Quadratic Hedging Approaches and Indifference Pricing in Insurance

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Preface

This thesis has been prepared in partial fulfillment of the requirements for the Ph.D. degree at the Laboratory of Actuarial Mathematics, Institute for Mathematical Sciences at the University of Copenhagen. The work was carried out in the period from April 1997 to March 2000 under the supervision of Ragnar Norberg (University of Copenhagen until April 2000; now London School of Economics), and Martin Schweizer (Technical University of Berlin).

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Summary

This thesis is concerned with quadratic hedging approaches, indifference pricing principles and their applications in insurance. These techniques are of general interest in incomplete financial markets, that is, in models for financial markets where general contingent claims cannot be priced uniquely using no-arbitrage arguments alone. Thus, the situation differs from the complete case, where all prices are already determined from the simple and economically reasonable assumption of absence of arbitrage, i.e. absence of any possibilities to make riskless profits by trading on the financial markets.

The emphasis in this thesis is on the analysis of insurance contracts which combine traditional actuarial risk and financial risk. A simple example is a unit-linked pure endowment contract with guarantee. With this life insurance contract, the sum insured payable to the policy-holder at the term of the contract is contingent upon survival and not fixed a priori, but linked to the development of some stock index and guaranteed against falling below some amount. The actuarial risk in this contract stems from the uncertainty of not knowing whether or not the policy-holder will survive until the term of the contract, and the financial risk is related to the performance of the underlying index. Another example of an insurance contract with an inherent financial risk is a financial stop-loss contract. This reinsurance contract differs from traditional stop-loss contracts in that the insurer's total losses have both an insurance and a financial component.

In the first part of the thesis, we focus on the problem of hedging and pricing payment streams generated by unit-linked insurance contracts using the criterion of risk-minimization. A widely used approach is based on the assumption of risk-neutrality with respect to mortality, and we first demonstrate within simple discrete time models how this can be derived from the asymptotic behaviour of the insurer's loss from a portfolio of unit-linked contracts as the number of policies increases. Next, risk-minimizing hedging strategies are determined explicitly for a portfolio of independent identical unit-linked pure endowment contracts with guarantee in the special case where the financial market is described by the Cox-Ross-Rubinstein model. These results characterize the combined insurance and financial risk in the contracts and decompose this risk into a hedgeable part and a non-hedgeable part. In addition, we show how payment streams can be incorporated into the theory of risk-minimization in a continuous time set-up. This extension provides a framework for the analysis of insurance contracts that generate genuine payment streams. In this setting, risk-minimizing hedging strategies are worked out for general unit-linked life insurance contracts driven by a Markov jump process with a finite state space and for non-life insurance risk processes where claim amounts and premiums are affected by some traded price index, for example a claim inflation index.

The second part of the thesis deals with financial transformations of two classical actuarial premium calculation principles, the variance and standard deviation principles. The corresponding financial valuation principles were derived by Schweizer (1997) via indifference arguments which embedded their actuarial counterparts in a financial framework. Under the financial variance (or standard deviation) principle, the fair premium equals the expected value of the claim under the variance optimal martingale plus a safety-loading which is proportional to the variance (or standard deviation) of the non-hedgeable part of the claim. We complement existing results by deriving optimal hedging strategies for the two financial valuation principles when the discounted price process of the traded assets is a continuous semimartingale. For the variance principle, the optimal strategy differs from the mean-variance hedging strategy only by a correction term which is independent of the claim under consideration; for the standard deviation principle, the result is more complicated. Furthermore, we provide an alternative direct characterization of the financial standard deviation principle which does not involve an indifference argument.

A separate study is devoted to an investigation of the impact on the fair premiums of the amount of information available to the seller of the insurance contracts. This includes a comparison result of mean-variance hedging errors under two different filtrations, which is obtained via a projection argument for Hilbert spaces. In particular, this result allows the derivation of simple bounds on the fair premiums under the financial variance and standard deviation principles in the situation where the insurance claim involves two stochastically independent sources of randomness, purely financial risk and pure insurance risk. Explicit formulas are obtained for the fair premiums and the optimal trading strategies under different levels of information for unit-linked insurance contracts and for some reinsurance contracts with an inherent financial risk.

Resumé

Denne afhandling omhandler kvadratiske hedging metoder og indifferens-prisfastsættelse samt deres anvendelser indenfor forsikring. Disse metoder er af generel interesse i forbindelse med analyse af ufuldstændige finansielle markeder, dvs. i modeller for finansielle markeder hvor antagelsen om fravær af arbitrage ikke er tilstrækkelig til at sikre entydig prisfastsættelse af generelle afledte kontrakter. I fuldstændige finansielle markeder er prisen på enhver afledt kontrakt derimod entydigt bestemt alene ud fra den enkle og økonomisk rimelige antagelse om fravær af arbitragemuligheder (muligheder for risikofrie gevinster).

Afhandlingen fokuserer på analyse af forsikringskontrakter, som kombinerer traditionel forsikringsrisiko og finansiel risiko. Dette er eksempelvis relevant for unit-link kontrakter indenfor livsforsikring. Ved en unit-link ren oplevelsesforsikring med slutgaranti er forsikringssummen, der udbetales ved oplevelse, ikke fastsat fuldstændigt ved aftalens indgåelse, men knyttet direkte til et underliggende aktieindeks eller en pulje. Samtidig garanterer kontrakten et mindstebeløb, som sikrer forsikringstageren imod situationer, hvor det underliggende aktieindeks udvikler sig ugunstigt. Forsikringsrisikoen i sådanne kontrakter er relateret til usikkerheden omkring forsikringstagerens overlevelse, og den finansielle risiko er knyttet til udviklingen i det underliggende aktieindeks. Et andet eksempel på en forsikringskontrakt indeholdende finansiel risiko er en finansiel stop-loss kontrakt. Denne genforsikringskontrakt adskiller sig fra traditionelle stop-loss kontrakter, idet den kan konstrueres således, at den dækker forsikringsselskabets samlede finansielle og forsikringsmæssige tab.

I den første del af afhandlingen anvendes kriteriet risiko-minimering til prisfastsættelse og hedging (afdækning) af betalingsstrømme fra generelle unit-link livsforsikringskontrakter. Et ofte anvendt princip ved prisfastsættelse af unit-link livsforsikringskontrakter er baseret på antagelsen om "risiko-neutralitet med hensyn til dødsrisiko". Dette princip udledes først indenfor relativt enkle modeller med diskrete handelstidspunkter ved at betragte den asymptotiske opførsel af forsikringsselskabets tab knyttet til disse kontrakter. Derefter udledes eksplicitte udtryk for den risiko-minimerende strategi for en portefølje af uafhængige, identiske, unit-link rene oplevelsesforsikringer i den situation, hvor det finansielle marked beskrives ved en Cox-Ross-Rubinstein model. Resultaterne giver en karakteristik af den kombinerede finans- og forsikringsrisiko i disse kontrakter, idet risikoen dekomponeres i to dele, en del som kan elimineres ved at handle på det finansielle marked, og en del som ikke kan afdækkes. Det demonstreres yderligere hvorledes generelle betalingsstrømme kan implementeres i eksisterende teorier for risiko-minimering i modeller hvor handel og betalinger beskrives i kontinuert tid. Denne udvidelse muliggør behandling af forsikringskontrakter som genererer egentlige betalingsstrømme. Indenfor denne nye ramme bestemmes risiko-minimerende strategier for generelle unit-link kontrakter modelleret ved Markov-springprocesser med et endeligt tilstandsrum. Derudover betragtes en klasse af risiko-processer fra skadeforsikring, hvor både enkeltskadebeløb og præmier kan afhænge af et prisindeks, som kan handles på et marked.

I afhandlingens anden del undersøges nogle finansielle transformationer af de klassiske aktuarielle præmieberegningsprincipper, varians- og standardafvigelsesprincippet. Disse afledte finansielle principper blev opnået af Schweizer (1997) ved anvendelse af indifferens-argumenter samt ved at indføre de aktuarielle principper i en finansiel ramme. Under det finansielle variansprincip (henholdsvis standardafvigelsesprincip) bestemmes den fair præmie som summen af den forventede værdi under det variansoptimale martingalmål af de diskonterede betalinger fra kontrakten og et sikkerhedstillæg, som er proportionalt med variansen (henholdsvis standardafvigelsen) af den del af risikoen, som ikke kan hedges. Optimale investeringsstrategier er udledt for de to finansielle principper i den situation, hvor den diskonterede prisproces for de underliggende aktiver kan beskrives ved en kontinuert semimartingal. Forskellen mellem den optimale strategi under variansprincippet og mean-variance strategien udgøres af et korrektionsled, som ikke afhænger af hvilken kontrakt der betragtes. Sammenhængen mellem disse to strategier og den optimale strategi under det finansielle standardafvigelsesprincip er derimod mere kompleks. Der gives yderligere en alternativ og mere direkte udledning af det finansielle standardafvigelsesprincip, som ikke er baseret på indifferens-argumenter.

Endelig undersøges hvorledes mængden af tilgængelig information påvirker de fair præmier. Denne undersøgelse omfatter et generelt resultat, som er opnået ved hjælp af projektionsargumenter for Hilbertrum, og som omhandler forskellen på præcisionen af mean-variance strategierne under to forskellige filtreringer. Resultatet muliggør udledning af simple øvre og nedre grænser for de fair præmier under det finansielle varians- og standardafvigelsesprincip for forsikringskontrakter med to stokastisk uafhængige risikokomponenter, ren finansiel risiko og ren forsikringsrisiko. Eksplicitte udtryk for den fair præmie og den optimale investeringsstrategi er opnået for unit-link livsforsikringskontrakter samt for nogle genforsikringskontrakter med finansiel risiko under varierende antagelser vedrørende mængden af tilgængelig information.

Contents

Chapter 1

Introduction

This thesis investigates methods for hedging and valuation of insurance claims with an inherent financial risk. It thus focuses on aspects of the interplay between finance and insurance, an area which has gained considerable attention over the last couple of decades.

1 Insurance background

The two fields of insurance and finance started as separate areas. At its very origin, the theory of insurance was mainly concerned with the computation of premiums for life insurance contracts. An overview of the early history of life insurance can be found in Braun (1937), and, according to this exposition, the first known social welfare programs with elements of life insurance are the Roman *Collegia* which date back at least to AD 133. The first primitive mortality tables were published in 1662 by John Graunt (1620–1674), who worked with only 7 different age groups. The first mortality table, where the expected number of survivors from year to year is given, is due to the astronomer Edmund Halley (1656–1742). These tables allowed for more precise predictions about portfolios of independent lives and were essential for computation of premiums for various life insurance contracts. In his book on evaluation of annuities on life from 1725, Abraham de Moivre (1667–1754) suggested methods for evaluation of life insurance contracts, combining interest and mortality under very simple assumptions about the mortality.

In 1738, Daniel Bernoulli (1700–1782) argued that risks[1] should not be measured by their expectations and hence laid the foundation for modern utility theory. Using examples related to gambling, he explained that the preferences of an individual may depend on his economic situation and, more specifically, that in some situations it

[1] i.e. *uncertain payoffs*. The notion of *risk* is used in several different contexts in both the actuarial and the financial literature; often it is simply used vaguely describing the fact that there is some uncertainty, for example in *mortality risk* known from insurance and *credit risk* known from finance. However, the notion also appears in various more specific concepts. Examples are *insurance risk process*, which is typically defined as the accumulated premiums minus claims in an insurance portfolio, and *risk-minimization*, which is a theory from mathematical finance that can be used for determining trading strategies.

could be reasonable for a poorer individual to prefer one uncertain future payment to another (more) uncertain payment with a larger expected value, whereas a wealthier person would prefer the payment with the largest expected value. This observation was also of importance for insurance in general, since it explained for example why individuals may accept to buy insurance contracts at a price which exceeds the expected value of the payment from the contract.

2 Classical valuation of insurance contracts

In the actuarial literature, insurance theory is divided into the analysis of *life insurance* and *non-life insurance* contracts. In addition to historical aspects, there are fundamental differences between the two areas, for example in respect of time horizon (for life insurance extending up to 50 years, whereas for non-life insurance typically limited to one year). These are reflected e.g. in the principles that are applied for the calculation of premiums. In this section, we review some notions and key concepts of life and non-life insurance, placing focus on the valuation techniques used there.

2.1 Life insurance

We recall some classical and basic concepts from life insurance; introductory expositions to the area are Gerber (1986) and Norberg (2000).

Consider a portfolio of n lives aged y, say, to be insured at time 0 with i.i.d. remaining life times $T_1, \ldots, T_n$, and assume that there exists a continuous function (called the hazard rate function) μ_{y+t} such that the survival probability is of the form ${}_tp_y = P(T_1 > t) = \exp(-\int_0^t \mu_{y+u}\, du)$. A *pure endowment* contract with sum insured K and term T stipulates that the amount K (the insurance benefit) is payable at time T contingent on survival of the policy-holder. Assume that the contract is paid by a single premium κ, say, at time 0. Assume furthermore that the seller of the contract (the insurance company) invests the premium κ in some asset which pays a rate of return $\delta = (\delta_t)_{0 \le t \le T}$ during $[0, T]$. For the i'th policy-holder, the obligation of the insurance company is now given by the *present value*

$$H_i = 1_{\{T_i > T\}} K e^{-\int_0^T \delta_t\, dt}, \tag{2.1}$$

which is obtained by discounting the amount payable at T, $1_{\{T_i > T\}} K$, using the rate of return δ. Note that (2.1) is a random variable. The fundamental *principle of equivalence* now states that the premiums should be chosen such that the present values of premiums and benefits balance on average. If we assume in addition that δ is stochastically independent of the remaining life times, the principle of equivalence states that

$$\kappa = \mathrm{E}[H_i] = {}_Tp_y K \mathrm{E}[e^{-\int_0^T \delta_t\, dt}] \tag{2.2}$$

for the single premium case. Since life insurance portfolios are often very large, this principle can be partly justified by using the law of large numbers. Indeed, as the size n of the portfolio is increased, the relative number of survivors $\frac{1}{n}\sum_{i=1}^{n} 1_{\{T_i>T\}}$ converges a.s. towards the probability ${}_Tp_y$ of survival to T by the strong law of large numbers since the lifetimes $T_1, \ldots, T_n$ are stochastically independent. Thus, for n sufficiently large, the actual number of survivors $\sum_{i=1}^{n} 1_{\{T_i>T\}}$ will be "approximately" equal to the expected number, $n\,{}_Tp_y$. Accumulating the amount $n\kappa$ with interest now leads to

$$n\kappa e^{\int_0^T \delta_t\,dt} = n\,{}_Tp_y K\mathrm{E}[e^{-\int_0^T \delta_t\,dt}]e^{\int_0^T \delta_t\,dt} \approx \sum_{i=1}^{n} 1_{\{T_i>T\}} K\mathrm{E}[e^{-\int_0^T \delta_t\,dt}]e^{\int_0^T \delta_t\,dt}. \quad (2.3)$$

In particular, when δ is non-random, the expression on the right is equal to the amount to be paid to the policy-holders. So in the case of a deterministic rate of return, the principle of equivalence is justified directly by use of the law of large numbers which essentially guarantees that the actual number of survivors is "close" to the expected number.

The problem becomes much more delicate in the more realistic situation where δ is a stochastic process, and it follows immediately from (2.3) that the simple accumulation of the premium κ will not in general generate the amount to be paid, since $e^{-\int_0^T \delta_t\,dt}$ may differ considerably from its expected value. One way of dealing with this problem is to replace the "true" rate of return process δ in (2.2) with some deterministic rate of return process δ' which is such that the single premium $n\kappa$ accumulated by the true rate of return δ is larger than K times the expected number of survivors with a large probability. The excess (if any) should then be added to the amount paid to the policy-holder and is known as *bonus*, see e.g. Ramlau-Hansen (1991) and Norberg (1999) and references therein. However, this approach really raises the problem of whether it is reasonable to assume the existence of *any* deterministic and strictly positive δ' which over a very long time horizon has the property that it will be larger than the actual return on investments with a very large probability. In particular, this is an extremely relevant discussion when one thinks of the historically low interest rates observed in the late 1990s. An alternative to this approach is therefore to replace δ by the so-called short rate of interest and then replace the last term in (2.2) by the price on the financial market of a financial asset which pays one unit at time T, a so-called *zero coupon bond*; see Persson (1998) who obtained a general version of *Thiele's differential equation* within this framework.

2.2 Non-life insurance

In comparison to the valuation principles in life insurance, discounting plays a much less prominent role in the classical non-life insurance premium calculation principles; see e.g. Bühlmann (1970) and Gerber (1979) for standard textbooks on the mathematics of these principles. This difference can be partly explained by the relatively short time horizons that are associated with most non-life insurance contracts, which typically change from year to year.

Let H denote some claim payable at a fixed time T, say. A premium calculation principle is a mapping which assigns to each claim a number, called the premium. One class of classical actuarial valuation principles applied in non-life insurance can be directly and somewhat pragmatically motivated from the law of large numbers. These principles prescribe charging a premium $\tilde{u}(H)$ which is equal to the expected value $\mathrm{E}[H]$ of the claim augmented by some amount $A(H)$, the so-called *safety-loading*, i.e.

$$\tilde{u}(H) = \mathrm{E}[H] + A(H). \tag{2.4}$$

The most important examples of such premium calculation principles are: $A(H) = 0$ (*the net premium principle* or the *principle of equivalence*), $A(H) = a\mathrm{E}[H]$ (*the expected value principle*), $A(H) = a(\mathrm{Var}[H])^{1/2}$ (*the standard deviation principle*), $A(H) = a\mathrm{Var}[H]$ (*the variance principle*) and $A(H) = a\mathrm{E}[((H - \mathrm{E}[H])^+)^2]$ (*the semi-variance principle*). In practice, the standard deviation principle seems to be the most widely used principle of the above. Bühlmann (1970) mentions the fact that it is linear up to scaling as one possible explanation for its popularity, but judges its theoretical properties to be inferior to those of the variance principle.

Another interesting class of premium calculation principles consists of the so-called *zero increase expected utility principles*, which are derived as follows. Let u be a utility function, i.e. $u'(x) \geq 0$ and $u''(x) \leq 0$ for any $x \in \mathbb{R}$, and let V_0 denote the insurer's initial capital at time 0 (possibly random, e.g. depending on the result of other business). The zero (increase expected) utility premium of H under u and initial capital V_0 is the solution $\tilde{u}(H)$ to the equation

$$\mathrm{E}[u(V_0 + \tilde{u}(H) - H)] = \mathrm{E}[u(V_0)], \tag{2.5}$$

which states that the expected utility of the final wealth $V_0 + \tilde{u}(H) - H$ from selling the claim H at the premium $\tilde{u}(H)$ should equal the expected utility of V_0; the latter may be interpreted as the wealth associated with not selling the claim H. The zero utility premium defined by (2.5) is often also called the *fair premium*, since selling the claim leaves the expected utility unaffected, i.e. it leads neither to an increase nor a decrease in expected utility. The most prominent example is probably the so-called *exponential principle* which is obtained for the exponential utility function $u(x) = \frac{1}{a}(1 - e^{-ax})$. In particular, when V_0 is constant P-a.s., the solution to (2.5) does not depend on V_0 and is given by

$$\tilde{u}(H) = \frac{1}{a} \log\left(\mathrm{E}[e^{aH}]\right).$$

Another frequently used utility function is the quadratic utility function which is defined by $u(x) = x - \frac{x^2}{2s}$, $x \leq s$, and $u(x) = \frac{s}{2}$ for $x > s$. For a more complete survey of utility functions in insurance (and finance), see e.g. Gerber and Pafumi (1998).

An alternative principle is the so-called *Esscher principle*, which states that

$$\tilde{u}(H) = \frac{\mathrm{E}[He^{aH}]}{\mathrm{E}[e^{aH}]}. \tag{2.6}$$

This principle basically amounts to an exponential scaling of the claim H.

Other premium calculation principles worth mentioning are generalizations of the so-called *maximal loss principle*. For $\varepsilon \in [0,1]$ and $p \in [0,1]$, the (generalized) $(1-\varepsilon)$-*percentile principle* states that the premium should be computed as

$$\tilde{u}(H) = p\mathrm{E}[H] + (1-p)F^{-1}(1-\varepsilon),$$

where F is the distribution function of H and F^{-1} is its generalized inverse, i.e. $F^{-1}(y) = \inf\{x | F(x) \geq y\}$. Thus the premium is a weighted average of the expected value of H and the $(1-\varepsilon)$-percentile of the distribution of H. In particular, the maximal loss principle is obtained for $\varepsilon = 0$ and $p = 0$.

For a detailed investigation of the above mentioned principles and several other premium calculation principles, see e.g. Goovaerts et al. (1984) and Heilmann (1987).

3 Financial background

Bachelier (1900) proposed to describe fluctuations in the price of a stock by a Brownian motion by assuming that the change in the value of the stock in a time interval of length h was normally distributed with mean αh and variance $\sigma^2 h$ and that changes in non-overlapping intervals were stochastically independent. Samuelson (1965) advocated a framework where the stock price was modeled by a geometric Brownian motion, which had the advantage that it did not generate negative prices. Within this framework, Black and Scholes (1973) and Merton (1973) introduced the idea that options on stocks should really be priced such that no sure profits could arise from composing portfolios of long and short positions in the underlying stock and in the option itself. Assuming that the option price was a deterministic function of time and the current value of the stock, they obtained the celebrated Black-Scholes formula for European call options, and this pricing formula has the at first glance surprising feature that it does not involve the expected return of the underlying stock. Cox, Ross and Rubinstein (1979) investigated a simple discrete time model, where the change in the value of the stock between two trading times can attain two different values only. In that setting, they derived option prices and obtained the pricing formulas of Black, Scholes and Merton as limiting cases by letting the length of the time intervals between trading times converge towards 0. Building on concepts and ideas in Harrison and Kreps (1979) for discrete time models, Harrison and Pliska (1981) gave a mathematical theory for pricing of options under continuous trading and clarified the role of martingale theory in the pricing of options and its connection to key concepts such as absence of arbitrage and completeness.

4 Financial valuation principles

We recall some basic notation and concepts from financial mathematics that will be used throughout. Standard textbooks are Duffie (1996) and Lamberton and

Lapeyre (1996); see also Hull (1997) for an exposition including some more institutional aspects. Let T denote a fixed finite time horizon and consider a financial market consisting of two traded assets, a stock and a savings account with price processes $S = (S_t)_{0 \le t \le T}$ and $B = (B_t)_{0 \le t \le T}$, respectively, which are defined on some probability space $(\Omega, \mathcal{F}, P)$, and introduce the discounted price processes $X = S/B$ and $X^0 = B/B \equiv 1$. In this setting, a *trading strategy* (or dynamical portfolio strategy) is a 2-dimensional process $\varphi = (\vartheta_t, \eta_t)_{0 \le t \le T}$ satisfying certain integrability conditions (which will be given later), and where ϑ is predictable and η is adapted with respect to some filtration $I\!\!F = (\mathcal{F}_t)_{0 \le t \le T}$ which describes the evolution of available information. The pair $\varphi_t = (\vartheta_t, \eta_t)$ is the *portfolio* held at time t, that is, ϑ_t is the number of shares of the stock held at t and η_t is the discounted amount invested in the savings account. Thus, the discounted value at time t of φ_t is given by $V_t(\varphi) = \vartheta_t X_t + \eta_t$. A strategy φ is said to be *self-financing* provided that

$$V_t(\varphi) = V_0(\varphi) + \int_0^t \vartheta_s \, dX_s. \tag{4.1}$$

Here, $V_0(\varphi)$ can be interpreted as the amount invested at time 0 and $\int_0^t \vartheta_s \, dX_s$ as the accumulated trading gains generated by φ up to and including time t. Thus, for a self-financing strategy φ, the current value of the portfolio φ_t at time t is exactly the initially invested amount plus trading gains, so that no in- or outflow of capital has taken place during $(0, t]$. A contract (or *claim*) specifying the discounted ($\mathcal{F}_T$-measurable) payoff H at time T is said to be *attainable* if there exists a self-financing strategy φ such that $V_T(\varphi) = H$ a.s., that is, if H coincides with the terminal value of a self-financing strategy. Thus a claim is attainable if and only if it can be represented as a constant H_0 plus a stochastic integral with respect to the discounted stock price process

$$H = H_0 + \int_0^T \vartheta_s^H \, dX_s. \tag{4.2}$$

The initial investment $V_0(\varphi) = H_0$ needed for this perfect replication of H is also called the unique no-arbitrage price of H.

A financial market is said to be *complete* if all claims are attainable, that is, if all claims can be replicated by means of a self-financing strategy. One example of a complete market with continuous trading is the so-called Black-Scholes model which consists of two assets, a stock whose price process is described by a geometric Brownian motion and a savings account which pays a deterministic and constant rate of return. An example with discrete time trading is the Cox-Ross-Rubinstein model described above, which is also known as the binomial model. One important feature of complete markets admitting no arbitrage possibilities is the existence of a unique risk-neutral measure, i.e. a probability measure Q which is equivalent to P and which is such that X is a (local) Q-martingale. From the general theory of stochastic calculus it follows that $\int \vartheta^H \, dX$ is also a local Q-martingale under minimal assumptions on ϑ^H. Furthermore, if ϑ^H is sufficiently integrable for $\int \vartheta^H \, dX$ to be a true Q-martingale, then it follows from (4.2) that the no-arbitrage price of H is $H_0 = \mathrm{E}_Q[H]$, since in this case $\mathrm{E}_Q[\int_0^T \vartheta^H \, dX] = 0$.

If there exist claims which are not attainable, i.e. claims which do not allow a representation of the form (4.2) and hence cannot be replicated by means of any self-financing trading strategy, then the market is said to be *incomplete*; in this case there are infinitely many risk-neutral measures. For example, the completeness property is lost as soon as we move on to more general models than the ones described above. In the discrete time case, incompleteness occurs already if we replace the binomial model with a *trinomial* model, i.e. a model where the change in the value of the stock between two trading times can attain three different values. An example of an incomplete model under continuous trading is obtained by adding to the geometric Brownian motion a Poisson-driven jump component, say. Another class of examples of incomplete markets consists of models where claims are allowed to depend on more uncertainty than the one generated by the financial market. Pricing of non-attainable claims is much more delicate and typically requires a description of the preferences of the buyers and sellers. In the following we mention some different approaches to pricing in incomplete markets.

Super-replication
One approach to pricing in incomplete markets is super-replication, see e.g. El Karoui and Quenez (1995). For a given contingent claim H, this approach essentially consists in finding the smallest number V_0^*, say, such that there exists a self-financing strategy $\tilde{\varphi}$ with $V_0(\tilde{\varphi}) = V_0^*$ and

$$V_T(\tilde{\varphi}) \geq H, \ \ P\text{-a.s.}$$

By charging the price V_0^* and applying the strategy $\tilde{\varphi}$, the hedger can generate an amount which exceeds the needed amount H, P-a.s. Thus, the main advantage of this approach is that it leaves no risk to the hedger, since, after an initial investment, no additional capital is needed in order to pay the amount H to the buyer of the contract.

A (marginal) utility approach
An alternative is to derive fair prices from some utility function describing the preferences of the buyers and sellers, see Davis (1997) and references therein. Using a marginal utility argument, Davis (1997) defines the fair price of a claim H as the price which makes investors indifferent between investing "a little of their funds" in the contract and not investing in this contract. More precisely, let u be a utility function, c the investor's initial capital at time 0, p the price charged at time 0 per unit of some claim H, z the amount invested in H, and introduce

$$W(z,p,c) = \sup_{\vartheta} \mathrm{E}\left[u\left(c - z + \int_0^T \vartheta_u \, dX_u + \frac{z}{p} H \right) \right],$$

where the supremum is taken over all strategies ϑ from some suitable space of processes. The number $W(z,p,c)$ is the maximum obtainable expected utility for an investor with initial capital c who invests in z/p units of the risk H. The fair price $\tilde{u}(H;c)$ of H is then defined as the solution $\tilde{p}$ to the equation

$$\frac{\partial}{\partial z} W(0,p,c) = 0,$$

provided that the relevant quantities exist.

Quadratic approaches
A third class of approaches for pricing and hedging in incomplete markets consists of the so-called quadratic methods, see e.g. Schweizer (1999) for a survey. This class of approaches can be divided into (local) risk-minimization approaches, proposed by Föllmer and Sondermann (1986) for the case where X is a martingale and generalized to semimartingales by Schweizer (1988, 1991), and mean-variance hedging approaches, proposed by Bouleau and Lamberton (1989) and Duffie and Richardson (1991). With mean-variance hedging approaches, the main idea is essentially to "approximate" the claim H as closely as possible by the terminal value of a self-financing strategy using a quadratic criterion. More precisely, this amounts to finding a self-financing strategy $\varphi^* = (\vartheta^*, \eta^*)$ which minimizes

$$\mathrm{E}\left[(H - V_T(\varphi))^2\right] = \|H - V_T(\varphi)\|^2_{L^2(P)} \tag{4.3}$$

over all self-financing strategies φ, i.e. a strategy which approximates H in the L^2-sense. By (4.1), this strategy is completely determined by the pair $(V_0(\varphi^*), \vartheta^*)$, so that the solution to the problem of minimizing (4.3) is obtained in principle by projecting the random variable H in $L^2(P)$ on the subspace spanned by $\mathbb{R}$ and random variables of the form $\int_0^T \vartheta\, dX$. The optimal initial capital $V_0(\varphi^*)$ is often called the *approximation price* for H, and the optimal strategy is the *mean-variance hedging strategy*.

Let us now turn to the criterion of risk-minimization. For any (not necessarily self-financing) strategy $\varphi = (\eta, \vartheta)$ we define the *cost process* by

$$C_t(\varphi) = V_t(\varphi) - \int_0^t \vartheta_s\, dX_s. \tag{4.4}$$

This process keeps track of the hedger's accumulated costs associated with φ: At any time t, it is the current value $V_t(\varphi)$ of the strategy reduced by trading gains $\int_0^t \vartheta\, dX$. In particular, it follows by inserting (4.1) in (4.4) that the cost process of a self-financing strategy is P-a.s. constant. In contrast to (4.3), Föllmer and Sondermann (1986) proposed to drop the restriction to self-financing strategies but insisted on keeping the condition $V_T(\varphi) = H$. With their terminology, a strategy φ is now said to be *risk-minimizing* (for H) if $V_T(\varphi) = H$ and if it minimizes at any time t the conditional expected squared remaining costs

$$\mathrm{E}\left[(C_T(\varphi) - C_t(\varphi))^2 \middle| \mathcal{F}_t\right].$$

This optimality criterion amounts to keeping the fluctuations in the cost process as small as possible under the condition $V_T(\varphi) = H$; in particular, Föllmer and

Sondermann (1986) proved that the cost process of a risk-minimizing strategy is a martingale.

Quantile hedging and shortfall risk minimization
One possibly undesirable feature of the quadratic approaches is the fact that they punish losses and gains equally. An alternative is to use *quantile hedging*, see Föllmer and Leukert (1999), where the objective is to hedge the claim with a certain probability. Another alternative is the criterion of minimizing the expected shortfall risk, i.e. expected losses from hedging, which has been proposed by Föllmer and Leukert (2000) and Cvitanić (1998). They introduce a *loss function* $l : [0, \infty) \mapsto [0, \infty)$, which is taken to be an increasing convex function with $l(0) = 0$, and consider the problem of minimizing

$$\mathrm{E}\left[l\left((H - V_T(\varphi))^+\right)\right], \tag{4.5}$$

over the class of self-financing hedging strategies. Typical loss functions are power functions, $l(x) = x^p$, $p \geq 1$, and in this case, (4.5) is related to minimizing the so-called lower partial moments.

5 Interplay between insurance and finance

In this section we mention some specific areas of the interplay between finance and insurance that will be treated in this thesis. A survey of aspects of the growing interplay between the two fields is also given in Embrechts (1996), who mentions institutional issues such as the increasing collaboration between insurance companies and banks (e.g. the construction of so-called "financial supermarkets") and the deregulation of insurance markets as two important aspects. In addition, the emergence of products combining financial and insurance risk (e.g. so-called unit-linked insurance contracts, various catastrophe futures and options and financial stop-loss reinsurance contracts) has forced the two fields to search for combinations and unification of methodologies and basic principles.

5.1 Unit-linked insurance contracts

Unit-linked[2] insurance contracts seem to have been introduced for the first time in the Netherlands in the early fifties; in the United States the first unit-linked insurance contracts were offered around 1954, and, in the United Kingdom, unit-linked contracts appeared for the first time in 1957[3]. A unit-linked life insurance contract differs from traditional life insurance contracts in that benefits (and sometimes also premiums) depend explicitly on the development of some stock index or the value of some (more or less) specified portfolio. This construction allows for great flexibility

[2]These contracts are often also called *equity-linked* or *equity-based* insurance contracts; in the United States the contracts are known as *variable life* insurance contracts.

[3]See Turner (1971) for an overview of the early history of unit-linked life insurance products; a treatment of some institutional aspects of unit-linked insurance contracts is given in Squires (1986).

as compared with traditional life insurance products in that the policy-holder is offered the opportunity of deciding how his or her premiums are to be invested. Today, issuers of unit-linked life insurance contracts typically offer a variety of investment possibilities that include worldwide, European or country specific indices, and reference portfolios with specific investment profiles, e.g. investments in companies from certain branches or regions, or organizations with certain ethical codes.

Unit-linked contracts have been analysed by actuaries since the late sixties, see e.g. Turner (1969), Kahn (1971) and Wilkie (1978); the two last mentioned give simulation studies for an insurance company administrating portfolios of unit-linked insurance contracts. Using modern theories of financial mathematics, Brennan and Schwarz (1976, 1979a,b) proposed new valuation principles and investment strategies for unit-linked insurance contracts with so-called asset value guarantees (minimum guarantees). Their principles essentially consisted in combining traditional (law of large numbers) arguments from life insurance with the methods of Black and Scholes (1973) and Merton (1973). By appealing to the law of large numbers, Brennan and Schwartz (1979a,b) first replaced the uncertain courses of the insured lives by their expected values, so that the actual insurance claims including mortality risk as well as financial risk were replaced by modified claims, which only contained financial uncertainty. These modified claims were then recognized as essentially being options (with a very long maturity, though) which could in principle be priced and hedged using the basic principles of (modern) financial mathematics due to Black and Scholes (1973) and Merton (1973). More recently, the problem of pricing unit-linked life insurance contracts (under constant interest rates) has been addressed by Delbaen (1986), Bacinello and Ortu (1993a) and Aase and Persson (1994), among others, who combined the martingale approach of Harrison and Kreps (1979) and Harrison and Pliska (1981) with law of large numbers arguments. Whereas all the above mentioned papers assumed a constant interest rate, Bacinello and Ortu (1993b), Nielsen and Sandmann (1995) and Bacinello and Persson (1998), among others, generalized existing results to the case of stochastic interest rates.

In contrast to earlier approaches, Aase and Persson (1994) worked with continuous survival probabilities (i.e. with death benefits that are payable immediately upon the death of the policy-holder and not at the end of the year as would be implied by discrete time survival probabilities) and suggested investment strategies for unit-linked insurance contracts by methods similar to the ones proposed by Brennan and Schwartz (1979a,b) for discrete time survival probabilities. In contrast to Brennan and Schwartz (1979a,b), who considered a "large" portfolio of policy-holders and therefore worked with "deterministic mortality", Aase and Persson (1994) considered a portfolio consisting of one policy-holder only. However, in all the above papers, the uncertain courses of the insured lives were replaced at an early point with the expected courses in order to allow an application of standard financial valuation techniques for complete markets. The resulting strategies therefore did not account for the mortality uncertainty within a portfolio of unit-linked life insurance contracts, and the approach thus leaves open the question of how to quantify and manage the combined actuarial and financial risk inherent in these contracts. In particular, it leaves open the question to which extent this combined risk can be hedged on the financial markets.

In a recent paper by the author (Møller, 1998a) risk-minimizing hedging strategies were determined for a portfolio of unit-linked pure endowment contracts using the theory of risk-minimization due to Föllmer and Sondermann (1986). In contrast to the approaches of Brennan and Schwartz (1979a,b) and Aase and Persson (1994), Møller (1998a) did not average away the mortality risk (the uncertainty associated with not knowing the actual number of survivors), but analysed the insurance contracts as contingent claims in an incomplete market. Consequently, the resulting strategies reflect, and react to, the financial uncertainty as well the uncertainty associated with not knowing the actual number of survivors. In particular, it is clearly visible from these strategies how an insurer applying the risk-minimizing hedging strategy is adapting his portfolio of stocks and his deposit on the savings account to the actual development within the portfolio of insured lives. Using the results of Föllmer and Sondermann (1986), Møller (1998a) also derived measures for the part of the total risk in the unit-linked contracts that cannot be hedged away by trading on financial markets only, the so-called *intrinsic risk*. Furthermore, it was shown that this intrinsic risk could actually be completely eliminated by including in addition a dynamic reinsurance market. More precisely, it was assumed that the insurer could trade continuously, in addition to the stock and the savings account, a third asset with a price process which was, at any time, equal to the prospective reserve associated with a pure endowment insurance with sum insured 1. In this way, the insurance risk was essentially transformed into a traded asset or a security, a procedure which is known as *securitization*. In the model considered there, this additional asset was indeed sufficient to restore completeness, leading to unique prices and self-financing investment strategies.

5.2 Other insurance derivatives

In this section we describe some further specific products that have appeared in practice and that combine traditional insurance risk and financial derivatives. The best known examples are probably catastrophe futures, catastrophe-linked bonds, financial stop-loss contracts and stop-loss contracts with a barrier. These new products are really genuine combinations of financial derivatives and insurance products, and they are known as *insurance derivatives*. The emergence of such products has been serving as a catalyst for breaking down borders between traditional reinsurance and finance and has opened up the possibility of rethinking fundamental principles of reinsurance and investment. This development presents a challenge to direct insurers and reinsurers as well as to financial institutions in general.

Catastrophe insurance (CAT) futures
In the 1980s and early 1990s, several severe catastrophes impaired the capacity of reinsurers offering traditional catastrophe covers, and this lack caused an increase in reinsurance premiums. In 1992 the so-called catastrophe insurance (CAT) futures and options on CAT futures were introduced. These instruments standardized catas-

trophe insurance risk and transformed it into tradeable securities, thus providing a new tool for insurers seeking covers against catastrophe risk. This securitization was modified in 1995, but the underlying idea essentially remained the same. For an introduction to CAT futures, see e.g. Cummins and Geman (1995) and references therein; an overview of securitization of catastrophe insurance risk and an analysis of some of the problems associated with securitization can be found in Tilley (1997).

The basic idea is the following. Consider losses occurring in a specific area and caused by certain well defined catastrophic events, e.g. hurricanes with a certain wind speed or earthquakes of a certain magnitude. Clearly, different insurers will be subject to different exposures from such risks as a consequence of differences in the composition of their insurance portfolios, and with traditional reinsurance contracts, each company would purchase their own insurance covers against risk. Assume now that a number of (suitably chosen) insurance companies report premiums and claims related to the pre-specified type of catastrophes (during certain pre-specified periods) to some central office. Based on the reports, this office constructs a *loss index* $L = (L_t)_{0 \le t \le T}$ which is taken to be the underlying process for a futures price process. More precisely, this means that the index L is being reported regularly to the public and that a futures price process $F = (F_t)_{0 \le t \le T}$ is constructed by fixing the terminal value $F_T = \min(2, L_T/\kappa)$, where κ is the accumulated premiums for the reporting companies and T is some fixed finite time horizon. Insurance and reinsurance companies as well as other investors can now buy and sell this standardized catastrophe risk by purchasing and issuing options on this index on some stock exchange. For example, the *call spread* $H = (F_T - K_1)^+ - (F_T - K_2)^+$, $0 \le K_1 \le K_2 \le 2$, provides cover for relative losses (i.e. the ratio of losses over premiums) in the interval $[K_1, K_2]$. The main advantage of this construction lies in the standardization and securitization of the catastrophic risk, which serves to transform the risk related to individual insurance companies into one (common) quantity. Thus, this transformed risk may be more attractive and understandable to a group of investors which is larger than the one of traditional reinsurance companies, since it is relatively close in nature to existing financial derivatives. By attracting agents from a wider group of agents than just the traditional reinsurance companies, these instruments increased the financial capacity of the reinsurance market. On the other hand, the disadvantage for the direct insurers is that their own relative losses may differ considerably from the average relative losses of the reporting companies. Thus, for a given insurance company, the cover from the call spread on the CAT futures index will typically not correspond exactly the actual loss experienced by this company.

Catastrophe-linked bonds

Individual insurance companies can also choose to securitize part of their insurance risk directly, for example by issuing bonds that are linked to insurance losses from certain insurance portfolios. One example of such an arrangement is the so-called *Winterthur Insurance Convertible Bond*, also called WinCAT bond. This bond, which was introduced by Winterthur in 1997, is described and analysed in Schmock (1999); see also Gisler and Frost (1999). With this three year bond, investors receive annual coupons as long as certain catastrophic events related to one of Winterthur's own insurance portfolios have not occurred. Thus, the investors receive a return from the bond which exceeded the market interest rate as long as no catastrophes has occurred and a lower return in the case of a catastrophe. The difference between the return under no catastrophes and the interest rate on the market was essentially a premium that Winterthur paid investors for "putting their money at risk"; similarly, the low return in connection with a catastrophic event essentially implied that the investors had covered part of Winterthur's losses.

This type of product has the advantage over for example options on the CAT futures index, that it provides a much more tailor-made cover for the issuer, in that the trigger events that knock out the coupons are directly linked to the company's own insurance portfolio and not to some standardized index. The disadvantage is that there may be considerable costs associated with the selling of such bonds and that the seller will have to convince buyers that they are only subject to a minimal moral hazard and credit risk.

Financial stop-loss contracts
Whereas CAT futures and Catastrophe-linked bonds are aimed at a larger group of investors, new reinsurance contracts that combine elements of insurance and financial derivatives have also been introduced by traditional reinsurers. In Swiss Re (1998), several new contracts are described under the title "Integrated Risk Management Solutions". One example is the so-called financial stop-loss contract, which promises to pay at some fixed time T the amount

$$H = (U_T + Y_T - K)^+ \tag{5.1}$$

where U_T is the aggregate claim amount during $[0, T]$ on some insurance portfolio, Y_T is some financial loss and K is some retention limit. For $Y_T \equiv 0$ P-a.s., the contract is just a traditional stop-loss contract; however, the loss Y_T could for example be a put option on some underlying stock index S, that is $Y_T = (c - S_T)^+$ or it could simply be the loss associated with holding one unit of this index, that is, $Y_T = S_0 - S_T$. The financial stop-loss contracts provide a coverage not only for large losses due to fluctuations within the insurance portfolio (insurance risk) but also for adverse development of the financial markets (financial risk). In practice, reinsurance companies would typically sell spreads on the form: $(U_T + Y_T - K_1)^+ - (U_T + Y_T - K_2)^+$, where $0 \leq K_1 \leq K_2$, which covers losses $U_T + Y_T$ in the interval $(K_1, K_2]$.

The main idea behind the insurance contract (5.1) is that it provides cover for the insurer's *total risk*, i.e. the combined insurance risk from the insurance portfolio and the financial risk from the financial portfolio. With a traditional stop-loss contract, the reinsurer would cover insurance losses exceeding the level K. However, the financial stop-loss contract is designed so that the cover is only paid provided that the insurance loss augmented by the financial loss exceeds this level. Thus, a large financial gain $-Y_T$ may compensate for large insurance losses, and in this situation, the buyer does not really need additional compensation from the reinsurer. This feature is illustrated by figure 1.1. The area above the solid line represents pairs (Y_T, U_T) of financial losses Y_T and accumulated insurance claims U_T that generate a payment from the reinsurer. The area between the solid line and and the dashed line are pairs (Y_T, U_T) where (large) insurance claims U_T are partially compensated by financial gains $-Y_T$. The problem of pricing these contracts is a challenge to both

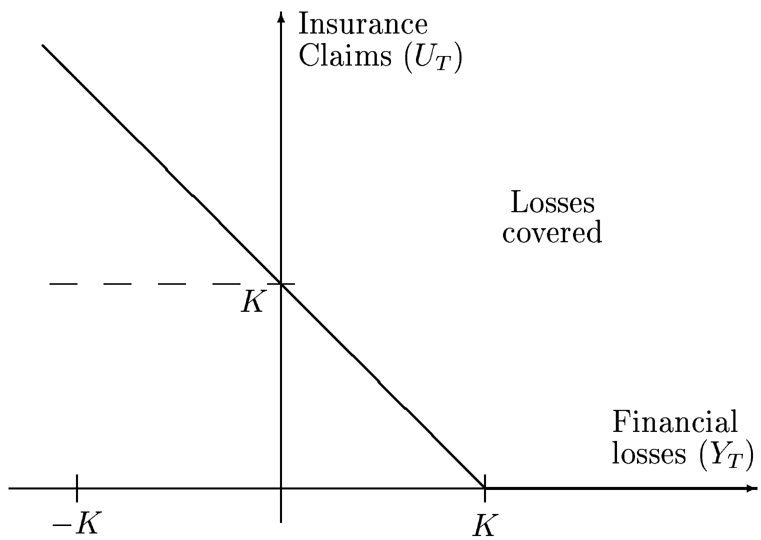

Figure 1.1: *Regions of cover under the financial stop-loss contract with retention K.*

actuaries and financial mathematicians. This fact is for example underscored by the following quotation from Swiss Re (1998, p. 15), "..., the risk-neutral valuation technique traditionally used for the pricing of financial derivatives cannot be applied directly but needs to be adjusted and complemented by actuarial methods".

The contract (5.1) should be compared to the alternative of buying a traditional stop-loss contract with retention level K' paying $(U_T - K')^+$ and a traditional financial derivative, which pays $(Y_T - K'')^+$; the constants K' and K'' could for example be chosen such that $K' + K'' = K$. It follows already from the inequality

$$(U_T + Y_T - K)^+ \leq (U_T - K')^+ + (Y_T - K'')^+, \tag{5.2}$$

which is satisfied provided that $K' + K'' \leq K$, that the cover from the financial stop-loss contract is dominated by combinations of a traditional stop-loss contract on U_T and a call option on Y_T. The region of cover under the stop-loss contract and the call option is depicted in Figure 1.2 as and the area above the solid lines. This figure shows that the region is indeed larger than the corresponding region under the financial stop-loss contract. In particular, it follows that the insurer will receive compensation from the reinsurer also in the situation where very large gains have arisen from investments. Thus, with the traditional instruments, the insurer has

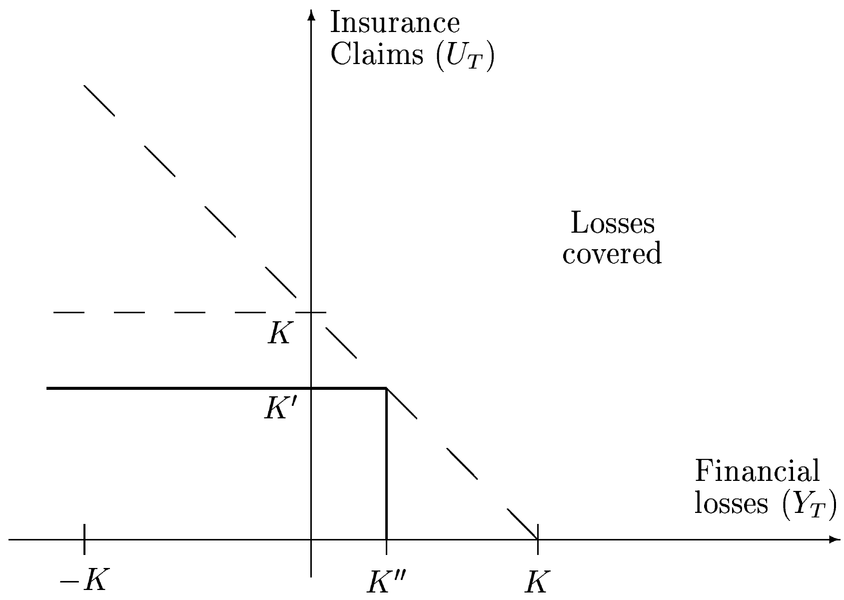

Figure 1.2: *Regions of cover under a traditional stop-loss contract* $(U_T - K')^+$ *and a call option* $(Y_T - K'')^+$.

actually bought too much insurance cover; the financial stop-loss contract suits the needs of the insurer better.

Finally, we emphasize that the inequality (5.2) indeed indicates that the premium for the financial stop-loss contract should be dominated by the sum of the price on the financial market of $(Y_T - K'')^+$ and the reinsurers' premium for $(U_T - K')^+$. However, the difference may be relatively small since financial stop-loss contracts have only appeared recently and since they are only bought and sold in very limited amounts. Another important point is that, whereas the call option is sold on the financial market, the (financial and traditional) stop-loss contracts are agreements between a reinsurer and an insurer, and such contracts are typically not traded on stock exchanges. Therefore it is not in general possible to make statements like "by no-arbitrage arguments" etc. about insurance premiums; see also the discussion on the difference between actuarial and financial valuation principles in Embrechts (1996).

5.3 Combining theories for financial and actuarial valuation

One fundamental difference between the financial valuation techniques, or, more precisely, pricing by no-arbitrage, and the classical actuarial valuation principles reviewed above is that the financial valuation principles are formulated within a framework which includes the possibility of trading certain assets, whereas several of the classical actuarial valuation principles are based on more or less ad hoc considerations involving the law of large numbers. While the financial valuation principles are based on *dynamic* trading, many decision problems in insurance, for example concerning the choice of optimal reinsurance plans and premiums, where traditionally analysed taking a *static* view. Several attempts have been made to bring together elements of the two theories, and this whole area is still very much "under construction". We do not aim at giving a complete overview of this process but rather at focusing on some specific developments of importance for the rest of the thesis.

Dynamic reinsurance markets (From financial to actuarial valuation principles)
Several authors have studied dynamic reinsurance markets in a continuous time framework using no-arbitrage conditions, see for example Sondermann (1991), Delbaen and Haezendonck (1989) and De Waegenaere and Delbaen (1992). For an equilibrium analysis of dynamic reinsurance markets, see e.g. Aase (1993) and references therein. The main idea underlying the above mentioned papers is to allow for dynamic rebalancing of proportional reinsurance covers. They all assume that some process related to an *insurance risk process* (accumulated premiums minus claims) of some insurance business is tradeable and that positions can be rebalanced continuously. For example, this can mean that reinsurers can change at any time (continuously) the fractions of the insurance business that they have accepted. Thus, the insurance risk process can essentially be viewed as a traded security, and this already imposes no-arbitrage bounds on premiums for other (traditional) reinsurance contracts such as stop-loss contracts.

Let us review the main results obtained by Sondermann (1991) and Delbaen and Haezendonck (1989) in more detail. As in the previous section, let U_t be the accumulated claims during $[0,t]$ in some insurance business. Let furthermore $p = (p_t)_{0 \leq t \leq T}$ be a predictable process related to the premiums on this business, and define a new process X by

$$X_t = U_t + p_t. \tag{5.3}$$

Sondermann (1991) takes $-p_t$ to be the premiums paid during $[0,t]$, so that $-X_t$ is in fact identical to the insurance risk process. Thus, one can think of X_t as the value at time t of an account where claims are added and premiums subtracted as they incur. In particular, in the special case where premiums are paid continuously at a fixed rate κ, $p_t = -\kappa t$. Reinsurers can now participate in the risk by trading the asset X, i.e. by holding a position in the asset with price process X. Sondermann (1991) points out that in this setting of a dynamic market for proportional reinsurance contracts, traditional reinsurance contracts such as stop-loss contracts can be viewed as contingent claims and that these claims should be priced so that no arbitrage possibilities arise. Delbaen and Haezendonck (1989) take p_t to be the premium at which the direct insurer can sell the remaining risk $U_T - U_t$ on the reinsurance market. Thus, in their framework, X_t represents the insurer's liabilities at time t. In the special case where the direct insurer receives continuously paid premiums at the rate κ and provided that this premium is identical to the one charged by the reinsurers, we obtain that $p_t = \kappa(T-t)$, so that p_t in this situation differs from

Sondermann's choice only by the constant κT. Delbaen and Haezendonck (1989) assume that U is a compound Poisson process, i.e. $U_t = \sum_{i=1}^{N_t} Z_i$, where N is a Poisson process and $Z_1, Z_2, \ldots$ is a sequence of i.i.d. non-negative random variables which are independent of N. They then focus on the set of equivalent measures Q which are such that U is also a Q-compound Poisson process. For each such measure Q, a predictable premium process p is obtained by requiring that X is a Q-martingale. This procedure is partly motivated by no-arbitrage considerations (assuming in addition that all amounts have been discounted with the interest rate on the market), since this guarantees that no arbitrage possibilities arise from trading in X. In this way, Delbaen and Haezendonck (1989) recover several traditional actuarial valuation principles on a certain subspace of claims from no-arbitrage considerations, namely the expected value principle, the variance principle and the Esscher principle. A more detailed account of the results of Delbaen and Haezendonck (1989) is also given by Embrechts (1996).

From actuarial to financial valuation principles
Gerber and Shiu (1996) among others consider the situation where the logarithm of the stock price process is a Levy process, i.e. a process with independent and stationary increments. For example, this class of processes includes the geometric Brownian motion and the geometric (shifted) compound Poisson process. Within this setting, they demonstrate how the Esscher transform (see (2.6)), can be used in the pricing of options. They give a very simple option pricing formula which involves Esscher transforms and which, for a European call option, indeed specializes to the well-known Black-Scholes formula in the case of a geometric Brownian motion. Furthermore, they demonstrate how this pricing formula can be derived via a simple utility indifference argument in the case of a power utility function $u(x) = \frac{x^{1-a}}{1-a}$ with parameter $a > 0$. This way Gerber and Shiu (1996) give a candidate for a martingale measure that could be used for pricing in incomplete markets also; they call the resulting martingale measure the *risk-neutral Esscher measure*. For further results on the relation between Esscher transforms, utility theory and equilibrium theory, see Bühlmann (1980, 1983) and references in Gerber and Shiu (1996). A treatment of some of the mathematical aspects associated with Esscher transforms for stochastic processes can be found in Bühlmann et al. (1996).

In Schweizer (1997), starting points are the traditional standard deviation and variance principles, which are of the form (2.4). These principles are taken as measures of riskiness, which assign to each claim a premium. It is then argued that the measures can equivalently be viewed as measures of preferences which operate on the insurer's terminal wealth by simply changing the sign on the loading factor. This way Schweizer (1997) obtains measures which to each outcome of the insurer's final wealth assign a number, and one can think of this number as the expected value of the insurer's utility of this wealth. These new measures are then embedded in a financial framework where the insurer can trade certain assets. Via an indifference argument, Schweizer (1997) derives financial counterparts of the actuarial standard deviation and variance principles. These financial valuation principles resemble their actuarial counterparts in that they consist of an expectation augmented by some safety-loading. However, for the financial valuation principle, the expected value is now computed under a specific martingale measure known as the variance optimal martingale measure. Furthermore, the loading factor is now a function of the variance of the so-called non-hedgeable part of the claim H which, in general, is smaller than the variance of H. These new financial valuations are in accordance with no-arbitrage pricing for attainable claims, and thus, they provide alternative approaches for the valuation of options and other derivatives in incomplete markets.

6 Overview and contributions of the thesis

The aim of this thesis is to analyse insurance claims which combine financial and insurance risk. The thesis can be divided into two main parts. The first part consists of Chapters 2 and 3 and gives applications to insurance of the theory of risk-minimization with special emphasis on hedging (and pricing) of unit-linked insurance contracts. The second part, Chapters 4, 5 and 6, deals with the financial variance and standard deviation principles of Schweizer (1997) and gives several actuarial applications of these principles. Below we give a brief account of the contents of the individual chapters.

Hedging unit-linked insurance contracts in discrete time
Chapter 2, based on Møller (1999a), gives an introduction to the necessary financial terminology and to the problem of pricing and hedging of unit-linked insurance contracts. The presentation, which is kept in a simple discrete time framework and hence requires only a minimum of stochastic analysis, discusses the application of various approaches for hedging and pricing in incomplete markets. The techniques of Brennan and Schwarz (1976, 1979a,b) are compared to the ones suggested by super-replication and risk-minimization, respectively. The Cox-Ross-Rubinstein model is considered as a main example. In this case, the financial market consists of two assets, a savings account with constant interest and a stock, whose change in value between two trading times can attain two different values only. Risk-minimizing hedging strategies are determined within this set-up for a portfolio of unit-linked life insurance contracts, and these strategies are briefly compared to the ones obtained by Møller (1998a) in a continuous time framework.

Hedging insurance payment processes
The theory of risk-minimization introduced by Föllmer and Sondermann (1986) focuses on the problem of hedging a contingent claim payable at a fixed time. However, insurance contracts often generate genuine payment streams where amounts are paid out over time. For example, with a so-called life annuity, payments are due yearly, say, from a certain time and as long as the policy-holder is still alive. Similarly, life insurance contracts in general are often paid by periodic premiums, e.g. premiums paid at the beginning of each year as long as the policy-holder is still alive. In Chapter 3, which is based on Møller (1998b), we incorporate general payment streams in to the theory of risk-minimization and hence provide a framework which allows for the analysis of (insurance) payment processes. This modified framework is applied

to the analysis of general unit-linked life insurance contracts, where the state of the policy is described by a Markov jump process with a finite state space. This generalizes previous results obtained in Møller (1998a). In addition, we consider payment processes with claims occurring according to an inhomogeneous Poisson process and where claim amounts are affected by some tradeable claim index, for example an index for claim inflation.

On transformations of actuarial valuation principles
In Chapter 4 we review the main results of Schweizer (1997) on financial transforms of the actuarial variance and standard deviation principles and recall the crucial indifference argument used there. We determine optimal investment strategies associated with these principles for the case where the discounted stock price process is described by a continuous semimartingale. Furthermore, we give an alternative and more direct characterization of the financial standard deviation principle, which does not involve an indifference argument. Finally, a numerical example demonstrates how the financial valuation principles can be applied for the valuation of unit-linked life insurance contracts and for determining optimal trading strategies. Chapter 4 is based on Møller (1999b).

Indifference pricing of insurance contracts: Theory
Chapters 5 and 6 are based on Møller (2000) and are devoted to a more detailed study of some properties of the financial variance and standard deviation principles. In particular, we focus on the dependence of the fair premiums on the amount of information available to the insurer. For reasons of length and in order to separate theory and examples, the presentation is divided into two chapters. Via a comparison result for mean-variance hedging errors in different filtrations, we obtain in Chapter 5 a natural ordering of the fair premiums. More precisely, we show that more actuarial information leads to lower premiums and characterize this difference further. The results allows for derivation of relatively simple upper and lower bounds for the fair premiums of reinsurance contracts under the assumption of independence between the traded assets and the insurance risk involved.

Indifference pricing of insurance contracts: Examples
In Chapter 6 we apply the results obtained in Chapter 5 to the analysis of some examples related to insurance. We determine the fair premiums and the optimal trading strategies under various scenarios corresponding to different amounts of information. Contracts considered include unit-linked insurance contracts and stop-loss contracts with a barrier. In addition, the chapter analyses a framework which is sufficiently general to allow for situations where the stock price process (of an insurance company, for example) is affected by certain catastrophic events.

Chapter 2

Hedging Unit-Linked Insurance Contracts in Discrete Time

(This chapter is an adapted version of Møller (1999a))

In this chapter we consider a portfolio of unit-linked life insurance contracts and determine risk-minimizing hedging strategies within a discrete time set-up. As a main example we consider the Cox-Ross-Rubinstein model and a unit-linked pure endowment contract under which the policy-holder receives $\max(S_T, K)$ at time T if he is then alive, where S_T is the value of a stock index at the term T of the contract and K is a guarantee stipulated by the contract. In contrast to most of the existing literature, we view the contracts as contingent claims in an incomplete model and discuss the problem of choosing an optimality criterion for hedging strategies. The subsequent analysis leads to a comparison of the risk (measured by the variance of the insurer's loss) inherent in unit-linked contracts in the two situations where (1) the insurer applies the risk-minimizing strategy and (2) the insurer does not hedge. The chapter includes numerical results which can be used to quantify the effect of hedging and to describe how this effect varies with the size of the insurance portfolio and assumptions concerning the mortality.

1 Introduction

A financial market is said to be *complete* if all contingent claims can be hedged perfectly and, hence, priced uniquely. This is the case for example in the well-known Black-Scholes model and the so-called CRR model, proposed by Cox, Ross and Rubinstein (1979). In the CRR model, the financial market consists of two basic traded assets, a stock and a savings account. Trading takes place at the end of fixed periods of equal length, and the change in the value of the stock between two trading times can attain *two* different values only; as a consequence of this simple structure, the model is also known as the *binomial model*. Between two trading times, a deterministic constant interest is earned on the savings account. To each contingent claim, there exists a unique self-financing trading strategy that duplicates the payment. This strategy, which specifies at any time a portfolio consisting of a certain number of stocks and a certain deposit on the savings account, requires an initial investment of some amount at time 0, say. Between time 0 and the fixed terminal time T, say, where the claim is payable, no additional inflow or outflow of capital is needed. Furthermore, at the terminal time T, the value of the strategy will be exactly equal to the amount payable in connection with the claim. However, in more general models, for example a trinomial model, where the change in the value of the stock between two trading times can attain *three* different values instead of two, this property is not preserved, and contingent claims can typically not be duplicated by self-financing strategies and hence cannot be priced uniquely by no-arbitrage arguments only. Such models are said to be *incomplete*.

Another simple class of incomplete models can be obtained from any a priori given complete model by allowing contingent claims to depend on an additional source of risk that is stochastically independent of the risk on the financial market. This extension is relevant e.g. for the analysis of unit-linked life insurance contracts. These insurance contracts typically include options on some underlying stock (index) that are payable to the policy-holder (the insured) provided that he or she survives to some agreed term, and thus, the insurance benefit is *linked* to the stock. For example, with a *unit-linked pure endowment contract with guarantee*, the policy-holder receives the maximum of the value of one (say) unit of a stock index and some guarantee provided that he is still alive at the term of the contract. By construction, the contracts include risk related to the future development of the stock index as well as uncertainty as to whether or not the policy-holder will survive. Unit-linked contracts have been analyzed by Brennan and Schwartz (1976, 1979a, 1979b), who proposed one pricing principle and investment strategies for insurers issuing these contracts. This pricing principle and these investment strategies were derived by combining no-arbitrage arguments and traditional arguments from insurance. More recently, several authors have dealt with the problem of pricing of various unit-linked contracts using ideas that originate from the ones proposed by Brennan and Schwartz (1976), see e.g. Delbaen (1990), Nielsen and Sandmann (1995), Aase and Persson (1994) and references therein. The principle proposed basically consists in replacing the unknown future course of the insured lives by the expected, a principle which can be justified using the law of large numbers since insurers are typically holding a large number of contracts. Thus, for the unit-linked pure endowment contract, the pricing problem is basically transformed into the problem of pricing a contract which specifies the payment of $\max(S_T, K)$ times the probability of survival to the terminal time T, where S_T denotes the terminal value of the stock index and where K denotes the guaranteed amount. This new claim can then typically be priced by using no-arbitrage arguments only.

In this chapter, we argue that the insurance contracts should really be viewed as contingent claims in an incomplete model. In the mere framework of no-arbitrage pricing, this leaves open the problem of pricing and hedging the contracts, since self-financing strategies which duplicate the claims do not exist. We discuss various approaches for determining optimal trading strategies and apply the criterion of *risk-minimization* instituted by Föllmer and Sondermann (1986) to suggest investment strategies for the contracts. These strategies minimize the variance of the insurer's future costs, which are defined as the difference between the insurance claim and the gains made from trading on the financial market. In Møller (1998a,b) risk-minimizing hedging strategies were determined for unit-linked life insurance contracts in a generalized Black-Scholes framework. In that framework, the financial market consisted of two basic assets, a stock and a savings account, which could be traded dynamically in continuous time, and the development of the price of the stock were modeled by a diffusion. Here, we demonstrate how the theory of risk-minimization works for unit-linked contracts in much more simple discrete time models, such as the CRR-model. This has the advantage over the previous approach that certain technical problems associated with stochastic processes in continuous time are avoided. Hence, we are able to keep the present chapter almost free of mathematical technicalities and still deal with the essentials of the problem of hedging unit-linked insurance contracts. To exemplify the incompleteness of the model, note that even if it is possible to replicate perfectly the option on the stock index, the combined contract where this option is payable provided that the policy-holder survives, cannot be hedged. For example, how would you replicate perfectly the contract that pays an amount equal to the value of unit of a stock to a policy-holder at a future date, provided that he or she is still alive at this date, by trading on the stock market only?

This chapter is organized as follows. In Section 2 we analyze the insurer's loss associated with a portfolio of identical unit-linked insurance contracts and examine the asymptotical properties as the size of the portfolio is increased. We consider first the situation where the insurer does not trade on the financial market, and we then show how risk can be reduced considerably by buying suitable options on the underlying stock. Section 3 is devoted to a brief introductory analysis of a two-period model, where trading takes place only at time 0 and time 1, say. A pure unit-linked insurance contract is considered as an example. In Section 4, we introduce the notion of trading in multi-period models and review briefly the theory of risk-minimization, see Föllmer and Sondermann (1986) and Föllmer and Schweizer (1988). We then mention some fundamental properties of the CRR model. In Section 5, we discuss different criteria for optimality of hedging strategies and compare briefly the criterion of superreplication, risk-minimization and the Brennan-Schwartz approach. Next, we determine risk-minimizing hedging strategies for unit-linked pure endowment contracts within the CRR-model. Some numerical results are presented in Section 6, and in Section 7 we finally compare the results from the present discrete time analysis to the ones obtained by the author in a continuous time framework (Møller, 1998a,b).

2 Unit-linked life insurance contracts

With a unit-linked life insurance contract, the sum insured typically depends on the development of some stocks or stock indices. We set out with some general considerations that do not rely on any specific assumptions concerning the choice of model for the stock market. The value of the stock at time t is denoted S_t and we shall refer to the entire development of the stock by simply writing S.

Consider a portfolio consisting of n policy-holders who buy the same form of unit-linked pure endowment contract at time 0. This contract specifies that an amount $f(S)$ is paid to the policy-holder at time T if he or she is still alive at this time; f is a function which prescribes some dependence on the development of the stock price. For example, the amount paid could be a function of the terminal value of the stock only, that is

$$f(S) \quad = \quad S_T, \tag{2.1}$$

or the terminal value guaranteed against falling short of some prefixed amount K

$$f(S) \quad = \quad \max(S_T, K). \tag{2.2}$$

The contract (2.1) is known as a *pure unit-linked* contract and (2.2) is called *unit-linked with guarantee* (the guarantee is in this case K). However, f could also specify more complex dependences, for example a guaranteed annual return is given by

$$f(S) = K \cdot \prod_{j=1}^{T} \max\left(1 + \frac{S_j - S_{j-1}}{S_{j-1}}, 1 + \delta_j\right). \tag{2.3}$$

Here, the fraction $(S_j - S_{j-1})/S_{j-1}$ is the return in year j on the asset S and δ_j is the guarantee in year j. At time 0 the amount payable at time T is guaranteed against falling short of $K \cdot \prod_1^T (1 + \delta_j)$.

2.1 The insurer's loss

In this section we consider the loss of an insurance company that is not trading on the financial market. Denote by $Y_t^{(n)}$ the number of survivors at time t, and let $f(S)$ denote the amount payable at time T if the policy-holder is still alive at that time. If each individual contract is paid by a single premium κ at time 0, the *present value* at time 0 of the insurer's loss associated with the contracts is

$$L_n = Y_T^{(n)} f(S) e^{-\delta T} - n\kappa, \tag{2.4}$$

i.e. the present value is taken to be the payments discounted at some constant interest rate δ (actuarial usage). Here, the random variables $Y_T^{(n)}$ and $f(S)$ are taken to be defined on some probability space $(\Omega, \mathcal{F}, P)$; Ω is a set which can be interpreted as all states of the world, and $\mathcal{F}$ is a σ-algebra of subsets of Ω. P is a probability measure which, in particular, describes the joint distribution of the pair $(Y_T^{(n)}, f(S))$.

We assume that the policy-holders' remaining lifetimes are stochastically independent of the development of the stock. This very natural assumption simplifies computations greatly. It is straightforward to compute the mean and the variance of the insurer's total loss by using standard rules for conditional expectations and variances:

$$\mathrm{E}[L_n] = e^{-\delta T}\mathrm{E}[f(S)]\,\mathrm{E}[Y_T^{(n)}] - n\kappa,$$

and

$$\begin{aligned}\mathrm{Var}[L_n] &= \mathrm{E}[\mathrm{Var}[L_n|S]] + \mathrm{Var}[\mathrm{E}[L_n|S]] \\ &= e^{-2\delta T}\mathrm{E}[f(S)^2]\,\mathrm{Var}[Y_T^{(n)}] + e^{-2\delta T}\mathrm{Var}[f(S)]\,\left(\mathrm{E}[Y_T^{(n)}]\right)^2.\end{aligned}$$

Assume in addition that the n policy-holders are aged x at time 0 with i.i.d. remaining lifetimes $T_1, \ldots, T_n$. The survival function is denoted

$$P[T_1 > t] = {}_t p_x,$$

where we have used standard actuarial notation. Formulas can now be made more explicit, since this additional assumption implies that

$$\mathrm{E}[Y_T^{(n)}] = \mathrm{E}\left[\sum_{i=1}^{n} 1_{\{T_i > T\}}\right] = \sum_{i=1}^{n}\mathrm{E}\left[1_{\{T_i>T\}}\right] = nP[T_1 > T] = n\,{}_T p_x,$$

and

$$\mathrm{Var}[Y_T^{(n)}] = \sum_{i=1}^{n}\mathrm{Var}[1_{\{T_i>T\}}] = n\,{}_T p_x(1 - {}_T p_x).$$

These formulas follow for example by noting that $1_{\{T_1>T\}}, \ldots, 1_{\{T_n>T\}}$ are i.i.d. Bernoulli variables (i.e. 0-1 variables) which attain the value 1 with probability ${}_T p_x$. By inserting the expressions for the mean and variance of $Y_T^{(n)}$ in the above formulas for the mean and variance of L_n, we find that

$$\mathrm{E}[L_n] = n\left({}_T p_x e^{-\delta T}\mathrm{E}[f(S)] - \kappa\right), \tag{2.5}$$

$$\mathrm{Var}[L_n] = e^{-2\delta T}\mathrm{E}[f(S)^2]\,n\,{}_T p_x(1 - {}_T p_x) + e^{-2\delta T}\mathrm{Var}[f(S)]\,n^2\,{}_T p_x{}^2. \tag{2.6}$$

We note the following properties: The expected present value of the loss is equal to 0 if and only if $\kappa = {}_T p_x e^{-\delta T}\mathrm{E}[f(S)]$. In addition, we see that $\mathrm{Var}[L_n]$ is asymptotically equivalent to $const \cdot n^2$ if $\mathrm{Var}[f(S)] > 0$ and asymptotically equivalent to $const \cdot n$ if $\mathrm{Var}[f(S)] = 0$. Since this variance is zero precisely when $f(S)$ is constant a.s. (i.e. with probability 1), we see that the second term in (2.6) does not appear for traditional life insurance contracts with deterministic benefits.

