ECONOMIC GROWTH
DIFFERENTIAL– AND DIFFERENCEEQUATIONS

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CHAPTER 1. An Interesting Case

Assume we have invested a kapital, K, in a bank obtaining a fixed interest of $r\%$ per year. If we denote the amount of the capital in the end of year n by k_n, then this means that we have

$$\Delta k_n = k_{n+1} - k_n = r \cdot k_n \tag{1.1}$$

This is the simplest case of a difference equation, we may imagine.

Well, we shall rewrite it as

$$k_{n+1} = (1+r)k_n \tag{1.2}$$

with the solution

$$k_n = K \cdot (1+r)^n \tag{1.3}$$

Now, we realize that our solutions allows a definition of the size of the capital to be calculated for real time – opposite to integral numbers of years – as the function of the real variable, t,

$$k(t) = K \cdot (1+r)^t = K \cdot e^{t \ln(1+r)} \tag{1.4}$$

We may consider this function as an interpolation between the known values for the integral values of the variable, t.

This solution (1.4) is differentiable with derivative

$$k'(t) = K \cdot \ln(1+r) \cdot e^{t \ln(1+r)} = \ln(1+r) \cdot k(t) \tag{1.5}$$

So, our interpolation satisfies the differential equation

$$k'(t) = \ln(1+r) \cdot k(t) \tag{1.6}$$

The difference equation (1.1) is replaced by a similar differential equation (1.6), with a solution interpolating the solution to the difference equation.

Savings and mortgages requires handling sums. Now, sums are particularly easy to handle, if the are telescoping. Consider

$$g_n = \Delta k_n = k_{n+1} - k_n \tag{1.7}$$

Then we find

$$\sum_{n=0}^{m} g_n = \sum_{n=0}^{m} k_{n+1} - \sum_{n=0}^{m} k_n = k_{m+1} - k_0 \tag{1.8}$$

We express this simplicity by saying that k_n is an indefinite sum of g_n, rewriting (1.7) as

$$\sum g_n \delta n = k_n \tag{1.9}$$

The difference equation (1.1) may be rewritten as

$$\sum k_n \delta n = \frac{1}{r} k_n \tag{1.10}$$

Now, if we pay an amount of a every year to be increased by the interest, r, then the capital obtained after the nth payment is

$$\sum_{k=0}^{n-1} a \cdot (1+r)^k = \frac{a}{r}\left((1+r)^n - 1\right) \tag{1.11}$$

In real life the capital, the payment and the time are agreed leaving the interest as the unknown. So, we are interested in solving the equation in r,

$$A \cdot r = (1+r)^n - 1 \tag{1.12}$$

where A is the ratio between the capital and the payment.

We want to find a zero of the polynomial

$$f(r) = (1+r)^n - Ar - 1 \tag{1.13}$$

The fastest way to do so, the socalled Newton method, takes an approximation, r_k, and draw the tangent to the function, f in that point and finds the cut with the r–axes as r_{k+1}.

This method works as finding the fixpoint of the function

$$g(r) = r - \frac{f(r)}{f'(r)} \tag{1.14}$$

by iteration g from a neighboring point to the solution. Let r^* be the fixpoint, then we have

$$|g(r) - r^*| = |g(r) - g(r^*)| = |g'(\hat{r})| \cdot |r - r^*| = \frac{|f(\hat{r}) f"(\hat{r})|}{f'(\hat{r})^2} \cdot |r - r^*| \tag{1.15}$$

for some intermediate point $\hat{r}$. If f is a polynomial, and the zero is not a zero for the derivative, then this fraction is small in a neighborhood of the wanted root.

CHAPTER 2. Stability

The material in this chapter is taken from [8, 7].

To be stable means to have the ability to recover from a disturbance. To define this concept of stability precisely, one must specify both the kind of disturbance and the criterion for an adequate recovery.

In the following, we study a motion as a real function of time, $\varphi(t)$ and a disturbed motion, $\psi(t)$. Then the stability means that, in the long run, $\varphi(t)$ and $\psi(t)$ are "equivalent" in some adequate sense that must be made precise in terms of a formal criterion. We present seven *criteria* for stability, each one containing the predecessors being contained in the successors. Hence, the criteria represent a succession from higher to lower *degrees* of stability. Usage differs among mathematicians, engineers and economists.

The *stability* of a motion against a disturbance will be called

1. Asymptotic if $\varphi(t) = \text{const.}$ and $\psi(t) - \varphi(t) \to 0$ for $t \to \infty$.
2. Strong absolute if $\psi(t) - \varphi(t) \to 0$ for $t \to \infty$.
3. Weak absolute if $|\psi(t) - \varphi(t)| < k < \infty$ for all t.
4. Strong relative if $\ln\psi(t) - \ln\varphi(t) \to 0$ for $t \to \infty$.
5. Weak relative if $|\ln\psi(t) - \ln\varphi(t)| < k < \infty$ for all t.
6. Strong logarithmic if $\ln\ln\psi(t) - \ln\ln\varphi(t) \to 0$ for $t \to \infty$.
7. Weak logarithmic if $|\ln\ln\psi(t) - \ln\ln\varphi(t)| < k < \infty$ for all t.

Remark. The criterion 4 reflects the requirement

$$\psi(t)/\varphi(t) \to 1 \quad \text{for} \quad t \to \infty \quad \text{or} \quad (\psi(t) - \varphi(t))/\varphi(t) \to 0 \quad \text{for} \quad t \to \infty\,,$$

which justifies the name "relative stability".

Similarly, criterion 6 reflects the requirement

$$\ln\psi(t)/\ln\varphi(t) \to 1 \quad \text{for} \quad t \to \infty\,.$$

The notion of *causality* refers in this context to systems of motions, where time paths of the variables depend only upon the *initial conditions* and the *time elapsed* since the establishment of such initial conditions, that is to say, the specification of similar given initial conditions at a later period would result in a similar evolution of the system, except at a constantly later time period. Thus, as used here, a system is said to be causal if, from an initial configuration, it determines its own behavior over time.

We now restrict ourselves to the study of a causal system in one variable. Furthermore, the variable will here – in a growth context – be required to increase monotonously in time.

Consider an increasing, differentiable function of time, $y = \varphi(t)$, moving in a range

$$-\infty \leq a < \varphi(t) < b \leq +\infty \, .$$

Then such a motion will be stable against delay, if there exists a transformation between the pair of functions $\varphi(t)$ and $\varphi(t+h)$ for *any* given value of h. This problem of *stability* of a *motion* against *delay* can always be transformed to the corresponding problem of *stability* with respect to the *initial position* of the set of *solutions* to a first-order *autonomous* differential equation.

Such correspondence is due to the fact that the family of functions $\{\varphi(t+h) | h \in \mathbb{R}\}$ is a set of solutions to the equation $\dot{y} = f(y), (\cdot = \mathrm{d}/\mathrm{d}t)$, with the function f defined by

$$f(x) = \varphi'(\varphi^{-1}(x)), \quad a < x < b \, .$$

Hence, it is sufficient to study the stability of sets of solutions to the differential equations with respect to the disturbance of the initial position.

Lemma 1. *Any solution $\varphi(t)$ to the equation $\dot{y} = f(y)$, with $f(x) > 0$ everywhere, is increasing and unbounded,*

(i) $\varphi'(t) > 0$ *(ii)* $\varphi(t) \to \infty$ *for* $t \to \infty$.

Proof. Let $\varphi(t)$ be a solution. Because $\varphi'(t) = f(\varphi(t)) > 0$, the function φ must be increasing.

Suppose $\varphi(t)$ is bounded. Then $\varphi(t) \to c$ for $t \to \infty$. Now

$$\lim_{t \to \infty} \varphi'(t) = \lim_{t \to \infty} f(\varphi(t)) = f(c) > 0 \, .$$

Hence, $\varphi'(t) > \varepsilon > 0$ for all values of t, and therefore

$$\varphi(t) > k + \varepsilon t$$

contradicting the limit of φ. Conclusion: The function φ must be unbounded.

Lemma 2. *Suppose the function $f(x)$ is positive and continuously differentiable everywhere. If $\varphi(t)$ is any solution to the equation $\dot{y} = f(y)$, then any other solution to this equation must take the form, $\varphi(t+h)$, for some constant h.*

Proof. Let $\psi(t)$ and $\varphi(t)$ be two solutions. Consider the time $t = 0$ and suppose $\psi(0) > \varphi(0)$. Now, as $\varphi(t) \to \infty$ for $t \to \infty$, according to Lemma 1, there exists $h > 0$ such that $\psi(0) = \varphi(h)$.

Since the function f is continuously differentiable, it satisfies the Lipschitz condition and, hence, the solution to the initial value problem is unique, i.e., $\psi(t) = \varphi(t+h)$ everywhere.

Corollary to Lemma 2. *Let $\varphi(t)$ be a bijective, twice continuously differentiable function satisfying $\varphi'(t) > 0$ for all t. Then the family $\{\varphi(t+h) | h \in \mathbb{R}\}$ is equal to the whole set of solutions to the equation $\dot{y} = f(y)$, where*

$$f(x) = \varphi'(\varphi^{-1}(x)) .$$

Proof. The function $f(x)$ is continuously differentiable and positive, so Lemma 2 applies. It is obvious (see above) that $\varphi(t)$ solves the equation.

Now we discuss the stability of the set of solutions to a differential equation of the form

$$\dot{y} = f(y) . \tag{2.1}$$

We relate the different criteria of stability mentioned above to the particular properties which must be satisfied by the function f.

Assumption (2.1). The function f is continuously differentiable everywhere.

Theorem 1. *The set of solutions to Equation (2.1) has asymptotic stability around the constant solution $y = c$, if and only if*

(i) $f(c) = 0$,
(ii) $f(x) > 0$ *for* $x < c$,
(iii) $f(x) < 0$ *for* $x > c$.

Proof. If. Let $\varphi(t)$ be a solution. If $\varphi(t) < c$, then $\varphi'(t) = f(\varphi(t)) > 0$, and, hence, φ is increasing. Now, $\varphi(t) < c$ for all t, due to the theorem of uniqueness of solutions to differential equations. Hence, $\varphi(t) \to d \leq c$ for $t \to \infty$. Now, for $t \to \infty$:

$$f(d) = f(\lim \varphi(t)) = \lim f(\varphi(t)) = \lim \varphi'(t) \geq 0 .$$

If $f(d) > 0$, then $\varphi'(t) > \varepsilon > 0$ for all t and, hence, $\varphi(t) \to \infty$ for $t \to \infty$. So $f(d) = 0$. Because c is the only zero for f, it must be equal to d. Conclusion: $\varphi(t) \to c$ for $t \to \infty$.

Only if. Clearly $f(c) = 0$ (because $y = c$ is supposed to be a solution), and f has no other zero d (since then $y = d$ would be a solution not tending to c).

If the function f has a zero, then Theorem 1 applies. If the function f has no zero, then following theorems shall prove useful. Hence, we shall from now on assume

Assumption (2.2). $f(x) > 0$ everywhere.

This assumption implies that the solution must be unbounded, a fact we need in the proofs below. For each of the six criteria of stability 2–7 defined above, we shall state for the *governing function* f some simple sufficient conditions that ensure the set of solutions to be either stable or unstable. The sufficient stability conditions below are also almost necessary.

Theorem 2. *The set of solutions to Equation (2.1) has strong absolute stability, if*

$$f(x) \to 0 \quad \text{for} \quad x \to \infty \tag{2.2+}$$

and cannot have strong absolute stability, if

$$f(x) > \varepsilon > 0 \qquad \text{everywhere}\,. \tag{2.2−}$$

Proof. Let $\varphi(t)$ be a solution. From Lemma 1 follows that $\varphi(t)$ is increasing and unbounded. From Lemma 2 follows that any pair of solutions can be of the form

$$\varphi(t) \quad \text{and} \quad \psi(t) = \varphi(t+h), \text{ for some constant, } h > 0\,.$$

So we have for some $\tau = \tau(t)$, with $t < \tau < t + h$

$$\psi(t) - \varphi(t) = \varphi(t+h) - \varphi(t) = \varphi'(\tau)h = f(\varphi(\tau))h\,.$$

Hence, under the condition (2+), we get

$$\psi(t) - \varphi(t) \to 0 \quad \text{for} \quad t \to \infty\,,$$

because with t both $\tau = \tau(t)$ and $\varphi(\tau)$ goes to infinity.

Similarly, under the condition (2−), we get

$$|\psi(t) - \varphi(t)| = f(\varphi(\tau))|h| > \varepsilon|h|\,.$$

Theorem 3. *The set of solutions to Equation (2.1) has weak absolute stability, if*

$$0 < f(x) < k \text{ everywhere}\,, \tag{2.3+}$$

and cannot have weak absolute stability, if

$$f(x) \to \infty \quad \text{for} \quad x \to \infty\,. \tag{2.3−}$$

Proof. According to Lemma 2, we can assume two solutions to be

$$\varphi(t) \quad \text{and} \quad \varphi(t+h) \quad \text{with} \quad h > 0\,,$$

and according to Lemma 1, they are both increasing and unbounded.

Then for $t < \tau < t + h$, we have

$$0 < \varphi(t+h) - \varphi(t) = \varphi'(\tau)h = f(\varphi(\tau))h < kh$$

under the condition (3+). This proves the weak absolute stability.

In the second case, let K be any number. We can find c, such that

$$f(x) > K \quad \text{for} \quad x > c\,,$$

according to (3−). Lemma 1 says that we can find t_c, such that for $t > t_c$, we have $\varphi(t) > c$ and hence

$$\varphi(t+h) - \varphi(t) = \varphi'(\tau)h = f(\varphi(\tau))h > Kh$$

because $\tau > t > t_c$ and $\varphi(\tau) > \varphi(t) > c$ and, hence, $f(\varphi(\tau)) > K$. Because K was arbitrary, we cannot have weak absolute stability.

Theorem 4. *The set of solutions to Equation (2.1) has strong relative stability, if*

$$f(x)/x \to 0 \quad \textit{for} \quad x \to \infty \tag{2.4+}$$

and cannot have strong relative stability, if

$$f(x)/x > \varepsilon > 0 \quad \textit{for} \quad x > 0\,. \tag{2.4−}$$

Proof. Let $\varphi(t)$ be a solution. Consider the function

$$\omega(t) = \ln\varphi(t)\,.$$

We shall prove that $\omega(t)$ solves the equation

$$\dot{y} = g(y) \quad \text{with} \quad g(y) = f(e^y)/e^y\,.$$

This follows from the straightforward computation below.

$$\omega'(t) = \varphi'(t)/\varphi(t) = f(\varphi(t))/\varphi(t) = f\left(e^{\omega(t)}\right)/e^{\omega(t)} = g(\omega(t))\,.$$

The condition (4+) and (4−) for f imply that g satisfies, respectively, the conditions (2+) and (2−) of Theorem 2. Hence, the pair $\omega(t)$ and $\omega(t+h)$ either have or cannot have strong absolute stability according to Theorem 2.

But strong *absolute* stability for $\omega(t) = \ln\varphi(t)$ and $\omega(t+h) = \ln\varphi(t+h)$ is by the definitions 2 and 4 strong *relative* stability for $\varphi(t)$ and $\varphi(t+h)$. This proves Theorem 4.

Theorem 5. *The set of solutions to Equation (2.1) has weak relative stability, if*

$$0 < f(x)/x < k \quad \textit{for} \quad x > 0 \tag{2.5+}$$

and cannot have weak relative stability, if

$$f(x)/x \to \infty \quad \textit{for} \quad x \to \infty\,. \tag{2.5−}$$

Proof. As in the proof of Theorem 4, this theorem is easily derived from Theorem 3.

Theorem 6. *The set of solutions to Equation (2.1) has strong logarithmic stability, if*

$$\frac{f(x)}{x\ln x} \to 0 \quad \textit{for} \quad x \to \infty\,(x > 1) \tag{2.6+}$$

and cannot have strong logarithmic stability, if

$$\frac{f(x)}{x \ln x} > \varepsilon > 0 \quad \textit{for} \quad x > 1\,. \tag{2.6$-$}$$

Proof. Let $\varphi(t)$ be a solution. Then $\omega(t) = \ln\varphi(t)$ solves the equation

$$\dot{y} = g(y), \quad \text{with } g(y) = f(e^y)/e^y\,.$$

Now the function g satisfies the conditions of Theorem 4, as

$$\frac{g(x)}{x} = \frac{f(e^x)}{xe^x} \to 0 \quad \text{for} \quad x \to \infty \quad \text{(from 6+)}$$
$$\frac{g(x)}{x} = \frac{f(e^x)}{xe^x} > \varepsilon > 0 \quad \text{for} \quad x > 0 \quad \text{(from 6$-$)}$$

Hence, Theorem 6 follows from Theorem 4.

Table I. Typical different equations and stability properties

Stability criteria	Differential equation	Solution	Theorem No.
1. Asymptotic	$\dot{y} = -y$	$\begin{cases} 0 \\ e^{-t} \end{cases}$	1
2. Strong absolute	$\dot{y} = y^{-1}$	$(2t)^{1/2}$	2
3. Weak absolute	$\dot{y} = 1$	t	3
4. Strong relative	$\dot{y} = 2(y)^{1/2}$	t^2	4
5. Weak relative	$\dot{y} = y$	e^t	5
6. Strong logarithmic	$\dot{y} = y(\ln y)^{1/2}$	$\exp(t^2)$	6
7. Weak logarithmic	$\dot{y} = y\ln y$	$\exp(\exp(t))$	7

Theorem 7. *The set of solutions to Equation (2.1) has weak logarithmic stability, if*

$$\frac{f(x)}{x \ln x} < k \quad \textit{for} \quad x > 1 \tag{2.7+}$$

and cannot have weak logarithmic stability, if

$$\frac{f(x)}{x \ln x} \to \infty \quad \textit{for} \quad x \to \infty \; (x > 1)\,. \tag{2.7$-$}$$

Proof. As Theorem 6, the theorem is easily derived from Theorem 5.

We have characterized the stability properties of the solutions to (1) in terms of appropriate *conditions* upon the *governing function*, f. Typical differential equations of the form (2.1), with their explicit solutions having stability according to these theorems, are given in Table I.

CHAPTER 3. Rapidity of Growth

The material in this chapter is taken from [8, 7].

The growth of a function is properly measures by its derivative – assuming the derivative to exist. The greater the derivative is, the faster is the growth. To distinguish different *speeds* of growth, we shall use the following terminology. The *growth* will be called

I.	Bounded	if	$\varphi'(t) > 0$ and $\varphi(\tau) < c$ everywhere.
II.	Unbounded	if	$\varphi'(t) > 0$ everywhere and $\varphi(t) \to \infty$ for $t \to \infty$.
III.	Linear	if	$\varphi'(t) > \varepsilon > 0$ everywhere.
IV.	Polynomial	if	$\varphi'(t) \to \infty$ for $t \to \infty$.
V.	Exponential	if	$\varphi'(t)/\varphi(t) > \varepsilon > 0$ everywhere.
VI.	Hyper-exponential	if	$\varphi'(t)/\varphi(t) \to \infty$ for $t \to \infty$.
VII.	Double-exponential	if	$\varphi'(t)/(\varphi(t)\ln\varphi(t)) > \varepsilon > 0$ everywhere.

Remark. Of course, growth can be more rapid than here considered, e.g., explosive growth means that there is a finite escape time, a, i.e.

$$\varphi(t) \to \infty \text{ for } t \to a \ (a < \infty)\,.$$

As examples of explosive growth, the equations $\dot{y} = y^2$, $\dot{y} = y^2 + 1$ have the respective solutions, $\varphi(t) = 1/(a-t)$, $\varphi(t) = \tan(t)$.

One of the main purposes of this paper is to underline how growth prevents stability. The *more rapid* is the growth, the *less* is the *degree* of *stability*, and conversely. We prove that as the speed of growth ascends from bounded to double-exponential, the degree of stability descends from asymptotic to weak logarithmic.

Suppose we have a growing function $\varphi(t)$ and want to know the stability with respect to changes of the initial conditions or, equivalently, the stability against time delays, i.e. the stability of the pair $\varphi(t+h)$ and $\varphi(t)$. For each of the situations, we have the following *complementarity* $(J, j) = (\mathrm{II}, 2), \ldots, (\mathrm{VII}, 7)$.

Theorem 1. *If the growing function $\varphi(t)$ has growth property $J \in \{II, \ldots, VII\}$, then any two functions $\varphi(t+h)$ and $\varphi(t)$ for $h \neq 0$, cannot have a degree of stability stronger than j.*

Proof. The conditions, II–VII, imply the applicability of the corollary to Lemma 2 and Theorems, 1–6 in chapter 11, as follows:

$J = \mathrm{II}$. The unbounded growth of $\varphi(t)$ gives $f(x) = \varphi'(\varphi^{-1}(x)) = \varphi'(t) > 0$ everywhere, preventing asymptotic stability by Theorem 1, (only if).

$J = \mathrm{III}$. The linear growth of $\varphi(t)$ gives $f(x) = \varphi'(\varphi^{-1}(x)) = \varphi'(t) > \varepsilon > 0$ everywhere, preventing strong absolute stability by Theorem 2, (2−).

$J = \text{IV}$. The polynomial growth of $\varphi(t)$ gives $f(x) = \varphi'(\varphi^{-1}(x)) = \varphi'(t) \to \infty$ for $x \to \infty$, preventing weak absolute stability by Theorem 3, (3−).

$J = \text{V}$. The exponential growth of $\varphi(t)$ gives

$$f(x)/x = \varphi'(\varphi^{-1}(x))/x = \varphi'(t)/\varphi(t) > \varepsilon > 0 \text{ everywhere},$$

preventing strong relative stability by Theorem 4, (4−).

$J = \text{VI}$. The hyper-exponential growth of $\varphi(t)$ gives

$$f(x)/x = \varphi'(\varphi^{-1}(x))/x = \varphi'(t)/\varphi(t) \to \infty \text{ for } x \to \infty,$$

preventing weak relative stability by Theorem 5, (5−).

