD'(0) & -D'(0) & -\frac{x}{\alpha}D'(0)+1 & \frac{1}{\alpha}D'(0)-\frac{1}{x} \\ 0 & \frac{1}{q}D'(0) & 0 & -\frac{1}{q\alpha}D'(0)+\frac{1}{qx} \end{pmatrix}$$

and since $D'(0)=1/(q;q)_\infty$, we have

$$\begin{pmatrix} A(xq) \\ B(xq) \\ C(xq) \\ D(xq) \end{pmatrix} = \begin{pmatrix} \frac{x((q;q)_\infty-1)}{(q;q)_\infty} & -1+\frac{1}{(q;q)_\infty} & -\frac{x((q;q)_\infty-1)}{\alpha(q;q)_\infty}-1 & \frac{((q;q)_\infty-1)}{\alpha(q;q)_\infty}+\frac{1}{x} \\ 0 & 1-\frac{1}{(q;q)_\infty} & 0 & -\frac{((q;q)_\infty-1)}{\alpha(q;q)_\infty}-\frac{1}{x} \\ \frac{x}{(q;q)_\infty} & -\frac{1}{(q;q)_\infty} & -\frac{x}{\alpha(q;q)_\infty}+1 & \frac{1}{\alpha(q;q)_\infty}-\frac{1}{x} \\ 0 & \frac{1}{(q;q)_\infty} & 0 & -\frac{1}{\alpha(q;q)_\infty}+\frac{1}{x} \end{pmatrix} \times \begin{pmatrix} A(x) \\ B(x) \\ C(x) \\ D(x) \end{pmatrix}.$$

This can also be written as

$$\begin{pmatrix} A(xq)+B(xq) \\ B(xq) \\ C(xq)+D(xq) \\ D(xq) \end{pmatrix} = \begin{pmatrix} \frac{x((q;q)_\infty-1)}{(q;q)_\infty} & 0 & -\frac{x((q;q)_\infty-1)}{\alpha(q;q)_\infty}-1 & 0 \\ 0 & 1-\frac{1}{(q;q)_\infty} & 0 & -\frac{((q;q)_\infty-1)}{\alpha(q;q)_\infty}-\frac{1}{x} \\ \frac{x}{(q;q)_\infty} & 0 & -\frac{x}{\alpha(q;q)_\infty}+1 & 0 \\ 0 & \frac{1}{(q;q)_\infty} & 0 & -\frac{1}{\alpha(q;q)_\infty}+\frac{1}{x} \end{pmatrix} \begin{pmatrix} A(x) \\ B(x) \\ C(x) \\ D(x) \end{pmatrix}$$

or

$$\begin{pmatrix} A(xq)+C(xq) \\ B(xq)+D(xq) \\ C(xq) \\ D(xq) \end{pmatrix} = \begin{pmatrix} x & -1 & -\frac{x}{\alpha} & \frac{1}{\alpha} \\ 0 & 1 & 0 & -\frac{1}{\alpha} \\ \frac{x}{(q;q)_\infty} & -\frac{1}{(q;q)_\infty} & -\frac{x}{\alpha(q;q)_\infty}+1 & \frac{1}{\alpha(q;q)_\infty}-\frac{1}{x} \\ 0 & \frac{1}{(q;q)_\infty} & 0 & -\frac{1}{\alpha(q;q)_\infty}+\frac{1}{x} \end{pmatrix} \begin{pmatrix} A(x) \\ B(x) \\ C(x) \\ D(x) \end{pmatrix}$$

or even

$$\begin{pmatrix} A(xq)+B(xq)+C(xq)+D(xq) \\ B(xq)+D(xq) \\ C(xq)+D(xq) \\ D(xq) \end{pmatrix} = \begin{pmatrix} x & 0 & -\frac{x}{\alpha} & 0 \\ 0 & 1 & 0 & -\frac{1}{\alpha} \\ \frac{x}{(q;q)_\infty} & 0 & -\frac{x}{\alpha(q;q)_\infty}+1 & 0 \\ 0 & \frac{1}{(q;q)_\infty} & 0 & -\frac{1}{\alpha(q;q)_\infty}+\frac{1}{x} \end{pmatrix} \begin{pmatrix} A(x) \\ B(x) \\ C(x) \\ D(x) \end{pmatrix}.$$

The last expression is equivalent to

$$\begin{pmatrix} B(xq)+D(xq) \\ D(xq) \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{\alpha} \\ \frac{1}{(q;q)_\infty} & -\frac{1}{\alpha(q;q)_\infty}+\frac{1}{x} \end{pmatrix} \begin{pmatrix} B(x) \\ D(x) \end{pmatrix} \tag{3.17}$$

and

$$\begin{pmatrix} A(xq)+B(xq)+C(xq)+D(xq) \\ C(xq)+D(xq) \end{pmatrix} = x \begin{pmatrix} 1 & -\frac{1}{\alpha} \\ \frac{1}{(q;q)_\infty} & -\frac{1}{\alpha(q;q)_\infty}+\frac{1}{x} \end{pmatrix} \begin{pmatrix} A(x) \\ C(x) \end{pmatrix}. \tag{3.18}$$

We are now ready to describe the transformation T at the level of Pick functions.

Theorem 3.6. *Suppose that $\mu \in V$ and let φ be the Pick function corresponding to μ. Then $\nu = \tau_q(qx\,d\mu(x)) \in V$ and the Pick function ψ corresponding to ν is given by*

$$\psi(x) = \frac{\frac{x}{(q;q)_\infty}\big(1-\frac{\varphi(x/q)}{\alpha}\big)+q\varphi(x/q)}{\frac{x}{(q;q)_\infty}((q;q)_\infty-1)\big(1-\frac{\varphi(x/q)}{\alpha}\big)-q\varphi(x/q)}.$$

Proof. The conclusion of Proposition 2.3 is that $\nu \in V$. Since

$$\int_0^\infty \frac{1}{qx-t}\,d\nu(t) = \int_0^\infty \frac{1}{qx-qt}qt\,d\mu(t) = \int_0^\infty \frac{t}{x-t}\,d\mu(t) = -1 + x\int_0^\infty \frac{1}{x-t}\,d\mu(t),$$

we have to show that

$$\frac{A(xq)\psi(xq)-C(xq)}{B(xq)\psi(xq)-D(xq)} = -1 + x\frac{A(x)\varphi(x)-C(x)}{B(x)\varphi(x)-D(x)}$$

and this is done by direct computations. With

$$\zeta(x) = \frac{x}{(q;q)_\infty}\left(1-\frac{\varphi(x)}{\alpha}\right)+\varphi(x) \quad \text{and} \quad \eta(x) = x\left(1-\frac{\varphi(x)}{\alpha}\right),$$

we have

$$\frac{A(xq)\psi(xq)-C(xq)}{B(xq)\psi(xq)-D(xq)} = \frac{\zeta(x)A(xq)+(\zeta(x)-\eta(x))C(xq)}{\zeta(x)B(xq)+(\zeta(x)-\eta(x))D(xq)}$$

and by (3.18) and (3.17), it follows that

$$\frac{\zeta(x)(A(xq)+C(xq))-\eta(x)C(xq)}{\zeta(x)(B(xq)+D(xq))-\eta(x)D(xq)} = -1 + \frac{\zeta(x)x\big(A(x)-\frac{1}{\alpha}C(x)\big)-\eta(x)x\big(\frac{1}{(q;q)_\infty}A(x)+\big(\frac{1}{x}-\frac{1}{\alpha(q;q)_\infty}\big)C(x)\big)}{\zeta(x)(B(xq)+D(xq))-\eta(x)D(xq)}$$

$$= -1 + \frac{\zeta(x)x\big(A(x) - \frac{1}{\alpha}C(x)\big) - \eta(x)x\big(\frac{1}{(q;q)_\infty}A(x) + \big(\frac{1}{x} - \frac{1}{\alpha(q;q)_\infty}\big)C(x)\big)}{\zeta(x)\big(B(x) - \frac{1}{\alpha}D(x)\big) - \eta(x)\big(\frac{1}{(q;q)_\infty}B(x) + \big(\frac{1}{x} - \frac{1}{\alpha(q;q)_\infty}\big)D(x)\big)}$$
$$= -1 + \frac{x\varphi(x)A(x) - \big(x\frac{\varphi(x)}{\alpha} + x\big(1 - \frac{\varphi(x)}{\alpha}\big)\big)C(x)}{\varphi(x)B(x) - \big(\frac{\varphi(x)}{\alpha} + \big(1 - \frac{\varphi(x)}{\alpha}\big)\big)D(x)}$$
$$= -1 + x\frac{A(x)\varphi(x) - C(x)}{B(x)\varphi(x) - D(x)}.$$

□

Let us list some consequences of Theorem 3.6. First of all, we see that T maps a N-extremal solution into another N-extremal solution or into a canonical solution of order 1. In general, T maps a canonical solution of order n into another canonical solution of order $\leqslant n+1$.

It is straightforward to verify that $T(\mu_0) = \mu_\alpha$ and $T(\mu_\alpha) = \mu_{-1}$. Actually, we can describe $T^{(n)}(\mu_0)$ for each $n \in \mathbb{N}$.

Theorem 3.7. *Let $T : V \mapsto V$ denote the map given by $T(\mu) = \tau_q(qx\,d\mu(x))$. For $n = 0, 1, \ldots,$ we have*

$$T^{(2n+1)}(\mu_0) = \mu_{R_n} \quad \textit{and} \quad T^{(2n+2)}(\mu_0) = \mu_{\widetilde{R}_n},$$

where R_n and $\widetilde{R}_n$ are real rational functions of order $\leqslant n$ given by

$$R_n(x) = \frac{\sum_{k=0}^n (-1)^{n-k} \big[{2n-k \atop k}\big]_q q^{(n-k)^2} x^k}{\sum_{k=0}^n (-1)^{n-k}\big((q;q)_\infty \big[{2n-k-1 \atop k-1}\big]_q q^{(n-k+1)^2-1} - \big[{2n-k \atop k}\big]_q q^{(n-k)^2}\big)x^k}$$

and

$$\widetilde{R}_n(x) = \left(\sum_{k=0}^n (-1)^{n-k} \begin{bmatrix} 2n-k+1 \\ k \end{bmatrix}_q q^{(n-k)(n-k+1)} x^k\right) \Bigg/ \left(\sum_{k=0}^n (-1)^{n-k}\left((q;q)_\infty \begin{bmatrix} 2n-k \\ k-1 \end{bmatrix}_q q^{(n-k+1)(n-k+2)-1} - \begin{bmatrix} 2n-k+1 \\ k \end{bmatrix}_q q^{(n-k)(n-k+1)}\right)x^k\right).$$

Proof. The proof is by induction. Start by noting that $R_0(x) = \alpha$ and $\widetilde{R}_0(x) = -1$. Suppose next that $T^{(2n+1)}(\mu_0) = \mu_{R_n}$ for some $n > 0$ and let $T^{(2n+2)}(\mu_0) = T(\mu_{R_n}) = \mu_\psi$, where ψ is a certain Pick function. The real rational function R_n has the form

$$R_n(x) = \frac{S_n(x)}{(q;q)_\infty T_n(x) - S_n(x)}$$

with

$$S_n(x) = \sum_{k=0}^n (-1)^{n-k} \begin{bmatrix} 2n-k \\ k \end{bmatrix}_q q^{(n-k)^2} x^k$$

and

$$T_n(x)=\sum_{k=0}^{n}(-1)^{n-k}\begin{bmatrix}2n-k-1\\k-1\end{bmatrix}_q q^{(n-k+1)^2-1}x^k.$$

So according to Theorem 3.6, we have

$$\begin{aligned}\psi(x)&=\frac{\frac{x}{(q;q)_\infty}\bigl(1-\frac{R_n(x/q)}{\alpha}\bigr)+qR_n(x/q)}{\frac{x}{(q;q)_\infty}((q;q)_\infty-1)\bigl(1-\frac{R_n(x/q)}{\alpha}\bigr)-qR_n(x/q)}\\&=\frac{x(T_n(x/q)-S_n(x/q))+qS_n(x/q)}{x((q;q)_\infty-1)(T_n(x/q)-S_n(x/q))-qS_n(x/q)}\\&=\frac{U_n(x)}{(q;q)_\infty V_n(x)-U_n(x)},\end{aligned}$$

where

$$V_n(x)=U_n(x)-qS_n(x/q)=x\bigl(T_n(x/q)-S_n(x/q)\bigr).$$

By collecting the terms, it follows that

$$\begin{aligned}V_n(x)&=\sum_{k=0}^{n}(-1)^{n-k}\left(\begin{bmatrix}2n-k-1\\k-1\end{bmatrix}_q q^{(n-k+1)^2-1}\right.\\&\qquad\left.-\begin{bmatrix}2n-k\\k\end{bmatrix}_q q^{(n-k)^2}\right)q^{-k}x^{k+1}\\&=\sum_{k=0}^{n-1}(-1)^{n-k}\begin{bmatrix}2n-k-1\\k\end{bmatrix}_q q^{(n-k)^2-k}\\&\qquad\times\left(q^{2(n-k)}\frac{1-q^k}{1-q^{2(n-k)}}-\frac{1-q^{2n-k}}{1-q^{2(n-k)}}\right)x^{k+1}\\&=\sum_{k=0}^{n-1}(-1)^{n-k+1}\begin{bmatrix}2n-k-1\\k\end{bmatrix}_q q^{(n-k)^2-k}x^{k+1}\\&=\sum_{k=1}^{n}(-1)^{n-k}\begin{bmatrix}2n-k\\k-1\end{bmatrix}_q q^{(n-k+1)^2-k+1}x^k\end{aligned}$$

and

$$\begin{aligned}U_n(x)&=\sum_{k=0}^{n}(-1)^{n-k}\left(\begin{bmatrix}2n-k\\k-1\end{bmatrix}_q q^{(n-k+1)^2}+\begin{bmatrix}2n-k\\k\end{bmatrix}_q q^{(n-k)^2}\right)q^{-k+1}x^k\\&=\sum_{k=0}^{n}(-1)^{n-k}\begin{bmatrix}2n-k+1\\k\end{bmatrix}_q q^{(n-k)(n-k+1)}\end{aligned}$$

$$\times\left(q^{n-2k+2}\frac{1-q^k}{1-q^{2n-k+1}}+q^{-n+1}\frac{1-q^{2(n-k)+1}}{1-q^{2n-k+1}}\right)x^k$$
$$=q^{-n+1}\sum_{k=0}^{n}(-1)^{n-k}\begin{bmatrix}2n-k+1\\k\end{bmatrix}_q q^{(n-k)(n-k+1)}x^k.$$

Hence

$$\psi(x)=\frac{q^{n-1}U_n(x)}{q^{n-1}(q;q)_\infty V_n(x)-q^{n-1}U_n(x)}=\widetilde{R}_n(x)$$

and this means that $T(\mu_{R_n})=\mu_{\widetilde{R}_n}$. In a similar way, one can prove that $T(\mu_{\widetilde{R}_n})=\mu_{R_{n+1}}$ and this completes the proof. □

An interesting question is what may happen when $n\to\infty$. In the light of Theorem 3.7, one should not expect $T^{(n)}(\mu_0)$ to converge. More likely $T^{(2n+1)}(\mu_0)$ and $T^{(2n+2)}(\mu_0)$ would converge and if so, the limit points would be fixed points of $T^{(2)}$ and possibly fit into the measures κ_s from (2.9). Since

$$S_n(x)=\sum_{k=0}^{n}(-1)^{n-k}\begin{bmatrix}2n-k\\k\end{bmatrix}_q q^{(n-k)^2}x^k=\sum_{j=0}^{n}(-1)^j\begin{bmatrix}n+j\\n-j\end{bmatrix}_q q^{j^2}x^{n-j}$$

and

$$T_n(x)=\sum_{k=0}^{n}(-1)^{n-k}\begin{bmatrix}2n-k-1\\k-1\end{bmatrix}_q q^{(n-k+1)^2-1}x^k$$
$$=\sum_{j=0}^{n}(-1)^j\begin{bmatrix}n+j-1\\n-j-1\end{bmatrix}_q q^{(j+1)^2-1}x^{n-j},$$

we see that $x^{-n}S_n(x)\to S(x)$ and $x^{-n}T_n(x)\to T(x)$ for $n\to\infty$, where

$$S(x)=\sum_{j=0}^{\infty}(-1)^j\frac{q^{j^2}}{(q;q)_{2j}}(1/x)^j\quad\text{and}$$
$$T(x)=\sum_{j=0}^{\infty}(-1)^j\frac{q^{(j+1)^2-1}}{(q;q)_{2j}}(1/x)^j.$$

Seeing that $T(x)=S(x/q^2)$, we thus find that

$$R_n(x)\to R_\infty(x)=\frac{S(x)}{(q;q)_\infty S(x/q^2)-S(x)}\quad\text{for } n\to\infty$$

and similarly

$$\widetilde{R}_n(x)\to\widetilde{R}_\infty(x)=\frac{\widetilde{S}(x)}{q(q;q)_\infty\widetilde{S}(x/q^2)-\widetilde{S}(x)}\quad\text{for } n\to\infty,$$

where

$$\widetilde{S}(x)=\sum_{j=0}^{\infty}(-1)^j\frac{q^{j(j+1)}}{(q;q)_{2j+1}}(1/x)^j.$$

Since the above convergence is uniform on compact subsets of $\mathbb{C}\setminus\{0\}$, it follows that R_∞ and $\widetilde{R}_\infty$ are Pick functions corresponding to solutions to the moment problem. In order to find the solutions μ_{R_∞} and $\mu_{\widetilde{R}_\infty}$ explicitly, we need the following result containing useful information about the supports of the measures $T^{(n)}(\mu_0)$.