By the strong law of large numbers, $\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} 1_{\{T_i>T\}} = \mathrm{E}[1_{\{T_1>T\}}] = {}_T p_x$ P-a.s., and hence it follows directly from (2.4) that

$$\frac{1}{n}L_n \to f(S)e^{-\delta T}\,{}_T p_x - \kappa, \quad P\text{-a.s.} \quad \text{as } n \to \infty. \tag{2.7}$$

This result has the following interpretation: By increasing the volume of the portfolio, the insurer can eliminate the risk associated with the uncertainty concerning the number of policy-holders that will survive to time T. However, it is crucial to realize that the financial uncertainty associated with the future development of the stock remains with the insurer, since all contracts are linked to the same stock; the limit in (2.7) is a random variable and is unknown at time 0. This could also be seen directly from (2.6), since the limit of L_n/n cannot be deterministic if the variance of L_n grows like n^2. The sign of the unknown quantity $f(S)e^{-\delta T}\,{}_Tp_x - \kappa$ has dramatic consequences for the loss L_n since it determines whether the loss L_n will converge towards ∞ or $-\infty$, that is, whether the insurer will suffer enormous losses or earn enormous profits as the size of the portfolio increases. One could argue that it should be possible to eliminate this remaining risk in a similar way (i.e. by the strong law of large numbers) by considering contracts linked to different stocks. However, this is simply impossible, since the insurance contracts are typically linked to a relatively small number of stocks, which furthermore are not stochastically independent.

2.2 Hedging by means of options on the underlying stock

Now assume that the insurer has access to a financial market and can purchase at time 0 a contract that pays the amount $f(S)$ at time T (an option on the stock S), and denote by π_0^f the price of such an contract. Assume for simplicity that the insurer buys exactly $n\,{}_Tp_x$ units of this contract at time 0, and that no further transactions are made on the financial market. This investment will give the payoff $n\,{}_Tp_x f(S)$ at time T, so that the present value at time 0, i.e. the amounts discounted by the interest rate δ, of the insurer's loss now equals

$$\tilde{L}_n = Y_T^{(n)} e^{-\delta T} f(S) - n\kappa - \left(n\,{}_Tp_x f(S) e^{-\delta T} - n\,{}_Tp_x \pi_0^f \right). \tag{2.8}$$

The first term is again the present value of the net amount paid to the surviving policy-holders, and the second term is the present value of the net loss ($-$ net gain) from buying the options. By comparison with (2.4), the loss $\tilde{L}_n$ is equal to the insurer's original loss L_n in the case with no trading, reduced by trading gains. It is straightforward to verify that $\mathrm{E}[\tilde{L}_n] = 0$ if and only if the single premium κ is determined such that

$$\kappa = {}_Tp_x\, \pi_0^f, \tag{2.9}$$

and in this case

$$\mathrm{Var}[\tilde{L}_n] = \mathrm{E}[\mathrm{Var}[\tilde{L}_n \mid S]] = e^{-2\delta T}\mathrm{E}[f(S)^2]\, n\,{}_Tp_x(1 - {}_Tp_x). \tag{2.10}$$

The equation (2.9) suggests that the single premium for the unit-linked pure endowment contract be computed by simply multiplying the option price π_0^f with the probability ${}_Tp_x$ of survival to time T. To comment on this principle, one introduces the notions of *mortality risk*, that is, risk associated with not knowing how many of the policy-holders will survive, and *financial risk*, that is, risk associated with not knowing the future development of stock prices. It is often said that the premium (2.9) is a consequence of the insurer being *risk-neutral with respect to mortality risk*. The notion of risk-neutrality with respect to mortality might be explained by the fact that the premium can be derived by a limit argument which neglects the mortality risk that will be present in any finite portfolio. Another explanation for the notion could be that the factor ${}_Tp_x$ in (2.9) is the true expectation of $1_{\{T_1>T\}}$, that is, the probability of survival to T, whereas the second factor π_0^f is the price of the option, which will typically differ from the expected present value of the pay-off. For comments on this premium principle and these notions, see e.g. Aase and Persson (1994) and references therein.

Comparing (2.10) with the variance (2.6) of the loss L_n, we see that the two variances differ by the second term in (2.6) which is proportional to n^2, that is, by the term essentially stemming from the uncertainty associated with S. Also from the computations leading to (2.6) it is seen that this difference equals $\mathrm{Var}[\mathrm{E}[Y_T^{(n)} f(S)e^{-\delta T}|S]]$. In particular, it can be verified by calculations similar to those leading to (2.7) that, when κ is given by (2.9), we have that $\lim_{n\to\infty}\frac{1}{n}\tilde{L}_n = 0$ P-a.s., so that the insurer is able to eliminate (at least in principle) the total risk associated with the portfolio of unit-linked life insurance contracts by buying standard options on the stock and by increasing the number of policies in the portfolio. In this sense, the situation is similar to the traditional case with deterministic benefits. Similar arguments show that if the single premium is larger than (2.9), then $\lim_{n\to\infty}\frac{1}{n}\tilde{L}_n = {}_Tp_x\pi_0^f - \kappa < 0$ P-a.s. and hence $\tilde{L}_n \to -\infty$ P-a.s. This means that the insurer will earn an infinite (positive) profit as the size of the portfolio is increased, if the single premium is larger than the corresponding option price times the probability of survival to T. Similarly, premiums lower than the corresponding option price lead to an (infinite) loss as the number of policy-holders increases.

Clearly we have neglected important facts of both theoretical and practical nature here, such as the difference between the time horizon for life insurance contracts (typically more than 15 years) and standard options (typically less than 1 year). Furthermore, life insurance contracts are normally priced using a so-called first order valuation basis, that is, a valuation principle which determines a premium on the safe side. Often the surplus that arises from this difference between the "fair price" and the charged premium is then returned in the form of bonuses. However, one might still ask the questions: Is it in some sense optimal to simply buy $n\,{}_Tp_x$ contracts which each pays the amount $f(S)$ at time T? Or do there exist better "strategies" for insurers who sell unit-linked life insurance contracts, for example by trading the stocks in a clever way? In particular, is it possible to currently update the investment strategy as more information about the insurance portfolio is revealed? Of course, answers to these vague questions depend on the criterion applied, and here we shall mainly apply a variant of the criterion of risk-minimization of Föllmer and Sondermann (1986).

3 Trading in a two period model

In this section we consider a financial market with two basic assets only, a stock and a savings account, where trading takes place at time 0 and at time 1, say. This will serve as an introduction to the general multi-period case, which will be treated in Section 4 below. Proofs of the results listed in this section can be found in Föllmer and Schweizer (1988).

The value of the stock at time $t \in \{0, 1\}$ is denoted S_t and the value at time t of 1 unit deposited on the savings account at time 0 is denoted B_t. We assume that $B_t = (1+r)^t$ for some constant $r > 0$, so that the interest on the savings account from time 0 to time 1 is known at time 0. The final outcome S_1 of the stock will not be known before time 1.

Consider an insurer (or a *hedger*) who has sold an insurance contract stipulating a pay-out at time 1 with present value H (at time 0), and assume that the insurer has access to this market and is interested in "reducing risk" as much as possible; we shall also refer to H as the *insurer's (discounted) liability*. We describe the insurer's actions by introducing a *trading strategy* φ. In the present two-period model, this basically amounts to saying that the number ξ of stocks bought at time 0 and held throughout the period $[0, 1]$ and the amount η_0 deposited on the savings account at time 0 are to be fixed at time 0. We introduce the discounted stock price $X_t = S_t/B_t$. The discounted value of one unit deposited on the savings account at time 0 is, trivially, $B_t/B_t = 1$. The (discounted) value at time 0 of the strategy φ after purchase of ξ stocks equals

$$V_0(\varphi) = \xi X_0 + \eta_0.$$

At time 1, the value of the stocks held has changed to ξX_1 and, if we allow the insurer to change his deposit on the savings account at time 1 to η_1, the value at time 1 is

$$V_1(\varphi) = \xi X_1 + \eta_1.$$

At time 1 a deposit or withdrawal must be made on the savings account to establish balance on the business, that is, η_1 must be chosen at time 1 such that $V_1(\varphi)$ exactly equals the insurer's liability H. Föllmer and Sondermann (1986) introduced the *cost process* associated with a strategy φ in a continuous time framework. In the present set-up, the cost process was introduced by Föllmer and Schweizer (1988) and Schweizer (1988); it is defined as $C_0(\varphi) = V_0(\varphi)$ and

$$C_1(\varphi) = V_1(\varphi) - \xi(X_1 - X_0) = H - \xi \Delta X_1. \tag{3.1}$$

In the second equality we have used the condition $V_1(\varphi) = H$ and introduced the notation $\Delta X_1 := X_1 - X_0$, which will be used throughout this chapter. The quantity $C_t(\varphi)$ is the *accumulated cost* at time t. At time 0 costs are exactly the initial value $V_0(\varphi)$ of the portfolio (ξ, η_0), and at time 1 costs are the value $V_1(\varphi) = H$ of the new portfolio (ξ, η_1) reduced by the trading gains $\xi\, \Delta X_1$ from the ξ stocks held. We also point out the analogy between this notion of costs and the notion of "insurer's loss" applied in the previous section: The loss (2.8) is the net amount paid to the policy-holders reduced by the net gains from trading on the financial market, and the costs (3.1) are the amount paid to the policy-holders reduced by net gains from trading on the financial market. In principle, this means that (2.8) and (3.1) differ only by the premium paid at time 0.

One possibility is now to choose the trading strategy so as to

$$\text{minimize } \mathrm{Var}[C_1(\varphi)]. \tag{3.2}$$

The problem (3.2) is so simple that it can be solved directly for a general claim H without introducing additional assumptions on the specific distribution of the increment ΔX_1 or the liability H. Direct calculations show that

$$\begin{aligned} \mathrm{Var}[C_1(\varphi)] &= \mathrm{Var}[H - \xi\Delta X_1] \\ &= \mathrm{Var}[H] - 2\xi\mathrm{Cov}(H, \Delta X_1) + \xi^2\mathrm{Var}[\Delta X_1] \\ &=: J(\xi). \end{aligned}$$

Provided that $\mathrm{Var}[\Delta X_1] > 0$, $J''(\xi) > 0$, and hence J attains its unique minimum for $\hat{\xi}$ satisfying $J'(\hat{\xi}) = 0$, that is

$$\hat{\xi} = \frac{\mathrm{Cov}(H, \Delta X_1)}{\mathrm{Var}[\Delta X_1]}. \tag{3.3}$$

Straightforward calculations show that the minimum obtainable variance as described by (3.2) is

$$\mathrm{Var}[H - \hat{\xi}\Delta X_1] = \mathrm{Var}[H] - \frac{\mathrm{Cov}(H, \Delta X_1)^2}{\mathrm{Var}[\Delta X_1]} = \mathrm{Var}[H]\left(1 - \mathrm{Corr}(H, \Delta X_1)^2\right), \tag{3.4}$$

where

$$\mathrm{Corr}(H, \Delta X_1) = \frac{\mathrm{Cov}(H, \Delta X_1)}{\sqrt{\mathrm{Var}[H]\,\mathrm{Var}[\Delta X_1]}}$$

is the correlation coefficient of H and ΔX_1. The solution (3.3) and the associated variance (3.4) are also recognized as the well-known solution to the problem of minimizing the variance of a linear estimator.

A related problem is to minimize the expected value of the square of the additional costs from time 0 to time 1, that is

$$\text{minimize } \mathrm{E}[(C_1(\varphi) - C_0(\varphi))^2] \tag{3.5}$$

as a function of ξ and η_0. Since $C_0(\varphi)$ is constant and equal to $V_0(\varphi)$, this minimum is attained for a strategy φ satisfying $C_0(\varphi) = \mathrm{E}[C_1(\varphi)]$, and hence the solution is also the solution for the problem (3.2). At first sight this adds no new insight to the understanding of the nature of the risk inherent in the contract. However, the problem (3.5) has the advantage over (3.2) that the initial value $V_0(\varphi)$ is obtained from the solution to (3.5), and this is given by

$$V_0(\hat{\varphi}) = C_0(\hat{\varphi}) = \mathrm{E}[H - \hat{\xi}\Delta X_1] = \mathrm{E}[H] - \hat{\xi}\mathrm{E}[\Delta X_1].$$

This quantity may be taken as a suggestion for the *fair premium* of the contract, see Föllmer and Schweizer (1988). In particular, this result shows that if $\mathrm{E}[\Delta X_1] = 0$, which actually means that X is a *martingale*, then $V_0(\hat{\varphi}) = \mathrm{E}[H]$. However, the fair premium will not be equal to $\mathrm{E}[H]$ in the general case.

Example 3.1 As an example, consider the so-called pure unit-linked pure endowment contract, where each policy-holder receives the value of one unit of the stock at time 1 if he is still alive at this time, that is

$$H = Y_1 X_1.$$

(With the notation of Section 2, Y_1 denotes the number of survivors at time 1.) By the independence between Y_1 and X, we find that

$$\begin{aligned} \mathrm{Cov}(Y_1X_1, \Delta X_1) &= \mathrm{E}[\mathrm{Cov}(Y_1X_1, \Delta X_1 \mid X_1)] + \mathrm{Cov}(\mathrm{E}[Y_1X_1 \mid X_1], \mathrm{E}[\Delta X_1 \mid X_1]) \\ &= 0 + \mathrm{Cov}(\mathrm{E}[Y_1]\, X_1, \Delta X_1) = \mathrm{E}[Y_1]\mathrm{Var}[\Delta X_1]. \end{aligned}$$

And by inserting this into the expression (3.3), we find the optimal number of stocks $\hat{\xi} = \mathrm{E}[Y_1]$, that is, it is optimal to hold a number of stocks corresponding to the expected number of survivors. The initial value of the strategy is

$$V_0(\hat{\varphi}) = \mathrm{E}[H] - \hat{\xi}\mathrm{E}[\Delta X_1] = \mathrm{E}[Y_1]\mathrm{E}[X_1] - \mathrm{E}[Y_1]\mathrm{E}[X_1 - X_0] = \mathrm{E}[Y_1]X_0.$$

This result is similar to what we obtained in Section 2.2, since the price at time 0 of one unit of the stock at time 1 is equal to $S_0 = X_0$ (simply buy the stock at time 0). However, the crucial difference is that this result is derived from a criterion and not just from an apparently ad hoc choice. □

4 Trading strategies in discrete time

In this section, we put up the notation and quantities needed for the introduction of the concept risk-minimizing trading strategies, which is due to Föllmer and Sondermann (1986) and Föllmer and Schweizer (1988). This account is inspired by the latter.

Let $T \in \mathbb{N}$ be a fixed finite time horizon and consider a financial market where trading is possible at discrete times $t = 0, 1, \ldots, T$, say. There exist two assets that can be traded, a stock $S = (S_t)_{t\in\{0,1,\ldots,T\}}$ and a savings account with price process $B_t = e^{\delta t}$, where δ is fixed and deterministic. We introduce at this point the discounted price process $X_t = S_t/B_t$; note that the discounted value of the savings account is constant and equal to 1.

These processes are defined on a probability space $(\Omega, \mathcal{F}, P)$ equipped with a *filtration* $\mathbb{F} = (\mathcal{F}_t)_{t\in\{0,1,\ldots,T\}}$ (i.e. an increasing sequence of σ-algebras $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_T$). The filtration describes the amount of information that is available to the insurer/hedger at any time. It is assumed that the price processes (S, B) are adapted, i.e. for each t, S_t is $\mathcal{F}_t$-measurable; B is deterministic. This has the usual interpretation that at time t we know the current outcome of the stock and the savings account. The probability measure P is called the physical measure, since it describes the true nature of the processes. We shall assume that there exists another probability measure P^* which is equivalent to P (i.e. for all $A \in \mathcal{F}$: $P(A) = 0 \Leftrightarrow P^*(A) = 0$) and which is such that X is a martingale under P^* (i.e. $\mathrm{E}^*[X_t \mid \mathcal{F}_{t-1}] = X_{t-1}$ for all $t = 1, \ldots, T$). The measure P^* is also called an *equivalent martingale measure.*

Formally, a *strategy* is a two-dimensional process $\varphi = (\xi, \eta)$ where ξ_t is $\mathcal{F}_{t-1}$-measurable and η_t is $\mathcal{F}_t$-measurable for each t. Here, ξ_t is the number of stocks held from time $t-1$ to time t, and the requirement that ξ_t should be $\mathcal{F}_{t-1}$-measurable amounts to saying that this number is fixed at time $t-1$ based on the knowledge about stock prices up to and including time $t-1$. In contrast, the deposit on the savings account η_t is only required to be $\mathcal{F}_t$-measurable, so that we at time t can fix the amount on the savings account based on the additional information available at time t also. The pair $\varphi_t = (\xi_t, \eta_t)$ is called the *portfolio* held at time t.

Let us consider the flow of capital which takes place during the time interval $(t-1, t]$, $t = 1, \ldots, T$. At time $t-1$ the hedger holds the portfolio $\varphi_{t-1} = (\xi_{t-1}, \eta_{t-1})$, that is, he holds ξ_{t-1} stocks and has a deposit on the savings account of $\eta_{t-1} B_{t-1}$. The discounted value of the portfolio φ_{t-1} at time $t-1$ is denoted $V_{t-1}(\varphi)$ and is given by

$$V_{t-1}(\varphi) = \xi_{t-1} X_{t-1} + \eta_{t-1}.$$

The process $(V_t(\varphi))_{t \in \{0,1,\ldots,T\}}$ will also be called the (discounted) *value process* of φ. Immediately after time $t-1$, the portfolio φ_{t-1} is adjusted so that the hedger now holds ξ_t stocks. This is achieved by buying additionally $\xi_t - \xi_{t-1}$ stocks, and this gives rise to the cost $(\xi_t - \xi_{t-1}) X_{t-1}$. The new portfolio (ξ_t, η_{t-1}) is held until time t where the new prices (S_t, B_t) are announced, and thus, the hedger receives the discounted gain $\xi_t(X_t - X_{t-1})$. Finally, the hedger may at time t decide to change the deposit on the savings account from $\eta_{t-1} B_t$ to $\eta_t B_t$ based on the additional information available at time t. So we see that

$$V_t(\varphi) - V_{t-1}(\varphi) = (\xi_t - \xi_{t-1}) X_{t-1} + \xi_t (X_t - X_{t-1}) + (\eta_t - \eta_{t-1}). \tag{4.1}$$

In (4.1), the first and the third terms on the right hand side represent costs to the hedger, whereas the second term is trading gains obtained from the strategy φ during $(t-1, t]$. And so, we introduce the *cost process* of the strategy φ given by

$$C_t(\varphi) = V_t(\varphi) - \sum_{j=1}^{t} \xi_j \Delta X_j. \tag{4.2}$$

This cost process was introduced by Föllmer and Sondermann (1986) in continuous time; Föllmer and Schweizer (1988) considered the discrete time analogue. It is simply the value of the strategy reduced by trading gains; in particular, the cost process $C(\varphi)$ satisfies the relation

$$V_t(\varphi) = V_{t-1}(\varphi) + \xi_t (X_t - X_{t-1}) + (C_t(\varphi) - C_{t-1}(\varphi)),$$

which corresponds to (4.1). We note that $C_0(\varphi) = V_0(\varphi)$, which says that the initial costs are exactly equal to the amount invested at time 0.

A strategy is said to be *self-financing* if the change (4.1) in the value process is generated by trading gains only, that is, the portfolio is not affected by any in- or outflow of capital during the period considered. This condition amounts to requiring that

$$V_t(\varphi) = V_0(\varphi) + \sum_{j=1}^{t} \xi_j \Delta X_j \tag{4.3}$$

for all t, and hence we see that this is the case precisely when the cost process is constant.

The insurer is trading on the financial market in order to control his risk associated with some liability H, that is, the insurer is *hedging* against some risk. In some situations it is possible to determine a self-financing strategy which generates or replicates the liability completely, that is, there exists a self-financing strategy φ which sets out with some amount $V_0(\varphi)$ and has terminal value $V_T(\varphi) = H$. In this case, the initial value $V_0(\varphi)$ is the only reasonable price for the liability H; such claims are also said to be *attainable*, and $V_0(\varphi)$ is called the no-arbitrage price of H. For example, this is the situation with the so-called Cox-Ross-Rubinstein model, which will be reviewed in Section 4.2 with emphasis on the existence and structure of a unique self-financing strategy to any (European) option on the stock. However, in many cases, the hedger's liabilities cannot be hedged perfectly by use of a self-financing strategy, and this leaves open the question of how to choose an optimal trading strategy. One very natural idea is to apply *superreplication*. This approach basically consists in looking for attainable claims H' which dominate the original claim H and which (by attainability) can be hedged perfectly by means of some self-financing strategy φ'. By definition, such strategies will have the property that $V_T(\varphi') \geq H$, and are called superreplicating strategies. The *superreplication price* is now defined as the smallest initial investment to which there exists a self-financing strategy which superreplicates H. The main advantage of the approach of superreplication is that it leaves no risk to the hedger in that, after the initial investment is made, no additional capital will be needed in order to pay the claim H. Unfortunately, a major drawback of the approach is that it leads in many situations to unreasonably high prices, since the superreplicating strategies often need a large investment at time 0, see El Karoui and Quenez (1995) for an amplification of this point. We shall also continue the discussion of this problem in Section 5. Another suggestion, which should be appealing to actuaries, is to minimize the variance of future liabilities; this theory is also called the theory of *risk-minimization*.

4.1 An introduction to risk-minimization in discrete time

In this section we very briefly review some results on risk-minimizing hedging strategies in discrete time; an introduction to this topic written especially for actuaries is found in Föllmer and Schweizer (1988).

We consider here the special case where, for each t, we seek to minimize

$$r_t(\varphi) = \mathrm{E}^* \left[(C_{t+1}(\varphi) - C_t(\varphi))^2 \middle| \mathcal{F}_t \right], \tag{4.4}$$

the conditional expected value under a martingale measure P^* of the square of the costs occurring during the next time interval. One can think of the approach of minimizing the quantities $r_t(\varphi)$ as a generalization of the problem (3.5) considered in the two-period case. However, in (4.4) expectations are with respect to a martingale measure, see Remark 4.1 below. The idea in (4.4) is to solve the problem by backwards induction starting from r_{T-1}. At time t, $r_t(\varphi)$ should then be minimized as a function of ξ_{t+1} and η_t for given $(\xi_{t+2}, \ldots, \xi_T)$ and $\eta_{t+1}, \ldots, \eta_T$. This minimization can be performed as in Section 3; see Föllmer and Schweizer (1988) for more details.

The structure of the solution can be related to the P^*-martingale V^* defined by

$$V_t^* = \mathrm{E}^*[H \mid \mathcal{F}_t].$$

This process has the unique decomposition

$$V_t^* = V_0^* + \sum_{j=1}^{t} \xi_j^H \Delta X_j + L_t^H, \tag{4.5}$$

where ξ^H is predictable (i.e. ξ_j^H is $\mathcal{F}_{j-1}$-measurable) and L^H is a P^*-martingale which is orthogonal to X, that is, the process XL^H is also a P^*-martingale.

From Föllmer and Schweizer (1988) we get that the optimal strategy $\hat{\varphi} = (\hat{\xi}, \hat{\eta})$ which minimizes the quantities r_t for each t is given by

$$\begin{aligned} \hat{\xi}_t &= \xi_t^H, \\ \hat{\eta}_t &= V_t^* - \hat{\xi}_t X_t. \end{aligned}$$

For this particular strategy, we see from the definition (4.2) of the cost process that

$$C_t(\hat{\varphi}) = V_t^* - \sum_{j=1}^{t} \hat{\xi}_j \Delta X_j = V_0^* + L_t^H, \tag{4.6}$$

and hence the minimum obtainable risk is

$$r_t(\hat{\varphi}) = \mathrm{E}^* \left[\left(\Delta L_{t+1}^H \right)^2 \middle| \mathcal{F}_t \right].$$

Thus, the problem reduces to determining the decomposition (4.5). In particular, we see that the cost process associated with the optimal, risk-minimizing strategy is a martingale under P^*. As a consequence, the strategy is said to be *mean-self-financing*. The class of mean-self-financing strategies includes the self-financing ones, since the cost process of a self-financing strategy is constant.

Remark 4.1 We point out that the expectations in (4.4) are with respect to a martingale measure P^* and not the physical measure P. It can be proven that the solution $\hat{\varphi}$ is also equal to the solution of the problem of minimizing

$$R_t(\varphi) = \mathrm{E}^*\left[(C_T(\varphi) - C_t(\varphi))^2 \middle| \mathcal{F}_t \right], \tag{4.7}$$

for all t, over all strategies φ. In the alternative formulation (4.7) of the problem, the term $C_T(\varphi) - C_t(\varphi)$ may be interpreted as the *future costs* associated with the strategy, and hence the criterion basically consists in minimizing the conditional variance of all future costs over all possible strategies φ.

This parallel cannot be drawn in the case where the expectation under P^* in (4.4) is replaced with an expectation under the physical measure P, and when P is not itself a martingale measure. In fact, the problem (4.7) does not have a solution in the general case when P^* is replaced by P and when X is not a P-martingale, see Schweizer (1988). However, the problem (4.4) can also be solved when the P^*-expectation is replaced by an expectation with respect to P, and this leads to the following recursion formula for the strategy:

$$\begin{aligned} \tilde{\xi}_t &= \frac{\mathrm{Cov}\left(H - \sum_{j=t+1}^T \tilde{\xi}_j \Delta X_j, \Delta X_t \middle| \mathcal{F}_{t-1} \right)}{\mathrm{Var}[\Delta X_t \mid \mathcal{F}_{t-1}]}, \\ \tilde{\eta}_t &= \mathrm{E}\left[H - \sum_{j=t+1}^T \tilde{\xi}_j \Delta X_j \middle| \mathcal{F}_t \right] - \tilde{\xi}_t X_t. \quad \square \end{aligned}$$

4.2 The Cox-Ross-Rubinstein model

With the Cox-Ross-Rubinstein model, the development of the stock price is given by

$$S_t = (1 + \rho_t) S_{t-1}, \tag{4.8}$$

where $\rho_1, \ldots, \rho_T$ is a sequence of i.i.d. random variables with $\rho_1 \in \{a, b\}$ and such that $0 < P(\rho_1 = b) < 1$; ρ_t is the return per unit of the stock during the time interval $(t-1, t]$ and this quantity is not known before time t. The savings account is often written on the form $B_t = (1 + r)^t$. A natural condition on the parameters a, b, r is that $-1 < a < r < b$, which amounts to saying that the return on the stock in each period should exceed the return on the savings account with a positive probability and vice versa. As above, we denote by X the discounted stock price process. We refer to Baxter and Rennie (1996) and Pliska (1997) for an introduction to this model; an exposition focusing on the mathematical aspects is Shiryaev et al. (1994).

Now introduce the natural filtration $\mathbb{G} = (\mathcal{G}_t)_{t \in \{0,1,\ldots,T\}}$ generated by S defined by $\mathcal{G}_t = \sigma\{S_1, \ldots, S_t\}$, and define a new probability measure P^* with $P^*(\rho_1 = b) = \frac{r-a}{b-a} =: p^*$ and such that $\rho_1, \ldots, \rho_T$ are i.i.d. under P^*. We note that with this specific choice of p^* the discounted stock price process X is a $(\mathbb{G}, P^*)$-martingale, since

$$\begin{aligned}\mathrm{E}^*\left[X_t \middle| \mathcal{G}_{t-1}\right] &= X_{t-1}\frac{1}{1+r}\mathrm{E}^*\left[(1+\rho_t) \middle| \mathcal{G}_{t-1}\right] = X_{t-1}\frac{1}{1+r}\left(1+\mathrm{E}^*[\rho_t]\right) \\ &= X_{t-1}\frac{1}{1+r}\left(1+p^*b+(1-p^*)a\right) = X_{t-1}.\end{aligned}$$

Martingales and martingale measures play an important role in mathematical finance. This is for example clear from the fundamental binomial representation property, which we here give the following formulation:

Theorem 4.2 *Let H be a $\mathcal{G}_T$-measurable P^*-integrable random variable. Then, the $(\mathbb{G}, P^*)$-martingale N defined by $N_t = \mathrm{E}^*[H \mid \mathcal{G}_t]$ admits a unique representation*

$$N_t = N_0 + \sum_{j=1}^{t} \alpha_j \Delta X_j, \tag{4.9}$$

where α_j is $\mathcal{G}_{j-1}$-measurable for each j.

Remark 4.3 It is pointed out that the representation property in Theorem 4.2 is very specific for the Cox-Ross-Rubinstein model. If, for example, $\rho_1, \ldots, \rho_T$ in (4.8) were replaced by a sequence of i.i.d. random variables with three different outcomes $\{a, b, c\}$ (a so-called trinomial model), then Theorem 4.2 would no longer hold and the representation (4.9) would have to be replaced by a more general decomposition of the form (4.5). The representation property may also be lost if the claim H depends on other sources of randomness than the one given by the stock. □

A proof of Theorem 4.2 can for example be found in Shiryaev et al. (1994). If we think of N as the discounted value process of some trading strategy φ and of H as the present value of some liability, we see from the definition of N and the property (4.9) that $V_T(\varphi) = H$ and

$$V_t(\varphi) = V_0(\varphi) + \sum_{j=1}^{t} \alpha_j \Delta X_j.$$

Now compare this with (4.3) to see that this shows that this strategy φ would be self-financing and that it would replicate H. The strategy $\varphi = (\xi, \eta)$ can actually be defined by letting $\xi_t = \alpha_t$, whereas η_t should be chosen such that the strategy is in fact self-financing, which is achieved by taking

$$\eta_t = N_0 + \sum_{j=1}^{t} \xi_j \Delta X_j - \xi_t X_t.$$

In particular, we see that the initial value of the self-financing strategy φ which replicates H is $N_0 = \mathrm{E}^*[H]$, and so, the price of the contract should be $\mathrm{E}^*[H]$. This implies that the price of the contract will typically differ from the expected present value $\mathrm{E}[H]$.

Example 4.4 As an example, consider a European-type option on the stock S with exercise time T which specifies the payment $g(S_T)$ at time T for some function g. The present value at time 0 of this payment is given by $H = g(S_T)/B_T$. Shiryaev et al. (1994) give a closed form solution for the process α for such claims. However, in general α can be computed as

$$\alpha_t = \frac{\Delta\langle N, X\rangle_t}{\Delta\langle X, X\rangle_t} = \frac{\mathrm{Cov}^*(\Delta N_t, \Delta X_t \mid \mathcal{F}_{t-1})}{\mathrm{Cov}^*(\Delta X_t, \Delta X_t \mid \mathcal{F}_{t-1})},$$

where N is defined by $N_t = \mathrm{E}^*[H \mid \mathcal{F}_t]$ and where $\langle .\,,.\rangle$ is the so-called predictable quadratic covariation process; for square-integrable P^*-martingales $\hat{X}$ and $\hat{Y}$ it is defined by

$$\Delta\langle \hat{X}, \hat{Y}\rangle_t = \mathrm{E}^*\left[\Delta(\hat{X}\hat{Y})_t \middle| \mathcal{F}_{t-1}\right] = \mathrm{Cov}^*(\Delta\hat{X}_t, \Delta\hat{Y}_t \mid \mathcal{F}_{t-1}). \ \Box$$

5 Hedging unit-linked contracts

5.1 Framework

As in Section 2, we consider a portfolio of n policy-holders aged x at time 0 and denote by Y_t the number of survivors at time t. We consider a unit-linked pure endowment contract payable at time T with present value

$$H = Y_T\, g(S_T)/B_T$$

at time 0. Here g is some function, see e.g. Example 4.4. It is assumed that S is defined by (4.8), that is, the financial market is described by the CRR-model, and that the remaining lifetimes of the policy-holders are stochastically independent of the stock price process S (or, equivalently, of the discounted stock price process X).

We assume that, at any time t, the insurer has current access to information concerning the number of surviving policy-holders as well as the development of the stock price up to and including time t. This is formalized by introducing the filtrations $\mathbb{G} = (\mathcal{G}_t)_{t\in\{0,1,\dots,T\}}$ defined by $\mathcal{G}_t = \sigma\{X_1, \dots, X_t\}$ and $\mathbb{H} = (\mathcal{H}_t)_{t\in\{0,1,\dots,T\}}$ defined by $\mathcal{H}_t = \sigma\{Y_1, \dots, Y_t\}$. The filtration $\mathbb{G}$ describes the information available about the development on the stock market and $\mathbb{H}$ contains information about the policy-holders. In addition, we introduce a third filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in\{0,1,\dots,T\}}$, given by $\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t = \sigma(\mathcal{G}_t \cup \mathcal{H}_t)$, which means that $\mathcal{F}_t$ is the smallest σ-algebra which includes both $\mathcal{G}_t$ and $\mathcal{H}_t$. Hence $\mathbb{F}$ includes all the available information.

We work here directly under a specific martingale measure P^*, which is closely related to the measure introduced in Section 4.2 (also denoted P^*) and which has the following properties:

1. X is a P^*-martingale and, under P^*, $\rho_1, \dots, \rho_T$ is a sequence of i.i.d. random variables with $P^*(\rho_1 = b) = \frac{r-a}{b-a}$.

2. $T_1, \dots, T_n$ are i.i.d. under P^* with $P^*(T_1 > t) = P(T_1 > t) = {}_tp_x$.

3. The two random sources are stochastically independent under P^*.

The first property states exactly that P^* is similar to the measure of Section 4.2, and, by the second property, the change of measure from P to P^* does not affect the marginal distribution of the remaining lifetimes. Finally, the last property states that the independence between the insured lives and the a priori given financial market is preserved under the measure P^*.

5.2 On the choice of criterion

Before working out the unique risk-minimizing strategy for the unit-linked pure endowment contract, we turn back to the more general discussion on the choice of optimality criterion for hedging strategies. One idea is to look for claims H' which are larger than H and which can be hedged by a self-financing strategy. For example, consider the claim $H' = n\, g(S_T)/B_T$, which is obtained by simply replacing the unknown number of survivors at time T with the number of policy-holders n entering the contract at time 0. Clearly, H' is larger than the actual claim $H = Y_T\, g(S_T)/B_T$, since $Y_T \leq n$. With the CRR-model, this new claim H' is attainable, since it is simply n times the European-type option $g(S_T)/B_T$, which depends only on the randomness from the financial market. Mathematically, this means that H' is $\mathcal{G}_T$-measurable. Thus H' can be hedged and priced uniquely, and we note that the strategy φ' which replicates H' satisfies $V_T(\varphi') = H' \geq H$, so that φ' is a superreplicating strategy for H. Theorem 4.2 determines the strategy as well as the unique price for the claim, and the latter is simply $n\, \mathrm{E}^*[g(S_t)/B_T]$. Actually it can be shown that this is the cheapest superreplicating strategy, that is, all other superreplicating strategies require an initial investment which exceeds $n\, \mathrm{E}^*[g(S_T)/B_T]$; see El Karoui and Quenez (1995, Example 3.4.2) for a similar problem. Thus, if we insist on applying superreplication, we obtain a superreplication price for H which is equal to the no-arbitrage price for $n\, g(S_T)/B_T$. This price is clearly too high since it basically corresponds to using a survival probability ${}_Tp_x$ which is equal to 1. Although superreplication has many appealing features, the perhaps most remarkable one being that it eliminates all risk for the hedger, we therefore conclude that superreplication does not seem to be the right tool for our situation, where claims depend on an additional source of risk which is stochastically independent of the financial market.

Another idea is the so-called Brennan-Schwartz approach which suggests to "replace" the original claim H with the claim $H'' = \mathrm{E}[Y_T]\, g(S_T)/B_T$, that is, to replace the unknown number of survivors by the expected number, and then consider the problem of pricing H'' instead of H. Here, H'' is again a constant times a European-type option and hence it is attainable and can be hedged and priced uniquely. Again, Theorem 4.2 determines the unique no-arbitrage price for H'', which is now given by $\mathrm{E}[Y_T]\, \mathrm{E}^*[g(S_T)/B_T]$. One candidate for the price of H could now be to simply use the price of H'' and, similarly, a candidate for the hedging strategy of H is the (self-financing) hedging strategy for H''. This choice could for example be motivated by the law of large numbers as was done in Section 2. Actually, most existing literature on pricing and hedging of unit-linked insurance contracts focus on the problem of pricing and hedging of H'' instead of H. The replacement of H with H'' is motivated by the assumption of risk-neutrality with respect to mortality, see e.g. Aase and Persson (1994) and references therein. We underline that the problem of pricing claims on the form H'' can be very complicated indeed and is extremely important for practical purposes since unit-linked contracts in practice often involve quite complex dependencies on the underlying stocks or stock indices. However, in comparison with the approach of superreplication, this "replacement approach" has the disadvantage that it leaves risk to the hedger, a risk which is related to not knowing the actual number of survivors Y_T. In addition, the approach gives no directions as to how to deal with or how to quantify this risk – it simply refers to the risk-neutrality and leaves to the law of large numbers to "do its job", which from a theoretical point of view is somewhat unsatisfactory. It is crucial to realize that there is an enormous difference between pricing and hedging of the original claim H, which cannot be done uniquely or perfectly, and pricing and hedging of H'', since this can be done uniquely and perfectly by use of Theorem 4.2.

Below we shall apply the results from Section 4.1 to determine the unique risk-minimizing hedging strategy for the claim H, an approach which basically amounts to minimizing the variance of the hedger's future costs. Thus, this approach really takes into account the total risk included in the claim in that the original claim H is not replaced with some modified claim. Although the idea of minimizing the variance of future costs seems to be a very natural one, it also has some undesired properties: For example, minimization of the variance (or the expected value of the square of the future costs) implies that relative losses and relative gains are treated equally. Consequently, being short of USD 100 at time T is as bad as having USD 100 too much! Of course, it would be more appealing to apply some non-symmetric criterion that does not punish losses and gains equally, but then explicit results are considerably harder to obtain. As an example of such a criterion we mention *quantile hedging*, see Föllmer and Leukert (1999), where the objective is to hedge the claim with a certain (given) probability. Another example is the criterion of minimizing the expected loss of hedging, i.e. the so-called shortfall risk; references are Föllmer and Leukert (2000) and Cvitanić (1998).

5.3 The unique risk-minimizing hedging strategy

In order to determine the risk-minimizing hedging strategy, we need to determine the decomposition (4.5). We define two processes π^g and M which are related to the financial market and the portfolio of insured lives, respectively. First, recall that, by Theorem 4.2, the unique (discounted) price at time t of a contract specifying the payment $g(S_T)$ at time T is

$$\pi_t^g := \mathrm{E}^*\left[\left.\frac{g(S_T)}{B_T}\right|\mathcal{G}_t\right] = \mathrm{E}^*\left[\frac{g(S_T)}{B_T}\right] + \sum_{j=1}^{t}\alpha_j^g \Delta X_j, \tag{5.1}$$

where the predictable process α^g is the *hedge* associated with $g(S_T)$. Second, we introduce a process M defined by

$$M_t := \mathrm{E}^*\left[Y_T | \mathcal{H}_t\right] = Y_t \, {}_{T-t}p_{x+t}, \tag{5.2}$$

which is the conditional expected number of survivors at time T; ${}_{T-t}p_{x+t}$ is the conditional probability of survival to time T given that the policy-holder is alive at time t. Note that M will typically fluctuate over time and that $M_T = Y_T$, that is, the terminal value (at time T) of M is equal to the number of survivors at time T. This approach does not rely on the specific structure of the filtration $I\!H$ and can be generalized to other choices of filtrations. For example, one could consider the situation where the number of deaths are not revealed before time T, that is, the investment department receives no information about the development within the portfolio of insured lives. In that case, M would be constant and equal to $\mathrm{E}^*[Y_T] = n \, {}_Tp_x$ until time T, when $M_T = Y_T$.

We shall see that π^g and M determine the decomposition (4.5). First note that by the independence between Y and S, we have that

$$\begin{aligned} V_t^* &:= \mathrm{E}^*\left[Y_T \frac{g(S_T)}{B_T} \middle| \mathcal{F}_t \right] = \mathrm{E}^*\left[\frac{g(S_T)}{B_T} \middle| \mathcal{F}_t \right] \mathrm{E}^*\left[Y_T | \mathcal{F}_t\right] \\ &= \mathrm{E}^*\left[\frac{g(S_T)}{B_T} \middle| \mathcal{G}_t \right] \mathrm{E}^*\left[Y_T | \mathcal{H}_t\right] = \pi_t^g \, M_t. \end{aligned}$$

In the second equality we have used the independence between Y_T and S_T, and in the third equality we have used the definition of the σ-algebra $\mathcal{F}_t$ and the independence between the remaining lifetimes of the policy-holders and the development on the stock market; the last equality is the definition of π^g and M. Now note that

$$\begin{aligned} \Delta V_t^* &= V_t^* - V_{t-1}^* = \pi_t^g \, M_t - \pi_{t-1}^g \, M_{t-1} \\ &= (\pi_t^g - \pi_{t-1}^g) M_{t-1} + \pi_t^g (M_t - M_{t-1}) = M_{t-1} \alpha_t^g \Delta X_t + \pi_t^g \Delta M_t, \end{aligned}$$

where we have used (5.1) in the last equality. Here we note that $M_{t-1}\alpha_t^g$ is $\mathcal{F}_{t-1}$-measurable and hence the process ξ^H defined by

$$\xi_t^H = M_{t-1} \alpha_t^g$$

is predictable. If we can show that the process L defined by

$$L_t = \sum_{j=1}^{t} \pi_j^g \Delta M_j$$

is a martingale and that LX is a martingale, then we have actually obtained that the decomposition (4.5) is

$$V_t^* = V_0^* + \sum_{j=1}^{t} M_{j-1} \alpha_j^g \Delta X_j + \sum_{j=1}^{t} \pi_j^g \Delta M_j. \tag{5.3}$$

To see that L is a martingale, note that, by the law of iterated expectations and the independence between $\mathbb{G}$ and $\mathbb{H}$, we find that

$$\mathrm{E}^*\left[\Delta L_t \middle| \mathcal{F}_{t-1}\right] = \mathrm{E}^*\left[\pi_t^g \Delta M_t \middle| \mathcal{F}_{t-1}\right] = \mathrm{E}^*\left[\pi_t^g \mathrm{E}^*\left[\Delta M_t \middle| \mathcal{G}_t \vee \mathcal{H}_{t-1}\right] \middle| \mathcal{F}_{t-1}\right] = 0,$$

since M is a martingale, and since M is stochastically independent of the filtration $\mathbb{G}$. Similar calculations show that also LX is a martingale: Since

$$\Delta(LX)_t = L_t X_t - L_{t-1} X_{t-1} = \Delta L_t \Delta X_t + L_{t-1} \Delta X_t + X_{t-1} \Delta L_t,$$

it is sufficient to show that

$$\mathrm{E}^*\left[\Delta L_t \Delta X_t \middle| \mathcal{F}_{t-1}\right] = 0.$$

This, in turn, follows by calculations similar to those used to show that L is a martingale. Thus, we have shown that (5.3) is the desired decomposition. From the results reviewed in the previous section, we find that the risk-minimizing hedging strategy is given by

$$\xi_t = Y_{t-1}\, {}_{T-(t-1)}p_{x+(t-1)}\, \alpha_t^g, \tag{5.4}$$

$$\eta_t = Y_t\, {}_{T-t}p_{x+t}\, \pi_t^g - Y_{t-1}\, {}_{T-(t-1)}p_{x+(t-1)}\, \alpha_t^g\, X_t. \tag{5.5}$$

After these calculations, some comments are in place: The optimal number ξ_t of stocks held in period t (during the interval $(t-1,t]$) is simply the hedge α_t^g from the underlying option $g(S_T)$ multiplied with the conditional expected number of survivors to T at time $t-1$. Furthermore, the deposit on the savings account is constantly adjusted, so that the value of the strategy at each time t exactly equals V_t^*, that is, the value at time t is equal to the fair premium for the portfolio of unit-linked contracts at time t. Let us also emphasize that the strategy defined above will typically not be self-financing. This follows by considering the cost process of the strategy, see (4.6), given by

$$C_t(\varphi) = V_0^* + L_t = V_0^* + \sum_{j=1}^{t} \pi_j^g \Delta M_j.$$

In particular, this means that the cost process is constant only if the process M defined by (5.2) is constant or if $\pi_t^g = 0$, and this will typically not be the case. The term ΔM_t is the change in the conditional expected number of survivors from time $t-1$ to time t. If the actual number of deaths during the interval $(t-1,t]$ is smaller than the expected number, then the expected number of survivors will increase and $\Delta M_t > 0$. This represents a loss to the insurer who has to pay the amount $g(S_T)$ to each survivor among the policy-holders. In contrast, if $\Delta M_t < 0$ then the expected number of survivors decreases, and the insurer can reduce his reserves associated with the contracts. Rewriting the survival probability ${}_{T-(t-1)}p_{x+(t-1)}$ as ${}_1p_{x+(t-1)}\, {}_{T-t}p_{x+t}$, we can also rewrite the loss during $(t-1,t]$ as

$$\Delta L_t = \pi_t^g\, {}_{T-t}p_{x+t} \left(Y_t - Y_{t-1}\, {}_1p_{x+(t-1)} \right),$$

which can be given the following interpretation: The insurer's loss is proportional to the factor $\pi_t^g \, {}_{T-t}p_{x+t}$ which is the discounted price at time t on the option $g(S_T)$ times the survival probability ${}_{T-t}p_{x+t}$. This quantity represents a reasonable reserve at time t for one policy-holder who is alive at time t. The second factor $(Y_t - Y_{t-1}\,{}_1p_{x+(t-1)})$ is exactly the difference between the actual number of survivors at time t and the conditional expected number calculated at time $t-1$.

We end this section by assessing the risk that remains with an insurer who applies the risk-minimizing strategy. For example, this can be done by considering the variance under P^* of the accumulated costs $C_T(\varphi)$ associated with the risk-minimizing strategy. Using that the martingale M has uncorrelated increments, the independence between π^g and M under P^* and the fact that the change of measure from P to P^* does not affect the distribution of the remaining lifetimes, we find that

$$\mathrm{Var}^*[C_T(\varphi)] = \sum_{t=1}^{T} \mathrm{E}^*[(\pi_t^g)^2 \Delta M_t^2] = \sum_{t=1}^{T} \mathrm{E}^*[(\pi_t^g)^2]\mathrm{E}[\Delta M_t^2]. \tag{5.6}$$

Here, the term involving ΔM_t can be expressed in terms of the survival probabilities. This can be seen from the following calculations:

$$\begin{aligned} \mathrm{E}[\Delta M_t^2] &= \mathrm{E}[\mathrm{Var}[\Delta M_t \mid \mathcal{F}_{t-1}]] + \mathrm{Var}[\mathrm{E}[\Delta M_t \mid \mathcal{F}_{t-1}]] \\ &= \mathrm{E}[\mathrm{Var}[{}_{T-t}p_{x+t}\, Y_t \mid \mathcal{F}_{t-1}]] \\ &= n\, {}_Tp_x\, {}_{T-t}p_{x+t}\, (1 - {}_1p_{x+(t-1)}), \end{aligned} \tag{5.7}$$

where the last equality follows by using that $Y_t \mid \mathcal{F}_{t-1} \sim \mathrm{Binomial}(Y_{t-1},\, {}_1p_{x+(t-1)})$. The variance (5.6) should be compared to the total variance of the claim H given by

$$\mathrm{Var}^*[Y_T g(S_T)/B_T] = \mathrm{E}^*[(\pi_T^g)^2]\, n\, {}_Tp_x\, (1 - {}_Tp_x) + \mathrm{Var}^*[\pi_T^g]\, (n\, {}_Tp_x)^2, \tag{5.8}$$

which is the variance of the insurer's loss with no trading, see also (2.6).

6 Numerical results

In this section, we give a numerical example with *five* trading times, $k = 0, 1, 2, 3, 4$, that is $T = 4$. For example, one can think of one year as being divided into four periods, each of length three months. We denote the length of each period by Δt, that is, $\Delta t = 1/4$. For simplicity, we assume that the remaining lifetimes of the policy-holders are independent and exponentially distributed with hazard rate μ, and consider various choices of μ. Thus, the survival probability is ${}_kp_x = \exp(-\mu k \Delta t)$ for all k (and x). In this case formulas (5.6) and (5.8) simplify to

$$\mathrm{Var}^*[C_T(\varphi)] = n \sum_{k=1}^{T} \mathrm{E}^*[(\pi_k^g)^2] e^{-2\mu T \Delta t + \mu k \Delta t}(1 - e^{-\mu \Delta t}), \tag{6.1}$$

and, with $H = Y_T g(S_T)/B_T$,

$$\mathrm{Var}^*[H] = \mathrm{E}^*[(\pi_T^g)^2]\, n e^{-\mu T \Delta t}(1 - e^{-\mu T \Delta t}) + \mathrm{Var}^*[\pi_T^g]\, n^2 e^{-2\mu T \Delta t}. \tag{6.2}$$

We shall assume in the following that the amount payable at T is given by $g(S_T) = \max(S_T, K)$, and we let the guarantee K be computed as $K = S_0(1+\frac{1}{2}r)^T$. Figure 2.1 shows the binomial tree for the stock price process S and the undiscounted price process $(B_t\pi_t^g)_{0\le t\le T}$ for the claim g; the parameters used are listed in Table 2.1.

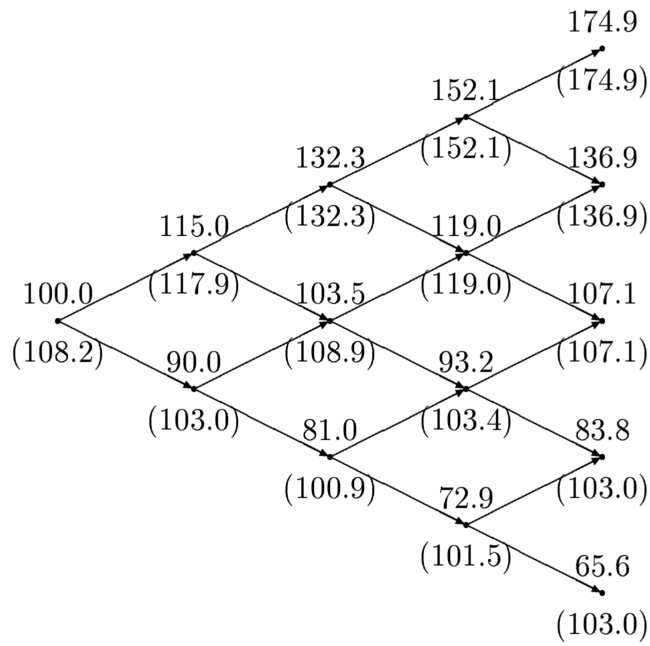

Figure 2.1: *Binomial tree for the stock price (the upper numbers) and the unique arbitrage-free price for the contract* $\max(S_T, K)$ *(the lower numbers).*

Δt	T	S_0	K	a	b	r	p^*	p
1/4	4	100	103.0	−0.10	0.15	0.015	0.46	0.50

Table 2.1: *The parameters used in the numerical example.*

It follows from Figure 2.1 that the terminal value of the stock is smaller than the guaranteed amount $K = 103.0$ if and only if the value of the stock has been decreasing during three or four of the periods. This corresponds to the terminal stock prices 83.8 and 65.6, respectively. As a consequence, the no-arbitrage free price for $\max(S_T, K)$ is equal to the discounted value of K if the value of the stock has decreased during each of the first three periods, since in this case, the terminal value cannot possibly exceed the guarantee. Similarly, if the value of the stock has increased at least twice during the first three periods, then the terminal value cannot fall short of the guarantee, and in this case, the no-arbitrage price for $\max(S_T, K)$ is exactly equal to the value of the stock. Between these two "extremes", it is not clear before the terminal time, whether S_T will be larger or smaller than K. This is reflected in the no-arbitrage price, which is now slightly more complicated.

The hedge α^g for $\max(S_T, K)$ and the number of stocks in the risk-minimizing hedging strategy for one policy-holder with hazard rate 1 can be found in Figure 2.2. The risk-minimizing strategy denotes the optimal number of stocks to be held for one policy-holder who is still alive at the time of consideration. At time 0, the risk-minimizing strategy consists of $\xi_1 = 0.219$ stocks and a deposit $\eta_0 = e^{-\mu}108.2 - 0.219 \cdot 100 = 17.9$ on the savings account, see (5.4) and (5.5). At time 1 these numbers will change in accordance with the development in the value of the stock and depending on whether or not the policy-holder is still alive at this time. If the policy-holder does not survive until time 1, then $\xi_2 = 0$ and $\eta_1 = 0$ (see (5.4) and (5.5) again). If he is still alive at time 1, and the value of the stock has increased (to 115.0), then $\xi_2 = 0.383$ whereas $\eta_1 = e^{-\mu 3/4}117.9 - 0.383 \cdot 115.0 = 11.6$.

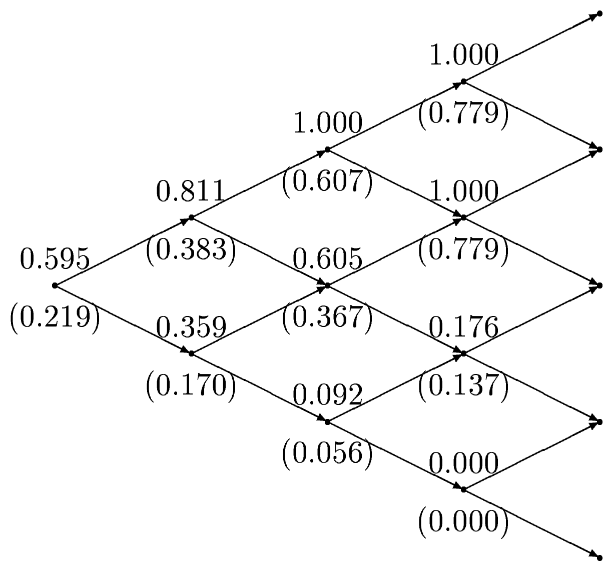

Figure 2.2: *The hedge for* $\max(S_T, K)$ *(the upper numbers) and the risk-minimizing hedging strategy for the pure endowment contract H for one policy-holder contingent on survival in the case where* $\mu = 1$ *(the lower numbers).*

In Table 2.2 we have listed the variance of $C_T(\varphi)$, the variance of H and the ratio between the two variances for various choices of hazard rate μ and number of policy-holders n. It follows immediately from (5.6) and (5.7) that $\text{Var}^*[C_T(\varphi)]$ is proportional to n, whereas the dependence of the hazard rate μ is more complex. Table 2.2 shows that the ratio between $\text{Var}^*[C_T(\varphi)]$ and $\text{Var}^*[H]$ seems to decrease as a function of n and increase as a function of μ. This property can for example be verified directly when $T = 1$ and $\Delta t = 1$, since in that case, it follows from (6.1) and (6.2) that the ratio can be rewritten as

$$\frac{1 - e^{-\mu}}{1 - e^{-\mu} + n\, e^{-\mu} \frac{\text{Var}^*[\pi_1^g]}{\text{E}^*[(\pi_1^g)^2]}} = \frac{1 - e^{-\mu}}{\left(1 - n \frac{\text{Var}^*[\pi_1^g]}{\text{E}^*[(\pi_1^g)^2]}\right)(1 - e^{-\mu}) + n \frac{\text{Var}^*[\pi_1^g]}{\text{E}^*[(\pi_1^g)^2]}}.$$

	hazard rate (μ)	$\text{Var}^*[C_T(\varphi)]$	$\text{Var}^*[H]$	$\text{Var}^*[C_T(\varphi)]/\text{Var}^*[H]$
	0.1	$1.02 \cdot 10^3$	$1.28 \cdot 10^3$	0.80
	0.5	$2.84 \cdot 10^3$	$2.98 \cdot 10^3$	0.95
$(n = 1)$	1	$2.77 \cdot 10^3$	$2.83 \cdot 10^3$	0.98
	0.1	$1.02 \cdot 10^4$	$3.49 \cdot 10^4$	0.29
	0.5	$2.84 \cdot 10^4$	$3.97 \cdot 10^4$	0.71
$(n = 10)$	1	$2.77 \cdot 10^4$	$3.20 \cdot 10^4$	0.87
	0.1	$1.02 \cdot 10^5$	$2.56 \cdot 10^6$	0.04
	0.5	$2.84 \cdot 10^5$	$1.39 \cdot 10^6$	0.20
$(n = 100)$	1	$2.77 \cdot 10^5$	$6.85 \cdot 10^5$	0.40

Table 2.2: *The variance* $\text{Var}^*[H]$ *of the discounted liabilities and the variance* $\text{Var}^*[C_T(\varphi)]$ *of the costs associated with the risk-minimizing strategy.*

Here, the first expression shows that the ratio decreases as a function of n and the second expression shows that the ratio increases as a function of μ. The dependency of μ has the following interpretation: When μ grows, the uncertainty with respect to not knowing the number of policy-holders who will survive becomes large in comparison with the financial uncertainty. Similarly, when n increases, the term appearing in $\text{Var}^*[H]$ which is proportional to n^2 becomes more dominating, and hence the ratio decreases.

7 Results in continuous time

Results similar to those in the previous section were obtained in a continuous time framework in Møller (1998a) for unit-linked insurance contracts payable at a fixed time T, for example the pure endowment contract. In Chapter 3 these results are extended to the situation where the insurer's liabilities are described by a genuine payment stream, for example general unit-linked insurance contracts where payments are due immediately upon certain insurance events. That paper also includes examples from non-life insurance, where the insurer's liabilities are modeled by a traditional risk process with the modification that claim amounts are affected by some tradeable price index. Both expositions (Møller (1998a), Chapter 3) consider as main example a so-called generalized Black-Scholes model, where prices are driven by the stochastic differential equations

$$\begin{aligned} dS_t &= \alpha(t,S_t)S_t\,dt + \sigma(t,S_t)S_t\,dW_t, \\ dB_t &= r(t,S_t)B_t\,dt, \end{aligned}$$

with $S_0 > 0$, $B_0 = 1$, where $W = (W_t)_{0 \le t \le T}$ is a standard Brownian motion on the time interval $[0,T]$. Also, let $\mathcal{G}_t = \sigma\{W_u,\, u \le t\}$. In notation similar to the one used in Section 5, the risk-minimizing strategy for a unit-linked pure endowment contract with present value $H = Y_T g(S_T)/B_T$ is

$$\xi_t = Y_{t-}\,{}_{T-t}p_{x+t} F_s^g(t,S_t), \tag{7.1}$$

$$\eta_t \quad = \quad Y_{t\,T-t}p_{x+t}\,B_t^{-1}F^g(t,S_t) - \xi_t X_t, \;\; 0 \le t \le T. \tag{7.2}$$

Here, the function F^g is defined by

$$F^g(t,S_t) = B_t\,\mathrm{E}^*\left[B_T^{-1}\,g(S_T)\,\middle|\,\mathcal{G}_t\right].$$

and $F^g_s(t,s)$ denotes the partial derivative of $F^g(t,s)$ with respect to s.

In contrast to the risk-minimizing strategy from the discrete time set-up, the strategy (7.1)–(7.2) requires that the portfolio is rebalanced constantly in a continuous manner. The analysis of such continuous time models requires more advanced stochastic analysis than the one we have used in this chapter.

Chapter 3

Hedging Insurance Payment Processes

(This chapter is an adapted version of Møller (1998b))

Föllmer and Sondermann (1986) proved the existence of a unique admissible risk-minimizing hedging strategy for any square-integrable contingent claim H in the martingale case. We extend this approach to the situation where the hedger's liabilities are described by a general payment process A and consider some examples related to insurance. These include a general unit-linked life insurance contract driven by a Markov jump process and a claim process from non-life insurance where the claim size distribution is affected by a traded price index.

1 Introduction

This chapter addresses the problem of determining risk-minimizing hedging strategies when the hedger's liabilities are described by a general payoff stream. For example, this would be relevant for a hedger facing not only one but several claims with different (fixed) maturities or claims with random payment times, which is often the case for insurance contracts. Life insurance contracts typically specify payment of some amount (called the benefit) immediately upon the occurrence of a specific insurance event, for example the death of a policy-holder. The benefit could be a fixed amount or a function of some financial assets. Similarly, non-life insurance contracts typically involve payments of random amounts at random times.

It is assumed that the financial market consists of two assets only: a riskless asset with a discounted price process equal to 1 and a risky asset whose discounted price process is given by a locally square-integrable local martingale. In this setting, Föllmer and Sondermann (1986) introduced the concept of mean-self-financing strategies and proved the existence of a unique risk-minimizing hedging strategy for any square-integrable contingent claim with a fixed maturity. Here, we propose an extension of their framework so as to allow for general payment streams, where the liabilities of the hedger are given by a square-integrable payment process. This process keeps track of the accumulated difference between (discounted) outgoes and incomes at any time within a fixed time period. By considering some specific payment processes, we show that this extension does indeed contain the original Föllmer-Sondermann approach.

In a recent paper by the author (Møller, 1998a) risk-minimizing hedging strategies for so-called unit-linked life insurance contracts were determined within the framework of Föllmer and Sondermann. With these contracts, the insurance benefits are dependent on the value of a stock index and are payable contingent on some specific events related to the stochastic life-length of the policy-holder. For example, the policy-holder could receive the value of a stock index at a fixed time T, if he is still alive at this time. In the present chapter, we determine risk-minimizing hedging strategies and intrinsic risk processes for more general unit-linked life insurance contracts with payments incurring at random times within the term of the contract. These intrinsic risk processes quantify the minimum risk that remains with a hedger who uses a risk-minimizing hedging strategy. Furthermore, we consider an example from non-life insurance, where claims are described by a marked point process with index-dependent claim amounts.

The chapter is organized as follows: In Section 2 we present the extension of risk-minimization for general payment processes. Some applications related to insurance are given in Section 3.

2 Risk-minimization for payment streams

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ satisfying the usual conditions of right-continuity and completeness and where $\mathcal{F}_0$ is trivial; T is a fixed finite time horizon. Let $\mathcal{P}$ denote the predictable σ-algebra on $\Omega \times [0,T]$ and $\mathcal{O}$ the optional σ-algebra on $\Omega \times [0,T]$. Consider a financial market consisting of two assets with discounted price processes $X = (X_t)_{0\leq t\leq T}$ and $Y = (Y_t)_{0\leq t\leq T}$, respectively. X is the discounted price process associated with some risky asset (a stock), and it is assumed that X is locally square-integrable, adapted to the filtration $\mathbb{F}$ and right-continuous with left limits (RCLL). The discounted price process Y associated with the riskless asset (henceforth called *savings account*) is assumed to be constant and equal to 1. Furthermore, it is assumed that P is a local martingale measure for the market (X, Y) so that X is a local P-martingale.

Let $\langle X\rangle$ be the sharp bracket process for X, i.e., the unique predictable process such that $X^2 - \langle X\rangle$ is a local martingale, and denote by $\mathcal{L}^2(P_X)$ the space of $\mathbb{F}$-predictable processes ξ satisfying

$$\mathrm{E}\left[\int_0^T \xi_u^2 \, d\langle X\rangle_u\right] < \infty.$$

A *(trading) strategy* is a process $\varphi = (\xi, \eta)$ satisfying some measurability and integrability conditions which will be given in Definition 2.1 below. Here, ξ_t is interpreted as the numbers of stocks held at time t, and η_t is the discounted deposit on the savings account at t; the pair $\varphi_t = (\xi_t, \eta_t)$ is called the *portfolio* held at time t. The *(discounted) value process* associated with the strategy φ is defined by

$$V_t(\varphi) = \xi_t X_t + \eta_t, \quad 0 \leq t \leq T. \tag{2.1}$$

Consider an agreement between two parties, a seller (henceforth called the hedger) and a buyer, specifying certain payments as given by an $\mathbb{F}$-adapted process $A = (A_t)_{0 \leq t \leq T}$. It is assumed throughout that A is square-integrable and RCLL. For $0 \leq s < t \leq T$, $A_t - A_s$ is taken to be the total discounted outgoes less incomes during the interval $(s, t]$ as seen from the hedger's point of view. Thus, A describes the hedger's discounted liabilities towards the buyer, and we shall refer to A as the *(discounted) payment process.* The simplest situation to be considered is where A is of the form

$$A_t = -\kappa + 1_{\{t \geq T\}} H, \quad 0 \leq t \leq T, \tag{2.2}$$

for some constant κ and $H \in \mathcal{L}^2(\mathcal{F}_T, P)$. By (2.2), payments take place at time 0 and time T only: The hedger receives the amount κ at time 0 and pays at time T the (discounted) amount H to the buyer of the contract.

Now consider a hedger whose liabilities in respect of a contract are given by the payment process A and who applies a trading strategy φ. Immediately upon signing the contract at time 0, the hedger receives $(-A_0)$ from the buyer of the contract and makes the initial investment $V_0(\varphi)$. Thus, the quantity

$$C_0(\varphi) = V_0(\varphi) + A_0 \tag{2.3}$$

is the hedger's *initial cost* associated with (φ, A). Note that in the special case (2.2), $C_0(\varphi)$ is equal to 0 if the price κ is equal to the amount $V_0(\varphi)$ invested at time 0. Similarly, additional costs may occur during $(0, T]$. One component is payments generated by the contract; another is additional investments currently made by the hedger in response to emergence of new information concerning future payments. This is described more explicitly by introducing a *cost process* $C(\varphi) = (C_t(\varphi))_{0 \leq t \leq T}$ with initial value (2.3), where at any time t, the quantity $C_t(\varphi)$ represents the hedger's accumulated costs during $[0, t]$. To motivate the formal definition given below, we consider the additional costs occurring during the infinitesimal interval $(t, t + dt]$:

$$dC_t(\varphi) = C_{t+dt}(\varphi) - C_t(\varphi) = (\xi_{t+dt} - \xi_t) X_{t+dt} + (\eta_{t+dt} - \eta_t) + A_{t+dt} - A_t. \tag{2.4}$$

Here, the term including ξ and X is the cost associated with a change in the stock position at time $t + dt$: $\xi_t X_{t+dt}$ is the value at time $t + dt$ of the stocks held from time t and $\xi_{t+dt} X_{t+dt}$ is the value of the new position. The second term is related to the change in the discounted deposit on the savings account, and $A_{t+dt} - A_t$ is the contractual payment made by the hedger during $(t, t + dt]$. Using the process $V(\varphi)$

defined by 2.1, we can rewrite (2.4) as

$$\begin{aligned} dC_t(\varphi) &= V_{t+dt}(\varphi) - V_t(\varphi) - \xi_t(X_{t+dt} - X_t) + A_{t+dt} - A_t \\ &= dV_t(\varphi) - \xi_t dX_t + dA_t. \end{aligned}$$

These considerations motivate the following definition:

Definition 2.1 *A strategy is any process $\varphi = (\xi, \eta)$ with $\xi \in \mathcal{L}^2(P_X)$ and η adapted such that the value process $V(\varphi)$ defined by (2.1) is RCLL and $V_t(\varphi) \in \mathcal{L}^2(P)$ for all $t \in [0, T]$.*
The cost process of the strategy φ (and the payment process A) is given by

$$C_t(\varphi) = V_t(\varphi) - \int_0^t \xi_u \, dX_u + A_t, \tag{2.5}$$

and the risk process of φ is defined by

$$R_t(\varphi) = \mathrm{E}\left[(C_T(\varphi) - C_t(\varphi))^2 \middle| \mathcal{F}_t \right]. \tag{2.6}$$

By Definition 2.1, the cost process $C(\varphi)$ is directly related to the hedger's liabilities as given by the payment process A. This means that $C_t(\varphi)$ comprises the hedger's accumulated costs during $[0, t]$ including the payments A_t, and hence, $V_t(\varphi)$ should be interpreted as the value of the portfolio $\varphi_t = (\xi_t, \eta_t)$ held at time t *after* the payments A_t. In particular, $V_T(\varphi)$ is the value of the portfolio φ_T upon settlement of all liabilities, and a natural condition is therefore to restrict to *0-admissible* strategies satisfying

$$V_T(\varphi) = 0, \ P\text{-a.s.} \tag{2.7}$$

By the square-integrability of the payment process A and the assumptions on the strategy φ, the cost process (2.5) is indeed adapted and square-integrable. Note that $R(\varphi)$ is defined as the conditional expected value of the squared future costs and is taken as a measure of the hedger's remaining risk. Observe also that the present usage of the term "risk process" differs from the actuarial one, where "risk process" typically refers to the accumulated payments. The criterion to be considered amounts to minimizing $R_t(\varphi)$ for all t, given some constraint for the value $V_T(\varphi)$ of the portfolio φ_T. This is made more precise in the following definition, which is taken from Schweizer (1994):

Definition 2.2 *A strategy $\varphi = (\xi, \eta)$ is called risk-minimizing if for any $t \in [0, T]$ and any strategy $\hat{\varphi} = (\hat{\xi}, \hat{\eta})$ satisfying*

$$V_T(\varphi) = V_T(\hat{\varphi}) \ P\text{-a.s.}, \ \xi_s = \hat{\xi}_s \text{ for } s \le t, \text{ and } \eta_s = \hat{\eta}_s \text{ for } s < t \,, \tag{2.8}$$

we have $R_t(\hat{\varphi}) \ge R_t(\varphi)$.

A strategy $\hat{\varphi}$ satisfying (2.8) is called an *admissible continuation* of φ at time t.

Remark 2.3 Let us briefly review the results of Föllmer and Sondermann (1986) (subsequently abbreviated as FS) and compare that framework to the one presented in this chapter. To distinguish the notions used in FS from the ones introduced here, we equip all FS-entities with a bar $\bar{\ }$.