$J = \text{VII}$. The double-exponential growth of $\varphi(t)$ gives

$$f(x)/x\ln x = \varphi'(\varphi^{-1}(x))/x\ln x = \varphi'(t)/\varphi(t)\ln\varphi(t) > \varepsilon > 0$$

everywhere, preventing strong logarithmic stability by Theorem 6, (6−).

Table II. Growth versus stability

Differential equation	Solution	Growth	Instability	Stability
$\dot{y} = -y$	$-e^{-t}$	bounded		asymptotic
$\dot{y} = y^{-1}$	$(2t)^{1/2}$	unbounded	not asymptotic	strong abs.
$\dot{y} = 1$	t	linear	not strong abs.	weak abs.
$\dot{y} = 2(y)^{1/2}$	t^2	polynomial	not weak abs.	strong rel.
$\dot{y} = y$	$\exp(t)$	exponential	not strong rel.	weak rel.
$\dot{y} = y(\ln y)^{1/2}$	$\exp(t^2)$	hyperexp.	not weak rel.	strong log.
$\dot{y} = y\ln y$	$\exp(\exp(t))$	doubleexp.	not strong log.	weak log.
$\dot{y} = y^2$	$-t^{-1}$	explosive	not weak log.	none

Remark. If the growing function $\varphi(t)$ shows explosive growth, then the functions $\varphi(t)$ and $\varphi(t+h)$ cannot have any type of stability for $h \neq 0$, because $\varphi(t) \to \infty$ for $t \to a$ and $\varphi(t+h) \to \infty$ for $t \to a - h$.

Using the examples and results from Table 1, we can give a table of behavior (Table II).

CHAPTER 4. Differential equations

The solution to a differential equation

$$y' = f(x, y) \tag{4.1}$$

is a differentiable function

$$y = \varphi(x) \tag{4.2}$$

satisfying

$$\varphi'(x) = f(x, \varphi(x)) \,. \tag{4.3}$$

Definition 1. The differential equation is called autonomous, if the function, f, of (4.1) is independent of x, i.e. the equation is

$$y' = f(y) \,. \tag{4.4}$$

In dynamics we consider x to be the time, so an equation is autonomous if it describes a behavior dependent of the state of the system, but not of exterior forces. The development is then assumed to be similar from a certain initial state, in dependent of the time when this situation emerges.

The solution of equations of form (4.4) is the following: Let A be the set of zeros of f,

$$A = \{\alpha \mid f(\alpha) = 0\} \,. \tag{4.5}$$

This gives rise to a set of constant solutions, if $\alpha \in A$, then

$$\varphi(x) = \alpha \tag{4.6}$$

solves (4.4).

If f is continuous and $\alpha, \beta \in A$ are consecutive roots of f, then we have $f(y)$ of the same sign for all $\alpha < y < \beta$. We may then write (4.4) as

$$\frac{y'}{f(y)} = 1 \quad \text{for} \quad \alpha < y < \beta \,. \tag{4.7}$$

Integration of (4.7) yields

$$\int \frac{y'}{f(y)}\, dx = \int dx \tag{4.8}$$

or, rather for some constant c:

$$F(y) := \int \frac{dy}{f(y)} = x + c\,, \tag{4.9}$$

and hence the solution

$$y = F^{-1}(x+c) \tag{4.10}$$

which exists, because $F'(y) = \frac{1}{f(y)}$ has the same sign in the interval $]\alpha, \beta[$ so that F is monotonic.

EXAMPLE.

$$y' = \sin y\,. \tag{4.11}$$

We find $A = \{\alpha \mid \sin\alpha = 0\} = \{p \cdot \pi \mid p \in \mathbb{Z}\}$. We choose two consecutive zeros, e.g. 0 and π. Now, for $0 < y < \pi$ we have $\sin y > 0$. Then we shall find

$$F(y) = \int \frac{dy}{\sin y}\,.$$

We substitute $y = 2t$ and get

$$\begin{aligned} F(2t) = \int \frac{dt}{\sin t \cos t} &= \int \frac{\cos^2 t + \sin^2 t}{\sin t \cos t}\,dt \\ &= \int \frac{\cos t}{\sin t}\,dt + \int \frac{\sin t}{\cos t}\,dt \\ &= \int \frac{d\sin t}{\sin t} - \int \frac{d\cos t}{\cos t} \\ &= \ln(\sin t) - \ln(\cos t) \\ &= \ln\tan t \end{aligned}$$

hence

$$F(y) = \ln\tan\left(\frac{y}{2}\right).$$

The solution to (4.11) then is

$$y = F^{-1}(x+c)\,, \qquad y = 2\operatorname{Arctan}(e^{x+c})\,.$$

The problem of solving (4.4) consists of one integration and one inversion.

If the function f is a rational function of y, then the integration is always possible. Suppose we have

$$f(y) = \frac{q(y)}{p(y)}$$

then we shall integrate a rational function,

$$F(y) = \int \frac{p(y)}{q(y)}\, dy\,.$$

If we divide p by q, we can write

$$\frac{p(y)}{q(y)} = r(y) + \frac{s(y)}{q(y)}$$

with r, s polynomials, the degree of s smaller than the degree of q. We can always integrate r. From the appendix we can write $\frac{s}{q}$ as a sum of terms of the form

$$\frac{\beta}{(y-\alpha)^m}\,.$$

And these are easily integrated.

Remark. If the polynomials are real, then the complex roots appear in conjugated pairs. E.g., we may have the terms

$$\begin{aligned}\frac{\beta}{y-\alpha} + \frac{\bar\beta}{y-\bar\alpha} &= \frac{\beta y - \beta\bar\alpha + \bar\beta y - \bar\beta\alpha}{y^2-(\alpha+\bar\alpha)y+\alpha\bar\alpha}\\ &= \frac{(\beta+\bar\beta)y - (\beta\bar\alpha+\bar\beta\alpha)}{y^2-(\alpha+\bar\alpha)y+\alpha\bar\alpha}\,.\end{aligned}$$

In this expression all terms are real. The integral of such a real rational function takes two steps:

$$\begin{aligned}\int \frac{by+c}{y^2-ay+d}\,dy &= \frac{b}{2}\int \frac{d(y^2-ay+d)}{y^2-ay+d} + \frac{b}{2}\int \frac{\frac{2c}{b}+a}{y^2-ay+d}\\ &= \frac{b}{2}\ln\left|y^2-ay+d\right| + \frac{1}{2}\int \frac{2c+ab}{y^2-ay+d}\,dy\,.\end{aligned}$$

We have the logarithm and need to find

$$\left(c+\frac{ab}{2}\right)\int \frac{dy}{y^2-ay+d}$$

omitting the factor we substitute $y = t + \frac{a}{2}$ to get

$$\int \frac{dy}{y^2-ay+d} = \int \frac{dt}{t^2+d-\frac{a^2}{4}} = \int \frac{dt}{t^2+\delta}\,.$$

Now we are done, if $\delta = 0$. Otherwise we substitute $t = \sqrt{|\delta|}\,s$ to get

$$\int \frac{\sqrt{|\delta|}ds}{\delta(s^2+\sigma(\delta))} = \frac{\sigma(\delta)}{\sqrt{|\delta|}}\int \frac{ds}{s^2+\sigma(\delta)}\,,$$

where $\sigma(\delta)$ is the sign of δ. And we have

$$\begin{aligned}
&\int \frac{ds}{s^2+1} = \operatorname{Arctan} s \\
&\int \frac{ds}{s^2-1} = \frac{1}{2}\int \frac{ds}{s-1} - \frac{1}{2}\int \frac{ds}{s+1} \\
&= \frac{1}{2}\ln|s-1| - \frac{1}{2}\ln|s+1| \\
&= \ln\sqrt{\left|\frac{s-1}{s+1}\right|}\,.
\end{aligned}$$

The last term is not necessary, because it will only appear for real roots in $q(y)$.

Appendix on partial fractions.

Given a rational function with the denominator of higher degree than the numerator, then it is always possible to write it as a sum of rational functions with numerator of degree zero (i.e. a constant), and denominator a power of a first degree polynomial. To be precise:

Theorem 1. *If $q(x)$ and $p(x) = (x-\alpha_1)^{\nu_1}\dots(x-\alpha_m)^{\nu_m}$ are polynomials, such that q have smaller degree then p, p and q sharing no roots, than the rational function $\frac{q(x)}{p(x)}$ is equal to a sum as follows:*

$$\frac{q(x)}{p(x)} = \sum_{\mu=1}^{m}\sum_{j=1}^{\nu_\mu} \frac{\beta_{\mu,j}}{(x-\alpha_\mu)^j}\,. \tag{4.12}$$

Definition 1. The terms of the double sum in (4.12) are called the partial fractions of the rational function $\frac{q}{p}$.

Proof. We want to find β such that

$$\frac{q(x)}{(x-\alpha)^\nu p(x)} - \frac{\beta}{(x-\alpha)^\nu} = \frac{\hat{q}(x)}{(x-\alpha)^{\nu-1}p(x)}$$

with $\hat{q}(x)$ of smaller degree than the denominator. Now, the difference equals

$$\frac{q(x)-\beta p(x)}{(x-\alpha)^\nu p(x)}\,.$$

If we choose $\beta = \frac{q(\alpha)}{p(\alpha)}$, then we may write

$$q(x) - \beta p(x) = (x-\alpha)\hat{q}(x)\,.$$

This $\hat{q}(x)$ solves the problem.

CHAPTER 5. Linear differential equations

The material in this chapter is taken from [9, 12, 7].

We want to analyze an initial value problem: a couple of linear first-order differential equations with constant real coefficients in order to find the real solutions. The system is

$$\dot{x}_1 = ax_1 + bx_2 \tag{5.1}$$

$$\dot{x}_2 = cx_1 + dx_2 \tag{5.2}$$

where $a, b, c, d \in \mathbb{R}$, and x_1, x_2 are functions with initial values

$$x_1(0) = x_1^0 \tag{5.3}$$

$$x_2(0) = x_2^0 \tag{5.4}$$

with $x_1^0, x_2^0 \in \mathbb{R}$. We shall prefer to write it in matrix form. We define vectors

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \mathbf{x}^0 = \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix}$$

and coefficient matrix

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{5.5}$$

Then we may replace (5.1), (5.2) by (5.6) and (5.3), (5.4) by (5.7):

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{5.6}$$

$$\mathbf{x}(0) = \mathbf{x}^0 \tag{5.7}$$

Motivation. In the traditional search for solutions we argue along the following lines: If $\mathbf{A}$ should happen to be a diagonal matrix, i.e., $b = c = 0$, then the system consists of two independent equations, namely

$$\dot{x}_1 = ax_1, \qquad \dot{x}_2 = dx_2,$$

with independent initial values

$$x_1(0) = x_1^0, \qquad x_2(0) = x_2^0.$$

If $\mathbf{A}$ is not of the wanted form, we look for a coordinate transformation

$$\mathbf{x} = \mathbf{S}\mathbf{y}$$

which changes the equation to

$$\dot{\mathbf{y}} = \mathbf{S}^{-1}A\mathbf{S}\mathbf{y}. \tag{5.8}$$

If the new matrix

$$\mathbf{B} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$$

happens to be diagonal, then we are through. Unfortunately, we might need to extend the problem into the complex domain in order to obtain this diagonalization, and even so, as the matrix

$$\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$$

shows, not all matrices can be diagonalized. In spite of the large amount of algebra employed we have hardly succeeded in finding the real solutions.

Alternative Analysis. The idea to be explained below is to argue slightly differently: If $\mathbf{A}^2$ should happen to be diagonal, then the system is easy to solve, even as an initial value problem. If $\mathbf{A}^2$ is not diagonal, then we are able to transform the problem, such that the new one has a coefficient matrix with diagonal square. As a matter of fact, neither of the above features needs complex numbers, and further, there are no exceptions to the procedure or even to the formulas for the solutions of the initial value problem.

The Solution with Trace Zero. If $\mathbf{A}$ is not already a diagonal matrix, then $\mathbf{A}^2$ is diagonal, if and only if the trace of $\mathbf{A}$ is zero. As we shall see, we can always transform the problem to the case where the trace of the new coefficient matrix is zero, even when $\mathbf{A}$ is diagonal. Hence we shall restrict our analysis to the case of trace zero.

Theorem 1. *If $\mathbf{A}$ has trace zero, then $-\mathbf{A}^2$ is the determinant of $\mathbf{A}$ times the unit matrix, i.e. $\mathbf{A}^2 = \Delta\mathbf{E}$ with $\Delta = a^2 + bc$.*

Theorem 2. *If $\mathbf{A}^2$ is diagonal, then either $\mathbf{A}$ is diagonal or the trace of $\mathbf{A}$ is zero.*

Proofs. Elementary. □

Under this assumption we shall analyze the solution of (5.6) and (5.7). Let $\mathbf{x}$ be a solution of (5.6)–(5.7). Then by Theorem 1:

$$\ddot{\mathbf{x}} = \Delta\mathbf{E}\mathbf{x}. \tag{5.9}$$

Let δ be the solution of the initial value problem

$$\ddot{\delta} = \Delta\delta; \tag{5.10}$$

$$\delta(0) = 0, \qquad \dot{\delta}(0) = 1. \tag{5.11}$$

Note that $\dot{\delta}$ solves (5.10), but not (5.11). Then $\dot{\delta}$ is not proportional to δ and hence the couple $(\delta, \dot{\delta})$ constitutes a basis for the solutions of (5.10). Because $\mathbf{x}$ solves (5.9), it must take the form

$$\mathbf{x} = \dot{\delta}\mathbf{v} + \delta\mathbf{w} \tag{5.12}$$

where $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^2$. As $\mathbf{x}$ satisfies (5.6), we have

$$\dot{\mathbf{x}} = \ddot{\delta}\mathbf{v} + \dot{\delta}\mathbf{w} = \mathbf{A}\mathbf{x} = \dot{\delta}\mathbf{A}\mathbf{v} + \delta\mathbf{A}\mathbf{w}.$$

Using (5.10) we get the equation

$$\dot{\delta}\mathbf{w} + \delta\Delta\mathbf{v} = \dot{\delta}\mathbf{A}\mathbf{v} + \delta\mathbf{A}\mathbf{w}.$$

At $t = 0$ we have, because of (5.11),

$$\mathbf{w} = \mathbf{A}\mathbf{v}. \tag{5.13}$$

Substitution of (5.13) in (5.12) yields

$$\mathbf{x} = (\dot{\delta}\mathbf{E} + \delta\mathbf{A})\mathbf{v}.$$

As $\mathbf{x}$ satisfies (5.7), we have

$$\mathbf{x}^0 = \mathbf{x}(0) = (1\mathbf{E} + 0\mathbf{A})\mathbf{v} = \mathbf{v}. \tag{5.14}$$

Hence the solution of (5.6) and (5.7) is of the form

$$\mathbf{x} = (\dot{\delta}\mathbf{E} + \delta\mathbf{A})\mathbf{x}^0. \tag{5.15}$$

This ends the analysis.

Now we can substitute (5.15) in (5.6) and (5.7) for verification. In the latter case we get (5.14), and in the former using (5.10)

$$\dot{\mathbf{x}} = (\ddot{\delta}\mathbf{E} + \dot{\delta}\mathbf{A})\mathbf{x}^0 = (\delta\Delta\mathbf{E} + \dot{\delta}\mathbf{A})\mathbf{x}^0$$

while using Theorem 1 yields

$$\mathbf{A}\mathbf{x} = \mathbf{A}(\dot{\delta}\mathbf{E} + \delta\mathbf{A})\mathbf{x}^0 = (\dot{\delta}\mathbf{A} + \delta\mathbf{A}^2)\mathbf{x}^0 = (\dot{\delta}\mathbf{A} + \delta\Delta\mathbf{E})\mathbf{x}^0.$$

The General Case. Without any assumptions about $\mathbf{A}$ we shall transform the equations (5.6) and (5.7) to the case of zero trace. This proves much easier than the transformation (5.8). Let $\Theta \in \mathbb{R}$ be a constant, and θ the solution of the initial value problem

$$\dot{\theta} = \Theta\theta; \qquad \theta(0) = 1 \tag{5.16}$$

(i.e. the exponential function $\theta(t) = e^{\Theta t}$).

We consider the coordinate transformation

$$\mathbf{x} = \theta\boldsymbol{\xi}.$$

Now (5.6) and (5.7) for $\mathbf{x}$ imply certain equations for $\boldsymbol{\xi}$. (5.7) is simple:

$$\mathbf{x}^0 = \mathbf{x}(0) = \theta(0)\boldsymbol{\xi}(0) = \boldsymbol{\xi}(0)$$

by (5.16). (5.6) is nicer:

$$\dot{\mathbf{x}} = \theta\dot{\boldsymbol{\xi}} + \dot{\theta}\boldsymbol{\xi} = \mathbf{A}\mathbf{x} = \theta\mathbf{A}\boldsymbol{\xi}.$$

Using (5.16) we get

$$\theta\dot{\boldsymbol{\xi}} + \Theta\theta\boldsymbol{\xi} = \theta\mathbf{A}\boldsymbol{\xi}.$$

Because $\theta \neq 0$, we can divide by it, hence

$$\dot{\boldsymbol{\xi}} = (\mathbf{A} - \Theta\mathbf{E})\boldsymbol{\xi}.$$

Now, if we choose Θ correctly, the new matrix will have trace zero. We define Θ as

$$\Theta = \frac{a+d}{2},$$

half of the trace of $\mathbf{A}$. Then the system gets the matrix

$$\mathbf{B} = (\mathbf{A} - \Theta\mathbf{E}) = \begin{pmatrix} \frac{a-d}{2} & b \\ c & \frac{d-a}{2} \end{pmatrix}$$

so $\boldsymbol{\xi}$ solves the initial value problem:

$$\dot{\boldsymbol{\xi}} = \mathbf{B}\boldsymbol{\xi}; \qquad \boldsymbol{\xi}(0) = \mathbf{x}^0 \tag{5.17}$$

of the type of trace zero. (5.17) is then solved by (5.15), where δ solves (5.10), (5.11) with

$$\Delta = \left(\frac{a-d}{2}\right)^2 + bc.$$

Conclusion. We can write down the solution of (5.6) and (5.7) explicitly. Let the half-trace Θ and the discriminant Δ of the matrix $\mathbf{A}$ be defined as

$$\Theta = \frac{a+d}{2}, \tag{5.18}$$

$$\Delta = \left(\frac{a-d}{2}\right)^2 + bc. \tag{5.19}$$

Let θ be the solution of the initial value problem

$$\dot{\theta} = \Theta\theta; \qquad \theta(0) = 1.$$

Let δ be the solution of the initial value problem

$$\ddot{\delta} = \Delta\delta; \quad \delta(0) = 0; \quad \dot{\delta}(0) = 1.$$

Then the solution of (5.6) and (5.7) is:

$$\mathbf{x} = \theta(\dot{\delta}\mathbf{E} + \delta(\mathbf{A} - \Theta\mathbf{E}))\mathbf{x}^0. \tag{5.20}$$

In coordinates this becomes

$$x_1 = \theta\left(x_1^0\dot{\delta} + \left(\frac{a-d}{2}x_1^0 + bx_2^0\right)\delta\right),$$

$$x_2 = \theta\left(x_2^0\dot{\delta} + \left(\frac{d-a}{2}x_2^0 + cx_1^0\right)\delta\right).$$

The functions θ and δ can be explicitly written down. They are

$$\begin{aligned} \theta(t) &= e^{\Theta t} = e^{\frac{a+d}{2}t}; \\ \delta(t) &= \begin{cases} \frac{1}{\sqrt{\Delta}}\sinh(\sqrt{\Delta}t) & \text{for } \Delta > 0, \\ t & \text{for } \Delta = 0, \\ \frac{1}{\sqrt{-\Delta}}\sin(\sqrt{-\Delta}t) & \text{for } \Delta < 0. \end{cases} \end{aligned} \tag{5.21}$$

Afterthought. From a higher point of view, the methods applied here are examples of more sophisticated analytic methods in algebraic disguise, to be compared with the standard sophisticated algebra. If Sophus Lie could have asked Jean B. J. Fourier to solve the equations, he would have done so as follows:

The system (5.1)–(5.2) should be transformed into one equation of second order, i.e.,

$$\ddot{x} - (a + d)\dot{x} + (ad - bc) = 0.$$

Fourier, of course, would have transformed the operator to a polynomial,

$$\xi^2 - (a + d)\xi + (ad - bc);$$

then he would have translated this by the distance Θ (from (5.18)) , say

$$\eta = \xi - \Theta, \tag{5.22}$$

and hence obtained

$$\eta^2 - \Theta^2 + (ad - bc) = \eta^2 - \Delta$$

with Δ defined by (5.19).

By the inverse Fourier transformation, η is transformed into y, satisfying

$$\ddot{y} = \Delta y$$

and related to x by the transform of (5.22), i.e.,

$$y = x \cdot e^{-\Theta t}.$$

Further, he would have formulated the results of his efforts in the form of (5.20). For then Sophus Lie could have extracted the matrix

$$\mathbf{C}(t) = \theta(t)(\dot{\delta}(t)\mathbf{E} + \delta(t)(\mathbf{A} - \Theta\mathbf{E})),$$

which is a handy representation of the *Lie group* of the flow of solutions of (5.1)–(5.4).

Hence $\mathbf{C}(t)$ must satisfy the relation

$$\mathbf{C}(t + s) = \mathbf{C}(t)\mathbf{C}(s) \tag{5.23}$$

We know this relation from the theory of Lie groups, but I shall leave the verification by the elementary trigonometric formulas for addition and their analogues as an exercise.