Theorem 3.8. *Let $T : V \mapsto V$ denote the map given by $T(\mu) = \tau_q(qx\,d\mu(x))$. For each $n \in \mathbb{N}$, the canonical solution $T^{(n)}(\mu_0)$ is supported on the zeros of $\Phi(x/q^{n-1})$.*

Proof. The proof is by induction. Start by noting that $T(\mu_0) = \mu_\alpha$ and recall that μ_α is supported on the zeros of $\Phi(x)$. As a matter of fact, by (3.14) we have

$$\int_0^\infty \frac{1}{x-t}\,d\mu_\alpha(t) = \frac{\Psi(x)}{\Phi(x)}.$$

Suppose next that

$$\int_0^\infty \frac{1}{x-t}\,dT^{(n)}(\mu_0)(t) = \frac{\Psi_n(x)}{\Phi(x/q^{n-1})}$$

for some entire function $\Psi_n(x)$ having no common zeros with $\Phi(x/q^{n-1})$. With $\sigma = T^{(n)}(\mu_0)$, we then have

$$\begin{aligned}\int_0^\infty \frac{1}{x-t}\,dT^{(n+1)}(\mu_0)(t) &= \int_0^\infty \frac{1}{x-t}\,dT(\sigma)(t) = \int_0^\infty \frac{1}{x-qt} qt\,d\sigma(t)\\ &= -1 + \frac{x}{q}\int_0^\infty \frac{1}{x/q-t}\,d\sigma(t) = -1 + \frac{x}{q}\frac{\Psi_n(x/q)}{\Phi(x/q^n)}\\ &= \frac{\frac{x}{q}\Psi_n(x/q) - \Phi(x/q^n)}{\Phi(x/q^n)}.\end{aligned}$$

Since $\Phi(x/q^{n-1})$ and $\Psi_n(x)$ are without common zeros, neither $\Phi(x/q^n)$ and

$$\Psi_{n+1}(x) = \frac{x}{q}\Psi_n\left(\frac{x}{q}\right) - \Phi\left(\frac{x}{q^n}\right)$$

have common zeros. For if $\Psi_{n+1}(y) = \Phi(y/q^n) = 0$ for some $y > 0$, then $\Psi_n(z) = \Phi(z/q^{n-1}) = 0$ with $z = y/q$. Consequently, $T^{(n+1)}(\mu_0)$ is supported on the zeros of $\Phi(x/q^n)$ and this proves the assertion. □

Since R_∞ and $\widetilde{R}_\infty$ are meromorphic functions in $\mathbb{C}\setminus\{0\}$, the solutions μ_{R_∞} and $\mu_{\widetilde{R}_\infty}$ are discrete and supported on the zeros of

$$B(x)R_\infty(x) - D(x) \quad \text{and} \quad B(x)\widetilde{R}_\infty(x) - D(x),$$

respectively. Being a discrete fixed point of $T^{(2)}$ means that $c > 0$ is a mass point of, say μ, exactly if cq^2 likewise is a mass point of μ and $\mu(\{cq^2\}) = q^3c^2\mu(\{c\})$. Recalling that the zeros of Φ are very well separated, Theorem 3.7 implies that if c and c' belong to the support of μ_{R_∞} (or $\mu_{\widetilde{R}_\infty}$) and $c > c'$, then $c/c' \geqslant q^{-2}$. Consequently, the supports of μ_{R_∞} and $\mu_{\widetilde{R}_\infty}$ have the form

$$\{cq^{2n} \mid n \in \mathbb{Z}\} \quad \text{and} \quad \{\tilde{c}q^{2n} \mid n \in \mathbb{Z}\}$$

for some $c, \tilde{c} > 0$. It is a natural conclusion that there may be a connection with the measures κ_{-1} and κ_1. To show that $c = q$, it suffices to prove that

$$B(q)R_\infty(q) - D(q) = 0 \tag{3.19}$$

and multiplying with $S(q) - (q;q)_\infty S(1/q) \neq 0$, it comes to prove that

$$\begin{aligned} 0 &= D(q)\big((q;q)_\infty S(1/q) - S(q)\big) - B(q)S(q) \\ &= q\Phi\big(q^2\big)S(1/q) - D(q)S(q) + \big(\Phi(1) + D(q)\big)S(q) \\ &= \Phi(1)S(q) + q\Phi\big(q^2\big)S(1/q). \end{aligned}$$

At this point, the identity

$$\begin{aligned} \big(-aq;q^2\big)_\infty \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-aq,q^2;q^2)_n} a^n &= \big(-aq^2;q^2\big)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-aq^2,q^2;q^2)_n} a^n \\ &= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} a^n \end{aligned} \tag{3.20}$$

due to Rogers [19] becomes useful. See also [2]. With $a = -1$ and $a = -1/q$ in (3.20), we get

$$\Phi(1) = \big(q;q^2\big)_\infty S(1/q) \quad \text{and} \quad \Phi(1/q) = \big(q;q^2\big)_\infty S(q)$$

which means that

$$\begin{aligned} \Phi(1)S(q) + q\Phi\big(q^2\big)S(1/q) &= S(1/q)\big(\big(q;q^2\big)_\infty S(q) + q\Phi\big(q^2\big)\big) \\ &= S(1/q)\big(\Phi(1/q) + q\Phi\big(q^2\big)\big). \end{aligned}$$

According to (3.15), we have

$$\Phi(1/q) + q\Phi\big(q^2\big) = \Phi(1/q) + \Phi(q) - \Phi(1) = 0$$

and this proves (3.19). Consequently, μ_{R_∞} is supported on $\{q^{2n+1} \mid n \in \mathbb{Z}\}$ and being a fixed point of $T^{(2)}$, it must coincide with κ_{-1}. In a similar way, we can prove that

$$B(1)\widetilde{R}_\infty(1) - D(1) = 0 \tag{3.21}$$

which implies that $\tilde{c} = 1$ and $\mu_{\widetilde{R}_\infty} = \kappa_1$. To sum up, we have established the following result.

Theorem 3.9. *Let R_∞ and $\widetilde{R}_\infty$ denote the Pick functions*

$$R_\infty(x) = \frac{\sum_{j=0}^{\infty}(-1)^j \frac{q^{j^2}}{(q;q)_{2j}}(1/x)^j}{(q;q)_\infty \sum_{j=0}^{\infty}(-1)^j \frac{q^{j(j+2)}}{(q;q)_{2j}}(1/x)^j - \sum_{j=0}^{\infty}(-1)^j \frac{q^{j^2}}{(q;q)_{2j}}(1/x)^j}$$

and

$$\widetilde{R}_\infty(x)=\frac{\sum_{j=0}^{\infty}(-1)^j\frac{q^{j(j+1)}}{(q;q)_{2j+1}}(1/x)^j}{q(q;q)_\infty\sum_{j=0}^{\infty}(-1)^j\frac{q^{j(j+3)}}{(q;q)_{2j+1}}(1/x)^j-\sum_{j=0}^{\infty}(-1)^j\frac{q^{j(j+1)}}{(q;q)_{2j+1}}(1/x)^j}.$$

The measures μ_{R_∞} and $\mu_{\widetilde{R}_\infty}$ are explicitly given by

$$\mu_{R_\infty}=\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}q^{\binom{2n+2}{2}}\varepsilon_{q^{2n+1}}$$

and

$$\mu_{\widetilde{R}_\infty}=\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}q^{\binom{2n+1}{2}}\varepsilon_{q^{2n}}.$$

Theorem 3.9 really brings the Nevanlinna parametrization into focus. As we have seen, finding the N-extremal solutions explicitly is out of reach and it is hardly possible to find the Pick functions corresponding to, for instance, the solutions v_c in (2.6). But for κ_{-1} and κ_1 we can determine the corresponding Pick function explicitly.

As a corollary, we can say somewhat about the asymptotic behaviour of the very well separated zeros of Φ.

Corollary 3.10. *Let $0<x_1<\cdots<x_m<x_{m+1}<\cdots$ denote the zeros of Φ. When $m\to\infty$, we have $x_{m+1}/x_m\to q^{-2}$.*

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A MOMENT PROBLEM AND A FAMILY OF INTEGRAL EVALUATIONS

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This paper is dedicated to Olav Njåstad on the occasion of his seventieth birthday.

Abstract. We study the Al-Salam-Chihara polynomials when $q > 1$. Several solutions of the associated moment problem are found and the orthogonality relations lead to explicit evaluations of several integrals. The polynomials are shown to have raising and lowering operators and a second order operator equation of Sturm-Liouville type whose eigenvalues are found explicitly. We also derive new measures with respect to which the Ismail-Masson system of rational functions is biorthogonal. An integral representation of the right inverse of a divided difference operator is obtained.

1. Introduction

In this work we shall follow the notation of Gasper and Rahman [10] or Andrews, Askey, and Roy [3] for basic hypergeometric series and use the theory of the moment problem as described in Akhiezer [1]. Other useful references are [19] and [21]. A modern treatment is in the interesting article by Simon [20].

The best example of an indeterminate moment problem on the real line is the moment problem studied by Ismail and Masson in [13]. The corresponding orthogonal polynomials, usually denoted $h_n(x|q)$, are called the q^{-1}-Hermite polynomials and satisfy the three-term recurrence relation

$$2xh_n(x|q) = h_{n+1}(x|q) + q^{-n}\,(1 - q^n)\,h_{n-1}(x|q), \quad n \geq 0 \tag{1.1}$$

with initial conditions $h_{-1} = 0$ and $h_0 = 1$.

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Askey [2] was the first to give an explicit weight function for the polynomials $h_n(x|q)$. Using the Askey–Roy q-beta integral [5]:

$$\int_0^\infty t^{c-1}\frac{(-at,-bq/t;q)_\infty}{(-t,-q/t;q)_\infty}\,dt=\frac{\left(ab,q^c,q^{1-c};q\right)_\infty}{\left(q,aq^{-c},bq^c;q\right)_\infty}\frac{\pi}{\sin\pi c},\tag{1.2}$$

$$\left(c>0,|a|<q^c,|b|<q^{-c}\right)$$

he proved that

$$\int_{-\infty}^{\infty}\frac{h_m(\sinh y|q)h_n(\sinh y|q)}{\left(-qe^{2y},-qe^{-2y};q\right)_\infty}dy=\log q^{-1}(q;q)_\infty(q;q)_n q^{-\binom{n+1}{2}}\delta_{m,n}.\tag{1.3}$$

Whenever y occurs we shall always assume that

$$x=\sinh y.$$

In 1994 Ismail and Masson [13] considered the q^{-1}-Hermite polynomials in details. They established the generating function

$$\sum_{n=0}^\infty\frac{q^{\binom{n}{2}}}{(q;q)_n}h_n(x|q)t^n=\left(-te^y,te^{-y};q\right)_\infty,\quad t\in\mathbb{C}\tag{1.4}$$

as well as the the Poisson kernel

$$\begin{aligned}&\sum_{n=0}^\infty\frac{q^{\binom{n}{2}}}{(q;q)_n}h_n(x|q)h_n\left(x'|q\right)t^n\\&=\frac{\left(-te^{y+y'},-te^{-y-y'},te^{y-y'},te^{-y+y'};q\right)_\infty}{(t^2/q;q)_\infty},\quad |t|<\sqrt{q}.\end{aligned}\tag{1.5}$$

Moreover they proved that

$$\int_{-\infty}^{\infty}\prod_{j=1}^4\left(-t_je^y,t_je^{-y};q\right)_\infty d\psi(x)=\frac{\prod_{1\le j<k\le 4}\left(-t_jt_k/q;q\right)_\infty}{\left(t_1t_2t_3t_4/q^3;q\right)_\infty},\tag{1.6}$$

$$|t_1t_3|,|t_2t_4|<q^{3/2}$$

whenever ψ is a solution to the moment problem. Since the integrand is the product of four generating functions for $\{h_n(x|q)\}$, the integral in (1.6) now plays the role played by the Askey–Wilson integral in the study of the continuous q-Hermite polynomials.

The Nevanlinna matrix was also computed in [13] and it is remarkable that all the N-extremal solutions were found explicitly. They have the form

$$\nu_a=\frac{1}{\left(-a^2,-q/a^2,q;q\right)_\infty}\sum_{n=-\infty}^\infty a^{4n}\left(1+a^2q^{2n}\right)q^{n(2n-1)}\varepsilon_{x_n(a)},\quad q<a\le 1\tag{1.7}$$

where

$$x_n(a)=\frac{1}{2}\left(\frac{1}{aq^n}-aq^n\right)$$

and ε_x denotes the measure having only a unit mass at the point x. In addition, the absolutely continuous solutions with densities

$$w(x;a)=\frac{a}{\pi i}\frac{\left(-a\overline{a},-q/a\overline{a},\overline{a}/a,qa/\overline{a},q;q\right)_\infty}{|(ae^y,-ae^{-y},-qe^y/a,qe^{-y}/a;q)_\infty|^2},\quad x=\sinh y\in\mathbb{R}\tag{1.8}$$

were derived along with more complicated solutions. To begin with, the parameter a in (1.8) belongs to the set

$$\{re^{i\theta} \mid r > 0,\ 0 < \theta < \pi/2\} \cup \{re^{i\theta} \mid 0 < r \leq 1,\ \theta = \pi/2\}$$

but since $w(x; aq) = w(x; a)$, it suffices to consider

$$a \in \{re^{i\theta} \mid q < r \leq 1,\ 0 < \theta \leq \pi/2\}.$$

We stress that no value of a gives the Askey weight function appearing in (1.3).

In the Nevanlinna parametrization, which gives a one-to-one correspondence between the set of Pick functions (including ∞) and the set of solutions to an indeterminate moment problem on the real line, the N-extremal solutions correspond to the Pick function being a real constant (or ∞). When the Pick function is a complex constant (in the open upper half-plane), the corresponding solutions are known to be absolutely continuous, see [6] and [13]. The solutions in (1.8) are exactly of that kind.