FS dealt with the problem of hedging (in an incomplete market) contingent claims $H \in \mathcal{L}^2(P, \mathcal{F}_T)$ payable at a fixed time T, so-called *T-claims*. In that framework, the cost process $\bar{C}(\bar{\varphi})$ associated with a trading strategy $\bar{\varphi}$ (defined as above) is given by

$$\bar{C}_t(\bar{\varphi}) = V_t(\bar{\varphi}) - \int_0^t \bar{\xi}_u \, dX_u, \quad 0 \le t \le T, \tag{2.9}$$

where $\bar{C}_t(\bar{\varphi})$ is the *accumulated costs* of $\bar{\varphi}$ up to t: It is the value $V_t(\bar{\varphi})$ of the portfolio φ_t reduced by the total trading gains during $[0, t]$. In contrast to the cost process of Definition 2.1, this cost process $\bar{C}(\bar{\varphi})$ is a priori independent of the hedger's liabilities. When considering a specific T-claim H, FS restricted to strategies $\bar{\varphi}$ which satisfy

$$V_T(\bar{\varphi}) = H, \tag{2.10}$$

a requirement which amounts to saying that the value of the portfolio φ_T at the time of payment of the liability H should equal H. In particular, the terminal costs associated with a strategy satisfying (2.10) are

$$\bar{C}_T(\bar{\varphi}) = H - \int_0^T \bar{\xi}_u \, dX_u. \tag{2.11}$$

Within the framework of FS, a strategy is said to be *self-financing* if its cost process is constant P-a.s. If there exists a self-financing strategy $\bar{\varphi}$ which satisfies (2.10), then the claim H is said to be *attainable*. Due to incompleteness of the financial market, the condition (2.10) cannot in general be fulfilled by a self-financing strategy. FS therefore considered a larger class of trading strategies, and minimized the conditional expected value of the square of the future costs at any time under the constraint (2.10). Introducing the notion of *mean-self-financing* hedging strategies, that is, strategies whose cost processes are martingales, FS proved that a risk-minimizing strategy is always mean-self-financing.

Note that the interpretation of the value process in FS differs from the one given in the extended framework, where $V_t(\varphi)$ was the value of the portfolio φ_t after payments A_t. Observe also that this way of defining the cost process as being independent of the contingent claim H was possible only due to the fact that trading and the actual payment of H were clearly separated in time in that no trading took place after payment of the claim. This separation is no longer possible in the extended set-up since intermediate payments can occur, possibly at random times and even in a continuous manner. Consequently, the cost process should be updated constantly with regard to the payments incurred by A, and this is achieved by Definition 2.1.

We end this remark by comparing explicitly the cost process in FS to the one of Definition 2.1 in the case where the payment process is of the form (2.2) and $\kappa = 0$, i.e. $A_t = 1_{\{t \geq T\}} H$. For any 0-admissible strategy φ the total cost at time T as measured by $C(\varphi)$ is

$$C_T(\varphi) = V_T(\varphi) - \int_0^T \xi_u \, dX_u + A_T = - \int_0^T \xi_u \, dX_u + H. \tag{2.12}$$

Define a strategy $\bar{\varphi}$ in terms of φ by $\bar{\xi} \equiv \xi$, $\bar{\eta}_t = \eta_t$ for $0 \leq t < T$ and $\bar{\eta}_T = H - \bar{\xi}_T X_T$, so that the strategies φ and $\bar{\varphi}$ differ by their terminal deposit on the savings account only. Clearly, this implies that $C_t(\varphi) = \bar{C}_t(\bar{\varphi})$ for $t < T$, and, by (2.11) and (2.12), also the terminal costs are equal. Thus, for payment processes of the form (2.2), the extended framework essentially reduces to the one of FS. □

It can be verified that also in our extended set-up, the cost process $C(\varphi)$ associated with a risk-minimizing strategy φ is a martingale. The proof is similar to the one given in FS and is omitted here.

Lemma 2.4 *If φ is a risk-minimizing strategy, then $C(\varphi)$ is a martingale.*

The notion of self-financing strategies and attainability for payment processes is introduced in the following definition:

Definition 2.5 *A strategy φ is called self-financing (for A) if $C_t(\varphi) \equiv C_0(\varphi)$ P-a.s. A payment process A is said to be attainable if there exists a strategy φ which is self-financing (for A) and $V_T(\varphi) = 0$ P-a.s.*

Note that this definition of self-financing strategies is only equivalent to the classical definition when $A_t \equiv A_0$ P-a.s. In the extended set-up, the appropriate interpretation of a self-financing strategy is that all fluctuations for the value process are either trading gains generated by the strategy or due to payments prescribed by A. It follows from the definition that a self-financing strategy can have negative terminal value $V_T(\varphi)$ with positive probability even if $V_0(\varphi) > 0$. However, if φ is self-financing for A and satisfies the condition (2.7) of 0-admissibility, then the payment process can be perfectly replicated by φ and we call A attainable. The next lemma relates the notion of attainability for the payment process A to classical attainability for contingent claims payable at time T.

Lemma 2.6 *The payment process A is attainable if and only if the T-claim $H = A_T$ is (classically) attainable.*

Proof. First assume that $C(\varphi)$ is constant and that $V_T(\varphi) = 0$ P-a.s. Then

$$C_0(\varphi) = C_T(\varphi) = - \int_0^T \xi_u \, dX_u + A_T$$

and, since $C_0(\varphi) = A_0 + V_0(\varphi)$,

$$A_T - A_0 = V_0(\varphi) + \int_0^T \xi_u \, dX_u.$$

Thus $A_T - A_0$ is (classically) attainable, and this is equivalent to A_T being attainable since $\mathcal{F}_0$ is assumed to be trivial.

Conversely, let $\bar{\varphi} = (\bar{\xi}, \bar{\eta})$ be a (classically) self-financing strategy with $V_T(\bar{\varphi}) = A_T - A_0$, and define a strategy $\varphi = (\xi, \eta)$ in terms of $\bar{\varphi}$ by $\xi \equiv \bar{\xi}$ and $\eta \equiv \bar{\eta} - (A - A_0)$. Clearly $V_T(\varphi) = 0$, and since

$$\begin{aligned} V_t(\varphi) &= \xi_t X_t + \eta_t = V_t(\bar{\varphi}) - (A_t - A_0) = V_0(\bar{\varphi}) + \int_0^t \bar{\xi}_u \, dX_u - (A_t - A_0) \\ &= V_0(\varphi) + \int_0^t \xi_u \, dX_u - (A_t - A_0), \end{aligned}$$

also $C_t(\varphi) \equiv C_0(\varphi)$. □

The construction of our risk-minimizing strategies is based on an application of the Galtchouk-Kunita-Watanabe decomposition. Define a martingale V^* by

$$V_t^* = \mathrm{E}\left[A_T \middle| \mathcal{F}_t\right], \ 0 \le t \le T. \tag{2.13}$$

This definition is similar to the one of FS which refers to V^* as the *intrinsic value process*. By use of the Galtchouk-Kunita-Watanabe decomposition, V^* can be uniquely decomposed as

$$V_t^* = \mathrm{E}[A_T \mid \mathcal{F}_t] = V_0^* + \int_0^t \xi_u^A \, dX_u + L_t^A, \tag{2.14}$$

where L^A is a zero-mean martingale which is orthogonal to X and ξ^A is a predictable process in $\mathcal{L}^2(P_X)$. This allows us to formulate and prove a result extending Theorem 2 of FS to the present framework.

Theorem 2.7 *There exists a unique 0-admissible risk-minimizing strategy* $\varphi = (\xi, \eta)$ *for* A *given by*

$$(\xi_t, \eta_t) = (\xi_t^A, V_t^* - A_t - \xi_t^A X_t), \ 0 \le t \le T.$$

The associated risk process is given by $R_t(\varphi) = \mathrm{E}\left[(L_T^A - L_t^A)^2 \middle| \mathcal{F}_t \right]$.

Proof. Note that by (2.14)

$$A_T = V_0^* + \int_0^T \xi_u^A \, dX_u + L_T^A = V_t^* + \int_t^T \xi_u^A \, dX_u + (L_T^A - L_t^A). \tag{2.15}$$

Using (2.15) and the fact that $V_T(\tilde{\varphi}) = 0$ for 0-admissible strategies, we see that

$$\begin{aligned} C_T(\tilde{\varphi}) - C_t(\tilde{\varphi}) &= V_T(\tilde{\varphi}) - \int_0^T \tilde{\xi}_u \, dX_u + A_T - \left(V_t(\tilde{\varphi}) - \int_0^t \tilde{\xi}_u \, dX_u + A_t \right) \\ &= (V_t^* - A_t - V_t(\tilde{\varphi})) + (L_T^A - L_t^A) + \int_t^T (\xi_u^A - \tilde{\xi}_u) \, dX_u. \end{aligned} \tag{2.16}$$

Since the martingales L^A and X are orthogonal and the first term in (2.16) is $\mathcal{F}_t$-measurable, we find

$$\begin{aligned} R_t(\tilde{\varphi}) &= \mathrm{E}\left[\left(L_T^A - L_t^A\right)^2 \middle| \mathcal{F}_t\right] + (V_t^* - A_t - V_t(\tilde{\varphi}))^2 \\ &\quad + \mathrm{E}\left[\int_t^T (\xi_u^A - \tilde{\xi}_u)^2 \, d\langle X\rangle_u \middle| \mathcal{F}_t\right]. \end{aligned}$$

The first term is independent of the strategy $\tilde{\varphi}$, whereas the other terms do depend on the strategy. The last two terms are eliminated by first choosing $\xi = \xi^A$ and then η such that $V_t(\varphi) = V_t^* - A_t$ for all t.

Uniqueness of the risk-minimizing strategy is proved as in FS: If a strategy $\hat{\varphi}$ is risk-minimizing and 0-admissible, then $\hat{\varphi}$ minimizes $R_0(\cdot)$, and hence $\hat{\xi} = \xi^A$. Furthermore, $C(\hat{\varphi})$ is a martingale by Lemma 2.4, and this implies that $V(\hat{\varphi}) = V^* - A$, that is $\hat{\eta}_t = V_t^* - A_t - \hat{\xi}_t X_t$. □

The structure of the solution is very natural: The number ξ of stocks held is exactly equal to the result obtained using the original framework of FS for the T-claim A_T, whereas the deposit η on the savings account is now being constantly reduced by the payments A. In particular, when A is of the form (2.2) with $\kappa = 0$, the risk-minimizing strategy of Theorem 2.7 differs from the corresponding strategy of FS only by the choice of η_T.

3 Applications

In this section we present some examples related to insurance with payment processes that are more general than the simple form (2.2). First, in Section 3.1, we consider a general life insurance contract specifying payments that are contingent on the life-length of the policy-holder (the buyer of the contract). The policy-holder pays premiums according to some predefined premium scheme, for example annual payments for 10 years or as long as he is alive. In return, he will receive some insurance benefits which could be payable at a fixed time contingent upon survival or immediately upon some specific event. In addition, we allow benefits and premiums to depend on the price of a traded stock (index). These contracts are known as unit-linked or equity-linked insurance contracts, and the problem of pricing such contracts has been analyzed by Brennan and Schwartz (1976, 1979a), Delbaen (1990), Aase and Persson (1994) and Nielsen and Sandmann (1995), among others. In Møller (1998a), risk-minimizing strategies for unit-linked life insurance contracts are determined for the case where the payment processes are of the form (2.2). In Section 3.2, an example from non-life insurance is considered. Here claims arising from some non-life insurance portfolio are described by a marked point process with claim size distribution that depends on the value of some traded price index.

3.1 Unit-linked life insurance contracts

Let $(\Omega, \mathcal{F}, P)$ and $\mathbb{F}$ be as in Section 2 and consider a financial market consisting of two assets with price processes (B, S) given by the P-dynamics

$$dS_t = r(t, S_t)S_t\,dt + \sigma(t, S_t)S_t\,dW_t, \tag{3.1}$$
$$dB_t = r(t, S_t)B_t\,dt, \tag{3.2}$$

$S_0 > 0$, $B_0 = 1$, where $W = (W_t)_{0\leq t\leq T}$ is a standard Brownian motion on the time interval $[0, T]$; r and σ are functions satisfying certain Lipschitz conditions ensuring the existence of a unique solution to (3.1). In addition it is assumed that r is bounded and non-negative, and that σ is strictly positive. Define, moreover, the discounted price processes $X = S/B$ and $B/B = 1$. Let $\mathbb{G} = (\mathcal{G}_t)_{0\leq t\leq T}$ denote the P-augmentation of the natural filtration of (B, S); it is defined by $\mathcal{G}_t = \mathcal{G}^0_{t+} \vee \mathcal{N}$, where $\mathcal{G}^0_t = \sigma\{(B_u, S_u);\ u \leq t\}$ and where $\mathcal{N}$ is the σ-algebra generated by all P-null-sets. Note that we work directly under the martingale measure P.

The development of some underlying life insurance contract is here described by the classical multi-state Markov model of Hoem (1969); see also Norberg (1992). There exists a finite set $\mathcal{J} = \{0, 1, \dots, J\}$ of possible states of the policy, 0 being the initial state. For example, $\mathcal{J}$ could consist of three states corresponding to *active*, *disabled* and *dead*. Let $Z = (Z_t)_{0\leq t\leq T}$ be an $\mathbb{F}$-adapted right-continuous Markov process with values in $\mathcal{J}$ and initial distribution $(1, 0, \dots, 0)$. The P-augmentation of the natural filtration of Z is denoted $\mathbb{H} = (\mathcal{H}_t)_{0\leq t\leq T}$. We shall assume that Z and (B, S) are stochastically independent under P and we take the filtration $\mathbb{F}$ to be the P-augmentation of the natural filtration of Z and (B, S).

Define in addition the multivariate counting process $N = (N^{jk})_{j\neq k}$ by

$$N^{jk}_t = \#\{s \mid s \in (0, t],\ Z_{s-} = j,\ Z_s = k\},$$

and introduce the processes $I^j_t = 1_{\{Z_t=j\}}$, $j \in \mathcal{J}$. The quantities N^{jk}_t can be interpreted as the number of transitions from state j to state k up to and including time t; I^j_t is equal to 1 if the policy is in the state j at time t and zero otherwise. The transition rates for the Markov chain Z are denoted λ^{jk} and are assumed to be of the form

$$\lambda^{jk}_t = I^j_{t-}\mu^{jk}_t,$$

where μ^{jk} are deterministic continuous functions. In particular, this implies that the processes

$$M^{jk}_t = N^{jk}_t - \int_0^t \lambda^{jk}_u\,du, \quad 0 \leq t \leq T, \tag{3.3}$$

are martingales so that the counting processes N^{jk} possess intensities. Since the N^{jk} by construction do not have any simultaneous jumps, the martingales M^{jk} are orthogonal and their predictable variance processes are given by

$$\langle M^{jk}\rangle_t = \int_0^t \lambda^{jk}_u\,du = \int_0^t I^j_u\mu^{jk}_u\,du.$$

Furthermore, we note that also the discounted stock price process X and M^{jk} defined by (3.3) are orthogonal. The transition probabilities for Z defined by $p_{jk}(t,u) = P(Z_u = k \mid Z_t = j)$ can be determined from Kolmogorov's backward differential equations

$$\frac{d}{dt}p_{jk}(t,u) = \sum_{\ell:\ell\neq j} \mu_t^{j\ell}\left(p_{jk}(t,u) - p_{\ell k}(t,u)\right), \tag{3.4}$$

for $0 \leq t \leq u$, subject to the conditions $p_{jk}(t,t) = 1_{\{j=k\}}$.

Remark 3.1 In order to underline the importance of the choice of filtration $\mathbb{F}$, we shall also refer to the market with the traded assets (B,S) by the triple $(B,S,\mathbb{F})$. On this market, a contingent claim is an $\mathcal{F}_T$-measurable, P-square-integrable random variable. Hence claims (and payment processes) can depend on the development of the traded assets (B,S) as well as uncertainty from the underlying insurance policy Z. As we shall see, the market $(B,S,\mathbb{F})$ is incomplete, and contingent claims are in general not attainable. However, contingent claims that are related to the traded assets only, that is, $H \in \mathcal{L}^2(\mathcal{G}_T,P) \subseteq \mathcal{L}^2(\mathcal{F}_T,P)$, are attainable and hence can be priced uniquely. To see this, note that by independence between $\mathcal{G}_t$ and $\mathcal{H}_t$ we have for such H that

$$V_t^* := \mathrm{E}[H \mid \mathcal{F}_t] = \mathrm{E}[H \mid \mathcal{G}_t \vee \mathcal{H}_t] = \mathrm{E}[H \mid \mathcal{G}_t].$$

Due to the regularity conditions on the functions σ and r in (3.1)–(3.2), the "small" financial market $(B,S,\mathbb{G})$ is complete, and hence the square-integrable $(\mathbb{G},P)$-martingale V^* admits a unique representation

$$V_t^* = \mathrm{E}[H] + \int_0^t \xi_u^H \, dX_u,$$

where ξ^H is $\mathbb{G}$-predictable and $\xi^H \in \mathcal{L}^2(P_X)$. Since $H \in \mathcal{L}^2(\mathcal{G}_T,P)$ is attainable on the small market it is also attainable on the larger market $(B,S,\mathbb{F})$.

We shall consider the problem of hedging on the market $(B,S,\mathbb{F})$ payment processes that are associated with unit-linked insurance contracts and are only adapted to the large filtration $\mathbb{F}$. In this model, the martingale measure for the discounted price process X is not unique and therefore unique prices for the contracts do not exist. But since P itself is already a martingale measure, it coincides trivially with the so-called minimal martingale measure, cf. Schweizer (1991, 1995). This measure is often used in the literature on pricing unit-linked insurance contracts, the motivation being the assumed insurer's risk-neutrality with respect to mortality; see for instance Aase and Persson (1994). Here, our aim is to derive risk-minimizing hedging strategies and we are not primarily dealing with the problem of pricing the claims. □

As in Norberg (1992), there are two basic forms of payments: Firstly, there are so-called general life insurances by which the amount $g_t^{jk} = g^{jk}(t,S_t)$ is payable immediately upon a transition from state j to state k at time t. Secondly, there are general state-wise life annuities payable continuously at rate $g_t^j = g^j(t, S_t)$ at time t contingent on the policy sojourning in state j. This assumption serves only to keep notation simple, and we refer to Remark 3.4 below for an extension to the case where the absolutely continuous annuity payments are replaced by piecewise continuous processes. Note that all payments specified by the contract are functions of the current value of the stock only. It is assumed that $(t,s) \mapsto g^{jk}(t,s)$ and $(t,s) \mapsto g^j(t,s)$ are measurable functions and that

$$\sup_{u\in[0,T]} \mathrm{E}[(B_u^{-1} g^{\cdot}(u, S_u))^2] < \infty \tag{3.5}$$

for all g^{jk} and g^j. This condition ensures that the processes $\int B^{-1} g^{jk}\, dM^{jk}$ are square-integrable martingales. Disregarding the random course of the policy, the $g^j(u, S_u)$ and $g^{jk}(u, S_u)$ are just simple u-claims, and, by Remark 3.1 above, these claims are attainable and hence can be uniquely priced on the financial market. Their unique arbitrage-free processes are denoted F^j and F^{jk}, respectively, and are given by

$$F^j(t, S_t, u) \;=\; \mathrm{E}\left[B_t B_u^{-1} g^j(u, S_u) \middle| \mathcal{G}_t \right], \tag{3.6}$$

$$F^{jk}(t, S_t, u) \;=\; \mathrm{E}\left[B_t B_u^{-1} g^{jk}(u, S_u) \middle| \mathcal{G}_t \right], \quad 0 \le t \le u \le T. \tag{3.7}$$

We assume that the functions $(t,s,u) \mapsto F^{\cdot}(t,s,u)$ are measurable, continuously differentiable with respect to t and twice continuously differentiable with respect to s. In addition, it will be assumed that the first partial derivatives with respect to s (henceforth denoted $F_s^{\cdot}$) are uniformly bounded, i.e., there exists a constant $K < \infty$ such that

$$\sup_{(t,s,u)} \left(\max_{j\neq k} \mid F_s^{jk}(t,s,u) \mid + \max_j \mid F_s^j(t,s,u) \mid \right) \le K < \infty. \tag{3.8}$$

By definition, the discounted price processes $B_t^{-1} F^{\cdot}(t, S_t, u)$ in (3.6) and (3.7) are martingales; Itô's formula and the uniqueness of the canonical decomposition then imply that

$$d\left(B_t^{-1} F^{\cdot}(t, S_t, u)\right) = F_s^{\cdot}(t, S_t, u) dX_t.$$

Note also that $F^j(t, S_t, t) = g^j(t, S_t)$, and similarly for F^{jk}.

The undiscounted payment process generated by the insurance contract is of the form

$$d\hat{A}_t = \sum_{j\in\mathcal{J}} I_t^j\, g_t^j dt + \sum_{j,k:j\neq k} g_t^{jk}\, dN_t^{jk},$$

and the discounted value of payments occurring during $[0,t]$ can be written as

$$A_t = A_0 + \int_0^t B_u^{-1}\, d\hat{A}_u = A_0 + \int_0^t B_u^{-1} \sum_{j\in\mathcal{J}} \left(I_u^j\, g_u^j\, du + \sum_{k:k\neq j} g_u^{jk}\, dN_u^{jk} \right). \tag{3.9}$$

The process (3.9) specifies payments that are contingent on the development of the policy (as described by Z) and are linked to the development on the financial market in that the amounts g^j_u and g^{jk}_u are time-dependent functions of the stock price.

Now consider the intrinsic value process V^* associated with A_T defined by

$$V^*_t = \mathrm{E}[A_T \mid \mathcal{F}_t] = A_t + \mathrm{E}[(A_T - A_t) \mid \mathcal{F}_t]. \tag{3.10}$$

Using the compensators for N^{jk}, the fact that $\int B^{-1} g^{jk}\, dM^{jk}$ are martingales and the independence between Z and (B,S), we find

$$\begin{aligned} V^*_t &= A_t + \mathrm{E}\left[\int_t^T B_u^{-1} \sum_{j\in\mathcal{J}} \left(I^j_u\, g^j_u\, du + \sum_{k:k\neq j} g^{jk}_u\, dN^{jk}_u \right) \middle| \mathcal{F}_t \right] \\ &= A_t + \mathrm{E}\left[\int_t^T B_u^{-1} \sum_{j\in\mathcal{J}} \left(I^j_u\, g^j_u\, du + \sum_{k:k\neq j} g^{jk}_u\, I^j_u \mu^{jk}_u\, du \right) \middle| \mathcal{F}_t \right] \\ &= A_t + B_t^{-1} \sum_{j\in\mathcal{J}} \int_t^T p_{Z_t,j}(t,u) \left(F^j(t,S_t,u) + \sum_{k:k\neq j} \mu^{jk}_u F^{jk}(t,S_t,u) \right) du. \end{aligned}$$

It is convenient to introduce auxiliary processes V^i, $i \in \mathcal{J}$, which represent the state-wise expected value under P of future benefits less premiums under the contract and are given by

$$V^i(t,S_t) = \sum_{j\in\mathcal{J}} \int_t^T p_{ij}(t,u) \left(F^j(t,S_t,u) + \sum_{k:k\neq j} \mu^{jk}_u F^{jk}(t,S_t,u) \right) du, \quad 0 \le t \le T.$$

The quantity $V^i(t,S_t)$ is often interpreted as the (undiscounted) market value of future benefits less premiums conditional on the policy being in state i at time t and on the value S_t of the stock. Formally, it is defined by the conditional expectation

$$V^i(t,s) = \mathrm{E}\left[B_t \int_t^T B_u^{-1}\, d\hat{A}_u \middle| Z_t = i,\ S_t = s \right].$$

However, since the insurance contracts cannot be priced uniquely on the market, justifying such an interpretation requires special assumptions concerning the market's attitude towards the pure insurance risk stemming from the underlying insurance policy. But in any case, the process V^* can now be written as

$$V^*_t = A_t + \sum_{i\in\mathcal{J}} I^i_t\, V^i(t,S_t) B_t^{-1}. \tag{3.11}$$

Lemma 3.2 *The Galtchouk-Kunita-Watanabe decomposition of V^* is given by*

$$V^*_t = V^*_0 + \int_0^t \left(\sum_{i\in\mathcal{J}} I^i_{u-} \xi^i_u \right) dX_u + \sum_{j,k:j\neq k} \int_0^t \nu^{jk}_u\, dM^{jk}_u, \tag{3.12}$$

where

$$\xi^i_t = \sum_{j\in\mathcal{J}} \int_t^T p_{ij}(t,u) \left(F^j_s(t,S_t,u) + \sum_{k:k\neq j} \mu^{jk}_u F^{jk}_s(t,S_t,u) \right) du, \tag{3.13}$$

$$\nu^{jk}_t = B_t^{-1}(V^k(t,S_t) + g^{jk}_t - V^j(t,S_t)). \tag{3.14}$$

Idea of proof of Lemma 3.2: We can easily verify (3.12) under the additional assumption that $V^i \in C^{1,2}$ for $i \in \mathcal{J}$, i.e., continuously differentiable with respect to t and twice continuously differentiable with respect to s. Note that the martingale V^* can be written as

$$V_t^* = A_t + f(t, S_t, Z_t)B_t^{-1},$$

where the function f is defined by $f(t,s,i) = V^i(t,s)$, $i \in \mathcal{J}$. Note also that the derivatives $F_s^{\cdot}$ are uniformly bounded due to (3.8) and that the functions μ^{jk} are deterministic and continuous, hence uniformly bounded on compacts. Thus, by dominated convergence

$$\begin{aligned} f_s(t,s,i) &= \frac{\partial}{\partial s} V^i(t,s) \\ &= \sum_{j \in \mathcal{J}} \int_t^T p_{ij}(t,u) \left(F_s^j(t,s,u) + \sum_{k:k \neq j} \mu_u^{jk} F_s^{jk}(t,s,u) \right) du. \end{aligned} \tag{3.15}$$

By the integration by parts formula, we get that

$$V_t^* = A_t + f(0, S_0, Z_0) + \int_0^t f(u, S_u, Z_u)\, dB_u^{-1} + \int_0^t B_u^{-1}\, df(u, S_u, Z_u). \tag{3.16}$$

We now use (3.16) and Itô's formula on f to rewrite the martingale V^* as the sum of a martingale and a predictable (even continuous) process of finite variation (henceforth called an FV-process). This immediately implies that the predictable FV-process is also a martingale and hence constant. Thus, it is only necessary to identify those terms in (3.16) that are not predictable FV-processes: From A we get the process $\sum_{j,k:j\neq k} \int B^{-1} g^{jk}\, dN^{jk}$, whereas the integral with respect to B^{-1} is a continuous FV-process since B is. The additional assumption $V^i \in C^{1,2}$ allows for an application of the Itô formula to the process $(f(t, S_t, Z_t))_{0 \leq t \leq T}$. The terms including the partial derivatives f_t and f_{ss} are continuous FV-processes and from the term including the partial derivative f_s, we get $\int f_s\, dX$ and a continuous FV-process. Moreover, by (3.15), the integrand f_s is

$$f_s(t, S_t, Z_{t-}) = \sum_{i \in \mathcal{J}} I_{t-}^i \frac{\partial}{\partial s} V^i(t, S_t) = \sum_{i \in \mathcal{J}} I_{t-}^i \xi_t^i,$$

which gives the first integral in (3.12). To obtain the last integral, note that by the continuity of V^i and S, the jump terms from the Itô formula are

$$B_t^{-1} \left(f(t, S_t, Z_t) - f(t-, S_t, Z_{t-}) \right) = \sum_{j,k:j\neq k} B_t^{-1} \left(V^k(t, S_t) - V^j(t, S_t) \right) dN_t^{jk},$$

which, combined with the contribution from A, gives the process $\sum_{j,k:j\neq k} \int \nu^{jk}\, dN^{jk}$. Now subtract and add the continuous FV-process $\sum_{j,k:j\neq k} \int \nu^{jk} \lambda^{jk}\, du$ to obtain the corresponding integrals with respect to the compensated counting processes M^{jk}.

The remaining terms now form a continuous FV-martingale, and hence are constant.
□

The above argument shows where the terms in the decomposition (3.12) come from. However, proving that $V^i \in C^{1,2}$ turns out to be rather laborious. The following proof does not use this assumption.

Proof of Lemma 3.2. We first show that $V^i(t, S_t)$ satisfies

$$B_t^{-1} V^i(t, S_t) = V^i(0, S_0) + \int_0^t \xi_\tau^i \, dX_\tau - \int_0^t \left(B_\tau^{-1} g_\tau^i + \sum_{k:k \neq i} \mu_\tau^{ik} \nu_\tau^{ik} \right) d\tau. \tag{3.17}$$

For each $i \in \mathcal{J}$ and $0 \leq t \leq u \leq T$, let

$$Y_t^{i,u} = \sum_{j \in \mathcal{J}} p_{ij}(t,u) B_t^{-1} \left(F^j(t, S_t, u) + \sum_{k:k \neq j} \mu_u^{jk} F^{jk}(t, S_t, u) \right).$$

By application of the integration by parts formula and Kolmogorov's backward differential equations for p_{ij} it can be shown that

$$dY_t^{i,u} = \alpha_t^{i,u} dt + \beta_t^{i,u} dX_t,$$

where

$$\begin{aligned} \alpha_t^{i,u} &= \sum_{k:k \neq i} \mu_t^{ik} \left(Y_t^{i,u} - Y_t^{k,u} \right), \\ \beta_t^{i,u} &= \sum_{j \in \mathcal{J}} p_{ij}(t,u) \left(F_s^j(t, S_t, u) + \sum_{k:k \neq j} \mu_u^{jk} F_s^{jk}(t, S_t, u) \right). \end{aligned}$$

Using

$$Y_t^{i,u} = Y_0^{i,u} + \int_0^t dY_\tau^{i,u} = Y_0^{i,u} + \int_0^t \alpha_\tau^{i,u} \, d\tau + \int_0^t \beta_\tau^{i,u} \, dX_\tau \tag{3.18}$$

and the expression for $V^i(t, S_t)$, we get

$$\begin{aligned} B_t^{-1} V^i(t, S_t) &= \int_t^T Y_t^{i,u}, du = \int_t^T \left(Y_0^{i,u} + \int_0^t \alpha_\tau^{i,u} \, d\tau + \int_0^t \beta_\tau^{i,u} \, dX_\tau \right) du \\ &= \int_0^T \left(Y_0^{i,u} + \int_0^t 1_{\{\tau \leq u\}} \alpha_\tau^{i,u} \, d\tau + \int_0^t 1_{\{\tau \leq u\}} \beta_\tau^{i,u} \, dX_\tau \right) du \\ &\quad - \int_0^t \left(Y_0^{i,u} + \int_0^u \alpha_\tau^{i,u} \, d\tau + \int_0^u \beta_\tau^{i,u} \, dX_\tau \right) du \\ &= \int_0^T Y_0^{i,u} \, du - \int_0^t Y_u^{i,u} \, du \\ &\quad + \int_0^T \int_0^t 1_{\{\tau \leq u\}} \alpha_\tau^{i,u} d\tau \, du + \int_0^T \int_0^t 1_{\{\tau \leq u\}} \beta_\tau^{i,u} \, dX_\tau \, du. \end{aligned} \tag{3.19}$$

In the last equality we have rearranged terms and used (3.18) for $t = u$. Consider the four terms in (3.19): By the definition of $Y^{i,u}$, we find that

$$\int_0^T Y_0^{i,u} \, du = V^i(0, S_0)$$

and

$$\int_0^t Y_u^{i,u}\,du = \int_0^t B_u^{-1}\left(g_u^i + \sum_{k:k\neq i}\mu_u^{ik}g_u^{ik}\right)du.$$

Since the functions $(\omega, t, u) \mapsto \alpha_t^{i,u}(\omega)$, $i \in \mathcal{J}$, are $\mathcal{O}\otimes\mathcal{B}([0,T])$-measurable and

$$\int_0^T \int_0^t 1_{\{\tau\leq u\}} \mid \alpha_\tau^{i,u} \mid d\tau\, du < \infty \quad P\text{-a.s},$$

the first double integral in (3.19) can be rewritten by the Fubini theorem as

$$\begin{aligned}\int_0^T \int_0^t 1_{\{\tau\leq u\}}\alpha_\tau^{i,u}\,d\tau\,du &= \int_0^t \int_\tau^T \alpha_\tau^{i,u}\,du\,d\tau \\ &= \int_0^t \left(\sum_{k:k\neq i}\mu_\tau^{ik}B_\tau^{-1}(V^i(\tau,S_\tau) - V^k(\tau,S_\tau))\right)d\tau.\end{aligned}$$

The second double integral in (3.19) involves a stochastic integral with respect to the square-integrable martingale X, and hence the standard Fubini theorem cannot be applied. However, note that $(\omega, t, u) \mapsto \beta_t^{i,u}(\omega)$ is $\mathcal{P}\otimes\mathcal{B}([0,T])$-measurable and uniformly bounded for each i by (3.8). Thus, by the Fubini theorem for stochastic integrals, see Protter (1990, Theorem 45), we obtain

$$\int_0^T \int_0^t 1_{\{\tau\leq u\}}\beta_\tau^{i,u}dX_\tau\,du = \int_0^t \int_\tau^T \beta_\tau^{i,u}du\,dX_\tau = \int_0^t \xi_\tau^i\,dX_\tau.$$

This proves (3.17). To prove that (3.12) is the Galtchouk-Kunita-Watanabe decomposition for V^*, note that

$$dI_t^i = \sum_{k:k\neq i}(dN_t^{ki} - dN_t^{ik}).$$

Since the processes V^i are continuous, an application of the integration by parts formula to (3.11) gives

$$\begin{aligned}dV_t^* &= dA_t + \sum_{i\in\mathcal{J}} I_{t-}^i d(B_t^{-1}V^i(t,S_t)) + \sum_{i\in\mathcal{J}} B_{t-}^{-1}V^i(t-,S_{t-})dI_t^i \\ &= B_t^{-1}\sum_{i\in\mathcal{J}}\left(I_t^i g_t^i dt + \sum_{k:k\neq i} g_t^{ik}dN_t^{ik}\right) - \sum_{i\in\mathcal{J}} I_{t-}^i\left(B_t^{-1}g_t^i + \sum_{k:k\neq i}\mu_t^{ik}\nu_t^{ik}\right)dt \\ &\quad + \sum_{i\in\mathcal{J}} I_{t-}^i\xi^i dX_t + \sum_{i,k:i\neq k} B_t^{-1}(V^k(t,S_t) - V^i(t,S_t))dN_t^{ik}.\end{aligned}$$

This proves (3.12) with index j replaced by i in the integrals with respect to the compensated counting processes M^{jk}. By the boundedness of $F_s^{\cdot}$, the integral with respect to X is a square-integrable martingale. Furthermore, (3.5) ensures that g^j, g^{jk} and V^j are square-integrable and that the integrals with respect to M^{jk} are square-integrable martingales. Since X and M^{jk} are orthogonal, this shows that (3.12) is the Galtchouk-Kunita-Watanabe decomposition. □

We can now use Theorem 2.7 to determine risk-minimizing strategies for the payment process A. The integrals with respect to the compensated counting processes M^{jk} are related to the non-hedgeable part of the payment processes. In particular, ν_t^{jk} represents the immediate extra costs for the insurer in connection with a transition from state j to k at time t: the insurer will have to pay the (discounted) amount $g_t^{jk}B_t^{-1}$, and the term $V^k(t,S_t)-V^j(t,S_t)$ denotes the difference between the "reserves" in the two states. In traditional life insurance, ν_t^{jk} is known as the *sum at risk* associated with a transition from state j to state k at time t. By orthogonality of the martingales M^{jk}, the intrinsic risk process associated with the payment process (3.9) is

$$\begin{aligned}R_t(\varphi) &= \mathrm{E}\left[\left(\int_t^T \sum_{j,k:j\neq k}\nu_u^{jk}\,dM_u^{jk}\right)^2\middle|\mathcal{F}_t\right]=\mathrm{E}\left[\int_t^T\sum_{j,k:j\neq k}(\nu_u^{jk})^2\lambda_u^{jk}\,du\middle|\mathcal{F}_t\right]\\ &= \sum_{i\in\mathcal{J}} I_t^i\int_t^T\sum_{j,k:j\neq k}\mathrm{E}[(\nu_u^{jk})^2\mid\mathcal{F}_t]\,p_{ij}(t,u)\mu_u^{jk}\,du. \qquad (3.20)\end{aligned}$$

In the last equality we have exploited the independence between (B,S) and Z and the fact that $(V_t^j)_{j\in\mathcal{J}}$ do not depend on Z. We have now proved:

Theorem 3.3 *For the payment process (3.9), the unique 0-admissible risk-minimizing hedging strategy is given by*

$$(\xi_t,\eta_t)=\left(\sum_{i\in\mathcal{J}}I_{t-}^i\xi_t^i,\ \sum_{i\in\mathcal{J}}I_t^iB_t^{-1}V^i(t,S_t)-X_t\sum_{i\in\mathcal{J}}I_{t-}^i\xi_t^i\right).$$

where ξ^i is given by (3.13). The intrinsic risk process is given by (3.20).

Remark 3.4 The result in Theorem 3.3 is also true with lump sum annuity payments at fixed deterministic times in $\mathcal{T}=\{\tau_1,\ldots,\tau_n\}$ for some $n\geq 1$. In that case, the state-wise annuity payments are described by the processes

$$G_t^j=\sum_{\tau\in\mathcal{T},\tau\leq t}\Delta G^j(\tau,S_\tau)+\int_0^t g^j(u,S_u)\,du,$$

and the auxiliary processes V^i are given by

$$\begin{aligned}V^i(t,S_t) &= \sum_{j\in\mathcal{J}}\int_t^T p_{ij}(t,u)\left(F^j(t,S_t,u)+\sum_{k:k\neq j}\mu_u^{jk}F^{jk}(t,S_t,u)\right)du\\ &\quad+\sum_{j\in\mathcal{J}}\sum_{\tau\in\mathcal{T},\tau>t}p_{ij}(t,\tau)\,F^{\Delta j}(t,S_t,\tau),\quad 0\leq t\leq T,\end{aligned}$$

with

$$F^{\Delta j}(t,S_t,\tau) = B_t\mathrm{E}\left[B_\tau^{-1}\Delta G^j(\tau,S_\tau)\middle|\mathcal{F}_t\right],\quad 0\leq t\leq\tau,\quad \tau\in\mathcal{T}.$$

The processes ξ^i appearing in the expression for ξ are modified similarly. Here, the processes $(V^i)_{i\in\mathcal{J}}$ are no longer continuous, but only piecewise continuous. However, as jumps can occur only at a fixed number of deterministic times, calculations are similar to the ones leading to Theorem 3.3. □

We will now consider some examples which illustrate the flexibility of the multistate Markov model. Assume for simplicity that the financial market is standard Black-Scholes, with constant volatility σ and interest rate r.

Example 3.5 Consider a so-called single life *term insurance* purchased by single premium. This contract specifies that the sum insured is payable immediately upon the death of the policy-holder if this occurs before time T. We will consider the state space $\mathcal{J} = \{0,1\}$, where 0 represents "policy-holder alive" and 1 "policy-holder dead". Denote by x the age of the policy-holder at time 0 and by T_x the policy-holder's remaining lifetime after time 0. Thus T_x is the time of death of the policy-holder and $N_t^{01} = 1_{\{T_x \le t\}}$. The intensity $\mu^{01} =: \mu$ is the hazard rate function, and the transition probabilities are the survival probability

$$p_{00}(t,u) = \exp\left(-\int_t^u \mu_\tau \, d\tau\right) \tag{3.21}$$

and the probability of death $p_{01} = 1 - p_{00}$. More precisely, $p_{00}(t,u)$ is the probability of survival to time u, given that the policy-holder is alive at time t; in actuarial literature, this probability is typically denoted ${}_{u-t}p_{x+t}$, and, similarly, $p_{01}(t,u) = {}_{u-t}q_{x+t}$. The contract functions are all equal to zero except g^{01} and ΔG_0^0 and we assume that $g^{01}(t, S_t) = \max(S_t, Ke^{\delta t})$ and $\Delta G_0^0 = -\kappa$, for some constants K, κ, δ. The constant κ represents a single premium paid by the policy-holder at time 0, and $Ke^{\delta t}$ is the minimum guaranteed amount payable in connection with a death at time t; δ could for example represent a constant inflation rate, which is used to adjust the guaranteed amount. If the policy-holder dies at time $t \in [0,T]$, the insurance company pays an amount equal to $\max(S_t, Ke^{\delta t})$. This type of contract is known as *unit-linked with guarantee*, see e.g. Aase and Persson (1994). In this case, the Black-Scholes formula gives

$$\begin{aligned}
F^{01}(t, S_t, u) &= \mathrm{E}[e^{-r(u-t)}(Ke^{\delta u} + (S_u - Ke^{\delta u})^+) \mid \mathcal{F}_t] \\
&= Ke^{\delta u} e^{-r(u-t)} + \left(S_t \Phi(z_t^{(u)}) - Ke^{\delta u} e^{-r(u-t)} \Phi\left(z_t^{(u)} - \sigma\sqrt{u-t}\right)\right) \\
&= Ke^{\delta u} e^{-r(u-t)} \Phi\left(-z_t^{(u)} + \sigma\sqrt{u-t}\right) + S_t \Phi(z_t^{(u)}),
\end{aligned}$$

where Φ is the standard normal distribution function and

$$z_t^{(u)} = \frac{\log(S_t/Ke^{\delta u}) + (r + \sigma^2/2)(u-t)}{\sigma\sqrt{u-t}}.$$

Using Theorem 3.3, we find the risk-minimizing strategy

$$\begin{aligned}
\xi_t &= 1_{\{T_x \ge t\}} \int_t^T p_{00}(t,u)\, \mu_u \, \Phi(z_t^{(u)}) \, du, \\
\eta_t &= 1_{\{T_x > t\}} \int_t^T p_{00}(t,u)\, \mu_u \, Ke^{-(r-\delta)u} \Phi(-z_t^{(u)} + \sigma\sqrt{u-t}) \, du \\
&\quad -1_{\{T_x = t\}} \int_t^T p_{00}(t,u)\, \mu_u \, \Phi(z_t^{(u)}) X_t \, du.
\end{aligned}$$

Note that the risk-minimizing strategy (ξ, η) does not depend on the premium κ paid at time 0. However, by (2.3) the initial value $C_0(\varphi)$ of the cost process is affected by κ, since

$$C_0(\varphi) = V_0(\varphi) - \kappa = \int_0^T p_{00}(0,u)\,\mu_u\,F^{01}(0,S_0,u)\,du - \kappa.$$

In particular, we find that the initial cost $C_0(\varphi)$ is 0 if and only if

$$\kappa = \int_0^T p_{00}(0,u)\,\mu_u\,F^{01}(0,S_0,u)\,du. \quad \square$$

Example 3.6 Now consider a portfolio consisting of n policy-holders with i.i.d. remaining lifetimes and common hazard rate function μ. The contracts considered are term insurance contracts paid by single premium at time 0, that is, the amount $g(t, S_t)$ is payable at time t if a death has occurred exactly at this time. We assume that g is of the same form as in Example 3.5. This contract can be embedded in the present framework by a state space $\mathcal{J} = \{0, 1, \ldots, n\}$, where state j corresponds to exactly j policy-holders having died. The transition rates for the Z-process are $\lambda_t^{jk} = I_t^j 1_{\{k=j+1\}}(n-j)\mu_t$ for $j, k = 0, \ldots, n-1$, and the transition probabilities are

$$p_{jj}(t,u) = \exp\left(-\int_t^u (n-j)\mu_\tau\,d\tau\right),$$

and $p_{jk} = 0$ for $j > k$; for $j < k$, p_{jk} are determined from the differential equations

$$\frac{d}{dt}p_{jk}(t,u) = (n-j)\mu_t(p_{jk}(t,u) - p_{j+1,k}(t,u)).$$

With this contract, the unique risk-minimizing hedging strategy is

$$\begin{aligned}
\xi_t &= \int_t^T \sum_{j=Z_{t-}}^n p_{Z_{t-},j}(t,u)(n-j)\mu_u\Phi(z_t^{(u)})\,du, \\
\eta_t &= \int_t^T \sum_{j=Z_t}^n p_{Z_t,j}(t,u)(n-j)\mu_u\left(Ke^{-(r-\delta)u}\Phi(-z_t^{(u)} + \sigma\sqrt{u-t}) + \Phi(z_t^{(u)})X_t\right)du \\
&\quad -X_t\int_t^T \sum_{j=Z_{t-}}^n p_{Z_{t-},j}(t,u)(n-j)\mu_u\Phi(z_t^{(u)})\,du.
\end{aligned}$$

Since the remaining lifetimes of the policy-holders are i.i.d. with hazard rate function μ, we see that for $u > t$ the conditional distribution of $(n - Z_u)$ given Z_t is binomial with parameters $((n - Z_t), p(t,u))$, where

$$p(t,u) = \exp\left(-\int_t^u \mu_\tau\,d\tau\right).$$

This implies that

$$\sum_{j=Z_t}^n p_{Z_t,j}(t,u)(n-j) = \mathrm{E}[(n-Z_u) \mid Z_t] = (n-Z_t)p(t,u),$$

and the risk-minimizing hedging strategy above reduces to

$$\begin{aligned}\xi_t &= (n - Z_{t-}) \int_t^T p(t,u)\mu_u \Phi(z_t^{(u)})\, du,\\ \eta_t &= (n - Z_t) \int_t^T p(t,u)\mu_u K e^{-(r-\delta)u} \Phi(-z_t^{(u)} + \sigma\sqrt{u-t})\, du\\ &\quad -\Delta Z_t \int_t^T p(t,u)\mu_u \Phi(z_t^{(u)}) X_t\, du.\end{aligned}$$

These strategies were also obtained in Møller (1998a) in a counting process set-up for payment processes of the form (2.2). However, all payments there were transformed into T-claims by deferring them to the final time T. These deferred insurance claims were then analyzed within the original FS-framework. □

3.2 Index-linked claim amounts in non-life insurance

The occurrence of insurance claims and their sizes are often described by a marked point process with an infinite mark space. In the actuarial literature, it is typically assumed that this process depends on unobservable random variables or some unknown structural parameters. Here, we assume that claim amounts are affected systematically by a price index which can actually be traded on the financial market. This opens the possibility of hedging the risk related to the development of this price index and thus controlling to some extent the uncertainty associated with future liabilities as measured by their variance.

Let $\gamma(dt, dy)$ be a marked point process on the mark space $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ with compensator $\nu(dt, dy)$ satisfying $\nu([0,t] \times A) < \infty$ P-a.s. for all $t \le T$ and $A \in \mathcal{B}(\mathbb{R}_+)$. That is, γ is an integer-valued random measure on $[0,T] \times \mathbb{R}_+$ and for each $A \in \mathcal{B}(\mathbb{R}_+)$

$$\gamma_A(t) = \gamma([0,t] \times A)$$

defines a counting process. We shall assume that the compensator is of the form

$$\nu(\omega, dt, dy) = \lambda_t\, dt\, G(\omega, t, dy), \tag{3.22}$$

where λ is a deterministic continuous function and G is a transition kernel from $(\Omega \times [0,T], \mathcal{P})$ into $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$. Furthermore, assume that $G(\omega, t, \cdot)$ is a probability measure for each (ω, t) and that $G(\omega, t, \cdot) = G(S_t(\omega), t, \cdot)$, where S_t is the price process associated with some underlying price index. This index and one riskless asset with price process B are assumed to be traded freely on the financial market. As in Section 3.1 the P-dynamics of the price processes (B, S) are given by (3.1)–(3.2). The interpretation of the marked point process γ is basically that the insurance claims occur in accordance with an inhomogeneous Poisson process and that claim amounts are affected by the index S. One way of specifying dependency between price index and claim amounts could be the situation where G is absolutely continuous with density $g(S_t, t, y)$ satisfying

$$g(S_t, t, y) = \frac{1}{S_t} g_0(y/S_t), \tag{3.23}$$

for some density g_0 so that S_t is simply a scale parameter. We will return to this example at the end of this section.

In this section we take $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ to be the P-augmentation of

$$\mathcal{F}_t^0 = \sigma\{(B_u, S_u),\ \gamma([0,u] \times A);\ u \leq t, A \in \mathcal{B}(\mathbb{R}_+)\}.$$

Denote the first and second moments of $G(s, t)$ by

$$m(t,s) = \int_{\mathbb{R}_+} y\, G(s,t,dy), \tag{3.24}$$

$$v(t,s) = \int_{\mathbb{R}_+} y^2\, G(s,t,dy), \tag{3.25}$$

and assume that

$$\mathrm{E}\left[\int_0^T v(u, S_u)\lambda_u\, du\right] < \infty. \tag{3.26}$$

For each predictable ($\mathcal{P} \otimes \mathcal{B}_T$-measurable) function h on $\Omega \times [0,T] \times \mathbb{R}_+$ we can define the integral process

$$h * \gamma_t = \int_{[0,t] \times \mathbb{R}_+} h(u,y)\, \gamma(du, dy)$$

if $|h| * \gamma_t$ is finite; $h * \nu_t$ is defined similarly if $|h| * \nu_t$ is finite. In particular, for

$$\tilde{h}(u,y) = y B_u^{-1}$$

the quantity $\tilde{h} * \gamma_t$ represents the discounted value of claims incurred during the interval $[0,t]$. For later use, we note that $\tilde{h} * (\gamma - \nu)$ is a square-integrable martingale under the above mentioned assumptions: By (3.22) the multivariate counting process γ possesses intensity $\lambda_t G(\omega, t, dy)$. In particular this implies that $\nu(\omega, \{t\}, dy) = 0$ for all ω, t, dy. Furthermore, (3.26) and boundedness of r imply that $\tilde{h}^2 * \nu$ is integrable, and hence, by Jacod and Shiryaev (1986, Theorem II.1.33), $\tilde{h} * (\gamma - \nu)$ is a square-integrable martingale. From that theorem is also follows that

$$\langle \tilde{h} * (\gamma - \nu), \tilde{h} * (\gamma - \nu)\rangle_t = \tilde{h}^2 * \nu_t = \int_0^t \int_{\mathbb{R}_+} \left(y B_u^{-1}\right)^2 \nu(du, dy). \tag{3.27}$$

Finally we note that since the martingale $\tilde{h} * (\gamma - \nu)$ is of finite variation, it is purely discontinuous and hence orthogonal to X and any local martingale which can be represented as a stochastic integral with respect to X.

We allow for continuously payable premiums with premium payment intensity $b_u = b(u, S_u)$ and a single premium κ paid at time 0 and consider the discounted payment process

$$\begin{aligned} A_t &= \tilde{h} * \gamma_t - \int_0^t b_u B_u^{-1}\, du - \kappa \\ &= \int_0^t \int_{I\!R_+} y B_u^{-1}\, \gamma(du, dy) - \int_0^t b_u B_u^{-1}\, du - \kappa. \end{aligned} \tag{3.28}$$

As in Section 3.1, we assume that

$$\sup_{u \in [0,T]} \mathrm{E}[(B_u^{-1} b(u, S_u))^2] < \infty, \tag{3.29}$$

and introduce the unique arbitrage-free price processes F^m and F^b associated with the claims specifying payment of the amounts $m(u, S_u)$ and $b(u, S_u)$, respectively, at time u, defined by

$$\begin{aligned} F^m(t, S_t, u) &= B_t \mathrm{E}\left[B_u^{-1} m(u, S_u) \middle| \mathcal{F}_t \right], \\ F^b(t, S_t, u) &= B_t \mathrm{E}\left[B_u^{-1} b(u, S_u) \middle| \mathcal{F}_t \right], \end{aligned}$$

for $0 \leq u \leq t \leq T$. We assume that F^m and F^b are measurable, continuously differentiable with respect to t, twice continuously differentiable with respect to s and that the first partial derivatives with respect to s are uniformly bounded, that is

$$\sup_{(s,t,u)} \left(\mid F_s^m(t, s, u) \mid + \mid F_s^b(t, s, u) \mid \right) \leq K.$$

Furthermore, note that

$$\begin{aligned} \mathrm{E}\left[\int_t^T \int_{I\!R_+} y B_u^{-1}\, \gamma(du, dy) \middle| \mathcal{F}_t \right] &= \mathrm{E}\left[\int_t^T B_u^{-1} \int_{I\!R_+} y\, G(S_u, u, dy) \lambda_u\, du \middle| \mathcal{F}_t \right] \\ &= \mathrm{E}\left[\int_t^T B_u^{-1} m(u, S_u) \lambda_u\, du \middle| \mathcal{F}_t \right] \\ &= B_t^{-1} \int_t^T F^m(t, S_t, u) \lambda_u\, du. \end{aligned}$$

For the intrinsic value process $V_t^* = \mathrm{E}[A_T \mid \mathcal{F}_t]$ associated with A_T we now have

$$\begin{aligned} V_t^* &= A_t + \mathrm{E}[(A_T - A_t) \mid \mathcal{F}_t] \\ &= A_t + \mathrm{E}\left[\int_t^T \int_{I\!R_+} y B_u^{-1}\, \gamma(du, dy) \middle| \mathcal{F}_t \right] - \mathrm{E}\left[\int_t^T b_u B_u^{-1}\, du \middle| \mathcal{F}_t \right] \\ &= A_t + \int_t^T B_t^{-1} \left(F^m(t, S_t, u) \lambda_u - F^b(t, S_t, u) \right) du. \end{aligned}$$

Lemma 3.7 *The Galtchouk-Kunita-Watanabe decomposition of V^* is given by*

$$V_t^* = V_0^* + \int_0^t \xi_u^A\, dX_u + \int_{[0,t] \times I\!R_+} \tilde{h}(u, y)(\gamma(du, dy) - \nu(du, dy)), \tag{3.30}$$

where

$$\xi_t^A = \int_t^T \left(F_s^m(t, S_t, u) \lambda_u - F_s^b(t, S_t, u) \right) du. \tag{3.31}$$

Proof: The proof of the decomposition (3.30) is analogous to the proof of Lemma 3.2 and involves application of the Fubini theorem for stochastic integrals. Therefore, we only show where the integrands come from, and we do this under the additional assumption that the function

$$f(t,s) := \int_t^T \Big(F^m(t,s,u)\lambda_u - F^b(t,s,u)\Big)\,du$$

is in $C^{1,2}$. Since F_s^m and F_s^b are uniformly bounded and $u \mapsto \lambda_u$ is continuous and deterministic, we find that

$$f_s(t,s) = \frac{\partial}{\partial s} f(t,s) = \int_t^T \Big(F_s^m(t,s,u)\lambda_u - F_s^b(t,s,u)\Big)\,du.$$

Now use the integration by parts formula to rewrite V^* as

$$V_t^* \;=\; A_t + f(0,S_0) + \int_0^t f(u,S_u)\,dB_u^{-1} + \int_0^t B_u^{-1}\,df(u,S_u), \tag{3.32}$$

and identify terms that are not predictable FV-processes: From A we get the process $\int_{[0,t]\times I\!R_+} yB_u^{-1}\,\gamma(du,dy)$, and application of the Itô-formula to $f(t,S_t)$ yields $\int f_s\,dX$. Now subtract and add the continuous FV-process $\int_{[0,t]\times I\!R_+} yB_u^{-1}\,\nu(du,dy)$ to obtain (3.30). Thus, V^* admits the decomposition (3.30) into an integral with respect to the discounted price process X and an integral with respect to the compensated random measure γ. Furthermore, since the integrals with respect to $(\gamma-\nu)$ and X are orthogonal, (3.30) is the Galtchouk-Kunita-Watanabe decomposition for V^*. □

Theorem 2.7 can now be used to find the unique 0-admissible risk-minimizing strategy for the payment process (3.28):

$$\xi_t \;=\; \xi_t^A, \tag{3.33}$$

$$\eta_t \;=\; B_t^{-1}\int_t^T F^m(t,S_t,u)\lambda_u\,du - B_t^{-1}\int_t^T F^b(t,S_t,u)\,du - \xi_t^A X_t. \tag{3.34}$$

By use of (3.27) the intrinsic risk process $R(\varphi)$ can be computed as

$$\begin{aligned} R_t(\varphi) &= \mathrm{E}\left[\left(\int_t^T\int_{I\!R_+}\tilde h(u,y)\,(\gamma(du,dy)-\nu(du,dy))\right)^2\middle|\,\mathcal{F}_t\right] \\ &= \mathrm{E}\left[\int_t^T\int_{I\!R_+}(yB_u^{-1})^2\,\nu(du,dy)\middle|\,\mathcal{F}_t\right] \\ &= \mathrm{E}\left[\int_t^T\int_{I\!R_+}(yB_u^{-1})^2 G(S_u,u,dy)\lambda_u\,du\middle|\,\mathcal{F}_t\right] \\ &= \int_t^T \mathrm{E}\left[(B_u^{-1})^2 v(u,S_u)\middle|\,\mathcal{F}_t\right]\lambda_u\,du. \end{aligned} \tag{3.35}$$

Note that the conditional variance of A_T given $\mathcal{F}_t$ can be computed by using the decomposition (3.30)

$$\mathrm{Var}\left[A_T\middle|\,\mathcal{F}_t\right] = \mathrm{Var}\left[(A_T-A_t)\middle|\,\mathcal{F}_t\right] \;=\; R_t(\varphi) + \mathrm{Var}\left[\int_t^T \xi_u^A\,dX_u\middle|\,\mathcal{F}_t\right], \tag{3.36}$$

where $R_t(\varphi)$ is given by (3.35). The second term in (3.36) is related to the hedgeable part of the payment process, whereas the first term equals the intrinsic risk $R_t(\varphi)$ at time t. The process (3.36) corresponds to the risk process associated with the strategy $(\xi, \eta) = (0, V^* - A)$.

Example 3.8 Consider a standard Black-Scholes market with constant r and σ. Assume that claims occur with a fixed intensity λ and that premiums are paid with a fixed constant intensity $b_t \equiv b$ during $[0,T]$. Assume moreover that the distribution $G(s,t,.)$ is absolutely continuous for each (t,s) with density

$$G(s,t,dy) = \frac{1}{s} g_0(y/s) dy$$

for some density g_0 on $\mathbb{R}_+$. This implies that the first and second moments of $G(t,s)$ are

$$m(t,s) = \int_{\mathbb{R}_+} y \frac{1}{s} g_0(y/s)\, dy = s \int_{\mathbb{R}_+} y g_0(y)\, dy =: s\, m_0,$$

and

$$v(t,s) = s^2 \int_{\mathbb{R}_+} y^2 g_0(y)\, dy =: s^2\, v_0.$$

Thus, $F^m(t, S_t, u) = S_t m_0$ and $F^m_s = m_0$, and so the risk-minimizing strategy is given by

$$\begin{aligned} \xi_t &= \int_t^T m_0 \lambda\, du = \lambda(T-t) m_0, \\ \eta_t &= -\int_t^T B_u^{-1} b\, du = -\frac{b}{r}(e^{-rt} - e^{-rT}). \end{aligned}$$

The intrinsic risk process is in this case

$$\begin{aligned} R_t(\varphi) &= \int_t^T \mathrm{E}\left[(B_u^{-1} S_u)^2 v_0 \middle| \mathcal{F}_t \right] \lambda\, du = \lambda v_0 X_t^2 \int_t^T e^{\sigma^2(u-t)}\, du \\ &= \lambda v_0 X_t^2 \frac{1}{\sigma^2} \left(e^{\sigma^2(T-t)} - 1 \right). \end{aligned}$$

The conditional variance of the hedgeable part of the risk can now be determined from (3.36):

$$\begin{aligned} \mathrm{Var}\left[A_T \middle| \mathcal{F}_t \right] - R_t(\varphi) &= \mathrm{Var}\left[\int_t^T \lambda(T-u) m_0 \sigma X_u\, dW_u \middle| \mathcal{F}_t \right] \\ &= \mathrm{E}\left[\int_t^T (\lambda(T-u)m_0)^2 \sigma^2 X_u^2\, du \middle| \mathcal{F}_t \right] \\ &= \lambda^2 m_0^2 X_t^2 \int_t^T \sigma^2 e^{\sigma^2(u-t)} (T-u)^2\, du. \end{aligned}$$

This quantity is the difference between the total risk associated with the payment process and its intrinsic risk process; it describes the risk-increase for an insurer who uses the strategy $\varphi = (0, V^* - A)$ instead of the risk-minimizing strategy. □

Chapter 4

On Transformations of Actuarial Valuation Principles

(This chapter is an adapted version of Møller (1999b))

In this chapter we determine optimal trading strategies associated with the financial variance and standard deviation principles proposed by Schweizer (1997). These principles take into consideration the possibilities of hedging on the financial market and are derived by an indifference argument, which embeds the traditional (actuarial) variance and standard deviation principles in a financial framework. We also investigate an alternative way of transforming actuarial principles and show that for the standard deviation principle this leads to the financial standard deviation principle. The principles are applied for the valuation and hedging of unit-linked life insurance contracts.

1 Introduction

Schweizer (1997) proposes a financial valuation principle that is derived from traditional actuarial premium calculation principles and at the same time takes into consideration the possibility of trading on a financial market. First, an a priori given actuarial valuation principle (which measures risk) is translated into a "measure of preferences". Then, this new measure is used in a so-called indifference argument to define a new financial premium principle, which can be viewed as a financial transformation of the a priori given actuarial principle. The financial counterparts of the actuarial variance and standard deviation principles are called the *financial variance principle* and the *financial standard deviation principle*, respectively.

The traditional variance and standard deviation principles are recalled in Section 2, and it is shown how these premium calculation principles can be translated into measures of preferences by use of an argument which is similar to a zero expected utility argument. In Section 3, we give some preliminaries and recall the definition of the so-called variance optimal martingale measure, which plays an important role for the financial valuation principles. The main results of Schweizer (1997) are reviewed in Section 4 in a simplified framework suitable for pricing of reinsurance contracts that combine traditional insurance risk and financial risk. In his framework, a liquid reinsurance market is not present and the only investment and trading possibilities are given on some financial market. Using the same set-up, we investigate in Chapter 5 how the premiums under the financial valuation principles depend on the amount of information available to the reinsurer and give upper and lower bounds for these premiums involving only conditional expectations and variances.

We then show in Sections 4.1 and 4.2 how optimal strategies associated with these transformed valuation principles can be determined in the case of the variance principle and the standard deviation principle, thus adding to the existing results. The results presented can basically be viewed as applications of the main results of Schweizer (1997) since most proofs are based on the techniques used there. For the variance principle, the optimal strategy, which maximizes the derived measure of preferences, differs from the *mean-variance hedging strategy* (see e.g. Schweizer (1999)) only by a correction term, which is independent of the contract considered. The optimal strategy under the standard deviation principle is also related to the mean-variance hedging strategy, but in this case, the difference between the two strategies is more complex. In Section 5, we consider an alternative modification of the same principles, which we call the *direct financial transformation*. This approach has the advantage that it does not involve a translation of the actuarial premium calculation principle into a measure of preferences, as it is defined directly in terms of the original actuarial principle. In the case of the standard deviation principle, this approach leads to premiums which are similar to the ones computed by using the financial standard deviation principle. For the variance principle, however, we show that the direct financial transformation does not lead to reasonable premiums. Finally, some applications and numerical results related to unit-linked life insurance contracts are presented in Section 6. The results are compared to the risk-minimizing strategies obtained in Møller (1998a) and in Chapter 2.

2 The actuarial premium calculation principles

In this section, we first introduce the two classical actuarial premium calculation principles which will be analyzed in the following: the variance and the standard deviation principles. Second, we recall that these valuation principles can be viewed as the solutions to certain simple indifference principles. This serves as a motivation for the results presented in Section 4.

Let H be a claim (or risk) which is to be valuated by an agent, henceforth called a reinsurer. The following actuarial valuation principles are widely used:

$$\tilde{u}_1(H) = \mathrm{E}[H] + a\mathrm{Var}[H], \tag{2.1}$$

$$\tilde{u}_2(H) = \mathrm{E}[H] + a\sqrt{\mathrm{Var}[H]}. \tag{2.2}$$

In the actuarial literature, (2.1) is called the variance principle and (2.2) is the standard deviation principle, see e.g. Goovaerts et al. (1984). The terms $a\mathrm{Var}[H]$ and $a\mathrm{Var}[H]^{1/2}$, respectively, are often called the safety loadings, and we shall refer to a as the safety loading parameter. It is convenient to work with the negative of H, $Y = -H$, which can be interpreted as the amount received by the reinsurer, and we introduce now the following slightly modified versions of the premium principles (2.1) and (2.2):

$$u_1(Y) = \mathrm{E}[Y] - a\mathrm{Var}[Y], \tag{2.3}$$

$$u_2(Y) = \mathrm{E}[Y] - a\sqrt{\mathrm{Var}[Y]}. \tag{2.4}$$

Note that u_i differs from $\tilde{u}_i$, $i = 1, 2$, by the sign on the loading factor and by the fact that $\tilde{u}_i$ operates on $-H$. Thus, we shall think of $\tilde{u}_i$ as a measure of risk, whereas u_i is taken as a "measure of preferences". More precisely we associate with u_i a preference relation specifying that the pair (p_i, H) (selling the claim H and receiving the premium p_i) is preferred to (p_i', H') if and only if

$$u_i(p_i - H) \geq u_i(p_i' - H').$$

In particular, the pair $(0, 0)$ corresponds to not selling any claims and not receiving any premiums, and the reinsurer is now indifferent in terms of u_i between not selling H and selling H at the premium p_i provided that

$$u_i(p_i - H) = u_i(0) = 0, \tag{2.5}$$

that is, provided that $p_i = \tilde{u}_i(H)$, $i = 1, 2$. Thus we can indeed obtain the original principles (2.1) and (2.2) from (2.3) and (2.4). This way of defining the premium is compatible with the *zero expected utility increase principle*; see e.g. Goovaerts et al. (1984) for more details. Note however that the equation (2.5) does not involve a proper utility function. Instead, we simply interpret (2.3) and (2.4) as quantities which describe the preferences of the reinsurer, and which lead to the well-known actuarial pricing principles (2.1) and (2.2).

It is relatively easy to construct examples which show that the actuarial valuation principles $\tilde{u}_1$ and $\tilde{u}_2$ do not satisfy the natural condition

$$H_1 \leq H_2 \ P\text{-a.s.} \ \Rightarrow \tilde{u}_i(H_1) \leq \tilde{u}_i(H_2), \tag{2.6}$$

$i = 1, 2$. Of course, this property would also not be satisfied if we replaced $\tilde{u}_i$ in (2.6) with the modified versions u_i.

3 Preliminaries

In this section we review some technical notions which are needed for the introduction of the financial market in the next section; for unexplained terminology, see Jacod and Shiryaev (1987).

Consider a complete filtered probability space $(\Omega, \mathcal{F}, P, \mathbb{F})$, where the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ satisfies the usual conditions and T is some fixed finite time horizon. It is not assumed that $\mathcal{F}_0$ is trivial. Let $X = (X_t)_{0 \le t \le T}$ be an $I\!R^d$-valued *continuous* semimartingale with respect to $I\!F$; X is taken to be the discounted price process for some financial assets. (In Section 6, we consider an example where X is a diffusion process, and hence in particular a continuous semimartingale.)

Before proceeding further, we give an outline of the rest of this section. We first define the *variance optimal martingale measure* $\tilde{P}$, which is here a probability measure that is equivalent to P and (basically) can be characterized by the following properties:

1. It is a martingale measure, i.e. the discounted price process X is a (local) $\tilde{P}$-martingale.
2. The Radon-Nikodym derivative $\frac{d\tilde{P}}{dP}$ with respect to the underlying measure P has minimum variance over all martingale measures.

Then we define a space $\tilde{\Theta}$ of $I\!R^d$-valued trading strategies which has the important property that the space $G_T(\tilde{\Theta})$ of (real-valued) stochastic integrals $G_T(\vartheta) := \int_0^T \vartheta_t \, dX_t$, $\vartheta \in \tilde{\Theta}$, is closed in $L^2(P)$; for $\vartheta \in \tilde{\Theta}$, ϑ_t^i is the number of shares of stock i held at t, and $G_t(\vartheta) = \int_0^t \vartheta_u \, dX_u$ is the trading gains generated from ϑ up to and including time t. This choice of space is crucial for Theorem 3.4 below and makes it possible to derive optimal trading strategies in the next section. References are Delbaen and Schachermayer (1996a,b), Schweizer (1996) and Rheinländer and Schweizer (1997).

Let $\mathcal{V}$ denote the linear space spanned by the random variables of the simple form $h^{tr}(X_{T_2} - X_{T_1})$, where $T_1 \le T_2 \le T$ are any stopping times such that the stopped process X^{T_2} is bounded, and h is a bounded $I\!R^d$-valued $\mathcal{F}_{T_1}$-measurable random variable. Denote by $\mathcal{M}^s(P)$ the space of signed measures $Q \ll P$ with $Q(\Omega) = 1$ and

$$\mathrm{E}\left[\frac{dQ}{dP} f\right] = 0, \tag{3.1}$$

for all $f \in \mathcal{V}$, and by $\mathcal{M}^e(P)$ the set of probability measures $Q \in \mathcal{M}^s(P)$ with $Q \sim P$. Furthermore, define spaces $\mathcal{D}^s$ and $\mathcal{D}^e$ by

$$\mathcal{D}^x = \left\{ \frac{dQ}{dP} \middle| Q \in \mathcal{M}^x(P) \right\},$$

for $x \in \{s, e\}$.

Definition 3.1 *The variance optimal martingale measure $\tilde{P}$ is the unique element of $\mathcal{M}^s(P)$ such that $\tilde{D} := \frac{d\tilde{P}}{dP} \in L^2(P)$ and such that $\tilde{D}$ minimizes $\|D\|_{L^2(P)}$ over all $D \in \mathcal{D}^s \cap L^2(P)$.*

We will be working under Assumption 3.2 below, which ensures the existence of the variance optimal martingale measure $\tilde{P}$ and guarantees that this measure is a probability measure which is equivalent to P, see Delbaen and Schachermayer (1996a, Theorem 1.3).

Assumption 3.2 $\mathcal{D}^e \cap L^2(P) \neq \emptyset$.

Let $\tilde{\Theta}$ denote the space of $\mathbb{R}^d$-valued $\mathbb{F}$-predictable processes ϑ which are such that $G(\vartheta) = \int \vartheta \, dX$ is a $\tilde{P}$-martingale and $\int_0^T \vartheta_t \, dX_t \in L^2(P)$, and define

$$G_T(\tilde{\Theta}) := \left\{ \int_0^T \vartheta_t \, dX_t \middle| \, \vartheta \in \tilde{\Theta} \right\}.$$

It was shown in Delbaen and Schachermayer (1996b) and in Gourieroux, Laurent and Pham (1998), that when X is continuous, Assumption 3.2 implies that $G_T(\tilde{\Theta})$ is closed and is equal to the closure in $L^2(P)$ of the space $\mathcal{V}$, see the remark following Proposition 15 of Rheinländer (1999). This remark also shows that a predictable process ϑ is in $\tilde{\Theta}$ if and only if the process $G(\vartheta)$ is a Q-martingale for any $Q \in \mathcal{M}^e(P)$ with $\frac{dQ}{dP} \in L^2(P)$.