Relation to difference equations. If we consider the equation (5.23) for $t = n \in \mathbb{N}$, we may deduce the formula

$$\mathbf{C}(n) = \mathbf{C}(1)^n \tag{5.24}$$

proving that the solutions for integral values of the argument satisfies the difference equation

$$\mathbf{x}(n + 1) = \mathbf{C}(1)\mathbf{x}(n) \tag{5.25}$$

Definition. This fact together with the functional equation (5.23) motivates the definition of the exponential function of a matrix:

$$\mathbf{C}(1) = exp(\mathbf{A}) \tag{5.26}$$

A system of differential equations allows the derivation of a system of difference equations with essential the same solutions. Exactly, the solutions to the latter belongs to the solutions to the former.

Nevertheless, difference equations are in a sense more general, because we may not take any logarithm to find the matrix $\mathbf{A}$ when we know $\mathbf{C}(1)$. The exponential will always have positive determinant, so for the most matrices is it a fact, that they are not exponential function values of any other matrix.

We shall analyze the stability of the solutions to difference equations in chapter 12. But we shall make an analysis here too, because of the fact that the solutions to the differential equations only corresponds to a simple part of the solutions to the difference equations.

Classification. It is obvious from (5.21) that the behavior of the solution vary with the signs of the *trace,* θ, and the *discriminant,* Δ. The third powerful criteria is the *determinant,*

$$D = ad - bc\,. \tag{5.27}$$

The three classifiers are related. We may notice, that

$$\Delta = \Theta^2 - D\,. \tag{5.28}$$

Proof.

$$\Delta = \left(\frac{a-d}{2}\right)^2 + bc = \left(\frac{a+d}{2}\right)^2 - ad + bc = \Theta^2 - D\,.$$

The forms of the solutions (5.21) shows, that for $\Delta < 0$ we see spirals, outgoing for $\theta > 0$ and ingoing for $\theta < 0$.

If $\Delta > 0$ we may expect asymptotic behavior. A natural question is, if there are solutions with trajectories as straight lines, such that the other solutions are attracted to or repelled from those ?

To show this more clearly, we assume b and c to be fixed. Then we consider the behavior as dependent of a and d, to be described in an (a, d)–plane. There are two essentially different pictures, 1) for $bc > 0$, e.g. b and c both positive, and 2) for $bc < 0$, e.g. $b > 0 > c$. In the diagrams the plane is divided into regions of similar behavior, which is drawn as examples in an (x_1, x_2)–plane.

In order to recognize possible straight line attractors we consider the ratio between the coordinates:

$$r = \frac{x_2}{x_1}\,. \tag{5.29}$$

If the trajectory is a straight line, it means that this ratio is constant, i.e.

$$r(t) = \alpha$$

or, equivalently,

$$\dot{r} = 0\,. \tag{5.30}$$

Now, we may find this derivative:

$$\dot{r} = \frac{x_1\dot{x}_2 - x_2\dot{x}_1}{x_1^2} = h(r) = c + (d-a)r - br^2 \tag{5.31}$$

using (5.1) and (5.2).

Hence for $\Delta > 0$ we have

$$\dot{r} = 0 \iff r = \frac{d - a \pm \sqrt{4\Delta}}{2b} = \alpha_\pm$$

giving two constant solutions.

The solutions must satisfy for $b \neq 0$

$$\alpha_+\alpha_- = -\frac{c}{b}$$

so, if b and c have the same sign, then the roots have opposite signs and vice versa.

A solution on one of these lines is easy to find. From (5.1) and (5.29) we get

$$\dot{x}_1 = a\,x_1 + b\alpha\,x_1 = (a + b\alpha)x_1 \tag{5.32}$$

with solutions

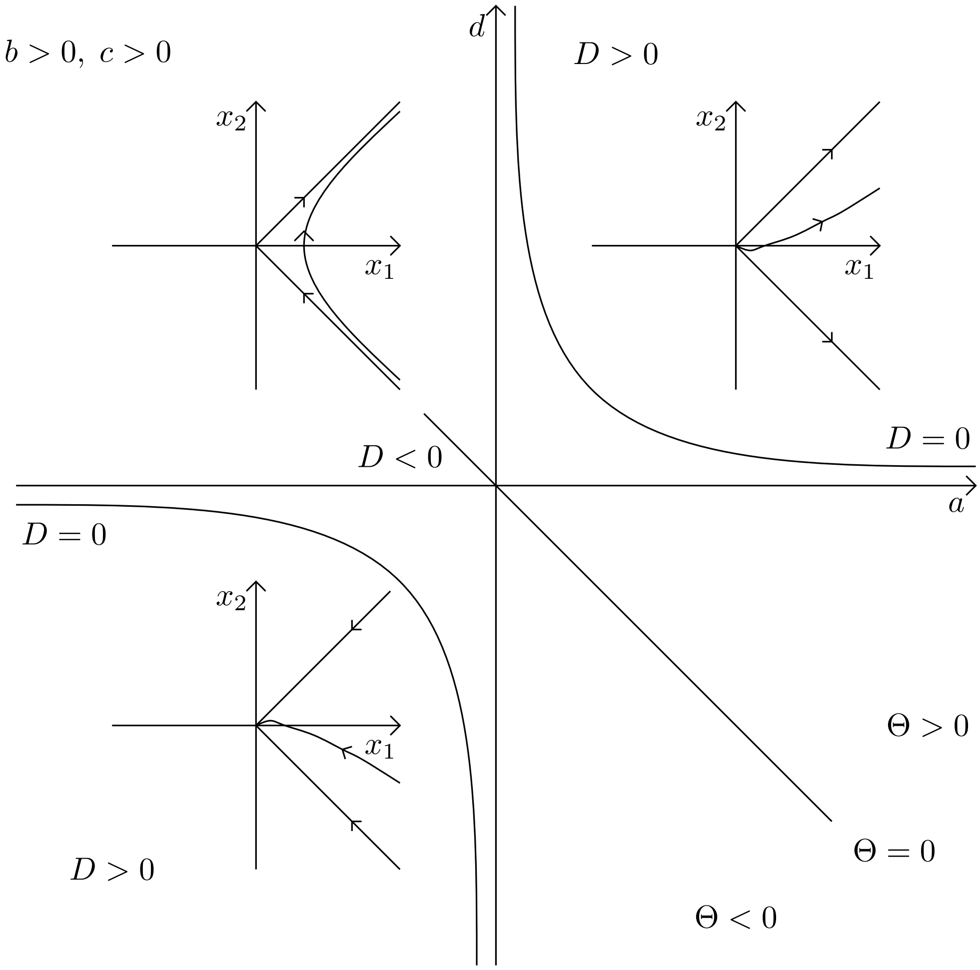
$$x_1 = \beta\,e^{(a+b\alpha)t}\,, \quad \beta \in \mathbb{R}\,. \tag{5.33}$$

These exponentials must increase or decline according to the sign of $(a+b\alpha)$. I.e.

$$\begin{aligned} a + b\alpha &= a + \frac{d-a}{2} \pm \sqrt{\Delta} = \frac{a+d}{2} \pm \sqrt{\Delta} \\ &= \Theta \pm \sqrt{\Delta}\,. \end{aligned} \tag{5.34}$$

Hence, they have opposite signs for (using (5.28))

$$\Theta^2 \leq \Delta \iff D < 0\,. \tag{5.35}$$



Stability of solutions

The trajectories may be found from (5.1) and (5.31) for $b \neq 0$ by

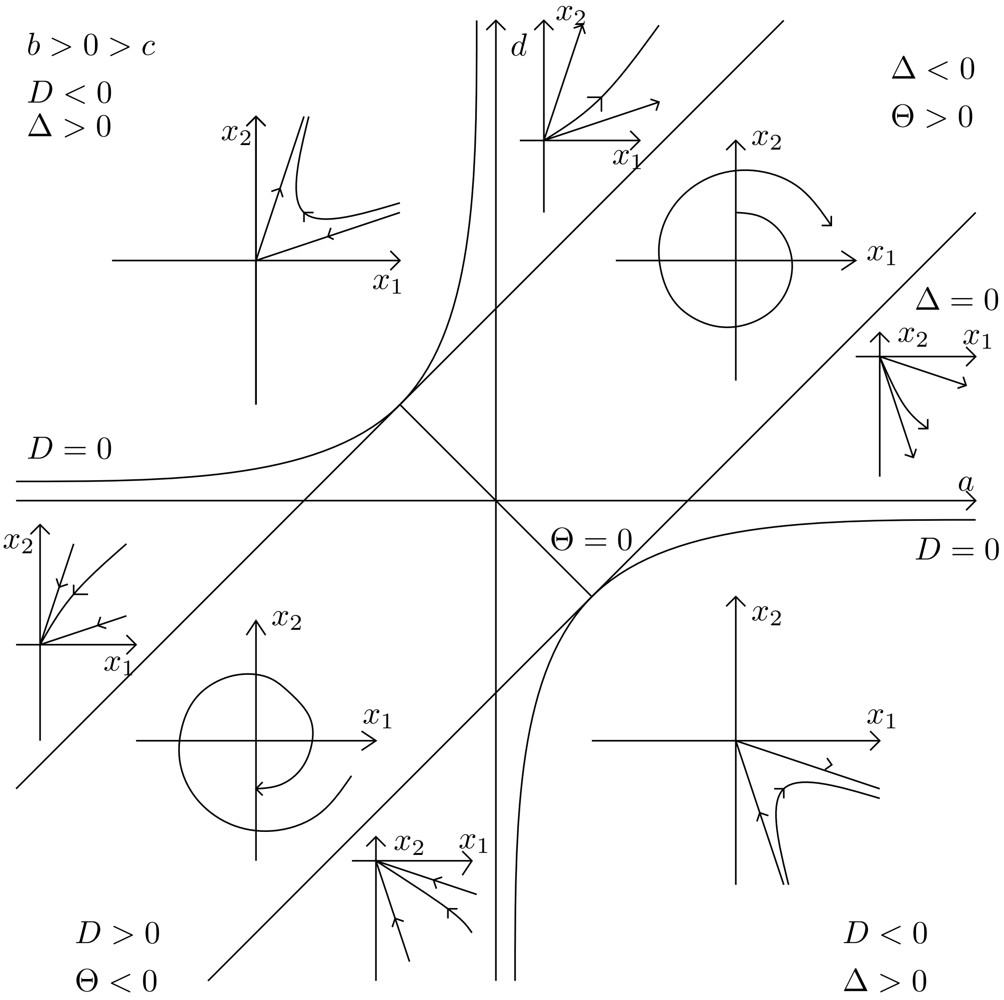
$$\frac{\dot{x}_1}{x_1} = \frac{a+br}{h(r)}\dot{r} \tag{5.36}$$

which may be integrated as

$$\ln|x_1| = -\frac{1}{2}\ln|h(r)| + \Theta \int \frac{dr}{h(r)} \tag{5.37}$$

or, equivalently

$$\ln|cx_1^2 + (d-a)x_1x_2 - bx_2^2| = (a+d)\int \frac{dr}{h(r)} \tag{5.38}$$



Stability of solutions

The integral depends on the sign of Δ.

If $\Delta > 0$ we may write

$$h(r) = -b(r - \alpha_+)(r - \alpha_-) \tag{5.39}$$

yielding the integral, cf. chapter 10,

$$\int \frac{dr}{h(r)} = \frac{1}{\Delta} \ln \left| \frac{r - \alpha_-}{r - \alpha_+} \right| \tag{5.40}$$

Substitution of (5.39) and (5.40) in (5.38) yields

$$|x_2 - \alpha_+ x_1|^{a + b\alpha_+} = \beta \, |x_2 - \alpha_- x_1|^{a + b\alpha_-} \tag{5.41}$$

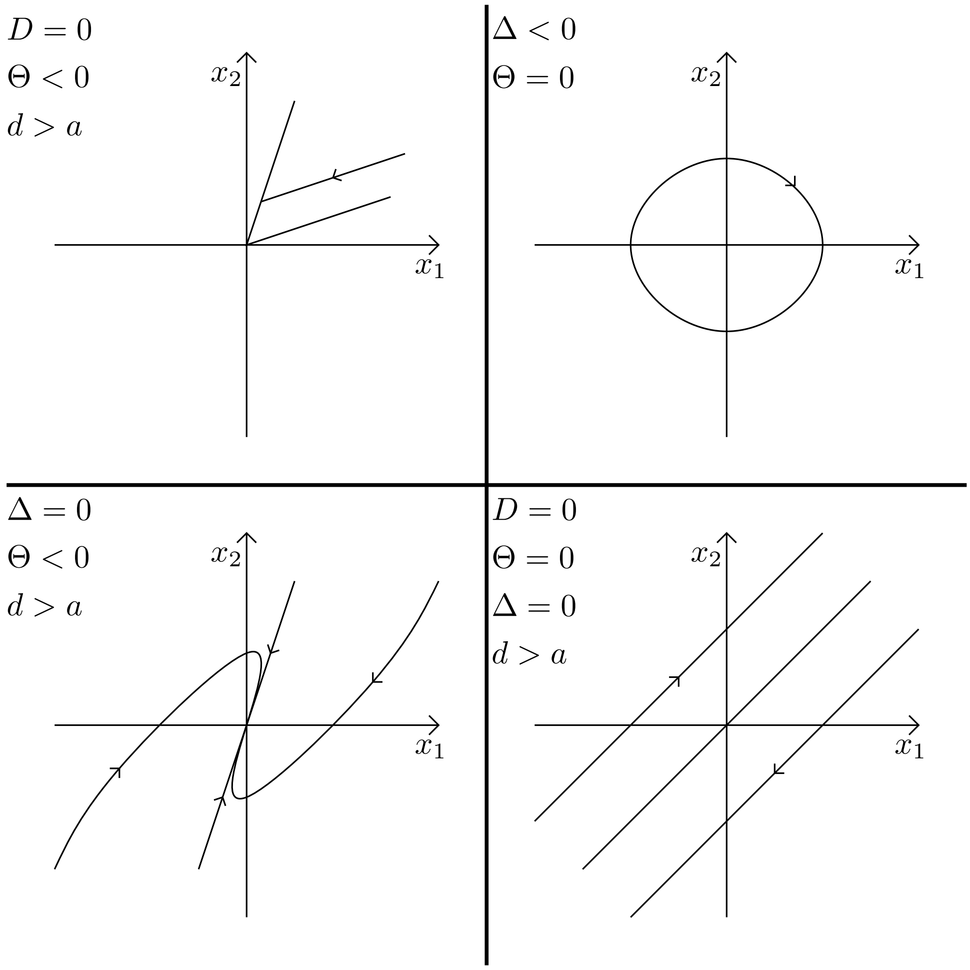
The exponents may be expressed as

$$a + b\alpha_\pm = \Theta \pm \frac{\sqrt{\Delta}}{2} \tag{5.42}$$

If $b, c > 0$ we have always $\Delta > 0$, the two straight line trajectories have slopes of different signs, and the behavior of the solution depends on the signs of D and Θ.

If $b > 0 > c$, the straight line trajectories have slopes of the same sign, both positive or both negative depending on the sign of $a - d$. The behavior around them depends on the signs of D and Θ. In this case it is possible to have $\Delta < 0$, in which case there are no straight line trajectories. In this case the solutions will spiral, outwards or inwards depending on the sign of Θ.

It is interesting to consider the limitcases between the possible typical behaviours. What is the interface between a spiral and a knot, or between an ellipse, a knot and a saddle? Four of these funny cases are depicted in the figure below.



Limitcases

Summary. The behavior of the solutions changes each time we pass one of the sets where $\Delta = 0$, $D = 0$ or $\Theta = 0$. If $\Delta > 0$, we have two straight lines of solutions, and if $\Delta < 0$ the solutions spiral around. If $D > 0$, the solutions on the lines run in the same direction in the case of $D > 0$, and if $D < 0$ they run in opposite directions. And if $\Theta > 0$, the solutions are repelled from the origin, and if $\Theta < 0$ they are attracted to the origin.

CHAPTER 6. Economical examples of differential equations

Example 1. Taken from [3]. Suppose that the demand, $D(t)$, and the supply, $S(t)$, both depend on the current price, $p(t)$, but in different ways. The demand depends directly of the price now,

$$D(t) = a + b\,p(t) \tag{6.1}$$

while the supply is governed by the expected price in the near future

$$\hat{p}(t) = p(t) + c\,p'(t)\,. \tag{6.2}$$

Hence the supply depends on the price and its trend

$$S(t) = a_1 + b_1\,\hat{p}(t) = a_1 + b_1\,p(t) + b_1\,cp'(t)\,. \tag{6.3}$$

Next, the assumption that the market forces equality between supply and demand

$$D(t) = S(t) \tag{6.4}$$

gives rise to the differential equation for the price,

$$p'(t) = \frac{1}{b_1\,c}\left((b-b_1)p(t) + a - a_1\right)\,. \tag{6.5}$$

This equation may be interpreted as follows. The market remains in equilibrium, if D, S and p are constant over time, and hence the derivative $p'(t) = 0$. If this is not the case, the three quantities vary over time, the price solves (6.5) and hence $D(t) = S(t)$ as these are derived from $p(t)$ by the use of (6.1) and (6.3).

If this variation over time converges towards the equilibrium value, we shall call it a *stable* equilibrium.

From the general theory of differential equations we know that this is the case iff

$$\frac{b-b_1}{b_1\,c} < 0\,. \tag{6.6}$$

Under normal circumstances we may assume $b < 0$ and $b_1 > 0$, so that (6.6) is fulfilled.

Example 2. Taken from [14]. W. Leontief considers the mutual demand and supply of several goods. Let us consider the output of two goods, X_i, $i = 1, 2$. Each output requires the input of the goods to a certain extend,

$$X_i = a_{i1}X_1 + a_{i2}X_2 + b_{i1}\dot{X}_1 + b_{i2}\dot{X}_2 + Y_i\,. \tag{6.7}$$

The a's represent the current use of the good, while the b's represents the future supply decision in analogy with (6.3). The quantity Y_i represents the consume of the good.

In matrix form the linear differential equation (6.7) looks as

$$\dot{X} = B^{-1}(E - A)X + B^{-1}Y\,. \tag{6.8}$$

If we let D be the determinant of B, i.e.

$$D = b_{11}b_{22} - b_{12}b_{21} \tag{6.9}$$

then the inverse matrix of B is

$$B^{-1} = \frac{1}{D}\begin{pmatrix} b_{22} & -b_{12} \\ -b_{21} & b_{11} \end{pmatrix} \tag{6.10}$$

To determine the behavior and stability of the solutions to (6.8), we need the trace and determinant of the matrix

$$B^{-1}(E - A) = \frac{1}{D}\begin{pmatrix} b_{22}(1 - a_{11}) + b_{12}a_{21} & -b_{22}a_{12} - b_{12}(1 - a_{22}) \\ -b_{21}(1 - a_{11}) - b_{11}a_{21} & b_{21}a_{12} + b_{11}(1 - a_{22}) \end{pmatrix} \tag{6.11}$$

The trace is

$$\Theta = \frac{1}{D}(b_{22} + b_{11} - a_{11}b_{22} - a_{22}b_{11} + b_{12}a_{21} + b_{21}a_{12}) \tag{6.12}$$

and the determinant is

$$\begin{aligned} \mathrm{Det} = \mathrm{Det}(B^{-1}(E - A)) &= \frac{\det(E - A)}{\det B} = \frac{(1 - a_{11})(1 - a_{22}) - a_{21}a_{12}}{D} \\ &= \frac{1 - \Theta(A) + \mathrm{Det}(A)}{D} \end{aligned} \tag{6.13}$$

Now we may find the discriminant by the formula (5.28) from chapter 5,

$$\Delta = \frac{1}{4}\Theta^2 - \mathrm{Det}\,. \tag{6.14}$$

For a reasonable system we assume that

$$1 - \Theta(A) > 0 \tag{6.15}$$

$$\mathrm{Det}(A) > 0 \tag{6.16}$$

while we do no assumptions about the sign of D.

The non–diagonal elements are

$$\text{"}b\text{"} = \frac{1}{D}\left(-b_{12} - b_{22}a_{12} + b_{12}a_{22}\right) \tag{6.17}$$

$$\text{"}c\text{"} = \frac{1}{D}\left(-b_{21} + b_{21}a_{11} - b_{11}a_{21}\right) \tag{6.18}$$

both having the opposite sign of D. Hence we know from chapter 5, that $\Delta > 0$.

We consider the two cases:

1) $D < 0$. Now "b" and "c" are both positive, and the determinant (6.13) is negative. So, according to the summary of chapter 5, the solutions form a saddle point of equilibrium.

2) $D > 0$. Now the two are both negative and the determinant is positive. Then the equilibrium is a knot. Now, the trace (6.12) is also positive, so the equilibrium is a repeller.

Example 3. Taken from [17]. Y. Shinkai has formulated a two sector growth model. The labor force, N, grows with a constant rate, n, i.e., it satisfies the equation

$$N' = n\,N\,. \tag{6.19}$$

The labor force splits into two sectors,

$$N = N_1 + N_2\,. \tag{6.20}$$

The capital, K, grows with a rate, K', proportional to the labor force in the capital good sector, (no. 1), and as the capital depends on the size of the labor force, i.e.

$$K = a_1 N_1 + a_2 N_2 \tag{6.21}$$

we get the equation

$$a_1 N_1' + a_2 N_2' = K' = m\,N_1\,. \tag{6.22}$$

Now, the two equations (6.19) and (6.22) form a system of linear differential equations of the form

$$\begin{pmatrix} a_1 & a_2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} N_1 \\ N_2 \end{pmatrix}' = \begin{pmatrix} m & 0 \\ n & n \end{pmatrix}\begin{pmatrix} N_1 \\ N_2 \end{pmatrix} \tag{6.23}$$

or,

$$A\,N' = B\,N \iff N' = A^{-1}BN\,. \tag{6.24}$$

Now, we need the three criteria-functions from $A^{-1}B$. First we find A^{-1}:

$$A^{-1} = \frac{1}{a_1 - a_2} \begin{pmatrix} 1 & -a_2 \\ -1 & a_1 \end{pmatrix} \tag{6.25}$$

And $A^{-1}B$:

$$A^{-1}B = \frac{1}{a_1 - a_2} \begin{pmatrix} m - na_2 & -na_2 \\ -m + na_1 & na_1 \end{pmatrix} \tag{6.26}$$

with the trace

$$\Theta = \frac{m + n(a_1 - a_2)}{a_1 - a_2} = \frac{m}{a_1 - a_2} + n \tag{6.27}$$

and determinant

$$D = \frac{mn}{a_1 - a_2}\,. \tag{6.28}$$

(The eigenvalues for the matrix are then obviously $\frac{m}{a_1 - a_2}$ and n, having the right sum and product.)