The continuous q-Hermite polynomials belong to the Askey-scheme as a special case of the Askey–Wilson polynomials when all four parameters are zero, see [17]. Halfway between Askey–Wilson and continuous q-Hermite we find the Al-Salam–Chihara polynomials with two free parameters. These polynomials are studied in [4] and when $q > 1$, the associated moment problem is determinate or indeterminate depending on the parameters. In the indeterminate case the Nevanlinna matrix was computed in [9] but no explicit solutions are derived. In this paper we restrict ourselves to consider the symmetric case of the Al-Salam–Chihara polynomials when $q > 1$. The analysis in the symmetric case simplifies a great deal and a simple weight function for the polynomials can be found directly from the Nevanlinna matrix.

The paper is organized as follows. In Section 2 we present the Al-Salam–Chihara polynomials and consider the symmetric case when $q > 1$. For convenience, we set $p = 1/q$ and besides p there is only one parameter left. This parameter will be called β and we point out how two special values, namely $\beta = 0$ and $\beta = 1/p$, lead to the polynomials $\{h_n(x|q)\}$. In Section 3 we give an explicit expression for the Nevanlinna matrix based on the results in [9]. On one hand the Nevanlinna matrix remains too complicated to give us the N-extremal solutions as will be explained in Section 4. On the other hand, the Nevanlinna matrix is simple enough to lead to an explicit weight function, the function $v(x; \beta)$ in (5.1), and the corresponding orthogonality relation is given in Section 5. In the same section we derive a new family of absolutely continuous solutions to the q^{-1}-Hermite moment problem.

For $x \in \mathbb{R}$, we use the parameterization $x = \sinh y$. When f is a function defined on $\mathbb{R}$, one can think of $f(x)$ as a function of e^y. We denote by $\breve{f}$ the function

$$\breve{f}(e^y) = f(x)$$

and the divided difference operator $\mathcal{D}_q$ given by

$$(\mathcal{D}_q f)(x) = \frac{\breve{f}(q^{1/2}e^y) - \breve{f}(q^{-1/2}e^y)}{(q^{1/2} - q^{-1/2})\cosh y} \tag{1.9}$$

was introduced by Ismail in [11]. Notice that if we set $e(x) = x$ then the denominator can also be written as

$$\breve{e}(q^{1/2}e^y) - \breve{e}(q^{-1/2}e^y).$$

It was proved in [11] that $\mathcal{D}_q$ is a lowering operator for the q^{-1}-Hermite polynomials.

We find a family of weight functions that lead to the same raising and lowering operators for the polynomials $\{h_n(x|q)\}$. Combining the lowering and raising operators one can obtain a q-Sturm–Liouville equation from which the orthogonality follows using Ismail's q-analogue of integration by parts [11]. Besides the Askey weight function (in (1.3)), the family also contains the special case $\beta = 1/p$ of the weight function $v(x;\beta)$.

In Section 6 we construct a family of discrete solutions from the weight function $v(x;\beta)$. These solutions are not N-extremal though they are supported on the same p-quadratic grid as the N-extremal solutions to the q^{-1}-Hermite moment problem. A possible way to verify this is through the Poisson kernel which we shall derive from a bilinear generating function established in Section 7.

One way of reaching the Al-Salam–Chihara polynomials when $q > 1$ is to start out with the q^{-1}-Hermite polynomials and use a simple procedure of attaching generating functions to measures. This procedure is explained in [7] and the next step takes us to the biorthogonal rational functions with four parameters studied by Ismail and Masson in [13]. In Section 8 we show how solutions to the moment problem lead to biorthogonality relations for the rational functions when certain restrictions on the last to parameters are fulfilled.

In Section 9 we obtain a p-Sturm–Liouville equation from lowering and raising operators. The weight function $v(x;\beta)$ appears in the raising operator but the p-Sturm–Liouville equation can be written in a form independent of $v(x;\beta)$. We use this form to derive a system of n nonlinear equations satisfied by the zeros of the polynomials. This is a typical example of Bethe Ansatz equations, see the Bethe Ansatz for the XXZ model in [18] and [12]. In Section 10 we consider the divided difference operator $\mathcal{D}_q$ as a bounded operator on the L^2-spaces of the N-extremal solutions. The right inverse $\mathcal{D}_q^{-1}$ is identified as an integral operator and we find the kernel explicitly. This is the $q > 1$ version of a result in [8]. The kernel for the inverse of the Askey-Wilson operator over the L^2-space weighted by the Askey-Wilson weight function is in [14].

2. The Al-Salam–Chihara polynomials

The Al-Salam–Chihara polynomials $Q_n(x) := Q_n(x;a,b|q)$ are defined by the three-term recurrence relation

$$2xQ_n(x) = Q_{n+1}(x) + (a+b)q^nQ_n(x) + (1-q^n)\left(1-abq^{n-1}\right)Q_{n-1}(x), \quad n \geq 0$$

with initial conditions $Q_{-1} = 0$ and $Q_0 = 1$, see for example [17]. These polynomials are orthogonal with respect to a positive measure (with bounded support) on $\mathbb{R}$ if $a+b \in \mathbb{R}$, $ab < 1$ and $0 < q < 1$. In the case $q > 1$, the polynomials are orthogonal on the imaginary axis (for suitable values of a and b) so we replace x by ix in order to obtain orthogonality on the real line. Indeed, with $p = 1/q$, the polynomials

$$\widetilde{Q}_n(x) = \frac{i^n p^{\binom{n}{2}}}{(p;p)_n}\, Q_n(ix/2;a,b|p) \tag{2.1}$$

satisfy the three-term recurrence relation

$$\left(1-p^{n+1}\right)\widetilde{Q}_{n+1}(x) = \left(-i(a+b) - xp^n\right)\widetilde{Q}_n(x) - \left(-ab+p^{n-1}\right)\widetilde{Q}_{n-1}(x)$$

and are therefore orthogonal with respect to a positive measure on $\mathbb{R}$ when $a+b \in i\mathbb{R}$, $ab \leq 0$ and $0 < p < 1$. The polynomials in (2.1) are a special case of the polynomials $v_n(x)$ studied in [4] and [9]. The parametrization, however, is slightly different and we have to identify a and b from [4] and [9] with $-i(a+b)$ and $-ab$, respectively. The parameter c is set to be -1 here.

In this paper we will study the special situation where $a=-b=\sqrt{\beta}$ for some $\beta \geq 0$. The motivation is simply to obtain symmetry. It is convenient to replace x by $-2x$ so our starting point is the polynomials $Q_n(x;\beta)$ generated by the three-term recurrence relation

$$2xp^n Q_n(x;\beta) = \left(1-p^{n+1}\right) Q_{n+1}(x;\beta) + \left(\beta + p^{n-1}\right) Q_{n-1}(x;\beta), \quad n \geq 0 \tag{2.2}$$

with initial conditions $Q_{-1}=0$ and $Q_0=1$. In accordance with [7] (set $t_1=-t_2=iq\sqrt{\beta}$ and replace q by p), the polynomials $Q_n(x;\beta)$ are explicitly given by

$$\begin{aligned} Q_n(x;\beta) &= \frac{\left(-ie^{-y}/\sqrt{\beta};p\right)_n}{(p;p)_n} \left(i\sqrt{\beta}\right)^n {}_2\phi_1\left(\begin{matrix} p^{-n}, -ie^{y}/\sqrt{\beta} \\ i\sqrt{\beta}e^{y}/p^{n-1} \end{matrix} \,\middle|\, p, -ip\sqrt{\beta}e^{y}\right) \\ &= \frac{(-1/\beta;p)_n}{(p;p)_n} \left(i\sqrt{\beta}\right)^n \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_p \frac{\left(ie^{-y}/\sqrt{\beta}, -ie^{y}/\sqrt{\beta};p\right)_k}{(-1/\beta;p)_k}, \end{aligned}$$

where $x=\sinh y$. Since the polynomials

$$h_n(x;\beta) = \frac{(p;p)_n}{p^{\binom{n}{2}}} Q_n(x;\beta)$$

satisfy the three-term recurrence relation

$$2xh_n(x;\beta) = h_{n+1}(x;\beta) + \left(p^{-2n+1}\beta + p^{-n}\right)(1-p^n)\, h_{n-1}(x;\beta),$$

we immediately see that the special case $\beta=0$ of our polynomials is $\{h_n(x|p)\}$. Furthermore, we observe that the special case $\beta=1/p$ corresponds to the polynomials $\left\{h_n(x|p^2)\right\}$. Throughout the paper we shall always try to have these special cases in mind. Certainly, this will throw more light on the q^{-1}-Hermite polynomials as well.

3. The Nevanlinna matrix

According to Theorem 3.2 in [4], the moment problem associated with the polynomials $Q_n(x;\beta)$ is indeterminate for $\beta \geq 0$. The entire functions from the Nevanlinna matrix were computed in [9]. The first step in the computation was to establish the generating functions

$$\sum_{n=0}^{\infty} Q_n(x;\beta)t^n = \frac{(te^{-y}, -te^{y};p)_\infty}{(-t^2\beta;p^2)_\infty}, \quad |t| < 1/\sqrt{\beta} \tag{3.1}$$

and

$$\begin{aligned} \sum_{n=0}^{\infty} Q_n^*(x;\beta)t^n &= \frac{2pt}{1+t^2\beta}\, {}_3\phi_2\left(\begin{matrix} te^{-y}, -te^{y}, p \\ ipt\sqrt{\beta}, -ipt\sqrt{\beta} \end{matrix} \,\middle|\, p, p\right) \\ &= 2t\sum_{n=0}^{\infty} \frac{(te^{-y}, -te^{y};p)_n}{(-t^2\beta;p^2)_{n+1}} p^{n+1}, \quad |t| < 1/\sqrt{\beta}, \end{aligned} \tag{3.2}$$

where $Q_n^*(x;\beta)$ denotes the numerator polynomials, that is, the polynomials generated by the three-term recurrence relation (2.2) with initial conditions $Q_0^*=0$ and $Q_1^*=2p/(1-p)$. Darboux's method was then applied to find the asymptotic behavior of Q_n and Q_n^* as $n \to \infty$. In our case, the expressions for the functions A, B, C, and D reduce to

$$\begin{aligned}
A(x;\beta) &= -\frac{2}{\sqrt{\beta}} \frac{(p;p)_\infty}{(-1/\beta;p)_\infty} S(0)S(x)\sin\left(\eta(x)-\eta(0)\right), \\
B(x;\beta) &= -\frac{(p;p)_\infty}{(-1/\beta;p)_\infty} S(0)R(x)\cos\left(\zeta(x)-\eta(0)\right), \\
C(x;\beta) &= \frac{(p;p)_\infty}{(-1/\beta;p)_\infty} R(0)S(x)\cos\left(\eta(x)-\zeta(0)\right), \\
D(x;\beta) &= -\frac{\sqrt{\beta}}{2} \frac{(p;p)_\infty}{(-1/\beta;p)_\infty} R(0)R(x)\sin\left(\zeta(x)-\zeta(0)\right),
\end{aligned}$$

where

$$R(x)e^{i\zeta(x)} = \frac{\left(ie^{-y}/\sqrt{\beta}, -ie^{y}/\sqrt{\beta};p\right)_\infty}{(p^2;p^2)_\infty}$$

and

$$\begin{aligned}
S(x)e^{i\eta(x)} &= {}_2\phi_1\left(\begin{matrix} ie^{-y}/\sqrt{\beta}, -ie^{y}/\sqrt{\beta} \\ -p \end{matrix} \,\middle|\, p, p\right) \\
&= \sum_{n=0}^{\infty} \frac{\left(ie^{-y}/\sqrt{\beta}, -ie^{y}/\sqrt{\beta};p\right)_n}{(p^2;p^2)_n} p^n
\end{aligned}$$

for $x \in \mathbb{R}$. It is assumed that $\zeta(x), \eta(x) \in \mathbb{R}$ and $R(x), S(x) > 0$. In particular, we have

$$R(0) = \frac{(-1/\beta;p^2)_\infty}{(p^2;p^2)_\infty}, \quad S(0) = \sum_{n=0}^{\infty} \frac{\left(-1/\beta;p^2\right)_n}{\left(p^2;p^2\right)_n} p^n = \frac{\left(-p/\beta;p^2\right)_\infty}{\left(p;p^2\right)_\infty}$$

and

$$\zeta(0) = \eta(0) = 0.$$

So the expressions reduce further to the more convenient forms

$$\begin{aligned}
A(x;\beta) &= -\frac{2}{\sqrt{\beta}} \frac{(p^2;p^2)_\infty}{(-1/\beta;p^2)_\infty} S(x)\sin(\eta(x)), \\
B(x;\beta) &= -\frac{(p^2;p^2)_\infty}{(-1/\beta;p^2)_\infty} R(x)\cos(\zeta(x)), \\
C(x;\beta) &= \frac{(p;p^2)_\infty}{(-p/\beta;p^2)_\infty} S(x)\cos(\eta(x)), \\
D(x;\beta) &= -\frac{\sqrt{\beta}}{2} \frac{(p;p^2)_\infty}{(-p/\beta;p^2)_\infty} R(x)\sin(\zeta(x)).
\end{aligned}$$

Hence, the Stieltjes transform of the solution μ_φ corresponding to the Pick function φ in the Nevanlinna parametrization is given by

$$\begin{aligned}
(3.3) \qquad & \int_{\mathbb{R}} \frac{1}{t-x} d\mu_\varphi(t) \\
&= \frac{S(x)}{R(x)} \frac{4\left(-p/\beta, p^2;p^2\right)_\infty \sin(\eta(x))\varphi(x) + 2\sqrt{\beta}\left(-1/\beta, p;p^2\right)_\infty \cos(\eta(x))}{\beta\left(-1/\beta, p;p^2\right)_\infty \sin(\zeta(x)) - 2\sqrt{\beta}\left(-p/\beta, p^2;p^2\right)_\infty \cos(\zeta(x))\varphi(x)}.
\end{aligned}$$

4. N-extremal solutions

In the search for the N-extremal solutions $\mu_t^{(\beta)}$, $t \in \mathbb{R} \cup \{\infty\}$ it is convenient to write the parameter t as

$$t = \frac{D(u;\beta)}{B(u;\beta)}$$

for u belonging to, say, the interval $(-x_1, x_1]$, where x_1 is the smallest positive zero of B. With this parametrization, the Stieltjes transform in (3.3) takes the form

$$\begin{aligned}\int_{\mathbb{R}} \frac{1}{t-x} d\nu_u^{(\beta)}(t) &= \frac{2}{\sqrt{\beta}} \frac{S(x)}{R(x)} \frac{\sin(\eta(x))\sin(\zeta(u)) + \cos(\eta(x))\cos(\zeta(u))}{\sin(\zeta(x))\cos(\zeta(u)) + \cos(\zeta(x))\sin(\zeta(u))} \\ &= \frac{2}{\sqrt{\beta}} \frac{S(x)}{R(x)} \frac{\cos(\eta(x)-\zeta(u))}{\sin(\zeta(x)+\zeta(u))}\end{aligned}$$

so the N-extremal solution $\nu_u^{(\beta)}$ is supported on the set of real x's for which

$$\zeta(x) + \zeta(u) \in \pi\mathbb{Z}.$$

In other words, if we set $u = \sinh v$ then $x = \sinh y$ belongs to the support of $\nu_u^{(\beta)}$ if and only if

$$\left(ie^{-y}/\sqrt{\beta}, ie^{-v}/\sqrt{\beta}, -ie^{y}/\sqrt{\beta}, -ie^{v}/\sqrt{\beta}; p\right)_\infty \in \mathbb{R}.$$

However, it seems impossible to solve the above equations explicitly. In the special case $u = 0$, for instance, we have to know exactly when

$$\left(ie^{-y}/\sqrt{\beta}, -ie^{y}/\sqrt{\beta}; p\right)_\infty \in \mathbb{R}$$

and even for $\beta = 1$, this comes to find the values of $t \in i\mathbb{R}_+$ for which

$$\operatorname{Im}\left((t, 1/t; p)_\infty\right) = 0.$$

5. Absolutely continuous solutions

For one particular Pick function we are able to find the corresponding solution explicitly. Observe that

$$R^2(x) = \frac{\left(-e^{2y}/\beta, -e^{-2y}/\beta; p^2\right)_\infty}{(p^2;p^2)_\infty^2}$$

so that B^2 and D^2 can be written as

$$B^2(x;\beta) = \frac{1}{(-1/\beta;p^2)_\infty^2} \left(-e^{2y}/\beta, -e^{-2y}/\beta; p^2\right)_\infty \cos^2(\zeta(x)).$$

and

$$D^2(x;\beta) = \frac{\beta}{4} \frac{\left(p;p^2\right)_\infty^2}{(-p/\beta, p^2; p^2)_\infty^2} \left(-e^{2y}/\beta, -e^{-2y}/\beta; p^2\right)_\infty \sin^2(\zeta(x)).$$

For the particular choice

$$\gamma = \frac{\sqrt{\beta}}{2} \frac{\left(-1/\beta, p; p^2\right)_\infty}{\left(-p/\beta, p^2; p^2\right)_\infty},$$

the absolutely continuous solution $\mu_{i\gamma}$ with density

$$\frac{\gamma/\pi}{D(x;\beta)^2 + \gamma^2 B(x;\beta)^2}, \quad x \in \mathbb{R}$$

has the form $d\mu_{i\gamma} = v(x;\beta)\,dx$, where

$$v(x;\beta) = \frac{(p^2;p^2)_\infty}{(p;p^2)_\infty}\frac{(-1/\beta;p)_\infty}{\pi\sqrt{\beta}}\frac{2}{(-e^{2y}/\beta,-e^{-2y}/\beta;p^2)_\infty}. \tag{5.1}$$

We state this result as a theorem.