Remark 3.3 Note that the spaces $\mathcal{M}^x(P)$, $x \in \{s, e\}$, depend on the filtration $\mathbb{F}$. Consequently, also the variance optimal martingale measure is affected by the choice of filtration; we refer to Chapter 5 for an investigation of this property. □

The following result is now a consequence of the projection theorem for Hilbert spaces:

Theorem 3.4 *Any random variable $H \in L^2(\mathcal{F}_T, P)$ admits a unique decomposition of the form*

$$H = c^H + \int_0^T \vartheta_t^H \, dX_t + N^H, \tag{3.2}$$

where $\vartheta^H \in \tilde{\Theta}$, $\mathrm{E}[N^H] = 0$, and $\mathrm{E}[N^H \int_0^T \vartheta_t \, dX_t] = 0$ for all $\vartheta \in \tilde{\Theta}$.

We denote by π the projection in $L^2(P)$ on $G_T(\tilde{\Theta})^\perp$. The following lemma, which is due to Delbaen and Schachermayer (1996a) and Schweizer (1996), relates the variance-optimal martingale measure $\tilde{P}$ to the projection π.

Lemma 3.5 *Under Assumption 3.2, the variance optimal martingale measure exists and is given by*

$$\frac{d\tilde{P}}{dP} = \frac{\pi(1)}{\mathrm{E}\,[\pi(1)]}.$$

Throughout, we let $\tilde{Z}_T = \frac{d\tilde{P}}{dP}$ and write $\tilde{\mathrm{E}}$ for $\mathrm{E}_{\tilde{P}}$.

4 The financial valuation principles

In this section, we introduce the financial market and recall the crucial indifference argument of Schweizer (1997) that leads to the definition of the fair premium. We consider a financial market consisting of $d+1$ basic traded assets: d stocks with price process $X = (X^1, \dots, X^d)^{tr}$ and a savings account with a price process which is constant and equal to 1. (One can think of X as the discounted value of the stocks expressed in terms of the savings account.) Here, a *self-financing trading strategy* is a $(d+1)$-dimensional $\mathbb{F}$-adapted process $\varphi = (\vartheta, \eta)$ such that $\vartheta \in \tilde{\Theta}$ and such that the value process defined by $V(\varphi) := \vartheta^{tr} X + \eta$ satisfies

$$V_t(\varphi) = V_0(\varphi) + \int_0^t \vartheta_u \, dX_u \tag{4.1}$$

for all $t \in [0, T]$. The pair $\varphi_t = (\vartheta_t, \eta_t)$ is the *portfolio* held at t: ϑ_t^i is the number of units of stock number i held, and η_t is the discounted deposit on the savings account at t. In (4.1), $V_0(\varphi)$ is the initial investment at time 0, and $\int_0^t \vartheta \, dX$ is trading gains from the strategy φ. Thus, (4.1) states that, at each time t, the value of the portfolio φ_t is the initial value plus trading gains, hence no additional in- or out-flow of capital occurs after time 0. A claim H is said to be *attainable* if there exists a self-financing strategy φ such that $V_T(\varphi) = H$ P-a.s., i.e.

$$H = V_0(\varphi) + \int_0^T \vartheta_t \, dX_t.$$

This implies that the claim H can be replicated perfectly by investing the amount $V_0(\varphi)$ at time 0 and thereafter following the self-financing strategy φ; the initial investment $V_0(\varphi)$ is called the (unique) no-arbitrage price for H. Since $\mathcal{F}_0$ is not assumed to be trivial, the above definition of a self-financing trading strategy allows for $\mathcal{F}_0$-measurable initial investments $V_0(\varphi)$, i.e. for random initial investments. However, in the following we shall restrict to strategies with a non-random initial investment.

The idea is now the following: Assume that the reinsurer applies one of the premium calculation principles $\tilde{u}_i$, $i = 1, 2$, or, equivalently, the corresponding valuation functions u_i, $i = 1, 2$ (henceforth called u). The reinsurer is considering the possibility of accepting (insuring) a fraction $\gamma \in \mathbb{R}$ of a claim due at time T with discounted value H (here $\gamma < 0$ corresponds to selling the fraction). We denote by c the reinsurer's (non-random) initial capital (basis capital at time 0) and consider the following possibilities: On the one hand, the reinsurer can choose to accept the risk γH, receive some premium $h(c, \gamma)$ and invest the amount $c + h(c, \gamma)$ on the financial market using a self-financing strategy $\varphi = (\vartheta, \eta)$. This will generate the discounted wealth

$$c + h(c, \gamma) + \int_0^T \vartheta_t \, dX_t - \gamma H,$$

where we have subtracted the term γH, which is to be paid at time T to the buyer of the contract. However, the reinsurer could also choose not to engage in the risk H

and simply invest the initial capital c on the market according to some self-financing strategy $\tilde{\varphi} = (\tilde{\vartheta}, \tilde{\eta})$ and generate the wealth

$$c + \int_0^T \tilde{\vartheta}_t \, dX_t.$$

The *fair premium* is now defined as the premium which, in terms of the valuation function u, makes the reinsurer indifferent between the two possibilities of accepting and not accepting the risk. From Schweizer (1997) we have the following formal definition of the fair premium for γ units of the risk H in the presence of a financial market

Definition 4.1 *$h(c, \gamma)$ is called a u-indifference price for γ units of H if it satisfies*

$$\sup_{\vartheta \in \tilde{\Theta}} u\left(c + h(c,\gamma) + \int_0^T \vartheta_t \, dX_t - \gamma H\right) = \sup_{\tilde{\vartheta} \in \tilde{\Theta}} u\left(c + \int_0^T \tilde{\vartheta}_t \, dX_t\right). \tag{4.2}$$

Remark 4.2 This definition of the fair premium specializes to the principle (2.5) in the case where the space of investment strategies is given by $\tilde{\Theta} = \{(0, \ldots, 0)^{tr}\}$, that is, no trading in the stocks is allowed, provided that $u(c+Y) = c + u(Y)$, for $c \in I\!R$. And hence, in the case of the variance or the standard deviation principles, (4.2) generalizes (2.5) to the situation where a financial market is present, since $u_i(c+Y) = c + u_i(Y)$, for $i = 1, 2$. Furthermore, it follows directly from this property and (4.2) that the fair premium $h(c, \gamma)$ will be independent of the initial capital c for the principles (2.3) and (2.4). From a mathematical point of view, the fraction γ in γH is redundant; we could as well work with a claim H. The inclusion of γ is motivated by the application we have in mind here, where the reinsurer participates in a fraction γ of the risk H. The question of how γ should be chosen is not addressed within this context, however. □

Schweizer (1997) proved that in the case of the variance and the standard deviation principles, the solutions to the problem (4.2) can be related to the decomposition (3.2). For completeness, we give here these two results (Theorem 9 and Theorem 12 of Schweizer (1997)). In the case of the variance principle, the solution is:

Theorem 4.3 (Schweizer (1997)) *For any $H \in L^2(P)$ and $\gamma, c \in I\!R$, the u_1-indifference price for γH is*

$$h_1(c,\gamma) = v_1(\gamma H) = \gamma \tilde{\mathrm{E}}[H] + a\gamma^2 \mathrm{Var}\left[N^H\right]$$

In the case of the standard deviation principle, we have the following result:

Theorem 4.4 (Schweizer (1997)) *For any $H \in L^2(P)$ and $\gamma, c \in I\!R$, the u_2-indifference price for γH is*

$$h_2(c,\gamma) = v_2(\gamma H) = \gamma \tilde{\mathrm{E}}[H] + a|\gamma| \sqrt{1 - \frac{\mathrm{Var}[\frac{d\tilde{P}}{dP}]}{a^2}} \sqrt{\mathrm{Var}[N^H]},$$

provided that $a^2 \geq \mathrm{Var}[\frac{d\tilde{P}}{dP}]$. If $a^2 < \mathrm{Var}[\frac{d\tilde{P}}{dP}]$, then the u_2-indifference price is undefined.

4.1 The optimal strategy for the variance principle

We will determine the optimal strategy for a reinsurer who is using the principle (4.2) in the case of the variance principle. That is, we determine ϑ^* so that

$$\sup_{\vartheta\in\tilde{\Theta}} u_1\left(c+h_1(c,\gamma)+\int_0^T \vartheta_t\,dX_t-\gamma H\right)=u_1\left(c+h_1(c,\gamma)+\int_0^T \vartheta_t^*\,dX_t-\gamma H\right),$$

where $h_1(c,\gamma)$ is the fair premium determined by Theorem 4.3, and express the maximum of u_1 in terms of the decomposition (3.2). The proof uses techniques from Schweizer (1997) and consists in first determining the optimal strategy for a fixed expected value m of the trading gains and then maximizing over all $m \in \mathbb{R}$. In particular, Theorem 4.3 and 4.4 will follow again from the results given in this and the next subsection.

First note that we can formulate (4.2) equivalently by taking supremum over all elements $g=\int_0^T \vartheta\,dX$ in the space $G_T(\tilde{\Theta})$, that is $h_1(c,\gamma)$ satisfies

$$\sup_{g\in G_T(\tilde{\Theta})} u_1\left(c+h_1(c,\gamma)+g-\gamma H\right)=\sup_{\tilde{g}\in G_T(\tilde{\Theta})} u_1\left(c+\tilde{g}\right). \tag{4.3}$$

Similarly, we let $g^H = \int_0^T \vartheta^H\,dX$ denote the term appearing in the decomposition (3.2) for H. The following lemma is crucial for determining the optimal strategy.

Lemma 4.5 *Assume that $1 \notin G_T(\tilde{\Theta})^{\perp}$. For any $m \in \mathbb{R}$, the solution to the problem*

$$\max_{g\in G_T(\tilde{\Theta})} u_1(g-N^H) \quad \textit{subject to } \mathrm{E}[g]=m, \tag{4.4}$$

is given by $g_m = c_m(1-\pi(1))$ where $c_m = \frac{m}{\mathrm{E}[(1-\pi(1))^2]}$.

Proof: We first note that, by the definition of N^H, we have that $\mathrm{E}[N^H]=\mathrm{E}[N^H g]=0$ for all $g\in G_T(\tilde{\Theta})$, so that

$$u_1(g-N^H) \;=\; \mathrm{E}[g-N^H]-a\mathrm{Var}[g-N^H]=\mathrm{E}[g]-a\mathrm{Var}[g]-a\mathrm{Var}[N^H],$$

and hence, we have to minimize $\|g\|^2 := \mathrm{E}[g^2]$ over all $g\in G_T(\tilde{\Theta})$ with $\langle g,1\rangle := \mathrm{E}[g]=m$. By the projection theorem for Hilbert spaces, any $g \in G_T(\tilde{\Theta})$ admits a unique decomposition

$$g=\alpha(1-\pi(1))+\hat{g},$$

where $\alpha\in\mathbb{R}$, $\hat{g}\in G_T(\tilde{\Theta})$, and $\hat{g}\perp(1-\pi(1))$ (i.e. $\mathrm{E}[\hat{g}(1-\pi(1))]=\langle\hat{g},(1-\pi(1))\rangle=0$). This implies that

$$\mathrm{E}[g^2]=\|g\|^2=\alpha^2\|(1-\pi(1))\|^2+\|\hat{g}\|^2,$$

and

$$\mathrm{E}[g]=\langle g,1\rangle=\alpha\langle(1-\pi(1)),1\rangle+\langle\hat{g},1\rangle=\alpha\|(1-\pi(1))\|^2,$$

since $\hat{g} \perp (1-\pi(1))$ and $\hat{g} \perp \pi(1)$. Thus the solution to (4.4) is obtained for $\hat{g} = 0$ and $\alpha = \frac{m}{\|(1-\pi(1))\|^2}$, that is

$$g_m = \frac{m}{\|(1-\pi(1))\|^2}(1-\pi(1)),$$

which is well-defined, since, by assumption, $1 \notin G_T(\tilde{\Theta})^\perp$, so that $\|1-\pi(1)\| > 0$. □

Lemma 4.6 *Assume that $1 \notin G_T(\tilde{\Theta})^\perp$. Let $m \in I\!R$ and let g_m be defined as in Lemma 4.5. Then*

$$\text{Var}[g_m] = \frac{m^2}{\text{Var}[\tilde{Z}_T]}.$$

Proof: This result follows from the proof of Lemma 10 from Schweizer (1997). However, for completeness, we give the proof here. Since $\langle (1-\pi(1)), \pi(1)\rangle = 0$, $\|\pi(1)\|^2 = \text{E}[\pi(1)^2] = \text{E}[\pi(1)]$, and hence $\|1-\pi(1)\|^2 = \text{E}[(1-\pi(1))^2] = 1 - \|\pi(1)\|^2$. Direct calculations now show that

$$\text{Var}[g_m] = \text{E}[g_m^2] - m^2 = \frac{m^2}{\|1-\pi(1)\|^2} - m^2 = m^2 \frac{\text{E}[\pi(1)^2]}{\text{E}[(1-\pi(1))^2]}.$$

Recall that $\tilde{Z}_T = \frac{\pi(1)}{\text{E}[\pi(1)]}$. Using the above properties, we have

$$\text{Var}[\tilde{Z}_T] = \frac{\text{E}[\pi(1)^2]}{(\text{E}[\pi(1)])^2} - 1 = \frac{1-\text{E}[\pi(1)]}{\text{E}[\pi(1)]} = \frac{\text{E}[(1-\pi(1))^2]}{\text{E}[\pi(1)^2]},$$

and this ends the proof. □

The next theorem essentially gives the solution to (4.3) in that it contains explicit expressions for the maximum obtainable value of u_1. The proof of this theorem is given after a subsequent corollary, which determines the optimal strategy associated to the financial variance principle, and a remark which relates this strategy to the mean-variance hedging strategy for H.

Theorem 4.7 *For any $H \in L^2(P)$ and $\gamma, c \in I\!R$,*

$$\sup_{g \in G_T(\tilde{\Theta})} u_1(c + h_1(c,\gamma) + g - \gamma H) = u_1(c + h_1(c,\gamma) + g^* - \gamma H), \tag{4.5}$$

where

$$g^* = \gamma g^H + \frac{1 + \text{Var}[\tilde{Z}_T]}{2a}(1-\pi(1)) \tag{4.6}$$

Furthermore, the value associated with $h_1(c,\gamma)$ and g^* is

$$u_1(c + h_1(c,\gamma) + g^* - \gamma H) = c + h_1(c,\gamma) - \gamma c^H + \frac{1}{4a}\text{Var}[\tilde{Z}_T] - a\gamma^2 \text{Var}[N^H]. \tag{4.7}$$

Corollary 4.8 *Let $H \in L^2(P)$ and let $1 - \pi(1) = \int_0^T \tilde{\beta}\, dX$. The optimal strategy ϑ^* for H under the financial variance principle is*

$$\vartheta^* \;=\; \vartheta^H + \frac{1 + \mathrm{Var}[\tilde{Z}_T]}{2a}\tilde{\beta}. \tag{4.8}$$

Proof of Corollary 4.8: This is an immediate consequence of Theorem 4.7. □

Remark 4.9 In the solution (4.8), the first term ϑ^H is exactly the *mean-variance hedging strategy* for H, see e.g. Schweizer (1999). The second term is related to the variance optimal martingale measure and the loading factor a and is independent of the claim H. In particular it is seen that as a is increased, the process (4.8) will converge towards the mean-variance hedging strategy. This is intuitively reasonable, since for very large a, (4.5) will essentially amount to minimizing the L^2-distance between H and $c^H + g$. Note also that ϑ^* is a linear combination of the mean-variance hedging strategy and the process related to the variance optimal martingale measure and that this combination does not depend on N^H. We point out that the proof of Theorem 4.7 is very similar to the one of Theorem 4.3. Furthermore, Theorem 4.3 follows directly from (4.7) since $c^H = \tilde{\mathrm{E}}[H]$. □

Proof of Theorem 4.7: First part of the proof is similar to the one of Schweizer (1997, proof of Theorem 9). Since $u_1(x + g - \gamma H) = x + u_1(g - \gamma H)$, we only consider $u_1(g - \gamma H)$, and as in the proof of Lemma 4.5 we find that

$$\begin{aligned} u_1(g - \gamma H) &= -\gamma c^H + u_1(g - \gamma(g^H + N^H)) \\ &= -\gamma c^H + \mathrm{E}\left[g - \gamma g^H - \gamma N^H\right] - a\mathrm{Var}\left[g - \gamma g^H - \gamma N^H\right] \\ &= -\gamma c^H + \mathrm{E}\left[g - \gamma g^H\right] - a\mathrm{Var}\left[g - \gamma g^H\right] - a\mathrm{Var}\left[\gamma N^H\right]. \end{aligned} \tag{4.9}$$

Introducing $g' = g - \gamma g^H$, we now have

$$u_1(g - \gamma H) = -\gamma c^H + u_1(g' - \gamma N^H),$$

and so, we can alternatively maximize $u_1(g' - \gamma N^H)$ over all $g' \in G_T(\tilde{\Theta})$.

We first assume that $1 \notin G_T(\tilde{\Theta})^\perp$ and solve this problem by first maximizing this term under the constraint that $\mathrm{E}[g] = m$ for a fixed $m \in I\!R$ and then maximizing over $m \in I\!R$. The first step follows by Lemma 4.5, and from Lemma 4.6 we find that we should maximize

$$u_1(g_m - \gamma N^H) = m - \frac{a\, m^2}{\mathrm{Var}[\tilde{Z}_T]} - a\gamma^2 \mathrm{Var}[N^H] =: f_1(m).$$

Note that $m \mapsto f_1(m)$ is just a negative definite quadratic function, and that its unique maximum is attained for m^* satisfying $f_1'(m^*) = 0$, i.e. $m^* = \mathrm{Var}[\tilde{Z}_T]/(2a)$. Thus, $u_1(g' - \gamma N^H)$ is maximized for

$$g_{m^*} = \frac{m^*(1 - \pi(1))}{\mathrm{E}\left[(1 - \pi(1))^2\right]} = \frac{\mathrm{Var}[\tilde{Z}_T](1 - \pi(1))}{2a\mathrm{E}\left[(1 - \pi(1))^2\right]}.$$

Since

$$\frac{1}{\mathrm{E}\left[(1-\pi(1))^2\right]} = \frac{\mathrm{E}[(1-\pi(1)+\pi(1))^2]}{\mathrm{E}\left[(1-\pi(1))^2\right]} = 1 + \frac{\mathrm{E}[(\pi(1))^2]}{\mathrm{E}\left[(1-\pi(1))^2\right]} = \frac{1+\mathrm{Var}[\tilde{Z}_T]}{\mathrm{Var}[\tilde{Z}_T]},$$

we finally get

$$g_{m^*} = \frac{1+\mathrm{Var}[\tilde{Z}_T]}{2a}(1-\pi(1)).$$

Hence, the solution to (4.5) is

$$g^* = \gamma g^H + g_{m^*},$$

which proves (4.6). Finally, (4.7) follows by inserting g^* into u_1:

$$\begin{aligned} u_1(g^*-\gamma H) &= -\gamma c^H + \mathrm{E}[g_{m^*}] - a\left(\mathrm{Var}[g_{m^*}] + \gamma^2\mathrm{Var}[N^H]\right) \\ &= -\gamma c^H + m^* - a(m^*)^2\frac{1}{\mathrm{Var}[\tilde{Z}_T]} - a\gamma^2\mathrm{Var}[N^H] \\ &= -\gamma c^H + \frac{\mathrm{Var}[\tilde{Z}_T]}{2a} - a\left(\frac{\mathrm{Var}[\tilde{Z}_T]}{2a}\right)^2\frac{1}{\mathrm{Var}[\tilde{Z}_T]} - a\gamma^2\mathrm{Var}[N^H] \\ &= -\gamma c^H + \frac{\mathrm{Var}[\tilde{Z}_T]}{4a} - a\gamma^2\mathrm{Var}[N^H]. \end{aligned}$$

Now assume that $1 \in G_T(\tilde{\Theta})^\perp$, so that $\pi(1) = 1$, $\mathrm{Var}[\tilde{Z}_T] = 0$ and $\mathrm{E}[g] = 0$ for all $g \in G_T(\tilde{\Theta})$. In this case P is a martingale measure, and (4.9) is maximized for $g^* = \gamma g^H$. It follows immediately that the associated optimal value for u_1 is given by (4.7). This ends the proof. □

4.2 The optimal strategy for the standard deviation principle

In the standard deviation case, we get a result which is similar to Theorem 4.7. This case has also been worked out by Schweizer (1997), and the following theorem can be proven by combining Lemma 10 and 11 and Theorem 12 from Schweizer (1997). Recall that $\tilde{Z}_T = \frac{d\tilde{P}}{dP}$.

Theorem 4.10 *Assume that* $a^2 > \mathrm{Var}[\tilde{Z}_T]$. *For any* $H \in L^2(P)$ *and* $\gamma, c \in \mathbb{R}$,

$$\sup_{g\in G_T(\tilde{\Theta})} u_2\left(c + h_2(c,\gamma) + g - \gamma H\right) = u_2\left(c + h_2(c,\gamma) + g^* - \gamma H\right), \tag{4.10}$$

where

$$g^* = \gamma g^H + \frac{1+\mathrm{Var}[\tilde{Z}_T]}{a\sqrt{1-\frac{\mathrm{Var}[\tilde{Z}_T]}{a^2}}}|\gamma|\sqrt{\mathrm{Var}[N^H]}(1-\pi(1)). \tag{4.11}$$

Furthermore, the value associated with $h_2(c,\gamma)$ and g^* is

$$\begin{aligned} u_2\left(c+h_2(c,\gamma)+g^*-\gamma H\right) &= c+h_2(c,\gamma)-\gamma c^H \\ &\quad -a\sqrt{\mathrm{Var}[\gamma N^H]}\sqrt{1-\frac{\mathrm{Var}[\tilde{Z}_T]}{a^2}}. \end{aligned} \tag{4.12}$$

Proof: As in the proof of Theorem 4.7, $u_2(x+g-\gamma H) = x + u_2(g-\gamma H)$, so that we only need to consider $u_2(g-\gamma H)$. Similarly,

$$\begin{aligned} u_2(g-\gamma H) &= \mathrm{E}\left[g-\gamma(c^H+g^H+N^H)\right] - a\sqrt{\mathrm{Var}\left[g-\gamma(c^H+g^H+N^H)\right]} \\ &= -\gamma c^H + \mathrm{E}\left[g'\right] - a\sqrt{\mathrm{Var}\left[g'\right]+\mathrm{Var}\left[\gamma N^H\right]}, \end{aligned} \tag{4.13}$$

where $g' = g-\gamma g^H$. This problem is very similar to the one considered in the case of the variance principle. We consider here only the situation where $1 \notin G_T(\tilde{\Theta})^\perp$; the case $1 \in G_T(\tilde{\Theta})^\perp$ can be treated as in the proof of Theorem 4.7. From the proof of Lemma 4.5 we have that subject to the constraint $\mathrm{E}[g]=m$, $\mathrm{Var}[g]$ is minimized by $g_m = \frac{m(1-\pi(1))}{\mathrm{E}[(1-\pi(1))^2]}$, and from Lemma 4.6 we have that $\mathrm{Var}[g_m] = m^2/\mathrm{Var}[\tilde{Z}_T]$. Thus, (4.13) is maximized by maximizing over $m \in I\!R$ the function f_2 defined by

$$u_2(g_m-\gamma N^H) = m - \sqrt{\frac{a^2}{\mathrm{Var}[\tilde{Z}_T]}m^2 + a^2\mathrm{Var}[\gamma N^H]} =: f_2(m). \tag{4.14}$$

This is a simple maximization problem, and it follows for example by Schweizer (1997, Lemma 11) that f_2 attains its unique maximum for

$$m^* = \sqrt{\frac{a^2\mathrm{Var}[\gamma N^H]}{C(C-1)}},$$

where $C = \frac{a^2}{\mathrm{Var}[\tilde{Z}_T]}$, provided that $C>1$. Thus, (4.13) attains its maximum for

$$\begin{aligned} g_{m^*} &= \frac{m^*(1-\pi(1))}{\mathrm{E}\left[(1-\pi(1))^2\right]} \\ &= \sqrt{\frac{a^2\mathrm{Var}[\gamma N^H]}{\frac{a^2}{\mathrm{Var}[\tilde{Z}_T]}\left(\frac{a^2}{\mathrm{Var}[\tilde{Z}_T]}-1\right)}}\frac{1+\mathrm{Var}[\tilde{Z}_T]}{\mathrm{Var}[\tilde{Z}_T]}(1-\pi(1)) \\ &= \frac{1+\mathrm{Var}[\tilde{Z}_T]}{a\sqrt{1-\frac{\mathrm{Var}[\tilde{Z}_T]}{a^2}}}|\gamma|\sqrt{\mathrm{Var}[N^H]}(1-\pi(1)), \end{aligned}$$

and this shows (4.11). To see (4.12), note that

$$\begin{aligned} u_2(g^*-\gamma H) &= -\gamma c^H + u_2(g_{m^*}-\gamma N^H) \\ &= -\gamma c^H + m^* - \sqrt{C(m^*)^2 + a^2\mathrm{Var}[\gamma N^H]} \\ &= -\gamma c^H + \sqrt{\frac{a^2\mathrm{Var}[\gamma N^H]}{C(C-1)}} - \sqrt{C\frac{a^2\mathrm{Var}[\gamma N^H]}{C(C-1)} + a^2\mathrm{Var}[\gamma N^H]} \\ &= -\gamma c^H - a\sqrt{\mathrm{Var}[\gamma N^H]}\sqrt{1-\frac{\mathrm{Var}[\tilde{Z}_T]}{a^2}}. \end{aligned}$$

This ends the proof. □

Remark 4.11 In Theorem 4.4 it was only assumed that $a^2 \geq \mathrm{Var}[\tilde{Z}_T]$. In Theorem 4.10, however, we need to assume that $a^2 > \mathrm{Var}[\tilde{Z}_T]$ in order to guarantee that the supremum (4.10) is attained for an element $g \in G_T(\tilde{\Theta})$. To see this, consider the case where $a^2 = \mathrm{Var}[\tilde{Z}_T]$, so that the function (4.14) is of the form

$$f_2(m) = m - \sqrt{m^2 + y},$$

where $y > 0$. In this case, $f_2(m) < 0$ for all $m \in \mathbb{R}$ and $f_2(m) \to 0$ for $m \to \infty$, so that the supremum is not attained for any $m \in \mathbb{R}$, and hence, (4.13) does not attain the supremum for any $g \in G_T(\tilde{\Theta})$. However, the supremum can be approximated e.g. by choosing a sequence $(g_{m_k})_{k \in \mathbb{N}}$, where g_m is defined in Lemma 4.5 and where $m_k \to \infty$ for $k \to \infty$. In this case, we obtain

$$\sup_{g \in G_T(\tilde{\Theta})} u_2\left(c + h_2(c,\gamma) + g - \gamma H\right) = c + h_2(c,\gamma) - \gamma c^H,$$

which extends (4.12) to the case where $a^2 = \mathrm{Var}[\tilde{Z}_T]$. For later use, we also note that when $a^2 < \mathrm{Var}[\tilde{Z}_T]$ then $f_2(m) \to \infty$ for $m \to \infty$, so that

$$\sup_{g \in G_T(\tilde{\Theta})} u_2\left(c + h_2(c,\gamma) + g - \gamma H\right) = \infty. \quad \square$$

As in the case of the variance principle, we obtain an explicit expression for the optimal strategy immediately as a straightforward consequence of the theorem:

Corollary 4.12 *Assume that* $a^2 > Var[\tilde{Z}_T]$. *Let* $H \in L^2(P)$ *and* $1 - \pi(1) = \int_0^T \tilde{\beta}\, dX$. *Then, the optimal strategy* ϑ^* *for* H *under the financial standard deviation principle is*

$$\vartheta^* = \vartheta^H + \frac{1 + \mathrm{Var}[\tilde{Z}_T]}{a\sqrt{1 - \frac{\mathrm{Var}[\tilde{Z}_T]}{a^2}}} \sqrt{\mathrm{Var}[N^H]}\, \tilde{\beta}. \tag{4.15}$$

Remark 4.13 Note that the factor on $\tilde{\beta}$ in (4.15) now depends on $\mathrm{Var}[N^H]$, whereas, for the variance principle, it was independent of N^H, see (4.8). This difference is a consequence of the fact that the standard deviation principle involves maximization of a function which includes the square-root of a sum of the variance of the trading gain and the variance of N^H. For the variance principle, this complex dependence is not present, since N^H is orthogonal to all trading gains $g \in G_T(\tilde{\Theta})$. □

We close this section with an investigation of the condition $a^2 > \mathrm{Var}[\tilde{Z}_T]$ in the case of a standard Black-Scholes market. In this case there is only one martingale measure, and hence this is trivially the variance optimal martingale measure. Let

ν denote the market price of risk; the Radon-Nikodym derivative of the variance optimal martingale measure with respect to P is then given by

$$\tilde{Z}_T = \exp\left(-\nu W_T - \frac{1}{2}\nu^2 T\right).$$

It is seen that

$$\tilde{Z}_T^2 = \exp\left(-2\nu W_T - \frac{1}{2}(2\nu)^2 T\right) \exp\left(\nu^2 T\right),$$

hence $\mathrm{Var}[\tilde{Z}_T] = \exp(\nu^2 T) - 1$. Thus, the standard deviation principle can be applied if $a^2 > \exp(\nu^2 T) - 1$, or equivalently

$$\sqrt{\frac{\ln(1+a^2)}{T}} > |\nu|.$$

If for example $T = 1$ and $\nu = 1/5$ (e.g. risk-free interest rate $r = 0.05$, rate of return on the stock $\alpha = 0.10$ and standard deviation $\sigma = 0.25$), then the standard deviation principle is well-defined provided that $a > 0.2020$.

5 Alternative financial transformations

In Section 4 we considered the financial variance and financial standard deviation principles as they were defined by Schweizer (1997). Either principle is derived from an actuarial premium principle $\tilde{u}$ by first changing sign on the loading factor to obtain a function u that measures the reinsurer's preferences. It is not immediately clear how similar modifications should be made for other premium principles, for example the so-called Esscher-transform principle. An alternative idea is, therefore, to define a new premium calculation principle directly in terms of the actuarial premium principle $\tilde{u}$ by

$$\hat{u}(H) = \inf_{g \in G_T(\tilde{\Theta})} \tilde{u}(H - g). \tag{5.1}$$

We shall refer to $\hat{u}$ as the *direct financial transformation* of $\tilde{u}$. The interpretation of the principle (5.1) is the following: For a given actuarial valuation principle, we look for the self-financing trading strategy ϑ which leads to the smallest possible premium for the claim $H - \int_0^T \vartheta\, dX$ using the original premium principle $\tilde{u}$.

Note that when H is an attainable claim, with $H = c^H + g^H$ for some $c^H \in \mathbb{R}$ and $g^H \in G_T(\tilde{\Theta})$, then the linearity of $G_T(\tilde{\Theta})$ implies that (5.1) can be rewritten as

$$\hat{u}(H) = \inf_{g \in G_T(\tilde{\Theta})} \tilde{u}(c^H + g^H - g) = \inf_{g \in G_T(\tilde{\Theta})} \tilde{u}(c^H + g).$$

In particular, we ask the questions: Does the principle (5.1) assign the no-arbitrage price c^H to an attainable claim? Is the principle equivalent to the indifference transformation principle proposed by Schweizer (1997) in the cases of the standard deviation and the variance principle?

Throughout this section, we work under the standing Assumption 3.2.

5.1 The variance principle

In the situation where $\tilde{u}$ is equal to $\tilde{u}_1$ we have the following negative result

Theorem 5.1 *The direct financial transformation of the variance principle is*

$$\hat{u}_1(H) = \tilde{\mathrm{E}}[H] + a\mathrm{Var}[N^H] - \frac{\mathrm{Var}[\tilde{Z}_T]}{4a}. \tag{5.2}$$

The associated optimal strategy $\hat{\vartheta}$ is given by (4.8).

Proof: From Theorem 3.4, we have the decomposition

$$H = c^H + g^H + N^H,$$

where we have used the notation $g^H = \int_0^T \vartheta^H \, dX$. Furthermore, it was shown in the proof of Theorem 4.7 that

$$u_1(g - H) = -c^H + \mathrm{E}[g - g^H] - a\mathrm{Var}[g - g^H] - a\mathrm{Var}[N^H]. \tag{5.3}$$

Since $\tilde{u}_1(H - g) = -u_1(g - H)$, minimizing $\tilde{u}_1(H - g)$ is equivalent to maximizing $u_1(g - H)$ over $g \in G_T(\tilde{\Theta})$, and hence, we find by (4.7) and (5.3) that

$$\begin{aligned} \inf_{g\in G_T(\tilde{\Theta})} \tilde{u}_1(H - g) &= -\sup_{g\in G_T(\tilde{\Theta})} u_1(g - H) \\ &= -u_1(g^* - H) = c^H + a\mathrm{Var}[N^H] - \frac{\mathrm{Var}[\tilde{Z}_T]}{4a}, \end{aligned}$$

where g^* is also given by Theorem 4.7. This proves (5.2). Since also $-u_1(g^* - H) = \tilde{u}_1(H - g^*)$, we find that the optimal strategy $\hat{\vartheta}$ is exactly equal to the strategy determined by Corollary 4.8. This completes the proof. □

Remark 5.2 It follows from Theorem 5.1 that the valuation principle $\hat{u}_1$ differs from the financial valuation principle of Schweizer (1997) by the additional term $-\mathrm{Var}[\tilde{Z}_T]/(4a)$. This has the consequence that $\hat{u}_1$ is only consistent with absence of arbitrage if $\mathrm{Var}[\tilde{Z}_T] = 0$, that is, if the measure P is a martingale measure. In fact, for the claim $H = 0$, we get $\hat{u}_1(0) = -\mathrm{Var}[\tilde{Z}_T]/(4a)$, and clearly this claim should have the price 0. The same applies to any claim on the form $H = \int_0^T \vartheta \, dX$. □

5.2 The standard deviation principle

In the situation where $\tilde{u}$ is equal to $\tilde{u}_2$ we have the following result

Theorem 5.3 *The direct financial transformation of the standard deviation principle is*

$$\hat{u}_2(H) = \tilde{\mathrm{E}}[H] + a\sqrt{1 - \frac{\mathrm{Var}[\frac{d\tilde{P}}{dP}]}{a^2}}\sqrt{\mathrm{Var}[N^H]}, \tag{5.4}$$

provided that $a^2 \geq \mathrm{Var}[\frac{d\tilde{P}}{dP}]$. If $a^2 < \mathrm{Var}[\frac{d\tilde{P}}{dP}]$, then the direct financial transformation is undefined. The associated optimal strategy $\hat{\vartheta}$ is given by (4.15).

Proof: The result follows directly from the proof of Theorem 4.10 and Remark 4.11 by using arguments similar to the ones in the proof of Theorem 5.1. □

Remark 5.4 For the standard deviation principle, the direct financial transform is equivalent to the indifference valuation principle proposed by Schweizer (1997), see Theorem 4.3. This gives an alternative characterization of this indifference pricing principle. □

5.3 A generalization

As shown above in Section 5.1 the direct financial transform (5.1) did not lead to a pricing principle with reasonable properties in the case of the variance principle in that the new principle would assign negative prices to attainable claims with no-arbitrage price 0. We also proved that the direct transform of the standard deviation principle was in fact identical to the indifference pricing principle examined in Section 4. In this section, we consider for every $\rho > 0$ principles

$$\tilde{v}_\rho(H) = \mathrm{E}[H] + a\,(\mathrm{Var}[H])^\rho\,, \tag{5.5}$$

noting that $\rho = 1$ and $\rho = \frac{1}{2}$ are the variance principle and the standard deviation principle, respectively. We basically show here by a simple argument that if P is not a martingale measure then the direct transform of $\tilde{v}_\rho$ will assign negative prices to any attainable claim with no-arbitrage price 0 if $\rho \neq \frac{1}{2}$. This implies that the standard deviation principle is the only principle from the class (5.5) that can be transformed directly into a pricing principle which is consistent with the unique no-arbitrage prices for attainable claims.

Assume that there exists $g = \int_0^T \vartheta\, dX \in G_T(\tilde{\Theta})$, with the property that $\mathrm{E}[g] < 0$; recall that g is the trading gain from some self-financing strategy $\vartheta \in \tilde{\Theta}$. Of course, if there exists a strategy with $\mathrm{E}[g] \neq 0$, then we can always get a strategy such that $\mathrm{E}[g] < 0$ by multiplying the strategy with -1 if the expected value is positive. Furthermore, we note that this automatically implies that $\mathrm{Var}[g] > 0$. To see this, assume that $\mathrm{Var}[g] = 0$, i.e. that g is constant and equal to $\mathrm{E}[g]$ P-a.s. Now, Assumption 3.2 guarantees the existence of an equivalent martingale measure Q such that $\mathrm{E}^Q[g] = 0$, since $g \in G_T(\tilde{\Theta})$. However, since $g = \mathrm{E}[g]$ a.s., this shows that $\mathrm{E}[g] = 0$, which contradicts our assumption. Hence, $\mathrm{Var}[g] > 0$.

For this strategy ϑ, we define for $x \in \mathbb{R}_+$, $g_x := xg$, which is the trading gain for the strategy $x\vartheta$. Then

$$\tilde{v}_\rho(g_x) = \mathrm{E}[g_x] + a\,(\mathrm{Var}[g_x])^\rho = x\left(\mathrm{E}[g] + a x^{2\rho-1}\,(\mathrm{Var}[g])^\rho\right). \tag{5.6}$$

Consider first the case where $\rho < \frac{1}{2}$. In this case, (5.6) immediately shows that

$$\tilde{v}_\rho(g_x) \to -\infty \text{ for } x \to \infty \text{ if } \rho < \frac{1}{2},$$

since $\mathrm{E}[g] < 0$ and since $x^{2\rho-1} \to 0$ for $x \to \infty$. This result implies that the direct financial transform is not well defined in the case where $\rho < \frac{1}{2}$, since

$$\inf_{g \in G_T(\tilde{\Theta})} \tilde{v}_\rho(H - g) = -\infty,$$

when H is attainable. Now assume that $\rho > \frac{1}{2}$. From (5.6) it is seen that $\tilde{v}_\rho(g_x) < 0$ for $x \in (0, x^*)$ where

$$x^* = \left(\frac{-\mathrm{E}[g]}{a(\mathrm{Var}[g])^\rho} \right)^{\frac{1}{2\rho-1}}.$$

This shows that there exist self-financing strategies $\vartheta \in \tilde{\Theta}$ so that $\tilde{v}_\rho(\int_0^T \vartheta \, dX) < 0$ and this has the consequence that the direct financial transform (5.1) will assign a strictly negative price to the trivial claim $H = 0$. By the above calculations combined with Theorem 4.10 and Remark 4.11, we obtain:

Theorem 5.5 *Assume in addition to Assumption 3.2 that* $\mathrm{E}[g] \neq 0$ *for some* $g \in G_T(\tilde{\Theta})$. *Consider for* $\rho > 0$ *the principles (5.5) and let* $H = c^H + \int_0^T \vartheta^H \, dX$ *be an attainable claim with no-arbitrage price* c^H. *Then*

$$\inf_{g \in G_T(\tilde{\Theta})} \tilde{v}_\rho(H - g) = \begin{cases} -\infty & \text{if } \rho < \frac{1}{2}, \\ -\infty & \text{if } \rho = \frac{1}{2} \text{ and } a^2 < \mathrm{Var}[\tilde{Z}_T], \\ c^H & \text{if } \rho = \frac{1}{2} \text{ and } a^2 \geq \mathrm{Var}[\tilde{Z}_T]. \end{cases}$$

Furthermore, if $\rho > \frac{1}{2}$, *then* $\inf_{g \in G_T(\tilde{\Theta})} \tilde{v}_\rho(H - g) < c^H$.

6 Applications to unit-linked life insurance

In this section we apply the financial valuation principles to the valuation of unit-linked life insurance contracts. With such a contract, benefits depend explicitly on the price of some specified assets; for more details see Aase and Persson (1994) and the references therein. In Møller (1998a), risk-minimizing hedging strategies were determined for a portfolio of unit-linked life insurance contracts. By applying these strategies, the insurer can reduce the combined financial and insurance risk (as measured by the variance of future losses under a specific martingale measure) inherent in these contracts. We employ the basic set-up of that paper, and this will allow us to draw on the results obtained there.

6.1 The basic model

In the following, all elements are defined on a probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$, and T is a fixed finite time horizon. We briefly review the basic model of Møller (1998a). Consider a life insurance portfolio consisting of n policy-holders aged y and denote by $T_1, \ldots, T_n$ the (unknown) remaining lifetimes. For simplicity, it is assumed that $T_1, \ldots, T_n$ are i.i.d. with survival function ${}_tp_y = \exp(-\int_0^t \mu_{y+\tau}\, d\tau)$, where μ is a deterministic continuous function (called the hazard rate). The process $N_t = \sum_{i=1}^n 1_{\{T_i \le t\}}$ counts the number of deaths up to time t and $(n - N_t)$ is the number of survivors.

Consider in addition a financial market consisting of two basic traded assets whose price processes are given by the dynamics

$$dS_t = \alpha(t, S_t) S_t\, dt + \sigma(t, S_t) S_t\, dW_t, \tag{6.1}$$

$$dB_t = r(t, S_t) B_t\, dt, \tag{6.2}$$

$S_0 > 0$, $B_0 = 1$, where $W = (W_t)_{0 \le t \le T}$ is a standard Brownian motion on the time interval $[0, T]$ and is assumed to be stochastically independent of N. Let $I\!F = (\mathcal{F}_t)_{0 \le t \le T}$ be the P-augmentation of the natural filtration generated by (N, W), that is $\mathcal{F}_t = \mathcal{F}_t^0 \vee \mathcal{N}$, where $\mathcal{F}_t^0 = \sigma\{(N_u, W_u); u \le t\}$ and $\mathcal{N}$ is the σ-algebra generated by all P-null-sets. The discounted stock price process is $X := S/B$, and we let $\lambda := \frac{\alpha - r}{\sigma^2 X}$. It is assumed that the functions α, r, σ are bounded and satisfy certain Lipschitz conditions, which in particular ensures the existence of a unique solution to (6.1), see e.g. Karatzas and Shreve (1991, Theorem 5.2.9). Furthermore, we assume that r and σ are non-negative and that σ is uniformly bounded away from 0.

Finally, we recall that the process $\tilde{M}$ defined by

$$\tilde{M}_t = N_t - \int_0^t (n - N_u) \mu_{y+u}\, du$$

is a P-martingale and is independent of W.

The filtration $I\!F$ describes the amount of information which is available to the insurer. With the present construction, the insurer has access to current information concerning the number of deaths within the insurance portfolio as well as to the development of the asset prices.

6.2 The variance optimal martingale measure

In the present set-up, the so-called market price of risk process

$$\nu_t := (\alpha(t, S_t) - r(t, S_t))/\sigma(t, S_t)$$

is bounded, and hence we can define a new measure $\hat{P} \in \mathcal{M}^e(P)$ by

$$D := \frac{d\hat{P}}{dP} = \exp\left(-\int_0^T \nu_u\, dW_u - \frac{1}{2}\int_0^T \nu_u^2\, du\right). \tag{6.3}$$

The measure $\hat{P}$ defined by (6.3) is known from the literature as the *minimal martingale measure*, see e.g. Schweizer (1995). For later use, introduce the likelihood process

$$Z_t := \mathrm{E}[D|\mathcal{F}_t] = \exp\left(-\int_0^t \nu_u\, dW_u - \frac{1}{2}\int_0^t \nu_u^2\, du\right).$$

In general, the variance optimal martingale measure $\tilde{P}$ differs from $\hat{P}$. As shown in Grandits and Rheinländer (1999), this would for instance be the case if the process ν would be a function of the triple (t, S_t, N_t). However, by exploiting the results of Pham, Rheinländer and Schweizer (1998, Section 4.3), we find that the two measures actually coincide in the present set-up: Since our model is a special case of what they call "an almost complete diffusion model", their argument justifies that D can be written on the form

$$D = D_0 + \int_0^T \tilde{\zeta}_u \, dX_u. \tag{6.4}$$

But Lemma 1 of Schweizer (1996) then implies that $\frac{d\tilde{P}}{dP} = D$, so that indeed $\tilde{P} = \hat{P}$. By Lemma 3.5, the density D can also be written as

$$\frac{d\tilde{P}}{dP} = \frac{\pi(1)}{\mathrm{E}[\pi(1)]} = \frac{1 - \int_0^T \tilde{\beta}_u \, dX_u}{\mathrm{E}[\pi(1)]},$$

and by equating the two expressions for D we find that $\mathrm{E}[\pi(1)] = \frac{1}{D_0}$ and $\tilde{\beta} = -\frac{\tilde{\zeta}}{D_0}$. The integrand $\tilde{\zeta}$ in (6.4) can now be determined as in Pham, Rheinländer and Schweizer (1998, Proposition 10), where $\tilde{\zeta}$ is expressed in terms of the solution to a second order PDE; note however that the present set-up differs slightly from their framework in that our coefficients α, r and σ depend on (t, S_t) instead of (t, X_t). In the special case where α, r and σ are functions of t only, we have that $D_0 = \tilde{\mathrm{E}}[D] = \exp(\int_0^T \nu^2(u) \, du)$, and

$$\tilde{\zeta}_t = -Z_t \lambda_t \exp\left(\int_t^T \nu^2(u) \, du\right) = -\frac{\nu(t)}{\sigma(t) X_t} \tilde{Z}_t, \tag{6.5}$$

where we have introduced the $\tilde{P}$-martingale $\tilde{Z}_t = \tilde{\mathrm{E}}[D|\mathcal{F}_t]$ and used that $\lambda_t = \frac{\nu(t)}{\sigma(t) X_t}$. Furthermore, when ν does not depend on S, it follows from (6.3) that

$$\tilde{Z}_t = Z_t \exp\left(\int_t^T \nu^2(u) \, du\right).$$

It was shown in Møller (1998a, Section 2.3) that $\tilde{M}$ is a $\tilde{P}$-martingale and that $(\tilde{M}, X)$ are stochastically independent under $\tilde{P}$.

6.3 The unit-linked pure endowment contract

We consider the discounted payoff

$$H = B_T^{-1} g(S_T)(n - N_T), \tag{6.6}$$

where g is some continuous function such that $\mathrm{E}[(g(S_T) B_T^{-1})^2] < \infty$. With this construction the benefit is linked to the financial asset S in that each of the $(n - N_T)$ survivors receives the amount $g(S_T)$ at time T. In addition, we introduce the unique no-arbitrage price process for $g(S_T)$ given by $F^g(t, S_t) := \tilde{\mathrm{E}}[B_T^{-1} B_t g(S_T) | \mathcal{F}_t]$ and assume that $F^g \in C^{1,2}$. We denote by F^g_s the partial derivative with respect to s and require that the function F^g_s is bounded, that is, that there exists a constant K_0 such that $|F^g(t,s)| \leq K_0$ for all $(t,s) \in [0,T] \times [0,\infty)$.

In order to apply the financial valuation principles, we need the decomposition (3.2) for the claim H. This decomposition can be expressed in terms of the Kunita-Watanabe decomposition (under $\tilde{P}$) for the $\tilde{P}$-martingale $\tilde{V}_t := \tilde{\mathrm{E}}[H|\mathcal{F}_t]$; see e.g. Schweizer (1999) for a general version of this result. The martingale $\tilde{V}$ was studied in Møller (1998a) for the unit-linked contract (6.6), and it was shown there that

$$\tilde{V}_t = (n - N_t)_{T-t}p_{y+t}\, B_t^{-1} F^g(t, S_t). \tag{6.7}$$

Furthermore, by Møller (1998a, Lemma 4.1), the Kunita-Watanabe decomposition for $\tilde{V}$ is

$$\tilde{V}_t = \tilde{V}_0 + \int_0^t \xi_u^H \, dX_u + \int_0^t \nu_u^H \, d\tilde{M}_u, \tag{6.8}$$

where (ξ^H, ν^H) are given by

$$(\xi_t^H,\ \nu_t^H) = \Big((n - N_{t-})_{T-t}p_{y+t} F_s^g(t,S_t),\ -B_t^{-1}F^g(t,S_t)_{T-t}p_{y+t}\Big). \tag{6.9}$$

From Schweizer (1999, Theorem 4.6) we now obtain the following expression for the integrand ϑ^H in the decomposition (3.2):

$$\vartheta_t^H = (n - N_{t-})_{T-t}p_{y+t}F_s^g(t,S_t) + \tilde{\zeta}_t \int_0^{t-} \frac{1}{\tilde{Z}_u} B_u^{-1} F^g(u, S_u)_{T-u}p_{y+u}\, d\tilde{M}_u. \tag{6.10}$$

Furthermore, by using (3.2), (6.8), (6.10) and the product rule, it follows that

$$N^H = \tilde{Z}_T \int_0^T \frac{1}{\tilde{Z}_u} \nu_u^H \, d\tilde{M}_u = -\tilde{Z}_T \int_0^T \frac{1}{\tilde{Z}_u} B_u^{-1} F^g(u,S_u)_{T-u}p_{y+u}\, d\tilde{M}_u.$$

6.4 Risk-minimizing hedging strategies

The main results of Møller (1998a) are briefly reviewed. In that paper, the criterion of risk-minimization, which is due to Föllmer and Sondermann (1986), is applied. This criterion essentially amounts to minimizing at any time t the variance (under some suitably chosen equivalent martingale measure) of future losses defined as the amount to be paid at time T reduced by future trading gains. The risk-minimizing strategy is determined by first computing the so-called intrinsic value process (6.7) and then determining the Kunita-Watanabe decomposition (6.8) of this process. In this way, $\tilde{V}$ is decomposed into an integral with respect to X which represents the hedgeable part of H and a martingale $L^H = \int \nu^H \, d\tilde{M}$ which is orthogonal to X and which represents the risk inherent in H that cannot be hedged away. It follows from Møller (1998a, Theorem 4.4) that the risk-minimizing hedging strategy $\varphi^* = (\xi^*, \eta^*)$ for (6.6) is given by

$$\begin{aligned}
\xi_t^* &= (n - N_{t-})_{T-t}p_{y+t}\, F_s^g(t, S_t), \\
\eta_t^* &= (n - N_t)_{T-t}p_{y+t}\, B_t^{-1} F^g(t, S_t) - \xi_t^* X_t,
\end{aligned}$$

and that the minimum obtainable $\tilde{P}$-variance associated with φ^* is

$$\begin{aligned} R_0^* &:= \operatorname{Var}_{\tilde{P}}\left[H - \int_0^T \xi_t^* \, dX_t\right] \\ &= \operatorname{Var}_{\tilde{P}}\left[L_T^H\right] = n \, {}_T p_y \int_0^T \tilde{\mathrm{E}}\left[(F^g(t,S_t)B_t^{-1})^2\right] {}_{T-t}p_{y+t} \, \mu_{y+t} \, dt. \end{aligned} \tag{6.11}$$

In Møller (1998a, Section 6) the quantities $\tilde{V}_0$ and R_0^* are evaluated numerically in the situation where the benefit is of the form $g(S_T) = \max(S_T, K)$ (K is a minimum guarantee) for various choices of volatility σ and guarantee K.

6.5 The variance of N^H

In the present situation, we have

$$\operatorname{Var}[N^H] = n \, {}_T p_y \int_0^T \mathrm{E}\left[\frac{Z_t}{\tilde{Z}_t}\left(F^g(t,S_t)B_t^{-1}\right)^2\right] {}_{T-t}p_{y+t} \, \mu_{y+t} \, dt. \tag{6.12}$$

We give here only an idea of how this result can be proved and refer to the analysis Section 4 (case 3) in Chapter 6 for a rigorous argument. Note that (6.12) specializes to (6.11) in the special case where $P = \tilde{P}$, that is, when the physical measure P is a martingale measure.

It follows already from Theorem 3.4 that $\mathrm{E}[N^H] = 0$, and hence

$$\operatorname{Var}[N^H] = \mathrm{E}[(N^H)^2] = \mathrm{E}\left[\left(\tilde{Z}_T \int_0^T \frac{1}{\tilde{Z}_u} \nu_u^H \, d\tilde{M}_u\right)^2\right] = \tilde{\mathrm{E}}\left[\tilde{Z}_T\left(\int_0^T \frac{1}{\tilde{Z}_u} \nu_u^H \, d\tilde{M}_u\right)^2\right].$$

We let $\tilde{L} = \int \frac{\nu^H}{\tilde{Z}} \, d\tilde{M}$ and apply Itô's formula to the process $\tilde{Z}\,\tilde{L}^2$ (see e.g. Jacod and Shiryaev (1987) for a version that applies in this generality). After some rearrangement of terms, we arrive at

$$\tilde{Z}_T \tilde{L}_T^2 = \int_0^T \tilde{L}_{t-}^2 \, d\tilde{Z}_t + 2\int_0^T \tilde{Z}_t \tilde{L}_{t-} \, d\tilde{L}_t + \int_0^T \tilde{Z}_t \left(\frac{\nu_t^H}{\tilde{Z}_t}\right)^2 dN_t.$$

The first two terms are integrals with respect to the $\tilde{P}$-martingales $\tilde{Z}$ and $\tilde{L}$, respectively, and hence, they are *likely* to be $\tilde{P}$-martingales. However, we can only guarantee that they are *local* $\tilde{P}$-martingales, and, in particular, this implies that their expected values under $\tilde{P}$ need not be equal to 0. It therefore requires some additional work to show that this is actually the case! But *provided* that the two processes are indeed $\tilde{P}$-martingales, we have now obtained that

$$\operatorname{Var}[N^H] = \tilde{\mathrm{E}}\left[\int_0^T \frac{(\nu_t^H)^2}{\tilde{Z}_t} \, dN_t\right].$$

And *provided* that also the local $\tilde{P}$-martingale $\int \frac{(\nu^H)^2}{\tilde{Z}} d\tilde{M}$ is a $\tilde{P}$-martingale, this can be rewritten as

$$\begin{aligned}
\mathrm{Var}[N^H] &= \tilde{\mathrm{E}}\left[\int_0^T \frac{(\nu_t^H)^2}{\tilde{Z}_t}(n-N_{t-})\mu_{y+t}\,dt\right] \\
&= \int_0^T \tilde{\mathrm{E}}\left[\frac{(\nu_t^H)^2}{\tilde{Z}_t}\right]\tilde{\mathrm{E}}\left[(n-N_{t-})\right]\mu_{t+y}\,dt \\
&= \int_0^T \mathrm{E}\left[\frac{Z_t}{\tilde{Z}_t}(F^g(t,S_t)B_t^{-1})^2\right]({}_{T-t}p_{y+t})^2\mathrm{E}[(n-N_t)]\mu_{y+t}\,dt \\
&= n\,{}_Tp_y\int_0^T \mathrm{E}\left[\frac{Z_t}{\tilde{Z}_t}(F^g(t,S_t)B_t^{-1})^2\right]{}_{T-t}p_{y+t}\,\mu_{y+t}\,dt.
\end{aligned}$$

The second equality follows by the Fubini theorem and the $\tilde{P}$-independence between S and N, and the third equality is obtained by applying the definition of the measure $\tilde{P}$ and using the explicit expression for ν^H given in (6.9). This verifies (6.12) under the additional assumption that the local $\tilde{P}$-martingales involved are true $\tilde{P}$-martingales.

6.6 The financial variance principle

The fair premium for (6.6) under the financial variance principle can now be determined by Theorem 4.3 and is given by

$$v_1(H) = n\,{}_Tp_y\left(F^g(0,S_0) + a\int_0^T \mathrm{E}\left[\frac{Z_t}{\tilde{Z}_t}(F^g(t,S_t)B_t^{-1})^2\right]{}_{T-t}p_{y+t}\,\mu_{y+t}\,dt\right). \quad (6.13)$$

Here, the first term is the number of survivors $n\,{}_Tp_y$ times the market value $F^g(0,S_0)$ at time 0 of the benefit $g(S_T)$. The second term is more difficult to interpret. However, as noted in Section 6.5, it specializes to the variance of $\int_0^T \nu^H\,d\tilde{M}$ when $P=\tilde{P}$. Note also that the premium is here proportional to n; Section 2.2 of Chapter 2 contains a discussion on this choice of premium for unit-linked life insurance contracts. The optimal strategy ϑ^* for the seller of H can be obtained by applying Corollary 4.8 and (6.10);

$$\begin{aligned}
\vartheta_t^* &= (n-N_{t-})_{T-t}p_{y+t}F_s^g(t,S_t) \\
&\quad +\tilde{\zeta}_t\int_0^{t-}\frac{1}{\tilde{Z}_u}B_u^{-1}F^g(u,S_u)_{T-u}p_{y+u}\,d\tilde{M}_u + \frac{1+\mathrm{Var}[\tilde{Z}_T]}{2a}\tilde{\beta}_t. \quad (6.14)
\end{aligned}$$

The first term in (6.14) is recognized as the risk-minimizing strategy for H under $\tilde{P}$, see Møller (1998a). The second term is a "correction term", which is related to the seller's loss; we also refer to Møller (1998a) for an interpretation of the integral with respect to $\tilde{M}$ in the martingale case. The third term is independent of the claim H and is related to the quadratic criterion applied.

Explicit expressions for $\tilde{\zeta}$, $\tilde{\beta}$ and $\frac{Z_t}{\tilde{Z}_t}$ are given in Section 6.2 for the case where the coefficients α, r and σ are functions of t only.

6.7 The financial standard deviation principle

The fair premium under the financial standard deviation principle is found by using Theorem 4.4, which applies provided that $a^2 \geq \mathrm{Var}[\frac{d\tilde{P}}{dP}]$. In this case, the fair premium under the financial standard deviation principle is given by

$$v_2(H) = n\,{}_Tp_y F^g(0,S_0) + \tilde{a}\left(n\,{}_Tp_y \int_0^T \mathrm{E}\left[\frac{Z_t}{\tilde{Z}_t}(F^g(t,S_t)B_t^{-1})^2\right] {}_{T-t}p_{y+t}\,\mu_{y+t}\,dt\right)^{\frac{1}{2}},$$

where we have introduced

$$\tilde{a} = a\sqrt{1 - \frac{\mathrm{Var}[\tilde{Z}_T]}{a^2}}.$$

The optimal strategy for the seller of H is obtained from Corollary 4.12 and is now given by

$$\begin{aligned}\vartheta_t^* \;=\;& (n - N_{t-})_{T-t}p_{y+t} F_s^g(t,S_t) + \tilde{\zeta}_t \int_0^{t-} \frac{1}{\tilde{Z}_u} B_u^{-1} F^g(u,S_u)_{T-u}p_{y+u}\,d\tilde{M}_u \\ &+ \frac{1+\mathrm{Var}[\tilde{Z}_T]}{\tilde{a}}\left(n\,{}_Tp_y \int_0^T \mathrm{E}\left[\frac{Z_t}{\tilde{Z}_t}(F^g(t,S_t)B_t^{-1})^2\right] {}_{T-t}p_{y+t}\,\mu_{y+t}\,dt\right)^{\frac{1}{2}} \tilde{\beta}_t.\end{aligned}$$

6.8 Numerical results

We consider a numerical example with the same parameters as the ones used in the numerical example of Møller (1998a); for the insurance portfolio we take $y = 45$, $T = 15$, $n = 100$ and

$$\mu_{y+t} = 0.0005 + 0.000075858 \cdot 1.09144^{y+t}, \quad t \geq 0. \tag{6.15}$$

We apply a standard Black-Scholes market with parameters $S_0 = B_0 = 1$, $\alpha = 0.10$, $r = 0.06$ and $\sigma = 0.25$, and consider in addition the cases of low volatility ($\sigma = 0.15$) and high volatility ($\sigma = 0.35$). Furthermore, we take $g(S_T) = \max(S_T, K)$, which is known as a unit-linked contract with guarantee, and consider various choices of guarantee K. It follows by the well-known Black-Scholes formula that

$$F^g(t,S_t) = Ke^{-r(T-t)}\Phi\left(-z_t + \sigma\sqrt{T-t}\right) + S_t\Phi(z_t), \tag{6.16}$$

where Φ is the standard normal distribution function and

$$z_t = \frac{\log(S_t/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.$$

Furthermore, the first partial derivative is $F_s^g(t,S_t) = \Phi(z_t)$.

With $\sigma = 0.25$, we find that $\nu = 0.16$ and hence it follows from the investigation at the end of Section 4 that the financial standard deviation principle is only well-

	Guarantee (K)	$\tilde{\mathrm{E}}[H]$	$\mathrm{Var}[N^H]$	(std. dev.)
$\sigma = 0.15$	0	0.8796	0.224	–
	$0.5\exp(rT)$	0.8996	0.224	(0.0004)
	$\exp(rT)$	1.0807	0.238	(0.0004)
($M = 500{,}000$)	$2\exp(rT)$	1.7993	0.379	(0.0003)
$\sigma = 0.25$	0	0.8796	0.415	–
	$0.5\exp(rT)$	0.9580	0.422	(0.0015)
	$\exp(rT)$	1.2066	0.460	(0.0015)
($M = 1{,}000{,}000$)	$2\exp(rT)$	1.9161	0.671	(0.0015)
$\sigma = 0.35$	0	0.8796	0.873	–
	$0.5\exp(rT)$	1.0255	0.883	(0.005)
	$\exp(rT)$	1.3213	0.940	(0.005)
($M = 5{,}000{,}000$)	$2\exp(rT)$	2.0511	1.197	(0.005)

Table 4.1: *Values for* $\tilde{\mathrm{E}}[H]$, $\mathrm{Var}[N^H]$ *for one policy-holder and various choices of volatility* σ *and guarantee* K.

defined for $a > 0.6842$. Since this value is probably too high for applications, we have chosen only to apply the financial variance principle in our numerical example.

We set out by computing the fair premium (6.13). Thus we need to determine the variance of N^H, which here simplifies to

$$n\,{}_Tp_y\mathrm{E}\left[\int_0^T e^{-\nu^2(T-t)}\left(F^g(t,S_t)e^{-rt}\right)^2{}_{T-t}p_{y+t}\,\mu_{y+t}\,dt\right], \tag{6.17}$$

since $\frac{Z_t}{\tilde{Z}_t} = e^{-\nu^2(T-t)}$. The evaluation of this variance is very similar to the computation of (6.11). Note however, that the two quantities differ by the factor $e^{-\nu^2(T-t)}$ and in that (6.17) involves a P-expectation whereas R_0^* involves expectation with respect to a martingale measure. We apply here the same numerical method as in Møller (1998a), that is, we use Monte Carlo simulation for S and discretize the integral in (6.17) by using the summed Simpson rule, see e.g. Schwarz (1989). Throughout, we use the step size $\Delta t = 1/100$. In Table 4.1 we have listed the quantities $\tilde{V}_0 = \tilde{\mathrm{E}}[H]$, which have been computed directly from (6.15) and (6.16) without the use of simulation, and the estimates of $\mathrm{Var}[N^H]$ for various choices of σ and K. This table also gives standard errors of the estimates of $\mathrm{Var}[N^H]$ and the number M of simulated paths used. Since the premium under the financial variance principle is linear in n, we have furthermore fixed $n = 1$. In Table 4.2 we have fixed $\sigma = 0.25$ and $K = \exp(rT)$ and computed the fair premium for various choices of safety loading parameter a. These numbers illustrate the impact of a on the fair premium, which attains values from 1.211 to 2.127, and this corresponds to a relative loading (computed as $(v_1(H) - \tilde{\mathrm{E}}[H])/\tilde{\mathrm{E}}[H]$) between 0.004 and 0.76.

We consider in the rest of this section an insurance portfolio with $n = 100$ and present some simulation results for N, S and ϑ^*. We take $a = 0.25$ and fix $\sigma =$

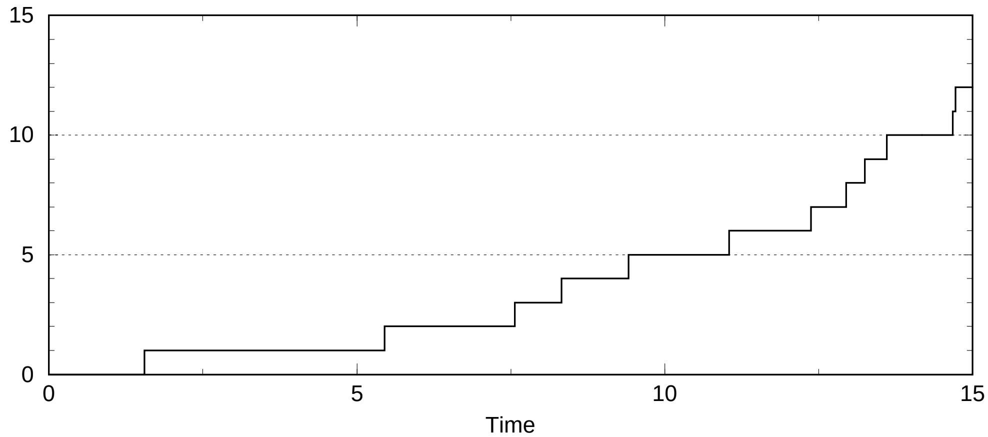

Figure 4.1: *Simulation of the process N in the case $n = 100$.*

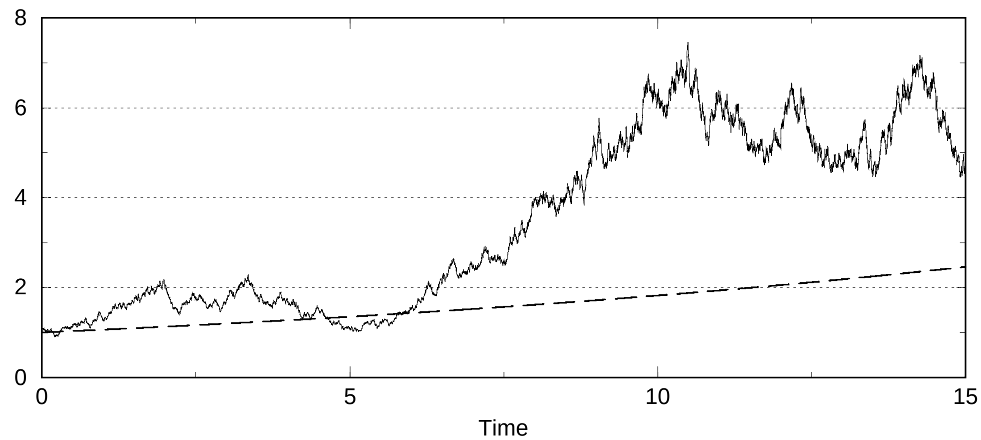

Figure 4.2: *Simulation of the price process S (solid line) and the deterministic savings account B (dashed line).*

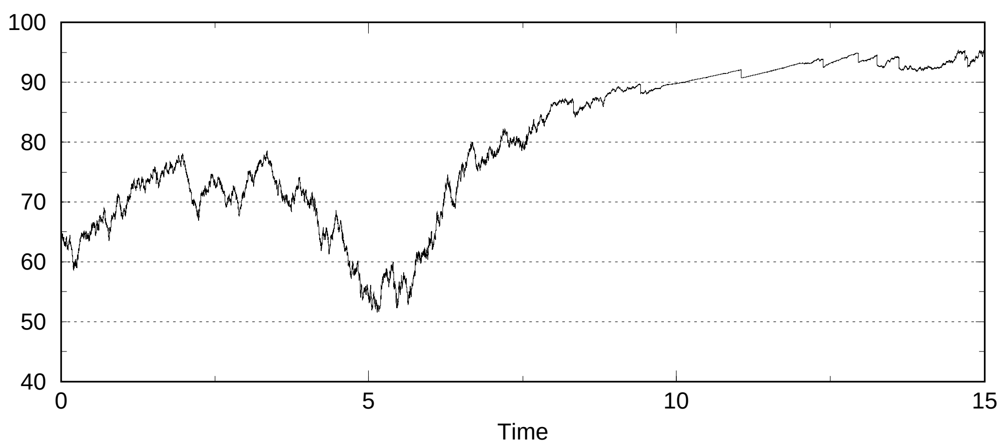

Figure 4.3: *The optimal trading strategy corresponding to the realizations in Figures 4.1 and 4.2.*

Safety loading (a)	0.01	0.1	0.25	0.5	1	2
Premium	1.211	1.253	1.322	1.437	1.667	2.127

Table 4.2: *The fair premium under the financial variance principle for* $n = 1$, $\sigma = 0.25$, $K = \exp(rT)$, *and various choices of safety loading* a.

0.25 and $K = \exp(rT)$. Figure 4.1 shows a possible realization for the counting process N. The first death occurs after approximately 1.5 years, and the total number of deaths is 12, which is close to the expected number $\mathrm{E}[N_T] = n(1 - {}_{15}p_{45}) = 100 \cdot (1 - 0.8796) \approx 12$. Figure 4.2 gives a possible realization for the stock price process S; for comparison we have included the deterministic savings account $B_t = e^{rt}$. The value of the stocks falls below the savings account only shortly after time 0 and after approximately 5 years. Finally, Figure 4.3 shows the optimal investment strategy (6.14) corresponding to the outcome of the insurance portfolio from Figure 4.1 and the stock price process from Figure 4.2. We draw attention to some interesting features: The drop in the price process close to time 5 is reflected in the optimal strategy, which falls from 70 at time 4 to 55 at time 5. The optimal number of stocks increases to 90 at time 10, which is close to the conditional expected number of survivors, $(100 - N_{10})_5p_{55} = 90 \cdot 0.9408 \approx 89$; this can be partly explained by the fact that the value of the stock is at a level, where the probability of falling below the guarantee $K = e^{rT}$ is very small. After time 10, deaths occurring within the insurance portfolio are clearly visible in the strategy, which shows sharp jumps downwards in connection with each death. These jumps can be described explicitly by considering a jump time τ for the process N: Letting $\vartheta^*_{t+} := \lim_{h\searrow 0} \vartheta^*_{t+h}$, we obtain from (6.14)

$$\vartheta^*_{\tau+} - \vartheta^*_{\tau} = -{}_{T-\tau}p_{y+\tau} F^g_s(\tau, S_\tau) + \tilde{\zeta}_\tau \frac{1}{\tilde{Z}_\tau} B_\tau^{-1} F^g(\tau, S_\tau)\,{}_{T-\tau}p_{y+\tau}$$

on the set $\{\tau < T\}$. Here, both terms are negative since $\tilde{\zeta} < 0$. Furthermore, this shows that the jump for ϑ^* is big when the value of S is big, since in this case $F^g_s \approx 1$ and $F^g(t, S_t) \approx S_t$.