The discriminant is always positive

$$\Delta = \frac{1}{4}\left(\Theta^2 - 4D\right) = \frac{1}{4}\left(\frac{m}{a_1 - a_2} - n\right)^2 > 0\,. \tag{6.29}$$

If we assume $a_1 < a_2$, then the determinant $a_1 - a_2$ becomes negative. The off diagonal element, $\frac{-n\,a_2}{a_1 - a_2}$ becomes positive, while the other one

$$\frac{-m + n\,a_1}{a_1 - a_2} \tag{6.30}$$

becomes positive for $n\,a_1 < m$ and negative for $n\,a_1 > m$. In both cases the negative determinant and the positive discriminant gives rise to a saddle point of equilibrium.

The two asymptotes have slopes $\frac{m - na_1}{na_2}$ and -1, the fraction having the sign of (6.30). The solution is increasing on the first asymptote and decreasing on the second. The speeds of the solutions will be, respectively, $\frac{2n}{a_2 - a_1}$ and $\frac{2m}{a_1 - a_2}$.

CHAPTER 7. Homogeneous Autonomous Dynamical Systems

The material in this chapter is taken from [10, 7].

Consider an autonomous *dynamic* system in the normal (explicit) form

$$\dot{x} = dx/dt = F(x, y), \quad (x, y) \in \mathbb{R}^2 \tag{7.1.1}$$

$$\dot{y} = dy/dt = G(x, y), \quad (x, y) \in \mathbb{R}^2 \tag{7.1.2}$$

We shall throughout make use of the assumption, that F and G are homogeneous functions of the same degree, $m \in \mathbb{R}$, with continuous partial derivatives.

Now, introducing the *ratio* variable

$$r = y/x, \qquad x \neq 0 \tag{7.2}$$

we have from the homogeneity

$$\dot{x} = x|x|^{m-1}F(1, r), \quad x \neq 0 \tag{7.3.1}$$

$$\dot{y} = x|x|^{m-1}G(1, r), \quad x \neq 0 \tag{7.3.2}$$

or, in simplified notation on the same domain

$$\dot{x} = x|x|^{m-1}f(r); \quad x \neq 0 \tag{7.4.1}$$

$$\dot{y} = x|x|^{m-1}g(r); \quad x \neq 0 \tag{7.4.2}$$

where

$$f(r) = F(1, r), \quad g(r) = G(1, r) \tag{7.5}$$

Then the *ratio* of the *individual coordinate solutions,*

$$r = \rho(t) = \varphi_2(t)/\varphi_1(t) \tag{7.6}$$

has the derivative, cf. (7.2), (7.1),

$$\begin{aligned} \dot{r} &= dr/dt = d(y/x)/dt = (x\dot{y} - y\dot{x})/x^2 \\ &= |x|^{m-1}g(r) - |x|^{m-1}rf(r) \\ &= |x|^{m-1}h(r) \end{aligned} \tag{7.7}$$

where

$$h(r) = g(r) - rf(r), \tag{7.8}$$

The *director root set,* $\mathcal{A}$, of $h(r) = 0$, (7.8), gives a *partition* of $\mathbb{R} = \mathcal{A} \cup \complement\mathcal{A}$, i.e.,

$$\mathcal{A} = \{\alpha \in \mathbb{R} \,|\, h(\alpha) = 0\}; \ \overline{\mathcal{A}} = \mathcal{A} \cup \{\pm\infty\}; \ \complement\mathcal{A} = \{\alpha \in \mathbb{R} \,|\, h(\alpha) \neq 0\} \tag{7.9}$$

where $\overline{\mathcal{A}}$ is its closure (in the *extended* real line, $\mathbb{R}^* = \mathbb{R} \cup \{\pm\infty\}$) and $\complement\mathcal{A}$ its complement. The complement is *open*; hence $\complement\mathcal{A}$ is the *disjoint union* of an at most countable set of *open intervals* of $\mathbb{R}$, and each interval may be characterized as a *maximal interval* of $\complement\mathcal{A}$, or, equivalently, an *open interval,* $]\underline{\alpha}, \overline{\alpha}[\subseteq \complement\mathcal{A}$, which has *endpoints,* $\underline{\alpha}, \overline{\alpha} \in \overline{\mathcal{A}}$.

Definition. The set $\overline{\mathcal{A}}$ gives a division of the punctuated plane, $\mathbb{R}^2 \setminus \{(0,0)\}$ into the *rays,* $R_\alpha = \{(x,y)|y = \alpha x\}$, for $\alpha \in \overline{\mathcal{A}}$, and the *cones,* $C_{\underline{\alpha},\overline{\alpha}} = \{(x,y)|\underline{\alpha}x < y < \overline{\alpha}x\}$, where $\underline{\alpha}, \overline{\alpha} \in \overline{\mathcal{A}}, \quad]\underline{\alpha}, \overline{\alpha}[\in \complement\mathcal{A}$.

Solutions remain on these rays and in these cones.

Definition. On a *maximal* interval, $]\underline{\alpha}, \overline{\alpha}[\subseteq \complement\mathcal{A}$, we define a function, H, by

$$dH/dr = H'(r) = f(r)/h(r)\ , \quad r \in]\underline{\alpha}, \overline{\alpha}[\subseteq \complement\mathcal{A}, \tag{7.10}$$

$H(r)$ cannot be extended continuously to points of $\mathcal{A}$.

Lemma 1. *Let $H(r)$ be defined on the maximal interval $]\underline{\alpha}, \overline{\alpha}[$ by (7.10). If $f(\overline{\alpha}) \neq 0$ ($f(\underline{\alpha}) \neq 0$), then $H(r)$ is unbounded in the neighborhood of $\overline{\alpha}$ ($\underline{\alpha}$). Precisely, if $h(r)$ and $f(\overline{\alpha})$ ($f(\underline{\alpha})$) have the same sign, then $H(r) \to \infty$ as $r \to \overline{\alpha}$ ($\underline{\alpha}$), otherwise $H(r) \to -\infty$ as $r \to \overline{\alpha}$ ($\underline{\alpha}$).*

Proof. Let $h(r) > 0$ and $f(\overline{\alpha}) > 0$. Then for some r_0 and all r, $r_0 \leq r < \overline{\alpha}$, we have for some positive constant k,

$$\frac{f(r)}{h(r)} > \frac{k}{\overline{\alpha} - r}$$

Hence,

$$\begin{aligned} H(r) &= H(r_0) + \int_{r_0}^{r} \frac{f(r)}{h(r)} dr > H(r_0) + \int_{r_0}^{r} \frac{k}{\overline{\alpha} - r} dr \\ &= H(r_0) + k\log(\overline{\alpha} - r_0) - \log(\overline{\alpha} - r) \to \infty \text{ as } r \to \overline{\alpha} \end{aligned} \tag{7.11}$$

The general behavior of the dynamic system (7.1) - apart from boundary ray behavior, is given by the following theorem.

Theorem 1. *For any solution $\phi(t) = (\phi_1(t), \phi_2(t))$, to (7.1), with initial data (x_0, y_0) in a cone, every ratio solution $\rho(t) = \phi_2(t)/\phi_1(t)$, (7.6) must solve the fundamental autonomous differential equation*

$$\dot{r} = Q(r) = K_0 \cdot e^{(m-1)H(r)} h(r) \tag{7.12}$$

with the constant K_0 depending on (x_0, y_0), precisely,

$$K_0 = x_0^{m-1} e^{-(m-1)H(r_0)}, \quad r_0 = y_0/x_0. \tag{7.13}$$

The individual coordinate solutions are

$$x = \phi_1(t) = k_0 e^{H(\rho(t))} \tag{7.14}$$

$$y = \phi_2(t) = \rho(t)\phi_1(t) \tag{7.15}$$

where the constant k_0 is

$$k_0 = x_0 e^{-H(r_0)}, \quad r_0 = y_0/x_0. \tag{7.16}$$

The trajectory is given implicitly by the equation

$$x = k_0 e^{H(y/x)} \tag{7.17}$$

Proof. Without loss of generality we may assume $x_0, y_0 \in \mathbb{R}_+$. By (7.4.1), we have

$$\dot{x}/x = x^{m-1} f(r) \tag{7.18}$$

Eliminating x^{m-1} from (7.18) and (7.7), we get

$$\frac{\dot{x}}{x} = \frac{f(r)\dot{r}}{h(r)} \tag{7.19}$$

Integration of (7.19) with respect to time gives

$$\int \frac{\dot{x}dt}{x} = \int \frac{f(r)\dot{r}dt}{h(r)} \tag{7.20}$$

or equivalently, cf. (7.10)

$$\int \frac{dx}{x} = \int \frac{f(r)dr}{h(r)} = \int H'(r)dr \tag{7.21}$$

and so

$$\log x = H(r) + k \tag{7.22}$$

where the constant of integration k has to satisfy the initial condition

$$k = \log x_o - H(r_0). \tag{7.23}$$

Thus, we can express x as a function of r

$$x = x_0 e^{-H(r_0)} e^{H(r)}, \tag{7.25}$$

and further

$$x^{m-1} = x_0^{m-1} e^{-(m-1)H(r_0)} e^{(m-1)H(r)} \tag{7.26}$$

Substituting this expression for x^{m-1} into (7.7) establishes (7.12) and (7.13).
Inserting the ratio solution $\rho(t)$, obtained from (7.12), (7.13) into (7.25) establishes (7.14)–(7.16).
Finally, the equation (7.17) follows by substitution of (7.2) in (7.14).

Corollary 1.1. *In a maximal interval,* $]\underline{\alpha}, \overline{\alpha}[$*, (7.7), any ratio solution* $\rho(t)$ *of (7.12) monotonously increases/decreases towards, respectively,* $\overline{\alpha}$ *and* $\underline{\alpha}$*, according to the sign of* $h(r)$*. The trajectories of all solutions* $\phi(t)$ *in* $C_{\underline{\alpha},\overline{\alpha}}$ *are attracted by the same boundary ray and repulsed by the other boundary ray.*

Proof. It is seen by the definition of a maximal interval that the governing function $Q(r)$, (7.12), cannot change *sign* in $]\underline{\alpha}, \overline{\alpha}[$. It must remain either positive or negative and the corollary follows.

For different initial values (x_0, y_0), it is clear from (7.12), (7.13) that the governing function of the ratio solutions are the same function, except for the value of the positive constant K_0. To compare the set of ratio solutions in $]\underline{\alpha}, \overline{\alpha}[$, we need the following lemma.

Lemma 2. *Consider two autonomous differential equations with a positive proportionality factor*

$$(i) \quad \dot{u} = Q(u), \quad (ii) \quad \dot{v} = \beta Q(v); \quad \beta > 0$$

Let $u(t)$ *and* $v(t)$ *be, respectively, a solution of (i) and (ii). Then there always exists a constant* τ *such that* $u(t)$ *and* $v(t)$ *are related by*

$$v(t) = u(\beta t + \tau) \tag{7.27}$$

Proof. The lemma is confirmed most easily by differentiation of the solutions, (7.27), i.e.,

$$\dot{v} = \beta \dot{u}(\beta t + \tau) = \beta Q[u(\beta t + \tau)] = \beta Q(v).$$

Corollary 1.2. *In* $]\underline{\alpha}, \overline{\alpha}[$*, any pair of ratio solutions,*

$$\rho_1(t) = \phi_2(t)/\phi_1(t), \quad \rho_2(t) = \psi_2(t)/\psi_1(t) \tag{7.28}$$

are related by an affine transformation of the argument

$$\rho_2(t) = \rho_1(\beta t + \tau), \quad \beta > 0. \tag{7.29}$$

Proof. Lemma 1 applied to (7.12), (7.13).

On the boundary rays, the solutions of the system (7.1) can be expressed explicitly.

Theorem 2. *For any solution $\phi(t) = (\phi_1(t), \phi_2(t))$, to (7.1) with initial data $(x_0, y_0) \neq (0,0)$ on a ray with the slope α, given by $h(\alpha) = 0$, (7.8), every ratio solution is the constant solution*

$$\rho(t) = \alpha = y_0/x_0. \tag{7.30}$$

The trajectory of any $\phi(t)$ remains on the α–ray, $y = \alpha x$,

$$\phi_2(t) = \alpha\phi_1(t) \tag{7.31}$$

where the coordinate solution $\phi_1(t)$ may take the forms

$$\begin{aligned}
&(7.32) \quad \textit{Case 1.} \quad && f(\alpha) = 0: \quad \phi_1(t) = x_0\\
&\qquad\quad\ \textit{Case 2.} && f(\alpha) \neq 0, \quad f(\alpha) > 0 \text{ or } f(\alpha) < 0:\\
&(7.33) \quad (i), && m = 1: \phi_1(t) \text{ grows/declines exponentially}\\
& && x = \phi_1(t) = x_0 e^{f(\alpha)t}\\
&(7.34) \quad (ii), && m < 1: \phi_1(t) \text{ grows/declines polynomially}\\
& && \qquad\qquad \text{(of degree 1/(1-m))}\\
& && x = \phi_1(t) = [x_0^{1-m} + (1-m)f(\alpha)t]^{1/(1-m)}\\
&(7.35) \quad (iii), && m > 1: \phi_1(t) \text{ grows/declines explosively}\\
& && x = \phi_1(t) = 1/[x_0^{1-m} - (m-1)f(\alpha)t]^{1/(m-1)}
\end{aligned}$$

Proof. Without loss of generality we may assume $x_0, y_0 \in \mathbb{R}_+$. By $r_0 = y_0/x_0 = \alpha$ and $h(\alpha) = 0$, equation (7.7) becomes $\dot{r} = 0$, hence $\rho(t) = \textit{constant} = \phi_2(t)/\phi_1(t)$, which with the initial condition give (7.30), (7.31).

Since a solution $\phi(t) = (\phi_1(t), \phi_2(t))$ must here always satisfy the differential equation, $\dot{r} = 0$, the trajectory of any $\phi(t)$ must then remain on the (initial) ray $y = \alpha x$. Next, we get from (7.4) that

$$\dot{x} = x^m f(\alpha). \tag{7.36}$$

Case 1, with $f(\alpha) = 0$, gives $\dot{x} = 0$, hence (7.32).

In case 2, with $f(\alpha) \neq 0$, we see that (7.36) can be solved by separation of variables

$$\int x^{-m} dx = \int f(\alpha) dt \tag{7.37}$$

For $m = 1$, (7.37) gives

$$\log x = f(\alpha)t + k, \quad k = \log x_0$$

which establishes (7.33).

For $m \neq 1$, (7.37) gives

$$x^{1-m}/(1-m) = f(\alpha)t + k, \quad k = x_0^{1-m}/(1-m) \tag{7.38}$$

Thus, (7.38) establishes (7.34), (7.35) for respectively, $m < 1$ and $m > 1$. An explosive solution means that the coordinates become infinite in finite time (the escape time)

Definition. We shall call the magnitude

$$f(\alpha) \tag{7.39}$$

the *directrix value* for the *directrix,* $y = \alpha x$.

Example. Although the primary use of theorem 1 was intended for the *qualitative* study of homogeneous dynamics, the basic idea of deriving the system solutions indirectly through the ratio solutions may also be useful as a *technical method* for explicitly solving the differential equations.

Obviously, the integral, $H(r)$, cf. (7.11) can seldom be written in closed form. Nevertheless, our procedure might work successfully in obtaining the explicit solutions. Consider the simple homogeneous system of degree $m = 2$, cf. (7.1):

$$\dot{x} = xy, \tag{7.40}$$

$$\dot{y} = x^2. \tag{7.41}$$

The trajectories in the phase plane are easily, cf. (7.17), derived from, $\dot{y}/\dot{x} = dy/dx = y/x$, as

$$y^2 - x^2 = c, \tag{7.42}$$

i.e. rectangular hyperbolas ($c \neq 0$, $\quad c = 0$: two straight lines (nodal rays), $y/x = \pm 1$ and the origin. As to the evolution in time and the speed of motion, however, we need the integral curves. By (7.40)–(7.41), our function $h(r)$ becomes

$$h(r) = g(r) - rf(r) = 1 - r^2 \tag{7.43}$$

which has the root values, $r = \pm 1$. Next, we have that

$$H'(r) = \frac{f(r)}{h(r)} = \frac{r}{1-r^2}, \quad r \neq \pm 1. \tag{7.44}$$

The integral of (7.44) is easily found to be

$$H(r) = \int H'(r)dr = -\log \sqrt{|1-r^2|} + k, \quad r \neq \pm 1 \tag{7.45}$$

with the constant of integration satisfying the initial condition

$$k = H(r_0) + \log\sqrt{|1-r_0^2|}, \quad r_0 \neq 1. \tag{7.46}$$

Hence, the governing function $Q(r)$ of the ratio solutions is, with $m = 2$, obtained from (7.12), (7.13), (7.45), as

$$\begin{aligned}
\dot{r} = Q(r) &= x_0 e^{-H(r_0)} e^{H(r)} h(r), \quad r_0 \neq 1 && (7.47)\\
&= x_0 e^{\log\sqrt{|1-r_0^2|}-k} e^{-\log\sqrt{|1-r^2|}+k} \left(1-r^2\right)\\
&= x_0 \left(|1-r_0^2|\right)^{1/2} \left(|1-r^2|\right)^{-1/2} \left(1-r^2\right)\\
&= \pm x_0\sqrt{|1-r_0^2|}\sqrt{|1-r^2|} = \pm A\sqrt{|1-r^2|}, \quad A > 0, && (7.48)
\end{aligned}$$

or else

$$\dot{r} = A\sqrt{1-r^2}, \quad r < 1, \tag{7.49}$$

$$\dot{r} = A\sqrt{r^2-1}, \quad r > 1. \tag{7.50}$$

For (7.49), let the initial values at $t = 0$ be, $(x_0, y_0) = (1, 0)$, implying $r_0 = 0$, $A = 1$ and thereby

$$\dot{r} = \sqrt{1-r^2}, \quad r < 1. \tag{7.51}$$

which gives the ratio solution

$$\rho(t) = \sin(t), \quad 0 < t < \pi/2. \tag{7.52}$$

For (7.50), let the initial values at $t = -\infty$ be $(x_{-\infty}, y_{-\infty}) = (0, 1)$, implying $A = -1$, and thereby

$$\dot{r} = -\sqrt{r^2-1}, \quad r > 1, \tag{7.53}$$

which gives the ratio solution

$$\rho(t) = \cosh(t), \quad t < 0. \tag{7.54}$$

The components of the coordinate solution $\phi(t)$ – corresponding to (7.52) and $x_0 = 1$, $r_0 = 0$ – become cf. (7.14), (7.16), (7.45)

$$\begin{aligned}
\phi_1(t) &= x_0 e^{-H(r_0)} e^{H(\rho(t))}, \quad r_0 < 1 && (7.55)\\
&= 1 \cdot e^{-\log\sqrt{1-\rho^2(t)}}\\
&= \left(1-\rho^2(t)\right)^{-1/2} = 1/\cos t, \quad 0 < t < \frac{\pi}{2}, && (7.56)
\end{aligned}$$

$$\phi_2(t) = \rho(t)\phi_1(t) = \sin(t)/\cos(t) = \tan t, \quad 0 < t < \frac{\pi}{2}. \tag{7.57}$$

The components of the coordinate solution $\psi(t)$ – corresponding to (7.54) and, $A = -1$, become cf. (7.45), (7.55)

(7.58)
$$\psi_1(t) = x_0 e^{\log\sqrt{r_0^2-1}} e^{-\log\sqrt{\rho^2(t)-1}}, \quad r_0 > 1,$$
$$= x_0\sqrt{r_0^2-1}\,\big(\rho^2(t)-1\big)^{-1/2} = \sqrt{y_0^2-x_0^2}\,\big(\rho^2(t)-1\big)^{-1/2}$$

(7.59)
$$= -\,\big(\rho^2(t)-1\big)^{-1/2} = -1/\sinh t, \quad t<0,$$

(7.60)
$$= \psi_2(t) = \rho(t)\psi_1(t) = \cosh(t)/(-\sinh(t)) = -\coth t, \quad t<0$$

Thus, the explicit coordinate solutions along the trajectories (7.45) with $a = 1$, are cf. (7.56)–(7.57), (7.59)–(7.60)

(7.61)
$$\phi(t) = (1/\cos t, \tan t), \quad 0 < t < \frac{\pi}{2},$$

(7.62)
$$\psi(t) = (-1/\sinh t, -\coth t), \quad t < 0.$$

The *complete* set of *ratio* solutions are obtained from (7.49)–(7.50), (7.52)–(7.53), using corollary 1.2, (7.29), i.e.