Theorem 5.1. *The polynomials $Q_n(x;\beta)$ are orthogonal with respect to the weight function*

$$\frac{1}{(-e^{2y}/\beta,-e^{-2y}/\beta;p^2)_\infty}, \qquad x = \sinh y \in \mathbb{R}$$

and the orthogonality relation is

$$2\int_{\mathbb{R}} \frac{Q_n(\sinh y;\beta)Q_m(\sinh y;\beta)}{(-e^{2y}/\beta,-e^{-2y}/\beta;p^2)_\infty}\cosh y\,dy$$
$$= \frac{\pi\sqrt{\beta}}{(-1/\beta;p)_\infty}\frac{(p;p^2)_\infty}{(p^2;p^2)_\infty}\frac{\beta^n(-1/\beta;p)_n}{p^n(p;p)_n}\delta_{n,m}.$$

Proof. We only have to check that the orthogonality relation is correct. It follows from the three-term recurrence relation (2.2) that the polynomials

$$P_n(x;\beta) = \sqrt{\frac{p^n(p;p)_n}{\beta^n(-1/\beta;p)_n}}\,Q_n(x;\beta) \tag{5.2}$$

are orthonormal. As a matter of fact, they satisfy the three-term recurrence relation

$$xP_n(x;\beta) = \frac{\sqrt{(1-p^{n+1})(\beta+p^n)}}{2p^{n+1/2}}P_{n+1}(x;\beta) + \frac{\sqrt{(1-p^n)(\beta+p^{n-1})}}{2p^{n-1/2}}P_{n-1}(x;\beta)$$

with initial conditions

$$P_0(x;\beta) = 1 \quad \text{and} \quad P_1(x;\beta) = \frac{2x\sqrt{p}}{\sqrt{(1-p)(1+\beta)}}.$$

Therefore, we have the orthogonality relation

$$\int_{\mathbb{R}} P_n(x;\beta)P_m(x;\beta)v(x;\beta)\,dx = \delta_{nm}$$

or, equivalently,

$$\int_{\mathbb{R}} Q_n(x;\beta)Q_m(x;\beta)v(x;\beta)\,dx = \frac{\beta^n(-1/\beta;p)_n}{p^n(p;p)_n}\delta_{nm}$$

and the result follows immediately. □

The special case $\beta = 1/p$ of (5.1) leads directly to an absolutely continuous solution to the q^{-1}-Hermite moment problem. Replace p^2 by q to obtain the density

$$w(x) = q^{1/4}\frac{1}{\pi}\frac{(q;q)_\infty}{(\sqrt{q};q)_\infty^2}\frac{2}{\left(-\sqrt{q}e^{2y},-\sqrt{q}e^{-2y};q\right)_\infty}. \tag{5.3}$$

Moreover, if we set $\beta = cp^{2n}$ for fixed $c \in (0,1]$ and let $n \to \infty$, we obtain the densities

$$w_c(x) = \frac{1}{\pi\sqrt{c}}\frac{\left(q^2;q^2\right)_\infty}{\left(q;q^2\right)_\infty}\frac{2(-1/c,-cq;q)_\infty}{(-e^{2y}/c,-ce^{2y}q^2,-e^{-2y}/c,-ce^{-2y}q^2;q^2)_\infty} \tag{5.4}$$

after replacing p by q. To see this, notice that

$$\begin{aligned}
&\frac{1}{\sqrt{cp^{2n}}}\frac{(-1/cp^{2n};p)_\infty}{(-e^{2y}/cp^{2n},-e^{-2y}/cp^{2n};p^2)_\infty}\\
&=\frac{1}{\sqrt{c}}\frac{(-1/c;p)_\infty}{(-e^{2y}/c,-e^{-2y}/c;p^2)_\infty}\frac{\left(-1/cp^{2n};p\right)_{2n}}{p^n\left(-e^{2y}/cp^{2n},-e^{-2y}/cp^{2n};p^2\right)_n}\\
&=\frac{1}{\sqrt{c}}\frac{(-1/c;p)_\infty}{(-e^{2y}/c,-e^{-2y}/c;p^2)_\infty}\frac{(-cp;p)_{2n}}{\left(-ce^{2y}p^2,-ce^{-2y}p^2;p^2\right)_n}\\
&\to\frac{1}{\sqrt{c}}\frac{(-1/c,-cp;p)_\infty}{(-e^{2y}/c,-e^{-2y}/c,-ce^{2y}p^2,-ce^{-2y}p^2;p^2)_\infty}
\end{aligned}$$

as $n\to\infty$. By construction, we have $w_{cq^2}(x)=w_c(x)$ (or $w_{cq}(x)=w_{c/q}(x)$) and since

$$a(-1/a,-aq;q)_\infty=(-a,-q/a;q)_\infty \quad \text{for} \quad a\neq 0,$$

we also have $w_{1/c}(x)=w_c(x)$. Therefore, it suffices to consider the case $q\le c\le 1$. Note that $w_c(x)$ reduces to $w(x)$ when $c=\sqrt{q}$ (or $1/\sqrt{q}$).

The probability densities $w_c(x)$ are not new solutions. They are special cases of the densities $w(x;a)$ in (1.8). Set $a=iq^{1/4}$ to obtain the density $w(x)$ and note that the densities $w_c(x)$ exactly correspond to $w(x;a)$ when $a=i\gamma$ with $q<\gamma\le 1$.

The orthogonality relation in (1.3) contains the probability density

$$\tilde{w}(x)=\frac{1}{\log q^{-1}(q;q)_\infty}\frac{2e^y}{(-e^{2y},-qe^{-2y};q)_\infty} \tag{5.5}$$

and the similarity to $w(x)$ in (5.3) is striking. It turns out that

$$h_{n+1}(x|q)=-\frac{1-q}{2q^{1+n/2}f(x)}\mathcal{D}_q\left(f(x)h_n(x|q)\right),\quad n\ge 0 \tag{5.6}$$

for $f=w$ as well as $f=\tilde{w}$, cf. Theorem 5.2. So both $w(x)$ and $\tilde{w}(x)$ give rise to a raising operator for the q^{-1}-Hermite polynomials. With respect to the inner product

$$\langle f,g\rangle=\int_{\mathbb{R}} f(x)\overline{g(x)}\frac{dx}{\sqrt{1+x^2}}$$

on $L^2\left(\mathbb{R},1/\sqrt{1+x^2}\right)$, the following rule for integration by parts applies

$$\langle\mathcal{D}_q f,g\rangle=-\left\langle f,\sqrt{1+x^2}\,\mathcal{D}_q\left(g(x)/\sqrt{1+x^2}\right)\right\rangle, \tag{5.7}$$

see [11] for details. Combining the raising operator in (5.6) with the lowering operator

$$\mathcal{D}_q h_n(x|q)=2q^{(1-n)/2}\frac{1-q^n}{1-q}h_{n-1}(x|q),\quad n\ge 0, \tag{5.8}$$

which can be obtained from the generating function in (1.4), we are led to the q-Sturm–Liouville equation

$$\mathcal{D}_q\left(f(x)\mathcal{D}_q h_n(x|q)\right)+4q\frac{1-q^n}{(1-q)^2}f(x)h_n(x|q)=0,\quad n\ge 0. \tag{5.9}$$

Again, $f=w$ or $f=\tilde{w}$. The eigenvalues $4q(1-q^n)/(1-q)^2$ are distinct. This indicates that the operator T defined by

$$T\phi(x)=-\frac{1}{f(x)}\mathcal{D}_q\left(f(x)\mathcal{D}_q\phi(x)\right),$$

is positive on the weighted Hilbert space $L^2(\mathbb{R}, f(x))$. Indeed this follows from (5.7).

The fact that the q^{-1}-Hermite polynomials are orthogonal with respect to $\tilde{w}(x)$ and $w(x)$ can now be obtained from (5.9) just by using integration by parts as described in (5.7). For more details, the reader is referred to the proof of Theorem 2.4 in [11].

We shall now describe a more general set up.

Theorem 5.2. *Let f_c denote the function given by*

$$f_c(x) = \frac{e^{y(2c-1)}}{(-q^{1-c}e^{2y}, -q^c e^{-2y}; q)_\infty}, \qquad x = \sinh y \in \mathbb{R}.$$

For each $c \in \mathbb{R}$, we have the following raising operator for the q^{-1}-Hermite polynomials

$$h_{n+1}(x|q) = -\frac{1-q}{2q^{1+n/2} f_c(x)} \mathcal{D}_q \left(f_c(x) h_n(x|q) \right), \quad n \geq 0.$$

Proof. We have to prove that (5.6) remains valid when f is replaced by f_c. This is more or less a repetition of the proof of Theorem 2.1 in [11]. From the generating function in (1.4) a straightforward computation gives

$$\begin{aligned} &\frac{1}{f_c(x)} \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{(q;q)_n} t^n \mathcal{D}_q \left(f_c(x) h_n(x|q) \right) \\ &\quad = \frac{2q^{3/2}}{t(1-q)} \left\{ \left(-te^y\sqrt{q}, te^{-y}\sqrt{q}; q\right)_\infty - \left(-te^y/\sqrt{q}, te^{-y}/\sqrt{q}; q\right)_\infty \right\} \end{aligned}$$

or

$$\begin{aligned} &\frac{1}{f_c(x)} \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{(q;q)_n} t^{n+1} \mathcal{D}_q \left(f_c(x) h_n(x|q) \right) \\ &\quad = \frac{2q^{3/2}}{1-q} \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{(q;q)_n} h_n(x|q) t^n \left(q^{n/2} - q^{-n/2} \right) \\ &\quad = -\frac{2q^{3/2}}{1-q} \sum_{n=1}^{\infty} \frac{q^{n^2/2-n}}{(q;q)_{n-1}} h_n(x|q) t^n \\ &\quad = -\frac{2q}{1-q} \sum_{n=0}^{\infty} \frac{q^{n^2/2}}{(q;q)_n} h_{n+1}(x|q) t^{n+1}. \end{aligned}$$

Equating the coefficients of t^{n+1} now leads to the desired raising operator. $\square$

Corollary 5.3. *The absolutely continuous measure with density*

$$v_c(x) = q^{c(1-c)} \frac{\sin \pi c}{\pi} \frac{(q;q)_\infty}{(q^c, q^{1-c}; q)_\infty} \frac{2e^{y(2c-1)}}{(-q^{1-c}e^{2y}, -q^c e^{-2y}; q)_\infty}$$

is solution to the q^{-1}-Hermite moment problem.

Proof. Since v_c satisfies the q-Sturm–Liouville equation (5.9), it is only left to verify that

$$\int_{\mathbb{R}} v_c(x) dx = 1.$$

By the Askey–Roy q-beta integral (1.2), we have

$$\begin{aligned}\int_{\mathbb{R}} \frac{2e^{y(2c-1)}}{(-q^{1-c}e^{2y},-q^c e^{-2y};q)_\infty}\,dx &= \int_{\mathbb{R}} \frac{e^{2yc}+e^{2y(c-1)}}{(-q^{1-c}e^{2y},-q^c e^{-2y};q)_\infty}\,dy \\ &= \frac{1}{2}\int_0^\infty \frac{t^{c-1}+t^{c-2}}{(-q^{1-c}t,-q^c/t;q)_\infty}\,dt \\ &= q^{c(c-1)}\frac{\pi}{\sin \pi c}\frac{\left(q^c,q^{1-c};q\right)_\infty}{(q;q)_\infty}\end{aligned}$$

because the integral

$$\int_0^\infty \frac{t^{c-2}}{(-q^{1-c}t,-q^c/t;q)_\infty}\,dt = \int_0^\infty \frac{s^{-c}}{(-q^{1-c}/s,-q^c s;q)_\infty}\,ds$$

is symmetric in c and $1-c$. So v_c is indeed the density of a probability measure. □

Since $v_{c+1}=v_c$, it suffices to consider v_c for $0<c\le 1$. Besides the special cases $c=1$ and $c=1/2$, which lead to $\tilde{w}(x)$ and $w(x)$, the solutions presented in Corollary 5.3 are new. We notice that the integral in (1.6) takes the form

$$\begin{aligned}&\int_{\mathbb{R}} \frac{e^{2cy}+e^{2(c-1)y}}{(-q^{1-c}e^{2y},-q^c e^{-2y};q)_\infty}\prod_{j=1}^4\left(-t_je^y,t_je^{-y};q\right)_\infty dy \\ &\quad= q^{c(c-1)}\frac{\pi}{\sin \pi c}\frac{\left(q^c,q^{1-c};q\right)_\infty}{(q;q)_\infty}\frac{\prod_{1\le j<k\le 4}(-t_jt_k/q;q)_\infty}{(t_1t_2t_3t_4/q^3;q)_\infty}\end{aligned}\tag{5.10}$$

when $d\psi = v_c(x)dx$.

It is known that the Pick function being equal to the constant

$$i\frac{\sqrt{c}}{2}\frac{\left(-1/c,-cq^2,q;q^2\right)_\infty}{\left(-q/c,-cq,q^2;q^2\right)_\infty}$$

in the open upper half-plane corresponds to the solution with density (5.4) in the Nevanlinna parametrization. Therefore, the Pick function corresponding to the density $w(x)$ equals the constant

$$i\frac{3}{2\sqrt{2}}\frac{\left(-2q^2,-q^2/2,q;q^2\right)_\infty}{\left(-2q,-q/2,q^2;q^2\right)_\infty}$$

in the open upper half-plane. It seems to be hard to determine the Pick functions corresponding to solutions from Corollary 5.3 when $c\neq 1/2$.