Chapter 5

Indifference Pricing of Insurance Contracts: Theory

(This chapter is an adapted version of the first part of Møller (2000))

We apply the financial variance and standard deviation principles of Schweizer (1997) for the valuation of insurance contracts. These principles are financial transformations of the classical actuarial variance and standard deviation principles and take into consideration the possibilities of hedging on financial markets. We focus on the role of the information available to the insurer and study its impact on the fair premiums and the optimal trading strategies. This leads to a general comparison result for the hedging errors in the mean-variance hedging problem under two different filtrations. Via a projection argument for Hilbert spaces, we obtain an explicit expression for the increase in the hedging error that arises from restricting the information from one filtration to a smaller filtration. These results are applied in a separate study of insurance contracts that depend on two stochastically independent sources of randomness representing *purely financial risk* and *pure insurance risk*, respectively. By considering different filtrations for the pure insurance risk, we then arrive at simple upper and lower bounds for the fair premiums. Examples considered include unit-linked life insurance contracts, financial stop-loss contracts and stop-loss contracts with barrier.

1 Introduction

One of the classical issues in actuarial mathematics is the valuation of insurance contracts. The main problem consists in determining reasonable principles that can be used to calculate *insurance premiums*. A *premium calculation principle* is a function, which assigns to each contract H (within a certain class) a number, a so-called premium. Traditional examples of such principles are

$$\tilde{u}_1(H) = \mathrm{E}[H] + a\mathrm{Var}[H], \tag{1.1}$$

$$\tilde{u}_2(H) = \mathrm{E}[H] + a\sqrt{\mathrm{Var}[H]}, \tag{1.2}$$

$$\tilde{u}_3(H) = \mathrm{E}[H](1+a), \tag{1.3}$$

$$\tilde{u}_4(H) = \frac{\mathrm{E}[He^{aH}]}{\mathrm{E}[e^{aH}]}, \tag{1.4}$$

provided that the involved quantities exist. These principles are known from the actuarial literature as the *variance*, *standard deviation*, *expected value* and *Esscher transform* principles; see e.g. Goovaerts et al. (1984) for a thorough treatment of these and other actuarial premium principles.

One striking feature of the principles (1.1)–(1.4) is the absence of a *market*. More precisely, they are formulated within a framework which does not allow the seller (or buyer) to trade on financial markets or on reinsurance markets. With this set-up, the premium is calculated at time 0, say, where the contract is sold. At the term T of the contract, the buyer of the contract then informs the seller (henceforth also called the reinsurer) about his claims and the seller pays the amount prescribed by the contract. Between time 0 and time T no action takes place! This is in contrast to financial valuation principles, see e.g. Duffie (1996), which are formulated within a framework which allows for trading between the time of issue of the contract and the term of the contract. As a consequence, the principles (1.1)–(1.4) cannot be expected to be consistent with e.g. no-arbitrage pricing theory. This fundamental difference between the two types of principles puts limits on the class of contracts for which it makes sense to apply these actuarial valuation principles. As a trivial example, consider (for instance) a standard Black-Scholes market and let the claim H be equal to the value at some fixed time T of the stock. Then it follows by simple no-arbitrage arguments, that the unique no-arbitrage price for H at time 0 is the initial value of the stock; this can also be obtained as the expected value of the discounted claim with respect to the unique martingale measure. In particular, this price will be independent of the drift-coefficient of the stock, and it is not difficult to verify, that this will not be the case in general if we apply the principles (1.1)–(1.4).

The problem becomes much more subtle if we instead consider for example a so-called *financial stop-loss contract*, which is a reinsurance contract that promises to pay at some fixed time T, say, the amount $H = (U_T + Y_T - K)^+$, where U_T denotes the aggregate claim amounts from some insurance portfolio during the period $[0,T]$, Y_T is a financial loss, and K is some retention limit. (Information on contracts of this type can be found for example in Swiss Re (1998).) Note that for $Y_T \equiv 0$, the contract is just a traditional stop-loss contract. The loss Y_T could for example be a put option on some underlying stock index S, that is $Y_T = (c - S_T)^+$ or it could simply be the loss associated with holding 1 unit of this index, that is, $Y_T = S_0 - S_T$. The financial stop-loss contracts provide a coverage not only for large losses due to fluctuations within the insurance portfolio (insurance risk) but also for adverse development on the financial markets (financial risk). Contracts of this form combine insurance and financial risk, and, typically, they cannot be priced uniquely by no-arbitrage arguments alone. On the other hand, applying the actuarial valuation principles directly to such a contract would completely neglect the fact that the seller and buyer of the contract can actually trade on the financial markets and in this way possibly reduce the financial risk. So the question is really: Should we use the financial valuation principles (which would only lead to very wide bounds for the premiums) or the actuarial ones (neglecting the existence of the financial market) for such a contract? The approach taken here to address this issue has been suggested by Schweizer (1997). It combines the two different approaches and leads to modifications of the actuarial valuation principles that are in a sense more consistent with the valuation on the financial markets than the original principles. The starting point for these new valuation principles is here the variance or the standard deviation principle introduced above. Taking these principles as a description of the reinsurer's preferences, a modified premium calculation principle is derived by use of an indifference argument, which takes explicitly into account the existence of the financial markets. In Chapter 4 optimal trading strategies are determined for reinsurers whose preferences are described by the variance and standard deviation principles. The issue of relating actuarial pricing principles to the one from financial mathematics has been addressed by Bühlmann (1980, 1984), Delbaen and Haezendonck (1989) and Sondermann (1991) among others. Embrechts (1996) gives an overview of literature on this area and discusses some important developments.

The insurers' possibilities for trading on the financial markets are in general constrained by many factors, such as legislation, transactions costs and the amount of information available. We shall in particular focus on the role of the last mentioned and analyze how the premium principles under consideration are affected by the information. In Section 3 we give a general comparison result for the so-called hedging errors (the minimum obtainable L^2-distance between the claim H and the terminal value of a self-financing strategy) under different filtrations. The financial valuation principles of Schweizer (1997) are recalled in Section 4. In Section 5 we set up a product space model which is used for the analysis of claims that depend on two stochastically independent sources of uncertainty, called *pure insurance risk* and *purely financial risk*, respectively. Within this framework, our comparison result allows us to derive upper and lower bounds for the fair premiums for a broad class of insurance contracts; these bounds are given in Section 6. The upper bound is obtained when the seller receives no information concerning the insurance risk; the lower bound corresponds to the situation where all information is revealed immediately after the selling of the contract. These bounds are relevant for the valuation of reinsurance contracts, since reinsurers often receive only summary information after the selling of the contracts.

Section 7 contains a separate study of contracts which are the product of two stochastically independent factors. For such contracts, more explicit expressions for the difference in hedging errors under two different filtrations are obtained. This class of contracts include unit-linked pure endowment contracts and so-called stop-loss contracts with barrier. Examples are given in Chapter 6; in Section 5 of Chapter 6, we give a framework which allows for dependence between the financial risk and the insurance risk. As a main example, we consider the situation where the drift and volatility parameters of the stock price process of an insurance company are affected by a Poisson process which is taken to describe the occurrence of certain catastrophes. It is demonstrated how the fair premiums and the optimal trading strategies can be obtained in an explicit example.

2 Preliminaries

In this section we introduce some general notation that will be used in the rest of the chapter.

Consider a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where the filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ satisfies the usual conditions of right-continuity and completeness and $\mathcal{F} = \mathcal{F}_T$; T is some fixed finite time horizon. We do not assume that $\mathcal{F}_0$ is trivial. Let X be a d-dimensional *continuous* semimartingale with respect to $\mathbb{F}$ with canonical decomposition

$$X = X_0 + M + A,$$

where, X_0 is $\mathcal{F}_0$-measurable, M is a continuous local P-martingale and A is continuous and of finite variation. A natural no-arbitrage condition on X is to require that A be absolutely continuous with respect to $\langle M \rangle$ and that there exists an $\mathbb{R}^d$-valued predictable process λ so that $A_t = \int_0^t d\langle M \rangle_s \lambda_s$ and $\int_0^T \lambda_s^{tr} d\langle M \rangle_s \lambda_s < \infty$ P-a.s. This condition is necessary for the existence of a measure Q which is equivalent to P and such that X is a Q-martingale, see Ansel and Stricker (1992, Théorème 4). In particular, the so-called *minimal martingale measure* $\hat{P}$ is defined[1] by

$$\frac{d\hat{P}}{dP} = \mathcal{E}\left(-\int \lambda \, dM\right)_T,$$

see Föllmer and Schweizer (1990). Note, however, that $\hat{P}$ need not exist in general. In the following, we shall mainly work with another martingale measure, namely the *variance optimal martingale measure.* Before defining this notion precisely, we mention some fundamental concepts.

Let $\mathcal{V}$ denote the linear space spanned by the random variables of the simple form $h^{tr}(X_{T_2} - X_{T_1})$, where $T_1 \leq T_2 \leq T$ are $\mathbb{F}$-stopping times such that the stopped process X^{T_2} is bounded and where h is a bounded $\mathbb{R}^d$-valued $\mathcal{F}_{T_1}$-measurable random variable. We introduce the class of (signed) martingale measures and the variance optimal martingale measure, see Delbaen and Schachermayer (1996a), Schweizer (1996) and Rheinländer and Schweizer (1997).

Let $\mathcal{M}^s(P)$ denote the space of all signed measures $Q \ll P$ with $Q(\Omega) = 1$ and

$$\mathrm{E}\left[\frac{dQ}{dP} f\right] = 0, \tag{2.1}$$

[1]Recall that the process $\hat{Z} = \mathcal{E}(-\int \lambda \, dM)$ is the unique semimartingale which solves the stochastic differential equation $\hat{Z}_t = 1 - \int_0^t \hat{Z}_{s-} \lambda_s dM_s$; in the special case where M is a continuous local martingale, we have $\mathcal{E}(-\int \lambda \, dM)_t = \exp(-\int_0^t \lambda_s \, dM_s - \frac{1}{2} \int_0^t \lambda_s \, d\langle M \rangle_s \lambda_s)$. Recall also for later use that for any two semimartingales $\hat{X}$ and $\hat{Y}$, the quadratic covariation of $\hat{X}$ and $\hat{Y}$ is the process $[\hat{X}, \hat{Y}] = \hat{X}\hat{Y} - \hat{X}_0\hat{Y}_0 - \int \hat{X}_- d\hat{Y} - \int \hat{Y}_- d\hat{X}$.

for all $f \in \mathcal{V}$, and $\mathcal{M}^e(P)$ the set of all probability measures $Q \in \mathcal{M}^s(P)$ with $Q \sim P$. In addition, we introduce the spaces $\mathcal{D}^s$ and $\mathcal{D}^e$ by

$$\mathcal{D}^x = \left\{ \frac{dQ}{dP} \middle| Q \in \mathcal{M}^x(P) \right\}, \tag{2.2}$$

for $x \in \{s, e\}$.

Definition 2.1 *The variance optimal martingale measure $\tilde{P}$ is the unique element of $\mathcal{M}^s(P)$ such that $\tilde{D} := \frac{d\tilde{P}}{dP} \in L^2(P)$ and such that $\tilde{D}$ minimizes $\|D\|_{L^2(P)}$ over all $D \in \mathcal{D}^s \cap L^2(P)$.*

Remark 2.2 The classes $\mathcal{M}^s(P)$ and $\mathcal{M}^e(P)$ depend on the filtration, and so does the variance optimal martingale measure. Thus, the symbols $\mathcal{V}$, $\mathcal{M}^x(P)$, $\mathcal{D}^x$ etc. should properly be equipped with an $\mathbb{F}$, but to save notation we will only mention explicitly the filtration in situations where we are working with more than one filtration. □

In the following, we will be working under Assumption 2.3, which ensures the existence of the variance optimal martingale measure. In general, $\tilde{P}$ exists if and only if $\mathcal{D}^s \cap L^2(P) \neq \emptyset$ since $\mathcal{D}^s \cap L^2(P)$ is a convex closed set. This does not guarantee that $\tilde{P}$ is equivalent to P or that $\tilde{P}$ is a probability measure, however, but Delbaen and Schachermayer (1996a, Theorem 1.3) proved that if X is continuous and provided that $\mathcal{D}^e \cap L^2(P) \neq \emptyset$ then $\tilde{P} \in \mathcal{M}^e(P)$, that is, $\tilde{P}$ is a probability measure and $\tilde{P} \sim P$. In particular, this means that $\tilde{D} > 0$ P-a.s. So we impose

Assumption 2.3 *$\mathcal{D}^e \cap L^2(P) \neq \emptyset$.*

Introduce now the space $\tilde{\Theta}(\mathbb{F})$ of $\mathbb{R}^d$-valued $\mathbb{F}$-predictable processes ϑ which are such that the real-value process $\int \vartheta \, dX$ is a $\tilde{P}$-martingale on $[0, T]$ and $\int_0^T \vartheta_t \, dX_t \in L^2(P)$, and let

$$G_T(\tilde{\Theta}(\mathbb{F})) := \left\{ \int_0^T \vartheta_t \, dX_t \middle| \vartheta \in \tilde{\Theta}(\mathbb{F}) \right\}.$$

It was shown in Delbaen and Schachermayer (1996b) and in Gouriéroux, Laurent and Pham (1998) that, when X is continuous, Assumption 2.3 implies that $G_T(\tilde{\Theta}(\mathbb{F}))$ is closed and is indeed the closure of $\mathcal{V}$ in $L^2(P)$;

$$\overline{\mathcal{V}}^{L^2(P)} = G_T(\tilde{\Theta}(\mathbb{F})),$$

compare also Rheinländer (1999). In fact, the remark following Proposition 15 of Rheinländer (1999) shows that $\tilde{\Theta}(\mathbb{F})$ is identical to the space of all $\mathbb{F}$-predictable processes ϑ such that $\int_0^T \vartheta dX \in L^2(P)$ and $\int \vartheta dX$ is a uniformly integrable Q-martingale for all $Q \in \mathcal{M}^e(P)$ with $\frac{dQ}{dP} \in L^2(P)$. We shall exploit this relation in Sections 3 and 5.

Introduce the strictly positive process $\tilde{Z}$ defined by

$$\tilde{Z}_t = \tilde{\mathrm{E}}[\tilde{D} \mid \mathcal{F}_t] = \frac{\mathrm{E}[\tilde{D}^2 \mid \mathcal{F}_t]}{\mathrm{E}[\tilde{D} \mid \mathcal{F}_t]}, \tag{2.3}$$

where $\tilde{D} = \frac{d\tilde{P}}{dP}$ and where we have used the short-hand notation $\tilde{\mathrm{E}}$ for the $\tilde{P}$-expectation $\mathrm{E}_{\tilde{P}}$. From Delbaen and Schachermayer (1996a) it follows that $\tilde{Z}$ is of the form

$$\tilde{Z}_t = \tilde{Z}_0 + \int_0^t \tilde{\zeta}_u \, dX_u, \tag{2.4}$$

for some $\tilde{\zeta} \in \tilde{\Theta}(\mathbb{F})$ and a constant $\tilde{Z}_0$. Note that $\tilde{Z}$ is continuous. Since $\tilde{D} \in L^2(P)$, we can also define a new probability measure $\tilde{R}$ by $\frac{d\tilde{R}}{dP} = \tilde{D}^2/c$, with $c = \mathrm{E}[\tilde{D}^2]$. In particular, this implies that $\frac{d\tilde{R}}{d\tilde{P}} = \tilde{D}/c$. Define in addition a $(d+1)$-dimensional process $Y = (Y^0, Y^1, \ldots, Y^d)^{tr}$ by $Y^0 = 1/\tilde{Z}$ and $Y^i = X^i/\tilde{Z}$, $i = 1, \ldots, d$. Note that Y is a continuous $\mathbb{R}^{d+1}$-valued local $\tilde{R}$-martingale, since X is a local $\tilde{P}$-martingale. The following theorem corresponds to Rheinländer and Schweizer (1997, Proposition 8); the extension to non-trivial initial σ-algebra $\mathcal{F}_0$ has been proved by Schweizer (1999).

Theorem 2.4 *Let $L^2(Y, \tilde{R})$ denote the space of $\mathbb{F}$-predictable processes ψ such that $\int \psi dY$ is an $\tilde{R}$-square-integrable $\tilde{R}$-martingale. Then*

$$\frac{1}{\tilde{Z}_T} G_T(\tilde{\Theta}(\mathbb{F})) := \left\{ \frac{1}{\tilde{Z}_T} \int_0^T \vartheta_u \, dX_u \middle| \vartheta \in \tilde{\Theta}(\mathbb{F}) \right\} = \left\{ \int_0^T \psi_u \, dY_u \middle| \psi \in L^2(Y, \tilde{R}) \right\}. \tag{2.5}$$

Furthermore, for given $\vartheta \in \tilde{\Theta}(\mathbb{F})$,

$$\frac{1}{\tilde{Z}_T} \int_0^T \vartheta_u \, dX_u = \int_0^T \psi_u \, dY_u, \tag{2.6}$$

where $\psi^0 = \int \vartheta \, dX - \vartheta^{tr} X$ and $\psi^i = \vartheta^i$, $i = 1, \ldots, d$.

It follows from Schweizer (1997, Lemma 2) that the space

$$\mathbb{R} + G_T(\tilde{\Theta}(\mathbb{F})) = \{c + G_T(\vartheta) | c \in \mathbb{R}, \, \vartheta \in \tilde{\Theta}(\mathbb{F})\}$$

is closed in $L^2(P)$ under Assumption 2.3. The following crucial result is now a consequence of the projection theorem for Hilbert spaces (used for the space $L^2(P)$ with inner product $\langle H^{(1)}, H^{(2)} \rangle := \mathrm{E}[H^{(1)} H^{(2)}]$):

Theorem 2.5 *Any random variable $H \in L^2(\mathcal{F}_T, P)$ admits a unique decomposition on the form*

$$H = c^H + \int_0^T \vartheta_t^H \, dX_t + N^H, \tag{2.7}$$

where $c^H \in \mathbb{R}$, $\vartheta^H \in \tilde{\Theta}(\mathbb{F})$, $\mathrm{E}[N^H] = 0$, and $\mathrm{E}[N^H \int_0^T \vartheta_t \, dX_t] = 0$ for all $\vartheta \in \tilde{\Theta}(\mathbb{F})$.

In addition, we recall the so-called feed-back formula of Schweizer (1999, Theorem 4.6) for the projection ϑ^H, which is related to the Galtchouk-Kunita-Watanabe decomposition for H under $\tilde{P}$:

Theorem 2.6 *Assume that $H \in L^2(\mathcal{F}_T, P)$ and consider the Galtchouk-Kunita-Watanabe decomposition of the $\tilde{P}$-martingale $\tilde{V}_t = \tilde{\mathrm{E}}[H \mid \mathcal{F}_t]$ given by*

$$\tilde{V}_t = \tilde{\mathrm{E}}[H \mid \mathcal{F}_t] = \tilde{\mathrm{E}}[H] + \int_0^t \xi_u^{H,\tilde{P}}\, dX_u + L_t^{H,\tilde{P}},$$

for $0 \le t \le T$. Then, the integrand ϑ^H in (2.7) is determined by

$$\begin{aligned} \vartheta_t^H &= \xi_t^{H,\tilde{P}} - \frac{\tilde{\zeta}_t}{\tilde{Z}_t}\left(\tilde{V}_{t-} - \tilde{\mathrm{E}}[H] - \int_0^t \vartheta_u^H\, dX_u\right) \\ &= \xi_t^{H,\tilde{P}} - \tilde{\zeta}_t\left(\frac{\tilde{V}_0 - \tilde{\mathrm{E}}[H]}{\tilde{Z}_0} + \int_0^{t-} \frac{1}{\tilde{Z}_u}\, dL_u^{H,\tilde{P}}\right). \end{aligned} \tag{2.8}$$

In particular, the constant c^H in (2.7) is determined by $c^H = \tilde{\mathrm{E}}[H]$, and it follows by applying the explicit formula for ϑ^H in (2.8) that the term N^H in the decomposition (2.7) is determined by

$$N^H = \frac{\tilde{Z}_T}{\tilde{Z}_0}(\tilde{V}_0 - \tilde{\mathrm{E}}[H]) + \tilde{Z}_T \int_0^T \frac{1}{\tilde{Z}_u}\, dL_u^{H,\tilde{P}}. \tag{2.9}$$

To see this, note that by (2.7), the Galtchouk-Kunita-Watanabe decomposition for $\tilde{V}$, (2.8), and the product rule

$$\begin{aligned} N^H &= H - c^H - \int_0^T \vartheta_t^H dX_t \\ &= \tilde{\mathrm{E}}[H] + \int_0^T \xi_t^{H,\tilde{P}} dX_t + L_T^{H,\tilde{P}} - c^H \\ &\quad - \int_0^T \left(\xi_t^{H,\tilde{P}} - \left(\frac{\tilde{V}_0 - \tilde{\mathrm{E}}[H]}{\tilde{Z}_0} + \int_0^{t-} \tilde{Z}^{-1} dL^{H,\tilde{P}}\right)\tilde{\zeta}_t\right) dX_t \\ &= (\tilde{Z}_T - \tilde{Z}_0)\frac{\tilde{V}_0 - \tilde{\mathrm{E}}[H]}{\tilde{Z}_0} + L_T^{H,\tilde{P}} \\ &\quad + \tilde{Z}_T \int_0^T \tilde{Z}_t^{-1} dL_t^{H,\tilde{P}} - \int_0^T \tilde{Z}_t \tilde{Z}_t^{-1}\, dL_t^{H,\tilde{P}} \\ &= L_0^{H,\tilde{P}} - (\tilde{V}_0 - \tilde{\mathrm{E}}[H]) + \frac{\tilde{Z}_T}{\tilde{Z}_0}(\tilde{V}_0 - \tilde{\mathrm{E}}[H]) + \tilde{Z}_T \int_0^T \tilde{Z}_t^{-1}\, dL_t^{H,\tilde{P}} \\ &= \frac{\tilde{Z}_T}{\tilde{Z}_0}(\tilde{V}_0 - \tilde{\mathrm{E}}[H]) + \tilde{Z}_T \int_0^T \tilde{Z}_t^{-1}\, dL_t^{H,\tilde{P}}. \end{aligned} \tag{2.10}$$

As pointed out by Gouriéroux, Laurent and Pham (1998), the process $\tilde{Z}$ can be viewed as the value process of a certain self-financing strategy, see e.g. (1.4.1) and (2.4). The first term in (2.10) can therefore be interpreted as the extra initial capital $(\tilde{V}_0 - \tilde{\mathrm{E}}[H])$ accumulated by $\tilde{Z}_T/\tilde{Z}_0$. Similarly, the second term represents accumulated increments of the non-hedgeable part $L^{H,\tilde{P}}$.

The *hedging error* $J_0(x)$ associated with the claim H is defined as

$$J_0(x) := \min_{\vartheta \in \tilde{\Theta}(\mathbb{F})} \mathrm{E}\left[\left(H - x - \int_0^T \vartheta_t\, dX_t\right)^2\right]. \tag{2.11}$$

Here, $x + \int_0^T \vartheta_t \, dX_t$ represents the value at time T of some self-financing strategy ϑ with initial value x. Thus, we are looking for the strategy with the minimum L^2-distance for given initial value x. In addition, we introduce the quantity

$$J_0 = \min_{x \in \mathbb{R}} J_0(x), \tag{2.12}$$

which is the hedging error associated with the "optimal" initial capital.

Remark 2.7 The problem of determining (2.11) and (2.12) can be used to give a very useful characterization of the process ϑ^H defined by (2.8). Note that by Theorem 2.5, $\int_0^T \vartheta_t^H \, dX_t$ is just the projection of $H - c^H$ on to the space $G_T(\tilde{\Theta})$ and, thus, ϑ^H is also the solution to the minimization problem (2.11) for $x = c^H$. In general (for x not necessarily equal to c^H) the solution to (2.11) can be found in Rheinländer and Schweizer (1997, Theorem 6) and is given by

$$\begin{aligned} \vartheta_t^{x,H} &= \xi_t^{H,\tilde{P}} - \frac{\tilde{\zeta}_t}{\tilde{Z}_t}\left(\tilde{V}_{t-} - x - \int_0^t \vartheta_u^{x,H} \, dX_u\right) \\ &= \xi_t^{H,\tilde{P}} - \tilde{\zeta}_t \left(\frac{\tilde{V}_0 - x}{\tilde{Z}_0} + \int_0^{t-} \frac{1}{\tilde{Z}_u} \, dL_u^{H,\tilde{P}}\right). \end{aligned} \tag{2.13}$$

In particular, we will apply this formula to $x = 0$ and let $\tilde{\vartheta}^H := \vartheta^{0,H}$ denote the projection of $H - 0$ on $G_T(\tilde{\Theta})$. The process $\vartheta^{x,H}$ differs from (2.8) only in that $\tilde{\mathrm{E}}[H]$ has been replaced by x. □

As noted in Rheinländer and Schweizer (1997), we have that

$$\begin{aligned} \mathrm{E}\left[\left(H - x - \int_0^T \vartheta_t \, dX_t\right)^2\right] &= \tilde{Z}_0 \mathrm{E}\left[\frac{\tilde{Z}_T^2}{\tilde{Z}_0}\left(\frac{H - x}{\tilde{Z}_T} - \frac{G_T(\vartheta)}{\tilde{Z}_T}\right)^2\right] \\ &= \tilde{Z}_0 \mathrm{E}_{\tilde{R}}\left[\left(\frac{H - x}{\tilde{Z}_T} - \frac{G_T(\vartheta)}{\tilde{Z}_T}\right)^2\right]. \end{aligned}$$

In the last expression, we can apply the one-to-one correspondence between the two spaces $\frac{1}{\tilde{Z}_T} G_T(\tilde{\Theta})$ and $\left\{\int_0^T \psi_u \, dY_u \,\middle|\, \psi \in L^2(Y, \tilde{R})\right\}$ which was established in Theorem 2.4. This implies that

$$J_0(x) = \tilde{Z}_0 \min_{\psi \in L^2(Y, \tilde{R})} \mathrm{E}_{\tilde{R}}\left[\left(\frac{H - x}{\tilde{Z}_T} - \int_0^T \psi_t \, dY_t\right)^2\right] \tag{2.14}$$

and thus the problem of determining the minimum obtainable quadratic risk $J_0(x)$ (or the minimum $L^2(P)$-distance) has been transformed into a related problem under the measure $\tilde{R}$. This result is due to Gouriéroux, Laurent and Pham (1998). This observation allows us to prove the following corollary, which corresponds to Pham, Rheinländer and Schweizer (1998, Corollary 9); see also Schweizer (1996, Lemma 15).

Corollary 2.8 *The hedging error (2.11) is determined by*

$$\begin{aligned} J_0(x) &= \frac{\mathrm{E}[(\tilde{V}_0 - x)^2]}{\tilde{Z}_0} + \tilde{\mathrm{E}}\left[\int_0^T \frac{1}{\tilde{Z}_t}\, d[L^{H,\tilde{P}}]_t\right] \\ &= \frac{\mathrm{E}[(\tilde{V}_0 - x)^2] - \mathrm{E}[(\tilde{V}_0 - c^H)^2]}{\tilde{Z}_0} + \mathrm{E}[(N^H)^2]. \end{aligned} \qquad (2.15)$$

Proof: Consider the Galtchouk-Kunita-Watanabe decomposition of $\frac{H}{\tilde{Z}_T}$ under $\tilde{R}$ given by

$$\frac{H}{\tilde{Z}_T} = \mathrm{E}_{\tilde{R}}\left[\frac{H}{\tilde{Z}_T}\bigg|\,\mathcal{F}_0\right] + \int_0^T \psi_t^{H,\tilde{R}}\, dY_t + L_T^{H,\tilde{R}},$$

where $\psi \in L^2(Y, \tilde{R})$, $L^{H,\tilde{R}} \in \mathcal{M}_0^2(\tilde{R})$ and where Y and $L^{H,\tilde{R}}$ are strongly $\tilde{R}$-orthogonal, i.e. the process $YL^{H,\tilde{R}}$ is a local $\tilde{R}$-martingale. Since $Y^0 = 1/\tilde{Z}$, the Galtchouk-Kunita-Watanabe decomposition for $\frac{x}{\tilde{Z}_T}$ is simply

$$\frac{x}{\tilde{Z}_T} = \frac{x}{\tilde{Z}_0} + \int_0^T x\, dY_t^0 = \frac{x}{\tilde{Z}_0} + \int_0^T \psi_t^x\, dY_t,$$

where $\psi_t^x = (x, 0, \ldots, 0)^{tr}$. Thus

$$\frac{H - x}{\tilde{Z}_T} = \mathrm{E}_{\tilde{R}}\left[\frac{H-x}{\tilde{Z}_T}\bigg|\,\mathcal{F}_0\right] + \int_0^T (\psi_t^{H,\tilde{R}} - \psi_t^x) dY_t + L_T^{H,\tilde{R}} \qquad (2.16)$$

is the Galtchouk-Kunita-Watanabe decomposition for $\frac{H-x}{\tilde{Z}_T}$. By the strong $\tilde{R}$-orthogonality of Y and $L^{H,\tilde{R}}$,

$$\begin{aligned} &\mathrm{E}_{\tilde{R}}\left[\left(\frac{H-x}{\tilde{Z}_T} - \int_0^T \psi_t\, dY_t\right)^2\right] \\ &= \mathrm{E}_{\tilde{R}}\left[\left(\mathrm{E}_{\tilde{R}}\left[\frac{H-x}{\tilde{Z}_T}\bigg|\,\mathcal{F}_0\right]\right)^2\right] + \mathrm{E}_{\tilde{R}}\left[\left(L_T^{H,\tilde{R}}\right)^2\right] \\ &\quad + \mathrm{E}_{\tilde{R}}\left[\int_0^T (\psi_t^{H,\tilde{R}} - \psi_t^x - \psi_t)^{tr}\, d\langle Y\rangle_t\, (\psi_t^{H,\tilde{R}} - \psi_t^x - \psi_t)\right], \end{aligned}$$

which is minimized for $\psi = \psi^{H,\tilde{R}} - \psi^x$. To obtain the first term of (2.15), use (2.14) and note that

$$\mathrm{E}_{\tilde{R}}\left[\frac{H-x}{\tilde{Z}_T}\bigg|\,\mathcal{F}_0\right] = \frac{1}{\tilde{Z}_0}\left(\tilde{\mathrm{E}}[H|\mathcal{F}_0] - x\right) = \frac{1}{\tilde{Z}_0}\left(\tilde{V}_0 - x\right).$$

Furthermore, by Rheinländer and Schweizer (1997, Proposition 10), we obtain that $L^{H,\tilde{R}} = \int \tilde{Z}^{-1} dL^{H,\tilde{P}}$, and since $L^{H,\tilde{R}} \in \mathcal{M}_0^2(\tilde{R})$ we find

$$\mathrm{E}_{\tilde{R}}\left[\left(L_T^{H,\tilde{R}}\right)^2\right] = \mathrm{E}_{\tilde{R}}\left[[L^{H,\tilde{R}}]_T\right]$$

$$
\begin{aligned}
&= \mathrm{E}_{\tilde{R}}\left[\int_0^T \tilde{Z}_t^{-2}\, d[L^{H,\tilde{P}}]_t\right] \\
&= \frac{1}{\tilde{Z}_0}\tilde{\mathrm{E}}\left[\tilde{Z}_T \int_0^T d[\int \tilde{Z}^{-1} dL^{H,\tilde{P}}]_t\right] \\
&= \frac{1}{\tilde{Z}_0}\tilde{\mathrm{E}}\left[\int_0^T \tilde{Z}_t\, d[\int \tilde{Z}^{-1} dL^{H,\tilde{P}}]_t\right] \\
&= \frac{1}{\tilde{Z}_0}\tilde{\mathrm{E}}\left[\int_0^T \tilde{Z}_t^{-1}\, d[L^{H,\tilde{P}}]_t\right].
\end{aligned}
$$

Here, the fourth equality follows by the optional projection theorem, see He, Wang and Yan (1992, Theorem 5.16), since $[\int \tilde{Z}^{-1} dL^{H,\tilde{P}}]$ is an increasing process. The second equality in (2.15) follows from (2.9) by similar calculations:

$$
\begin{aligned}
\mathrm{E}[(N^H)^2] &= \mathrm{E}\left[\frac{\tilde{Z}_T^2}{\tilde{Z}_0^2}(\tilde{V}_0 - \tilde{\mathrm{E}}[H])^2\right] + \mathrm{E}\left[\tilde{Z}_T^2\left(\int_0^T \tilde{Z}_t^{-1}\, dL_t^{H,\tilde{P}}\right)^2\right] \\
&\quad +2\mathrm{E}\left[\frac{\tilde{Z}_T^2}{\tilde{Z}_0}(\tilde{V}_0 - \tilde{\mathrm{E}}[H])\left(\int_0^T \tilde{Z}_t^{-1}\, dL_t^{H,\tilde{P}}\right)\right] \\
&= \frac{\mathrm{E}[(\tilde{V}_0 - \tilde{\mathrm{E}}[H])^2]}{\tilde{Z}_0} + \tilde{Z}_0 \mathrm{E}_{\tilde{R}}\left[\left(L_T^{H,\tilde{R}}\right)^2\right] + 2\mathrm{E}_{\tilde{R}}\left[(\tilde{V}_0 - \tilde{\mathrm{E}}[H]) L_T^{H,\tilde{R}}\right] \\
&= \frac{\mathrm{E}[(\tilde{V}_0 - \tilde{\mathrm{E}}[H])^2]}{\tilde{Z}_0} + \tilde{\mathrm{E}}\left[\int_0^T \tilde{Z}_t^{-1} d[L^{H,\tilde{P}}]_t\right].
\end{aligned}
$$

In the last equality we have used the fact that $L^{H,\tilde{R}} \in \mathcal{M}_0^2(\tilde{R})$. This completes the proof. □

3 A comparison of hedging errors for two filtrations

In this section we give a comparison result for the hedging errors under two different filtrations by applying a projection argument for Hilbert spaces. The framework is the one of Gouriéroux, Laurent and Pham (1998); see the previous section for the necessary notation and definitions. We consider a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where the filtration $\mathbb{F}$ satisfies the usual conditions, $\mathcal{F}_0$ is trivial and $\mathcal{F} = \mathcal{F}_T$. Let X be a d-dimensional continuous semimartingale with respect to $\mathbb{F}$, and consider in addition a smaller filtration $\mathbb{F}^\circ \subseteq \mathbb{F}$ satisfying the usual conditions and which is so that X is adapted with respect to $\mathbb{F}^\circ$; hence, X is also an $\mathbb{F}^\circ$-semimartingale, see Stricker (1977). $\mathbb{F}^\circ$ could for example be the P-augmentation of the natural filtration generated by X. The role of $\mathbb{F}^\circ$ is only to restrict the space of strategies and not the space of claims, and we require in addition that $\mathcal{F}_T^\circ = \mathcal{F}_T$; for a comment on this assumption, see Remark 3.2 below. Throughout this section, we work under the standing Assumption 2.3.

As in Section 2, we let $\mathcal{V}(\mathbb{F}^\circ)$ denote the span of random variables of the form

$h^{tr}(X_{T_2} - X_{T_1})$, where $T_1 \leq T_2 \leq T$ are $\mathbb{F}^\circ$-stopping times such that the stopped process X^{T_2} is bounded, and h is a bounded $\mathcal{F}^\circ_{T_1}$-measurable random variable. It is clear that $\mathcal{V}(\mathbb{F}^\circ) \subseteq \mathcal{V}(\mathbb{F})$. Recall that by Assumption 2.3, the variance optimal martingale measure $\tilde{P}$ under $\mathbb{F}$ exists. We first show that this assumption also guarantees the existence of a variance optimal martingale measure $\tilde{P}^\circ$ under $\mathbb{F}^\circ$. To see this, note that by (2.1) and (2.2), and since $\mathcal{V}(\mathbb{F}^\circ) \subseteq \mathcal{V}(\mathbb{F})$, we have that $\mathcal{D}^e(\mathbb{F}) \subseteq \mathcal{D}^e(\mathbb{F}^\circ)$. Thus, Assumption 2.3 implies that $\mathcal{D}^e(\mathbb{F}^\circ) \cap L^2(P) \neq \emptyset$ and hence the arguments used in Section 2 show that the variance optimal martingale measure $\tilde{P}^\circ$ under $\mathbb{F}^\circ$ exists. Second, we denote by $\tilde{\Theta}(\mathbb{F}^\circ)$ the space of $\mathbb{F}^\circ$-predictable processes ϑ where $\int \vartheta dX$ is a $\tilde{P}^\circ$-martingale and $\int_0^T \vartheta dX \in L^2(P)$. Since $\tilde{P}$ and $\tilde{P}^\circ$ may differ, it is not immediately clear that $\tilde{\Theta}(\mathbb{F}^\circ) \subseteq \tilde{\Theta}(\mathbb{F})$. However, the property $\mathcal{D}^e(\mathbb{F}^\circ) \cap L^2(P) \neq \emptyset$ already implies that $G_T(\tilde{\Theta}(\mathbb{F}^\circ))$ is closed and that $G_T(\tilde{\Theta}(\mathbb{F}^\circ)) = \overline{\mathcal{V}(\mathbb{F}^\circ)}$, and thus

$$G_T(\tilde{\Theta}(\mathbb{F}^\circ)) = \overline{\mathcal{V}(\mathbb{F}^\circ)} \subseteq \overline{\mathcal{V}(\mathbb{F})} = G_T(\tilde{\Theta}(\mathbb{F})).$$

The inclusion $\tilde{\Theta}(\mathbb{F}^\circ) \subseteq \tilde{\Theta}(\mathbb{F}^\circ)$ can be proved directly using the alternative characterization of $\tilde{\Theta}(\mathbb{F})$ (and $\tilde{\Theta}(\mathbb{F}^\circ)$) which was given by Rheinländer (1999) and also recalled in Section 2. The argument goes as follows. Assume that $\vartheta \in \tilde{\Theta}(\mathbb{F}^\circ)$, so that $\int \vartheta\, dX$ is a $\tilde{P}^\circ$-martingale and $\int_0^T \vartheta\, dX \in L^2(P)$. By the observation of Rheinländer (1999) this implies that $\int \vartheta\, dX$ is a Q-martingale for all $Q \in \mathcal{M}^e(\mathbb{F}^\circ)$ with $\frac{dQ}{dP} \in L^2(P)$. Since $\mathcal{D}^e(\mathbb{F}) \subseteq \mathcal{D}^e(\mathbb{F}^\circ)$, or, equivalently, $\mathcal{M}^e(\mathbb{F}) \subseteq \mathcal{M}^e(\mathbb{F}^\circ)$, this shows that $\int \vartheta\, dX$ is a Q-martingale for all $Q \in \mathcal{M}^e(\mathbb{F})$ with $\frac{dQ}{dP} \in L^2(P)$, and hence $\vartheta \in \tilde{\Theta}(\mathbb{F})$.

Finally, we introduce the Radon-Nikodým derivative $\tilde{D}^\circ = \frac{d\tilde{P}^\circ}{dP}$ and the process

$$\tilde{Z}^\circ_t = \mathrm{E}^{\tilde{P}^\circ}[\tilde{D}^\circ \mid \mathcal{F}^\circ_t] = \frac{\mathrm{E}[(\tilde{D}^\circ)^2 \mid \mathcal{F}^\circ_t]}{\mathrm{E}[\tilde{D}^\circ \mid \mathcal{F}^\circ_t]} = \tilde{Z}^\circ_0 + \int_0^t \tilde{\zeta}^\circ{}_s\, dX_s, \tag{3.1}$$

with $\tilde{\zeta}^\circ \in \tilde{\Theta}(\mathbb{F}^\circ)$. Note that (3.1) is analogous to (2.3) and (2.4). In the rest of this section, we consider the following problem:

Question: What can be said about the difference between the hedging errors $J_0(\mathbb{F}, x)$ and $J_0(\mathbb{F}^\circ, x)$ defined by

$$J_0(\mathbb{G}, x) := \min_{\vartheta \in \tilde{\Theta}(\mathbb{G})} \mathrm{E}\left[\left(H - x - \int_0^T \vartheta_t\, dX_t\right)^2\right], \tag{3.2}$$

$\mathbb{G} \in \{\mathbb{F}, \mathbb{F}^\circ\}$, where $H \in L^2(P, \mathcal{F}_T)$ is a contingent claim?

Remark 3.1 For the computations in connection with the valuation principles, we shall actually consider $H - c^H$ instead of $H - x$, where c^H is given as in (2.7). Since the constants c^H may be different for the two choices of filtrations, this implies that in the general case, it will not be sufficient to compare $J_0(\mathbb{F}, x)$ and $J_0(\mathbb{F}^\circ, x)$ for fixed x. Instead, one should compare quantities $J_0(\mathbb{F}, x)$ and $J_0(\mathbb{F}, y)$, where x and

y may differ. However, when the variance optimal martingale measures under $\mathbb{F}$ and $\mathbb{F}^\circ$ coincide, $J_0(\mathbb{F}, x) - J_0(\mathbb{F}^\circ, x)$ is the relevant quantity for describing the change in the hedging error. □

Remark 3.2 We show that the assumption $\mathcal{F}_T^\circ = \mathcal{F}_T$ is not essential for the inclusion $\tilde{\Theta}(\mathbb{F}^\circ) \subseteq \tilde{\Theta}(\mathbb{F})$. To see this, let $\mathbb{G}^\circ = (\mathcal{G}_t^\circ)_{0 \le t \le T}$ be a filtration satisfying the usual conditions and which is such that X is $\mathbb{G}^\circ$-adapted and $\mathbb{G}^\circ \subseteq \mathbb{F}$. We do not assume here that $\mathcal{G}_T^\circ = \mathcal{F}_T$ and verify that $\tilde{\Theta}(\mathbb{G}^\circ) \subseteq \tilde{\Theta}(\mathbb{F})$. Define another filtration $\mathbb{G} = (\mathcal{G}_t)_{0 \le t \le T}$ by $\mathcal{G}_t = \mathcal{G}_t^\circ$, $t < T$, and $\mathcal{G}_T = \mathcal{F}_T$. We first show that $\mathcal{P}(\mathbb{G}) = \mathcal{P}(\mathbb{G}^\circ)$, that is, the σ-algebra generated by left-continuous $\mathbb{G}$-adapted processes is equal to the one generated by the left-continuous $\mathbb{G}^\circ$-adapted ones: Note that the $\mathbb{G}$-predictable σ-algebra $\mathcal{P}(\mathbb{G})$ is generated by the sets

$$\{A \times \{0\} \,|A \in \mathcal{G}_0\} \cup \{A \times (s,t] \,|s < t \le T,\, A \in \mathcal{G}_s\}.$$

This follows from Jacod and Shiryaev (1987, Theorem I.2.2) and the fact that any left-continuous adapted process X can be obtained as the point-wise limit of a sequence (X^n) defined by

$$X_t^n = X_0 1_{[\![0]\!]} + \sum_{k \in \mathbb{N}_0} X_{k/2^n} 1_{]\!]k/2^n,(k+1)/2^n \wedge T]\!]}.$$

In particular, this implies that $\mathcal{P}(\mathbb{G})$ does not depend on $\mathcal{G}_T$ and therefore $\mathcal{P}(\mathbb{G}) = \mathcal{P}(\mathbb{G}^\circ)$.

We now show that $\tilde{\Theta}(\mathbb{G}^\circ) = \tilde{\Theta}(\mathbb{G}) \subseteq \tilde{\Theta}(\mathbb{F})$, that is, the space of strategies under $\mathbb{G}^\circ$ and $\mathbb{G}$ coincide; the last inclusion follows by the direct proof given above. Since $\mathcal{P}(\mathbb{G}) = \mathcal{P}(\mathbb{G}^\circ)$, it is sufficient to verify that the variance optimal martingale measures for $\mathbb{G}$ and $\mathbb{G}^\circ$ coincide. To see the latter, we use that $\mathcal{V}(\mathbb{G}^\circ) = \mathcal{V}(\mathbb{G})$, which will be proved below, so that in fact $\mathcal{D}^x(P,\mathbb{G}^\circ) \subseteq \mathcal{D}^x(P,\mathbb{G})$, $x \in \{e,s\}$. Furthermore, it follows from this property and (2.1) that for any $D \in \mathcal{D}^x(P,\mathbb{G})$, $\mathrm{E}[D \mid \mathcal{G}_T^\circ] \in \mathcal{D}^x(P,\mathcal{G}^\circ)$. Thus, the minimum for $\|D\|_{L^2(P)}$ over all $D \in \mathcal{D}^s(P,\mathbb{G}) \cap L^2(P)$ is obtained for some $D \in \mathcal{D}^s(P,\mathbb{G}^\circ) \cap L^2(P)$ and hence, the variance optimal martingale measures for $\mathbb{G}$ and $\mathbb{G}^\circ$ coincide.

We finally verify that $\mathcal{V}(\mathbb{G}^\circ) = \mathcal{V}(\mathbb{G})$. The inclusion "$\subseteq$" is trivial, since $\mathbb{G}^\circ \subseteq \mathbb{G}$. To see "$\supseteq$", assume that $f = h^{tr}(X_{T_2} - X_{T_1}) \in \mathcal{V}(\mathbb{G})$, where $T_1 \le T_2 \le T$ are $\mathbb{G}$-stopping times such that X^{T_2} is bounded and h is a bounded $\mathcal{G}_{T_1}$-measurable random variable. Then clearly, $T_1 \le T_2 \le T$ are also $\mathbb{G}^\circ$-stopping times. Define $h_0 := h 1_{\{T_1 < T\}}$, which is bounded and $\mathcal{G}_{T_1}^\circ$-measurable, since for any $B \in \mathcal{B}(\mathbb{R}^d)$ with $0 \notin B$:

$$\begin{aligned} \{h_0 \in B\} \cap \{T_1 \le T\} &= \{h \in B\} \cap \{T_1 < T\} \\ &= \bigcup_{n \in \mathbb{N}} \left(\{h \in B\} \cap \{T_1 \le T - \frac{1}{n}\} \right) \in \mathcal{G}_{T-} \subseteq \mathcal{G}_T^\circ. \end{aligned}$$

Thus, $f_0 := h_0^{tr}(X_{T_2} - X_{T_1}) \in \mathcal{V}(\mathbb{G}^\circ)$. Since $f = f_0$, we see that $f \in \mathcal{V}(\mathbb{G}^\circ)$. □

We set out by proving Lemma 3.3 whose assumptions are satisfied under Assumption 2.3, in the case where X is continuous. The proof is based on a projection argument for Hilbert spaces and uses only the closedness of the spaces $G_T(\tilde{\Theta}(\mathbb{F}^\circ)) \subseteq G_T(\tilde{\Theta}(\mathbb{F}))$. In particular, it does not rely on the continuity of X. Denote by $\int \tilde{\vartheta}^H dX$ and $\int \tilde{\vartheta}^{H,\circ} dX$ the projections of H on the two spaces $G_T(\tilde{\Theta}(\mathbb{F}))$ and $G_T(\tilde{\Theta}(\mathbb{F}^\circ))$, respectively, see Remark 2.7. We consider the situation where $x = 0$, which basically amounts to replacing $H - x$ by H. Recall that if $x = c^H$, then the projection $\int \tilde{\vartheta}^H dX$ for $H - x$ is exactly the term appearing in (2.7). A similar remark applies for $\mathbb{F}^\circ$.

Lemma 3.3 *Assume only that the two linear spaces $G_T(\tilde{\Theta}(\mathbb{F}^\circ)) \subseteq G_T(\tilde{\Theta}(\mathbb{F}))$ are closed. Then*

$$J_0(\mathbb{F}^\circ, 0) - J_0(\mathbb{F}, 0) \quad = \quad \mathrm{E}\left[\left(\int_0^T (\tilde{\vartheta}_t^{H,\circ} - \tilde{\vartheta}_t^H)\, dX_t\right)^2\right] \geq 0. \tag{3.3}$$

Proof: Denote by $g^H = \int_0^T \tilde{\vartheta}^H dX$ and $g^{H,\circ} = \int_0^T \tilde{\vartheta}^{H,\circ} dX$ the projection of H on $G_T(\tilde{\Theta}(\mathbb{F}))$ and $G_T(\tilde{\Theta}(\mathbb{F}^\circ))$, respectively. Since $G_T(\tilde{\Theta}(\mathbb{F}^\circ)) \subseteq G_T(\tilde{\Theta}(\mathbb{F}))$, it follows that $g^{H,\circ}$ can also be viewed as the projection of g^H on $G_T(\tilde{\Theta}(\mathbb{F}^\circ))$. Furthermore, $(H - g^H) \in G_T(\tilde{\Theta}(\mathbb{F}))^\perp$ and $g^H - g^{H,\circ} \in G_T(\tilde{\Theta}(\mathbb{F}^\circ))^\perp \cap G_T(\tilde{\Theta}(\mathbb{F}))$. Now write H on the form

$$H - g^{H,\circ} = (H - g^H) + (g^H - g^{H,\circ}),$$

and note that by Pythagoras' theorem:

$$\|H - g^{H,\circ}\|^2 = \|H - g^H\|^2 + \|g^H - g^{H,\circ}\|^2.$$

This shows that

$$\begin{aligned} J_0(\mathbb{F}^\circ, 0) - J_0(\mathbb{F}, 0) &= \|H - g^{H,\circ}\|^2 - \|H - g^H\|^2 \\ &= \|g^H - g^{H,\circ}\|^2 = \mathrm{E}\left[\left(\int_0^T (\tilde{\vartheta}_t^{H,\circ} - \tilde{\vartheta}_t^H)\, dX_t\right)^2\right], \end{aligned}$$

which proves the result. □

As an important corollary to Lemma 3.3 we have the following:

Corollary 3.4 *Assume that $\mathcal{D}^e(\mathbb{F}) \cap L^2(P) \neq \emptyset$. Then* $\mathrm{Var}[N^{H,\circ}] \geq \mathrm{Var}[N^H]$.

Proof: The result follows by repeating the arguments in the proof of Lemma 3.3 for the spaces $\mathbb{R} + G_T(\tilde{\Theta}(\mathbb{F}^\circ)) \subseteq \mathbb{R} + G_T(\tilde{\Theta}(\mathbb{F}))$ which are closed provided that $\mathcal{D}^e(\mathbb{F}) \cap L^2(P) \neq \emptyset$, see Section 2. □

The above lemma gives a first expression for the difference $J_0(\mathbb{F}^\circ, 0) - J_0(\mathbb{F}, 0)$; we shall also refer to the quantity (3.3) as the *risk-increase (from $\mathbb{F}$ to $\mathbb{F}^\circ$)*. The following calculations are concerned with the explicit form of this risk-increase, and will relate (3.3) to the variance optimal martingale measure. First note that since $\tilde{P} \sim P$, $\tilde{Z}_T > 0$ P-a.s., and hence

$$\mathrm{E}\left[\left(\int_0^T (\tilde{\vartheta}_t^{H,\circ} - \tilde{\vartheta}_t^H)\, dX_t\right)^2\right] = \tilde{\mathrm{E}}\left[\frac{\left(\int_0^T (\tilde{\vartheta}_t^{H,\circ} - \tilde{\vartheta}_t^H)\, dX_t\right)^2}{\tilde{Z}_T}\right].$$

Here is the main result:

Theorem 3.5 *Assume that $\mathcal{D}^e(I\!\!F) \cap L^2(P) \neq \emptyset$. Then*

$$J_0(I\!\!F^\circ, 0) - J_0(I\!\!F, 0) = \tilde{\mathrm{E}}\left[\int_0^T \frac{1}{\tilde{Z}_t} \varrho_t^{tr}\, d[X]_t\, \varrho_t\right], \tag{3.4}$$

where

$$\varrho_t := (\tilde{\vartheta}_t^H - \tilde{\vartheta}_t^{H,\circ}) - \frac{\tilde{\zeta}_t}{\tilde{Z}_t}\left(\int_0^t (\tilde{\vartheta}_u^H - \tilde{\vartheta}_u^{H,\circ})\, dX_u\right). \tag{3.5}$$

Idea of proof of Theorem 3.5: Consider the case where X is 1-dimensional and let $\tilde{\vartheta} = \tilde{\vartheta}^H - \tilde{\vartheta}^{H,\circ}$. By Itô's formula:

$$\begin{aligned}\frac{\left(\int \tilde{\vartheta}\, dX\right)^2}{\tilde{Z}} &= \int 2\frac{\left(\int \tilde{\vartheta}\, dX\right)}{\tilde{Z}}\tilde{\vartheta}\, dX - \int \frac{\left(\int \tilde{\vartheta}\, dX\right)^2}{\tilde{Z}^2}\, d\tilde{Z} + \int \frac{1}{\tilde{Z}}\tilde{\vartheta}^2\, d[X] \\ &\quad + \int \frac{\left(\int \tilde{\vartheta}\, dX\right)^2}{\tilde{Z}^3}\, d[\tilde{Z}] - 2\int \frac{\left(\int \tilde{\vartheta}\, dX\right)}{\tilde{Z}^2}\, d[\tilde{Z}, \int \tilde{\vartheta}\, dX].\end{aligned}$$

Here, the first two terms are local $\tilde{P}$-martingales. If we assume that these processes are true $\tilde{P}$-martingales, then the expected value of the above quantity evaluated at time T simplifies to

$$\tilde{\mathrm{E}}\left[\int_0^T \frac{1}{\tilde{Z}}\left(\tilde{\vartheta}^2 + \frac{\left(\int \tilde{\vartheta} dX\right)^2}{\tilde{Z}^2}\tilde{\zeta}^2 - 2\frac{\left(\int \tilde{\vartheta} dX\right)}{\tilde{Z}}\tilde{\vartheta}\tilde{\zeta}\right) d[X]\right] = \tilde{\mathrm{E}}\left[\int_0^T \frac{1}{\tilde{Z}}\varrho^2 d[X]\right],$$

which proves the result under the additional assumption that the two local $\tilde{P}$-martingales $\int (\int \tilde{\vartheta} dX)/\tilde{Z}\, \tilde{\vartheta}\, dX$ and $\int (\int \tilde{\vartheta} dX)^2/\tilde{Z}^2 d\tilde{Z}$ are true $\tilde{P}$-martingales. □

It is not clear how it could be proved directly that the local $\tilde{P}$-martingales appearing in the expression for $(\int \tilde{\vartheta}\, dX)^2/\tilde{Z}$ are true $\tilde{P}$-martingales. We shall, therefore, prove Theorem 3.5 by using the identity (2.5) and working under the measure $\tilde{R}$ instead of $\tilde{P}$. As pointed out in Gouriéroux, Laurent and Pham (1998) and Rheinländer and Schweizer (1997) this has the advantage that for $\psi \in L^2(Y, \tilde{R})$, $\int \psi\, dY$ is by definition a square-integrable $\tilde{R}$-martingale.

Proof of Theorem 3.5: Let $\tilde{\vartheta} = \tilde{\vartheta}^H - \tilde{\vartheta}^{H,\circ}$ and note that $\tilde{\vartheta} \in \tilde{\Theta}(I\!\!F)$. By definition of the measures $\tilde{P}$ and $\tilde{R}$ we have that

$$\tilde{\mathrm{E}}\left[\frac{1}{\tilde{Z}_T}\left(\int_0^T \tilde{\vartheta}\, dX\right)^2\right] = \mathrm{E}\left[\left(\int \tilde{\vartheta}\, dX\right)^2\right]$$

$$
\begin{aligned}
&= c\mathrm{E}_{\tilde{R}}\left[\left(\frac{1}{\tilde{Z}_T}\int_0^T \tilde{\vartheta}\, dX\right)^2\right] \\
&= c\mathrm{E}_{\tilde{R}}\left[\left(\int_0^T \psi\, dY\right)^2\right],
\end{aligned}
$$

where the last equality follows from Theorem 2.4, with $\psi^0 = \int \tilde{\vartheta} dX - \tilde{\vartheta}^{tr} X$ and $\psi^i = \tilde{\vartheta}^i$, $i = 1, \ldots, d$. Since $\int \psi dY$ is a square-integrable martingale in $\mathcal{M}^2(\tilde{R})$, we obtain that

$$
\begin{aligned}
c\mathrm{E}_{\tilde{R}}\left[\left(\int_0^T \psi\, dY\right)^2\right] &= c\mathrm{E}_{\tilde{R}}\left[\int_0^T d\left[\int \psi dY\right]\right] \\
&= \mathrm{E}_{\tilde{P}}\left[\tilde{Z}_T \int_0^T d\left[\int \psi dY\right]\right] \\
&= \mathrm{E}_{\tilde{P}}\left[\int_0^T \tilde{Z} d\left[\int \psi dY\right]\right] \\
&= \mathrm{E}_{\tilde{P}}\left[\int_0^T \tilde{Z}\psi^{tr} d[Y]\psi\right], \qquad (3.6)
\end{aligned}
$$

where the third equality follows by the optional projection theorem, see He, Wang and Yan (1992, Theorem 5.16), since $[\int \psi dY]$ is an increasing process. The theorem now follows from (3.6) by some rather tedious direct calculations: We consider only the situation where X is 1-dimensional, so that $Y = (Y^0, Y^1)^{tr}$. By the definition of Y, the Itô formula, and the continuity of the local $\tilde{P}$-martingales X and $\tilde{Z}$, we find that

$$
Y^0 = \frac{1}{\tilde{Z}} = \frac{1}{\tilde{Z}_0} - \int \frac{\tilde{\zeta}}{\tilde{Z}^2} dX + \int \frac{\tilde{\zeta}^2}{\tilde{Z}^3} d[X]
$$

and

$$
Y^1 = \frac{X}{\tilde{Z}} = \frac{X_0}{\tilde{Z}_0} + \int \left(\frac{1}{\tilde{Z}} - \frac{X\tilde{\zeta}}{\tilde{Z}^2}\right) dX + \int \left(\frac{X\tilde{\zeta}^2}{\tilde{Z}^3} - \frac{\tilde{\zeta}}{\tilde{Z}^2}\right) d[X].
$$

Hence,

$$
\begin{aligned}
\left[\frac{1}{\tilde{Z}}\right] &= \int \frac{\tilde{\zeta}^2}{\tilde{Z}^4} d[X], \\
\left[\frac{X}{\tilde{Z}}\right] &= \int \left(\frac{1}{\tilde{Z}} - \frac{X\tilde{\zeta}}{\tilde{Z}^2}\right)^2 d[X] = \int \left(\frac{1}{\tilde{Z}^2} + \frac{X^2\tilde{\zeta}^2}{\tilde{Z}^4} - 2\frac{X\tilde{\zeta}}{\tilde{Z}^3}\right) d[X], \\
\left[\frac{1}{\tilde{Z}}, \frac{X}{\tilde{Z}}\right] &= -\int \frac{\tilde{\zeta}}{\tilde{Z}^2}\left(\frac{1}{\tilde{Z}} - \frac{X\tilde{\zeta}}{\tilde{Z}^2}\right) d[X] = \int \left(\frac{X\tilde{\zeta}^2}{\tilde{Z}^4} - \frac{\tilde{\zeta}}{\tilde{Z}^3}\right) d[X].
\end{aligned}
$$

Now rewrite (3.6) by

$$
\begin{aligned}
\mathrm{E}_{\tilde{P}}\left[\int_0^T \tilde{Z}\psi^{tr} d[Y]\psi\right] &= \mathrm{E}_{\tilde{P}}\left[\int_0^T \tilde{Z}(\psi^0)^2 d[1/\tilde{Z}] + 2\int_0^T \tilde{Z}\psi^0\psi^1 d[1/\tilde{Z}, X/\tilde{Z}]\right. \\
&\quad \left. + \int_0^T \tilde{Z}(\psi^1)^2 d[X/\tilde{Z}]\right]. \qquad (3.7)
\end{aligned}
$$

To prove (3.4), simply insert $\psi^1 = \tilde{\vartheta}$ and $\psi^0 = \int \tilde{\vartheta} dX - \tilde{\vartheta} X$ to find that the three terms under the $\tilde{P}$-expectation in (3.7) are

$$
\begin{aligned}
&\int \left((\int \tilde{\vartheta}\, dX)^2 + (\tilde{\vartheta} X)^2 - 2(\int \tilde{\vartheta}\, dX)\tilde{\vartheta} X \right) \frac{\tilde{\zeta}^2}{\tilde{Z}^3} d[X] \\
&+2 \int (\int \tilde{\vartheta}\, dX - \tilde{\vartheta} X)\tilde{\vartheta} \left(\frac{X\tilde{\zeta}^2}{\tilde{Z}^3} - \frac{\tilde{\zeta}}{\tilde{Z}^2} \right) d[X] \\
&+ \int \tilde{\vartheta}^2 \left(\frac{1}{\tilde{Z}} + \frac{X^2\tilde{\zeta}^2}{\tilde{Z}^3} - 2\frac{X\tilde{\zeta}}{\tilde{Z}^2} \right) d[X] \\
&= \int \frac{1}{\tilde{Z}} \left(\frac{\tilde{\zeta}^2}{\tilde{Z}^2} (\int \tilde{\vartheta}\, dX)^2 - 2\frac{\tilde{\zeta}}{\tilde{Z}} \tilde{\vartheta} (\int \tilde{\vartheta}\, dX) + \tilde{\vartheta}^2 \right) d[X].
\end{aligned}
$$

This proves (3.4). □

Remark 3.6 We note that (3.4) of Theorem 3.5 establishes a connection between the difference $J_0(\mathbb{F}^\circ, 0) - J_0(\mathbb{F}, 0)$ and the quadratic variation process of the local $\tilde{P}$-martingale X under mild assumptions on the filtrations $\mathbb{F}^\circ$ and $\mathbb{F}$. Thus, we only require that the $\mathbb{F}$-semimartingale X is also $\mathbb{F}^\circ$-adapted and that $\mathcal{D}^e(\mathbb{F}) \cap L^2(P) \neq \emptyset$. This result is not an immediate consequence of Corollary 2.8, since the hedging errors $J_0(\mathbb{F}^\circ, 0)$ and $J_0(\mathbb{F}, 0)$ are given in terms of the processes L^H from the Galtchouk-Kunita-Watanabe decompositions under the variance optimal martingale measures with respect to $\mathbb{F}^\circ$ and $\mathbb{F}$, respectively. An alternative approach to compare hedging errors could therefore be to apply projection results for the corresponding Galtchouk-Kunita-Watanabe decompositions under $\tilde{P}$. Föllmer and Schweizer (1990) have shown how the Galtchouk-Kunita-Watanabe decomposition under one filtration can be related to the Galtchouk-Kunita-Watanabe decomposition under a certain larger filtration. However, these results seem difficult to apply in the full generality of Theorem 3.5. □

We can now apply the feed-back formula from Theorem 2.6, see also Remark 2.7, for the projections $\tilde{\vartheta}^H$ and $\tilde{\vartheta}^{H,\circ}$ to obtain an expression for ϱ in terms of the Galtchouk-Kunita-Watanabe decompositions for H under the variance optimal martingale measures. We work under the additional assumption:

Assumption 3.7 *Assume that the variance optimal martingale measures w.r.t. $\mathbb{F}$ and $\mathbb{F}^\circ$ coincide, that is $\tilde{Z}^\circ = \tilde{Z}$.*

In section 5 we analyze a model where the two filtrations differ only by some additional independent risk, and we shall see that in this case, Assumption 3.7 is satisfied.

Introduce now the Galtchouk-Kunita-Watanabe decompositions for H under $\tilde{P}$ with respect to $\mathbb{F}$ and $\mathbb{F}^\circ$ given by

$$H = H_0 + \int_0^T \xi_t^{H,\tilde{P}}\, dX_t + L_T^{H,\tilde{P}},$$

and

$$H = H_0^\circ + \int_0^T \xi_t^{H,\circ,\tilde{P}}\, dX_t + L_T^{H,\circ,\tilde{P}},$$

respectively, and let $\tilde{V} = \mathrm{E}_{\tilde{P}}[H \mid \mathbb{F}]$ and $\tilde{V}^\circ = \mathrm{E}_{\tilde{P}}[H \mid \mathbb{F}^\circ]$; by Assumption 3.7, $H_0^\circ = H_0$. Note first that by Theorem 3.5

$$\varrho_t = \left(\tilde{\vartheta}_t^H - \frac{\tilde{\zeta}_t}{\tilde{Z}_t}\int_0^t \tilde{\vartheta}_s^H\, dX_s\right) - \left(\tilde{\vartheta}_t^{H,\circ} - \frac{\tilde{\zeta}_t}{\tilde{Z}_t}\int_0^t \tilde{\vartheta}_s^{H,\circ}\, dX_s\right).$$

By the feed-back formula of Theorem 2.6, see also Remark 2.7, we can rewrite the first term by

$$\begin{aligned}\tilde{\vartheta}_t^H - \frac{\tilde{\zeta}_t}{\tilde{Z}_t}\int_0^t \tilde{\vartheta}_s^H dX_s &= \left(\xi_t^{H,\tilde{P}} - \frac{\tilde{\zeta}_t}{\tilde{Z}_t}\left(\tilde{V}_{t-} - \int_0^t \tilde{\vartheta}_s^H dX_s\right)\right) - \frac{\tilde{\zeta}_t}{\tilde{Z}_t}\int_0^t \tilde{\vartheta}_s^H dX_s \\ &= \xi_t^{H,\tilde{P}} - \frac{\tilde{\zeta}_t}{\tilde{Z}_t}\tilde{V}_{t-}.\end{aligned}$$

By Assumption 3.7, $\tilde{Z}$ and $\tilde{Z}^\circ$ as well as $\tilde{\zeta}$ and $\tilde{\zeta}^\circ$ coincide, and hence we obtain similarly that

$$\begin{aligned}\tilde{\vartheta}_t^{H,\circ} - \frac{\tilde{\zeta}_t}{\tilde{Z}_t}\int_0^t \tilde{\vartheta}_s^{H,\circ} dX_s &= \left(\xi_t^{H,\circ,\tilde{P}} - \frac{\tilde{\zeta}_t^\circ}{\tilde{Z}_t^\circ}\left(\tilde{V}_{t-}^\circ - \int_0^t \tilde{\vartheta}_s^{H,\circ} dX_s\right)\right) - \frac{\tilde{\zeta}_t}{\tilde{Z}_t}\int_0^t \tilde{\vartheta}_s^{H,\circ} dX_s \\ &= \xi_t^{H,\circ,\tilde{P}} - \frac{\tilde{\zeta}_t}{\tilde{Z}_t}\tilde{V}_{t-}^\circ.\end{aligned}$$

Thus, we have proved the following corollary to Theorem 3.5:

Corollary 3.8 *Assume that $\mathcal{D}^e(\mathbb{F}) \cap L^2(P) \neq \emptyset$ and that Assumption 3.7 is satisfied. Then*

$$J_0(\mathbb{F}^\circ, 0) - J_0(\mathbb{F}, 0) = \tilde{\mathrm{E}}\left[\int_0^T \frac{1}{\tilde{Z}_t}\varrho_t^{tr}\, d[X]_t\, \varrho_t\right], \tag{3.8}$$

where

$$\varrho = \xi_t^{H,\tilde{P}} - \xi_t^{H,\circ,\tilde{P}} - \frac{\tilde{\zeta}_t}{\tilde{Z}_t}\left(\tilde{V}_{t-} - \tilde{V}_{t-}^\circ\right). \tag{3.9}$$

Remark 3.9 Theorem 3.5 shows how the hedging error increases when the filtration is made smaller. It follows from (3.9) that the increase in the general case can be expressed in terms of the Kunita-Watanabe decompositions under the filtrations $\mathbb{F}$ and $\mathbb{F}^\circ$. In the examples considered in the next sections, these will be easier to obtain than the corresponding integrands appearing in the projections g^H and $g^{H,\circ}$.

It is noted that an expression similar to (3.9) cannot be obtained in the case where the variance optimal martingale measures w.r.t. $\mathbb{F}$ and $\mathbb{F}^\circ$ are not identical. In that case, the above calculations show that

$$\varrho_t = \xi_t^{H,\tilde{P}} - \xi_t^{H,\circ,\tilde{P}} - \left(\frac{\tilde{\zeta}_t}{\tilde{Z}_t}\tilde{V}_{t-} - \frac{\tilde{\zeta}_t^\circ}{\tilde{Z}_t^\circ}\tilde{V}_{t-}^\circ\right) + \left(\frac{\tilde{\zeta}_t}{\tilde{Z}_t} - \frac{\tilde{\zeta}_t^\circ}{\tilde{Z}_t^\circ}\right)\int_0^t \tilde{\vartheta}_u^{H,\circ}\, dX_u.$$

We also point out that in the martingale case, $\tilde{\zeta} \equiv 0$ and $\tilde{Z} \equiv 1$, so that $\varrho = (\xi^{H,\tilde{P}} - \xi^{H,\circ,\tilde{P}})$, and so Theorem 3.5 simplifies to

$$\begin{aligned} J_0(\mathbb{F}^\circ, 0) - J_0(\mathbb{F}, 0) &= \mathrm{E}\left[\int_0^T \left(\xi_t^{H,\tilde{P}} - \xi_t^{H,\circ,\tilde{P}}\right)^{tr} d[X]_t \left(\xi_t^{H,\tilde{P}} - \xi_t^{H,\circ,\tilde{P}}\right)\right] \\ &= \|\xi^{H,\tilde{P}} - \xi^{H,\circ,\tilde{P}}\|^2_{L^2(P_X)}. \end{aligned}$$

A similar result can be found in Schweizer (1988, Section II.2) for the case where the continuous-time model is compared with a discrete-time model. □

4 The financial premium calculation principles

In this section we very briefly review the indifference pricing principles proposed by Schweizer (1997) and examined further in Chapter 4. Let $H \in L^2(P)$ be a claim and consider the so-called variance principle and standard deviation principles defined by

$$\tilde{u}_1(H) = \mathrm{E}[H] + a\mathrm{Var}[H], \tag{4.1}$$
$$\tilde{u}_2(H) = \mathrm{E}[H] + a\sqrt{\mathrm{Var}[H]}. \tag{4.2}$$

These two principles are classical actuarial premium calculation principles and often used for the pricing of reinsurance contracts, for example stop-loss contracts; the terms $a\mathrm{Var}[H]$ and $a\sqrt{\mathrm{Var}[H]}$ are often called safety loadings. The indifference principles of Schweizer (1997) are derived by an indifference argument and allow for modifications of (4.1) and (4.2). It is convenient to work with $Y = -H$, which can be interpreted as the gain or final wealth at some fixed time T from selling the claim H. Furthermore, we introduce the functions $u_i(Y) = -\tilde{u}_i(H)$, that is

$$u_1(Y) = \mathrm{E}[Y] - a\mathrm{Var}[Y], \tag{4.3}$$
$$u_2(Y) = \mathrm{E}[Y] - a\sqrt{\mathrm{Var}[Y]}. \tag{4.4}$$

The functions u_1 and u_2 describes the preferences of the insurer. We shall assume that the insurer's objective is to maximize $u_i(Y)$.