(7.63)
$$\rho(t) = \sin(At - t_0),$$

(7.64)
$$\rho(t) = -\cosh(At - t_0).$$

Hence, by (7.63)–(7.64), (7.44), (7.14), (7.16), the *complete* set of *coordinate solutions* are given by the family pair

(7.65)
$$\phi(t) = (A/\cos(At+t_0), \quad A\tan(At+t_0)),$$

(7.66)
$$\psi(t) = (-A/\sinh(At+t_0), \quad -A\coth(At+t_0)),$$

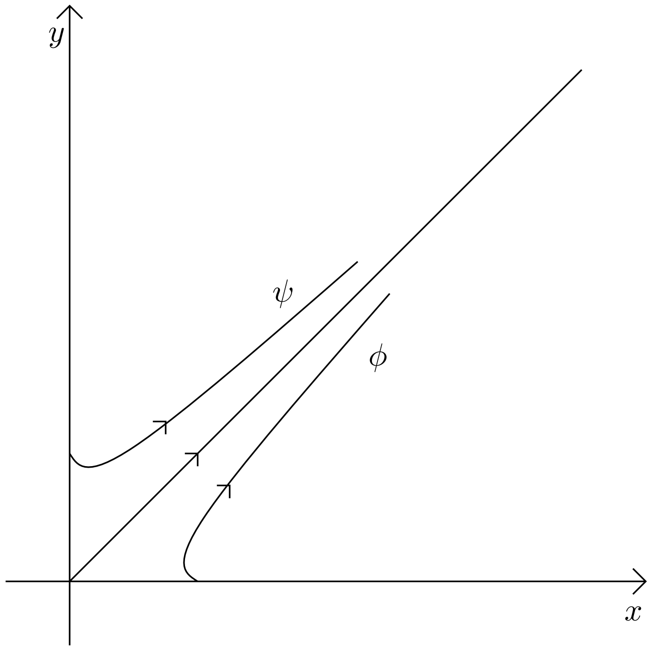
where $B = A = x_0|1-r_0^2|^{-1/2}$.

It follows from (7.49), (7.42), (7.40) and theorem 4 that the root $\alpha = +1$ represent the attractive nodal ray – upon which the coordinate solutions

(7.67)
$$\phi(t) = [(x_0^{-1}-t)^{-1}, (x_0^{-1}-t)^{-1}]$$

are growing explosively, as $f(1) = 1 > 0$ and $m = 2$, cf. (7.35).

The integral curves (7.61), (7.62) and the corresponding trajectories ($A = 1$) are depicted.



Trajectories

Appendix on l'Hospital.

Theorem of l'Hospital. *Let f and g be differentiable functions of a real variable in some interval $I =]\beta, \alpha[$ with $g'(x) \neq 0$ for $x \in I$. Suppose $\lim_{x\to\alpha} f(x) = \infty$ and $\lim_{x\to\alpha} g(x) = \infty$ and that*

$$\lim_{x\to\alpha} \frac{f'(x)}{g'(x)} = k$$

Then we have

$$\lim_{x\to\alpha} \frac{f(x)}{g(x)} = k$$

Proof. As $g'(x) > 0$ for some $x \in I$, we have $g'(x) > 0$ for all $x \in I$. Suppose now $k' < k$. Then we have

$$k' < \frac{f'(x)}{g'(x)}$$

for x close to α. This means that $f'(x) - k'g'(x) > 0$ such that the function $f(x) - k'g(x)$ is increasing. For any $\epsilon > 0$ the function $\epsilon g(x) \to \infty$ for $x \to \alpha$, so that

$$f(x) - k'g(x) + \epsilon g(x) \to \infty \quad \text{for} \quad x \to \alpha$$

hence, we have

$$f(x) - k'g(x) + \epsilon g(x) > 0$$

for x close to α, or equivalently

$$k' - \epsilon < \frac{f(x)}{g(x)}$$

Similarly is shown that

$$\frac{f(x)}{g(x)} < k'' + \epsilon$$

Hence the theorem follows.

CHAPTER 8. Stability for Homogeneous Dynamical Systems

The material in this chapter is taken from [10, 7].

We shall consider the behavior of solutions in the neighborhood of an isolated boundary ray with increasing solutions. Such a ray will play the role of an asymptote for the vicinity solutions.

Theorem 1. *Suppose α is an isolated zero for $h(r)$, h, defined by (7.8), decreasing through α, and that $f(\alpha) > 0$, f defined by (7.5). Then the solutions around will satisfy*

$$x(r) \to \infty \quad \text{for} \quad r \to \alpha. \tag{8.1}$$

Proof. According to lemma 1 of Chapter 7we have

$$H(r) \to +\infty \text{ for } r \to \alpha - . \tag{8.2}$$

By (7.25) we have

$$x(t) = B \cdot e^{H(\rho(t))},$$

from which (8.1) follows.

If the degree of homogeneity is at most one, then this behavior exhibits for time going to infinity. Precisely we have:

Theorem 2. *Suppose α is an isolated zero for $h(r)$, h defined by (7.8), and thus $f(\alpha) > 0$, f defined by (7.5). If further $m \leq 1$, then the solutions from the cone below $y = \alpha x$ must satisfy*

$$x(t) \to \infty \text{ for } t \to \infty$$
$$\rho(t) \to \alpha \text{ for } t \to \infty$$

with a speed not greater than

$$\rho(t) \leq \alpha - K \cdot e^{-A \cdot kt}$$

for some fixed K and A depending on the point of start, while k is just an upper bound for $h'(\rho)$.

Proof. The solution–ratio $\rho(t)$ satisfies (7.12):

$$\dot{\rho} = A \cdot e^{(m-1)H(\rho)} h(\rho).$$

With $m \leq 1$ the exponential is limited according to theorem 1. The mean value theorem says, that

$$h(\rho) = h'(\theta) \cdot (\rho - \alpha) \leq k \cdot (\alpha - \rho),$$

if k is an upper bound for h'.

Hence

$$\dot{\rho} \leq A \cdot k \cdot (\alpha - \rho),$$

and

$$\frac{\dot{\rho}}{\alpha - \rho} \leq A \cdot k$$

So

$$\int_0^t \frac{\dot{\rho}}{\alpha - \rho} \leq \int_0^t A \cdot k dt$$

or

$$-\log(\alpha - \rho) + \log(\alpha - \rho(0)) \leq A \cdot k \cdot t$$

From this follows with $K = \alpha - \rho(0) > 0$

$$\frac{K}{\alpha - \rho} \leq e^{A \cdot kt}$$

or

$$\rho(t) \leq \alpha - K e^{-A \cdot kt}.$$

In case of $m > 1$ it is expected, that

$$x(t) \to \infty \quad \text{and} \quad \rho(t) \to \alpha \quad \text{for} \quad t \to T < \infty.$$

We look at (7.12) again. From (8.2) we have

$$H(\rho) \geq L - K \log(\alpha - \rho)$$

and hence

$$e^{(m-1)H(\rho)} \geq e^{(m-1)L} \cdot \left(\frac{1}{\alpha - \rho}\right)^{(m-1) \cdot K}.$$

Now, suppose that $h(\rho) \geq (\alpha - \rho)^p$ for some $p \geq 1$. Then (7.1) gives

$$\dot{\rho} = A \cdot e^{(m-1)H(\rho)} h(\rho) \geq A \cdot e^{(m-1)L} \cdot (\alpha - \rho)^{p-(m-1)K}.$$

As soon as

$$m > 1 + \frac{p-1}{K}$$

we must have $\rho(t) \to \alpha$ for $t \to T < \infty$.

Remark. The normal case is of course $p = 1$ reducing the inequality to $m > 1$.

We shall now discuss the different types of stability of the coordinate solutions that are associated with the global asymptotic ratio stability conditions.

To start with, we observe a simple fact as stated in Lemma 3.

Lemma 1. *The sets of the individual coordinate solutions* $\varphi_1(t)$ *and* $\varphi_2(t)$ *to (7.1) will – under the conditions of global asymptotic ratio stability,* $h'(\alpha) < 0$, *and with a director root* $\alpha \neq 0$ *– always have the same stability properties.*

Proof. By theorem 1, we have for all the solutions of (7.12) that $\rho(t) \to \alpha$ for $t \to \infty$. Hence, as $t \to \infty$, $\varphi_1(t)$ and $\varphi_2(t)$ evolve proportionally by (7.6), and the lemma follows.

Consider any pair of coordinate solutions to (7.1),

$$\varphi(t) = [\varphi_1(t), \varphi_2(t)] \text{ and } \psi(t) = [\psi_1(t), \psi_2(t)] \tag{8.3}$$

Theorem 3. *The family of coordinate solutions to (7.1) has – with lemma 1 and a negative or zero directrix value, (7.39), – respectively, asymptotic and weak absolute stability*

$$f(\alpha) < 0 : \varphi_i(t) = 0, \quad \psi_i(t) \to 0 \quad \text{as } t \to \bar{t}_i \leq \infty, \quad i = 1, 2 \tag{8.4}$$

$$f(\alpha) = 0 : |\psi_i(t) - \varphi_i(t)| < k < \infty, \forall t, \qquad i = 1, 2 \tag{8.5}$$

Proof. By a change of variables, $\xi = \frac{1}{x}$ and $\eta = \frac{1}{y}$, the theorem follows from the corresponding theorem with $f(\alpha) \geq 0$ and degree of homogeneity equal to $2 - m$.

The actual *value* of the degree of homogeneity will play a crucial role in various stability conditions that refer to classification of families of coordinate solutions which ultimately *increase* towards infinity.

Theorem 4. *The family of non–stationary coordinate solutions to (7.1) has – with* h *decreasing through* α *and a positive directrix value (7.39) and either* $m < 1$ *or* $m = 1$ *– respectively, strong relative and strong logarithmic stability, i.e.,*

$$f(\alpha) > 0,\ m = 1 : \log \psi_i(t) / \log \varphi_i(t) \to 1 \text{ as } t \to \infty, \quad i = 1, 2 \tag{8.6}$$

$$f(\alpha) > 0,\ m < 1 : \psi_i(t)/\varphi_i(t) \to 1 \qquad \text{as } t \to \infty, \quad i = 1, 2 \tag{8.7}$$

Proof. To prove (8.6–8.7), it suffices to examine two solutions, referred to in (8.3), a directrix solution $\varphi(t)$ of the form (7.33) or (7.34) and a non–directrix solution $\psi(t)$ of the form (7.14). Furthermore, by lemma 1, it is enough to compare their first coordinate solutions.

ad $m = 1$. From (7.33) and (7.14), we have

$$\frac{\log \psi_1(t)}{\log \varphi_1(t)} = \frac{\log(k_0 e^{H[\rho(t)]})}{\log(\hat{x}_0 e^{f(\alpha)t})} = \frac{\log |k_0| + H[\rho(t)]}{\log |\hat{x}_0| + f(\alpha)t} \tag{8.8}$$

Due to lemma 1 of ch. 7 and theorem 1, the rule of l'Hospital can be used to evaluate the limit of (8.8), i.e.,

(8.9)
$$\lim_{t\to\infty}\frac{\log\psi_1(t)}{\log\varphi_1(t)}=\lim_{t\to\infty}\frac{d\log\psi_1(t)/dt}{d\log\varphi_1(t)/dt}=\lim_{t\to\infty}\frac{H'[\rho(t)]\dot\rho(t)}{f(\alpha)}=$$
$$\lim_{t\to\infty}\frac{(f[\rho(t)]/h[\rho(t)])h[\rho(t)]}{f(\alpha)}=\lim_{t\to\infty}\frac{f[\rho(t)]}{f(\alpha)}=\frac{f(\alpha)}{f(\alpha)}=1$$

using (7.10), (7.12) and $m=1$. Thus, (8.9) establishes the *strong logarithmic* stability property of the family of coordinate solutions $\varphi(t)$, (8.6).

ad $m<1$. From (7.34) and (7.14), we have

(8.10)
$$\frac{\psi_1(t)}{\varphi_1(t)}=\frac{k_0e^{H[\rho(t)]}}{[\hat x_0^{1-m}+(1-m)f(\alpha)t]^{\frac{1}{1-m}}}=\frac{k_0e^{H[\rho(t)]}}{[(1-m)f(\alpha)(t+t_0)]^{\frac{1}{1-m}}}$$

Due to lemma 1, we may also use the rule of l'Hospital to evaluate the limit of (8.10). Thus, one gets, cf. (7.10), (7.12)

(8.11)
$$\lim_{t\to\infty}\frac{\psi_1(t)}{\varphi_1(t)}=\lim_{t\to\infty}\frac{\dot\psi_1(t)}{\dot\varphi_1(t)}=\lim_{t\to\infty}\frac{k_0e^{H[\rho(t)]}H'[\rho(t)]\dot\rho(t)}{[(1-m)f(\alpha)(t+t_0)]^{\frac{m}{1-m}}\,f(\alpha)}=$$
$$\lim_{t\to\infty}\frac{k_0e^{H[\rho(t)]}\,(f[\rho(t)]/h[\rho(t)])\,|k_0e^{H[\rho(t)]}|^{m-1}h[\rho(t)]}{[(1-m)f(\alpha)(t+t_0)]^{\frac{m}{1-m}}\,f(\alpha)}=$$
$$\lim_{t\to\infty}\left[\frac{|k_0e^{H[\rho(t)]}|^m}{[(1-m)f(\alpha)(t+t_0)]^{\frac{m}{1-m}}}\right]\left[\frac{f[\rho(t)]}{f(\alpha)}\right]=$$
$$\lim_{t\to\infty}\left[\frac{|k_0|^m}{[(1-m)f(\alpha)]^{\frac{m}{1-m}}}\right]\left[\frac{e^{(1-m)H[\rho(t)]}}{t+t_0}\right]^{\frac{m}{1-m}}\left[\frac{f[\rho(t)]}{f(\alpha)}\right]$$

In (8.11), both the numerator (cf. lemma 1 of ch. 7. and theorem 1) and the denominator of the middle fraction go to infinity for $t\to\infty$, so we apply the l'Hospital rule once more. Thus, using again (7.10), (7.12), we obtain

(8.12)
$$\lim_{t\to\infty}\frac{e^{(1-m)H[\rho(t)]}}{t+t_0}=\lim_{t\to\infty}e^{(1-m)H[\rho(t)]}[(1-m)H'[\rho(t)]\dot\rho(t)=$$
$$\lim_{t\to\infty}e^{(1-m)H[\rho(t)]}(1-m)(f[\rho(t)]/h[\rho(t)])|k_0e^{H[\rho(t)]}|^{m-1}h[\rho(t)]=$$
$$\lim_{t\to\infty}(1-m)|k_0|^{m-1}f[\rho(t)]=(1-m)|k_0|^{m-1}f(\alpha)$$

as the last results follows from the assumption, $\rho(t)\to\alpha$ for $t\to\infty$. Finally, the standard theorem on *composite* limits is applied to (8.11–8.12), i.e.,
(8.13)
$$\lim_{t\to\infty}\frac{\psi_1(t)}{\varphi_1(t)}=\left[\frac{|k_0|^m}{[(1-m)f(\alpha)]^{\frac{m}{1-m}}}\right]\left[(1-m)|k_0|^{m-1}f(\alpha)\right]^{\frac{m}{1-m}}\left[\frac{f(\alpha)}{f(\alpha)}\right]=1$$

which establishes the *strong relative* stability property of the family of coordinate solutions, (8.7).

Theorem 4 is the most general and powerful stability result bringing – in regard to the values of m – the stability properties for families of *increasing* (but non–explosive) solutions to the general homogeneous dynamic system (7.1), under a simple, all–embracing rule.

However, it is possible – for $m = 1$, $m = 0$ and $m < 0$ – to sharpen theorem 4 by adding a very mild assumptions to the conditions in (8.6.–8.7).

Theorem 5. *The family of non–stationary coordinate solutions to (7.1) has – with $h'(\alpha) < 0$, a positive directrix value, (7.39), and $m = 1$ – weak relative stability, i.e.,*

$$f(\alpha) > 0,\ m = 1,\ h'(\alpha) < 0 : \psi_i(t)/\varphi_i(t) < k\ < \infty,\ \forall t, \quad i = 1, 2 \tag{8.14}$$

Proof. From (7.33), (7.14) and lemma 1, we have

$$\frac{\psi_1(t)}{\varphi_1(t)} = \frac{k_0 e^{H[\rho(t)]}}{\hat{x}_0 e^{f(\alpha)t}} = \frac{k_0}{\hat{x}_0} e^{H[\rho(t)] - f(\alpha)t} \tag{8.15}$$

We consider the difference

$$\chi(t) = H[\rho(t)] - f(\alpha)t \tag{8.16}$$

and next, by (7.10), (7.12), $m = 1$ and the mean value theorem on f

$$\begin{aligned} |\dot{\chi}(t)| &= |H'[\rho(t)]\dot{\rho}(t) - f(\alpha)| = |f[\rho(t)] - f(\alpha)| = f'(\xi)|\rho(t) - \alpha| \\ &< B_1|\rho(t) - \alpha| \end{aligned} \tag{8.17}$$

where the constant B_1 is an *upper* bound for $|f'(r)|$, i.e., $B_1 > f'(\xi)$, $\forall \xi$.

By $h'(\alpha) < 0$ and the mean value theorem on h, there exists a constant $B_2 > 0$, being a *lower* bound for $-h'(r)$ in a *neighborhood* $N(\alpha)$ of α, such that for r, $\xi \in N(\alpha)$,

$$|h(r)| = |h'(\xi)(r - \alpha)| \geq B_2|r - \alpha| \tag{8.18}$$

Hence, by (7.12), $m = 1$, $r = \rho(t)$ and (8.18), we get

$$|\dot{\rho}(t)| = |h[\rho(t)]| \geq B_2|\rho(t) - \alpha| \tag{8.19}$$

Integrating (8.19) by separation of variables yields

$$\log|r_0 - \alpha| - \log|\rho(t) - \alpha| \geq B_2 t \ \Leftrightarrow\ |\rho(t) - \alpha| \leq |r_0 - \alpha| e^{-B_2 t} \tag{8.20}$$

Substituting (8.20) in (8.17) and integrating gives, cf. (8.16)

(8.21)

$$\begin{aligned}|\chi(t)| &\leq |\chi(0)| + \int_0^t |\dot{\chi}(t)|dt \leq H(r_0) + B_1 \int_0^t |r_0 - \alpha| e^{-B_2 t} dt \\ &= H(r_0) + B_1 |r_0 - \alpha| \left[\frac{1 - e^{-B_2 t}}{B_2} \right]_0^t \\ &\leq H(r_0) + B_1 |r_0 - \alpha| / B_2\end{aligned}$$

Finally, by (8.15–8.16) and (8.21), we obtain

(8.22) $$\frac{\psi_1(t)}{\varphi_1(t)} = \frac{k_0}{\hat{x}_0} e^{H[\rho(t)] - f(\alpha)t} \leq \frac{k_0}{\hat{x}_0} e^{H(r_0) + B_1 |r_0 - \alpha| / B_2} < \infty$$

which establishes the *weak relative* stability property (8.14).

Corollary 5. *The ratio* $\frac{\psi_1(t)}{\varphi_1(t)}$ *converges towards a limit for* $t \to \infty$.

Proof. The ratio is limited according to theorem 5. And from (8.16) follows, that it is monotonic, because

$$\dot{\chi}(t) = f(\rho(t)) - f(\alpha) \simeq \frac{1}{n!} f^{(n)}(\alpha)(\rho(t) - \alpha)^n$$

does not change sign, provided f is analytic.

Theorem 6. *The family of non–stationary coordinate solutions to (7.1) has – with* $h'(\alpha) < 0$*, a positive directrix value (7.39), and the assumption,* $g'(\alpha) - \alpha f'(\alpha) < 0$ *– for* $m = 0$ *and* $m < 0$*, – weak and strong absolute stability, i.e.,*

(8.23) $f(\alpha) > 0,\ m = 0,\ h'(\alpha) < 0 : |\psi_i(t) - \phi_i(t)| < k < \infty,\ \forall t,\ i = 1, 2$

(8.24) $f(\alpha) > 0,\ m < 0,\ h'(\alpha) < 0 : \psi_i(t) - \phi_i(t) \to 0$ *as* $t \to \infty,\ i = 1, 2$

Proof. From (7.34), (7.14) and lemma 1, we have

(8.25) $$\psi_1(t) - \varphi_1(t) = x_0 e^{H(\rho) - H(r_0)} - [\hat{x}_0^{1-m} + (1 - m) f(\alpha) t]^{\frac{1}{1-m}}$$

And from (7.10) we approximate

(8.26) $$H'(r) = \frac{f(r)}{h(r)} \approx \frac{f(\alpha)}{h'(\alpha)(r - \alpha)}$$

Integrating (8.26) yields

(8.27) $$\begin{aligned}H(\rho) - H(r_0) &\approx \int_{r_0}^{\rho} \frac{f(\alpha)}{h'(\alpha)} \cdot \frac{1}{|\rho - \alpha|} d\rho \\ &= \frac{f(\alpha)}{h'(\alpha)} [\log |\rho - \alpha| - \log |r_0 - \alpha|] \\ &= \frac{f(\alpha)}{h'(\alpha)} \log \left| \frac{\rho - \alpha}{r_0 - \alpha} \right|\end{aligned}$$

By (7.12) we have

$$\begin{aligned} \dot{\rho} &= x_0^{m-1} e^{(m-1)(H(\rho)-H(r_0))} h(\rho) \\ &\approx x_0^{m-1} \left| \frac{\rho-\alpha}{r_0-\alpha} \right|^{(m-1)\frac{f(\alpha)}{h'(\alpha)}} h'(\alpha)(\rho-\alpha) \end{aligned} \tag{8.28}$$

Since the ratio solution $\rho(t)$ have asymptotic stability, our approximation, (8.28), is useful only for $h'(\alpha) < 0$.

Integration by separation of variables yields

$$\left[\frac{|\rho-\alpha|^{(1-m)\frac{f(\alpha)}{h'(\alpha)}}}{(1-m)\frac{f(\alpha)}{h'(\alpha)}} \right]_{r_0}^{\rho} = x_0^{m-1} |r_0-\alpha|^{(1-m)\frac{f(\alpha)}{h'(\alpha)}} h'(\alpha) t \tag{8.29}$$

or, equivalently

$$\left| \frac{\rho-\alpha}{r_0-\alpha} \right|^{\frac{f(\alpha)}{h'(\alpha)}} = x_0^{-1} \left(x_0^{1-m} + (1-m) f(\alpha) t \right)^{\frac{1}{1-m}} \tag{8.30}$$

By (7.14), (8.27) and (8.30), we obtain

$$\begin{aligned} \psi_1(t) &= x_0 e^{H(\rho)-H(r_0)} \\ &\approx \left(x_0^{1-m} + (1-m) f(\alpha) t \right)^{\frac{1}{1-m}} \end{aligned} \tag{8.31}$$

ad $m = 0$. We have by (8.25), (8.31), $m = 0$ and together with $\lim_{t\to\infty} \rho(t) = \alpha$ due to $h'(\alpha) < 0$,

$$\lim_{t\to\infty} [\psi_1(t) - \varphi_1(t)] = x_0 - \hat{x}_0 \tag{8.32}$$

which establishes the *weak absolute* stability property of the coordinate solutions, (8.23).

ad $m < 0$. We have by (8.25), (8.31), $m < 0$ and together with $\lim_{t\to\infty} \rho(t) = \alpha$ due to $h'(\alpha) < 0$,

$$\lim_{t\to\infty} [\psi_1(t) - \varphi_1(t)] = 0 \tag{8.33}$$

which establishes the *strong absolute* stability property of the coordinate solutions, (8.24).