6. Discrete solutions

Recall from the proof of Theorem 5.1 that the orthogonality relation has the form

$$\int_{\mathbb{R}} Q_n(x;\beta)Q_m(x;\beta)v(x;\beta)dx = \frac{\beta^n(-1/\beta;p)_n}{p^n(p;p)_n}\,\delta_{nm}.$$

According to Proposition 1.1 in [7] and the q-binomial theorem [10, II. 3], we thus have

$$\int_{\mathbb{R}} \sum_{n=0}^{\infty} Q_n(x;\beta)t_1^n \sum_{m=0}^{\infty} Q_m(x;\beta)t_2^m v(x;\beta)\,dx = \sum_{n=0}^{\infty} \frac{(-1/\beta;p)_n}{(p;p)_n}(t_1t_2\beta/p)^n = \frac{(-t_1t_2/p;p)_\infty}{(t_1t_2\beta/p;p)_\infty}, \tag{6.1}$$

whenever $|t_1|,|t_2| < \sqrt{p/\beta}$. In view of (3.1) and since the right-hand side of (6.1) only depends on t_1t_2, a positive measure μ is solution to the moment problem if and only if

$$\begin{aligned}&\int_{\mathbb{R}} \left(t_1e^{-y}, -t_1e^{y}, t_2e^{-y}, -t_2e^{y};p\right)_\infty d\mu(x)\\ &\quad= \frac{(-t_1t_2/p;p)_\infty}{(t_1t_2\beta/p;p)_\infty}\left(-t_1^2\beta, -t_2^2\beta;p^2\right)_\infty, \quad |t_1t_2\beta/p| < 1.\end{aligned} \tag{6.2}$$

For $a > 0$ consider the discrete measure $\lambda_a^{(\beta)}$ supported on $\{x_n(a) \mid n \in \mathbb{Z}\}$ and defined by

$$\lambda_a^{(\beta)}(\{x_n(a)\}) = \frac{\sqrt{1+x_n^2(a)}}{L(a)}\tilde{v}(x_n(a);\beta), \quad n \in \mathbb{Z},$$

where

$$x_n(a) := x_n(a;p) = \frac{1}{2}\left(\frac{1}{ap^n} - ap^n\right), \quad n \in \mathbb{Z},$$

$$\tilde{v}(x;\beta) = \frac{1}{\left(-e^{2y}/\beta, -e^{-2y}/\beta;p^2\right)_\infty}, \quad x = \sinh y \in \mathbb{R}$$

and $L(a)$ is a certain constant so that $\lambda_a^{(\beta)}$ becomes a probability measure. As we shall see below, these measures turn out to be discrete solutions to the moment problem. It is remarkable that they are constructed in the same way as one can obtain the N-extremal solutions to the q^{-1}-Hermite moment problem from the weight function in (1.3) or any other function from Theorem 5.2.

Direct computations lead to

$$\begin{aligned}&\int_{\mathbb{R}} \left(t_1e^{-y}, -t_1e^{y}, t_2e^{-y}, -t_2e^{y};p\right)_\infty d\lambda_a^{(\beta)}(x)\\ &= \frac{1}{L(a)}\sum_{n=-\infty}^{\infty} \frac{1}{2}\left(\frac{1}{ap^n} + ap^n\right)\frac{(t_1ap^n, -t_1/ap^n, t_2ap^n, -t_2/ap^n;p)_\infty}{(-1/a^2p^{2n}\beta, -a^2p^{2n}/\beta;p^2)_\infty}\\ &= \frac{1}{2L(a)}\frac{(t_1a, -t_1/a, t_2a, -t_2/a;p)_\infty}{(-1/a^2\beta, -a^2/\beta;p^2)_\infty}\\ &\quad\times \sum_{n=-\infty}^{\infty}\left(1+a^2p^{2n}\right)\frac{\left(-a^2/\beta;p^2\right)_n}{\left(-1/a^2p^{2n}\beta;p^2\right)_n}\frac{\left(-t_1/ap^n, -t_2/ap^n;p\right)_n}{\left(t_1a, t_2a;p\right)_n}\frac{1}{ap^n}\end{aligned}$$

and due to Bailey's ${}_6\psi_6$ sum [10, II. 33], we have

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty}(1+a^2p^{2n})\frac{\left(-a^2/\beta;p^2\right)_n}{\left(-1/a^2p^{2n}\beta;p^2\right)_n}\frac{(-t_1/ap^n,-t_2/ap^n;p)_n}{(t_1a,t_2a;p)_n}\frac{1}{ap^n}\\
&\quad=\frac{1+a^2}{a}\sum_{n=-\infty}^{\infty}\frac{\left(-a^2p^2,-a^2/\beta;p^2\right)_n}{\left(-a^2,-a^2p^2\beta;p^2\right)_n}\frac{(-ap/t_1,-ap/t_2;p)_n}{(t_1a,t_2a;p)_n}(t_1t_2\beta/p)^n\\
&\quad=\frac{1+a^2}{a}\frac{\left(-a^2p,-p/a^2,-p\beta,-t_1t_2/p,p;p\right)_\infty}{(t_1a,-t_1/a,t_2a,-t_2/a,t_1t_2\beta/p;p)_\infty}\frac{\left(-t_1^2\beta,-t_2^2\beta;p^2\right)_\infty}{\left(-a^2p^2\beta,-p^2\beta/a^2;p^2\right)_\infty}
\end{aligned}$$

for $|t_1t_2\beta/p|<1$. Consequently,

$$\begin{aligned}
&\int_{\mathbb{R}}\left(t_1e^{-y},-t_1e^{y},t_2e^{-y},-t_2e^{y};p\right)_\infty d\lambda_a^{(\beta)}(x)\\
&\quad=\frac{1+a^2}{2aL(a)}\frac{\left(-a^2p,-p/a^2,-p\beta,p;p\right)_\infty}{\left(-a^2/\beta,-1/a^2\beta,-a^2p^2\beta,-p^2\beta/a^2;p^2\right)_\infty}\\
&\quad\quad\times\frac{(-t_1t_2/\beta;p)_\infty}{(t_1t_2\beta/p;p)_\infty}\left(-t_1^2\beta,-t_2^2\beta;p^2\right)_\infty
\end{aligned}$$

and with

$$L(a)=\frac{1+a^2}{2a}\frac{\left(a^2p,-p/a^2,-p\beta,p;p\right)_\infty}{\left(-a^2/\beta,-1/a^2\beta,-a^2p^2\beta,-p^2\beta/a^2;p^2\right)_\infty},$$

the following result is obtained.

Theorem 6.1. *The discrete measures*

$$\lambda_a^{(\beta)}=\frac{\left(-a^2p^2\beta,-p^2\beta/a^2;p^2\right)_\infty}{\left(-a^2,-p/a^2,-p\beta,p;p\right)_\infty}\sum_{n=-\infty}^{\infty}\frac{\left(-a^2/\beta;p^2\right)_n}{\left(-a^2p^2\beta;p^2\right)_n}a^{2n}\beta^n\left(1+a^2p^{2n}\right)p^{n^2}\varepsilon_{x_n(a)}$$

are solutions to the moment problem

Since $\lambda_{ap}^{(\beta)}=\lambda_a^{(\beta)}$, it suffices to consider $p<a\le 1$. In the special case $\beta=0$, the measures in Theorem 6.1 reduce to

$$\nu_a=\frac{1}{\left(-a^2,-q/a^2,q;q\right)_\infty}\sum_{n=-\infty}^{\infty}a^{4n}\left(1+a^2q^{2n}\right)q^{n(2n-1)}\varepsilon_{x_n(a;q)}\tag{6.3}$$

if we replace p by q. That is, we obtain the N-extremal solutions to the q^{-1}-Hermite moment problem. So for a moment one may believe that the solutions $\lambda_a^{(\beta)}$ are N-extremal. However, the special case $\beta=1/p$ reads

$$\tilde{\nu}_a=\frac{1}{2\left(-a^2,-p^2/a^2,p^2;p^2\right)_\infty}\sum_{n=-\infty}^{\infty}a^{2n}\left(1+a^2p^{2n}\right)p^{n^2-n}\varepsilon_{x_n(a;p)}\tag{6.4}$$

and since

$$\begin{aligned}
\sum_{n=-\infty}^{\infty}a^{2n}\left(1+a^2p^{2n}\right)p^{n^2-n}\varepsilon_{x_n(a;p)}&=\sum_{n=-\infty}^{\infty}a^{4n}\left(1+a^2p^{4n}\right)p^{4n^2-2n}\varepsilon_{x_n(a;p^2)}\\
&\quad+\sum_{n=-\infty}^{\infty}a^{4n+2}\left(1+a^2p^{4n+2}\right)p^{4n^2+2n}\varepsilon_{x_n(ap;p^2)},
\end{aligned}$$

we see after replacing p^2 with q that

$$\tilde{\nu}_a = \frac{1}{2}\left(\nu_a + \nu_{a\sqrt{q}}\right),$$

where ν_a is defined in (6.3) (or (1.7)). It is plausible that the supports of the N-extremal solutions in some way should depend on β.

7. A bilinear generating function

In this section we shall derive a bilinear generating function for the polynomials $Q_n(x;\beta)$. In particular, an expression for the Poisson kernel will be obtained. We follow more or less the same procedure as Ismail and Stanton in [15] and [16]. Unless otherwise stated, it is assumed that $\beta > 0$.

Lemma 7.1. *The polynomials $Q_n(x;\beta)$ have the p-integral representation*

$$\begin{aligned}(-1)^n p^n \frac{(p;p)_n}{(-1/\beta;p)_n} Q_n(x;\beta) = &\frac{\left(-e^{2y}/\beta, -e^{-2y}/\beta; p^2\right)_\infty}{2ip\sqrt{\beta}(-1/\beta;p)_\infty} \frac{\left(p;p^2\right)_\infty}{\left(p^2;p^2\right)_\infty} \\ &\times \frac{1}{1-p}\int_{-ip\sqrt{\beta}}^{ip\sqrt{\beta}} t^n \frac{\left(it/\sqrt{\beta}, -it/\sqrt{\beta}; p\right)_\infty}{\left(te^y/p\beta, -te^{-y}/p\beta; p\right)_\infty} d_p t.\end{aligned}$$

Proof. By definition of the p-integral, we have

$$\begin{aligned}&\frac{1}{1-p}\int_{-ip\sqrt{\beta}}^{ip\sqrt{\beta}} t^n \frac{\left(it/\sqrt{\beta}, -it/\sqrt{\beta}; p\right)_\infty}{\left(te^y/p\beta, -te^{-y}/p\beta; p\right)_\infty} d_p t \\ &= \left(ip\sqrt{\beta}\right)^{n+1} \sum_{k=0}^{\infty} \frac{\left(-p^{k+1}, p^{k+1}; p\right)_\infty}{\left(ip^k e^y/\sqrt{\beta}, -ip^k e^{-y}/\sqrt{\beta}; p\right)_\infty} \left(p^{n+1}\right)^k \\ &\quad - \left(-ip\sqrt{\beta}\right)^{n+1} \sum_{k=0}^{\infty} \frac{\left(p^{k+1}, -p^{k+1}; p\right)_\infty}{\left(-ip^k e^y/\sqrt{\beta}, ip^k e^{-y}/\sqrt{\beta}; p\right)_\infty} \left(p^{n+1}\right)^k \\ &= \left(ip\sqrt{\beta}\right)^{n+1} \left\{ \frac{\left(p^2;p^2\right)_\infty}{\left(ie^y/\sqrt{\beta}, -ie^{-y}/\sqrt{\beta}; p\right)_\infty} {}_2\phi_1\left(\left.\begin{matrix} ie^y/\sqrt{\beta}, -ie^{-y}/\sqrt{\beta} \\ -p \end{matrix}\,\right|\, p, p^{n+1}\right)\right. \\ &\quad \left. + (-1)^n \frac{\left(p^2;p^2\right)_\infty}{\left(-ie^y/\sqrt{\beta}, ie^{-y}/\sqrt{\beta}; p\right)_\infty} {}_2\phi_1\left(\left.\begin{matrix} -ie^y/\sqrt{\beta}, ie^{-y}/\sqrt{\beta} \\ -p \end{matrix}\,\right|\, p, p^{n+1}\right)\right\} \\ &= (-1)^n \frac{\left(ip\sqrt{\beta}\right)^{n+1}\left(p^2;p^2\right)_\infty}{\left(-ie^y/\sqrt{\beta}, ie^{-y}/\sqrt{\beta}; p\right)_\infty} \left\{ {}_2\phi_1\left(\left.\begin{matrix} ie^{-y}/\sqrt{\beta}, -ie^y/\sqrt{\beta} \\ -p \end{matrix}\,\right|\, p, p^{n+1}\right)\right. \\ &\quad \left. + (-1)^n \frac{\left(-ie^y/\sqrt{\beta}, ie^{-y}/\sqrt{\beta}; p\right)_\infty}{\left(ie^y/\sqrt{\beta}, -ie^{-y}/\sqrt{\beta}; p\right)_\infty} {}_2\phi_1\left(\left.\begin{matrix} -ie^{-y}/\sqrt{\beta}, ie^y/\sqrt{\beta} \\ -p \end{matrix}\,\right|\, p, p^{n+1}\right)\right\}.\end{aligned}$$

According to [10, III. 31], the combination of ${}_2\phi_1$'s in the brackets reduces to

$$\begin{aligned}
&\frac{(-p^n/\beta,-1;p)_\infty}{\left(-ip^ne^{-y}/\sqrt{\beta},-ipe^y\sqrt{\beta};p\right)_\infty}\,{}_2\phi_1\left(\begin{matrix}ipe^y\sqrt{\beta},-p^{1-n}\beta\\ ie^yp^{1-n}\sqrt{\beta}\end{matrix}\;\middle|\;p,ie^y/\sqrt{\beta}\right)\\
&=\frac{2(-1/\beta,-p;p)_\infty}{\left(-ie^{-y}/\sqrt{\beta},-ipe^y\sqrt{\beta};p\right)_\infty}\frac{\left(-ie^{-y}/\sqrt{\beta};p\right)_n}{(-1/\beta;p)_n}\\
&\times{}_2\phi_1\left(\begin{matrix}ipe^y\sqrt{\beta},-p^{1-n}\beta\\ ie^yp^{1-n}\sqrt{\beta}\end{matrix}\;\middle|\;p,ie^y/\sqrt{\beta}\right)
\end{aligned}$$

and by Heine's transformation formula [10, III. 3], the above ${}_2\phi_1$ can be written as

$$\frac{\left(-ipe^y\sqrt{\beta};p\right)_\infty}{\left(ie^y/\sqrt{\beta};p\right)_\infty}\,{}_2\phi_1\left(\begin{matrix}p^{-n},-ie^y/\sqrt{\beta}\\ ie^yp^{1-n}\end{matrix}\;\sqrt{\beta}\;\middle|\;p,-ipe^y\sqrt{\beta}\right).$$

Recalling the explicit form of the polynomials $Q_n(x;\beta)$, the representation follows easily. □

Theorem 7.2. *For $|z|<1$, the polynomials $Q_n(x;\beta)$ satisfy the bilinear generating function*

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(p;p)_n}{(-1/\beta;p)_n}Q_n(x;\beta)Q_n\left(x';\beta\right)(z/\beta)^n\\
&=\frac{\left(-ie^y/\sqrt{\beta},ie^{-y}/\sqrt{\beta},ize^{y'}/\sqrt{\beta},-ize^{-y'}/\sqrt{\beta};p\right)_\infty}{2(-1/\beta;p)_\infty}\\
&\times\frac{\left(p;p^2\right)_\infty}{\left(z^2;p^2\right)_\infty}\,{}_4\phi_3\left(\begin{matrix}ie^y/\sqrt{\beta},-ie^{-y}/\sqrt{\beta},z,-z\\ ize^{y'}/\sqrt{\beta},-ize^{-y'}/\sqrt{\beta},-p\end{matrix}\;\middle|\;p,p\right)\\
&+\text{ a similar term with } y \text{ replaced by } -y \text{ and } y' \text{ replaced by } -y'.
\end{aligned}$$