Let c denote the insurer's initial capital. The u_i-indifference price for H is now defined as the solution h_i to the equation

$$\sup_{\vartheta\in\tilde{\Theta}} u_i\left(c + h_i + \int_0^T \vartheta_t\, dX_t - H\right) = \sup_{\tilde{\vartheta}\in\tilde{\Theta}} u_i\left(c + \int_0^T \tilde{\vartheta}_t\, dX_t\right).$$

The solution is also called the *fair premium*, and the associated maximizing strategy ϑ^* is called the *optimal strategy*. The term on the left side of the equality is the maximum obtainable value assigned to the wealth $c + h_i + \int_0^T \vartheta\, dX - H$, which is simply the initial capital c augmented by the premium h_i and trading gains $\int_0^T \vartheta\, dX$ from some self-financing strategy ϑ, and reduced by the claim H. The term on the right is the maximum obtainable value assigned to the wealth from not selling the claim and simply investing the initial capital in a self-financing strategy.

Denote by $\pi(\cdot)$ the projection in $L^2(P)$ on $G_T(\tilde{\Theta})^\perp$ and let $1-\pi(1)=\int_0^T \tilde{\beta}\,dX$. We formulate the main results which relate the indifference pricing principles to the quantities introduced in the previous section. First, for the variance principle:

Theorem 4.1 (Schweizer (1997), Møller (1999b)) *For any $H \in L^2(P)$ and $c \in \mathbb{R}$, the u_1-indifference price for H is*

$$h_1(c) = v_1(H) = \tilde{\mathrm{E}}\left[H\right] + a\mathrm{Var}\left[N^H\right],$$

and the optimal strategy is

$$\vartheta^* = \vartheta^H + \frac{1+\mathrm{Var}[\tilde{Z}_T]}{2a}\tilde{\beta}.$$

Second, for the standard deviation principle:

Theorem 4.2 (Schweizer (1997), Møller (1999b)) *For any $H \in L^2(P)$ and $c \in \mathbb{R}$, the u_2-indifference price for H is*

$$h_2(c) = v_2(H) = \tilde{\mathrm{E}}\left[H\right] + a\sqrt{1-\frac{\mathrm{Var}[\tilde{Z}_T]}{a^2}}\sqrt{\mathrm{Var}[N^H]},$$

provided that $a^2 \geq \mathrm{Var}[\tilde{Z}_T]$. If $a^2 < \mathrm{Var}[\tilde{Z}_T]$, then the u_2-indifference price is undefined. If $a^2 > \mathrm{Var}[\tilde{Z}_T]$, then the optimal strategy is

$$\vartheta^* = \vartheta^H + \frac{1+\mathrm{Var}[\tilde{Z}_T]}{a\sqrt{1-\frac{\mathrm{Var}[\tilde{Z}_T]}{a^2}}}\sqrt{\mathrm{Var}[N^H]}\tilde{\beta}.$$

5 A product space model

We consider in the following a model which describes a financial market with some additional insurance risk. The aim is to investigate the premium principles reviewed in the previous section and apply them to the problem of pricing certain reinsurance contracts and unit-linked life insurance contracts. We work in the rest of this chapter under the following assumption

Assumption 5.1 *The financial market and the additional source of risk are stochastically independent.*

Remark 5.2 This assumption provides a simple set-up which is sufficiently general for the analysis of several interesting problems. We can think of the additional risk as an underlying *pure insurance risk* in nominal values, which will be discounted by deflating the nominal values using some tradeable or non-tradeable financial asset. On the other hand, the assumption rules out the possibility of including problems where for example jumps in stock prices are triggered by certain insurance events. Such problems would have to be considered separately within the more general framework of Section 3. □

The model is constructed by setting out with two separate probability spaces, one for the financial market and one for the additional risk, and then embedding these in a product space. This leads to a new market consisting of the same financial assets, but where the class of claims is extended in that claims can now depend on the additional insurance risk as well as on the financial assets. This construction has the advantage that it allows to merge directly classical insurance risk models with models for financial markets. We show that important properties for the a priori given financial market are preserved by this extension. In particular, we will be able to vary the amount of available information from the additional source of risk and obtain simple results for its impact on the financial valuation principles.

The financial market

Let $(\Omega_1, \mathcal{F}^1, P_1)$ be a complete probability space with a filtration $\bar{\mathbb{F}}^1 = (\bar{\mathcal{F}}^1_t)_{0\le t\le T}$ satisfying the usual conditions of right-continuity and completeness. We assume that the σ-algebra $\bar{\mathcal{F}}^1_0$ is trivial, $\mathcal{F}^1 = \bar{\mathcal{F}}^1_T$, and fix a finite time horizon T. Consider a d-dimensional process $\bar{X} = (\bar{X}^1, \dots, \bar{X}^d)$ which is taken to describe the evolution of the discounted prices of d tradeable stocks. To emphasize the fact that $\bar{X}$ is defined on the space $(\Omega_1, \mathcal{F}^1)$, we will also write $\bar{X}(\omega_1)$ for the path of $\bar{X}$ associated with $\omega_1 \in \Omega_1$. A *purely financial derivative* is a random variable $H^1 \in L^2(P_1, \bar{\mathcal{F}}^1_T)$.

It is assumed that $\bar{X}$ is a *continuous* semimartingale with canonical decomposition

$$\bar{X} = \bar{X}_0 + \bar{M} + \bar{A}, \tag{5.1}$$

where $\bar{M}$ is a continuous local martingale on $(\Omega_1, \mathcal{F}^1, P_1)$ and $\bar{A}$ is a continuous adapted process of finite variation. Moreover, the model is assumed to be free of arbitrage; a necessary condition is the existence of a predictable process $\bar{\lambda}$ such that $\bar{A} = \int d\langle \bar{M} \rangle \bar{\lambda}$ and

$$\int_0^T \bar{\lambda}^{tr} d\langle \bar{M} \rangle \bar{\lambda} < \infty \quad P\text{-a.s.}$$

Denote by $\mathcal{M}^e(P_1)$ the space of all equivalent local martingale measures Q_1 and let $\mathcal{D}^e(\bar{\mathbb{F}}^1)$ be the space of their densities $\bar{D}$ on $\mathcal{F}^1$, see Section 2. Throughout, we work under the following assumption which corresponds to Assumption 2.3 and ensures the existence of the variance optimal martingale measure $\tilde{P}_1$ for $\bar{X}$.

Assumption 5.3 $\mathcal{D}^e(\bar{\mathbb{F}}^1) \cap L^2(P_1) \neq \emptyset$.

As in Section 2, we denote by $\tilde{\Theta}(\bar{\mathbb{F}}^1)$ the space of $\bar{\mathbb{F}}^1$-predictable processes $\bar{\vartheta}$ such that $\int_0^T \bar{\vartheta} d\bar{X} \in L^2(P_1)$ and $\int \bar{\vartheta} d\bar{X}$ is an $(\bar{\mathbb{F}}^1, \tilde{P}_1)$-martingale. We shall, however, use the fact that $\bar{\vartheta} \in \tilde{\Theta}(\bar{\mathbb{F}}^1)$ if and only if $\int_0^T \bar{\vartheta} d\bar{X} \in L^2(P_1)$ and $\int \bar{\vartheta} d\bar{X}$ is an $(\bar{\mathbb{F}}^1, Q_1)$-martingale for any $Q_1 \in \mathcal{M}^e(P_1)$ with $\frac{dQ_1}{dP_1} \in L^2(P_1)$; see the remark following Proposition 15 of Rheinländer (1999). Recall also that the space $G_T(\tilde{\Theta}(\bar{\mathbb{F}}^1))$ is closed in $L^2(P_1)$ under Assumption 5.3.

The financial market $(\bar{X}, \bar{\mathbb{F}}^1)$ is said to be *complete* if any purely financial derivative $\bar{H}$ admits a representation as a constant plus a stochastic integral with respect to

$\bar{X}$, that is, if there exist a constant $\bar{H}_0$ and $\bar{\vartheta}^H \in \Theta(\bar{\mathbb{F}}^1)$ such that

$$\bar{H} = \bar{H}_0 + \int_0^T \bar{\vartheta}_t^H d\bar{X}_t \ P\text{-a.s.}$$

The additional source of risk
Consider another complete, filtered probability space $(\Omega_2, \mathcal{F}^2, \bar{\mathbb{F}}^2, P_2)$, where the filtration $\bar{\mathbb{F}}^2$ is right-continuous but not necessarily complete and where $\bar{\mathcal{F}}_0^2$ is not necessarily trivial. For example, we can think of this space as carrying a pure insurance risk process which describes the occurrence of insurance claims and the development of the nominal amounts paid. To emphasize that an insurance risk process $\bar{U}$ is defined on the space $(\Omega_2, \mathcal{F}^2)$ we may write $\bar{U}(\omega_2)$ for the path of $\bar{U}$ associated with $\omega_2 \in \Omega_2$. Similarly to the notion of a purely financial derivative, a *pure insurance contract* is a random variable $H^{(2)} \in L^2(P_2, \bar{\mathcal{F}}_T^2)$.

The combined model
The two separate models are merged by introducing a new filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ as the product space of $(\Omega_1, \mathcal{F}^1, \bar{\mathbb{F}}^1, P_1)$ and $(\Omega_2, \mathcal{F}^2, \bar{\mathbb{F}}^2, P_2)$. Since this leads to some technical issues, we shall in the following discuss this construction in detail. We let $\Omega = \Omega_1 \times \Omega_2$ and $P = P_1 \otimes P_2$; in order to obtain a complete probability space, we introduce the σ-algebra $\mathcal{N}$ generated by all subsets of null-sets from $\mathcal{F}^1 \otimes \mathcal{F}^2$, that is

$$\mathcal{N} = \sigma\{F \subseteq \Omega^1 \times \Omega^2 \,|\, \exists G \in \mathcal{F}^1 \otimes \mathcal{F}^2 : F \subseteq G, (P^1 \otimes P^2)(G) = 0\}.$$

We then let

$$\mathcal{F} \;=\; \left(\mathcal{F}^1 \otimes \mathcal{F}^2\right) \vee \mathcal{N}, \tag{5.2}$$

and define the filtrations $\mathbb{F}^1$ and $\mathbb{F}^2$ on the product space by

$$\mathcal{F}_t^1 \;=\; \left(\bar{\mathcal{F}}_t^1 \otimes \{\emptyset, \Omega_2\}\right) \vee \mathcal{N}, \tag{5.3}$$

$$\mathcal{F}_t^2 \;=\; \left(\{\emptyset, \Omega_1\} \otimes \bar{\mathcal{F}}_t^2\right) \vee \mathcal{N}. \tag{5.4}$$

The filtrations defined by (5.3) and (5.4) correspond to the original filtrations $\bar{\mathbb{F}}^1$ and $\bar{\mathbb{F}}^2$, since they basically contain the same "amount of information". However, they differ from $\bar{\mathbb{F}}^1$ and $\bar{\mathbb{F}}^2$ in that (5.3) and (5.4) are defined on the new product space. Some results on these filtrations are gathered in the next lemma:

Lemma 5.4

1. *$\mathbb{F}^1$ and $\mathbb{F}^2$ satisfy the usual conditions.*

2. *$\mathbb{F}^1$ and $\mathbb{F}^2$ are independent.*

3. *The filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ defined by $\mathcal{F}_t = \mathcal{F}_t^1 \vee \mathcal{F}_t^2$ satisfies the usual conditions. Furthermore*

$$\mathcal{F}_t = (\bar{\mathcal{F}}_t^1 \otimes \bar{\mathcal{F}}_t^2) \vee \mathcal{N}. \tag{5.5}$$

Proof: 1. Since $\mathcal{N} \subseteq \mathcal{F}_t^i$ for all $t \in [0,T]$, $i = 1,2$, $\mathbb{F}^i$ is complete. We verify that $\mathbb{F}^1$ is right-continuous; the proof for the right-continuity of $\mathbb{F}^2$ is symmetric and uses only the right-continuity of $\bar{\mathbb{F}}^2$. Define for $s \in [0,T]$

$$\mathcal{D}_s = \{F_1 \times \Omega_2 \mid F_1 \in \bar{\mathcal{F}}_s^1\}.$$

By definition, $\sigma(\mathcal{D}_s) = \bar{\mathcal{F}}_s^1 \otimes \{\emptyset, \Omega_2\}$, and since $\mathcal{D}_s$ is also a σ-algebra we obtain that

$$\mathcal{D}_s = \bar{\mathcal{F}}_s^1 \otimes \{\emptyset, \Omega_2\}.$$

This implies that

$$\bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon}^1 = \bigcap_{\varepsilon>0} (\mathcal{D}_{t+\varepsilon} \vee \mathcal{N}) = (\bigcap_{\varepsilon>0} \mathcal{D}_{t+\varepsilon}) \vee \mathcal{N} = \mathcal{D}_t \vee \mathcal{N},$$

where the second equality follows from Kallenberg (1997, Lemma 6.8). The last equality is a consequence of the right-continuity of $\bar{\mathbb{F}}^1$: Assume that $F \in \mathcal{D}_{t+\varepsilon}$ for all $\varepsilon > 0$, then $F = F_1 \times \Omega_2$, where $F_1 \in \bigcap_{\varepsilon>0} \bar{\mathcal{F}}_{t+\varepsilon}^1 = \bar{\mathcal{F}}_t^1$.

2. We have to show that, for any $F_1 \in \mathcal{F}_T^1$ and $F_2 \in \mathcal{F}_T^2$,

$$P(F_1 \cap F_2) = P(F_1)P(F_2), \tag{5.6}$$

that is, the σ-algebras $\mathcal{F}_T^1$ and $\mathcal{F}_T^2$ are stochastically independent under P. We only need to verify the property (5.6) for $F_1 \in \bar{\mathcal{F}}_T^1 \otimes \{\emptyset, \Omega_2\}$ and $F_2 \in \{\emptyset, \Omega_1\} \otimes \bar{\mathcal{F}}_T^2$, since the result then follows for general F_1 and F_2 by choosing $G_1 \in \bar{\mathcal{F}}_T^1 \otimes \{\emptyset, \Omega_2\}$ and $G_2 \in \{\emptyset, \Omega_1\} \otimes \bar{\mathcal{F}}_T^2$, so that $P((G_i \setminus F_i) \cup (F_i \setminus G_i)) = 0$, $i = 1,2$. So we consider F_1, F_2 on the form $F_1 = \bar{F}_1 \times O_2$ and $F_2 = O_1 \times \bar{F}_2$, where $\bar{F}_i \in \bar{\mathcal{F}}_i$ and $O_i \in \{\emptyset, \Omega_i\}$; hence $P_i(O_i) \in \{0,1\}$. We now get

$$\begin{aligned} P(F_1 \cap F_2) &= P\Big((\bar{F}_1 \times O_2) \cap (O_1 \times \bar{F}_2)\Big) = P\Big((\bar{F}_1 \cap O_1) \times (\bar{F}_2 \cap O_2)\Big) \\ &= P_1\Big(\bar{F}_1 \cap O_1\Big) P_2\Big(\bar{F}_2 \cap O_2\Big) = P_1(\bar{F}_1)P_1(O_1)P_2(\bar{F}_2)\, P_2(O_2) \\ &= P(F_1)P(F_2), \end{aligned}$$

where the third equality is the definition of the product measure $P = P_1 \otimes P_2$ and the fourth equality follows since $P_i(O_i) \in \{0,1\}$.

3. Since $\mathbb{F}^1$ and $\mathbb{F}^2$ are independent and satisfy the usual conditions, it follows from He and Wang (1982, Theorem 1) that $\mathbb{F}$ satisfies the usual conditions. To verify (5.5), note that

$$\Big(\bar{\mathcal{F}}_t^1 \otimes \{\emptyset, \Omega_2\}\Big) \vee \Big(\{\emptyset, \Omega_1\} \otimes \bar{\mathcal{F}}_t^2\Big) = \Big(\bar{\mathcal{F}}_t^1 \otimes \bar{\mathcal{F}}_t^2\Big).$$

Here, the inclusion "$\subseteq$" follows immediately. To see "$\supseteq$", note that for any $F = F_1 \times F_2$, with $F_i \in \bar{\mathcal{F}}_t^i$, we have

$$F_1 \times F_2 = (F_1 \times \Omega_2) \cap (\Omega_1 \times F_2).$$

This completes the proof. □

Remark 5.5 For filtrations $\bar{I\!H}^1$ and $\bar{I\!H}^2$ on $(\Omega_1, \mathcal{F}^1)$ and $(\Omega_2, \mathcal{F}^2)$, respectively, we shall henceforth use the notation $\bar{I\!H}^1 \otimes \bar{I\!H}^2$ for the complete filtration $I\!H = (\mathcal{H}_t)_{0 \le t \le T}$ defined by

$$\mathcal{H}_t = (\bar{\mathcal{H}}_t^1 \otimes \bar{\mathcal{H}}_t^2) \vee \mathcal{N}. \quad \square$$

The processes $\bar{X}$ and $\bar{U}$ are now embedded in the product space by

$$\begin{aligned} X(\omega_1, \omega_2) &:= \bar{X}(\omega_1), \\ U(\omega_1, \omega_2) &:= \bar{U}(\omega_2), \end{aligned}$$

and a natural question is then: Is X a semimartingale on $(\Omega, \mathcal{F}, I\!F, P)$? The following lemma ensures that this is the case and relates the canonical decomposition of X to the one of $\bar{X}$.

Lemma 5.6 *Assume that $\bar{X}$ is a continuous semimartingale on $(\Omega_1, \mathcal{F}^1, \bar{I\!F}^1, P_1)$ with canonical decomposition (5.1). Then X is a continuous semimartingale on $(\Omega, \mathcal{F}, I\!F, P)$ with decomposition*

$$X = X_0 + M + A,$$

where $X_0(\omega_1, \omega_2) := \bar{X}_0(\omega_1)$, $M(\omega_1, \omega_2) := \bar{M}(\omega_1)$ and $A(\omega_1, \omega_2) := \bar{A}(\omega_1)$. In addition

$$A = \int d\langle M \rangle \lambda,$$

with $\lambda(\omega_1, \omega_2) := \bar{\lambda}(\omega_1)$.

Proof: We verify that M is a continuous $(I\!F, P)$-local martingale and that A is continuous, adapted and of finite variation. The latter is clear, since $\bar{A}$ is continuous and simply the difference between two increasing processes, and path properties are not affected by the embedding of the process in the product space. Similarly, M is continuous, since $\bar{M}$ is continuous.

By localization, we may assume that $\bar{M}$ is an $(\bar{I\!F}^1, P_1)$-martingale. We prove that M is then an $(I\!F, P)$-martingale by showing that for all $0 \le s \le t$ and $F \in \mathcal{F}_s = (\bar{\mathcal{F}}_s^1 \otimes \bar{\mathcal{F}}_s^2) \vee \mathcal{N}$:

$$\mathrm{E}_{P_1 \otimes P_2}[M_t 1_F] = \mathrm{E}_{P_1 \otimes P_2}[M_s 1_F]. \tag{5.7}$$

To verify this, recall that

$$(\bar{\mathcal{F}}_s^1 \otimes \bar{\mathcal{F}}_s^2) \vee \mathcal{N} = \left\{ F \in \mathcal{F} \,\middle|\, \exists \bar{F} \in \bar{\mathcal{F}}_s^1 \otimes \bar{\mathcal{F}}_s^2 : \ P(F \Delta \bar{F}) = 0 \right\},$$

where Δ is the symmetric difference, i.e. $F \Delta \bar{F} = (F \setminus \bar{F}) \cup (\bar{F} \setminus F)$. Thus, it is sufficient to verify (5.7) for F on the form $F = F_1 \times F_2$, where $F_1 \in \bar{\mathcal{F}}_s^1$ and $F_2 \in \bar{\mathcal{F}}_s^2$, and this follows by

$$\begin{aligned} &\mathrm{E}_{P_1 \otimes P_2}[1_F (M_t - M_s)] \\ &= \mathrm{E}_{P_1 \otimes P_2}[1_{F_1 \times \Omega_2} 1_{\Omega_1 \times F_2} (M_t - M_s)] \\ &= \int_{\Omega_2} 1_{F_2}(\omega_2) \left(\int_{\Omega_1} 1_{F_1}(\omega_1) (\bar{M}_t(\omega_1) - \bar{M}_s(\omega_1)) dP_1(\omega_1) \right) dP_2(\omega_2) \\ &= 0, \end{aligned}$$

where the last equality follows by noting that the inner integral is 0 by the $(\bar{\mathbb{F}}^1, P_1)$-martingale property for $\bar{M}$. $\Box$

The following simple lemma establishes the existence of an equivalent measure Q on $(\Omega, \mathcal{F})$ which satisfies the property (2.1) and which has a square-integrable density.

Lemma 5.7 *Assume that Assumption 5.3 is satisfied. Then* $\mathcal{D}^e(\mathbb{F}) \cap L^2(P) \neq \emptyset$.

Proof: By Assumption 5.3, there exists a measure $Q_1 \in \mathcal{M}^e(P_1, \bar{\mathbb{F}}^1)$ with $\frac{dQ_1}{dP_1} \in L^2(P_1)$. For this measure, we define a measure Q on $(\Omega, \mathcal{F})$ by $Q := Q_1 \otimes P_2$, and show that $Q \in \mathcal{M}^e(P, \mathbb{F})$ and $\frac{dQ}{dP} \in L^2(P)$. Since by definition $\frac{dQ}{dP}(\omega_1, \omega_2) = \frac{dQ_1}{dP_1}(\omega_1)$, we immediately obtain that

$$\mathrm{E}_P\left[\left(\frac{dQ}{dP}\right)^2\right] = \mathrm{E}_{P_1}\left[\left(\frac{dQ_1}{dP_1}\right)^2\right] < \infty.$$

We next use the fact that $Q \in \mathcal{M}^e(P, \mathbb{F})$ with $\frac{dQ}{dP} \in L^2(P)$ if and only if X is a local $(Q, \mathbb{F})$-martingale and $\frac{dQ}{dP} \in L^2(P)$; see e.g. the comments following the proof of Proposition 4.2 in Schweizer (1999). With this result, we only need to verify that Q is an equivalent local martingale measure for X, and this follows immediately by applying arguments similar to the ones used in the proof of Lemma 5.6, so that the local Q_1-martingale $\bar{X}$ on $(\Omega_1, \mathcal{F}_1)$ can be extended to a local Q-martingale X on $(\Omega, \mathcal{F})$. $\Box$

In particular, this result guarantees the existence of the variance optimal martingale measure $\tilde{P}$ for X. In fact we will prove below that $\tilde{P} = \tilde{P}_1 \otimes P_2$, which is intuitively reasonable. However, to obtain this result we first need to introduce the space of integrands on $(\Omega, \mathcal{F})$ and to establish a connection to the integrands on the space $(\Omega_1, \mathcal{F}_1)$.

Recall that $\tilde{\Theta}(\mathbb{F})$ is the space of $\mathbb{F}$-predictable processes ϑ on $(\Omega, \mathcal{F})$ such that $G_T(\vartheta) \in L^2(P)$ and $G(\vartheta)$ is a Q-martingale for each $Q \in \mathcal{M}^e(P, \mathbb{F})$ with $\frac{dQ}{dP} \in L^2(P)$. Define in addition the space $\tilde{\Theta}(\mathbb{F}^1)$ for the smaller filtration $\mathbb{F}^1$, and note that $\tilde{\Theta}(\mathbb{F}^1) \subseteq \tilde{\Theta}(\mathbb{F})$ by Remark 3.2. We then show that for any $\bar{\vartheta} \in \tilde{\Theta}(\bar{\mathbb{F}}^1)$, we can define a process $\vartheta \in \tilde{\Theta}(\mathbb{F}^1)$ by $\vartheta(\omega_1, \omega_2) := \bar{\vartheta}(\omega_1)$. This result gives a relation between the spaces $\tilde{\Theta}(\bar{\mathbb{F}}^1)$ and $\tilde{\Theta}(\mathbb{F}^1)$, which is very useful for our further analysis and will be used extensively throughout this section. Our argument uses the so-called local character of the stochastic integral, see Dellacherie and Meyer (1982).

Let $\bar{\vartheta} \in \tilde{\Theta}(\bar{\mathbb{F}}^1)$ and consider the semimartingale $\int \bar{\vartheta} d\bar{X}$ on $(\Omega_1, \mathcal{F}^1)$. By Lemma 5.6 we can extend this process to a semimartingale $G(\bar{\vartheta})$ on the product space $(\Omega, \mathcal{F})$ by

$$G(\bar{\vartheta})(\omega_1, \omega_2) := \left(\int \bar{\vartheta}\, d\bar{X}\right)(\omega_1).$$

Now let $\vartheta(\omega_1, \omega_2) := \bar{\vartheta}(\omega_1)$ and note that ϑ is $\mathbb{F}^1$-predictable, since $\bar{\vartheta}$ is $\bar{\mathbb{F}}^1$-predictable. Furthermore, since M and $\bar{M}$ are indistinguishable, Theorem VIII.23

of Dellacherie and Meyer (1982) implies that

$$(\int \vartheta \, dM)(\omega_1, \omega_2) = (\int \bar{\vartheta} \, d\bar{M})(\omega_1)$$

for P-a.a. (ω_1, ω_2). Similarly, $\int \vartheta dA = \int \bar{\vartheta} d\bar{A}$ P-a.s. since A and $\bar{A}$ are of finite variation and hence

$$(\int \vartheta \, dX)(\omega_1, \omega_2) = G(\bar{\vartheta})(\omega_1, \omega_2)$$

for P-a.a. (ω_1, ω_2).

Using Lemma 5.6 and the local character of the stochastic integral, we obtain

Lemma 5.8 *Assume that Assumption 5.3 is satisfied. For any $\bar{\vartheta} \in \tilde{\Theta}(\bar{\mathbb{F}}^1)$, the process ϑ defined by*

$$\vartheta(\omega_1, \omega_2) := \bar{\vartheta}(\omega_1) \tag{5.8}$$

is in $\tilde{\Theta}(\mathbb{F}^1) \subseteq \tilde{\Theta}(\mathbb{F})$.

Proof: Assume that $\bar{\vartheta} \in \tilde{\Theta}(\bar{\mathbb{F}}^1)$ and let ϑ be defined by (5.8). By the above arguments, ϑ is $\mathbb{F}^1$-predictable and $G(\vartheta)$ is well-defined. Furthermore, it follows immediately that $G_T(\vartheta) \in L^2(P)$, since $G_T(\bar{\vartheta}) \in L^2(P_1)$.

We consider $Q \in \mathcal{M}^e(P, \mathbb{F}^1)$ with $\frac{dQ}{dP} \in L^2(P)$ and verify that $G(\vartheta)$ is indeed a Q-martingale, so that $\vartheta \in \tilde{\Theta}(\mathbb{F}^1)$. Define first a measure Q_1 on $(\Omega_1, \mathcal{F}_1)$ by $Q_1(A_1) := Q(A_1 \times \Omega_2)$ for $A_1 \in \mathcal{F}_1$, that is

$$\begin{aligned} Q_1(A_1) &= \int_{A_1} \frac{dQ_1}{dP_1} dP_1 := Q(A_1 \times \Omega_2) \\ &= \int_{A_1 \times \Omega_2} \frac{dQ}{dP} dP_2 dP_1 = \int_{A_1} \left(\int_{\Omega_2} \frac{dQ}{dP} dP_2 \right) dP_1. \end{aligned}$$

By Cauchy-Schwarz, $\frac{dQ_1}{dP_1} \in L^2(P_1)$. Furthermore, we claim that $Q_1 \in \mathcal{M}^e(\bar{\mathbb{F}}^1)$, that is

$$\mathrm{E}_{P_1}\left[\frac{dQ_1}{dP_1} \bar{h}^{tr} (\bar{X}_{S_2} - \bar{X}_{S_1}) \right] = 0,$$

for any bounded $\bar{\mathbb{F}}^1$-stopping times $S_1 \leq S_2 \leq T$ such that $\bar{X}^{S_2}$ is bounded and for any $\bar{\mathcal{F}}^1_{S_1}$-measurable bounded $\mathbb{R}^d$-valued random variable $\bar{h}$. To see this, define bounded $\mathbb{F}^1$-stopping times $T_i(\omega_1, \omega_2) := S_i(\omega_1)$, $i = 1, 2$, and a bounded $\mathcal{F}^1_{T_1}$-measurable random variable $h(\omega_1, \omega_2) := \bar{h}(\omega_1)$. Now since by assumption $Q \in \mathcal{M}^e(P, \mathbb{F}^1)$,

$$\mathrm{E}_P\left[\frac{dQ}{dP} h^{tr} (X_{T_2} - X_{T_1}) \right] = 0. \tag{5.9}$$

The claim now follows from the following equalities

$$\begin{aligned}\mathrm{E}_P\left[\frac{dQ}{dP}h^{tr}(X_{T_2}-X_{T_1})\right] &= \mathrm{E}_P\left[\frac{dQ}{dP}\bar{h}^{tr}(\bar{X}_{S_2}-\bar{X}_{S_1})\right] \\ &= \mathrm{E}_{P_1}\left[\frac{dQ_1}{dP_1}\bar{h}^{tr}(\bar{X}_{S_2}-\bar{X}_{S_1})\right],\end{aligned}$$

where the first equality follows from the definition of h and T_i, and the second equality is a consequence of the definition of Q_1.

We finally verify that $G(\vartheta)$ is a $(Q,\mathbb{F}^1)$-martingale, that is for all $0 \le s < t \le T$ and all $A \in \mathcal{F}^1_s$:

$$\mathrm{E}_Q\left[1_A\left(G_t(\vartheta)-G_s(\vartheta)\right)\right] = 0. \tag{5.10}$$

By definition of $\mathcal{F}^1_s$ it is sufficient to consider sets of the form $A = A_1 \times \Omega_2$, where $A_1 \in \bar{\mathcal{F}}^1_s$, and (5.10) now follows again by switching between $G(\vartheta)$ and $G(\bar{\vartheta})$ and exploiting that $G(\bar{\vartheta})$ is a Q_1-martingale:

$$\begin{aligned}\mathrm{E}_Q\left[1_{A_1\times\Omega_2}(G_t(\vartheta)-G_s(\vartheta))\right] &= \mathrm{E}_P\left[\frac{dQ}{dP}1_{A_1}(\omega_1)1_{\Omega_2}(\omega_2)(G_t(\bar{\vartheta})-G_s(\bar{\vartheta}))\right] \\ &= \mathrm{E}_{P_1}\left[\frac{dQ_1}{dP_1}1_{A_1}(\omega_1)(G_t(\bar{\vartheta})-G_s(\bar{\vartheta}))\right] \\ &= 0.\end{aligned}$$

This ends the proof. □

The following result deals with the structures of the minimal and the variance optimal martingale measures for X.

Proposition 5.9 *Let $\hat{P}_1$ and $\tilde{P}_1$ denote the minimal and the variance optimal martingale measures for $\bar{X}$.*

1. *The minimal martingale measure $\hat{P}$ for X is given by $\hat{P} = \hat{P}_1 \otimes P_2$.*
2. *The variance optimal martingale measure $\tilde{P}$ for X is given by $\tilde{P} = \tilde{P}_1 \otimes P_2$.*

Proof: By definition, the minimal martingale measure for $\bar{X}$ has density on $\mathcal{F}^1$

$$\frac{d\hat{P}_1}{dP_1} = \mathcal{E}\left(-\int \bar{\lambda}\, d\bar{M}\right)_T,$$

and similarly, the minimal martingale measure for X can be written as

$$\begin{aligned}\frac{d\hat{P}}{dP} &= \mathcal{E}\left(-\int \lambda\, dM\right)_T = \mathcal{E}\left(-\int \bar{\lambda}\, d\bar{M}\right)_T \cdot 1 = \frac{d\hat{P}_1}{dP_1}\cdot 1 \\ &= \frac{d\hat{P}_1}{dP_1}\cdot\frac{dP_2}{dP_2} = \frac{d\left(\hat{P}_1\otimes P_2\right)}{dP},\end{aligned}$$

where the second equality follows by using the local character of the stochastic integral. Now for any $F = F_1 \times F_2$ with $F_1 \in \mathcal{F}^1$ and $F_2 \in \mathcal{F}^2$, we find that

$$\begin{aligned} \hat{P}(F) &= \int_F \frac{d\hat{P}}{dP} dP = \int_{F_2} \left(\int_{F_1} \frac{d\hat{P}_1}{dP_1} \cdot 1 \, dP_1 \right) dP_2 \\ &= \int_{F_2} \left(\int_{F_1} d\hat{P}_1 \right) dP_2 = \hat{P}_1(F_1) P_2(F_2). \end{aligned}$$

This proves that the measures $\hat{P}$ and $\hat{P}_1 \otimes P_2$ coincide on $\mathcal{F}^1 \otimes \mathcal{F}^2$, and hence they will also coincide on the completed σ-field $\mathcal{F}$.

The density on $\mathcal{F}^1$ for the variance optimal martingale measure $\tilde{P}_1$ for $\bar{X}$ admits the representation

$$\frac{d\tilde{P}_1}{dP_1} = c + \int_0^T \bar{\zeta}_t \, d\bar{X}_t,$$

for some $c \in (0, \infty)$ and $\bar{\zeta} \in \tilde{\Theta}(\bar{\mathbb{F}}^1)$. Now define a measure P^* on $(\Omega, \mathcal{F})$ by $P^* = \tilde{P}_1 \otimes P_2$. By calculations similar to those in the proof of Lemma 5.6, it is seen that any $(\bar{\mathbb{F}}^1, \tilde{P}_1)$-martingale on $(\Omega_1, \mathcal{F}^1)$ can be extended to an $(\mathbb{F}, P^*)$-martingale on $(\Omega_1, \mathcal{F}^1)$. And so, P^* is a martingale measure for X, see the proof of Lemma 5.7. Furthermore, the density for the measure P^* can be written as

$$\frac{dP^*}{dP} = \frac{d\tilde{P}_1}{dP_1} 1 = c + \int_0^T \bar{\zeta}_t \, d\bar{X}_t = c + \int_0^T \zeta_t \, dX_t \ P\text{-a.s.}, \tag{5.11}$$

where $\zeta(\omega_1, \omega_2) = \bar{\zeta}(\omega_1)$, and where we have used the local character of the stochastic integral. Furthermore, since $\bar{\zeta} \in \tilde{\Theta}(\bar{\mathbb{F}}^1)$ Lemma 5.8 implies that also $\zeta \in \tilde{\Theta}(\mathbb{F})$ and hence by Schweizer (1996, Lemma 1), we have that $P^* = \tilde{P}$. □

As a corollary to Proposition 5.9 we have the following:

Corollary 5.10 *Assume that the martingale measure for $(\bar{X}, \bar{\mathbb{F}}^1)$ is unique. Then $\tilde{P} = \hat{P}$.*

Proof: Denote by P_1^* the unique martingale measure for $(\bar{X}, \bar{\mathbb{F}}^1)$. Then $\hat{P}_1 = \tilde{P}_1 = P_1^*$, and hence, by Proposition 5.9, $\hat{P} = \tilde{P}$. □

As another consequence of Proposition 5.9 we see that due to the independence between the two sources of risk the minimal martingale measure and the variance optimal martingale measure are not affected by the choice of filtration $\bar{\mathbb{F}}^2$ on the space $(\Omega_2, \mathcal{F}^2, P_2)$ for the additional risk, that is, the density does not depend on the filtration $\bar{\mathbb{F}}^2$. This allows us to vary $\bar{\mathbb{F}}^2$ without affecting these two martingale measures, and in particular we see that the variance optimal martingale measures under $\mathbb{F}^1$ and $\mathbb{F}$ coincide.

The next result states that completeness for $(\bar{X}, \bar{\mathbb{F}}^1)$ is preserved when $\bar{X}$ and $\bar{\mathbb{F}}^1$ are embedded in the product space.

Lemma 5.11 *Assume that $(\bar{X}, \bar{\mathbb{F}}^1)$ is complete. Then $(X, \mathbb{F}^1)$ is complete in the sense that for all $H \in L^2(P, \mathcal{F}_T^1)$ there exists a process $\vartheta^H \in \tilde{\Theta}(\mathbb{F}^1)$ and $H_0 \in \mathbb{R}$ so that*

$$H = H_0 + \int_0^T \vartheta_t^H dX_t.$$

Proof: Let $H \in L^2(P, \mathcal{F}_T^1)$ and introduce the random variables

$$\begin{aligned} H' &= \mathrm{E}[H \mid \bar{\mathcal{F}}_T^1 \otimes \{\emptyset, \Omega_2\}], \\ H'' &= \int_{\Omega_2} H'(\omega_1, \omega_2) dP_2(\omega_2). \end{aligned}$$

We show that $H = H' = H''$ P-a.s. and then use that H'' does not depend on ω_2, so that $H'' \in L^2(P, \bar{\mathcal{F}}_T^1)$. Hence, H'' admits a decomposition as a constant plus a stochastic integral with respect to $\bar{X}$.

We first verify that $H = H'$ P-a.s.: Since $\mathcal{F}_T^1 = (\bar{\mathcal{F}}_T^1 \otimes \{\emptyset, \Omega_2\}) \vee \mathcal{N}$, we have by the $\mathcal{F}_T^1$-measurability of H and the definition of H' that

$$H = \mathrm{E}[H | (\bar{\mathcal{F}}_T^1 \otimes \{\emptyset, \Omega_2\}) \vee \mathcal{N}] = \mathrm{E}[H | \bar{\mathcal{F}}_T^1 \otimes \{\emptyset, \Omega_2\}] = H' \quad P\text{-a.s.}$$

To see that $H' = H''$ P-a.s., note that by definition H' is $\bar{\mathcal{F}}_T^1 \otimes \{\emptyset, \Omega_2\}$-measurable. Thus for fixed $\omega_1 \in \Omega_1$, $H'(\omega_1, .)$ is $\{\emptyset, \Omega_2\}$-measurable, that is, constant. In particular, this implies that $H' = \int_{\Omega_2} H' dP_2$, so that $H' = H''$.

Since $H'' \in L^2(P_1, \bar{\mathcal{F}}_T^1)$, the completeness assumption implies that there exist a constant H_0'' and $\bar{\vartheta}^H \in \tilde{\Theta}(\bar{\mathbb{F}}^1)$ so that

$$H'' = H_0'' + \int_0^T \bar{\vartheta}_t^H d\bar{X}_t.$$

Define $H_0 := H_0''$ and $\vartheta^H(\omega_1, \omega_2) := \bar{\vartheta}^H(\omega_1)$. Since $H = H''$ P-a.s., we now find that

$$H = H'' = H_0'' + \int_0^T \bar{\vartheta}_t^H d\bar{X}_t = H_0 + \int_0^T \vartheta_t^H dX_t \quad P\text{-a.s.},$$

where the last equality is a consequence of the local character of the stochastic integral. Since $\vartheta^H \in \tilde{\Theta}(\mathbb{F})$ this ends the proof. □

The following result will play an important role in the derivation of upper bounds for the fair premium of insurance contracts. This bound will correspond to the situation where the hedger receives no information about the additional source of risk before the terminal time T.

Proposition 5.12 *Let P_1^* be a martingale measure for $(\bar{X}, \bar{\mathbb{F}}^1)$, and define a measure P^* by $P^* = P_1^* \otimes P_2$. Then for any $H \in L^1(P, \mathcal{F}_T) \cap L^1(P^*, \mathcal{F}_T)$:*

$$\mathrm{E}_{P^*}\left[H \,\middle|\, \mathcal{F}_T^1\right] = \mathrm{E}_P\left[H \,\middle|\, \mathcal{F}_T^1\right] \quad \text{a.s.} \tag{5.12}$$

This result states that the two conditional distributions $P_{|\mathcal{F}_T^1}$ and $P^*_{|\mathcal{F}_T^1}$ coincide, since for the random variables $H = 1_F$, $F \in \mathcal{F}$, (5.12) specializes to

$$P^*(F \mid \mathcal{F}_T^1) = P(F \mid \mathcal{F}_T^1).$$

This property is intuitively reasonable, since the two measures $P = P_1 \otimes P_2$ and $P^* = P_1^* \otimes P_2$ are product measures, which differ only by the choice of measure on the space $(\Omega_1, \mathcal{F}^1)$. Thus, if we condition on $\mathcal{F}_T^1$ (the financial uncertainty), then (basically) only the additional uncertainty remains, and this additional source of risk is not affected by the change of measure from P to P^*.

Proof of Proposition 5.12: We verify (5.12) by a monotone class argument; consider the multiplicative class

$$\mathcal{M} = \left\{ H^{(1)}H^{(2)} \,\middle|\, H^{(i)} \text{ is } \mathcal{F}_T^i\text{-measurable and bounded, } i = 1,2 \right\}.$$

Clearly, $\mathcal{M}$ is multiplicative, since for any $H, \tilde{H} \in \mathcal{M}$, we have that

$$H\tilde{H} \;=\; \left(H^{(1)}H^{(2)}\right)\left(\tilde{H}^1\tilde{H}^2\right) = \left(H^{(1)}\tilde{H}^1\right)\left(H^2\tilde{H}^2\right) \in \mathcal{M}.$$

Now, let $\mathcal{H}$ be the space of bounded random variables satisfying (5.12), that is

$$\mathcal{H} = \{ H \mid H \text{ is bounded, } \mathcal{F}_T\text{-measurable, and satisfying (5.12)} \}$$

It is clear that $\mathcal{H}$ is a vector space over $I\!R$ and that $1_\Omega \in \mathcal{H}$. Furthermore, $\mathcal{H}$ is closed under monotone convergence for bounded elements in the sense that for $H_n \in \mathcal{H}$ with $0 \le H_1 \le H_2 \le ...$, $H_n \to H$ where also H is bounded, we have that $H \in \mathcal{H}$. To see this, use monotone convergence twice:

$$\begin{aligned}
\mathrm{E}_P\left[\lim_{n\to\infty} H_n \,\middle|\, \mathcal{F}_T^1\right] &= \lim_{n\to\infty} \mathrm{E}_P\left[H_n \,\middle|\, \mathcal{F}_T^1\right] = \lim_{n\to\infty} \mathrm{E}_{P^*}\left[H_n \,\middle|\, \mathcal{F}_T^1\right] \\
&= \mathrm{E}_{P^*}\left[\lim_{n\to\infty} H_n \,\middle|\, \mathcal{F}_T^1\right],
\end{aligned}$$

where the second equality follows from (5.12). Furthermore, we see that $\mathcal{M} \subseteq \mathcal{H}$, since for any $H = H^{(1)}H^{(2)} \in \mathcal{M}$

$$\begin{aligned}
\mathrm{E}_P\left[H \,\middle|\, \mathcal{F}_T^1\right] &= \mathrm{E}_P\left[H^{(1)}H^{(2)} \,\middle|\, \mathcal{F}_T^1\right] = H^{(1)}\mathrm{E}_P\left[H^{(2)} \,\middle|\, \mathcal{F}_T^1\right] \\
&= H^{(1)}\mathrm{E}_P\left[H^{(2)}\right] = H^{(1)} \int_{\Omega_1}\int_{\Omega_2} H^{(2)} dP_2 dP_1 = H^{(1)}\mathrm{E}_{P_2}\left[H^{(2)}\right].
\end{aligned}$$

where we have used that $H^{(1)}$ is $\mathcal{F}_T^1$-measurable in the second equality; the third equality is a consequence of the independence between $\mathcal{F}_T^1$ and $\mathcal{F}_T^2$. Similarly, we obtain for $P^* = P_1^* \otimes P_2$:

$$\begin{aligned}
\mathrm{E}_{P^*}\left[H \,\middle|\, \mathcal{F}_T^1\right] &= H^{(1)}\mathrm{E}_{P^*}\left[H^{(2)}\right] = H^{(1)} \int_{\Omega_1}\int_{\Omega_2} H^{(2)} dP_2 dP_1^* \\
&= H^{(1)}\left(\int_{\Omega_1} dP_1^*\right)\left(\int_{\Omega_2} H^{(2)} dP_2\right) = H^{(1)}\mathrm{E}_{P_2}\left[H^{(2)}\right].
\end{aligned}$$

Now the monotone class theorem gives that $\mathcal{G} \subseteq \mathcal{H}$, where $\mathcal{G}$ is the set of all bounded random variables that are measurable with respect to the σ-algebra

$$\mathcal{T} = \sigma\left\{(H \in B)\,|B \in \mathcal{B}(I\!R),\ H \in \mathcal{M}\right\}.$$

Since, by definition, $\mathcal{F}_T^1 = (\bar{\mathcal{F}}_T^1 \otimes \{\emptyset, \Omega_2\}) \vee \mathcal{N}$ and $\mathcal{F}_T^2 = (\{\emptyset, \Omega_1\} \otimes \bar{\mathcal{F}}_T^2) \vee \mathcal{N}$, we find that $\mathcal{F}_T = \mathcal{F}_T^1 \vee \mathcal{F}_T^2$. Furthermore, since $\mathcal{F}_T^i \subseteq \mathcal{T}$, $i = 1, 2$, we conclude that $\mathcal{H}$ contains all bounded $\mathcal{F}_T$-measurable random variables.

For general (non-bounded) $H \in L^1(P, \mathcal{F}_T) \cap L^1(P^*, \mathcal{F}_T)$, define $H_m = (H \wedge m) \vee (-m)$, and use dominated convergence to verify that H satisfies (5.12). This follows by noting that the sequence H_n is dominated by $|H|$ and hence, by dominated convergence:

$$\lim_{m\to\infty} \mathrm{E}_P\left[H_m \left|\mathcal{F}_T^1\right.\right] = \mathrm{E}_P\left[\lim_{m\to\infty} H_m \left|\mathcal{F}_T^1\right.\right]. \ \square$$

In the rest of this section we consider the problem of determining the decomposition (2.7) for some special choices of filtrations on the space $(\Omega_2, \mathcal{F}^2)$. In particular, we investigate the two "extremes" where (a) the filtration for the insurance risk is trivial and (b) all information concerning the insurance risk is revealed at time 0.

The trivial case

We consider briefly the situation, where the filtration $\bar{I\!F}^2 = (\bar{\mathcal{F}}_t^2)_{0\le t\le T}$ is given by

$$\bar{\mathcal{F}}_t^2 = \begin{cases} \{\emptyset, \Omega_2\}, & t < T, \\ \mathcal{F}^2, & t = T. \end{cases} \tag{5.13}$$

Remark 5.13 In this case, the filtration $I\!F$ is given by

$$\mathcal{F}_t = \begin{cases} \left(\bar{\mathcal{F}}_t^1 \otimes \{\emptyset, \Omega_2\}\right) \vee \mathcal{N}, & t < T, \\ (\mathcal{F}^1 \otimes \mathcal{F}^2) \vee \mathcal{N}, & t = T, \end{cases}$$

and hence $\mathcal{F}_t = \mathcal{F}_t^1$ for $t < T$. Thus, Remark 3.2 shows that $\mathcal{P}(I\!F) = \mathcal{P}(I\!F^1)$. $\square$

As mentioned above, (5.13) will correspond to the situation where the reinsurer receives no information concerning the additional risk during the time interval $[0, T)$. Under this assumption, we have the following result for the decomposition (2.7):

Theorem 5.14 *Assume that $\bar{I\!F}^2$ is given by (5.13) and that the model $(\bar{X}, \bar{I\!F}^1)$ is complete. Then, for any $H \in L^2(P, \mathcal{F}_T)$, the unique decomposition (2.7) is given by*

$$N^H = H - \mathrm{E}_{\tilde{P}}\left[H \left|\mathcal{F}_T^1\right.\right] = H - \mathrm{E}_P\left[H \left|\mathcal{F}_T^1\right.\right], \tag{5.14}$$

and ϑ^H is determined such that

$$\mathrm{E}_{\tilde{P}}\left[H \left|\mathcal{F}_T^1\right.\right] = H_0 + \int_0^T \vartheta_t^H dX_t, \tag{5.15}$$

where H_0 is some constant.

Thus, in the situation where $(\bar{X}, \bar{\mathbb{F}}^1)$ is complete and no information concerning the risk from the space $(\Omega_2, \bar{\mathcal{F}}^2)$ is available before time T, the projection of H on the space $\mathbb{R} + G_T(\tilde{\Theta}(\mathbb{F}))$ is exactly equal to (the projection on $\mathbb{R} + G_T(\tilde{\Theta}(\mathbb{F}))$ of) $\mathrm{E}_{\tilde{P}}[H|\mathcal{F}_T^1]$, which is an $\mathcal{F}_T^1$-measurable square-integrable random variable and hence an attainable claim by the completeness assumption.

Proof of Theorem 5.14: The second equality in (5.14) is a direct consequence of Proposition 5.12. Note that since $(\bar{X}, \bar{\mathbb{F}}^1)$ is complete, $(X, \mathbb{F}^1)$ is complete by Lemma 5.11, and hence the $\mathcal{F}_T^1$-measurable random variable $\mathrm{E}_{\tilde{P}}[H\,|\mathcal{F}_T^1]$ can be uniquely represented as a constant plus a stochastic integral $\int_0^T \vartheta^H dX$, where $\vartheta^H \in \tilde{\Theta}(\mathbb{F}^1) \subseteq \tilde{\Theta}(\mathbb{F})$. Also note that by Proposition 5.12, we have that

$$\mathrm{E}_P\left[N^H\left|\mathcal{F}_T^1\right.\right] \;=\; \mathrm{E}_P\left[\left(H - \mathrm{E}_{\tilde{P}}\left[H\left|\mathcal{F}_T^1\right.\right]\right)\left|\mathcal{F}_T^1\right.\right] = \mathrm{E}_P\left[\left(H - \mathrm{E}_P\left[H\left|\mathcal{F}_T^1\right.\right]\right)\left|\mathcal{F}_T^1\right.\right] = 0,$$

so that $\mathrm{E}[N^H] = 0$. Hence, we only need to verify that

$$\mathrm{E}_P\left[\int_0^T \vartheta\, dX \left(H - \mathrm{E}_P\left[H\left|\mathcal{F}_T^1\right.\right]\right)\right] = 0, \tag{5.16}$$

for all $\vartheta \in \tilde{\Theta}(\mathbb{F})$. This follows immediately for $\vartheta \in \tilde{\Theta}(\mathbb{F}^1)$, since $\int_0^T \vartheta\, dX$ is then $\mathcal{F}_T^1$-measurable. However, by Remark 5.13 above, $\mathcal{P}(\mathbb{F}) = \mathcal{P}(\mathbb{F}^1)$, so that $\int_0^T \vartheta\, dX$ is actually $\mathcal{F}_T^1$-measurable for any $\vartheta \in \tilde{\Theta}(\mathbb{F})$, and hence the left side of (5.16) equals

$$\mathrm{E}_P\left[\left(\int_0^T \vartheta\, dX\right) H - \mathrm{E}_P\left[\left(\int_0^T \vartheta\, dX\right) H\right| \mathcal{F}_T^1\right]\right] = 0,$$

which completes the proof. □

The general case

In the general case, the decomposition (2.7) is not determined by Theorem 5.14. Even if $(\bar{X}, \bar{\mathbb{F}}^1)$ is complete, the decompositions may not be on this form. This can be seen from the following example.

Example 5.15 Let $\mathbb{F}^1 = \mathbb{F}^{W^{(1)}}$ and $\mathbb{F}^2 = \mathbb{F}^{W^{(2)}}$, where $W^{(1)}$ and $W^{(2)}$ are independent standard Brownian motions on $(\Omega, \mathcal{F})$, and assume that $X = W^{(1)}$. Consider H on the form

$$H = \int_0^T \vartheta_t^{(2)} dW_t^{(1)} + \int_0^T \vartheta_t^{(1)} dW_t^{(2)},$$

where $\vartheta^{(i)}$ is $\mathbb{F}^i$-predictable, bounded and simple. The solution (5.14)–(5.15) of Theorem 5.14 is

$$\begin{aligned} N^H &= H - \mathrm{E}_{\tilde{P}}\left[H\left|\mathcal{F}_T^1\right.\right] = H - \mathrm{E}_P\left[H\left|\mathcal{F}_T^1\right.\right] \\ &= \int_0^T (\vartheta_t^{(2)} - \mathrm{E}[\vartheta_t^{(2)}])dW_t^1 + \int_0^T \vartheta_t^{(1)} dW_t^{(2)}. \end{aligned}$$

Now let $\vartheta^* = \vartheta^{(2)} - \mathrm{E}[\vartheta^{(2)}]$. Then, ϑ^* is $\mathbb{F}$-predictable and bounded, hence $\vartheta^* \in \tilde{\Theta}(\mathbb{F})$, but

$$\begin{aligned}\mathrm{E}\left[\int_0^T \vartheta_t^*\, dW_t^{(1)}\, N^H\right] &= \mathrm{E}\left[\int_0^T \vartheta_t^*\, dW_t^{(1)} \left(\int_0^T \vartheta_t^*\, dX_t + \int_0^T \vartheta_t^{(1)} dW_t^{(2)}\right)\right] \\ &= \mathrm{E}\left[\int_0^T (\vartheta_t^*)^2 dt\right],\end{aligned}$$

which differs from 0 if $\vartheta^{(2)}$ is not constant $P \otimes \lambda_{[0,T]}$-a.s., where $\lambda_{[0,T]}$ denotes the Lebesgue measure on $[0,T]$. □

The "complete" case
We get another extreme case, when $\bar{\mathbb{F}}^2$ is defined by

$$\bar{\mathcal{F}}_t^2 = \mathcal{F}^2,\ 0 \leq t \leq T, \tag{5.17}$$

which implies that $\mathcal{F}_t = \mathcal{F}_t^1 \vee \mathcal{F}_T^2$. Thus, this is the case where all information about the additional risk is revealed at time 0, that is, the filtration $\mathbb{F}$ is constructed by an initial enlargement of $\mathbb{F}^1$. Under this assumption we can use results on initially enlarged filtrations to show the following result:

Theorem 5.16 *Assume that $\bar{\mathbb{F}}^2$ is given by (5.17) and that the model $(\bar{X}, \bar{\mathbb{F}}^1)$ is complete. Then any $H \in L^2(P, \mathcal{F}_T)$ admits a unique decomposition*

$$H = \tilde{H}_0 + \int_0^T \vartheta_t^H dX_t, \tag{5.18}$$

where $\tilde{H}_0 \in L^2(P, \mathcal{F}_0)$ and $\vartheta \in \tilde{\Theta}(\mathbb{F})$.

Proof: By Lemma 5.11, $(X, \mathbb{F}^1)$ is complete. Since $\mathcal{F}_T^1$ and $\mathcal{F}_T^2$ are independent, Theorem 3.9 of Amendinger and Becherer (2000) now shows that the model with the initially enlarged filtration is also complete. □

Remark 5.17 It is noted that Theorem 5.16 is not really a statement about completeness in the usual sense since H_0 is allowed to be $\mathcal{F}_0$-measurable and hence is not necessarily constant. □

6 Simple bounds for the fair premium

The results obtained in the previous sections can be used to derive simple bounds for the fair premiums under the financial valuation principles reviewed in Section 4. We apply the product space set-up from the previous section and keep the filtration $\bar{\mathbb{F}}^1$ from the financial market fixed. Throughout this section, we work under the assumption that the financial market is complete, that is, the model consisting of $(\Omega_1, \mathcal{F}^1, \bar{\mathbb{F}}^1, P_1)$ and $\bar{X}$ is complete. In Section 3 it was shown that the hedging error for two different filtrations $\mathbb{F}^\circ \subseteq \mathbb{F}$ satisfies the inequality

$\mathrm{E}[(N^{H,\circ})^2] = J_0(\mathbb{F}^\circ, c^{H,\circ}) \geq J_0(\mathbb{F}, c^H) = \mathrm{E}[(N^H)^2]$, where $N^{H,\circ}$, N^H, $c^{H,\circ}$ and c^H are the quantities given in (2.7). In general, $c^H \neq c^{H,\circ}$. However, in the present product space framework they coincide, when the two filtrations differ by the choice of filtration for the insurance risk; see the remark following the proof of Corollary 5.10. Upper and lower bounds for the premiums arising from all possible choices of filtration on the space $(\Omega_2, \mathcal{F}^2)$ for the insurance risk are now determined by introducing a *minimal* and a *maximal* filtration on this space. The upper bound for the fair premium is obtained for the minimal filtration, which corresponds to the situation where the seller of the contract does not receive any information about the insurance risk; the lower bound is obtained in the case where all information about the insurance risk is revealed immediately after the the contract is sold.

It is clear that for any filtration $\bar{\mathbb{F}}^2$ on $(\Omega_2, \mathcal{F}^2)$ such that $\bar{\mathcal{F}}^2_T = \mathcal{F}^2$, we have that $\bar{\mathbb{G}}^\circ \subseteq \bar{\mathbb{F}}^2 \subseteq \bar{\mathbb{G}}$ where $\bar{\mathbb{G}}^\circ$ is the trivial filtration given by

$$\bar{\mathcal{G}}^\circ_t = \begin{cases} \{\emptyset, \Omega_2\}, & t < T, \\ \mathcal{F}^2, & t = T. \end{cases}$$

and where the filtration $\bar{\mathbb{G}}$ is defined by $\bar{\mathcal{G}}_t = \mathcal{F}^2$, for all $t \in [0,T]$. Here, $\bar{\mathbb{G}}^\circ$ is the minimal filtration on $(\Omega_2, \mathcal{F}^2)$ which satisfies the condition $\bar{\mathcal{G}}^\circ_T = \mathcal{F}^2$; $\bar{\mathbb{G}}$ is the maximal filtration on $(\Omega_2, \mathcal{F}^2)$ with $\bar{\mathcal{G}}_T = \mathcal{F}^2$. Introduce also the corresponding complete and right-continuous filtrations $\mathbb{G}^\circ := \bar{\mathbb{F}}^1 \otimes \bar{\mathbb{G}}^\circ$ and $\mathbb{G} := \bar{\mathbb{F}}^1 \otimes \bar{\mathbb{G}}$, see Remark 5.5. By the results presented in Section 3 we hence find that the upper bound for the financial variance principle is obtained for the minimal filtration $\mathbb{G}^\circ$ whereas the lower bound is obtained for the maximal filtration $\mathbb{G}$. Using the result in Theorem 5.14 and the formula for conditioning for variances, we get the upper bound for the fair premium under the variance principle:

$$\begin{aligned} v_{1,max}(H) &= \tilde{\mathrm{E}}[H] + a\mathrm{Var}[H - \mathrm{E}[H \mid \mathcal{F}^1_T]] \\ &= \tilde{\mathrm{E}}[H] + a\mathrm{E}[\mathrm{Var}[H \mid \mathcal{F}^1_T]]. \end{aligned} \tag{6.1}$$

For the lower bound, note that by Theorem 5.16 $(X, \mathbb{G})$ is "complete" and hence any $H \in L^2(P, \mathcal{F}_T)$ admits the unique decomposition

$$H = \tilde{H}_0 + \int_0^T \vartheta^H_t dX_t,$$

where $\tilde{H}_0 = \mathrm{E}[H \mid \mathcal{G}_0]$. This can be rewritten as

$$H = c^H + \int_0^T \vartheta^H_t dX_t + L^{H,\tilde{P}}_T,$$

where $c^H = \tilde{\mathrm{E}}[H]$ and $L^{H,\tilde{P}}_t = \tilde{\mathrm{E}}[H \mid \mathcal{G}_0] - \tilde{\mathrm{E}}[H]$, $t \in [0,T]$. Thus, by Corollary 2.8 applied for non-trivial initial σ-algebra $\mathcal{G}_0$ (see also Pham, Rheinländer and Schweizer (1998, Corollary 9)), we find that

$$J_0(\mathbb{G}) := J_0(\mathbb{G}, c^H) = \mathrm{E}[(N^{H,\mathbb{G}})^2] = \frac{\mathrm{E}[(L^{H,\tilde{P}}_0)^2]}{\mathrm{E}[\tilde{Z}^2_T]}. \tag{6.2}$$

Furthermore, by the independence between $\tilde{Z}_T$ and $\mathcal{G}_0$ we find that

$$\tilde{\mathrm{E}}[H] = \tilde{\mathrm{E}}[\tilde{\mathrm{E}}[H \mid \mathcal{G}_0]] = \mathrm{E}[\tilde{\mathrm{E}}[H \mid \mathcal{G}_0]],$$

hence

$$\mathrm{E}[(L_0^{H,\tilde{P}})^2] = \mathrm{E}[(\tilde{\mathrm{E}}[H \mid \mathcal{G}_0] - \tilde{\mathrm{E}}[H])^2] = \mathrm{Var}[\tilde{\mathrm{E}}[H \mid \mathcal{G}_0]].$$

This shows that the lower bound for the fair premium under the variance principle is

$$v_{1,min}(H) = \tilde{\mathrm{E}}[H] + a\frac{\mathrm{Var}[\tilde{\mathrm{E}}[H \mid \mathcal{G}_0]]}{\mathrm{E}[\tilde{Z}_T^2]}. \tag{6.3}$$

Note that $\mathcal{G}_0 = (\mathcal{N}^1 \otimes \mathcal{F}^2) \vee \mathcal{N}$, so that the loading is related to the variance of the term $\tilde{\mathrm{E}}[H \mid \mathcal{N}^1 \otimes \mathcal{F}^2]$, which can be interpreted as the price for H given the complete information about the future development of the insurance portfolio.

Remark 6.1 We briefly compare the premium calculation principle (6.3) with the approach proposed by Brennan and Schwartz (1976) for valuation of unit-linked contracts; see e.g. Section 5 in Chapter 2 for a detailed description of their approach. Using Proposition 5.12, we first obtain that

$$\tilde{\mathrm{E}}[H] = \mathrm{E}[\tilde{\mathrm{E}}[H \mid \mathcal{G}_0]].$$

Thus, (6.3) can be rewritten as

$$v_{1,min}(H) = \mathrm{E}[\tilde{\mathrm{E}}[H \mid \mathcal{G}_0]] + \frac{a}{\mathrm{E}[\tilde{Z}_T^2]}\mathrm{Var}[\tilde{\mathrm{E}}[H \mid \mathcal{G}_0]], \tag{6.4}$$

and this result can be given the following interpretation: The lower bound for the fair premium is obtained by first computing $\tilde{\mathrm{E}}[H \mid \mathcal{G}_0]$, which is the unique market price of H given the complete information about the future development of the insurance risk, and then using the traditional variance principle with loading parameter $a_{min} := a/(\mathrm{E}[\tilde{Z}_T^2]) \le a$ on $\tilde{\mathrm{E}}[H \mid \mathcal{G}_0]$. Thus, (6.4) is a relatively simple combination of pricing by no-arbitrage and the actuarial variance principle.

The valuation principle of Brennan and Schwarz (1976) consists in first replacing the claim H by $H' = \mathrm{E}[H \mid \mathcal{F}_T^1]$ and then pricing this attainable claim H' by no-arbitrage arguments. This procedure neglects the insurance risk and is based on the assumption that the insurance risk is diversifiable, i.e. it can be eliminated, for example by increasing the size of the insurance portfolio. In particular, this approach is widely used for the pricing of unit-linked life insurance contracts, where the number of policy-holders is typically large. □

For the standard deviation principle, the corresponding bounds are

$$v_{2,max}(H) = \tilde{\mathrm{E}}[H] + a\sqrt{1 - \frac{\mathrm{Var}[\tilde{Z}_T]}{a^2}}\sqrt{\mathrm{E}[\mathrm{Var}[H \mid \mathcal{F}_T^1]]}, \tag{6.5}$$

$$v_{2,min}(H) = \tilde{\mathrm{E}}[H] + a\sqrt{1 - \frac{\mathrm{Var}[\tilde{Z}_T]}{a^2}}\sqrt{\frac{\mathrm{Var}[\tilde{\mathrm{E}}[H \mid \mathcal{G}_0]]}{\mathrm{E}[\tilde{Z}_T^2]}}. \tag{6.6}$$

We summarize these results by:

Theorem 6.2 *Assume that the financial market is complete and that Assumption 5.3 is satisfied. For a claim $H \in L^2(P)$, and any filtration $\bar{\mathbb{F}}^2$ for the insurance risk, the fair premium $v_i(H)$ satisfies*

$$v_{i,min}(H) \leq v_i(H) \leq v_{i,max}(H),$$

$i = 1, 2$, where $v_{i,min}$ and $v_{i,max}$ are determined by (6.1), (6.3), (6.5) and (6.6).

For a given choice of filtration for the insurance risk, the crucial decomposition (2.7) may be complicated, and it may therefore be difficult to determine the corresponding fair premium. In this situation, the bounds given by the above theorem could be used to provide important information about the fair premium. Furthermore, these bounds should be relatively simple to compute since they involve only conditional expectations and variances; this observation motivates the title of this section.

Remark 6.3 We give a simple direct argument, which shows by use of the Cauchy-Schwarz inequality that $v_{i,max}(H)$ is not smaller than $v_{i,min}(H)$. We restrict our attention to claims of the form $H = H^{(1)}\,H^{(2)}$, where $H^{(1)}, H^{(2)} \in L^2(P)$, $H^{(1)}$ is $\mathcal{F}^1_T$-measurable and $H^{(2)}$ is $\mathcal{G}_0$-measurable; the next section is devoted to a separate study of claims of this form. By Lemma 5.4, $\mathcal{F}^1_T$ and $\mathcal{G}_0$ are stochastically independent and hence

$$\mathrm{E}[\mathrm{Var}[H \mid \mathcal{F}^1_T]] = \mathrm{E}\left[\left(H^{(1)}\right)^2\right] \mathrm{Var}\left[H^{(2)}\right],$$

and similarly, since $\tilde{Z}_T$ is $\mathcal{F}^1_T$-measurable

$$\mathrm{Var}[\tilde{\mathrm{E}}[H \mid \mathcal{G}_0]] = \mathrm{Var}\left[H^{(2)}\mathrm{E}[\tilde{Z}_T H^{(1)}]\right] = \mathrm{Var}\left[H^{(2)}\right]\left(\mathrm{E}\left[\tilde{Z}_T H^{(1)}\right]\right)^2.$$

Now, by the Cauchy-Schwarz inequality, we find that

$$\left(\mathrm{E}\left[\tilde{Z}_T H^{(1)}\right]\right)^2 \leq \mathrm{E}\left[\left(H^{(1)}\right)^2\right] \mathrm{E}\left[\left(\tilde{Z}_T\right)^2\right],$$

which confirms that indeed $v_{i,max}(H) \geq v_{i,min}(H)$. □

We end this section by determining the process ϑ^H appearing in the decomposition (2.7) for the minimal and maximal filtrations in the case where the claim is of the form $H = H^{(1)}H^{(2)}$, where $H^{(1)}, H^{(2)} \in L^2(P) \cap L^2(\tilde{P})$, $H^{(1)}$ is $\mathcal{F}^1_T$-measurable and $H^{(2)}$ is $\mathcal{G}_0$-measurable; see also the above remark. Since $(\bar{X}, \bar{\mathbb{F}}^1)$ is complete by assumption, $H^{(1)} = H^{(1)}_0 + \int_0^T \xi^{H^{(1)}}_t dX_t$ for some constant $H^{(1)}_0$ and some predictable process $\xi^{H^{(1)}}$ which is such that $\int \xi^{H^{(1)}} dX$ is a square-integrable $\tilde{P}$-martingale. For the minimal filtration, Theorem 5.14 gives that

$$\begin{aligned}\tilde{\mathrm{E}}[H|\mathcal{F}^1_T] &= \tilde{\mathrm{E}}[H^{(2)}]\left(H^{(1)}_0 + \int_0^T \xi^{H^{(1)}}_t dX_t\right)\\ &= \tilde{\mathrm{E}}[H^{(2)}]H^{(1)}_0 + \int_0^T \left(\tilde{\mathrm{E}}[H^{(2)}]\xi^{H^{(1)}}_t\right) dX_t,\end{aligned}$$

so that in this case $\vartheta_t^H = \tilde{\mathrm{E}}[H^{(2)}]\xi_t^{H^{(1)}}$. For the maximal filtration, the decomposition (5.18) of Theorem 5.16 is

$$H^{(1)}H^{(2)} = H_0^{(1)}H^{(2)} + \int_0^T \left(H^{(2)}\xi_t^{H^{(1)}}\right) dX_t,$$

so that in this case $\vartheta_t^H = H^{(2)}\xi_t^{H^{(1)}}$.

7 A special class of insurance contracts

We consider in this section the following class of fairly general reinsurance contracts:

$$\mathcal{A} = \left\{ H^{(1)}H^{(2)} \,\middle|\, H^{(i)} \in L^2(\mathcal{F}_T^i, P) \cap L^2(\mathcal{F}_T^i, \tilde{P}),\ i = 1, 2 \right\}. \tag{7.1}$$

The product form $H = H^{(1)}H^{(2)}$ of the claim is convenient under the product space model where $H^{(1)}$ and $H^{(2)}$ are stochastically independent (under P) and where $H^{(1)}$ can be viewed as a purely financial derivative and $H^{(2)}$ as a pure insurance contract. The class $\mathcal{A}$ is a rather simple subset of $L^2(\mathcal{F}_T, P) \cap L^2(\mathcal{F}_T, \tilde{P})$, but still rich enough to include for example unit-linked pure endowment contracts, where an amount depending on certain stock prices is payable at a fixed time provided that a policy-holder is still alive at this time. This can be obtained by considering the situation where the probability space $(\Omega_2, \mathcal{F}^2, P_2)$ carries the random life-time T_x of a policy-holder. Then, $H^{(1)}$ and $H^{(2)}$ could be chosen such that $H^{(1)}$ is some function of the path of the stock and $H^{(2)} = 1_{\{T_x > T\}}$. Other examples of contracts included in the class $\mathcal{A}$ are knocked in/out stop-loss contracts, that is, stop-loss contracts which are payable only if some specific financial event has occurred. This event could for example be the value of the stock dropping below some critical value.