CHAPTER 9. Economical examples of differential equations

R. M. Solow [18, 19, 20] refines the model of Shinkai, [17], in the following way:

We assume the labor force, L, to grow with a rate, n, as in Chapter 12, (25)

$$L' = n\,L\,. \tag{9.1}$$

But the capital grows with a speed of

$$K' = G(L, K) \tag{9.2}$$

where G is some homogeneous function of L and K of degree 1, but not necessarily linear.

One assumption due to Solow, is that G takes the form

$$G(L, K) = s\,(L^p + aK^p)^{\frac{1}{p}} \tag{9.3}$$

with $0 < a$, $0 < p < 1$, and $0 < s < 1$.

Another assumption due to C. W. Cobb and P. H. Douglas, is that G takes the form

$$G(L, K) = sK^\alpha L^{1-\alpha} \tag{9.4}$$

for some $0 < s < 1$ and $0 < \alpha < 1$.

Then we define our functions as in Chapter 7, (7.5), (7.6) and (7.8):

$$k = \frac{K}{L} \tag{9.5}$$

$$f(k) = n \tag{9.6}$$

$$g(k) = G(1, k) \tag{9.7}$$

$$h(k) = g(k) - nk\,. \tag{9.8}$$

In the examples we get for Solow

$$g(k) = s(1 + ak^p)^{\frac{1}{p}} \tag{9.9}$$

$$h(k) = s(1 + ak^p)^{\frac{1}{p}} - nk \tag{9.10}$$

and for Cobb–Douglas

$$g(k) = sk^\alpha \tag{9.11}$$

$$h(k) = sk^\alpha - nk\,. \tag{9.12}$$

The first question is, whether h has a root? Hence we shall solve for Solow

$$s(1 + ak^p)^{\frac{1}{p}} = nk\,. \tag{9.13}$$

This equation has a solution, if

$$a < \left(\frac{n}{s}\right)^p \tag{9.14}$$

and then the solution is

$$k_0 = \frac{1}{\left(\left(\frac{n}{s}\right)^p - a\right)^{\frac{1}{p}}} . \tag{9.15}$$

For Cobb–Douglas the solution is

$$k_0 = \left(\frac{s}{n}\right)^{\frac{1}{1-\alpha}} . \tag{9.16}$$

It is obvious that h is decreasing through k_0, because from (9.10) we have $h(0) = s > 0$, and from (9.12) we have $h(k) = k^\alpha(s - nk^{1-\alpha}) > 0$ for $0 < k < k_0$.

In both cases we get

$$f(k_0) = n > 0 , \tag{9.17}$$

and further in case of Solow

$$h'(k) = \frac{1}{p}(1 + ak^p)^{\frac{1}{p}-1} \cdot apk^{p-1} - n , \tag{9.18}$$

and using (9.15) and (9.14) we find

$$h'(k_0) = as^p n^{1-p} - n < 0 , \tag{9.19}$$

while in the case of Cobb–Douglas

$$h'(k) = s\alpha k^{\alpha-1} - n , \tag{9.20}$$

and using (9.16) we find

$$h'(k_0) = s\alpha\left(\frac{s}{n}\right)^{\frac{1}{1-\alpha}\cdot(\alpha-1)} - n = n(\alpha - 1) < 0 . \tag{9.21}$$

Hence we may apply theorem 5 to conclude, that the solutions satisfy the weak relative stability, (8.14) of Chapter 8.

Even the corollary 5 of Chapter 8 applies, telling that the ratio between two different solutions for L or K will converge towards a finite non–zero constant.

CHAPTER 10. The calculus of finite sums and differences

In analogy to the operators of differentiation and integration, we shall introduce the operators of differences and sums. We shall see how far the analogy shall reach with respect to theorems. The results of this chapter are mainly stolen from [5], but the terminology is different.

Definition 1. The difference operator, Δ, is defined by the equation

$$\Delta f(x) = f(x+1) - f(x) \tag{10.1}$$

This means that a new function, Δf, is defined in analogy to the derivative, $Df = f'$, of the function, f.

This operator may be iterated, e.g.

$$\Delta^2 f(x) = \Delta(f(x+1) - f(x)) = f(x+2) - 2f(x+1) + f(x)\,. \tag{10.2}$$

The difference operator is not in harmony with the powers as the derivative is. E.g.

$$\Delta x^3 = (x+1)^3 - x^3 = 3x^2 + 3x + 1\,. \tag{10.3}$$

Rather than using the powers or monomials as the bases for the space of polynomials, we shall introduce the (descending) factorials, $[x]_n = x(x-1)\cdots(x-n+1)$, with n factors. It is convenient to define them for negative length as well.

Definition 2. The descending factorial of length $n \in \mathbb{Z}$ is

$$[x]_n = \begin{cases} \prod_{j=0}^{n-1}(x-j) & n \in \mathbb{N} \\ 1 & n = 0 \\ \prod_{j=1}^{-n}\dfrac{1}{x+j} & -n \in \mathbb{N} \end{cases} . \tag{10.4}$$

The factorials $[x]_n$ for $n \in \mathbb{N}_0$ constitute a base for the polynomials.

There are a series of rules of calculation for the factorials. The most important are

$$[x]_n = [-x+(n-1)]_n(-1)^n \tag{10.5}$$
$$[x]_n = [x]_m[x-m]_{n-m} \tag{10.6}$$
$$[x]_n = 1/[x-n]_{-n} \tag{10.7}$$
$$[x]_n = [x-1]_n + n[x-1]_{n-1} \tag{10.8}$$

Together with the difference operator it obeys

$$\Delta[x]_n = n[x]_{n-1}\,. \tag{10.9}$$

To show (10.9), use (10.8) on $[x+1]_n$ and (10.1).

Rather than defining a sum by addition we shall define an indefinite sum or anti–difference in analogy to the indefinite integral or anti–derivative in ordinary calculus.

Definition 3.

$$g(x) = \Delta f(x) \Leftrightarrow f(x) = \sum g(x)\delta x\,. \tag{10.10}$$

REMARK. If g is given, $f(x)$ is only uniquely determined up to a periodic function $C(x)$ with period 1. But if $g(x)$ is a polynomial, then $f(x)$ is uniquely determined as a polynomial up to a constant, because the constant is the only periodic polynomial.

The definite sum shall be

Definition 4. For $b - a \in \mathbb{N}_0$ we define

$$\begin{aligned} &\sum\nolimits_a^b g(x)\delta x = f(b) - f(a), \text{ where} \\ &f(x) = \sum g(x)\delta x\,. \end{aligned} \tag{10.11}$$

REMARK. This definition is unique, because it is independent of the periodic function $C(x)$.

Now, this sum may be computed by some additions in analogy to the determination of the definite integral as some area.

Theorem 1. *The definite sum is*

$$\sum\nolimits_a^b g(x)\delta x = \sum_{k=a}^{b-1} g(k)\,. \tag{10.12}$$

Proof. Induction after $b-a$. For $b-a=0$ we have

$$\sum\nolimits_a^a g(x)\delta x = f(a) - f(a) = 0 = \sum_{k=a}^{a-1} g(k)\,.$$

For $b-a=1$ we have

$$\sum\nolimits_a^{a+1} g(x)\delta x = f(a+1) - f(a) = \Delta f(a) = g(a) = \sum_{k=a}^{a} g(k)\,.$$

The general step is

$$\sum\nolimits_a^{b+1} g(x)\delta x = \sum\nolimits_a^{b} g(x)\delta x + \sum\nolimits_b^{b+1} g(x)\delta x =$$
$$\sum_{k=a}^{b-1} g(k) + g(b) = \sum_{k=a}^{b} g(k)\,.$$

EXAMPLE: Use (10.10) on (10.9):

$$[x]_n = \sum n[x]_{n-1}\delta x\,.$$

Then (10.12) and (10.11) yield

$$[m]_n - [0]_n = \sum\nolimits_0^m n[x]_{n-1}\delta x = n\sum_{k=0}^{m-1}[k]_{n-1}\,.$$

Or, rather

$$\sum_{k=0}^{m}[k]_n = \frac{[m+1]_{n+1}}{n+1}\,. \tag{10.13}$$

For $n = 1$ we have

$$\sum_{k=0}^{m} k = \frac{[m+1]_2}{2} = \frac{m(m+1)}{2}$$

and as we may write $k^2 = [k]_2 + [k]_1$, we deduce

$$\sum_{k=0}^{m} k^2 = \sum_{k=0}^{m}[k]_2 + \sum_{k=0}^{m}[k]_1 =$$
$$\frac{[m+1]_3}{3} + \frac{[m+1]_2}{2} =$$
$$\frac{2(m-1)[m+1]_2 + 3[m+1]_2}{6} =$$
$$\frac{m(m+1)(2m+1)}{6}\,.$$

ANOTHER EXAMPLE: Let $x \in \mathbb{Z}$, then we define

$$f(x) = \frac{[a]_x}{[b]_{x-1}}\,.$$

Now

$$\Delta f(x) = (a-b-1)\frac{[a]_x}{[b]_x}\,.$$

So that

$$\sum_{k=m}^{n} \frac{[a]_k}{[b]_k} = \sum\nolimits_m^{n+1} \frac{[a]_x}{[b]_x}\delta x = \frac{1}{a-b-1}\left(\frac{[a]_{n+1}}{[b]_n} - \frac{[a]_m}{[b]_{m-1}}\right)\,. \tag{10.14}$$

In order to establish the analogy to the integration by parts we shall need the trivial operation of a shift.

Definition 5. The shift operator, E, is defined by

$$E\,f(x) = f(x+1) = \Delta f(x) + f(x)\,. \tag{10.15}$$

This allows a simple formulation of the difference of a product.

Theorem 2. *For any functions $u(x)$ and $v(x)$ we have*

$$\Delta(uv) = u\,\Delta v + Ev\,\Delta u \tag{10.16}$$

or, symmetric in u and v

$$\Delta(uv) = u\,\Delta v + v\,\Delta u + \Delta u\,\Delta v\,. \tag{10.17}$$

Proof. Obvious.

Now we are able to state the Abelian summation formulas:

Theorem 3. *For any functions u and v we have*

$$\sum u(x)\Delta v(x)\delta x = u(x)v(x) - \sum Ev(x)\Delta u(x)\delta x \tag{10.18}$$

and

$$\sum\nolimits_a^b u(x)\Delta v(x)\delta x = u(b)v(b) - u(a)v(a) - \sum\nolimits_a^b v(x+1)\Delta u(x)\delta x\,. \tag{10.19}$$

Proof. Follows immediately from (10.16).

EXAMPLE: $\Delta 2^x = 2^{x+1} - 2^x = 2^x(2-1) = 2^x$. Hence

$$\begin{aligned}\sum x2^x\delta x &= x2^x - \sum 2^{x+1}\Delta x\delta x\\ &= x2^x - \sum 2^{x+1}\delta x\\ &= x2^x - 2^{x+1}\end{aligned}$$

such that – using (10.12) and (10.11) –

$$\sum_{k=0}^{n} k2^k = (n+1)2^{n+1} - 2^{n+2} + 2 = (n-1)2^{n+1} + 2$$

or

$$\sum_{k=2}^{n} k2^k = (n-1)2^{n+1}\,.$$

In the formula (10.13) we have assumed $n \neq -1$. In analogy to the natural logarithms we introduce harmonic numbers as partial sums of the harmonic series.

Definition 6. The harmonic number H_x is for $x \in \mathbb{N}$

$$H_x = \sum_{n=1}^{x} \frac{1}{n}\,. \tag{10.20}$$

Then we have $\Delta H_x = \frac{1}{x+1} = [x]_{-1}$, such that

$$\sum\nolimits_a^b [x]_{-1}\delta x = H_b - H_a\,. \tag{10.21}$$

EXAMPLE: From (10.9) with $n = 2$ we get

$$\begin{aligned}\sum xH_x\delta x &= \frac{[x]_2}{2}H_x - \sum \frac{[x+1]_2[x]_{-1}}{2}\delta x \\ &= \frac{1}{2}[x]_2 H_x - \frac{1}{2}\sum [x]_1\delta x\end{aligned}$$

by using (10.6). Furthermore,

$$\sum [x]_1\delta x = \frac{[x]_2}{2}$$

so we obtain

$$\sum xH_x\delta x = \frac{1}{4}[x]_2(2H_x - 1)\,.$$

The most important rule from differential calculus, the rule of substitution, does not carry over. Unless $f(x)$ or $g(x)$ are linear, the function

$$\Delta f(g(x)) = f(g(x+1)) - f(g(x)) \tag{10.22}$$

is not related to neither Δf nor Δg. Hence non-linear difference equations are in general much harder to solve than the corresponding differential equations.

The simplest difference equations are solved similarly to the corresponding differential equations. We may introduce an exponential function, $f(x) = c^x$ for $c \neq 1$, and remark that

$$\Delta f(x) = c^{x+1} - c^x = (c-1)c^x\,. \tag{10.23}$$

Hence, the difference equation for $a \neq 0$

$$\Delta f(x) = a \cdot f(x) \tag{10.24}$$

has the solutions with C arbitrary

$$f(x) = C \cdot (1+a)^x\,. \tag{10.25}$$

CHAPTER 11. Linear difference equations

The treatment of this subject is mainly taken from [5].

A linear difference equation with unknown solution $f(x)$ and known data $g(x)$ is often given in the form

$$f(x+1) - af(x) = g(x) \tag{11.1}$$

rather than in the difference operator form

$$\Delta f(x) - (a-1)f(x) = g(x)\,. \tag{11.2}$$

We have seen that the corresponding homogeneous equations

$$f(x+1) - af(x) = 0 \tag{11.3}$$

has the general solution

$$f(x) = k \cdot a^x\,. \tag{11.4}$$

To solve (11.1) or (11.2) we apply the same method, to guess a function of the form

$$f(x) = \varphi(x)a^x \tag{11.5}$$

and then look for restrictions on the function $\varphi(x)$. Substitution of (11.5) in (11.2) gives the equation

$$\Delta(\varphi a^x) - (a-1)\varphi a^x = g(x)\,.$$

Now we apply straightforward computation to get

$$\begin{gathered}(\Delta a^x)\varphi + (Ea^x)\Delta\varphi - (\Delta a^x)\varphi = g(x) \\ \Delta\varphi = a^{-1-x}g(x),\end{gathered} \tag{11.6}$$

from which by (11.10) we get $\varphi(x)$

$$\varphi(x) = \sum a^{-x-1}g(x)\delta x \tag{11.7}$$

and finally

$$f(x) = a^x\varphi(x) = a^x \sum a^{-x-1}g(x)\delta x\,. \tag{11.8}$$

Example 1. The equation

$$f(x+1) - 2f(x) = 2^x$$

has $a = 2$ and $g(x) = 2^x$, so

$$f(x) = 2^x \sum 2^{-x-1} 2^x \delta x = 2^x \sum \frac{1}{2} \delta x = \\ = 2^{x-1}(x+k)\,.$$

Example 2. The equation

$$f(x+1) - 2f(x) = 2^x x^3$$

has $a = 2$ and $g(x) = 2^x x^3$, so by the definition of the Bernoulli polynomials, (18.11), and their table, (18.20), we get

$$\begin{aligned} f(x) &= 2^x \sum 2^{-x-1} 2^x x^3 \delta x = 2^x \frac{1}{2} \sum x^3 \delta x = \\ &= 2^{x-1} \frac{B_4(x) - B_4}{4} = 2^{x-3}\left(B_4(x) - B_4\right) = \\ &= 2^{x-3}\left(x^4 - 2x^3 + x^2\right). \end{aligned}$$

Suppose a first order difference operator, A_i, has the form

$$A_i f(x) = f(x+1) - \alpha_i f(x)\,. \tag{11.9}$$

If we iterate two operators of the form (11.9), we get

$$A_i A_j f(x) = f(x+2) - (\alpha_i + \alpha_j) f(x+1) + \alpha_i \alpha_j f(x)\,. \tag{11.10}$$

Given a second order operator, B

$$B f(x) = f(x+2) - a f(x+1) + b f(x) \tag{11.11}$$

we may find B as a composite

$$B = A_1 A_2 \tag{11.12}$$

with A_i defined by (11.9) and α_1, α_2 roots in the polynomial

$$\xi^2 - a\xi + b\,. \tag{11.13}$$

Theorem 1. *A linear operator, B, of the form*

$$B f(x) = f(x+n) + a_{n-1} f(x+n-1) + \cdots + a_0 f(x) \tag{11.14}$$

can be written as the composition of n operators of first order,

$$B = A_1 A_2 \ldots A_n \tag{11.15}$$

with A_i defined by (11.9) and $\alpha_1, \ldots, \alpha_n$ the roots of the polynomial

$$p(\xi) = \xi^n + a_{n-1}\xi^{n-1} + \cdots + a_0\,. \tag{11.16}$$

Proof. The fundamental theorem of algebra.

Definition 1. The polynomial (11.16) is called the characteristic polynomial of the operator (11.14).

Application. An equation in f for given g

$$Bf(x) = g(x) \tag{11.17}$$

is solved by repeated application of (11.8), i.e.

$$f(x) = \alpha_1^x \sum \alpha_1^{-x-1} \alpha_2^x \sum \alpha_2^{-x-1} \cdots \sum \alpha_n^{-x-1} g(x) \delta x \,. \tag{11.18}$$

Example. Consider the pattern in Figure 1.

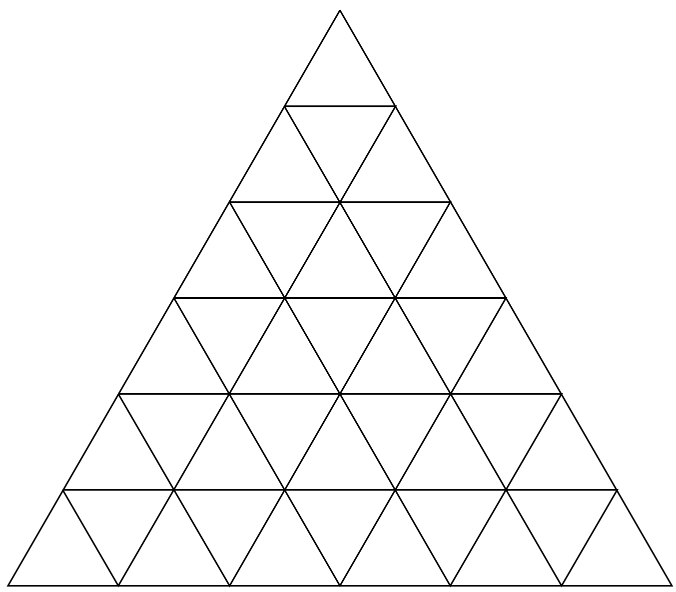


Figure 1

This problem is taken from [13]. We may ask how many triangles there are in this figure? To simplify matters we shall restrict ourselves to triangles, which are right-way-up, and to compensate for this simplification we shall ask for the formula for $f(n)$, the number of triangles in a triangle of size n. Then we have $f(1) = 1$, $f(2) = 4$, etc.

First solution. We ask for a function, $g(n)$, such that

$$\Delta f(n) = g(n) \,.$$

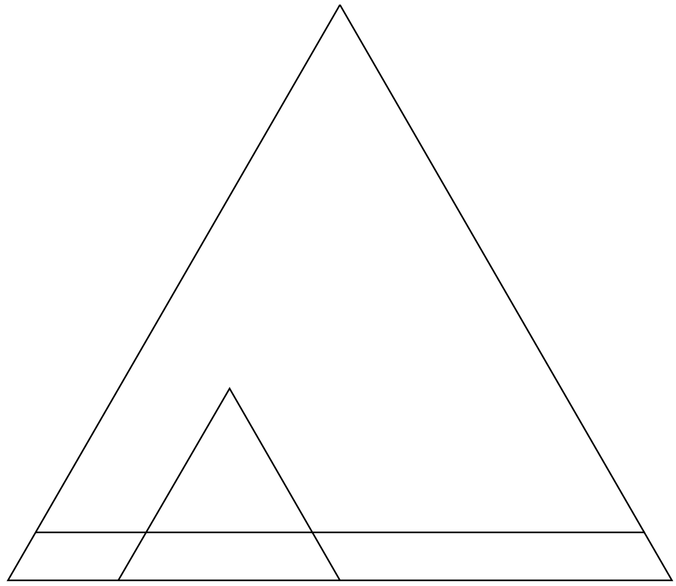


Figure 2

We take a look at figure 2, and decide that $\Delta f(n)$ counts exactly those triangles which have their basis on the basis of the big triangle. And their number is the number of points in the big triangle of size n, i.e.

$$g(n) = \sum_{k=1}^{n+1} k = \binom{n+2}{2}.$$

Hence $f(n) = \sum g(n)\delta n = \sum \frac{[n+2]_2}{2}\delta n = \frac{[n+2]_3}{6} + c$. Since $f(1) = 1$, we find $c = 0$, such that

$$f(n) = \frac{[n+2]_3}{6} = \binom{n+2}{3}.$$

Second solution. We ask for an operator, B, such that for some constant, k,

$$Bf(n) = k\,.$$

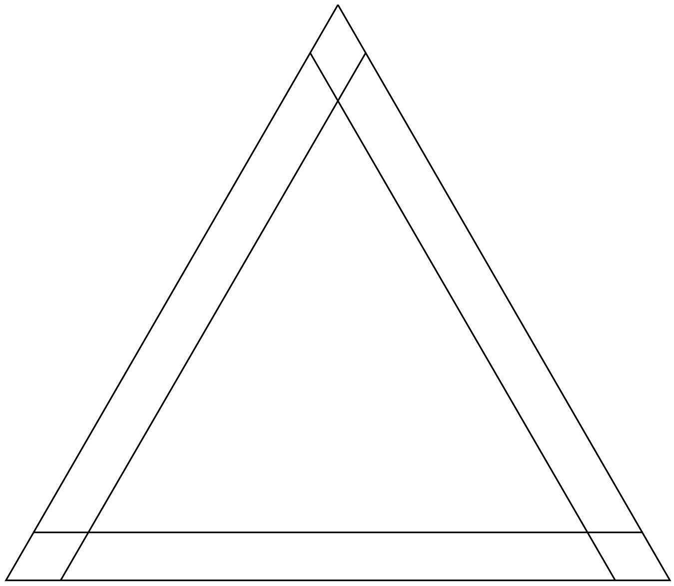


Figure 3

We take a look at figure 3. The triangle, T_n, of size n, contains three triangles, T^j_{n-1}, of size $n-1$. The intersections of two of these is a triangle of size $n-2$, $T^{ij}_{n-2} = T^i_{n-1} \cap T^j_{n-1}$. There are three such intersections, and the intersection of all six triangles is a triangle of size $n-3$, T_{n-3}. Their relations are illustrated in diagram 1.