Proof. By the previous lemma, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(p;p)_n}{(-1/\beta;p)_n}Q_n(x;\beta)Q_n\left(x';\beta\right)(z/\beta)^n\\
&=\frac{\left(-e^{2y}/\beta,-e^{-2y}/\beta;p^2\right)_\infty}{2ip\sqrt{\beta}(-1/\beta;p)_\infty}\frac{\left(p;p^2\right)_\infty}{\left(p^2;p^2\right)_\infty}\sum_{n=0}^{\infty}Q_n\left(x';\beta\right)(-z/p\beta)^n\\
&\times\frac{1}{1-p}\int_{-ip\sqrt{\beta}}^{ip\sqrt{\beta}}t^n\frac{\left(it/\sqrt{\beta},-it/\sqrt{\beta};p\right)_\infty}{(te^y/p\beta,-te^{-y}/p\beta;p)_\infty}\,d_pt.
\end{aligned}$$

Interchanging the order of summation and integration, the above sum reduces to

$$\begin{aligned}
&\frac{1}{1-p}\int_{-ip\sqrt{\beta}}^{ip\sqrt{\beta}}\frac{\left(it/\sqrt{\beta},-it/\sqrt{\beta};p\right)_\infty}{(te^y/p\beta,-te^{-y}/p\beta;p)_\infty}\sum_{n=0}^{\infty}Q_n\left(x';\beta\right)(-zt/p\beta)^n d_pt\\
&=\frac{1}{1-p}\int_{-ip\sqrt{\beta}}^{ip\sqrt{\beta}}\frac{\left(it/\sqrt{\beta},-it/\sqrt{\beta},zte^{y'}/p\beta,-zte^{-y'}/p\beta;p\right)_\infty}{\left(te^y/p\beta,-te^{-y}/p\beta,izt/p\sqrt{\beta},-izt/p\sqrt{\beta};p\right)_\infty}d_pt
\end{aligned}$$

for $|zt/p| < \sqrt{\beta}$. Here the p-integral can be written as

$$
\begin{aligned}
&ip\sqrt{\beta}\sum_{n=0}^{\infty}\frac{\left(-p^{n+1},p^{n+1},izp^{n}e^{y'}/\sqrt{\beta},-izp^{n}e^{-y'}/\sqrt{\beta};p\right)_{\infty}}{\left(ip^{n}e^{y}/\sqrt{\beta},-ip^{n}e^{-y}/\sqrt{\beta},-zp^{n},zp^{n};p\right)_{\infty}}p^{n}\\
&+ip\sqrt{\beta}\sum_{n=0}^{\infty}\frac{\left(p^{n+1},-p^{n+1},-izp^{n}e^{y'}/\sqrt{\beta},izp^{n}e^{-y'}/\sqrt{\beta};p\right)_{\infty}}{\left(-ip^{n}e^{y}/\sqrt{\beta},ip^{n}e^{-y}/\sqrt{\beta},zp^{n},-zp^{n};p\right)_{\infty}}p^{n}\\
&=ip\sqrt{\beta}\frac{(p^2;p^2)_\infty}{(z^2;p^2)_\infty}\frac{\left(ize^{y'}/\sqrt{\beta},-ize^{-y'}/\sqrt{\beta};p\right)_{\infty}}{\left(ie^{y}/\sqrt{\beta},-ie^{-y}/\sqrt{\beta};p\right)_{\infty}}\\
&\quad\times{}_4\phi_3\left(\begin{matrix}ie^{y}/\sqrt{\beta},-ie^{-y}/\sqrt{\beta},z,-z\\ ize^{y'}/\sqrt{\beta},-ize^{-y'}/\sqrt{\beta},-p\end{matrix}\,\middle|\,p,p\right)\\
&\quad+\text{a similar term with } y \text{ replaced by } -y \text{ and } y' \text{ replaced by } -y'
\end{aligned}
$$

and the theorem is proved. □

Corollary 7.3. *For $|z| < 1/p$, the Poisson kernel is given by*

$$
\begin{aligned}
\sum_{n=0}^{\infty}P_n(x;\beta)P_n(x';\beta)\,z^n &= \frac{\left(-ie^{y}/\sqrt{\beta},ie^{-y}/\sqrt{\beta},izpe^{y'}/\sqrt{\beta},-izpe^{-y'}/\sqrt{\beta};p\right)_{\infty}}{2(-1/\beta;p)_\infty}\\
&\times\frac{(p;p^2)_\infty}{(z^2p^2;p^2)_\infty}{}_4\phi_3\left(\begin{matrix}ie^{y}/\sqrt{\beta},-ie^{-y}/\sqrt{\beta},zp,-zp\\ izpe^{y'}/\sqrt{\beta},-izpe^{-y'}/\sqrt{\beta},-p\end{matrix};p,p\right)\\
&+\text{ a similar term with } y \text{ and } y' \text{ replaced by } -y \text{ and } -y'.
\end{aligned}
$$

In particular, we have

$$
\begin{aligned}
\sum_{n=0}^{\infty}P_n^2(x;\beta) =&\frac{(-e^{2y}/\beta,-e^{-2y}/\beta;p^2)_\infty}{(-1/\beta;p)_\infty}\frac{(p;p^2)_\infty}{(p^2;p^2)_\infty}\\
&\times\sum_{n=0}^{\infty}\frac{1+p^{2n}/\beta}{(1+p^{2n}e^{2y}/\beta)(1+p^{2n}e^{-2y}/\beta)}p^n.
\end{aligned}
$$

Proof. The first part follows immediately from (5.2). With $z = 1$ and $x' = x$, the Poisson kernel reduces to

$$
\begin{aligned}
&\frac{(-e^{2y}/\beta,-e^{-2y}/\beta;p^2)_\infty}{2(-1/\beta;p)_\infty}\frac{(p;p^2)_\infty}{(p^2;p^2)_\infty}\\
&\times\sum_{n=0}^{\infty}\left\{\frac{p^n}{(1-ip^ne^{y}/\sqrt{\beta})(1+ip^ne^{-y}/\sqrt{\beta})}+\frac{p^n}{(1+ip^ne^{y}/\sqrt{\beta})(1-ip^ne^{-y}/\sqrt{\beta})}\right\}\\
&=\frac{(-e^{2y}/\beta,-e^{-2y}/\beta;p^2)_\infty}{(-1/\beta;p)_\infty}\frac{(p;p^2)_\infty}{(p^2;p^2)_\infty}\sum_{n=0}^{\infty}\frac{1+p^{2n}/\beta}{(1+p^{2n}e^{2y}/\beta)(1+p^{2n}e^{-2y}/\beta)}p^n
\end{aligned}
$$

and this proves the second part. □

With an explicit expression for the Poisson kernel at hand one should be able to explain that the discrete solutions $\lambda_a^{(\beta)}$ in Theorem 6.1 are not N-extremal. Recall that the masses of the N-extremal solutions are given by the function

$$
\rho(x;\beta) = \left(\sum_{n=0}^{\infty}P_n^2(x;\beta)\right)^{-1}, \quad x \in \mathbb{R}.
$$

When $x = 0$, the value is

$$\rho(0;\beta) = \frac{(p^2;p^2)_\infty}{(p;p^2)_\infty} \frac{(-p/\beta;p^2)_\infty}{(-1/\beta;p^2)_\infty} \left(\sum_{n=0}^\infty \frac{p^n}{1+p^{2n}/\beta} \right)^{-1}$$

and hence

$$\begin{aligned} &\frac{1}{\rho(0;\beta)} + \frac{1}{\rho(0;1/\beta)} \\ &= \frac{(p;p^2)_\infty}{(p^2;p^2)_\infty} \left(\frac{(-1/\beta;p^2)_\infty}{(-p/\beta;p^2)_\infty} \sum_{n=0}^\infty \frac{p^n}{1+p^{2n}/\beta} + \frac{(-\beta;p^2)_\infty}{(-p\beta;p^2)_\infty} \sum_{n=0}^\infty \frac{p^n}{1+p^{2n}\beta} \right). \end{aligned}$$

By Ramanujan's ${}_1\psi_1$ sum [10, II. 29], we have

$$\sum_{n=-\infty}^\infty \frac{(-1/\beta;p^2)_n}{(-p^2/\beta;p^2)_n} p^n = \frac{(p^2;p^2)_\infty^2}{(p;p^2)_\infty^2} \frac{(-p/\beta,-p\beta;p^2)_\infty}{(-p^2/\beta,-p^2\beta;p^2)_\infty} \tag{7.1}$$

and thus

$$\begin{aligned} &\frac{(-p/\beta;p^2)_\infty}{(-p^2/\beta;p^2)_\infty} \frac{1}{\rho(0;\beta)} + \frac{(-p\beta;p^2)_\infty}{(-p^2\beta;p^2)_\infty} \frac{1}{\rho(0;1/\beta)} \\ &\quad = \frac{(p;p^2)_\infty}{(p^2;p^2)_\infty} \left(1 + \frac{(p^2;p^2)_\infty^2}{(p;p^2)_\infty^2} \frac{(-p/\beta,-p\beta;p^2)_\infty}{(-p^2/\beta,-p^2\beta;p^2)_\infty} \right) \\ &\quad = \frac{(p;p^2)_\infty}{(p^2;p^2)_\infty} + \frac{(p^2;p^2)_\infty}{(p;p^2)_\infty} \frac{(-p/\beta,-p\beta;p^2)_\infty}{(-p^2/\beta,-p^2\beta;p^2)_\infty}. \end{aligned} \tag{7.2}$$

On the other hand,

$$\lambda_1^{(\beta)}(\{0\}) = \frac{(p;p^2)_\infty}{(p^2;p^2)_\infty} \frac{(-p^2\beta;p^2)_\infty}{(-p\beta;p^2)_\infty}$$

so that

$$\begin{aligned} &\frac{(-p/\beta;p^2)_\infty}{(-p^2/\beta;p^2)_\infty} \frac{1}{\lambda_1^{(\beta)}(\{0\})} + \frac{(-p\beta;p^2)_\infty}{(-p^2\beta;p^2)_\infty} \frac{1}{\lambda_1^{(1/\beta)}(\{0\})} \\ &\quad = 2\frac{(p^2;p^2)_\infty}{(p;p^2)_\infty} \frac{(-p/\beta,-p\beta;p^2)_\infty}{(-p^2/\beta,-p^2\beta;p^2)_\infty}. \end{aligned} \tag{7.3}$$

Since

$$\sum_{n=-\infty}^\infty \frac{(-1/\beta;p^2)_n}{(-p^2/\beta;p^2)_n} p^n > 1,$$

we get from (7.1) that

$$\frac{(-p/\beta,-p\beta;p^2)_\infty}{(-p^2/\beta,-p^2\beta;p^2)_\infty} > \frac{(p;p^2)_\infty^2}{(p^2;p^2)_\infty^2}$$

and as a consequence, the expression in (7.2) is > the expression in (7.3). So for each $\beta > 0$, we either have

$$\rho(0;\beta) > \lambda_1^{(\beta)}(\{0\})$$

or

$$\rho(0;1/\beta) > \lambda_1^{(1/\beta)}(\{0\}).$$

In particular, this means that $\rho(0;1) > \lambda_1^{(1)}(\{0\})$ and at least when $\beta = 1$, the solution $\lambda_1^{(\beta)}$ is not N-extremal.

8. Some biorthogonal rational functions

In [7] Berg and Ismail have shown how to systematically build the classical q-orthogonal polynomials from the q-Hermite polynomials using a simple procedure of attaching generating functions to measures. As an example, the attachment procedure for the q^{-1}-Hermite polynomials leads to the polynomials

$$u_n(x;t_1,t_2) = \frac{(-qe^{-y}/t_2;q)_n}{(q;q)_n}(-t_2/q)^n {}_2\phi_1\left(\begin{matrix} q^{-n}, qe^y/t_1 \\ -t_2e^y/q^n \end{matrix}\,\middle|\, q, -t_1e^y\right) \tag{8.1}$$

which are special cases of the Al-Salam–Chihara polynomials $v_n(x)$ (corresponding to $q>1$) from [4]. If we set $t_1 = -t_2 = iq\sqrt{\beta}$ for some $\beta \geq 0$ and replace q by p, the polynomials in (8.1) reduce to $Q_n(x;\beta)$. At the second stage, the attachment procedure is applied to $\{u_n\}$ and leads to the biorthogonal rational functions

$$\varphi_n(x;t_1,t_2,t_3,t_4) = {}_4\phi_3\left(\begin{matrix} q^{-n}, -t_1t_2q^{n-2}, -t_1t_3/q, -t_1t_4/q \\ -t_1e^y, t_1e^{-y}, t_1t_2t_3t_4/q^3 \end{matrix}\,\middle|\, q, q\right) \tag{8.2}$$

studied by Ismail and Masson in [13].

It is known that the rational functions $\varphi_n(x;t_1,t_2,t_3,t_4)$ are biorthogonal with respect to any measure μ of the form

$$d\mu(x) = \prod_{j=1}^{4}(t_je^{-y}, -t_je^y;q)_\infty d\psi(x), \tag{8.3}$$

where ψ is a solution to the q^{-1}-Hermite moment problem. In this section we show how solutions to the moment problem associated with the polynomials $Q_n(x;\beta)$ lead to biorthogonality relations for the special case $t_3 = -t_4 = iq\sqrt{\beta}$ of the rational functions in (8.2).

Theorem 8.1. *Suppose that ν is a positive measure such that*

$$\int_{\mathbb{R}} Q_n(x;\beta)Q_m(x;\beta)d\nu(x) = \frac{\beta^n(-1/\beta;p)_n}{p^n(p;p)_n}\delta_{nm}.$$

Then the rational functions

$$\varphi_n(x;t_1,t_2,\beta) = {}_4\phi_3\left(\begin{matrix} p^{-n}, it_1\sqrt{\beta}, -it_1\sqrt{\beta}, -t_1t_2p^n \\ -t_1e^y, t_1e^{-y}, t_1t_2\beta/p \end{matrix}\,\middle|\, p, p\right)$$

are biorthogonal with respect to the measure μ given by

$$d\mu(x) = \prod_{j=1}^{2}(t_je^{-y}, -t_je^y;q)_\infty d\nu(x)$$

and the biorthogonality relation is

$$\int_{\mathbb{R}} \varphi_n(x;t_1,t_2,\beta)\varphi_m(x;t_2,t_1,\beta)d\mu(x) = \frac{1+t_1t_2p^{n-2}}{1+t_1t_2p^{2n-2}}\frac{(-1/\beta,p;p)_n(t_1t_2\beta/p)^n}{(t_1t_2\beta/p;p)_n}$$
$$\times\frac{(-t_1t_2p^{n-1};p)_\infty(-t_1^2\beta,-t_2^2\beta;p^2)_\infty}{(t_1t_2\beta/p;p)_\infty}\delta_{nm}.$$

Proof. To show the biorthogonality, it is sufficient to prove that

$$\int_{\mathbb{R}} \frac{\varphi_n(x;t_1,t_2,\beta)}{(t_2e^{-y},-t_2e^y;p)_m}d\mu(x) = 0 \quad \text{for} \quad 0 \leq m < n.$$

According to (6.2), we have

$$\begin{aligned}\int_{\mathbb{R}} &(t_1p^ke^{-y}, -t_1p^ke^y, t_2p^me^{-y}, -t_2p^me^y; p)_\infty d\nu(x) \\ &= \frac{(-t_1t_2p^{k+m-1}; p)_\infty}{(t_1t_2\beta p^{k+m-1}; p)_\infty}(-t_1^2\beta p^{2k}, -t_2^2\beta p^{2m}; p^2)_\infty\end{aligned}$$

and thus

$$\begin{aligned}\int_{\mathbb{R}} \frac{\varphi_n(x; t_1, t_2, \beta)}{(t_2e^{-y}, -t_2e^y; p)_m} d\mu(x) &= \sum_{k=0}^{n} \frac{(p^{-n}, it_1\sqrt{\beta}, -it_1\sqrt{\beta}, -t_1t_2p^{n-2}; p)_k}{(t_1t_2\beta/p, p; p)_k} p^k \\ &\times \int_{\mathbb{R}} (t_1p^ke^{-y}, -t_1p^ke^y, t_2p^me^{-y}, -t_2p^me^y; p)_\infty d\nu(x) \\ &= \frac{(-t_1t_2p^{m-1}; p)_\infty}{(t_1t_2\beta p^{m-1}; p)_\infty}(-t_1^2\beta, -t_2^2\beta p^{2m}; p^2)_\infty \\ &\times \sum_{k=0}^{n} \frac{(p^{-n}, -t_1t_2p^{n-2}, t_1t_2\beta p^{m-1}; p)_k}{(t_1t_2\beta/p, -t_1t_2p^{m-1}, p; p)_k} p^k.\end{aligned}$$

By the q-Saalschütz sum [10, II. 12], the above sum is equal to

$$\frac{(p^{m-n+1}, -1/\beta; p)_n}{(-t_1t_2p^{m-1}, p^{2-n}/t_1t_2\beta; p)_n}$$

and

$$(p^{m-n+1}; p)_n = 0 \quad \text{for} \quad 0 \le m < n.$$

The case $m = n$ reads

$$\begin{aligned}\int_{\mathbb{R}} \frac{\varphi_n(x; t_1, t_2, \beta)}{(t_2e^{-y}, -t_2e^y; p)_n} d\mu(x) &= \frac{(p, -1/\beta; p)_n}{(-t_1t_2p^{n-1}, p^{2-n}/t_1t_2\beta; p)_n} \\ &\times \frac{(-t_1t_2p^{n-1}; p)_\infty}{(t_1t_2\beta p^{n-1}; p)_\infty}(-t_1^2\beta, -t_2^2\beta p^{2n}; p^2)_\infty\end{aligned}$$

and the biorthogonality relation is established after multiplication by

$$\frac{(p^{-n}, it_2\sqrt{\beta}, -it_2\sqrt{\beta}, -t_1t_2p^{n-2}; p)_n}{(t_1t_2\beta/p, p; p)_n} p^n.$$

□

The special cases $d\nu = v(x; \beta)dx$ and $\nu = \lambda_a^{(\beta)}$ are not leading to new measures of biorthogonality for the rational functions $\varphi_n(x; t_1, t_2, t_3, t_4)$. In the first case, μ is of the form (8.3) with $d\psi = w_\beta(x)dx$, see (5.4), and in the second case, a more general result without the restrictions on t_3 and t_4 is contained in Theorem 4.2 in [13].