7.1 Basic properties

The following simple argument shows that the σ-algebras $\mathcal{F}_T^1$ and $\mathcal{F}_T^2$ are indeed stochastically independent under $\tilde{P}$. Recall that

$$\begin{aligned} \mathcal{F}_T^1 &= \left(\bar{\mathcal{F}}_T^1 \otimes \{\emptyset, \Omega_2\}\right) \vee \mathcal{N}, \\ \mathcal{F}_T^2 &= \left(\{\emptyset, \Omega_1\} \otimes \bar{\mathcal{F}}_T^2\right) \vee \mathcal{N}. \end{aligned}$$

As in the previous sections, $\tilde{P} = \tilde{P}_1 \otimes P_2$ is the variance optimal martingale measure for X. We note that by Lemma 5.4 the σ-algebras $\mathcal{F}_T^1$ and $\mathcal{F}_T^2$ are stochastically independent under P, and similarly under the product measure $\tilde{P} = \tilde{P}_1 \otimes P_2$. As an immediate consequence of this result, we find that the $\tilde{P}$-martingales associated with such $H^{(1)}$ and $H^{(2)}$ are stochastically independent under $\tilde{P}$ and under P. This is formulated as a lemma (we write $\tilde{\mathrm{E}}$ for $\mathrm{E}_{\tilde{P}}$):

Lemma 7.1 *For any $H^{(i)} \in L^2(\mathcal{F}_T^i, \tilde{P})$ define $\tilde{P}$-martingales $N^{(1)}$, $N^{(2)}$ and N by*

$$N_t^{(i)} = \tilde{\mathrm{E}}\left[H^{(i)} \middle| \mathcal{F}_t \right], \tag{7.2}$$

$i = 1, 2$, and $N_t = \tilde{\mathrm{E}}\left[H^{(1)}H^{(2)}\middle| \mathcal{F}_t\right]$. Then $N^{(1)}$ and $N^{(2)}$ are stochastically independent (under P and under $\tilde{P}$), and $N = N^{(1)}N^{(2)}$.

Proof: Since $\mathbb{F}^1$ and $\mathbb{F}^2$ are stochastically independent under $\tilde{P}$, we find from Lemma 5.4 (part 3) that

$$N_t^{(i)} \;=\; \tilde{\mathrm{E}}[H^{(i)} \mid \mathcal{F}_t] = \tilde{\mathrm{E}}[H^{(i)} \mid \mathcal{F}_t^1 \vee \mathcal{F}_t^2] = \tilde{\mathrm{E}}[H^{(i)} \mid \mathcal{F}_t^i],$$

P-a.s. Since $\mathbb{F}^1$, $\mathbb{F}^2$ and $\mathbb{F}$ satisfy the usual conditions, all processes can be chosen to be right-continuous. This shows that $N^{(1)}$ and $N^{(2)}$ are stochastically independent under P and under $\tilde{P}$. By applying the independence again, we finally obtain

$$N_t = \tilde{\mathrm{E}}[H^{(1)}H^{(2)} \mid \mathcal{F}_t] = \tilde{\mathrm{E}}[H^{(1)} \mid \mathcal{F}_t]\tilde{\mathrm{E}}[H^{(2)} \mid \mathcal{F}_t] = N_t^{(1)}N_t^{(2)}. \quad \square$$

The above lemma is used for the derivation of the Galtchouk-Kunita-Watanabe decomposition for $N^{(1)}N^{(2)}$. Moreover, in the next section we use extensively that $N^{(1)}$ and $N^{(2)}$ are stochastically independent under $\tilde{P}$ in order to obtain more explicit results. As a consequence of the above lemma, we also obtain that the martingale $N^{(1)}$ is not affected by the choice of filtration on the space $(\Omega_2, \mathcal{F}^2)$ for the additional risk.

Now for $H \in \mathcal{A}$, define an $(\mathbb{F}, \tilde{P})$-martingale by $\tilde{V} = \mathrm{E}_{\tilde{P}}[H \mid \mathbb{F}]$. By Lemma 7.1, $\tilde{V}$ satisfies

$$\tilde{V}_t = N_t^{(1)}N_t^{(2)},$$

where $N^{(1)}$ and $N^{(2)}$ are given by (7.2). Since X is continuous, the Kunita-Watanabe decomposition under $\tilde{P}$ for $N^{(1)}$ exists, and we write

$$N_t^{(1)} = N_0^{(1)} + \int_0^t \xi_u^{H^{(1)},\tilde{P}} dX_u + L_t^{H^{(1)},\tilde{P}}, \tag{7.3}$$

where $N_0^{(1)}$ is a constant, $\xi^{H^{(1)},\tilde{P}} \in \tilde{\Theta}(\mathbb{F})$ and $L^{H^{(1)},\tilde{P}}$ is a $\tilde{P}$-martingale which is strongly orthogonal to X. Here we have also used that $H^{(1)}$ is stochastically independent of $\mathcal{F}_0^2$ under $\tilde{P}$, so that $N_0^{(1)} := \tilde{\mathrm{E}}[H^{(1)} \mid \mathcal{F}_0] = \tilde{\mathrm{E}}[H^{(1)}]$. We can now express the Kunita-Watanabe decomposition for $\tilde{V}$ in terms of (7.3).

Lemma 7.2 *The Kunita-Watanabe decomposition for $\tilde{V}$ under $\tilde{P}$ is given by*

$$\tilde{V}_t \;=\; \tilde{\mathrm{E}}[H] + \int_0^t \xi_u^{H,\tilde{P}} dX_u + L_t^{H,\tilde{P}}, \tag{7.4}$$

where

$$\begin{aligned}
\xi^{H,\tilde{P}} &= \xi^{H^{(1)},\tilde{P}} N_-^{(2)},\\
L^{H,\tilde{P}} &= \int N_-^{(1)} dN^{(2)} + \int N_-^{(2)} dL^{H^{(1)},\tilde{P}} + \left(\tilde{\mathrm{E}}[H \mid \mathcal{F}_0] - \tilde{\mathrm{E}}[H]\right).
\end{aligned}$$

Proof: By the product rule applied to $N^{(1)}N^{(2)}$ and the strong $\tilde{P}$-orthogonality between $N^{(1)}$ and $N^{(2)}$, we find

$$\begin{aligned} N^{(1)}N^{(2)} &= N_0^{(1)}N_0^{(2)} + \int N_-^{(2)} dN^{(1)} + \int N_-^{(1)} dN^{(2)} + [N^{(1)}, N^{(2)}] \\ &= N_0^{(1)}N_0^{(2)} + \int N_-^{(2)} \xi^{H^{(1)},\tilde{P}} dX + \int N_-^{(1)} dL^{H^{(1)},\tilde{P}} + \int N_-^{(1)} dN^{(2)}. \end{aligned}$$

Here, X and $L^{H^{(1)},\tilde{P}}$ are strongly $\tilde{P}$-orthogonal by definition; furthermore, $N^{(2)}$ and X are strongly $\tilde{P}$-orthogonal as a consequence of Lemma 7.1 (and this implies that also $N^{(2)}$ and $L^{H^{(1)},\tilde{P}}$ are strongly $\tilde{P}$-orthogonal, since $L^{H^{(1)},\tilde{P}} = N^{(1)} - \int \xi^{H^{(1)},\tilde{P}} dX - N_0^{(1)}$). In addition, we note that

$$N_0^{(1)}N_0^{(2)} = \tilde{\mathrm{E}}[H^{(1)} \mid \mathcal{F}_0]\, \tilde{\mathrm{E}}[H^{(2)} \mid \mathcal{F}_0] = \tilde{\mathrm{E}}[H^{(1)}H^{(2)} \mid \mathcal{F}_0] = \tilde{\mathrm{E}}[H \mid \mathcal{F}_0],$$

so that the process Y defined by

$$Y_t = Y_0 := \tilde{\mathrm{E}}[H \mid \mathcal{F}_0] - \tilde{\mathrm{E}}[H]$$

is also strongly $\tilde{P}$-orthogonal to X. Thus, (7.4) gives a decomposition of $\tilde{V}$ in terms of two local $\tilde{P}$-martingales $\int \xi^{H,\tilde{P}} dX$ and $L^{H,\tilde{P}}$, which are strongly orthogonal under $\tilde{P}$. Since X is continuous, we can apply the results of Ansel and Stricker (1993), where it is shown that there exists a unique Kunita-Watanabe decomposition for $\tilde{V}$ with exactly these properties, and hence, (7.4) is the Kunita-Watanabe decomposition for $\tilde{V}$. As in Rheinländer and Schweizer (1998, proof of Theorem 3), we finally note that by the continuity of X, $[X] = \langle X \rangle$ and $0 = \langle X, L^{H,\tilde{P}} \rangle = [X, L^{H,\tilde{P}}]$, and this implies that

$$\begin{aligned} [\tilde{V}] &= [\int \xi^{H,\tilde{P}} dX] + [L^{H,\tilde{P}}] + [\int \xi^{H,\tilde{P}} dX, L^{H,\tilde{P}}] \\ &= \int (\xi^{H,\tilde{P}})^{tr} d[X] \xi^{H,\tilde{P}} + [L^{H,\tilde{P}}] + \int \xi^{H,\tilde{P}} d[X, L^{H,\tilde{P}}] \\ &= \int (\xi^{H,\tilde{P}})^{tr} d\langle X \rangle \xi^{H,\tilde{P}} + [L^{H,\tilde{P}}]. \end{aligned}$$

Since $H \in L^2(\tilde{P})$, the $\tilde{P}$-martingale $\tilde{V}$ is in the space $\mathcal{M}^2(\tilde{P})$, that is, $\sup_{0 \le t \le T} |\tilde{V}_t| \in L^2(\tilde{P})$, and by the Burkholder-Davis-Gundy inequality this means that $[\tilde{V}]_T \in L^1(\tilde{P})$. Thus, $[\int \xi^{H,\tilde{P}} dX]_T, [L^{H,\tilde{P}}]_T \in L^1(\tilde{P})$, which implies via the Burkholder-Davis-Gundy inequality that the local $\tilde{P}$-martingales $\int \xi^{H,\tilde{P}} dX$ and $L^{H,\tilde{P}}$ are true $\tilde{P}$-martingales. This ends the proof. □

We use the above lemma to obtain the following expression for the hedging error $J_0(\mathbb{F}) := J_0(\mathbb{F}, c^H)$, where $c^H = \tilde{\mathrm{E}}[H]$ is the optimal initial capital.

Theorem 7.3 *Assume that $H \in \mathcal{A}$ and that X is continuous. Then*

$$\begin{aligned} J_0(\mathbb{F}) &= \tilde{\mathrm{E}}\left[\int_0^T \frac{1}{\tilde{Z}_s} \left(N_{s-}^{(1)}\right)^2 d[N^{(2)}]_s\right] + \tilde{\mathrm{E}}\left[\int_0^T \frac{1}{\tilde{Z}_s} \left(N_{s-}^{(2)}\right)^2 d[L^{H^{(1)},\tilde{P}}]_s\right] \\ &\quad + \frac{\mathrm{E}\left[\left(\tilde{\mathrm{E}}[H \mid \mathcal{F}_0] - \tilde{\mathrm{E}}[H]\right)^2\right]}{\mathrm{E}\left[\tilde{Z}_T^2\right]}. \end{aligned}$$

Proof: First note that

$$\begin{aligned}[L^{H,\tilde{P}}] &= [L^{H,\tilde{P}} - L_0^{H,\tilde{P}}] = [\int N_-^{(2)} dL^{H^{(1)},\tilde{P}} + \int N_-^{(1)} dN^{(2)}] \\ &= \int \left(N_-^{(2)}\right)^2 d[L^{H^{(1)},\tilde{P}}] + \int \left(N_-^{(1)}\right)^2 d[N^{(2)}],\end{aligned}$$

where the third equality follows by using that $N^{(2)}$ and $L^{H^{(1)},\tilde{P}}$ are strongly $\tilde{P}$-orthogonal (see proof of Lemma 7.1), so that $[N^{(2)}, L^{H^{(1)},\tilde{P}}] = 0$. The result now follows by using Corollary 2.8 in the case where the initial σ-algebra is non-trivial, see also Pham, Rheinländer and Schweizer (1998, Corollary 9). □

7.2 A comparison of hedging errors

In particular, if the model $(\bar{X}, \bar{\mathbb{F}}^1)$ is complete and $\mathcal{F}_0$ is trivial, then Theorem 7.3 says that the hedging error $J_0(\mathbb{F})$ is

$$J_0(\mathbb{F}) = \tilde{\mathrm{E}}\left[\int_0^T \frac{1}{\tilde{Z}_s} \left(N_{s-}^{(1)}\right)^2 d[N^{(2)}]_s\right]. \tag{7.5}$$

In this section we will ask the question: How does the quantity $J_0(\mathbb{F})$ depend on the choice of filtration on the space $(\Omega_2, \mathcal{F}^2)$ for the additional risk? More specifically, we will fix one filtration $\bar{\mathbb{F}}^1$ on the space for the financial assets and consider different choices of filtrations for the additional insurance risk. For $H \in \mathcal{A}$, where $\mathcal{A}$ is defined by (7.1), we give a description of the change in the quantity $J_0(\mathbb{F})$ when the filtration on the space $(\Omega_2, \mathcal{F}^2)$ for the additional risk is changed. We consider two filtrations on $(\Omega_2, \mathcal{F}^2)$ denoted by $\bar{\mathbb{F}}^2$ and $\bar{\mathbb{F}}^{2,\circ}$, respectively, and we shall assume that $\bar{\mathbb{F}}^{2,\circ} \subseteq \bar{\mathbb{F}}^2$, that is $\bar{\mathcal{F}}_t^{2,\circ} \subseteq \bar{\mathcal{F}}_t^2$ for all $0 \le t \le T$. For simplicity, we assume that $\bar{\mathcal{F}}_0^{2,\circ}$ and $\bar{\mathcal{F}}_0^2$ are trivial and that $\mathcal{F}^2 = \bar{\mathcal{F}}_T^{2,\circ} = \bar{\mathcal{F}}_T^2$; the last assumption says that the two filtrations contain the same information at the terminal time T. Define the corresponding product space filtrations $\mathbb{F}$ and $\mathbb{F}^\circ$ as in Section 5, see in particular Remark 5.5. By Proposition 5.9, the variance optimal martingale measures $\tilde{P}$ and $\tilde{P}^\circ$ under $\mathbb{F}$ and $\mathbb{F}^\circ$, respectively, coincide, that is $\tilde{Z} = \tilde{Z}^\circ$. We consider only the situation, where the model $(\bar{X}, \bar{\mathbb{F}}^1)$ is complete.

In addition to the $\tilde{P}$-martingales $N^{(1)}$ and $N^{(2)}$ defined by (7.2), we introduce $(\mathbb{F}^\circ, \tilde{P}^\circ)$-martingales $N^{(i),\circ}$ given by

$$N_t^{(i),\circ} := \mathrm{E}_{\tilde{P}^\circ}[H^{(i)} \mid \mathbb{F}^\circ],$$

$i = 1, 2$. First note that since $\tilde{P}$ and $\tilde{P}^\circ$ coincide, $N^{(i),\circ} = \tilde{\mathrm{E}}[H^{(i)} \mid \mathbb{F}^\circ]$, where we have written $\tilde{\mathrm{E}}$ for $\mathrm{E}_{\tilde{P}}$. Furthermore, we see from the proof of Lemma 7.1 that

$$N_t^{(1),\circ} = \tilde{\mathrm{E}}[H^{(1)} \mid \mathcal{F}_t^\circ] = \tilde{\mathrm{E}}[H^{(1)} \mid \mathcal{F}_t^1] = \tilde{\mathrm{E}}[H^{(1)} \mid \mathcal{F}_t] = N_t^{(1)}, \quad P\text{-a.s.},$$

so that $N^{(1),\circ} \equiv N^{(1)}$; henceforth we only write $N^{(1)}$. Now, by Theorem 7.3, the difference between the hedging errors $J_0(\mathbb{F})$ and $J_0(\mathbb{F}^\circ)$ can be written as

$$J_0(\mathbb{F}^\circ) - J_0(\mathbb{F}) = \tilde{\mathrm{E}}\left[\int_0^T \frac{1}{\tilde{Z}_s} \left(N_{s-}^{(1)}\right)^2 d[N^{(2),\circ}]_s\right] - \tilde{\mathrm{E}}\left[\int_0^T \frac{1}{\tilde{Z}_s} \left(N_{s-}^{(1)}\right)^2 d[N^{(2)}]_s\right]$$

$$= \tilde{\mathrm{E}}\left[\int_0^T \frac{1}{\tilde{Z}_s}\left(N_{s-}^{(1)}\right)^2 d\left([N^{(2),\circ}]_s - [N^{(2)}]_s\right)\right].$$

We have the following result for the variance processes of $N^{(2)}$ and $N^{(2),\circ}$:

Lemma 7.4 *Let $N^{(2)}$ and $N^{(2),\circ}$ be defined as above. Then*

$$\tilde{\mathrm{E}}\left[[N^{(2),\circ}]_t\right] \leq \tilde{\mathrm{E}}\left[[N^{(2)}]_t\right], \quad \textit{for } t \in [0,T],$$

with equality for $t = T$.

Proof: Since $N^{(2)}$ and $N^{(2),\circ}$ are square-integrable, we obtain from the definition of $[N^{(2)}]$ that

$$[N^{(2)}] = \left(N^{(2)}\right)^2 - \left(N_0^{(2)}\right)^2 - 2\int N_-^{(2)} dN^{(2)},$$

and similarly for $[N^{(2),\circ}]$. Hence

$$\begin{aligned}
\tilde{\mathrm{E}}\left[[N^{(2),\circ}]_t\right] &= \tilde{\mathrm{E}}\left[\left(N_t^{(2),\circ}\right)^2\right] - \left(N_0^{(2),\circ}\right)^2 - 2\tilde{\mathrm{E}}\left[\int_0^t N_-^{(2),\circ} dN^{(2),\circ}\right] \\
&= \tilde{\mathrm{E}}\left[\left(\tilde{\mathrm{E}}\left[H^{(2)}\middle|\mathcal{F}_t^\circ\right]\right)^2\right] - \left(N_0^{(2),\circ}\right)^2 \\
&= \tilde{\mathrm{E}}\left[\left(\tilde{\mathrm{E}}\left[\tilde{\mathrm{E}}\left[H^{(2)}\middle|\mathcal{F}_t\right]\middle|\mathcal{F}_t^\circ\right]\right)^2\right] - \left(\mathrm{E}[H^{(2)}]\right)^2 \\
&\leq \tilde{\mathrm{E}}\left[\left(\tilde{\mathrm{E}}\left[H^{(2)}\middle|\mathcal{F}_t\right]\right)^2\right] - \left(N_0^{(2)}\right)^2 = \tilde{\mathrm{E}}\left[[N^{(2)}]_t\right],
\end{aligned}$$

where the second equality follows by using that $N^{(2),\circ}$ is square-integrable. The third equality follows by the law of iterated expectations, and the inequality is a consequence of Jensen's inequality for conditional expectations. Since $\mathcal{G}_T^\circ = \mathcal{G}_T$, $N_T^{(2),\circ} = N_T^{(2)}$, and hence the equality for $t = T$ follows immediately. □

The next lemma allows us to change the order of integration in (7.5) in the following "operational manner":

$$\tilde{\mathrm{E}}\left[\int_0^T \frac{1}{\tilde{Z}_s}\left(N_{s-}^{(1)}\right)^2 d[N^{(2)}]_s\right] = \int_0^T \tilde{\mathrm{E}}\left[\frac{\left(N_{s-}^{(1)}\right)^2}{\tilde{Z}_s}\right] d\left(\tilde{\mathrm{E}}\left[[N^{(2)}]_s\right]\right).$$

This result is closely connected to the independence (under P) between $N^{(1)}$ and the pair $(N^{(2)}, N^{(2),\circ})$ which was established in Lemma 7.1.

Lemma 7.5 *Let A and B be two stochastic processes which are stochastically independent under P and assume that B is increasing. Assume furthermore that $B_T - B_0 \in L^1(P)$ and that*

$$A_T^* := \sup_{0 \leq u \leq T} \mid A_u \mid \in L^1(P).$$

Then

$$\mathrm{E}\left[\int_0^T A_u \, dB_u\right] = \int_0^T \mathrm{E}\left[A_u\right] d\left(\mathrm{E}\left[B_u\right]\right). \tag{7.6}$$

Proof: Since B is increasing, so is the function $u \mapsto \mathrm{E}[B_u]$, and hence both integrals in (7.6) exist as Lebesgue-Stieltjes integrals. In particular, for any sequence of partitions $0 = t_0^{(n)} \leq t_1^{(n)} \leq \cdots \leq t_{N^{(n)}}^{(n)} = T$ of $[0, T]$ so that

$$\lim_{n\to\infty} \sup_j |\, t_j^{(n)} - t_{j-1}^{(n)} \,| = 0,$$

we have that

$$\int_0^T A_u \, dB_u = \lim_{n\to\infty} \sum_{i=1}^{N^n} A_{i-1}^{(n)} \Delta B_i^{(n)}, \quad P-\text{a.s.},$$

where $A_i^{(n)} := A_{t_i^n}$ and $\Delta B_i^{(n)} := B_{t_i^{(n)}} - B_{t_{i-1}^{(n)}}$. Note that

$$\sum_{i=1}^{N^n} A_{i-1}^{(n)} \Delta B_i^{(n)} \leq \sum_{i=1}^{N^n} A_T^* \Delta B_i^{(n)} = A_T^*(B_T - B_0),$$

which is integrable since the two factors $B_T - B_0$ and A_T^* are stochastically independent and integrable by assumption. Thus, by dominated convergence and the independence between A and B we get

$$\begin{aligned}
\mathrm{E}\left[\int_0^T A_u \, dB_u\right] &= \mathrm{E}\left[\lim_{n\to\infty} \sum_{i=1}^{N^n} A_{i-1}^{(n)} \Delta B_i^{(n)}\right] \\
&= \lim_{n\to\infty} \mathrm{E}\left[\sum_{i=1}^{N^n} A_{i-1}^{(n)} \Delta B_i^{(n)}\right] \\
&= \lim_{n\to\infty} \sum_{i=1}^{N^n} \mathrm{E}\left[A_{i-1}^{(n)}\right] \mathrm{E}\left[\Delta B_i^{(n)}\right] \\
&= \int_0^T \mathrm{E}\left[A_u\right] d\left(\mathrm{E}\left[B_u\right]\right).
\end{aligned}$$

Here, the second equality follows by dominated convergence and the third equality is a consequence of the independence between A and B; the last equality follows since $u \mapsto \mathrm{E}[B_u]$ is increasing, so that the integral with respect to $\mathrm{E}[B]$ is a Lebesgue-Stieltjes integral. □

Define an $(\mathbb{F}, P)$-martingale Z by

$$Z_t = \mathrm{E}\left[\left.\frac{d\tilde{P}}{dP}\right| \mathcal{F}_t\right].$$

Using Lemma 7.5, we obtain that

$$\begin{aligned}
J_0(\mathbb{F}) &= \tilde{\mathrm{E}}\left[\int_0^T \frac{1}{\tilde{Z}_s} \left(N_{s-}^{(1)}\right)^2 d[N^{(2),\circ}]_s\right] \\
&= \mathrm{E}\left[\tilde{Z}_T \int_0^T d\left[\int \frac{N_-^{(1)}}{(\tilde{Z})^{1/2}} dN^{(2),\circ}\right]_s\right]
\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{E}\left[\int_0^T Z_s\, d\left[\int \frac{N^{(1)}_-}{(\tilde{Z})^{1/2}} dN^{(2),\circ}\right]_s\right] \\
&= \mathrm{E}\left[\int_0^T \frac{Z_s}{\tilde{Z}_s}\left(N^{(1)}_{s-}\right)^2 d[N^{(2),\circ}]_s\right] \\
&= \int_0^T \mathrm{E}\left[\frac{Z_s}{\tilde{Z}_s}\left(N^{(1)}_{s-}\right)^2\right] d\left(\mathrm{E}\left[[N^{(2),\circ}]_s\right]\right),
\end{aligned}
$$

where the third equality follows by the optional projection theorem, see He, Wang and Yan (1992, Theorem 5.16), since the process $\int(\tilde{Z})^{-1/2}N^{(1)}_- dN^{(2),\circ}$ is of finite variation and since $\mathrm{E}[\tilde{Z}_T \mid \mathcal{F}_t] = Z_t$. The last equality follows by Lemma 7.5 provided that

$$
\sup_{0\le s\le T}\left(\frac{Z_s}{\tilde{Z}_s}\left(N^{(1)}_{s-}\right)^2\right) \in L^1(P). \tag{7.7}
$$

We have now shown:

Corollary 7.6 *Assume that (7.7) is satisfied. Then*

$$
J_0(\mathbb{F}) = \int_0^T \mathrm{E}\left[\frac{Z_s}{\tilde{Z}_s}\left(N^{(1)}_{s-}\right)^2\right] d\left(\mathrm{E}\left[[N^{(2),\circ}]_s\right]\right).
$$

Remark 7.7 The condition (7.7) is satisfied if for example $H^{(1)} \in L^{2+\varepsilon}(P)$, for some $\varepsilon > 0$. To see this note first that $N^{(1)}$ is continuous since $(X, \mathbb{F}^1)$ is complete and since X is continuous. Furthermore

$$
\begin{aligned}
\frac{Z_s}{\tilde{Z}_s}\left(N^{(1)}_{s-}\right)^2 &= \frac{Z_s}{\tilde{Z}_s}\left(\tilde{\mathrm{E}}[H^{(1)} \mid \mathcal{F}_s]\right)^2 \\
&= \frac{Z_s}{\tilde{Z}_s}\left(\frac{\mathrm{E}[\tilde{D}\, H^{(1)} \mid \mathcal{F}_s]}{Z_s}\right)^2 \\
&= \frac{\left(\mathrm{E}[\tilde{D}\, H^{(1)} \mid \mathcal{F}_s]\right)^2}{Z_s \tilde{Z}_s} \\
&= \frac{\left(\mathrm{E}[\tilde{D}\, H^{(1)} \mid \mathcal{F}_s]\right)^2}{\mathrm{E}[\tilde{D} \mid \mathcal{F}_s]} \\
&\le \mathrm{E}[(H^{(1)})^2 \mid \mathcal{F}_s] =: L_s,
\end{aligned}
$$

where the inequality is a consequence of the Cauchy-Schwarz inequality. It now follows by Doob's inequality (see e.g. He, Wang and Yan (1992, Theorem 2.49)) that a sufficient condition for $\sup_{0\le s\le T} L_s \in L^1(P)$ is that $H^{(1)} \in L^{2+\varepsilon}(P)$, for some $\varepsilon > 0$; another sufficient condition is that $\sup_{0\le s\le T}\mathrm{E}[L_s\,(\log(L_s))^+] < \infty$. □

The next lemma allows us to derive the final comparison result for the hedging errors.

Lemma 7.8 *The function*

$$s \mapsto f(s) := \mathrm{E}\left[\frac{Z_s}{\tilde{Z}_s}\left(\tilde{\mathrm{E}}\left[H^{(1)}\middle|\mathcal{F}_s^1\right]\right)^2\right]$$

is increasing.

Proof: Consider first the martingale case, i.e. the case where $\tilde{P} = P$, so that $Z = \tilde{Z} \equiv 1$. By Jensen's inequality for conditional expectations we get for $s < t$:

$$\begin{aligned} f(s) &= \mathrm{E}\left[\left(\tilde{\mathrm{E}}\left[H^{(1)}\middle|\mathcal{F}_s^1\right]\right)^2\right] \\ &= \mathrm{E}\left[\left(\tilde{\mathrm{E}}\left[\tilde{\mathrm{E}}\left[H^{(1)}\middle|\mathcal{F}_t^1\right]\middle|\mathcal{F}_s^1\right]\right)^2\right] \\ &\leq \mathrm{E}\left[\tilde{\mathrm{E}}\left[\left(\tilde{\mathrm{E}}\left[H^{(1)}\middle|\mathcal{F}_t^1\right]\right)^2\middle|\mathcal{F}_s^1\right]\right] = f(t). \end{aligned}$$

In the general case, define a new measure $\tilde{R}$ by

$$\frac{d\tilde{R}}{dP} = \frac{1}{c}\left(\tilde{Z}_T\right)^2,$$

where $c = \mathrm{E}[(\tilde{Z}_T)^2]$. Note also that by the abstract Bayes formula and the independence between $\mathbb{F}^1$ and $\mathbb{F}^2$

$$\tilde{Z}_t = \tilde{\mathrm{E}}\left[\tilde{Z}_T\middle|\mathcal{F}_t^1\right] = \frac{\mathrm{E}\left[(\tilde{Z}_T)^2\middle|\mathcal{F}_t^1\right]}{\mathrm{E}\left[\tilde{Z}_T\middle|\mathcal{F}_t^1\right]} = \frac{1}{Z_t}\mathrm{E}\left[(\tilde{Z}_T)^2\middle|\mathcal{F}_t^1\right].$$

(Recall that $Z_T = \tilde{Z}_T$ and $Z_t = \mathrm{E}[\tilde{Z}_T \mid \mathcal{F}_t]$). Now use this result together with the abstract Bayes formula again, to get

$$\begin{aligned} f(s) &= \mathrm{E}\left[\frac{Z_s}{\tilde{Z}_s}\left(\tilde{\mathrm{E}}\left[H^{(1)}\middle|\mathcal{F}_s^1\right]\right)^2\right] \\ &= \mathrm{E}\left[\frac{(Z_s)^2}{\mathrm{E}\left[(\tilde{Z}_T)^2\middle|\mathcal{F}_s^1\right]}\left(\frac{\mathrm{E}\left[(\tilde{Z}_T)^2 H^{(1)}\frac{1}{\tilde{Z}_T}\middle|\mathcal{F}_s^1\right]}{\mathrm{E}\left[\tilde{Z}_T\middle|\mathcal{F}_s^1\right]}\right)^2\right] \\ &= \mathrm{E}\left[\mathrm{E}\left[(\tilde{Z}_T)^2\middle|\mathcal{F}_s^1\right]\left(\frac{\frac{1}{c}\mathrm{E}\left[(\tilde{Z}_T)^2 H^{(1)}\frac{1}{\tilde{Z}_T}\middle|\mathcal{F}_s^1\right]}{\frac{1}{c}\mathrm{E}\left[(\tilde{Z}_T)^2\middle|\mathcal{F}_s^1\right]}\right)^2\right] \\ &= c\,\mathrm{E}_{\tilde{R}}\left[\left(\mathrm{E}_{\tilde{R}}\left[\frac{H^{(1)}}{\tilde{Z}_T}\middle|\mathcal{F}_s^1\right]\right)^2\right]. \end{aligned}$$

It now follows by calculations similar to the ones used in the martingale case that f is increasing. This completes the proof. □.

Finally, the above lemmas allow us to express the difference between hedging errors in an alternative way and to quantify the increase in risk which occurs when the filtration is reduced. This is possible provided that (7.7) is satisfied, that is, provided that the process A defined by

$$A_s := \frac{Z_s}{\tilde{Z}_s}\left(\tilde{\mathrm{E}}\left[H^{(1)}\middle|\mathcal{F}_s^1\right]\right)^2$$

satisfies $A_T^* := \sup_{0\le s\le T}|A_s| \in L^1(P)$; see also Remark 7.7.

Theorem 7.9 *Assume that (7.7) is satisfied. Then*

$$J_0(\mathbb{F}^\circ) - J_0(\mathbb{F}) = \int_0^T \left(\mathrm{E}\left[[N^{(2)}]_s\right] - \mathrm{E}\left[[N^{(2),\circ}]_s\right]\right) d(\mathrm{E}[A_s]) \ge 0. \tag{7.8}$$

Proof: The inequality in (7.8) follows immediately, since the function $s \mapsto f_s = \mathrm{E}[A_s]$ is increasing by Lemma 7.8 and since the integrand is non-negative by Lemma 7.4. The equality follows by applying the change of variable formula. First introduce the increasing functions $g_s := \mathrm{E}[[N^{(2)}]_s]$ and $g_s^\circ := \mathrm{E}[[N^{(2),\circ}]_s]$. By Corollary 7.6 and the change of variable formula we now find

$$\begin{aligned} J_0(\mathbb{F}^\circ) - J_0(\mathbb{F}) &= \int_0^T f_s\,dg_s^\circ - \int_0^T f_s\,dg_s = \int_0^T f_s\,d(g_s^\circ - g_s) \\ &= (g_T^\circ - g_T)f_T - (g_0^\circ - g_0)f_0 - \int_0^T (g_{u-}^\circ - g_{u-})df_u \\ &= \int_0^T (g_{u-} - g_{u-}^\circ)df_u, \end{aligned}$$

since $g_0^\circ = g_0$ and $g_T^\circ = g_T$. Furthermore, since X is continuous and $(\bar{X}, \bar{\mathbb{F}}^1)$ is assumed to be complete, f is continuous, so that we can replace $(g_{u-} - g_{u-}^\circ)$ by $(g_u - g_u^\circ)$. □.

Remark 7.10 We establish a connection between Theorem 3.5 and the above Theorem 7.9 by demonstrating how the general results from Section 3 specialize to the ones obtained in the present set-up, since this is not immediately clear. For simplicity, we only consider the case where X is 1-dimensional, that is, we take $d = 1$. Since we are within the framework of Section 5, the variance optimal martingale measures under $\mathbb{F}$ and $\mathbb{F}^\circ$ coincide by Proposition 5.9, and so Corollary 3.8 states that

$$J_0(\mathbb{F}^\circ, 0) - J_0(\mathbb{F}, 0) = \tilde{\mathrm{E}}\left[\int_0^T \frac{1}{\tilde{Z}}\varrho^2 d[X]\right], \tag{7.9}$$

where

$$\varrho = \xi^H - \xi^{H,\circ} - \frac{\tilde{\zeta}}{\tilde{Z}}(\tilde{V}_- - \tilde{V}_-^\circ) = \left(\xi^{H^{(1)}} - \frac{\tilde{\zeta}}{\tilde{Z}}N_-^{(1)}\right)(N_-^{(2)} - N_-^{(2),\circ}), \tag{7.10}$$

and where the second equality follows from Lemma 7.2 and the fact that $\tilde{V} = N^{(1)}N^{(2)}$. As in the proof of Lemma 7.4, we also note that

$$\tilde{\mathrm{E}}\left[\left(N_t^{(2)} - N_t^{(2),\circ}\right)^2\right] = \mathrm{E}\left[\left(N_t^{(2)} - N_t^{(2),\circ}\right)^2\right] = \mathrm{E}\left[[N^{(2)}]_t - [N^{(2),\circ}]_t\right]. \tag{7.11}$$

We impose the following additional assumptions:

$$|H^{(1)}|,\ |H^{(2)}| \text{ and } |\xi^{H^{(1)}}| \text{ are bounded,}$$
$$\int_0^T \frac{Z}{\tilde{Z}} d[X],\ \int_0^T \frac{Z\tilde{\zeta}^2}{\tilde{Z}^3} d[X],\ \text{and} \sup_{0\le s\le T} \frac{1}{\tilde{Z}_s} \in L^1(P).$$

In particular, this is sufficient to guarantee that

$$\int_0^T \frac{Z}{\tilde{Z}}\left(\xi^{H^{(1)}} - \frac{\tilde{\zeta}}{\tilde{Z}} N_-^{(1)}\right)^2 d[X] \in L^1(P),$$

and that $\xi^{H^{(1)}}$, $N^{(1)}$, $N^{(2)}$, $N^{(2),\circ}$ are bounded by some constant C; these conditions ensure that Lemma 7.5 can be applied to (7.9). Using the optional projection theorem, (7.10), Lemma 7.5 and (7.11), we can hence rewrite (7.9) as

$$\begin{aligned}
&J_0(\mathbb{F}^\circ, 0) - J_0(\mathbb{F}, 0)\\
&= \mathrm{E}\left[\tilde{Z}_T \int_0^T \frac{1}{\tilde{Z}} \varrho^2 d[X]\right]\\
&= \mathrm{E}\left[\int_0^T \frac{Z}{\tilde{Z}}\left(\xi^{H^{(1)}} - \frac{\tilde{\zeta}}{\tilde{Z}} N_-^{(1)}\right)^2 (N_-^{(2)} - N_-^{(2),\circ})^2 d[X]\right]\\
&= \int_0^T \mathrm{E}\left[[N^{(2)}]_t - [N^{(2),\circ}]_t\right] d\left(\mathrm{E}\left[\int_0^t \frac{Z}{\tilde{Z}}\left(\xi^{H^{(1)}} - \frac{\tilde{\zeta}}{\tilde{Z}} N_-^{(1)}\right)^2 d[X]\right]\right).
\end{aligned}$$

In order to verify that this is identical to the result given in Theorem 7.9, we only need to show that

$$\mathrm{E}\left[\frac{Z_t}{\tilde{Z}_t}(N_t^{(1)})^2 - \int_0^t \frac{Z}{\tilde{Z}}\left(\xi^{H^{(1)}} - \frac{\tilde{\zeta}}{\tilde{Z}} N_-^{(1)}\right)^2 d[X]\right] = K_0,$$

for all $t \in [0,T]$ and for some constant K_0, or, equivalently, that

$$\tilde{\mathrm{E}}\left[\frac{1}{\tilde{Z}_t}(N_t^{(1)})^2 - \int_0^t \frac{1}{\tilde{Z}}\left(\xi^{H^{(1)}} - \frac{\tilde{\zeta}}{\tilde{Z}} N_-^{(1)}\right)^2 d[X]\right] = K_0. \tag{7.12}$$

Now apply the Itô formula to $\frac{1}{\tilde{Z}}(N^{(1)})^2$ to obtain (after some rearrangement of terms)

$$\begin{aligned}
\frac{(N^{(1)})^2}{\tilde{Z}} &= \frac{(N_0^{(1)})^2}{\tilde{Z}_0} + \int \left(2\frac{N_-^{(1)}}{\tilde{Z}}\xi^{H^{(1)}} - \frac{(N_-^{(1)})^2}{\tilde{Z}^2}\tilde{\zeta}\right) dX\\
&\quad + \int \frac{1}{\tilde{Z}}\left(\xi^{H^{(1)}} - \frac{N_-^{(1)}}{\tilde{Z}}\tilde{\zeta}\right)^2 d[X].
\end{aligned} \tag{7.13}$$

Here, the last term on the right side of the equality cancels with the second term in (7.12), whereas the second term is a local $\tilde{P}$-martingale; to see that it is indeed a true $\tilde{P}$-martingale, note first that

$$\left[\int \left(2\frac{N_-^{(1)}}{\tilde{Z}}\xi^{H^{(1)}} - \frac{(N_-^{(1)})^2}{\tilde{Z}^2}\tilde{\zeta}\right) dX\right]_T = \int_0^T (N_-^{(1)})^2 \frac{1}{\tilde{Z}^2}\left(2\xi^{H^{(1)}} - \frac{N_-^{(1)}}{\tilde{Z}}\tilde{\zeta}\right)^2 d[X].$$

Hence

$$\begin{aligned}
&\tilde{\mathrm{E}}\left[\left(\left[\int \left(2\frac{N_-^{(1)}}{\tilde{Z}}\xi^{H^{(1)}} - \frac{(N_-^{(1)})^2}{\tilde{Z}^2}\tilde{\zeta}\right) dX\right]_T\right)^{1/2}\right] \\
&= \mathrm{E}\left[\left(\int_0^T (N_-^{(1)})^2 \frac{Z}{\tilde{Z}^2}\left(2\xi^{H^{(1)}} - \frac{N_-^{(1)}}{\tilde{Z}}\tilde{\zeta}\right)^2 d[X]\right)^{1/2}\right] \\
&\leq C\sqrt{2}\mathrm{E}\left[\left(\int_0^T \frac{Z}{\tilde{Z}^2}(2\xi^{H^{(1)}})^2 d[X]\right)^{1/2} + \left(\int_0^T \frac{Z}{\tilde{Z}^2}\frac{(N_-^{(1)})^2}{\tilde{Z}^2}\tilde{\zeta}^2 d[X]\right)^{1/2}\right] \\
&\leq 2\sqrt{2}C^2\mathrm{E}\left[\left(\int_0^T \frac{Z}{\tilde{Z}} d[X]\right)^{1/2}\left(\sup_{0\leq t\leq T}\frac{1}{\tilde{Z}_t}\right)^{1/2}\right] \\
&\quad +\sqrt{2}C^2\mathrm{E}\left[\left(\int_0^T \frac{Z\tilde{\zeta}^2}{\tilde{Z}^3} d[X]\right)^{1/2}\left(\sup_{0\leq t\leq T}\frac{1}{\tilde{Z}_t}\right)^{1/2}\right],
\end{aligned}$$

which is finite by Cauchy-Schwarz, since all factors are integrable by assumption. This implies via the Burkholder-Davis-Gundy inequality that the local $\tilde{P}$-martingale in (7.13) is a true $\tilde{P}$-martingale, and this completes the proof of (7.12). □

Chapter 6

Indifference Pricing of Insurance Contracts: Examples

(This chapter is an adapted version of the second part of Møller (2000))

In this chapter, we consider several examples of insurance contracts which will be evaluated by the financial variance principle. We set out by keeping the assumption of independence between the pure insurance risk and the financial risk. The chapter is organized as follows. We first mention in Section 1 some well-known results that are related to the standard Black-Scholes model and which are needed for the subsequent analysis of our examples. In Section 2 we then consider the situation where the additional risk is modeled by a homogeneous Poisson process N. Explicit formulas are given for the fair premium and the optimal trading strategy under the financial variance principle for the claim $H = N_T X_T$, where X is the discounted price process associated with a traded asset. In a simple numerical example, we find that the relative difference between the upper and lower bounds for the fair premiums is less than 5 per cent for some specific choices of parameters. Section 3 is devoted to a study of stop-loss contracts with barrier. With these contracts, a stop-loss cover on the pure insurance risk is payable contingent on the occurrence of some event related to the financial market. In Section 4 we focus on hedging and valuation of unit-linked insurance contracts under the financial variance and standard-deviation principles. In contrast to the results obtained in Chapter 4, we here focus on the role of the amount of information available to the seller of the contract and determine the fair premium and the optimal strategy under various choices of filtration for the insurance risk. In Section 5 we give a framework, which allows for dependence between the traded assets and the insurance risk and which is sufficiently general to include stochastic volatility models. Within this set-up, we briefly turn to the problem of valuating the financial stop-loss contract. As another example, we consider the situation where the volatility and drift of the stock price process for an insurance company are affected by the occurrence of certain catastrophic events.

1 The Black-Scholes model

Let $\Omega_1 = \mathcal{C}[0,T]$ be the space of all continuous functions on $[0,T]$, and let, for $\omega_1 \in \Omega_1$, $W_t : \omega_1 \mapsto \omega_1(t)$ be the coordinate process. Furthermore, let $\mathcal{F}^1$ be the σ-algebra generated by $W = (W_t)_{0\le t\le T}$ and denote by $\bar{I\!\!F}^1$ the P-augmentation of the natural filtration of W. Finally, P_1 is taken as the Wiener measure on Ω_1, so that W is indeed a Brownian motion.
The standard Black-Scholes market consists of two assets whose price processes are given by $B_t = \exp(rt)$, for some $r > 0$ and

$$S_t = S_0 + \int_0^t \alpha S_u\, du + \int_0^t \sigma S_u\, dW_u,$$

where $S_0, \alpha \in I\!\!R$ and $\sigma \in (0,\infty)$. The discounted prices are $B/B = 1$ and $X := S/B$, and it is straightforward to verify that $X = X_0 + M + \int \lambda\, d\langle M\rangle$, where $M = \int \sigma X\, dW$ and $\lambda = \frac{\alpha - r}{\sigma^2 X} = \frac{\nu}{\sigma X}$ and where $\nu = (\alpha - r)/\sigma$ is the so-called market price of risk. It is well known that the Black-Scholes model is free of arbitrage and complete, and the unique martingale measure $\tilde{P}_1$ for X is determined by Girsanov's theorem

$$\begin{aligned}\frac{d\tilde{P}_1}{dP_1} &= \mathcal{E}\left(-\int \lambda\, dM\right)_T = \mathcal{E}\left(-\int \lambda\, dX + \int \lambda^2\, d\langle M\rangle\right)_T \\ &= \mathcal{E}\left(-\int \lambda\, dX\right)_T \exp(\nu^2 T).\end{aligned}$$

In particular, this implies that the process $\tilde{Z}$ defined by $\tilde{Z} = \mathrm{E}_{\tilde{P}_1}[\frac{d\tilde{P}_1}{dP_1} \mid \bar{I\!\!F}^1]$ can be written as

$$\tilde{Z}_t = \exp(\nu^2 T) - \int_0^t \mathcal{E}\left(-\int \lambda\, dX\right)_u \exp(\nu^2 T)\lambda_u\, dX_u = \tilde{Z}_0 + \int_0^t \tilde{\zeta}_u\, dX_u,$$

where $\tilde{Z}_0 = \exp(\nu^2 T)$ and $\tilde{\zeta} = -\mathcal{E}\left(-\int \lambda\, dX\right)\exp(\nu^2 T)\lambda$. Since furthermore $\tilde{Z}_t = \mathcal{E}\left(-\int \lambda\, dX\right)_t \exp(\nu^2 T)$, we also see that

$$\begin{aligned}Z_t &= \mathrm{E}\left[\mathcal{E}\left(-\int \lambda\, dM\right)_T \middle| \bar{\mathcal{F}}_t^1\right] = \mathcal{E}\left(-\int \lambda\, dM\right)_t = \mathcal{E}\left(-\int \lambda\, dX\right)_t \exp(\nu^2 t) \\ &= \tilde{Z}_t \exp(-\nu^2 (T-t)),\end{aligned}$$

which shows that $Z_t/\tilde{Z}_t = \exp(-\nu^2(T-t))$. With the notation used in Chapter 4, the projection on the space $G_T(\tilde{\Theta})^{\perp}$ is denoted by $\pi(\cdot)$. Furthermore, $\pi(1) = 1 - \int_0^T \tilde{\beta}\, dX$, and since by Lemma 3.5 of Chapter 4, $\frac{d\tilde{P}_1}{dP_1} = \frac{\pi(1)}{\mathrm{E}[\pi(1)]}$, we see that

$$\tilde{\beta} = \lambda \mathcal{E}\left(-\int \lambda\, dX\right) = \lambda \tilde{Z} e^{-\nu^2 T}.$$

Finally, we note that since $\mathcal{E}\left(-\int \lambda\, dM\right) = \mathcal{E}\left(-\nu W\right)$, we have that

$$\mathrm{E}[(\tilde{Z}_T)^2] = \mathrm{E}\left[\exp\left(-2\nu W_T - \nu^2 T\right)\right] = \exp(\nu^2 T),$$

and hence $\mathrm{Var}[\tilde{Z}_T] = \exp(\nu^2 T) - 1$.

2 An additional Poisson process

Consider the standard Black-Scholes market reviewed in Section 1 and take as additional risk a homogeneous Poisson process N with intensity Λ. This process is defined on some probability space $(\Omega_2, \mathcal{F}^2, P_2)$, which is assumed to carry the natural filtration $\bar{\mathbb{F}}^N$ of N. We shall also use the notation

$$\bar{\mathcal{F}}_t^2 = \bar{\mathcal{F}}_t^N = \sigma\{N_u,\, u \leq t\} \tag{2.1}$$

and assume that $\mathcal{F}^2 = \bar{\mathcal{F}}_T^N$. As in Section 5 of Chapter 5 we embed the processes N and X in the product space and also define the filtrations $\mathbb{F}^1$ and $\mathbb{F}^2$ by (5.5.3) and (5.5.4), respectively. The Poisson process N is taken to describe the occurrence of some insurance claims, and by this construction, N is independent of the financial market.
In this section we consider the claim

$$H = N_T X_T, \tag{2.2}$$

which for example could be used in the situation where claims are subject to some inflation as described by the process X. We derive fair premiums and optimal strategies for a reinsurer under four different scenarios. Each of the scenarios considered is connected to a specific filtration on the space $(\Omega_2, \mathcal{F}^2)$ and represents a different level of information about the Poisson process:

1. The trivial filtration $\bar{\mathbb{F}}^{2,\circ} = (\bar{\mathcal{F}}_t^{2,\circ})_{0\leq t\leq T}$ defined by $\bar{\mathcal{F}}_t^{2,\circ} = \{\emptyset, \Omega_2\}$, $0 \leq t < T$, and $\bar{\mathcal{F}}_T^{2,\circ} = \bar{\mathcal{F}}_T^N$ is the situation where no information concerning the Poisson process N is available to the reinsurer before time T.

2. The piecewise constant filtration $\bar{\mathbb{F}}^{2,p} = (\bar{\mathcal{F}}_t^{2,p})_{0\leq t\leq T}$ defined by $\bar{\mathcal{F}}_t^{2,p} = \{\emptyset, \Omega_2\}$, $0 \leq t < t_0$, $\bar{\mathcal{F}}_t^{2,p} = \bar{\mathcal{F}}_{t_0}^N$, $t_0 \leq t < T$ and $\bar{\mathcal{F}}_T^{2,p} = \bar{\mathcal{F}}_T^N$ is the situation where information concerning the past development is revealed only at some fixed time t_0 during $(0, T)$.

3. The filtration $\bar{\mathbb{F}}^2 = (\bar{\mathcal{F}}_t^2)_{0\leq t\leq T}$ defined by (2.1) is the natural filtration of N. This means that the reinsurer is observing the Poisson process during the period $[0, T]$.

4. The revealing filtration $\bar{\mathbb{F}}^{2,r} = (\bar{\mathcal{F}}_t^{2,r})_{0\leq t\leq T}$ defined by $\bar{\mathcal{F}}_t^{2,r} = \bar{\mathcal{F}}_T^N$, $0 \leq t \leq T$, is the hypothetical situation where the reinsurer knows the final outcome of the Poisson process immediately after the signing of the contract at time 0.

We introduce the associated filtrations on the product space defined (as in Remark 5.5.5) by $\mathbb{F}^\circ = \bar{\mathbb{F}}^1 \otimes \bar{\mathbb{F}}^{2,\circ}$, $\mathbb{F}^p = \bar{\mathbb{F}}^1 \otimes \bar{\mathbb{F}}^{2,p}$, $\mathbb{F} = \bar{\mathbb{F}}^1 \otimes \bar{\mathbb{F}}^2$ and $\mathbb{F}^r = \bar{\mathbb{F}}^1 \otimes \bar{\mathbb{F}}^{2,r}$, respectively, and deal with the four cases separately. Recall first however, that by Theorem 5.4.1, the optimal strategy for the variance principle is given by

$$\vartheta^* = \vartheta^H + \frac{1 + \mathrm{Var}[\tilde{Z}_T]}{2a} \tilde{\beta} = \vartheta^H + \frac{\tilde{Z}\lambda}{2a}, \tag{2.3}$$

where ϑ^H is determined by Theorem 5.2.5 and 5.2.6 for the filtration of interest. For the standard deviation principle, the optimal strategy is

$$\vartheta^* = \vartheta^H + \frac{1+\mathrm{Var}[\tilde{Z}_T]}{a\sqrt{1-\frac{\mathrm{Var}[\tilde{Z}_T]}{a^2}}}\sqrt{\mathrm{Var}[N^H]}\tilde{\beta} = \vartheta^H + \frac{\tilde{Z}\lambda}{\tilde{a}}\sqrt{\mathrm{Var}[N^H]}, \tag{2.4}$$

where

$$\tilde{a} = a\sqrt{1-\frac{\mathrm{Var}[\tilde{Z}_T]}{a^2}}. \tag{2.5}$$

As we shall see in our examples, ϑ^H and N^H depend on the choice of filtration, whereas the processes $\tilde{Z}$ and λ are given in Section 1 and do not depend on the filtration on $(\Omega_2, \mathcal{F}^2)$. In addition, note that with the notation of Section 7 of Chapter 5,

$$N_t^{(1)} = \tilde{\mathrm{E}}[X_T \mid \mathcal{F}_t^1] = X_t.$$

This process will appear in all four cases considered below.

Case 1. (No information) We apply the notation of Section 7 of Chapter 5 and note that as in the proof of Lemma 5.7.1 and by the definition of $\bar{\mathbb{F}}^{2,\circ}$

$$N_t^{(2),\circ} := \tilde{\mathrm{E}}[N_T \mid \mathcal{F}_t^\circ] = \begin{cases} \Lambda T, & t < T, \\ N_T, & t = T, \end{cases}$$

which implies that

$$[N^{(2),\circ}]_t = (N_T - \Lambda T)^2 1_{\{t \geq T\}}.$$

Thus, by Lemma 5.7.2, the Kunita-Watanabe decomposition of $\tilde{V}^\circ = \tilde{\mathrm{E}}[H \mid \mathbb{F}^\circ]$ is

$$\tilde{V}_t^\circ = \Lambda T X_0 + \int_0^t \Lambda T \, dX_u + X_T(N_T - \Lambda T) 1_{\{t \geq T\}}. \tag{2.6}$$

Using (5.7.5), we obtain that

$$J_0(\mathbb{F}^\circ) = \mathrm{E}\left[\frac{Z_T}{\tilde{Z}_T} X_T^2 (N_T - \Lambda T)^2\right] = \mathrm{E}\left[X_T^2\right] \mathrm{Var}[N_T] = \Lambda T X_0^2 e^{2(\alpha - r)T + \sigma^2 T},$$

which gives the fair premium for this case

$$v_1^\circ(H) = \Lambda T X_0 + a \Lambda T X_0^2 e^{2(\alpha - r)T + \sigma^2 T}.$$

From the Kunita-Watanabe decomposition (2.6) and Theorem 5.2.6, we find that $\vartheta^H = \Lambda T$, so that the optimal strategy is

$$\vartheta_t^* = \Lambda T + \frac{\tilde{Z}_t \lambda_t}{2a}.$$

The first term ΛT is the conditional expected number of claims during $[0,T]$ at any time $t \in [0,T)$, since no information about the occurrence of claims is available before time T. The second term $\tilde{Z}\lambda/(2a)$ is a correction term which is related to the financial variance principle, see e.g. (2.3).

Case 2. (Piecewise constant filtration) In this case

$$N_t^{(2),p} := \tilde{\mathrm{E}}[N_T \mid \mathcal{F}_t^p] = \begin{cases} \Lambda T, & t < t_0, \\ N_{t_0} + \Lambda(T - t_0), & t_0 \le t < T, \\ N_T, & t = T, \end{cases}$$

so that

$$[N^{(2),p}]_t = (N_{t_0} - \Lambda t_0)^2 1_{\{t \ge t_0\}} + (N_T - N_{t_0} - \Lambda(T - t_0))^2 1_{\{t \ge T\}}.$$

By (5.7.5), this implies that

$$\begin{aligned} J_0(\mathbb{F}^p) &= \mathrm{E}\left[\frac{Z_{t_0}}{\tilde{Z}_{t_0}} X_{t_0}^2 (N_{t_0} + \Lambda(T - t_0) - \Lambda T)^2\right] \\ &\quad + \mathrm{E}\left[\frac{Z_T}{\tilde{Z}_T} X_T^2 (N_T - N_{t_0} - \Lambda(T - t_0))^2\right] \\ &= \Lambda t_0 X_0^2 e^{-\nu^2(T-t_0)+2(\alpha-r)t_0+\sigma^2 t_0} + \Lambda(T - t_0) X_0^2 e^{2(\alpha-r)T+\sigma^2 T}. \end{aligned}$$

We have now shown that the fair premium is given by

$$\begin{aligned} v_1^p(H) &= \Lambda T X_0 \\ &\quad + a\left(\Lambda t_0 X_0^2 e^{-\nu^2(T-t_0)+(2(\alpha-r)+\sigma^2)t_0} + \Lambda(T - t_0) X_0^2 e^{(2(\alpha-r)+\sigma^2)T}\right). \end{aligned}$$

By Lemma 5.7.2 and Theorem 5.2.6

$$\vartheta_t^H = \begin{cases} \Lambda T, & t \le t_0, \\ N_{t_0} + \Lambda(T - t_0) - \tilde{\zeta}_t X_{t_0}(N_{t_0} - \Lambda t_0), & t_0 < t < T, \end{cases}$$

and this implies that

$$\vartheta_t^* = \begin{cases} \Lambda T + \frac{\tilde{Z}_t \lambda_t}{2a}, & t \le t_0, \\ N_{t_0} + \Lambda(T - t_0) - \tilde{\zeta}_t X_{t_0}(N_{t_0} - \Lambda t_0) + \frac{\tilde{Z}_t \lambda_t}{2a}, & t_0 < t < T. \end{cases}$$

This strategy coincides up to and including time t_0 with the optimal strategy in the case where no information is available. At time t_0, the reinsurer is informed about the current total number of claims N_{t_0} and adjusts his strategy according to this additional information. The term $N_{t_0} + \Lambda(T - t_0)$ is the conditional expected number of survivors computed after time t_0 and $X_{t_0}(N_{t_0} - \Lambda t_0)$ is the difference in the reinsurer's estimate (at time t_0 and immediately before t_0) of the final outcome of the claim $N_T X_T$.

Case 3. (Natural filtration) For this filtration,

$$N_t^{(2)} = \tilde{\mathrm{E}}[N_T \mid \mathcal{F}_t] = N_t + \Lambda(T - t) = \Lambda T + M_t^u,$$

where $M_t^u := N_t - \Lambda t$. This shows that $[N^{(2)}]_t = N_t$. By Lemma 5.7.2, the Kunita-Watanabe decomposition for $\tilde{V} := \tilde{\mathrm{E}}[H \mid I\!\!F]$ is

$$\tilde{V}_t = \Lambda T X_0 + \int_0^t (N_{s-} + \Lambda(T-s))dX_s + \int_0^t X_s\, dM_s^u.$$

By (5.7.5), the martingale property of $\int \tilde{Z}^{-1} X^2\, dM$ and the Fubini theorem, we obtain that

$$\begin{aligned} J_0(I\!\!F) &= \tilde{\mathrm{E}}\left[\int_0^T \frac{1}{\tilde{Z}_s} X_s^2\, dN_s\right] \\ &= \int_0^T \mathrm{E}\left[\frac{Z_s}{\tilde{Z}_s} X_s^2\right] \Lambda\, ds \\ &= \int_0^T e^{-\nu^2(T-s)} X_0^2 e^{2(\alpha-r)s+\sigma^2 s} \Lambda\, ds \\ &= \Lambda e^{-\nu^2 T} X_0^2 \frac{1}{\nu^2 + 2(\alpha - r) + \sigma^2}\left(e^{(\nu^2+2(\alpha-r)+\sigma^2)T} - 1\right). \end{aligned}$$

Hence, the fair premium is

$$v_1(H) = \Lambda T X_0 + a \frac{\Lambda e^{-\nu^2 T} X_0^2}{\nu^2 + 2(\alpha - r) + \sigma^2}\left(e^{(\nu^2+2(\alpha-r)+\sigma^2)T} - 1\right).$$

Finally, Theorem 2.6 combined with (2.3) shows that

$$\vartheta_t^* = N_{t-} + \Lambda(T-t) - \tilde{\zeta}_t \int_0^{t-} \tilde{Z}_s^{-1} X_s\, dM_s^u + \frac{\tilde{Z}_t \lambda_t}{2a},$$

where again the last term is related to the financial variance principle. The first two terms $N_{t-} + \Lambda(T-t)$ are the conditional expected number of survivors just before time t, and the integral with respect to M^u is related to the change in the reinsurer's predictions for the final value of the claim.

Case 4. (Full initial information) In this case, $N_t^{(2),r} = \tilde{\mathrm{E}}[N_T \mid \mathcal{F}_t^r] = N_T$ and hence, by Lemma 5.7.2, the Kunita-Watanabe decomposition for $\tilde{V}^r := \tilde{\mathrm{E}}[H \mid I\!\!F^r]$ is

$$\tilde{V}_t^r = \Lambda T X_0 + \int_0^t N_T\, dX_s + (N_T X_0 - \Lambda T X_0).$$

By the results obtained in Section 6 of Chapter 5 (see (5.6.2) and (5.6.3)) we obtain that

$$J_0(I\!\!F^r) = \frac{\mathrm{Var}[\tilde{\mathrm{E}}[H \mid \mathcal{G}_0]]}{\mathrm{E}[\tilde{Z}_T^2]} = e^{-\nu^2 T} X_0^2 \Lambda T,$$

where $\mathcal{G}_0 = (\mathcal{N}^1 \otimes \mathcal{F}^2) \vee \mathcal{N}$, see Section 6 of Chapter 5. Thus, the fair premium is in this case given by

$$v_1^r(H) = \Lambda T X_0 + a e^{-\nu^2 T} X_0^2 \Lambda T.$$

The optimal strategy is determined via Theorem 5.2.6

$$\vartheta_t^H = N_T - \tilde{\zeta}_t \tilde{Z}_0^{-1} X_0 (N_T - \Lambda T),$$

and consequently the optimal strategy is determined by

$$\vartheta_t^* = N_T - \tilde{\zeta}_t \tilde{Z}_0^{-1} X_0 (N_T - \Lambda T) + \frac{\tilde{Z}_t \lambda_t}{2a}.$$

Here, the first term N_T is the final outcome of the Poisson process; the second term is related to the difference between the reinsurer's estimate for $N_T X_T$ before time 0 and at time 0.

We end this section with a numerical example which quantifies the range of the premiums determined under the different filtrations. We fix $T = 1$, $\Lambda = 1$, $t_0 = 0.5$, $X_0 = 1$, $a = 0.25$ and take $\alpha = 0.10$ and $r = 0.06$. The hedging errors $J_0(\mathbb{F}^\circ)$, $J_0(\mathbb{F}^p)$, $J_0(\mathbb{F})$ and $J_0(\mathbb{F}^r)$ are listed in Table 6.1 together with the relative difference $\delta^{\circ,r} = (J_0(\mathbb{F}^\circ) - J_0(\mathbb{F}^r))/J_0(\mathbb{F}^\circ)$ between the maximal and minimum hedging errors and the upper and lower bounds for the fair premiums under the financial variance principle. It follows from these numbers that the hedging error under the revealing filtration is between 15 percent and 19 percent smaller than the hedging error under the trivial filtration. The corresponding relative differences between the upper and lower bounds for the fair premiums are less than 5 percent in this example.

Volatility	$J_0(\mathbb{F}^\circ)$	$J_0(\mathbb{F}^p)$	$J_0(\mathbb{F})$	$J_0(\mathbb{F}^r)$	$\delta^{\circ,r}$	v_1°	v_1^r
$\sigma = 0.15$	1.1079	1.0619	1.0171	0.9314	0.1594	1.2770	1.2328
$\sigma = 0.25$	1.1532	1.1067	1.0614	0.9747	0.1547	1.2883	1.2437
$\sigma = 0.35$	1.2245	1.1619	1.1015	0.9870	0.1939	1.3061	1.2468

Table 6.1: *The hedging errors under the four different scenarios, the relative difference between the maximal and minimal hedging errors and the upper and lower bounds for the fair premiums under the financial variance principle for various choices of volatility.*

3 Stop-loss contracts with barrier

As a very simple class of examples we consider traditional stop-loss contracts that are payable contingent on the occurrence of some event on the financial market, for example, the event that the terminal value of a certain stock lies within a certain interval. Let $F \in \mathcal{F}_T^1$ represent some financial event, and note that this event is related to the development on the financial market only. We denote by $X = (X_t)_{0 \le t \le T}$ the discounted price process of the financial asset (a stock) and by $U = (U_t)_{0 \le t \le T}$ an insurance claim process, which is stochastically independent of the financial market; we consider the contract given by

$$H = 1_F \, (U_T - K)^+. \tag{3.7}$$

We shall henceforth refer to a reinsurance contract of this form as a *stop-loss contract with barrier*. As a main example, we shall consider the subset F of Ω where the terminal value X_T of the stock is within a set $B \in \mathcal{B}(\mathbb{R}_+)$, that is $F = \{X_T \in B\}$. For example, we might have that $B = [0, c]$ or $B = [c, \infty)$ for some $c > 0$; the first case can be compared to a so-called knock-out option known from finance (see e.g. Musiela and Rutkowski (1997)), and the second case is similar to a knock-in option. These contracts could be relevant for an insurer who has invested in the stock X and who is interested in a cover against the risk U: If the insurer holds a long position, then he may decide that stop-loss cover is only necessary if the value of the stock has not exceeded some value c, and this is obtained for $B = [0, c]$. Similarly, if the insurer has a short position in the stock, $B = [c, \infty)$ may be relevant.

Under the assumption that the financial market is complete, the results from Section 6 of Chapter 5 can be applied to determine an upper bound for the fair premium for the contract (3.7); we find that

$$\begin{aligned} v_{1,max}(H) &= \mathrm{E}[\tilde{Z}_T\, 1_F\, (U_T - K)^+] + a\mathrm{E}[\mathrm{Var}[1_F\, (U_T - K)^+ \mid \mathcal{F}_T^1]] \\ &= \mathrm{E}[\tilde{Z}_T\, 1_F]\, \mathrm{E}[(U_T - K)^+] + aP(F)\, \mathrm{Var}[(U_T - K)^+] \\ &= \tilde{P}(F)\, \mathrm{E}[(U_T - K)^+] + aP(F)\, \mathrm{Var}[(U_T - K)^+], \end{aligned} \tag{3.8}$$

where we have used the independence between $(X, \tilde{Z}_T)$ and U twice in the second equality. From (3.8) we see that the expression for the premium for the stop-loss contract with barrier is very similar to the premium for the original stop-loss contract $(U_T - K)^+$ computed by means of the traditional actuarial variance principle. It is also noted that the premium will not in general be equal to $P(F)$ times the premium for $(U_T - K)^+$ computed using the traditional variance principle; this difference between the two premiums is related to the fact that $P(F) \neq \tilde{P}(F)$ in the general case. Similarly, the lower bound for the fair premium is

$$\begin{aligned} v_{1,min}(H) &= \tilde{P}(F)\, \mathrm{E}[(U_T - K)^+] + a\frac{\mathrm{Var}[\tilde{\mathrm{E}}[1_F\, (U_T - K)^+ \mid \mathcal{F}_T^2]]}{\mathrm{E}[\tilde{Z}_T^2]} \\ &= \tilde{P}(F)\, \mathrm{E}[(U_T - K)^+] + a\frac{(\tilde{P}(F))^2\, \mathrm{Var}[(U_T - K)^+]}{\tilde{Z}_0}. \end{aligned}$$

Let us compare more explicitly the upper bound for the fair premium for H to the premium computed using the traditional variance principle; the latter can be computed by using the standard rule for conditioning for variances and the independence between U and X:

$$\begin{aligned} \tilde{u}_1(H) &= \mathrm{E}[1_F\, (U_T - K)^+] + a\mathrm{E}[\mathrm{Var}[1_F\, (U_T - K)^+ \mid \mathcal{F}_T^1]] \\ &\quad + a\mathrm{Var}[\mathrm{E}[1_F\, (U_T - K)^+ \mid \mathcal{F}_T^1]] \\ &= P(F)\, \mathrm{E}[(U_T - K)^+] + aP(F)\, \mathrm{Var}[(U_T - K)^+] \\ &\quad + aP(F)(1 - P(F))\, (\mathrm{E}[(U_T - K)^+])^2. \end{aligned} \tag{3.9}$$

By comparing (3.8) and (3.9) we immediately get that $\tilde{u}_1(H) > v_{1,max}(H)$ if and only if

$$a\mathrm{E}[(U_T - K)^+] > \frac{\tilde{P}(F) - P(F)}{P(F)(1 - P(F))},$$

provided that $P(F) \notin \{0,1\}$. In particular, this implies that $\tilde{u}_1(H) > v_{1,max}(H)$ if for example $\tilde{P}(F) - P(F) \leq 0$; however, this is not a necessary condition.

As a naive example where more explicit formulas can be obtained, let us assume that U_T is log-normally distributed under P and parametrize this as follows

$$U_T := u_0 \exp(-\frac{1}{2}\kappa^2 T + \kappa \hat{W}_T), \tag{3.10}$$

where $\hat{W}$ is a standard Brownian motion which is independent of the financial market. This implies that $\log(U_T)$ is normally distributed under P with parameters $(\log(u_0) - \frac{1}{2}\kappa^2 T, \kappa^2 T)$. Since U_T would typically represent the total accumulated claim amounts on some insurance portfolios or the ratio between claims and premiums, this is not a very realistic assumption. It would indeed be much more suitable to let U_T be for example a compound Poisson variable. However, in that setting explicit formulas would generally not be within reach. More modestly, one could choose to approximate the ratio between claims and premiums by a sum of independent log-normal random variables; also in that situation, explicit formulas cannot be obtained, and one would have to apply some numerical method in order to determine the fair premiums.

Assume furthermore that the financial market is described by a standard Black-Scholes model, where the discounted stock price is given by the dynamics

$$dX_t = (\alpha - r)X_t\,dt + \sigma X_t\,dW_t,$$

and $X_0 = x_0$. (r is the risk-free interest rate, α the expected rate of return and σ the volatility on the stock). It then follows by direct calculations that

$$\begin{aligned} \mathrm{E}[(U_T - K)^+] &= u_0\Phi(c_2) - K\Phi(c_2 - \kappa\sqrt{T}), \\ \mathrm{E}[((U_T - K)^+)^2] &= K^2\Phi(c_2 - \kappa\sqrt{T}) + u_0^2 e^{\kappa^2 T}\Phi(c_2 + \kappa\sqrt{T}) - 2Ku_0\Phi(c_2), \end{aligned}$$

where

$$c_2 = \frac{\log\frac{u_0}{K} + \frac{1}{2}\kappa^2 T}{\kappa\sqrt{T}},$$

and so, the two first central moments of $(U_T - K)^+$ can be expressed explicitly in terms of the parameters u_0, T, κ. Furthermore, we find that for $F = \{X_T \in [0,c]\}$:

$$\begin{aligned} P(F) &= \Phi\left(-\frac{\log\frac{x_0}{c} + (\alpha - r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right), \\ \tilde{P}(F) &= \Phi\left(-\frac{\log\frac{x_0}{c} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right). \end{aligned}$$

This shows the obvious inequality $\tilde{P}(X_T \le c) > P(X_T \le c)$ in the natural situation where $\alpha > r$ and the reverse inequality in the case where $\alpha < r$. Hence, we have obtained explicit expressions for the premiums (3.8) and (3.9).