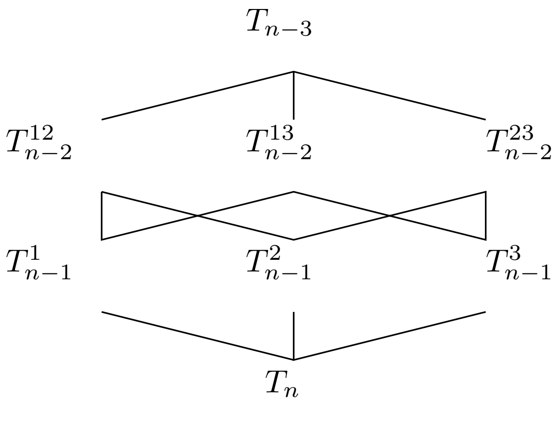


Diagram 1.

The function $f(n)$ must satisfy the difference equation

$$f(n) = 1 + 3f(n-1) - 3f(n-2) + f(n-3)\,. \tag{11.19}$$

The interpretation is as follows: We count the big triangle T_n only in $f(n)$. The rest of the triangles counted in $f(n)$ will appear in at least one of the triangles

T^j_{n-1}. A triangle in $T^i_{n-1} \cap T^j_{n-1}$, is counted at least twice. Hence we subtract the number of triangles in T^{ij}_{n-2}. Now, in the counting $3f(n-1) - 3f(n-2)$ we have counted every triangle in T_{n-3}, $3 - 3 = 0$ times. Therefore we must add $f(n-3)$ to get the result correct.

So, with the operator

$$Bf(x) = f(x+3) - 3f(x+2) + 3f(x+1) - f(x) \tag{11.20}$$

we must solve the difference equation equivalent to (11.19)

$$Bf(x) = 1\,. \tag{11.21}$$

The characteristic polynomial for B is

$$p(\xi) = \xi^3 - 3\xi^2 + 3\xi - 1 = (\xi - 1)^3 \tag{11.22}$$

hence, the operator may be written as

$$B = \Delta^3 \tag{11.23}$$

and the difference equation (11.21) is

$$\Delta^3 f(x) = 1\,. \tag{11.24}$$

The solution, (11.18), is

$$\begin{aligned} f(x) &= \sum\sum\sum \delta x = \sum\sum (x + k_1)\delta x \\ &= \sum\big(\frac{[x]_2}{2} + k_1 x + k_2\big)\delta x \\ &= \frac{[x]_3}{6} + k_1 \frac{[x]_2}{2} + k_2 x + k_3\,. \end{aligned} \tag{11.25}$$

From $f(0) = 0$, $f(1) = 1$ and $f(2) = 4$ we get $k_3 = 0$, $k_2 = 1$ and $k_1 = 2$. Consequently, the solution is

$$f(x) = \frac{[x]_3}{6} + [x]_2 + x = \frac{[x+2]_3}{6} = \binom{x+2}{3}\,. \tag{11.26}$$

Theorem 2. *The homogeneous difference equation*

$$f(x+n) + a_{n-1}f(x+n-1) + \cdots + a_0 f(x) = 0 \tag{11.27}$$

with characteristic polynomial

$$p(\xi) = \xi^n + a_{n-1}\xi^{n-1} + \cdots + a_0 \tag{11.28}$$

and eigenvalues $\alpha_1, \ldots, \alpha_m$ *such that*

$$p(\xi) = (\xi - \alpha_1)^{\nu_1} \ldots (\xi - \alpha_m)^{\nu_m} \tag{11.29}$$

has the complete solution

$$f(x) = \sum_{j=1}^{m} p_j(x)\alpha_j^x \tag{11.30}$$

where $p_j(x)$ *is any polynomial of degree* $\nu_j - 1$ *and* c *an arbitrary constant.*

Proof by induction after n. For $n = 1$ (11.30) follows from (11.4).

Suppose the theorem is true, and that we shall prove it for $(\xi - \alpha)p(\xi)$. This means that we shall solve the difference equation

$$ABf(x) = 0$$

where $Af(x) = f(x+1) - \alpha f(x)$ and B is the operator in (11.27). Now, $AB = BA$, so we must solve

$$B(Af(x)) = 0$$

or

$$Af(x) = \sum_{j=1}^{m} p_j(x)\alpha_j^x\,.$$

Because $f(x)$ is uniquely determined up to a solution to the homogeneous equation, $Af(x) = 0$, i.e. a function of the form $k\alpha^x$, it is enough to solve each of the equations

$$Af(x) = [x]_k\alpha_j^x$$

for $k = 0, 1, \ldots, \nu_j - 1$.

We treat the two cases: 1) $\alpha = \alpha_j$ 2) $\alpha \neq \alpha_j$.

1). By (11.8) we have

$$f(x) = \alpha^x \sum \alpha^{-x-1}[x]_k\alpha^x\delta x = \alpha^{x-1}\sum [x]_k\delta x = \alpha^{x-1}\frac{[x]_{k+1}}{k+1}$$

which is af function of the right form.

2). We use induction after k. Try with $f(x) = [x]_k\alpha_j^x$. Then by (1.8) we get

$$\begin{aligned} Af(x) &= [x+1]_k\alpha_j^{x+1} - \alpha[x]_k\alpha_j^x = \\ &= ([x]_k + k[x]_{k-1})\alpha_j\alpha_j^x - \alpha[x]_k\alpha_j^x = \\ &= (\alpha_j - \alpha)[x]_k\alpha_j^x + k\alpha_j[x]_{k-1}\alpha_j^x \end{aligned}$$

so that

$$\frac{1}{\alpha_j - \alpha}[x]_k\alpha_j^x$$

reduces the problem to $k - 1$, and solves the problem for $k = 0$.

Theorem 2 allows the solution of (11.17) by guesssing. It is sometimes possible to guess just one solution to (11.17). Then the complete solution is obtained by adding the set of solutions to (11.27).

CHAPTER 12. Systems of linear difference equations

A system of n first order difference equations in the n unknown functions $f_1, \dots, f_n$, consists of n equations

$$\begin{aligned} f_1(x+1) &= a_{11} f_1(x) + \cdots + a_{1n} f_n(x) + g_1(x) \\ &\vdots \\ f_n(x+1) &= a_{n1} f_1(x) + \cdots + a_{nn} f_n(x) + g_n(x) \end{aligned} \tag{12.1}$$

where the functions $g_1, \dots, g_n$ are given. Of course, this equation is equivalent to the vector equation

$$f(x+1) = Af(x) + g(x) \tag{12.2}$$

with A an $n \times n$-matrix and f, g vector valued functions of x.

A SIMPLE EXAMPLE. Let f satisfy an n'th order equation

$$f(x+n) = \alpha_1 f(x+n-1) + \cdots + \alpha_n f(x)\,. \tag{12.3}$$

If we define

$$f_i(x) = f(x+i-1) \qquad i = 1, \dots, n \tag{12.4}$$

then the vector function $(f_1, \dots, f_n)$ satisfies the system

$$\begin{aligned} f_1(x+1) &= f_2(x) \\ &\vdots \\ f_{n-1}(x+1) &= f_n(x) \\ f_n(x+1) &= \alpha_n f_1(x) + \cdots + \alpha_1 f_n(x)\,, \end{aligned} \tag{12.5}$$

or, in matrix form

$$f(x+1) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ \alpha_n & \alpha_{n-1} & & & \alpha_1 \end{pmatrix} f(x)\,. \tag{12.6}$$

The solution to (12.2) takes two steps. For $g(x) = 0$, the homogeneous equation

$$f(x+1) = Af(x) \tag{12.7}$$

has the general solution

$$f(x) = A^x \cdot k \tag{12.8}$$

with k an arbitrary constant vector. To solve (12.2) we try with an arbitrary vector valued function $\varphi(x)$

$$f(x) = A^x \varphi(x) \tag{12.9}$$

which must satisfy (12.2), i.e.

$$A^{x+1}\varphi(x+1) - AA^x\varphi(x) = g(x)$$

or, if A is regular,

$$\varphi(x+1) - \varphi(x) = A^{-x-1} g(x)$$

with solution

$$\varphi(x) = \sum A^{-x-1} g(x)\delta x$$

to be substituted in (12.9) to give

$$f(x) = A^x \sum A^{-x-1} g(x)\delta x\,. \tag{12.10}$$

We might enjoy other methods to find the inconvenient solution, A^x, to (12.7). In fact, we may apply the transfer to a higher order system. In order to do so we need to remember a small, but beautifyl piece of linear algebra.

Let A be any $n \times n$–matrix, $A = (a_{ij})$. Then the determinant may be computed by the development after a column or row, e.g.,

$$\det A = \sum_{j=1}^{n} a_{ij}(-1)^{i+j} \det A^{(i,j)}$$

where $A^{(i,j)}$ is the (i,j)–th complement, i.e., the matrix obtained by omitting the i–th row and the j–th column from the matrix, A. This means, that if we define a matrix, $B = (b_{jk})$, as

$$b_{jk} = (-1)^{k+j} \det A^{(k,j)}$$

then we have traced the inverse of the matrix, A, provided it exists. At least we may write

$$\sum_{j=1}^{n} a_{ij} b_{jk} = \delta_{ik} \det A$$

or, in matrix form,

$$AB = (\det A)E \tag{12.11}$$

We have derived the values in the diagonal above, but the zeros are coming from the fact, that if $i \neq k$ then it corresponds to the replacement of the k–th row with the i–th row, in which case the matrix becomes singular and the determinant zero.

(In the case of a regular matrix A, we have derived the inverse by dividing by $\det A$.)

In the formula (12.11) we shall enjoy that the elements of the matrix, B, are obtained as products of the elements from A.

Now we are able to state and prove

The Cayley–Hamilton theorem. *If*

$$p(\xi) = \det(\xi E - A) = \xi^n + a_{n-1}\xi^{n-1} + \cdots + a_0$$

is the characteristic polynomial for the matrix A, then the matrix $p(A)$ satisfies

$$p(A) = A^n + a_{n-1}A^{n-1} + \cdots + a_0 E = 0$$

Proof. We shall apply (12.11) to the matrix $\lambda E - A$ to get

$$p(\lambda)E = (\lambda E - A)\, B(\lambda) \tag{12.12}$$

where $B(\lambda) = (b_{ij}(\lambda))$ is a matrix of polynomials in λ, defined as the (j,i)–th complement of the matrix $\lambda E - A$. Hence we may write

$$B(\lambda) = \lambda^{n-1}B_{n-1} + \cdots + \lambda B_1 + B_0$$

as a polynomial in λ with coefficients which are matrices independent of λ.

For any $k \geq 1$ we may write

$$A^k - \lambda^k E = (A - \lambda E)\left(A^{k-1} + \lambda A^{k-2} + \cdots + \lambda^{k-1}E\right)$$

Hence we can write

(12.13)
$$\begin{aligned} p(A) - p(\lambda)E &= A^n - \lambda^n E + a_{n-1}\left(A^{n-1} - \lambda^{n-1}E\right) + \cdots + a_1\left(A^1 - \lambda^1 E\right) \\ &= (A - \lambda E)\, C(\lambda) = (A - \lambda E)\left(\lambda^{n-1}E + \lambda^{n-2}C_{n-2} + \cdots + C_0\right) \end{aligned}$$

where $C(\lambda)$ is a polynomial in λ with coefficients which are matrices independent of λ.

Adding (12.12) and (12.13) we get

$$\begin{aligned} p(A) &= (A - \lambda E)\,(C(\lambda) - B(\lambda)) \\ &= (A - \lambda E)\left(\lambda^{n-1}(E - B_{n-1}) + \cdots + \lambda(C_1 - B_1) + C_0 - B_0\right) \\ &= \lambda^{k+1}(C_k - B_k) + \lambda^k (C \cdots \end{aligned}$$

where k is the degree of the second factor to the right.

But this polynomial in λ can only be constant, i.e., independent of λ, if the second factor is zero. But in that case it is all zero, or $p(A) = 0$.

Theorem 1. *Let $p_A(\xi)$ be the characteristic polynomial for A,*

$$p_A(\xi) = \xi^n + \alpha_1 \xi^{n-1} + \cdots + \alpha_n$$

Then each component $f_1(x), \dots, f_n(x)$ of the solution $f(x)$ to (12.7) satisfies

$$f_i(x+n) + \alpha_1 f_i(x+n-1) + \cdots + \alpha_n f_i(x) = 0\,. \tag{12.14}$$

Proof. The Cayley–Hamilton theorem states that

$$p_A(A) = A^n + \alpha_1 A^{n-1} + \cdots + \alpha_n E = 0\,.$$

Hence

$$A^n f(x) + \alpha_1 A^{n-1} f(x) + \cdots + \alpha_n f(x) = 0\,,$$

or

$$f(x+n) + \alpha_1 f(x+n-1) + \cdots + \alpha_n f(x) = 0$$

by (12.7). This gives (12.14).

The conclusion is that in principle systems and higher order equations are essentially the same.

If we solve (12.14) and find solutions $h_i(x)$, then this function does not immediately give a solution (12.10). But it may be convenient, if we are able to guess a solution to (12.2), because all other solutions must differ from this with a solution to (12.7).

CHAPTER 13. Stability for linear difference equations

One of the main questions is how will different solutions to (5.1) differ in the long run?

As the difference between two solutions to (5.1) is a solution to (5.7), we are concerned with the general problem, will solutions to (5.7) vanish, stay bounded or diverge unbounded for $x \to \infty$?

Theorem 1 of Ch. 12 tells us that we shall apply the roots of the characteristic polynomial of A, also called the eigenvalues of A. The eigenvalues are the clues to the stability.

Theorem 1. *The solutions will all vanish if and only if each eigenvalue, α, satisfies $|\alpha| < 1$. If one eigenvalue, α, satisfies $|\alpha| > 1$, there is a solution, which diverges unbounded.*

Proof. For each eigenvalue, α, α^x is a solution to (12.7), so for $|\alpha| > 1$ it diverges.

The general solution (10.30) is

$$f(x) = \sum_j p_j(x)\alpha_j^x$$

these functions approach zero for $x \to \infty$, if and only if $|\alpha_i| < 1$.

Remark. If the eigenvalues α with $|\alpha| = 1$ all have multiplicity 1 and all eigenvalues satisfy $|\alpha| \leq 1$, then the solutions remain bounded, but if there is one eigenvalue, α, with $|\alpha| = 1$ and multiplicity at least 2, then there is a solution diverging towards infinity.

Example 1. For which coefficients $a, b \in \mathbb{R}$ will the solutions to the second order equation

$$f(x+2) + af(x+1) + bf(x) = 0 \tag{13.1}$$

approach zero for $x \to \infty$?

The characteristic equation is

$$\xi^2 + a\xi + b = 0\,. \tag{13.2}$$

Now, if the discriminant

$$\Delta = a^2 - 4b \tag{13.3}$$

is negative, the roots are λ, $\overline{\lambda}$, such that for each root, $|\lambda|^2 = \lambda\,\overline{\lambda} = b$. Hence, it is necessary and sufficient, that $|b| < 1$ or $b < 1$. Otherwise, the roots are

$$\left.\begin{matrix}\alpha\\ \beta\end{matrix}\right\} = \frac{-a \pm \sqrt{\Delta}}{2}\,.$$

The conditions are then

$$-1 \leq -\frac{a}{2} \pm \frac{1}{2}\sqrt{\Delta} \leq 1$$

or

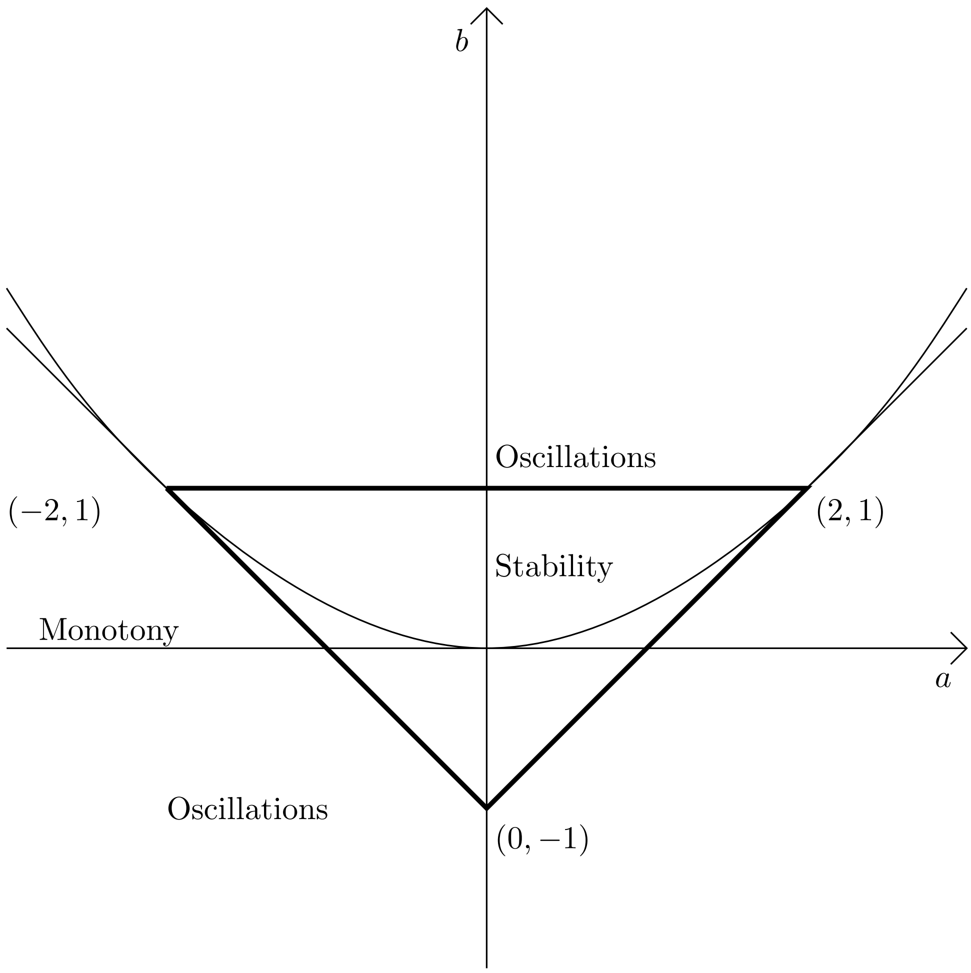
$$a - 2 \leq \pm\sqrt{\Delta} \leq a + 2$$

or

$$a^2 - 4b \leq a^2 \pm 4a + 4$$

or

$$|a| \leq 1 + b$$



The triangle of stability

The stable area is a triangle in the (a, b)–plane, characterized by the inequalities

$$|a| - 1 \leq b \leq 1\,. \qquad (13.5)$$

The discriminant (13.3) changes sign on a parabola, which means that the solutions are oscillating above this parabola. Below it they are either monotonic or alternating, depending on whether both of the roots are positive or not. If both roots are positive, so are their sum, $-a$, and their product, b. Hence the solutions behave monotonic, iff

$$a \leq 0 \leq b \leq \left(\frac{a}{2}\right)^2 \tag{13.6}$$

EXAMPLE 2. For which coefficients $a, b, c, d \in \mathbb{R}$ will the solutions to a system of two equations,

$$\begin{aligned} f_1(x+1) &= af_1(x) + bf_2(x) \\ f_2(x+1) &= cf_1(x) + df_2(x) \end{aligned} \tag{13.7}$$

be stable?