9. A p-Sturm–Liouville equation

The main result in this section is the p-Sturm–Liouville equation in Theorem 9.3. As an application, we give an easy proof of the fact that the polynomials $Q_n(x; \beta)$ are orthogonal with respect to the weight function

$$\tilde{v}(x; \beta) = \frac{1}{(-e^{2y}/\beta, -e^{-2y}/\beta; p^2)_\infty}, \quad x = \sinh y \in \mathbb{R}$$

and the discrete measures

$$\tilde{\lambda}_a^{(\beta)} = \sum_{n=-\infty}^{\infty} \tilde{v}(x_n(a);\beta)\sqrt{1+x_n^2(a)}\;\varepsilon_{x_n(a)},$$

where

$$x_n(a) = \frac{1}{2}\Big(\frac{1}{ap^n} - ap^n\Big).$$

But we also use the p-Sturm–Liouville equation to derive a Bethe Ansatz type relation satisfied by the zeros of $Q_n(x;\beta)$.

The first step is to establish a lowering operator.

Lemma 9.1. *A lowering operator for $Q_n(x;\beta)$ is given by*

$$\mathcal{D}_p Q_{n+1}(x;\beta) = \frac{2p^{n/2}}{1-p} Q_n(x;\beta/p), \quad n \geq 0. \tag{9.1}$$

Proof. Apply $\mathcal{D}_p$ to both sides of the generating function in (3.1) to get

$$\sum_{n=0}^{\infty} \mathcal{D}_p Q_{n+1}(x;\beta) t^{n+1} = \frac{2}{1-p} \sum_{n=0}^{\infty} Q_n(x;\beta/p) p^{n/2} t^{n+1}.$$

Equating the coefficients of t^{n+1} now leads to (9.1). □

The next step is to find an appropriate raising operator.

Lemma 9.2. *A raising operator for $Q_n(x;\beta)$ is given by*

$$\frac{1}{\tilde{v}(x;\beta p)} \mathcal{D}_p\big(\tilde{v}(x;\beta) Q_{n-1}(x;\beta)\big) = \frac{2(p^{n/2} - p^{-n/2})}{\beta(1-p)} Q_n(x;\beta p), \quad n \geq 1. \tag{9.2}$$

Proof. A direct computation using the generating function in (3.1) shows that

$$\sum_{n=1}^{\infty} \mathcal{D}_p\big(\tilde{v}(x;\beta) Q_{n-1}(x;\beta)\big) t^{n-1}$$
$$= \frac{2}{\beta(1-p)} \tilde{v}(x;\beta p) \sum_{n=1}^{\infty} Q_n(x;\beta p)(p^{n/2} - p^{-n/2}) t^{n-1}$$

and (9.2) follows by equating the coefficients of t^{n-1}. □

Combining the lowering and raising operators in Lemma 9.1 and Lemma 9.2, we get the following result.

Theorem 9.3. *The polynomials $Q_n(x;\beta)$ satisfy the p-Sturm–Liouville equation*

$$\mathcal{D}_p\big(\tilde{v}(x;\beta/p)\mathcal{D}_p Q_n(x;\beta)\big) + \frac{4\sqrt{p}(1-p^n)}{\beta(1-p)^2} \tilde{v}(x;\beta) Q_n(x;\beta) = 0, \quad n \geq 0. \tag{9.3}$$

An alternative proof of the first statement in Theorem 5.1 now goes as follows. Set

$$h_n = -\frac{4\sqrt{p}}{\beta} \frac{1-p^n}{(1-p)^2} \tag{9.4}$$

and note that this sequence is strictly decreasing in n. According to (9.3), we have

$$\begin{aligned}
&(h_n - h_m) \int\limits_{\mathbb{R}} Q_n(x;\beta) Q_m(x;\beta) \tilde{v}(x;\beta) dx \\
&\quad = \int\limits_{\mathbb{R}} \mathcal{D}_p\big(\tilde{v}(x;\beta/p) \mathcal{D}_p Q_n(x;\beta)\big) Q_m(x;\beta) dx \\
&\qquad - \int\limits_{\mathbb{R}} \mathcal{D}_p\big(\tilde{v}(x;\beta/p) \mathcal{D}_p Q_m(x;\beta)\big) Q_n(x;\beta) dx \\
&\quad = \Big\langle \mathcal{D}_p\big(\tilde{v}(x;\beta/p)\mathcal{D}_p Q_n(x;\beta)\big), \sqrt{1+x^2} Q_m(x;\beta) \Big\rangle \\
&\qquad - \Big\langle \mathcal{D}_p\big(\tilde{v}(x;\beta/p)\mathcal{D}_p Q_m(x;\beta)\big), \sqrt{1+x^2} Q_n(x;\beta) \Big\rangle
\end{aligned}$$

and using integration by parts as described in (5.7), this expression reduces to

$$\begin{aligned}
&-\Big\langle \tilde{v}(x;\beta/p)\mathcal{D}_p Q_n(x;\beta), \sqrt{1+x^2}\mathcal{D}_p Q_m(x;\beta) \Big\rangle \\
&+ \Big\langle \tilde{v}(x;\beta/p)\mathcal{D}_p Q_m(x;\beta), \sqrt{1+x^2}\mathcal{D}_p Q_n(x;\beta) \Big\rangle \\
&= \int\limits_{\mathbb{R}} \mathcal{D}_p Q_n(x;\beta) \mathcal{D}_p Q_m(x;\beta) \tilde{v}(x;\beta/p) dx \\
&\quad - \int\limits_{\mathbb{R}} \mathcal{D}_p Q_m(x;\beta) \mathcal{D}_p Q_n(x;\beta) \tilde{v}(x;\beta/p) dx = 0.
\end{aligned}$$

Since $h_n \neq h_m$ for $n \neq m$, it is proved that $Q_n(x;\beta)$ are orthogonal with respect to $\tilde{v}(x;\beta)$.

Let $\mathbb{Z}_a$ denote the set $\{x_k(a) \mid k \in \mathbb{Z}\}$. Since $\mathbb{Z}_{ap} = \mathbb{Z}_a$, we only consider the case $p < a \leq 1$. With respect to the inner product

$$\langle f, g \rangle_a = \sum_{k=-\infty}^{\infty} f\big(x_k(a)\big) g\big(x_k(a)\big) \sqrt{1 + x_k^2(a)}$$

on $\ell^2(\mathbb{Z}_a, \sqrt{1+x^2})$, integration by parts can be carried out by following the rule

$$\langle \mathcal{D}_p f, g \rangle_a = - \langle f, \mathcal{D}_p g \rangle_{a\sqrt{p}}.$$

Hence, the p-Sturm–Liouville equation (9.3) leads to

$$\begin{aligned}
&(h_n - h_m) \sum_{k=-\infty}^{\infty} Q_n(x_k(a);\beta) Q_m(x_k(a);\beta) \tilde{v}(x_k(a);\beta) \sqrt{1 + x_k^2(a)} \\
&\quad = \Big\langle \mathcal{D}_p\big(\tilde{v}(x;\beta/p)\mathcal{D}_p Q_n(x;\beta)\big), Q_m(x;\beta) \Big\rangle_a \\
&\qquad - \Big\langle \mathcal{D}_p\big(\tilde{v}(x;\beta/p)\mathcal{D}_p Q_m(x;\beta)\big), Q_n(x;\beta) \Big\rangle_a \\
&\quad = -\Big\langle \tilde{v}(x;\beta/p)\mathcal{D}_p Q_n(x;\beta), \mathcal{D}_p Q_m(x;\beta) \Big\rangle_{a\sqrt{p}} \\
&\qquad + \Big\langle \tilde{v}(x;\beta/p)\mathcal{D}_p Q_m(x;\beta), \mathcal{D}_p Q_n(x;\beta) \Big\rangle_{a\sqrt{p}} = 0
\end{aligned}$$

and we have proved that $Q_n(x;\beta)$ are orthogonal with respect to $\tilde{\lambda}_a^{(\beta)}$. Exactly the same method can be used to prove that the polynomials $h_n(x|q)$ are orthogonal with respect to the discrete measures in (1.7).

The essence of Theorem 9.3 is the fact that the polynomials $Q_n(x;\beta)$ are eigenfunctions of the second order divided difference operator $\mathcal{T}$ defined to act on functions in $L^2(\mathbb{R}, v(x;\beta))$ by

$$\mathcal{T}f(x) = -\frac{1}{\tilde{v}(x;\beta)}\mathcal{D}_p\big(\tilde{v}(x;\beta/p)\mathcal{D}_p f(x)\big).$$

The corresponding eigenvalues are given in (9.4). The operator $\mathcal{T}$ is clearly positive since

$$\begin{aligned}
(\mathcal{T}f(x), f(x))_v &= -\int_{\mathbb{R}} \mathcal{D}_p\big(\tilde{v}(x;\beta/p)\mathcal{D}_p f(x)\big)\overline{f(x)}dx \\
&= -\Big\langle \mathcal{D}_p\big(\tilde{v}(x;\beta/p)\mathcal{D}_p f(x)\big), \sqrt{1+x^2}f(x)\Big\rangle \\
&= \Big\langle \tilde{v}(x;\beta/p)\mathcal{D}_p f(x), \sqrt{1+x^2}\mathcal{D}_p f(x)\Big\rangle \\
&= \int_{\mathbb{R}} |\mathcal{D}_p f(x)|^2 \tilde{v}(x;\beta/p)dx \geq 0.
\end{aligned}$$

However, the polynomials $Q_n(x;\beta)$ are not dense in $L^2(\mathbb{R}, v(x;\beta))$ because the absolutely continuous solution with density $v(x;\beta)$ is not N-extremal. Therefore, we do not have an explicit orthonormal basis for the Hilbert space $L^2(\mathbb{R}, v(x;\beta))$.

The average operator $\mathcal{A}_p$ is defined to act on functions on $\mathbb{R}$ by

$$\mathcal{A}_p f(x) = \frac{1}{2}\Big(\breve{f}(p^{1/2}e^y) + \breve{f}(p^{-1/2}e^y)\Big),$$

where again $\breve{f}(e^y) = f(x)$. The reason for introducing $\mathcal{A}_p$ is to obtain the p-Leibniz rule

$$\mathcal{D}_p fg = \mathcal{D}_p f \mathcal{A}_p g + \mathcal{A}_p f \mathcal{D}_p g,$$

which follows from the fact that

$$(a+b)(c-d) + (a-b)(c+d) = 2(ac-bd) \quad \text{for} \quad a,b,c,d \in \mathbb{R}.$$

It is straightforward to see that

$$\mathcal{D}_p\tilde{v}(x;\beta) = -\frac{4x}{\beta\sqrt{p}(1-p)}\tilde{v}(x;p\beta)$$

and

$$\mathcal{A}_p\tilde{v}(x;\beta) = \Big(\frac{2x^2+1}{p\beta} + 1\Big)\tilde{v}(x;p\beta)$$

so the p-Sturm–Liouville equation (9.3) can be written as

$$(2x^2+1+\beta)\mathcal{D}_p^2 Q_n(x;\beta) - \frac{4x\sqrt{p}}{1-p}\mathcal{A}_p\mathcal{D}_p Q_n(x;\beta) + \frac{4\sqrt{p}(1-p^n)}{(1-p)^2}Q_n(x;\beta) = 0. \tag{9.5}$$

Note that the weight function $\tilde{v}(x;\beta)$ has disappeared completely and, as such, Eq. (9.5) is more general than Eq. (9.3).

We are now in a position to derive Bethe Ansatz equations satisfied by the zeros of $Q_n(x;\beta)$, cf. [18].