Note that (3.7) with $F = \{X_T \in B\}$ is on the form

$$H = b(U_T)g(X_T), \tag{3.11}$$

where b, g are measurable functions such that $b(U_T),\ g(X_T) \in L^2(P) \cap L^2(\tilde{P})$. Thus, we can also apply the results obtained in Section 7 of Chapter 5. We assume that the insurance claims process with terminal value (3.10) is given by

$$U_t = u_0 \exp(-1/2\kappa^2 t + \kappa \hat{W}_t).$$

We interpret U_t as the current estimate at time t for the terminal value U_T which appears in the claim (3.11). For example, U_T could be the ratio between claims and premiums during the period $[0,T]$ and hence, U_t is the estimate at t for this ratio. As in Section 7 of Chapter 5 we assume that $\hat{W}$ and U a priori are defined on a separate probability space $(\Omega_2, \mathcal{F}^2, P_2)$ equipped with the natural filtration of U, which is denoted $\bar{\mathbb{F}}^2$. In addition, we consider the filtration $\bar{\mathbb{F}}^{2,\circ}$ defined by $\bar{\mathcal{F}}_t^{2,\circ} = \{\emptyset, \Omega_2\}$, $0 \le t < T$ and $\bar{\mathcal{F}}_T^{2,\circ} = \bar{\mathcal{F}}_T^2 = \sigma\{U_s,\ 0 \le s \le T\}$. These two filtrations represent the situation where the reinsurer observes the process U and where the process is not observed, respectively. We proceed as in Section 7 of Chapter 5 and deal with the two cases separately.

Case 1. (No information) Consider first the situation described by the filtration $\mathbb{F}^\circ = \bar{\mathbb{F}}^1 \otimes \bar{\mathbb{F}}^{2,\circ}$. In this case

$$N_t^{(2),\circ} := \tilde{\mathrm{E}}[b(U_T) \mid \mathcal{F}_t^\circ] = \begin{cases} \mathrm{E}[b(U_T)], & t < T, \\ b(U_T), & t = T, \end{cases}$$

which implies that

$$[N^{(2),\circ}]_t = \left(b(U_T) - \mathrm{E}[b(U_T)]\right)^2 1_{\{t \ge T\}}.$$

Since X is a Markov process, we obtain by the Itô formula and the uniqueness of the canonical decomposition

$$N_t^{(1)} := \tilde{\mathrm{E}}[g(X_T) \mid \mathcal{F}_t^\circ] = F^g(t, X_t) = F^g(0, X_0) + \int_0^t F_x^g(s, X_s)\, dX_s, \tag{3.12}$$

provided that $F^g \in C^{1,2}$. Thus, by (5.7.5) the hedging error is

$$J_0(\mathbb{F}^\circ) = \mathrm{E}\left[\frac{Z_T}{\tilde{Z}_T}\left(g(X_T)\right)^2 \left(b(U_T) - \mathrm{E}[b(U_T)]\right)^2\right] = \mathrm{E}\left[\left(g(X_T)\right)^2\right] \mathrm{Var}\left[b(U_T)\right].$$

In particular, this leads to (3.8) when $g(X_T) = 1_{\{X_T \in B\}}$ and $h(U_T) = (U_T - K)^+$. It follows immediately from Lemma 5.7.2 that $\xi_t^{H,\tilde{P}} = \mathrm{E}[b(U_T)]\, F_x^g(t, X_t)$ for $0 \le t \le T$, and this lemma together with Theorem 5.2.6 shows that $\vartheta^H = \xi^{H,\tilde{P}}$. Thus, the optimal strategy is in this case

$$\vartheta_t^* = \mathrm{E}[b(U_T)]\, F_x^g(t, X_t) + \frac{\tilde{Z}_t \lambda_t}{2a}. \tag{3.13}$$

Case 2. (Natural filtration) We consider the filtration $\mathbb{F} = \bar{\mathbb{F}}^1 \otimes \bar{\mathbb{F}}^2$. By arguments similar to the ones leading to (3.12)

$$N_t^{(2)} := \tilde{\mathrm{E}}[b(U_T) \mid \mathcal{F}_t] = F^b(t, U_t) = F^b(0, U_0) + \int_0^t F_u^b(s, U_s)\, dU_s,$$

provided that $F^b \in C^{1,2}$. This implies that

$$[N^{(2)}]_t = \int_0^t \left(F_u^b(s, U_s)\right)^2 d[U]_s = \int_0^t \left(F_u^b(s, U_s) U_s\right)^2 \kappa^2\, ds.$$

Thus, it follows from Corollary 5.7.6 and Remark 5.7.7 that

$$J_0(\mathbb{F}) = \int_0^T \mathrm{E}\left[\frac{Z_t}{\tilde{Z}_t} \left(F^g(t, X_t)\right)^2\right] \mathrm{E}\left[\left(F_u^b(t, U_t) U_t\right)^2\right] \kappa^2\, dt,$$

provided that $g(X_T) \in L^{2+\varepsilon}(P)$, for some $\varepsilon > 0$.

In most examples, this expression has to be evaluated numerically, for example by Monte-Carlo simulation. The optimal strategy is again determined by applying Theorem 5.2.6 combined with Lemma 5.7.2. This gives

$$\vartheta_t^H = F^b(t, U_t)\, F_x^g(t, S_t) - \tilde{\zeta}_t \int_0^t \tilde{Z}_s^{-1} F^g(s, X_s) F_u^b(s, U_s)\, dU_s$$

which gives the optimal strategy for this situation

$$\vartheta_t^* = F^b(t, U_t)\, F_x^g(t, S_t) - \tilde{\zeta}_t \int_0^t \tilde{Z}_s^{-1} F^g(s, X_s) F_u^b(s, U_s)\, dU_s + \frac{\tilde{Z}_t \lambda_t}{2a}.$$

4 Unit-linked life insurance contracts

We apply the financial variance principle to the pricing of unit-linked pure endowment life insurance contracts for a portfolio consisting of n policy-holders aged y with i.i.d. remaining lifetimes with hazard rate functions μ_y; the survival probability is denoted ${}_tp_y = \exp(-\int_0^t \mu_{y+\tau}\, d\tau)$ in accordance with standard actuarial notation. Similarly, ${}_tq_y = 1 - {}_tp_y$. Let $T_1, \ldots, T_n$ denote the remaining lifetimes, and $N_t = \sum 1_{\{T_i \le t\}}$ the number of deaths up to and including time t. We apply the set-up from the product space model and take N to be defined on a separate probability space $(\Omega_2, \mathcal{F}^2)$. We denote by $\bar{\mathbb{F}}^N$ the augmented natural filtration of N, and take $\mathcal{F}^2 = \sigma(1_{\{T_i \le t\}},\, t \le T,\, i = 1, \ldots, n)$. In addition we consider the standard Black-Scholes market defined in Section 1 on the complete probability space $(\Omega_1, \mathcal{F}^1)$.The contracts considered here will be on the form

$$H = (n - N_T) g(X_T), \tag{4.14}$$

where g is taken to be some continuous function. If we choose for example $g(x) = (x-K)^+$, for some $K>0$, then we may think of (4.14) as a reinsurance contract which pays the amount $(X_T - K)^+$ (a European call option with strike price K) for each surviving policy-holder. Similarly, (4.14) corresponds to the present value of a unit-linked pure endowment contract with guarantee when $g(x) = \max(x, K)$, see e.g. Aase and Persson (1994) and Møller (1998a). The financial variance and standard deviation principles were also applied for the valuation of the contract (4.14) in Section 6 of Chapter 4. However, in contrast to the analysis given there, we here apply the results obtained in Chapter 5 and focus on the role of the amount of information available to the insurer.

First note that applying the traditional variance principle to the valuation of the contract (4.14) gives the premium

$$\begin{aligned}\tilde{u}_1(H) &= n\,{}_Tp_y\mathrm{E}[g(X_T)] + an\,{}_Tp_y(1-{}_Tp_y)\mathrm{E}[(g(X_T))^2] + an^2({}_Tp_y)^2\mathrm{Var}[g(X_T)] \\ &= n\,{}_Tp_y\Big(\mathrm{E}[g(X_T)] + a\,{}_Tq_y\mathrm{E}[(g(X_T))^2] + an\,{}_Tp_y\mathrm{Var}[g(X_T)]\Big), \qquad (4.15)\end{aligned}$$

where we have used the independence between N and X, standard rules for conditional expectations and variances and the fact that $(n - N_T) \sim \mathrm{Binomial}(n, {}_Tp_y)$. For more details and comments on this, we refer to Chapter 2.

Note that the claim (4.14) remains in the class $\mathcal{A}$ of contracts considered in Section 7 of Chapter 5, so that we can apply these results in this example. Hence we can investigate how the safety loading depends on the information available about the development within the portfolio of insured lives. In addition, we give the optimal strategies determined in Chapter 4 under the four different scenarios considered. These scenarios correspond to the following filtrations for the insurance portfolio and their associated filtrations on the product space:

1. The reinsurer receives no information about the number of deaths within the portfolio of insured lives before time T. This is described by the filtration $\bar{\mathbb{F}}^{2,\circ} = (\bar{\mathcal{F}}_t^{2,\circ})_{0\le t\le T}$ where $\bar{\mathcal{F}}_t^{2,\circ} = \{\emptyset, \Omega_2\}$, $0 \le t < T$ and $\bar{\mathcal{F}}_T^{2,\circ} = \bar{\mathcal{F}}_T^N$. The filtration on the product space is denoted $\mathbb{F}^\circ = \bar{\mathbb{F}}^1 \otimes \bar{\mathbb{F}}^{2,\circ}$, see Remark 5.5.5.

2. Information about the past development is revealed only at some intermediate time t_0, and after this time no additional information is available before time T. This corresponds to the following piecewise constant filtration $\bar{\mathbb{F}}^{2,p} = (\bar{\mathcal{F}}_t^{2,p})_{0\le t\le T}$

$$\bar{\mathcal{F}}_t^{2,p} = \begin{cases} \{\emptyset, \Omega_2\}, & t < t_0, \\ \mathcal{F}_{t_0}^N, & t_0 \le t < T, \\ \mathcal{F}_T^N, & t = T, \end{cases}$$

   and the corresponding product filtration is $\mathbb{F}^p = \bar{\mathbb{F}}^1 \otimes \bar{\mathbb{F}}^{2,p}$.

3. The reinsurer receives information about any death at the time of the death, that is $\bar{\mathbb{F}}^2 = (\bar{\mathcal{F}}_t^2)_{0\le t\le T}$. Similarly, $\mathbb{F} = \bar{\mathbb{F}}^1 \otimes \bar{\mathbb{F}}^2$.

4. All information about the future development within the insurance portfolio is known to the reinsurer at time 0, that is, immediately after the contract is sold. This situation corresponds to the filtration $\bar{\mathbb{F}}^{2,r} = (\bar{\mathcal{F}}_t^{2,r})_{0\leq t\leq T}$ where $\bar{\mathcal{F}}_t^{2,r} = \bar{\mathcal{F}}_T^N$, $0 \leq t \leq T$. We denote by $\mathbb{F}^r = \bar{\mathbb{F}}^1 \otimes \bar{\mathbb{F}}^{2,r}$ the corresponding filtration.

The filtrations $\mathbb{F}^\circ$ and $\mathbb{F}^r$ correspond to the minimal and maximal filtrations considered in Section 6 of Chapter 5, and hence they lead to the upper and lower bound, respectively, for the fair premium.

It follows from straightforward calculations (see e.g. Møller (1998a)) that

$$(n - N_T) \quad = \quad n\,{}_Tp_y - \int_0^T {}_{T-u}p_{y+u}(dN_u - \lambda_u\,du), \tag{4.16}$$

where $\lambda_u = (n-N_{u-})\mu_{y+u}$ is the predictable intensity for N. For later use, introduce the compensated counting process $\tilde{M} = N - \int \lambda_t\,dt$ which is a square-integrable martingale. Also, let $\hat{M}_t = \mathrm{E}[(n - N_T) \mid \mathcal{F}_t^N] = (n - N_t){}_{T-t}p_{y+t}$. For $0 \leq s < t \leq T$, (4.16) together with the Fubini theorem gives that

$$\begin{aligned}
\mathrm{E}\left[\left(\hat{M}_t - \hat{M}_s\right)^2\right] &= \mathrm{E}\left[\left(\int_s^t {}_{T-u}p_{y+u}\,d\tilde{M}_u\right)^2\right] \\
&= \mathrm{E}\left[\int_s^t ({}_{T-u}p_{y+u})^2\lambda_u\,du\right] \\
&= n\,{}_Tp_y \int_s^t {}_{T-u}p_{y+u}\,\mu_{y+u}\,du \\
&= n\,{}_Tp_y\left({}_{T-t}p_{y+t} - {}_{T-s}p_{y+s}\right).
\end{aligned}$$

Also, it follows by the independence between N and X that

$$\tilde{\mathrm{E}}[(n - N_T)g(X_T)] = \mathrm{E}[(n - N_T)]\,\tilde{\mathrm{E}}[g(X_T)] = n\,{}_Tp_y\,\tilde{\mathrm{E}}[g(X_T)]$$

We now deal with the four cases separately and determine the premiums and the associated optimal strategies ϑ^*. Recall that by Lemma 5.5.11 there exists a unique pair (c^g, ξ^g) where $c^g \in \mathbb{R}$ and $\xi^g \in \tilde{\Theta}(\bar{\mathbb{F}}^1)$ such that

$$g(X_T) = c^g + \int_0^T \xi_t^g\,dX_t. \tag{4.17}$$

Furthermore, since X is a Markov process w.r.t. $\tilde{P}$

$$F^g(t, X_t) = \tilde{\mathrm{E}}[g(X_T) \mid \bar{\mathcal{F}}_t^1] = c^g + \int_0^t \xi_u^g\,dX_u,$$

and provided that $F^g \in C^{1,2}$ it follows by application of Itô's formula and the uniqueness of the canonical decomposition, that $\xi_t^g = F_x^g(t, X_t)$, where $F_x^g(t,x)$ denotes the partial derivative of the function $F^g(t,x)$ w.r.t. x.

Case 1. (No information) The filtration $\mathbb{F}^\circ$ is exactly the minimal filtration considered in Section 6 of Chapter 5, and from this we immediately get the premium

$$\begin{aligned}
v_1^\circ(H) &= n\,{}_Tp_y\,\tilde{\mathrm{E}}[g(X_T)] + a\mathrm{E}[\mathrm{Var}[H \mid \mathcal{F}_T^1]] \\
&= n\,{}_Tp_y\,\tilde{\mathrm{E}}[g(X_T)] + a\mathrm{E}[(g(X_T))^2]\,\mathrm{Var}[(n - N_T)],
\end{aligned}$$

where we have used the independence between N and X in the second inequality. Again, since $(n-N_T) \sim \text{Bin}(n, {}_Tp_y)$, we find that $\text{Var}[(n-N_T)] = n\,{}_Tp_y(1-{}_Tp_y)$, so that we have shown that the fair premium for (4.14) in the case of no information before time T is

$$v_1^\circ(H) = n\,{}_Tp_y\left(\tilde{\text{E}}[g(X_T)] + a\text{E}[(g(X_T))^2]\,{}_Tq_y\right). \tag{4.18}$$

The optimal strategy can be determined by use of Theorem 5.5.14, since the filtration $\mathbb{F}^\circ$ is identical to the one considered there. By that theorem it follows that ϑ^H is determined such that

$$\tilde{E}\left[H\,\middle|\mathcal{F}_T^1\right] = H_0 + \int_0^T \vartheta_t^H\,dX_t.$$

Thus, by using (4.17) we see that

$$\tilde{E}\left[H\,\middle|\mathcal{F}_T^1\right] = n\,{}_Tp_y\,g(X_T) = n\,{}_Tp_y\,c^g + \int_0^T n\,{}_Tp_y\,\xi_t^g\,dX_t,$$

so that $\vartheta^H = n\,{}_Tp_y\,\xi^g$. The optimal strategy is now obtained by inserting this into the expression given in (2.3):

$$\vartheta_t^* = n\,{}_Tp_y\,\xi_t^g + \frac{\tilde{Z}_t\lambda_t}{2a}, \tag{4.19}$$

where $\lambda = \frac{\alpha-r}{\sigma^2 X}$. The first term in (4.19) is ϑ^H, and this is simply $n\,{}_Tp_y = \hat{M}_0$ times the process ξ^g. This is also equal to the hedge for the modified claim $H' = n\,{}_Tp_y\,g(X_T)$, where the unknown number of survivors has been replaced by the expected number $\hat{M}_0$; see Chapter 2 for comments in this direction. The second term in (4.19) is a general correction term which is closely related to the financial variance principle and which is always present (i.e. for any claim), see Chapter 4.

Remark 4.1 It was shown in Chapter 2 that charging ${}_Tp_y\tilde{\text{E}}[g(X_T)] + \varepsilon$, $\varepsilon > 0$, as single premium for each policy-holder will imply that the insurer's gain converges towards $+\infty$ a.s. as the size of the portfolio is increased. Thus, the result (4.18) indicates that the financial variance principle may not be a reasonable principle, since the premium is proportional to n. The reason for this may be that the variance principle is punishing large variances too hard and small variances too little, see Chapter 4 for a further investigation of the variance principle. □

Case 2. (Piecewise constant information) We apply Theorem 5.7.3 in order to determine the hedging error $J_0(\mathbb{F}^p)$. First, introduce processes $N^{(1)}$ and $N^{(2)}$ as in Section 7 of Chapter 5; $N^{(1)}$ is defined by

$$N_t^{(1)} = \tilde{\text{E}}[g(X_T) \mid \mathcal{F}_t^1] = c^g + \int_0^t \xi_u^g\,dX_u, \tag{4.20}$$

where $c^g = \tilde{\text{E}}[g(X_T)]$ and where $\xi^g \in \tilde{\Theta}(\bar{\mathbb{F}}^1)$. The existence and uniqueness of such a process ξ^g follows from Lemma 5.5.11. The process $N^{(2)}$ is given by

$$N_t^{(2)} = \tilde{\text{E}}[(n-N_T) \mid \mathcal{F}_t^p] = \text{E}[(n-N_T) \mid \mathcal{F}_t^p],$$

where the second equality is again a consequence of the independence between the two sources of risk, and hence

$$N_t^{(2)} = \begin{cases} n\,{}_Tp_y, & t < t_0, \\ (n - N_{t_0})\,{}_{T-t_0}p_{y+t_0}, & t_0 \le t < T, \\ n - N_T, & t = T, \end{cases} \tag{4.21}$$

so that $N_t^{(2)} = \hat{M}_t$ for $t \in \{0, t_0, T\}$ by definition of the martingale $\hat{M}$. From this result we get by definition of the square bracket process $[N^{(2)}]$ that

$$[N^{(2)}]_t = \left(\hat{M}_{t_0} - \hat{M}_0\right)^2 1_{\{t \ge t_0\}} + \left(\hat{M}_T - \hat{M}_{t_0}\right)^2 1_{\{t \ge T\}}.$$

Introduce the function $f : [0,T] \mapsto I\!R_+$ given by

$$f(t) := \mathrm{E}\left[\frac{Z_t}{\tilde{Z}_t}\left(\tilde{\mathrm{E}}[g(X_T) \mid \mathcal{F}_t^1]\right)^2\right], \tag{4.22}$$

for $0 \le t \le T$, so that in particular $f(T) = \mathrm{E}[(g(X_T))^2]$. Furthermore, Lemma 5.7.8 shows that $f(s) \le f(t)$ for $0 \le s < t \le T$. Thus, by Theorem 5.7.3 the hedging error (and hence the loading) is given by

$$\begin{aligned} J_0(I\!F^p) &= \tilde{\mathrm{E}}\left[\frac{(N_{t_0}^{(1)})^2}{\tilde{Z}_{t_0}}\left(\hat{M}_{t_0} - \hat{M}_0\right)^2\right] + \tilde{\mathrm{E}}\left[\frac{(N_T^{(1)})^2}{\tilde{Z}_T}\left(\hat{M}_T - \hat{M}_{t_0}\right)^2\right]. \\ &= \mathrm{E}\left[\frac{Z_{t_0}}{\tilde{Z}_{t_0}}\left(\tilde{\mathrm{E}}[g(X_T) \mid \mathcal{G}'_{t_0}]\right)^2\right] n\,{}_Tp_y\left({}_{T-t_0}p_{y+t_0} - {}_Tp_y\right) \\ &\quad + \mathrm{E}\left[g(X_T)^2\right] n\,{}_Tp_y\left(1 - {}_{T-t_0}p_{y+t_0}\right) \\ &= n\,{}_Tp_y\left(f(t_0)\left({}_{T-t_0}p_{y+t_0} - {}_Tp_y\right) + f(T)\,{}_{T-t_0}q_{y+t_0}\right). \end{aligned}$$

This also shows that the loading determined in this case is indeed smaller than the loading determined in Case 1 (No information). We have shown that

$$v_1^p(H) = n\,{}_Tp_y\left(\tilde{\mathrm{E}}[g(X_T)] + af(t_0)\left({}_{T-t_0}p_{y+t_0} - {}_Tp_y\right) + af(T)\,{}_{T-t_0}q_{y+t_0}\right).$$

The optimal strategy for the financial variance principle in the case of the piecewise constant filtration is here determined by applying Theorem 5.2.6. First we note that Lemma 5.7.2 allows us to express the Kunita-Watanabe decomposition of the $\tilde{P}$-martingale $\tilde{V}^p = \tilde{\mathrm{E}}[H \mid I\!F^p]$ in terms of the processes $N^{(1)}$ and $N^{(2)}$ defined by (4.20) and (4.21). With the notation of this lemma, we have that

$$\xi_t^{H,\tilde{P}} = \xi_t^g N_{t-}^{(2)} = \begin{cases} \xi_t^g n\,{}_Tp_y, & t \le t_0, \\ \xi_t^g (n - N_{t_0})\,{}_{T-t_0}p_{y+t_0}, & t_0 < t \le T, \end{cases}$$

and

$$L_t^{H,\tilde{P}} = N_{t_0}^{(1)}(\hat{M}_{t_0} - \hat{M}_0)1_{\{t \ge t_0\}} + N_T^{(1)}(\hat{M}_T - \hat{M}_{t_0})1_{\{t \ge T\}}.$$

Thus, by Theorem 5.2.6

$$\vartheta_t^H = \begin{cases} \xi_t^g n\,{}_T p_y, & t \le t_0, \\ \xi_t^g (n - N_{t_0})\,{}_{T-t_0}p_{y+t_0} - \tilde{\zeta}_t \tilde{Z}_{t_0}^{-1} N_{t_0}^{(1)} (\hat{M}_{t_0} - \hat{M}_0), & t_0 < t \le T. \end{cases}$$

This is now inserted in (2.3) in order to obtain the optimal strategy ϑ^*. We see that this strategy differs from the one obtained in case 1 on the interval $(t_0, T]$. This difference is explained by the fact that the reinsurer with filtration $\mathbb{F}^p$ receives information about the actual number of deaths up to and including time t_0 at time t_0. This allows him to adjust his strategy according to this new information, and the direct consequence of this is that $n\,{}_T p_y$ is replaced by $(n - N_{t_0})\,{}_{T-t_0}p_{y+t_0}$. However, in addition a correction term appears in ϑ_t^H for $t > t_0$ and this term is related to the adjustment in the expected number of survivors.

Case 3. (Natural filtration) Consider now the situation where the filtration is given by $\mathbb{F} = \bar{\mathbb{F}}^1 \otimes \bar{\mathbb{F}}^2$. In this case, we obtain from (4.16)

$$N_t^{(2)} = \hat{M}_t = (n - N_t)_{T-t}p_{y+t} = n\,{}_T p_y - \int_0^t {}_{T-u}p_{y+u}\, d\tilde{M}_u.$$

By standard rules for computation of the process [.,.] it follows that

$$[N^{(2)}]_t = \int_0^t ({}_{T-u}p_{y+u})^2 dN_u,$$

so that by the Fubini theorem

$$\mathrm{E}\left[[N^{(2)}]_t\right] = \mathrm{E}\left[\int_0^t ({}_{T-u}p_{y+u})^2 \lambda_u\, du\right] = n\,{}_T p_y \int_0^t {}_{T-u}p_{y+u}\, \mu_{y+u}\, du.$$

Thus, by Corollary 5.7.6 we find that

$$\begin{aligned} J_0(\mathbb{F}) &= \int_0^T \mathrm{E}\left[\frac{Z_u}{\tilde{Z}_u}\left(\tilde{\mathrm{E}}[g(X_T) \mid \mathcal{F}_u^1]\right)^2\right] n\,{}_T p_y\,{}_{T-u}p_{y+u}\, \mu_{y+u}\, du \\ &= n\,{}_T p_y \int_0^T f(u)\,{}_{T-u}p_{y+u}\, \mu_{y+u}\, du, \end{aligned}$$

under sufficient integrability conditions on $g(X_T)$. For example, it is sufficient that $g(X_T) \in L^{2+\varepsilon}(P)$, for some $\varepsilon > 0$, see Remark 5.7.7. We have now shown that

$$v_1(H) = n\,{}_T p_y \left(\tilde{\mathrm{E}}[g(X_T)] + a\int_0^T f(u)\,{}_{T-u}p_{y+u}\, \mu_{y+u}\, du\right).$$

The optimal strategy is again determined via Theorem 5.2.6 which states that

$$\vartheta_t^H = \xi_t^g (n - N_{t-})_{T-t}p_{y+t} + \tilde{\zeta}_t \int_0^t \tilde{Z}_u^{-1} N_u^{(1)}\,{}_{T-u}p_{y+u}\, d\tilde{M}_u,$$

so that the optimal strategy is

$$\vartheta_t^* = \xi_t^g (n - N_{t-})_{T-t}p_{y+t} + \tilde{\zeta}_t \int_0^t \tilde{Z}_u^{-1} N_u^{(1)}\,{}_{T-u}p_{y+u}\, d\tilde{M}_u + \frac{\tilde{Z}_t \lambda_t}{2a}.$$

These results are identical to the ones obtained (via heuristic calculations) in Section 6 in Chapter 4. We have included the results here for comparison and since they follow directly (up to the above mentioned integrability conditions) from the general results obtained in Chapter 5.

Note that the first term in ϑ^H is again of the structure which also appeared in Case 1 and 2, namely the process ξ^g multiplied by the conditional expected number of survivors. Here, the correction term is now an integral with respect to the compensated counting process $\tilde{M}$ and hence it really depends on the entire past development within the portfolio of insured lives. Consider now an infinitesimal interval $(t, t+dt]$. Note that if $d\tilde{M}_t > 0$, this means (loosely speaking) that $dN_t > \lambda_t\, dt$ which again can be interpreted as saying that the number dN_t of deaths during $(t, t+dt]$ exceeds the expected number of deaths $\lambda_t dt$. Since $\tilde{\zeta}_t < 0$, this means that a negative term is added to the correction term. Similarly, if $d\tilde{M}_t > 0$ then (4.16) shows that the expected number of survivors decreases and hence the factor on ξ^g will also decrease.

Case 4. (Full initial information) Consider the filtration $\mathbb{F}^r = \bar{\mathbb{F}}^1 \otimes \bar{\mathbb{F}}^{2,r}$, where $\bar{\mathcal{F}}_t^{2,r} = \bar{\mathcal{F}}_T^N$ for all $0 \leq t \leq T$. This represents the hypothetical situation where all information about the policy-holders' future life-times is available immediately after the signing of the contract at time 0. In this situation, we can apply the results obtained in Section 6 of Chapter 5, since $\mathbb{F}^r$ leads to the lower bound for the fair premium which was derived there. By (5.6.2) and (5.6.3), we find that

$$J_0(\mathbb{F}^r) = \frac{\mathrm{E}[(L_0^{H,\tilde{P}})^2]}{\mathrm{E}[\tilde{Z}_T^2]} = \frac{\mathrm{Var}[\tilde{\mathrm{E}}[H \mid \mathcal{G}_0]]}{\mathrm{E}[\tilde{Z}_T^2]},$$

where, by definition $\mathrm{E}[\tilde{Z}_T^2] = \tilde{Z}_0$, and where

$$\mathrm{Var}[\tilde{\mathrm{E}}[H \mid \mathcal{G}_0]] = \mathrm{Var}\left[(n - N_T)\tilde{\mathrm{E}}[g(X_T)]\right] = n\, {}_Tp_y(1 - {}_Tp_y)\left(\tilde{\mathrm{E}}[g(X_T)]\right)^2.$$

Now insert this expression into (5.6.3) and use (4.22) to obtain the premium

$$\begin{aligned} v_1^r(H) &= n\, {}_Tp_y\tilde{\mathrm{E}}[g(X_T)] + a\frac{1}{\tilde{Z}_0}\left(\tilde{\mathrm{E}}[g(X_T)]\right)^2 n\, {}_Tp_y(1 - {}_Tp_y) \\ &= n\, {}_Tp_y\left(\tilde{\mathrm{E}}[g(X_T)] + a\, {}_Tq_y\, f(0)\right). \end{aligned}$$

The optimal strategy can be determined by applying Theorem 5.2.6. In Section 6 of Chapter 5 we showed that $L_t^{H,\tilde{P}} = L_0^{H,\tilde{P}} = \tilde{\mathrm{E}}[H \mid \mathcal{G}_0] - \tilde{\mathrm{E}}[H]$ for all $0 \leq t \leq T$. Furthermore, by Lemma 5.7.2, $\xi_t^{H,\tilde{P}} = \xi_t^g(n - N_T)$, since $N_t^{(2)} = \tilde{\mathrm{E}}[(n - N_T) \mid \mathcal{F}_T^N] = (n - N_t)$ for all $0 \leq t \leq T$. Thus

$$\vartheta_t^H = \xi_t^g(n - N_T) - \tilde{\zeta}_t\tilde{\mathrm{E}}[g(X_T)]e^{-\nu^2 T}\left((n - N_T) - n\, {}_Tp_y\right),$$

which leads to the strategy

$$\vartheta_t^* = \xi_t^g(n - N_T) - \tilde{\zeta}_t\tilde{\mathrm{E}}[g(X_T)]e^{-\nu^2 T}\left((n - N_T) - n\, {}_Tp_y\right) + \frac{\tilde{Z}_t\lambda_t}{2a}.$$

This strategy differs from the previous ones in that ξ^g is now multiplied with the actual number of survivors $(n - N_T)$ since this is known to the reinsurer at time 0. However, he is not able to hedge the claim perfectly, since the valuation of the contract is not based on this information, that is, the premium is not allowed to be random.

Remark 4.2 From the previous calculations we can also determine the fair premiums for the standard deviation principle. By Theorem 5.4.2, this is given by

$$v_2^{\cdot}(H) = \tilde{\mathrm{E}}[H] + \tilde{a}\sqrt{J_0(\cdot)},$$

where $\tilde{a}$ is defined by (2.5) provided that $a^2 > \mathrm{Var}[\tilde{Z}_T] = e^{\nu^2 T} - 1$. By inserting the above obtained expressions, we find that

$$\begin{aligned}
v_2^{\circ}(H) &= n\,{}_Tp_y\tilde{\mathrm{E}}[g(X_T)] + \tilde{a}n^{1/2}\left(f(T)\,{}_Tp_y\,{}_Tq_y\right)^{1/2}, \\
v_2^{p}(H) &= n\,{}_Tp_y\tilde{\mathrm{E}}[g(X_T)] + \tilde{a}n^{1/2}\left({}_Tp_y\,f(t_0)({}_{T-t_0}p_{y+t_0} - {}_Tp_y)\right. \\
&\quad \left. + {}_Tp_y f(T)({}_{T-t_0}q_{y+t_0})\right)^{1/2}, \\
v_2(H) &= n\,{}_Tp_y\tilde{\mathrm{E}}[g(X_T)] + \tilde{a}n^{1/2}\left({}_Tp_y\int_0^T f(u)\,{}_{T-u}p_{y+u}\,\mu_{y+u}\,du\right)^{1/2}, \\
v_2^{r}(H) &= n_Tp_y\tilde{\mathrm{E}}[g(X_T)] + \tilde{a}n^{1/2}\left({}_Tp_y\,{}_Tq_y\,f(0)\right)^{1/2},
\end{aligned}$$

where the increasing function f is defined by (4.22). The associated optimal strategies are obtained directly by inserting ϑ^H into (2.4). In particular, it is noted that the ratio between the safety loading and the premium converges to 0 as n converges to $+\infty$. Thus, these premiums determined by the financial standard deviation principle will not necessarily lead to an infinite profit a.s. as the size of the portfolio increases; see Remark 4.1 for a similar comment on the financial variance principle. □

5 The general Markov case

In this section, we give a general framework which allows for dependence between the stock and the additional risk. We consider, in addition to a stock price process X, some insurance risk process U, which is not necessarily stochastically independent of X, and we assume that the pair (X, U) is a Markov process under the variance optimal martingale measure for X. In this setting, we focus on claims of the form $\Psi(X_T, U_T)$, where Ψ is some measurable function. The computation of the fair premiums and optimal strategies then essentially boils down to solving certain partial differential equations; explicit results seem to be difficult to obtain for realistic models. As an example, we investigate in more detail the situation where the drift and the volatility of the stock price process of an insurance company are affected by the occurrence of certain insurance events. We end this section with a very simple example, where the fair premium and the optimal trading strategy under the financial variance principle can be determined explicitly.

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be as in Section 2 of Chapter 5 and let X be a continuous semimartingale with canonical decomposition

$$X = X_0 + M^x + A^x.$$

Let U be another special semimartingale[1] defined on $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with canonical decomposition under P

$$U = U_0 + M^{u,c} + M^{u,d} + A^u,$$

where $M^{u,c}$ is a continuous local P-martingale, $M^{u,d}$ is a purely discontinuous local P-martingale and A^u is a predictable process of finite variation. We work under the following assumption

Assumption 5.1 $\langle M^x, M^{u,c} \rangle = 0$.

Thus, the continuous martingale parts of X and U are strongly P-orthogonal. As we shall see below, this may well include cases where X and U are not stochastically independent. Assume in addition that Assumption 5.2.3 is satisfied, so that the variance optimal martingale measure $\tilde{P}$ for X exists. In particular, X is a continuous local $\tilde{P}$-martingale. It follows by He, Wang, Yan (1992, Theorem 12.18) and the fact that $\tilde{P} \ll P$ since X is continuous that U is also a semimartingale under $\tilde{P}$ and we denote by $\tilde{M}^{u,c}$ its unique continuous martingale part under $\tilde{P}$. By He, Wang, Yan (1992, Theorem 12.14), $[X,U]^P$ and $[X,U]^{\tilde{P}}$ are $\tilde{P}$-indistinguishable. Furthermore, the continuity of X implies that

$$[X,U]^P = \langle M^x, M^{u,c} \rangle^P + \sum_{s>0} \Delta X_s \Delta U_s = \langle M^x, M^{u,c} \rangle^P,$$

and similarly

$$[X,U]^{\tilde{P}} = \langle X, \tilde{M}^{u,c} \rangle^{\tilde{P}} + \sum_{s>0} \Delta X_s \Delta U_s = \langle X, \tilde{M}^{u,c} \rangle^{\tilde{P}},$$

and hence $\langle X, \tilde{M}^{u,c} \rangle^{\tilde{P}} = \langle M^x, M^{u,c} \rangle^P = 0$. We proceed under the following assumption

Assumption 5.2 *Assume that (X,U) is a $\tilde{P}$-Markov process and that U is a special semimartingale under $\tilde{P}$.*

[1] Recall that a semimartingale U is called *special* if there exists a *predictable* process A^u of finite variation and a local martingale M^u such that $X = X_0 + M^u + A^u$; if such a decomposition exists, it is unique and it is called the *canonical decomposition*. In particular, any continuous semimartingale is special.

Remark 5.3 For example, U is a special semimartingale under $\tilde{P}$ if U is locally bounded or if $\sum(|\Delta U| 1_{\{|\Delta U|>1\}}) \in \mathcal{A}^+_{loc}(\tilde{P})$, see e.g. He, Wang, Yan (1992, Corollary 11.26). Another sufficient condition is that the process $Z_t = \mathrm{E}[\frac{d\tilde{P}}{dP} \mid \mathcal{F}_t]$ is continuous. In fact, this implies that $[U, Z]$ is continuous; thus $\int \frac{1}{Z}\, d[U, Z]$ is continuous and hence a special semimartingale, and He, Wang, Yan (1992, Theorem 12.18) gives that U is a special semimartingale under $\tilde{P}$. A sufficient condition for Z to be continuous is that the mean-variance tradeoff process $\hat{K}$ has a deterministic terminal value $\hat{K}_T$, see Schweizer (1999, Lemma 4.7). □

Denote by γ^u the jump measure of U, i.e. the integer valued random measure defined by

$$\gamma^u(\omega, dt, dy) = \sum_{s>0} 1_{\{\Delta U_s(\omega)\neq 0\}} \varepsilon_{(s,\Delta U_s(\omega))}(dt, dy),$$

where ε_y is the Dirac measure at y. Introduce in addition the dual predictable projection (the compensator) $\tilde{\gamma}^{u,p}$ of γ^u under $\tilde{P}$. The canonical decomposition of U under $\tilde{P}$ is then

$$U = U_0 + \tilde{M}^{u,c} + \int\int_{I\!R} y(\gamma^u - \tilde{\gamma}^{u,p}) + \tilde{A}^u,$$

where $\tilde{A}^u$ is predictable and of finite variation.

Consider a contract on the form

$$H = \Psi(X_T, U_T), \tag{5.1}$$

where Ψ is a bounded measurable function. Define now a $\tilde{P}$-martingale $\tilde{V}$ by

$$\begin{aligned} \tilde{V}_t &:= \tilde{\mathrm{E}}[\Psi(X_T, U_T) \mid \mathcal{F}_t] \\ &= \tilde{\mathrm{E}}[\Psi(X_T, U_T) \mid (X_t, U_t)] \\ &=: F^{\psi}(t, X_t, U_t), \end{aligned}$$

where we have used the Markov property in the second equality. We assume that $F^{\psi} \in C^{1,2,2}$. Using He, Wang, Yan (1992, Corollary 11.27), we can show the following representation result which is similar to Elliott and Föllmer (1991, Proposition 3.1). They consider a claim of the form $h(X_T)$ and assume that X is a (not necessarily continuous) local martingale with respect to some measure P and the P-augmentation of the natural filtration of X. Under the assumption that X is a Markov process, they give a representation result for the martingale $F(t, X_t) := \mathrm{E}_P[h(X_T) \mid \mathcal{F}_t]$ (assuming in addition that $F \in C^{1,2}$) which involves an integral with respect to the continuous local martingale part of X and an integral with respect to the compensated random measure associated with the jumps of X. In the present set-up, we consider the more simple situation where X is only a continuous local martingale, but assume in addition that the claim depends on the terminal value of the special semimartingale U. The Markov-assumption in Elliott and Föllmer (1991) is here replaced by the assumption that the pair (X, U) is a Markov process under $\tilde{P}$.

Proposition 5.4 *Assume that $F^{\psi} \in C^{1,2,2}$. Then the $\tilde{P}$-martingale $\tilde{V}$ admits the representation*

$$\begin{aligned}\tilde{V}_t &= F^{\psi}(0, X_0, U_0) + \int_0^t F_x^{\psi}(s, X_s, U_{s-}) dX_s + \int_0^t F_u^{\psi}(s, X_s, U_{s-}) d\tilde{M}_s^{u,c} \\ &\quad + \int_0^t \int_{I\!R} \Delta F^{\psi}(s, y)(\gamma^u(ds, dy) - \tilde{\gamma}^{u,p}(ds, dy)), \qquad (5.2)\end{aligned}$$

where

$$\Delta F^{\psi}(s, y) = F^{\psi}(s, X_s, U_{s-} + y) - F^{\psi}(s, X_s, U_{s-}).$$

Assume furthermore that $\langle X \rangle = \int \rho_s^x \, ds$, $\langle \tilde{M}^{u,c} \rangle = \int \rho_s^u \, ds$, $\tilde{A}^u = \int \tilde{\alpha}_s^u \, ds$ and $\tilde{\gamma}^{u,p} = \tilde{m}(s, dy)\tilde{\lambda}_s^u \, ds$. Then F^{ψ} solves the equation

$$\begin{aligned}0 &= F_t^{\psi}(t, X_t, U_{t-}) + \tilde{\alpha}_t^u F_u^{\psi}(t, X_t, U_{t-}) + \frac{1}{2}\rho_t^x F_{xx}^{\psi}(t, X_t, U_{t-}) + \frac{1}{2}\rho_t^u F_{uu}^{\psi}(t, X_t, U_{t-}) \\ &\quad + \int_{I\!R} (\Delta F^{\psi}(t, y) - y F_u^{\psi}(t, X_t, U_{t-}))\tilde{m}(t, dy)\tilde{\lambda}_t^u,\end{aligned}$$

with boundary condition

$$F^{\psi}(T, X_T, U_T) = \Psi(X_T, U_T).$$

Proof: We refer to He, Wang, Yan (1992, Chapter 11). Define

$$U' = U - U_0 - \int \int y 1_{(|y|>1)} \gamma^u(ds, dy),$$

which is a special semimartingale under $\tilde{P}$; the canonical decomposition of U' is denoted

$$U' = \tilde{M}^{u,c} + \int \int y 1_{(|y| \le 1)} (\gamma^u - \tilde{\gamma}^{u,p}) + \tilde{A}^{u'},$$

see proof of their Theorem 11.25. From He, Wang, Yan (1992, Corollary 11.27) it now follows that if $F^{\psi} \in C^{1,2,2}$ then $\tilde{V}$ is a special semimartingale with canonical decomposition

$$\tilde{V} = \tilde{V}_0 + \tilde{M}^{\psi} + \tilde{A}^{\psi},$$

where

$$\begin{aligned}\tilde{M}^{\psi} &= \int F_x^{\psi} \, dX + \int F_u^{\psi} \, d\tilde{M}^{u,c} + \int \int_{I\!R} \Delta F^{\psi} (\gamma^u - \tilde{\gamma}^{u,p}) \\ \tilde{A}^{\psi} &= \int F_t^{\psi} \, dt + \int F_u^{\psi} \, d\tilde{A}^{u'} + \frac{1}{2} \int F_{xx}^{\psi} \, d\langle X \rangle + \frac{1}{2} \int F_{uu}^{\psi} \, d\langle \tilde{M}^{u,c} \rangle \\ &\quad + \int \int_{I\!R} (\Delta F^{\psi} - y F_u^{\psi} 1_{(|y| \le 1)}) \tilde{\gamma}^{u,p}.\end{aligned}$$

Since $\tilde{V}$ by definition is a $\tilde{P}$-martingale, $\tilde{A}^{\psi} \equiv 0$, and this proves the first part of the proposition. Furthermore, He, Wang, Yan (1992, Corollary 11.26) establishes a connection between $\tilde{A}^u$ and $\tilde{A}^{u'}$, namely

$$\tilde{A}^u = \tilde{A}^{u'} + \int \int_{I\!R} (y 1_{(|y|>1)}) \tilde{\gamma}^{u,p}. \qquad (5.3)$$

Now simply add and subtract $\int F_x^\psi \, d\tilde{A}^u$ in the expression for $\tilde{A}^\psi$ obtained above and use (5.3) to see that

$$\begin{aligned}\tilde{A}^\psi &= \int F_t^\psi \, dt + \int F_u^\psi \, d\tilde{A}^u + \frac{1}{2}\int F_{xx}^\psi \, d\langle X\rangle + \frac{1}{2}\int F_{uu}^\psi \, d\langle \tilde{M}^{u,c}\rangle \\ &\quad + \int\int_{I\!R} (\Delta F^\psi - y F_u^\psi))\tilde{\gamma}^{u,p}.\end{aligned}$$

This proves the second part of the proposition by using the simplifying expressions for $\langle X\rangle$, $\langle \tilde{M}^{u,c}\rangle$, $\tilde{A}^u$ and $\tilde{\gamma}^{u,p}$. □

This representation result is very important for our applications, since it essentially determines the Galtchouk-Kunita-Watanabe decomposition for $\tilde{V}$ under $\tilde{P}$, which can be used together with Theorem 5.2.6 to obtain the crucial decomposition (5.2.7). For completeness, we state this result as a corollary.

Corollary 5.5 *Assume that $F^\psi \in C^{1,2,2}$. Then the Galtchouk-Kunita-Watanabe under $\tilde{P}$ for $\tilde{V}$ is*

$$\tilde{V}_t = F^\psi(0, X_0, U_0) + \int_0^t F_x^\psi(s, X_s, U_{s-})\, dX_s + L_t^\psi, \tag{5.4}$$

where

$$L_t^\psi = \int_0^t F_u^\psi(s, X_s, U_{s-}) d\tilde{M}_s^{u,c} + \int_0^t \int_{I\!R} \Delta F^\psi(s,y)(\gamma^u(ds,dy) - \tilde{\gamma}^{u,p}(ds,dy)). \tag{5.5}$$

Proof: The representation (5.4) and (5.5) follows immediately from Proposition 5.4. Hence we only need to verify that L^ψ and X are strongly $\tilde{P}$-orthogonal, and this can be shown as in the proof of Lemma 5.7.2: Since $\int F_x^\psi \, dX$ and $\int F_u^\psi \, d\tilde{M}^{u,c}$ are local $\tilde{P}$-martingales, it follows from the proof of Proposition 5.4 that $\int\int_{I\!R} \Delta F^\psi(\gamma^u - \tilde{\gamma}^{u,p})$ is also a local $\tilde{P}$-martingale. Hence, this process is a purely discontinuous local $\tilde{P}$-martingale, which, by definition, is strongly orthogonal to any continuous local martingale. This shows that X and L^ψ are indeed strongly $\tilde{P}$-orthogonal, since X and $\tilde{M}^{u,c}$ are strongly $\tilde{P}$-orthogonal by assumption. As in the proof of Lemma 5.7.2 we obtain

$$[\tilde{V}] = \int (F_x^\psi)^2 d\langle X\rangle + \int (F_u^\psi)^2 d\langle \tilde{M}^{u,c}\rangle + \left[\int\int_{I\!R} \Delta F^\psi(\gamma^u - \tilde{\gamma}^{u,p})\right].$$

Since $\tilde{V}$ is bounded, $\sup_{0\le t\le T} |\tilde{V}_t|^2 \in L^2(\tilde{P})$, which implies via the Burkholder-Davis-Gundy inequality that $[\tilde{V}]_T \in L^1(\tilde{P})$. Applying this inequality once more, we find that $\int F_x^\psi \, dX$, $\int F_u^\psi \, d\tilde{M}^{u,c}$ and $\int\int_{I\!R} \Delta F^\psi(\gamma^u - \tilde{\gamma}^{u,p})$ are actually $\tilde{P}$-square-integrable $\tilde{P}$-martingales. □

In order to compute the fair premiums of Theorem 5.4.1 and 5.4.2, we need in addition to determine the process $[L^\psi]$, see for instance Corollary 5.2.8. Since

$\int\int_{I\!R} \Delta F^\psi(\gamma^u - \tilde{\gamma}^{u,p})$ is a purely discontinuous local martingale, we find that

$$\begin{aligned}
&\left[\int\int_{I\!R} \Delta F^\psi(\gamma^u - \tilde{\gamma}^{u,p})\right]_t \\
&= \sum_{0<s\leq t} \left(\int_0^s \int_{I\!R} \Delta F^\psi(\gamma^u - \tilde{\gamma}^{u,p}) - \int_0^{s-} \int_{I\!R} \Delta F^\psi(\gamma^u - \tilde{\gamma}^{u,p})\right)^2 \\
&= \sum_{0<s\leq t} \left(\int_{I\!R} \Delta F^\psi(s,y)(\gamma^u(\{s\},dy) - \tilde{\gamma}^{u,p}(\{s\},dy))\right)^2. \qquad (5.6)
\end{aligned}$$

In general, $\tilde{\gamma}^{u,p}(\{s\}, I\!R)$ may differ from 0. This is for example the case, if U has jumps occurring at fixed deterministic times with a strictly positive probability. By Corollary II.1.19 of Jacod and Shiryaev (1986), $\tilde{\gamma}^{u,p}(\{s\}, I\!R) = 0$ for all $s \in [0,T]$ if and only if U is *quasi-left-continuous*, that is, if and only if $\Delta U_\tau = 0$ a.s. on the set $\{\tau < \infty\}$ for any predictable time τ. (In particular, $\tau = t$, $t \in I\!R$, is a predictable time.) If we in addition assume that U is quasi-left-continuous, then (5.6) simplifies to

$$\left[\int\int_{I\!R} \Delta F^\psi(\gamma^u - \tilde{\gamma}^{u,p})\right]_t = \int_0^t \int_{I\!R} (\Delta F^\psi(s,y))^2 \gamma^u(ds,dy).$$

In the general case, Corollary 5.2.8 gives that

$$\begin{aligned}
J_0(I\!F) &= \tilde{\mathrm{E}}\left[\int_0^T \tilde{Z}^{-1}(F_u^\psi)^2 d\langle \tilde{M}^{u,c}\rangle\right] \qquad (5.7) \\
&\quad + \tilde{\mathrm{E}}\left[\sum_{0<t\leq T} \tilde{Z}_t^{-1} \left(\int_{I\!R} \Delta F^\psi(\gamma^u(\{t\},dy) - \tilde{\gamma}^{u,p}(\{t\},dy))\right)^2\right].
\end{aligned}$$

Furthermore, by Theorem 5.2.6 we have that

$$\begin{aligned}
\vartheta_t^H &= F_x^\psi(t, X_t, U_{t-}) - \tilde{\zeta}_t \int_0^t \tilde{Z}_s^{-1} F^\psi(s, X_s, U_{s-})\, d\tilde{M}_s^{u,c} \qquad (5.8) \\
&\quad - \tilde{\zeta}_t \int_0^{t-} \tilde{Z}_s^{-1} \int_{I\!R} \Delta F^\psi(s,y)(\gamma^u(ds,dy) - \tilde{\gamma}^{u,p}(ds,dy)).
\end{aligned}$$

5.1 The financial stop-loss contract revisited

We briefly turn to the problem of valuating the financial stop-loss contract introduced in Chapter 1, see (1.5.1). We restrict here to the case where the financial loss component is of the form $Y_T = g(X_T)$ and consider, for $K_1 < K_2 < \infty$, the spread

$$(U_T + g(X_T) - K_1)^+ - (U_T + g(X_T) - K_2)^+ =: \Psi(X_T, U_T), \qquad (5.9)$$

which is of the special form (5.1) and bounded by $K_2 - K_1$. Thus, this contract can be analysed within the present framework.

The general case
Provided that the pair (X, U) satisfies Assumptions 5.1 and 5.2, one can apply the

results obtained in the present chapter in the valuation of (5.9). First of all this would involve a verification of the smoothness condition $F^{\psi} \in C^{1,2,2}$, and this seems to be a difficult problem in the general case; one possible approach is to apply techniques similar to the ones used in Lamberton and Lapeyre (1996, Chapter 7). Having verified this condition, one would then typically apply Proposition 5.4 to obtain a PDE for F^{ψ}, which then, in most situations, must be solved numerically.

Independence between X_T and U_T
Under the additional assumption of independence between X and U, we can of course apply the results obtained in Sections 5 and 6 of Chapter 5 by embedding all quantities in a product space model of the type considered there. If furthermore the model $(X, \mathbb{F}^1)$ is complete, see Lemma 5.5.11, then Theorem 5.6.2 provides upper and lower bounds for the fair premium which involve only on the distribution functions of X_T under P and $\tilde{P}$ and the distribution function of U_T under P. Thus, the premiums can be determined by evaluating simple double-integrals with respect to these distribution functions.

5.2 A stochastic volatility model

Let us consider a more explicit example within this framework. First note that if X and U are stochastically independent, then we are basically within the framework considered in Section 5 of Chapter 5 and hence we can apply the results presented there; see also the example at the end of Section 3 of the present chapter. Thus, we consider an example where U and X are not independent. This example is a special case of the main example of Grandits and Rheinländer (1999). Let $U = N$, where N is a homogeneous Poisson process with intensity Λ and let W be a standard Brownian motion which is independent of N under P. This is obtained by letting $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be the product space of two separate spaces, carrying W and N respectively (again, we use the notation $\bar{W}$ and $\bar{N}$ for the original processes on the underlying spaces). The filtration $\mathbb{F}$ is then defined as in Section 5 of Chapter 5, so that $\mathcal{F}_t = (\bar{\mathcal{F}}_t^W \otimes \bar{\mathcal{F}}_t^N) \vee \mathcal{N}$. Similarly to (5.5.3) and (5.5.4), we embed the filtrations $\bar{\mathbb{F}}^N$ and $\bar{\mathbb{F}}^W$ in the product space and denote by $\mathbb{F}^N$ and $\mathbb{F}^W$ the embedded filtrations; recall also Lemma 5.5.4. However, it is emphasized that the process X is now only defined on the product space and is here taken to be given by the dynamics

$$dX_t = \alpha(t, N_{t-})X_t\,dt + \sigma(t, N_{t-})X_t\,dW_t,$$

so that the drift and volatility of X depend on the additional risk N. With the notation above, $M^x = \int \sigma X\,dW$ and $M_t^u = N_t - \Lambda t$. We let $\lambda = \frac{\alpha}{\sigma^2 X}$ and introduce the mean-variance tradeoff process $\hat{K} = \int \lambda^2\,d\langle M^x\rangle = \int \frac{\alpha^2}{\sigma^2}dt$. We assume that $\nu := \frac{\alpha}{\sigma}$ is uniformly bounded and work under the conditions on parameters which are given in Grandits and Rheinländer (1999), henceforth abbreviated as GR, who then show that the variance optimal martingale measure is given by

$$\tilde{Z}_T = C\,\mathcal{E}\left(-\int \lambda\,dM^x\right)_T \exp(-\hat{K}_T) = C\,\mathcal{E}\left(-\int \lambda\,dX\right)_T, \tag{5.10}$$

for some $C \in I\!R_+$. Consider contracts of the form $H = \Psi(X_T, N_T)$, where Ψ is a bounded measurable function. In order to apply proposition 5.4 above it should be verified that the function F^ψ defined by $F^\psi(t, X_t, N_t) = \tilde{\mathrm{E}}[\Psi(X_T, N_T) \mid \mathcal{F}_t]$ is in $C^{1,2,2}$. Note that N is a special semimartingale under $\tilde{P}$ since it has bounded jumps, so that we only need to verify that N is a $\tilde{P}$-Markov process and to determine the canonical decomposition for N under $\tilde{P}$ (i.e. the compensator for N).

We characterize the change of measure from P to $\tilde{P}$ further by determining the density process $Z_t = \mathrm{E}[\frac{d\tilde{P}}{dP} \mid \mathcal{F}_t]$. This can be done by applying arguments similar to the ones applied in the example in GR, since

$$\begin{aligned} Z_t &= C\,\mathrm{E}\left[\exp(-\hat{K}_T)\mathcal{E}\left(-\int \lambda\, dM^x\right)_T \middle| \mathcal{F}_t\right] \\ &= C\,\mathrm{E}\left[\exp(-\hat{K}_T)\mathrm{E}\left[\mathcal{E}\left(-\int \lambda\, dM^x\right)_T \middle| \mathcal{F}_t^W \vee \mathcal{F}_T^N\right] \middle| \mathcal{F}_t\right]. \end{aligned}$$

Provided that $\mathcal{E}\left(-\int \lambda\, dM^x\right)$ is also a martingale with respect to the filtration $I\!F^* = (\mathcal{F}_t^*)_{0 \le t \le T}$, where $\mathcal{F}_t^* = \mathcal{F}_t^W \vee \mathcal{F}_T^N$, we have thus obtained

$$Z_t = C\,\mathcal{E}\left(-\int \lambda\, dM^x\right)_t \mathrm{E}\left[\exp(-\hat{K}_T) \middle| \mathcal{F}_t\right]. \tag{5.11}$$

We only need to show the following:

Lemma 5.6 $\mathcal{E}\left(-\int \lambda\, dM^x\right)$ *is an* $(I\!F^*, P)$*-martingale.*

Proof: It follows by arguments similar to the ones used in the proof of Lemma 5.5.6 that the standard Brownian motion $\bar{W}$ from the underlying space can be extended to a continuous $(I\!F^*, P)$-martingale W. Similarly, $W_t^2 - t$ is also an $(I\!F^*, P)$-martingale, and hence, W is an standard Brownian motion with respect to $(I\!F^*, P)$. By assumption, $\nu := \frac{\alpha}{\sigma}$ is uniformly bounded. Since α and σ are assumed to be measurable functions, ν is $I\!F^*$-predictable (actually ν_t is $\mathcal{F}_0^*$-measurable). It now follows by Novikov's condition (see e.g. Karatzas and Shreve (1991, Corollary 3.5.13)) that the process

$$\mathcal{E}\left(-\int \lambda\, dM^x\right) = \mathcal{E}\left(-\int \frac{\alpha}{\sigma}\, dW\right)$$

is indeed an $(I\!F^*, P)$-martingale. □

Let $k(s,n) := \frac{\alpha(s,n)^2}{\sigma(s,n)^2}$, so that $\hat{K} = \int k(s, N_{s-})ds$. By the Markov property for N and the independence between N and W, we get

$$\begin{aligned} M_t^k &:= \mathrm{E}\left[\exp(-\hat{K}_T) \middle| \mathcal{F}_t\right] \\ &= e^{-\hat{K}_t}\mathrm{E}\left[\exp\left(-\int_t^T k(s, N_{s-})\, ds\right) \middle| \mathcal{F}_t\right] \\ &=: e^{-\hat{K}_t} F^k(t, N_t). \end{aligned}$$

For later use, we note that it now follows directly from (5.11) and (5.10) that $\frac{1}{C} = \mathrm{E}[\exp(-\hat{K}_T)]$ and

$$\tilde{Z}_t = C\,\mathcal{E}\left(-\int \lambda\, dX\right)_t = C - \int_0^t \lambda C \mathcal{E}\left(-\int \lambda\, dX\right) dX. \tag{5.12}$$

Furthermore,

$$\begin{aligned} \frac{Z_t}{\tilde{Z}_t} &= \frac{C\,\mathcal{E}\left(-\int \lambda\, dM^x\right)_t \mathrm{E}\left[\exp(-\hat{K}_T) \middle| \mathcal{F}_t\right]}{C\,\mathcal{E}\left(-\int \lambda\, dX\right)_t} \\ &= \frac{\mathcal{E}\left(-\int \lambda\, dX\right)_t \exp(-\hat{K}_t)\mathrm{E}\left[\exp(-(\hat{K}_T - \hat{K}_t)) \middle| \mathcal{F}_t\right] \exp(\hat{K}_t)}{\mathcal{E}\left(-\int \lambda\, dX\right)_t} \\ &= F^k(t, N_t). \end{aligned}$$

We shall also use later that

$$\mathrm{Var}[\tilde{Z}_T] = \mathrm{E}[\tilde{Z}_T^2] - (\mathrm{E}[\tilde{Z}_T])^2 = \tilde{Z}_0 - 1 = C - 1,$$

where $C = 1/\mathrm{E}[\exp(-\hat{K}_T)] = 1/F^k(0,0)$. Letting $\tilde{\zeta}_t = -\lambda \tilde{Z}_t$, we obtain from (5.12) that $\tilde{\beta} = -\frac{\tilde{\zeta}}{C} = \lambda \mathcal{E}(-\int \lambda\, dX)$ ($\tilde{\beta}$ is defined in Section 4 of Chapter 5).

Assume now that F^k is continuously differentiable with respect to t and let $M_t^u = N_t - \Lambda t$. Then Itô's formula gives

$$\begin{aligned} dM_t^k &= -k(t, N_t) M_{t-}^k\, dt \\ &\quad + e^{-\hat{K}_t}\left(F_t^k(t, N_{t-})dt + (F^k(t, N_{t-}+1) - F^k(t, N_{t-}))dN_t\right) \\ &= e^{-\hat{K}_t}\left(F^k(t, N_{t-}+1) - F^k(t, N_{t-})\right) dM_t^u \\ &\quad + e^{-\hat{K}_t}\left(-k(t, N_t)F^k(t, N_t) + F_t^k(t, N_t) + (F^k(t, N_t+1) - F^k(t, N_t))\Lambda\right) dt \\ &= M_{t-}^k \beta_t^k\, dM_t^u, \end{aligned}$$

where

$$\beta_t^k = \frac{F^k(t, N_{t-}+1) - F^k(t, N_{t-})}{F^k(t, N_{t-})}. \tag{5.13}$$

Furthermore, these calculations show that the function F^k satisfies the equation

$$-k(t,n)F^k(t,n) + F_t^k(t,n) + \Lambda\left(F^k(t,n+1) - F^k(t,n)\right) = 0, \tag{5.14}$$

with terminal condition $F^k(T,n) = 1$ for all n. This shows that

$$Z = Z_0 - \int Z_{s-} \frac{\alpha(s, N_{s-})}{\sigma(s, N_{s-})}\, dW_s + \int Z_{s-} \beta_s^k\, dM_s^u, \tag{5.15}$$

and hence, by the Girsanov theorem, the compensator for N under $\tilde{P}$ is

$$\int (1 + \beta_s^k)\Lambda\, ds = \int \Lambda \frac{F^k(s, N_{s-}+1)}{F^k(s, N_{s-})}\, ds,$$

so that N is indeed a Markov process under $\tilde{P}$. (To see this, it is sufficient to verify that for any bounded function f, $\tilde{\mathrm{E}}[f(N_{t+s})|\mathcal{F}_s] = \tilde{\mathrm{E}}[f(N_{t+s}) \mid N_s]$, see e.g. Chung (1982, Section 1.1), and this can for instance be established by applying the abstract Bayes formula and Lemma 5.6.) In particular, with the notation of Proposition 5.4, $\tilde{\alpha}^u = (1+\beta^k)\Lambda$, $\gamma^u(ds,dy) = \varepsilon_1(dy)dN_s$ and $\tilde{\gamma}^{u,p}(ds,dy) = \varepsilon_1(dy)\Lambda(1+\beta^k)ds$. Provided that $F^\psi \in C^{1,2,2}$, Proposition 5.4 gives that this function is determined by the equation

$$F_t^\psi + \frac{1}{2}\sigma^2 x^2 F_{xx}^\psi + \Delta F^\psi \Lambda(1+\beta^k) = 0, \tag{5.16}$$

with terminal condition $F^\psi(T,x,n) = \Psi(x,n)$, and where

$$\Delta F^\psi(t,x,n) = F^\psi(t,x,n+1) - F^\psi(t,x,n).$$

Note that the equation (5.16) also involves the function F^k in that $1+\beta^k(t,n) = F^k(t,n+1)/F^k(t,n)$. The same proposition now also determines the Kunita-Watanabe decomposition for the $\tilde{P}$-martingale F^ψ and the hedging error $J_0(\mathbb{F})$, so that the fair premium and the optimal strategy can be computed by means of Theorems 5.4.1 and 5.4.2. In the present example, (5.7) and (5.8) specialize to

$$\begin{aligned} \vartheta_t^H &= F_x^\psi(t,X_t,N_{t-}) \\ &\quad -\tilde{\zeta}_t \int_0^{t-} \tilde{Z}_s^{-1} \left(F^\psi(s,X_s,N_{s-}+1) - F^\psi(s,X_s,N_{s-})\right)(dN_s - \Lambda(1+\beta_s^k)ds), \end{aligned} \tag{5.17}$$

where β^k is defined by (5.13), and

$$J_0(\mathbb{F}) = \tilde{\mathrm{E}}\left[\int_0^T \tilde{Z}_t^{-1}\left(F^\psi(t,X_t,N_{t-}+1) - F^\psi(t,X_t,N_{t-})\right)^2 dN_t\right]. \tag{5.18}$$

The above framework could for example be used in the situation, where N describes the occurrence of some catastrophic events which affect the stock price process of (say) an insurance company. This is definitely not unrealistic, since a severe catastrophe will affect the surplus of the insurance company and might cause speculations whether the insurance company will be able to cover its obligations or whether it will be ruined.

Example 5.7 Let us consider a very simple example where explicit formulas can be obtained relatively easily. We assume that $\alpha(t,n) = \alpha$ (independent of time and N) and that

$$\sigma(t,n) = \sigma_0 1_{\{n=0\}} + \sigma_1 1_{\{n\geq 1\}}.$$

Thus, the volatility of the stock jumps in connection with the occurrence of the first catastrophe. We let $k_0 = \frac{\alpha^2}{\sigma_0^2}$ and $k_1 = \frac{\alpha^2}{\sigma_1^2}$, and we assume for simplicity that $\Lambda + k_0 \neq k_1$; this is satisfied if for example $\sigma_1 > \sigma_0$. We shall determine the fair premium and the optimal strategy (under the financial variance principle) for the claim

$$\Psi(X_T, N_T) = 1_{\{N_T \geq 1\}}, \tag{5.19}$$

which pays one unit at time T if at least one catastrophe has occurred during $[0, T]$.

We need to determine the functions $F^k(t, n)$ and $F^\psi(t, n)$. The functions F^k can be computed directly or be determined via the equations (5.14). Since

$$F^k(t,n) = \mathrm{E}\left[\exp\left(-\int_t^T \left(k_0 1_{\{N_{s-}=0\}} + k_1 1_{\{N_{s-}\geq 1\}}\right) ds\right)\bigg| N_t = n\right],$$

we first note that

$$F^k(t,n) = F^k(t,1) = e^{-k_1(T-t)},$$

for $n \geq 1$. From (5.14) we now find that $F^k(t, 0)$ is the solution to

$$(k_0 + \Lambda)F^k(t,0) - F^k_t(t,0) = \Lambda e^{-k_1(T-t)},$$

with terminal condition $F^k(T, 0) = 1$. Hence

$$F^k(t,0) = \frac{k_0 - k_1}{\Lambda + k_0 - k_1} e^{-(\Lambda + k_0)(T-t)} + \frac{\Lambda}{\Lambda + k_0 - k_1} e^{-k_1(T-t)}.$$

We now turn to the problem of determining F^ψ. Using arguments similar to the ones leading to (5.11) and the abstract Bayes formula, we find for any bounded function f

$$\begin{aligned}
\tilde{\mathrm{E}}\left[f(N_T)\middle|\mathcal{F}_t\right] &= \frac{\mathrm{E}\left[\tilde{Z}_T f(N_T)\middle|\mathcal{F}_t\right]}{Z_t} \\
&= \frac{C\,\mathcal{E}\left(-\int \lambda\, dM^x\right)_t \exp(-\hat{K}_t)\mathrm{E}\left[\exp\left(-(\hat{K}_T - \hat{K}_t)\right) f(N_T)\middle|\mathcal{F}_t\right]}{C\,\mathcal{E}\left(-\int \lambda\, dM^x\right)_t \exp(-\hat{K}_t) F^k(t, N_t)} \\
&= \frac{\mathrm{E}\left[\exp\left(-(\hat{K}_T - \hat{K}_t)\right) f(N_T)\middle|\mathcal{F}_t\right]}{F^k(t, N_t)},
\end{aligned}$$

which shows that F^ψ is a function of (t, N_t) only, since N is a Markov process. Clearly, $F^\psi(t, n) = 1$, for $n \geq 1$, and $F^\psi(t, 0)$ could now be computed directly by using that the first jump time τ for N is exponentially distributed with mean $1/\Lambda$. We shall instead, however, use that $F^\psi(t, 0)$ solves the equation (5.16) with boundary condition $F^\psi(T, 0) = 0$. Since F^ψ does not depend on X, and since $F^\psi(t, 1) = 1$, this specializes to

$$F^\psi_t(t,0) - F^\psi(t,0)\Lambda(1 + \beta^k(t,0)) = -\Lambda(1 + \beta^k(t,0)),$$

which has solution

$$F^\psi(t,0) = 1 - \exp\left(-\int_t^T \Lambda(1 + \beta^k(s,0))\, ds\right).$$

(In fact, this also follows immediately from the form of the compensator for N under $\tilde{P}$ and the fact that $F^\psi(t,0) = 1 - \tilde{P}(N_T = 0 \mid N_t = 0)$.) After some tedious but simple calculations, we thus arrive at

$$F^\psi(t,0) = 1 - \frac{e^{-(\Lambda+k_0-k_1)(T-t)}}{\frac{k_0-k_1}{\Lambda+k_0-k_1}e^{-(\Lambda+k_0-k_1)(T-t)} + \frac{\Lambda}{\Lambda+k_0-k_1}}.$$

Thus, by (5.17) and Theorem 5.4.1, the optimal strategy under the variance principle is

$$\begin{aligned}\vartheta_t^* &= \lambda_t C\,\mathcal{E}(-\int \lambda\, dX)_t \int_0^{t-} \tilde{Z}_s^{-1}\left(1 - F^\psi(s,0)\right) 1_{\{N_{s-}=0\}}(dN_s - \Lambda\frac{F^k(s,1)}{F^k(s,0)}ds)\\ &\quad + \frac{\lambda_t \mathcal{E}(-\int \lambda\, dX)_t}{2a\, F^k(0,0)}.\end{aligned}$$

Similarly, we obtain from (5.18)

$$\begin{aligned}J_0(I\!\!F) &= \tilde{\mathrm{E}}\left[\int_0^T \tilde{Z}_t^{-1}\left(1 - F^\psi(t,0)\right)^2 1_{\{N_{t-}=0\}}\, dN_t\right]\\ &= \tilde{\mathrm{E}}\left[\int_0^T \tilde{Z}_t^{-1}\left(1 - F^\psi(t,0)\right)^2 1_{\{N_{t-}=0\}}\Lambda(1+\beta^k(t,0))\, dt\right]\\ &= \mathrm{E}\left[\int_0^T \frac{Z_t}{\tilde{Z}_t}\left(1 - F^\psi(t,0)\right)^2 1_{\{N_{t-}=0\}}\Lambda(1+\beta^k(t,0))\, dt\right]\\ &= \mathrm{E}\left[\int_0^T F^k(t,N_t)\left(1 - F^\psi(t,0)\right)^2 1_{\{N_{t-}=0\}}\Lambda\frac{F^k(t,1)}{F^k(t,0)}\, dt\right]\\ &= \int_0^T \left(1 - F^\psi(t,0)\right)^2 e^{-\Lambda t}\Lambda F^k(t,1)\, dt, \qquad (5.20)\end{aligned}$$

where the second equality follows by using the $\tilde{P}$-intensity for N and the third equality is a consequence of the optional projection theorem. The last equality is obtained via the Fubini theorem and by using the fact that $P(N_t = 0) = e^{-\Lambda t}$. The expression (5.20) can for example be evaluated by numerical integration. The fair premium under the financial variance principle is thus

$$\begin{aligned}v_1(1_{\{N_T\geq 1\}}) &= F^\psi(0,0) + aJ_0(I\!\!F)\\ &= 1 - \frac{e^{-(\Lambda+k_0-k_1)T}}{\frac{k_0-k_1}{\Lambda+k_0-k_1}e^{-(\Lambda+k_0-k_1)T} + \frac{\Lambda}{\Lambda+k_0-k_1}}\\ &\quad + a\int_0^T \left(1 - F^\psi(t,0)\right)^2 e^{-\Lambda t}\Lambda e^{-k_1(T-t)}\, dt.\end{aligned}$$

It is now also possible to give the optimal strategy and the fair premium under the financial standard deviation principle by applying the results of Section 4 of Chapter 5. This ends the example. □

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