Theorem 9.4. *Let $x_1 = \sinh y_1, \ldots, x_n = \sinh y_n$ denote the n simple zeros of $Q_n(x;\beta)$. If we set $p = e^{2\eta}$ and $\beta = e^{2\gamma}$, then the following n equations are satisfied*

$$\prod_{\substack{i=1\\ i\neq j}}^{n} \frac{\sinh\left(\frac{y_j-y_i}{2}+\eta\right)\cosh\left(\frac{y_j+y_i}{2}+\eta\right)}{\sinh\left(\frac{y_j-y_i}{2}-\eta\right)\cosh\left(\frac{y_j+y_i}{2}-\eta\right)} = e^{-2y_j}\frac{\cosh(y_j+\gamma)}{\cosh(y_j-\gamma)}, \qquad j=1,\ldots,n.$$

Proof. Let c_n denote the leading coefficient of $Q_n(x;\beta)$ and define

$$f(x) := \frac{1}{c_n} Q_n(x;\beta) = \prod_{i=1}^{n}(x-x_i).$$

A tiresome computation shows that

$$\begin{aligned}&\mathcal{D}_p^2 f(x)|_{x=x_j}\\ &= \frac{2}{\left(\sqrt{p}-1/\sqrt{p}\right)^2}\left(\frac{\prod_i\left(\frac{e^{y_j}p-e^{-y_j}/p}{2}-x_i\right)}{e^{y_j}\sqrt{p}-e^{-y_j}/\sqrt{p}} + \frac{\prod_i\left(\frac{e^{y_j}/p-e^{-y_j}p}{2}-x_i\right)}{e^{y_j}/\sqrt{p}-e^{-y_j}\sqrt{p}}\right)\frac{2}{e^{y_j}+e^{-y_j}}\end{aligned}$$

and similarly

$$\begin{aligned}&\mathcal{A}_p\mathcal{D}_p f(x)|_{x=x_j}\\ &= \frac{1}{\sqrt{p}-1/\sqrt{p}}\left(\frac{\prod_i\left(\frac{e^{y_j}p-e^{-y_j}/p}{2}-x_i\right)}{e^{y_j}\sqrt{p}-e^{-y_j}/\sqrt{p}} - \frac{\prod_i\left(\frac{e^{y_j}/p-e^{-y_j}p}{2}-x_i\right)}{e^{y_j}/\sqrt{p}-e^{-y_j}\sqrt{p}}\right).\end{aligned}$$

According to (9.5), we thus have

$$\begin{aligned}&(2x_j^2+1+\beta)\frac{2}{\left(\sqrt{p}-1/\sqrt{p}\right)^2}\\ &\times\left(\frac{\prod_i\left(\frac{e^{y_j}p-e^{-y_j}/p}{2}-x_i\right)}{e^{y_j}\sqrt{p}-e^{-y_j}/\sqrt{p}} + \frac{\prod_i\left(\frac{e^{y_j}/p-e^{-y_j}p}{2}-x_i\right)}{e^{y_j}/\sqrt{p}-e^{-y_j}\sqrt{p}}\right)\frac{2}{e^{y_j}+e^{-y_j}}\\ &= \frac{4x_j\sqrt{p}}{1-p}\,\frac{1}{\sqrt{p}-1/\sqrt{p}}\left(\frac{\prod_i\left(\frac{e^{y_j}p-e^{-y_j}/p}{2}-x_i\right)}{e^{y_j}\sqrt{p}-e^{-y_j}/\sqrt{p}} - \frac{\prod_i\left(\frac{e^{y_j}/p-e^{-y_j}p}{2}-x_i\right)}{e^{y_j}/\sqrt{p}-e^{-y_j}\sqrt{p}}\right)\end{aligned}$$

or

$$\begin{aligned}&\left(\frac{e^{2y_j}+e^{-2y_j}}{2}+\beta\right)\left(\frac{\prod_i\left(\frac{e^{y_j}p-e^{-y_j}/p}{2}-x_i\right)}{e^{y_j}\sqrt{p}-e^{-y_j}/\sqrt{p}} + \frac{\prod_i\left(\frac{e^{y_j}/p-e^{-y_j}p}{2}-x_i\right)}{e^{y_j}/\sqrt{p}-e^{-y_j}\sqrt{p}}\right)\\ &= \frac{e^{2y_j}-e^{-2y_j}}{2}\left(\frac{\prod_i\left(\frac{e^{y_j}p-e^{-y_j}/p}{2}-x_i\right)}{e^{y_j}\sqrt{p}-e^{-y_j}/\sqrt{p}} - \frac{\prod_i\left(\frac{e^{y_j}/p-e^{-y_j}p}{2}-x_i\right)}{e^{y_j}/\sqrt{p}-e^{-y_j}\sqrt{p}}\right)\end{aligned}$$

Hence,

$$\begin{aligned}&(e^{2y_j}+\beta)\left(e^{y_j}/\sqrt{p}+e^{-y_j}\sqrt{p}\right)\prod_{i=1}^{n}\left(\frac{e^{y_j}p-e^{-y_j}/p}{2}-x_i\right)\\ &\quad= -(e^{-2y_j}+\beta)\left(e^{y_j}\sqrt{p}+e^{-y_j}/\sqrt{p}\right)\prod_{i=1}^{n}\left(\frac{e^{y_j}/p-e^{-y_j}p}{2}-x_i\right),\end{aligned}$$

that is,

$$\prod_{i=1}^{n} \frac{\big(e^{y_j}p - e^{-y_j}/p\big)/2 - x_i}{\big(e^{y_j}/p - e^{-y_j}p\big)/2 - x_i} = -\frac{(e^{-2y_j}+\beta)\big(e^{y_j}\sqrt{p} + e^{-y_j}/\sqrt{p}\big)}{(e^{2y_j}+\beta)\big(e^{y_j}/\sqrt{p} + e^{-y_j}\sqrt{p}\big)}.$$

With $p = e^{2\eta}$ and $\beta = e^{2\gamma}$, the above expression can be written as

$$\prod_{i=1}^{n} \frac{\sinh\big(\frac{y_j-y_i}{2}+\eta\big)\cosh\big(\frac{y_j+y_i}{2}+\eta\big)}{\sinh\big(\frac{y_j-y_i}{2}-\eta\big)\cosh\big(\frac{y_j+y_i}{2}-\eta\big)} = -e^{-2y_j}\frac{\cosh(y_j+\gamma)\cosh(y_j+\eta)}{\cosh(y_j-\gamma)\cosh(y_j-\eta)}$$

and the theorem is proved once we remove the factor corresponding to $i = j$. □

For more information about the connection between q-Sturm–Liouville problems and Bethe Ansatz equations, the reader is referred to [12].

10. A right inverse to the divided difference operator $\mathcal{D}_p$

In Section 5 and Section 9 we have seen that the divided difference operator $\mathcal{D}_q$ (or $\mathcal{D}_p$) is a lowering operator for the polynomials $h_n(x|q)$ and $Q_n(x;\beta)$. In this section we establish a right inverse $\mathcal{D}_q^{-1}$ (or $\mathcal{D}_p^{-1}$) on appropriate L^2-spaces.

For an indeterminate moment problem on the real line it is well-known that the polynomials are dense in $L^2(\mathbb{R},\mu)$ if and only if the measure μ is N-extremal. So in the case of the q^{-1}-Hermite moment problem, the polynomials

$$P_n(x) = \frac{q^{n(n+1)/4}}{\sqrt{(q;q)_n}}h_n(x|q)$$

form an orthonormal basis for $L^2(\mathbb{R},\nu_a)$ exactly when ν_a has the form (1.7) (which we shall assume). Any function $f \in L^2(\mathbb{R},\nu_a)$ can thus be written as

$$f(x) \sim \sum_{n=0}^{\infty} f_n P_n(x) \tag{10.1}$$

for some sequence $(f_n) \in \ell^2$ and since

$$\mathcal{D}_q P_n(x) = -\frac{2\sqrt{1-q^n}}{\sqrt{q}-1/\sqrt{q}}P_{n-1}(x),$$

we see that

$$\mathcal{D}_q f(x) \sim \frac{-2}{\sqrt{q}-1/\sqrt{q}}\sum_{n=0}^{\infty} f_{n+1}\sqrt{1-q^{n+1}}P_n(x).$$

In other words, we can think of $\mathcal{D}_q$ as a bounded operator on $L^2(\mathbb{R},\nu_a)$. It is readily seen that $\mathcal{D}_q$ is onto for if $g \in L^2(\mathbb{R},\nu_a)$ has the form

$$g(x) \sim \sum_{n=0}^{\infty} g_n P_n(x), \tag{10.2}$$

then $\mathcal{D}_q f(x) = g(x)$ with f as in (10.1) and

$$f_n = -\frac{\sqrt{q}-1/\sqrt{q}}{2}\frac{g_{n-1}}{\sqrt{1-q^n}}.$$

But $\mathcal{D}_q$ is clearly not one-to-one. So we can only have hope of finding a right inverse to $\mathcal{D}_q$, that is, an operator $\mathcal{D}_q^{-1}$ on $L^2(\mathbb{R},\nu_a)$ so that

$$\mathcal{D}_q\mathcal{D}_q^{-1} = I.$$

It is straightforward how to define $\mathcal{D}_q^{-1}$. With g as in (10.2), the operator $\mathcal{D}_q^{-1}$ is given by

$$\begin{aligned}\mathcal{D}_q^{-1}g(x) &\sim -\frac{\sqrt{q}-1/\sqrt{q}}{2}\sum_{n=1}^{\infty}\frac{g_{n-1}}{\sqrt{1-q^n}}P_n(x)\\ &= -\frac{\sqrt{q}-1/\sqrt{q}}{2}\sum_{n=1}^{\infty}\frac{\int_{\mathbb{R}}g(x')P_{n-1}(x')d\nu_a(x')}{\sqrt{1-q^n}}P_n(x)\\ &= -\frac{\sqrt{q}-1/\sqrt{q}}{2}\int_{\mathbb{R}}g(x')\sum_{n=1}^{\infty}\frac{P_{n-1}(x')P_n(x)}{\sqrt{1-q^n}}d\nu_a(x')\end{aligned}$$

and this means that $\mathcal{D}_q^{-1}$ is an integral operator with kernel

$$K(x,x') = \sum_{n=1}^{\infty}\frac{P_n(x)P_{n-1}(x')}{\sqrt{1-q^n}} = \sum_{n=1}^{\infty}\frac{q^{n^2/2}}{(q;q)_n}h_n(x|q)h_{n-1}(x'|q). \tag{10.3}$$

Since $\big(P_n(x)\big)\in\ell^2$ for all $x\in\mathbb{C}$, the kernel in (10.3) is convergent in $L^2(\mathbb{R},\nu_a)$ as a function of x'. The change of summation and integration can therefore easily be justified.

Theorem 10.1. *The kernel in* (10.3) *is explicitly given by*

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{q^{n^2/2}}{(q;q)_n}h_n(x|q)h_{n-1}(x'|q)\\ &\quad=\frac{2(-\sqrt{q}e^{y+y'},-\sqrt{q}e^{-y-y'},\sqrt{q}e^{y-y'},\sqrt{q}e^{-y+y'};q)_\infty}{(q;q)_\infty}\\ &\qquad\times\sum_{n=0}^{\infty}\frac{(q;q)_{2n}(\sinh y-q^{n+1/2}\sinh y')q^{n+1/2}}{(-\sqrt{q}e^{y+y'},-\sqrt{q}e^{-y-y'},\sqrt{q}e^{y-y'},\sqrt{q}e^{-y+y'};q)_{n+1}}.\end{aligned}$$

Proof. The idea of the proof is to apply $\mathcal{D}_q$ with respect to x' to the Poisson kernel. Applying $\mathcal{D}_q$ (wrt. x') to the left-hand side in (1.5) leads to

$$\frac{-2}{\sqrt{q}-1/\sqrt{q}}\sum_{n=1}^{\infty}\frac{q^{n^2/2-n}}{(q;q)_{n-1}}h_n(x|q)h_{n-1}(x'|q)t^n$$

or

$$\frac{-2}{\sqrt{q}-1/\sqrt{q}}\big\{H(x,x';t/q)-H(x,x';t)\big\}$$

if we set

$$H(x,x';t)=\sum_{n=1}^{\infty}\frac{q^{n^2/2}}{(q;q)_n}h_n(x|q)h_{n-1}(x'|q)t^n.$$

Applying $\mathcal{D}_q$ (wrt. x') to the right-hand side gives after some computations

$$\begin{aligned}&\frac{-2}{\sqrt{q}-1/\sqrt{q}}\frac{2t}{\sqrt{q}}\Big(\sinh y-\frac{t}{\sqrt{q}}\sinh y'\Big)\\ &\quad\times\frac{(-t\sqrt{q}e^{y+y'},-t\sqrt{q}e^{-y-y'},t\sqrt{q}e^{y-y'},t\sqrt{q}e^{-y+y'};q)_\infty}{(t^2/q;q)_\infty}.\end{aligned}$$

In other words,

$$\begin{aligned}
&H(x,x';t/q)-H(x,x';t)\\
&=\frac{2t}{\sqrt{q}}\Big(\sinh y-\frac{t}{\sqrt{q}}\sinh y'\Big)\frac{(-t\sqrt{q}e^{y+y'},-t\sqrt{q}e^{-y-y'},t\sqrt{q}e^{y-y'},t\sqrt{q}e^{-y+y'};q)_\infty}{(t^2/q;q)_\infty}
\end{aligned}$$

and since $H(x,x';0)=0$, we have

$$\begin{aligned}
H(x,x';t)&=\sum_{n=1}^\infty\big\{H(x,x';tq^{n-1})-H(x,x';tq^n)\big\}\\
&=2\sum_{n=1}^\infty(\sinh y-tq^{n-1/2}\sinh y')tq^{n-1/2}\\
&\quad\times\frac{(-tq^{n+1/2}e^{y+y'},-tq^{n+1/2}e^{-y-y'},tq^{n+1/2}e^{y-y'},tq^{n+1/2}e^{-y+y'};q)_\infty}{(t^2q^{2n-1};q)_\infty}\\
&=\frac{2(-t\sqrt{q}e^{y+y'},-t\sqrt{q}e^{-y-y'},t\sqrt{q}e^{y-y'},t\sqrt{q}e^{-y+y'};q)_\infty}{(t^2q;q)_\infty}\\
&\quad\times\sum_{n=1}^\infty\frac{(t^2q;q)_{2n-2}(\sinh y-tq^{n-1/2}\sinh y')tq^{n-1/2}}{(-t\sqrt{q}e^{y+y'},-t\sqrt{q}e^{-y-y'},t\sqrt{q}e^{y-y'},t\sqrt{q}e^{-y+y'};q)_n}.
\end{aligned}$$

It is only left to set $t=1$ and shift the summation. □

For the moment problem associated with the polynomials $Q_n(x;\beta)$, the situation is almost the same. However, since

$$\mathcal{D}_pP_n(x;\beta)=\frac{2\sqrt{p}}{1-p}\sqrt{\frac{1-p^n}{\beta+1}}P_{n-1}(x;\beta/p),$$

the operator $\mathcal{D}_p$ maps $L^2(\mathbb{R},\nu_a^{(\beta)})$ into $L^2(\mathbb{R},\nu_a^{(\beta/p)})$ and the right inverse $\mathcal{D}_p^{-1}$ is defined on $L^2(\mathbb{R},\nu_a^{(\beta/p)})$ as the integral operator with kernel

$$K(x,x';\beta)=\sum_{n=1}^\infty\frac{P_n(x;\beta)P_{n-1}(x';\beta/p)}{\sqrt{1-p^n}}. \tag{10.4}$$

Theorem 10.2. *The kernel in* (10.4) *is explicitly given by*

$$\begin{aligned}
&\sum_{n=1}^\infty\frac{P_n(x;\beta)P_{n-1}(x';\beta/p)}{\sqrt{1-p^n}}\\
&=-\frac{i\sqrt{p}}{2}\frac{(-ie^y/\sqrt{\beta},ie^{-y}/\sqrt{\beta},i\sqrt{p}e^{y'}/\sqrt{\beta},-i\sqrt{p}e^{-y'}/\sqrt{\beta})_\infty}{(-p/\beta;p)_\infty\sqrt{1+1/\beta}}\frac{(p;p^2)_\infty}{(p^2;p^2)_\infty}\\
&\quad\times\sum_{n=0}^\infty\frac{(p^2;p^2)_n}{(i\sqrt{p}e^{y'}/\sqrt{\beta},-i\sqrt{p}e^{-y'}/\sqrt{\beta})_n}p^n\sum_{k=0}^n\frac{(ie^y/\sqrt{\beta},-ie^{-y}/\sqrt{\beta};p)_k}{(p^2;p^2)_k}p^k\\
&\quad+\text{ a similar term with } y \text{ replaced by } -y \text{ and } y' \text{ replaced by } -y'.
\end{aligned}$$

Proof. The same procedure as in the proof of Theorem 10.1 can be carried out. The details are left to the reader. □

If we compare $\mathcal{D}_q$ (or $\mathcal{D}_p$) with differentiation d/dx, it is remarkable that $\mathcal{D}_q$ is bounded on $L^2(\mathbb{R},\nu_a)$ whereas d/dx is unbounded. Also, the Askey–Wilson operator is known to be unbounded on the L^2-space weighted by the weight function for the Askey–Wilson polynomials.

Since $\mathcal{D}_q$ is a q-analogue of differentiation, we can think of $\mathcal{D}_q^{-1}$ as a q-analogue of integration. Thus, for $f \in L^2(\mathbb{R}, \nu_a)$ we have the following q-analogue of the definite integral

$$\int_a^b f(x)\breve{d}_q x = -\frac{\sqrt{q}-1/\sqrt{q}}{2} \int_{\mathbb{R}} f(x)\big(K(b,x) - K(a,x)\big)d\nu_a(x),$$

where the kernel K is computed in Theorem 10.3.

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