The characteristic equation is

$$\xi^2 - 2\Theta\xi + D = 0 \tag{13.8}$$

where Θ and D are respectively the half trace,

$$\Theta = \frac{a+d}{2} \tag{13.9}$$

and the determinant,

$$D = ad - bc\,. \tag{13.10}$$

In order to recognize the possible complex solutions to (13.8), we shall define the discriminant to observe its sign,

$$\Delta = \Theta^2 - D \tag{13.11}$$

So, the stability of the system requires from (13.5)

$$|2\Theta| - 1 \leq D \leq 1 \tag{13.12}$$

or

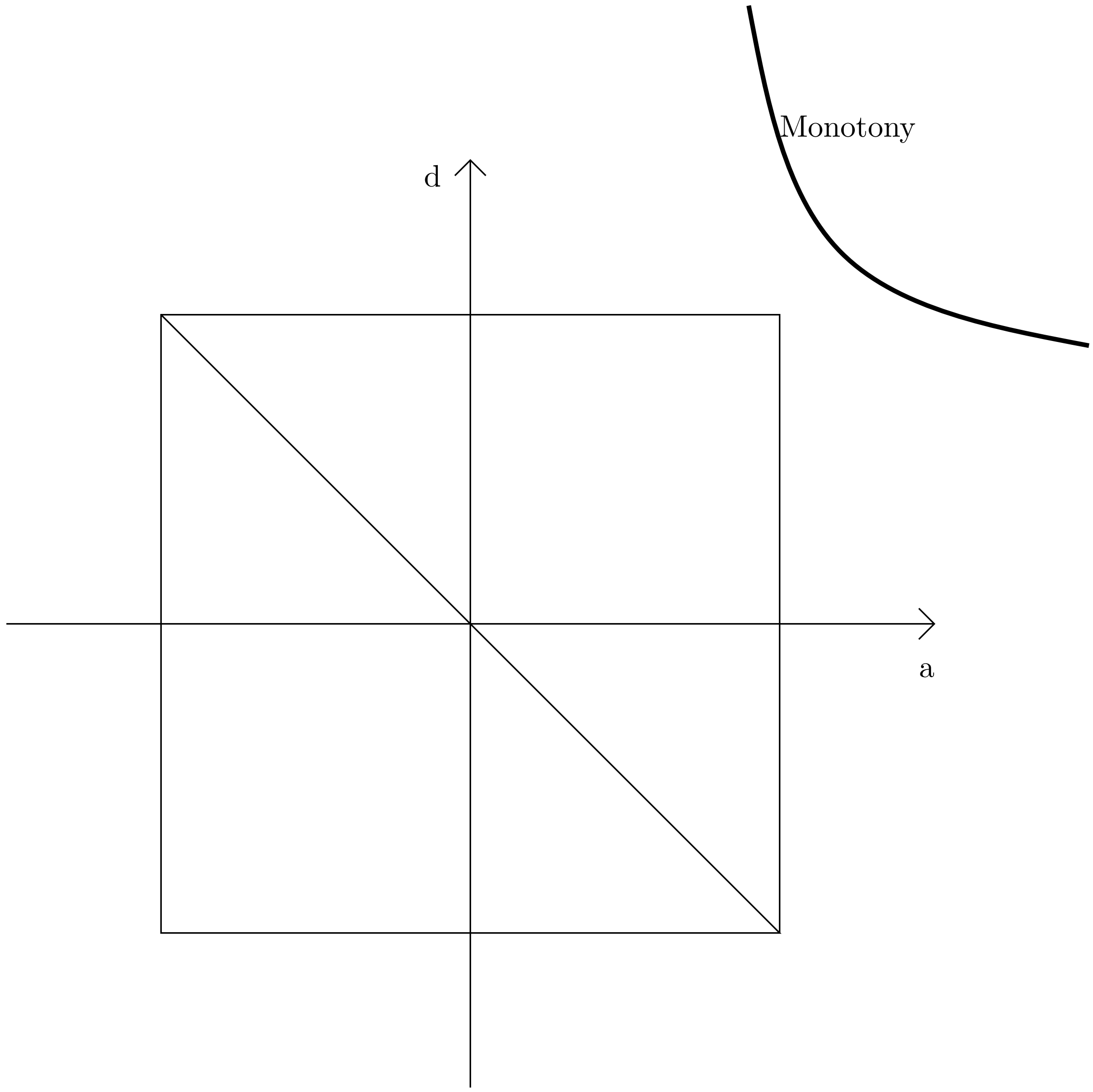
$$|a+d| - 1 \leq ad - bc \leq 1\,. \tag{13.13}$$

while the solutions are monotonic, according to (13.6), iff

$$-2\Theta \leq 0 \leq D \leq \Theta^2 \tag{13.14}$$

or

$$-a - d \leq 0 \leq ad - bc \leq \left(\frac{a+d}{2}\right)^2 \tag{13.15}$$

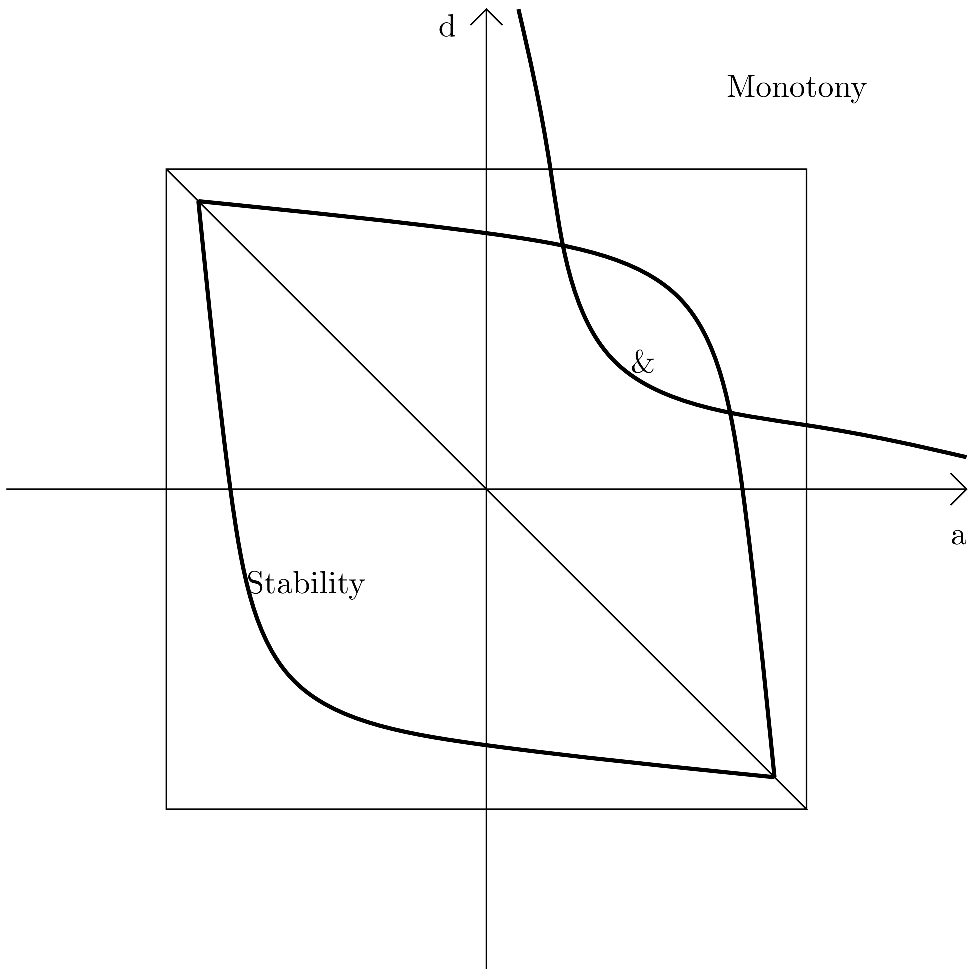


Monotony domain for $1 < bc$ outside arc

The inequalities (13.13) and (13.15) are slightly inconvenient. For bc fixed, we may describe the stable area in the (a,d)-plane. We may rewrite (13.13) for $a+d \geq 0$ as $(a-1)(d-1) \geq bc$ and $ad \leq 1+bc$, giving the stable area on the right side of some hyperbolas depending on the sign and size of bc. Further, we may rewrite (13.15) as $a+d \geq 0$, $bc \leq ad$ and $-bc \leq \left(\frac{a-d}{2}\right)^2$, giving the monotonic area on the upper–right side of the main diagonal, on the right side of some hyperbolas depending on the sign and size of bc and on the outside of a pair of straight lines in the case of $bc < 0$.

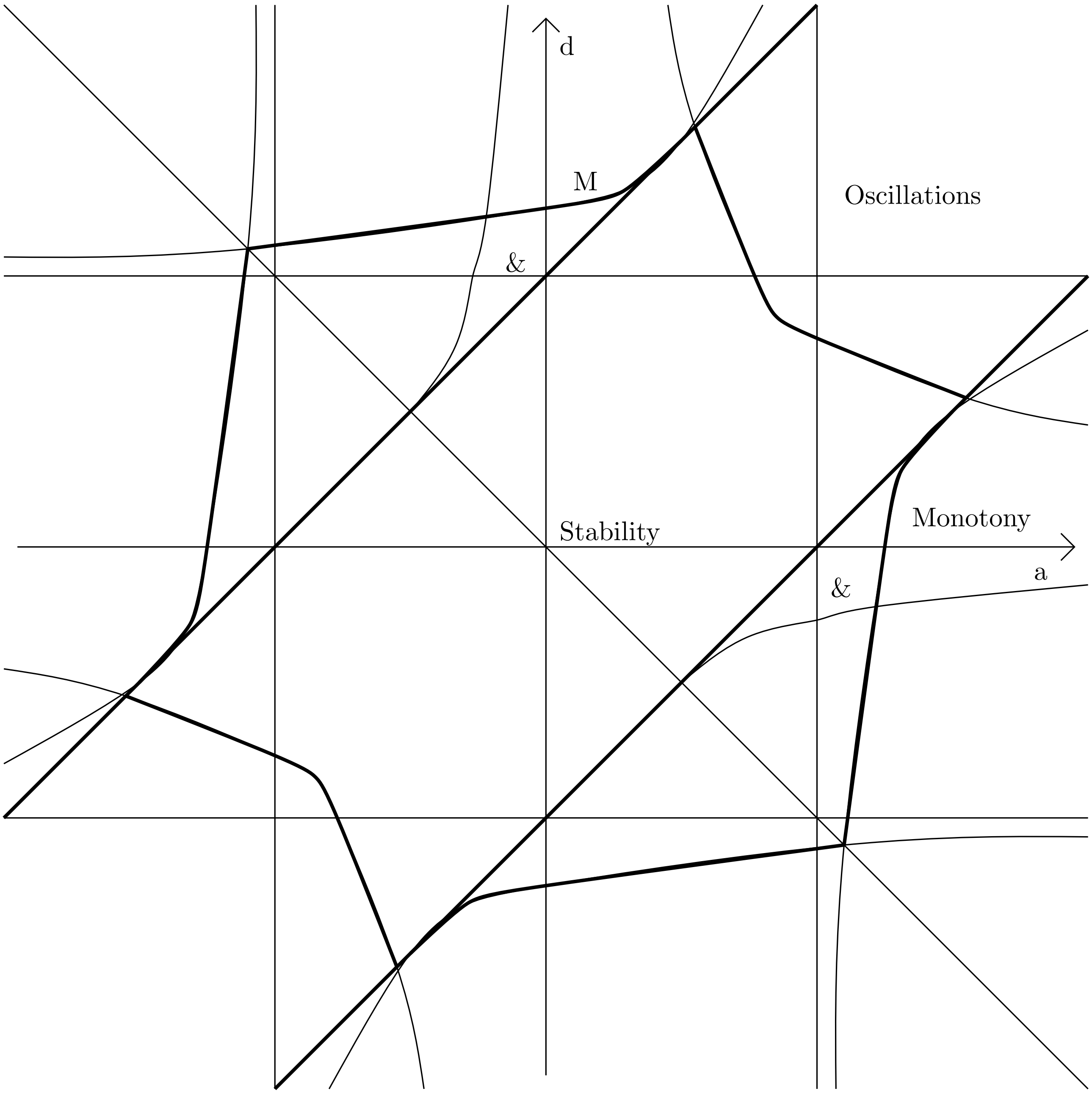
The figures depend on the size of bc. If $bc > 1$, then there is no stable area, and the monotonic area appears outside one single branch of a hyperbola.

But for $0 < bc < 1$ the stable area lies between two hyperbolas. In this case the monotonic area appears outside one single branch of a hyperbola too. For $0 < bc < \frac{1}{4}$ the two do not intersect, but for $\frac{1}{4} < bc < 1$ they do.



Stability & monotony domains for $0 < bc < \frac{1}{4}$ between arcs

If $-1 < bc < 0$ the stability appears inside a hexagon with sides of six hyperbolas. In this case oscillating solutions appear in a stripe between the dotted lines on the figure, corresponding to the inequality $\Delta < 0$. The monotonic area divides in two symmetric parts, both between the dotted lines and two half branches of one hyperbola.



Stability & monotony domains for $-1 < bc < 0$ between arcs and lines

When $bc < -1$, the area of stability divides in two symmetric parts, each a triangles of hyperbolas. The solutions will still be oscillating between the dotted lines on the figure. The monotonic area divides in two symmetric parts, both betwwen the dotted lines and two half branches of one hyperbola.

In the case of a system of two equations, we may also consider the question of stability of the ratio–solutions. It means that for a couple of solutions to (13.7) we shall study the behavior of the ratio,

$$r_n = \frac{f_2(n)}{f_1(n)} \tag{13.16}$$

We ask, if there are solutions, such that the ratio remains constant, i.e., solutions remaining on a straight line or ray. A possible ray is characterized by the ratio being constant

$$r_n = \alpha \tag{13.17}$$

This means that $r_{n+1} = r_n$, but we may compute r_{n+1} by means of r_n

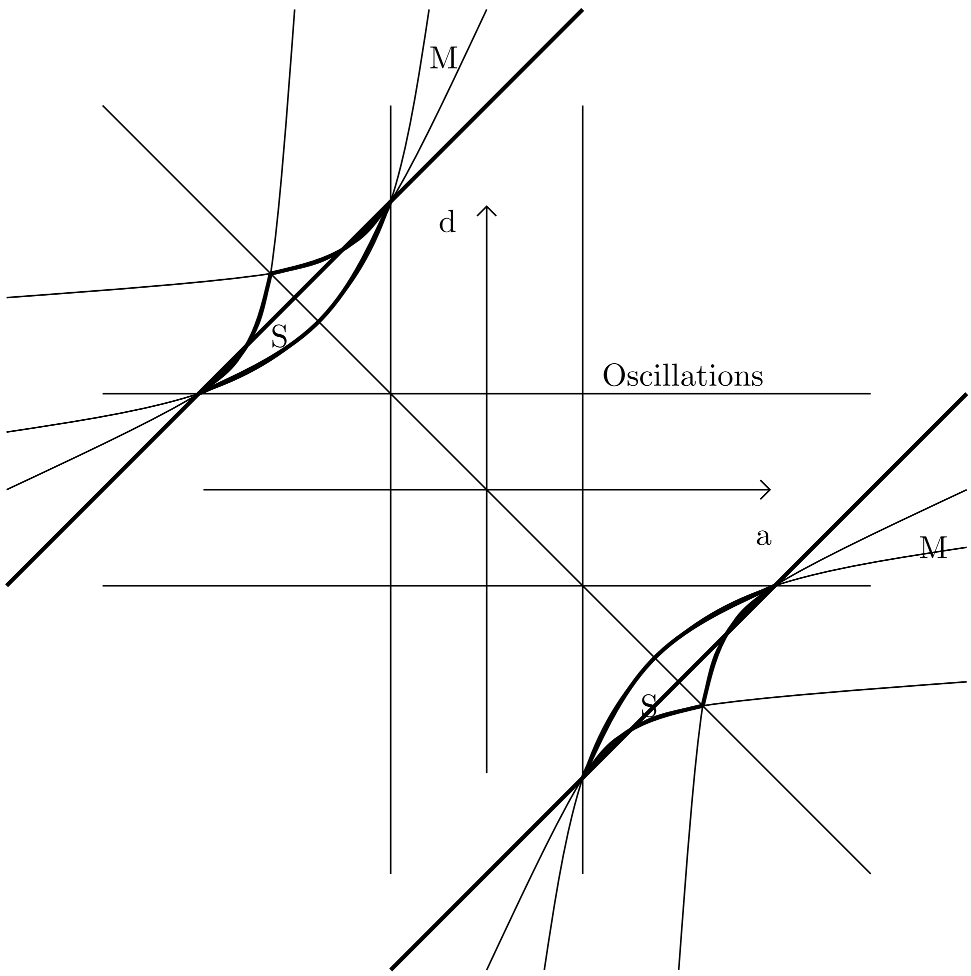
$$r_{n+1} = \frac{f_2(n+1)}{f_1(n+1)} = \frac{cf_1(n) + df_2(n)}{af_1(n) + bf_2(n)} = \frac{c + dr_n}{a + br_n} \tag{13.18}$$

which is equal to r_n, if and only if

$$-br_n^2 + (d-a)r_n + c = 0 \tag{13.19}$$

in which case, the constant value becomes

$$r_n = \alpha_\pm = \frac{1}{b}\left(\frac{d-a}{2} \pm \sqrt{\left(\frac{d-a}{2}\right)^2 + bc}\right) = \frac{1}{b}\left(\frac{d-a}{2} \pm \sqrt{\Delta}\right) \tag{13.20}$$



Stability and monotony domains for $bc < -1$ between arcs and lines

The question, what the solutions look like on the rays is also easy to answer, from (13.7) we get

$$f_1(n+1) = (a + b\alpha_\pm)\, f_1(n) = \left(\frac{a+d}{2} \pm \sqrt{\Delta}\right) f_1(n) = \left(\Theta \pm \sqrt{\Delta}\right) f_1(n) \tag{13.21}$$

with the solutions

$$f_1(n) = \lambda^n f_1(0) \text{ for } \lambda = \lambda_\pm \tag{13.22}$$
$$f_2(n) = \alpha\lambda^n f_2(0) \text{ for } \alpha \text{ and } \lambda \text{ corresponding.} \tag{13.23}$$

with

$$\lambda_\pm = \Theta \pm \sqrt{\Delta} \tag{13.24}$$

Following [3] we substitute $s_n = a + br_n$ in (13.18) and obtain the simple form

(13.25)
$$\begin{aligned} s_{n+1} &= a + br_{n+1} = a + b\frac{c + dr_n}{a + br_n} = a + b\frac{c + d\left(\frac{1}{b}(s_n - a)\right)}{s_n} = \\ &= a + \frac{bc - ad + ds_n}{s_n} = a + d - \frac{ad - bc}{s_n} = 2\Theta - \frac{D}{s_n} \end{aligned}$$

where we have used the notation from (13.9–13.10). The ratio–solution r_n is an affine transformation of the sequence s_n, which is obtained from the iteration of the function

$$y = f(x) = 2\Theta - \frac{D}{x} \tag{13.26}$$

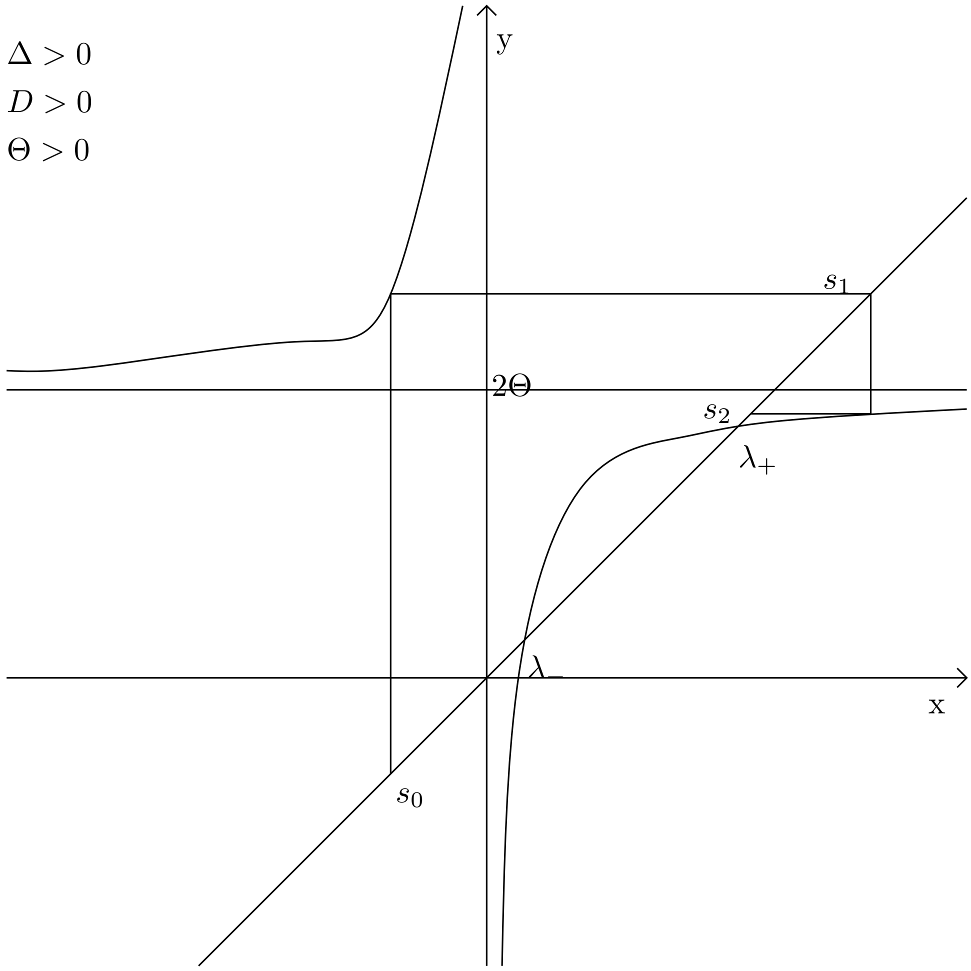
The natural questions are, if there are fixpoints, and eventually, will the derivative in some of them be numerically less than one?

A fixpoint is a solution to the equation

$$x^2 - 2\Theta x + D = 0 \tag{13.27}$$

with discriminant $\Theta^2 - D = \Delta$ using (13.11). So, the condition for fixpoints is the usual, $\Delta \geq 0$. Suppose this is the case.

Then the roots are as usual $\lambda_\pm = \Theta \pm \sqrt{\Delta}$ cf. (13.24).



Attraction of the ratios by iteration of $y = 2\Theta - \frac{D}{x}$

The derivative of the hyperbola is

$$f'(x) = \frac{dy}{dx} = \frac{D}{x^2} \tag{13.28}$$

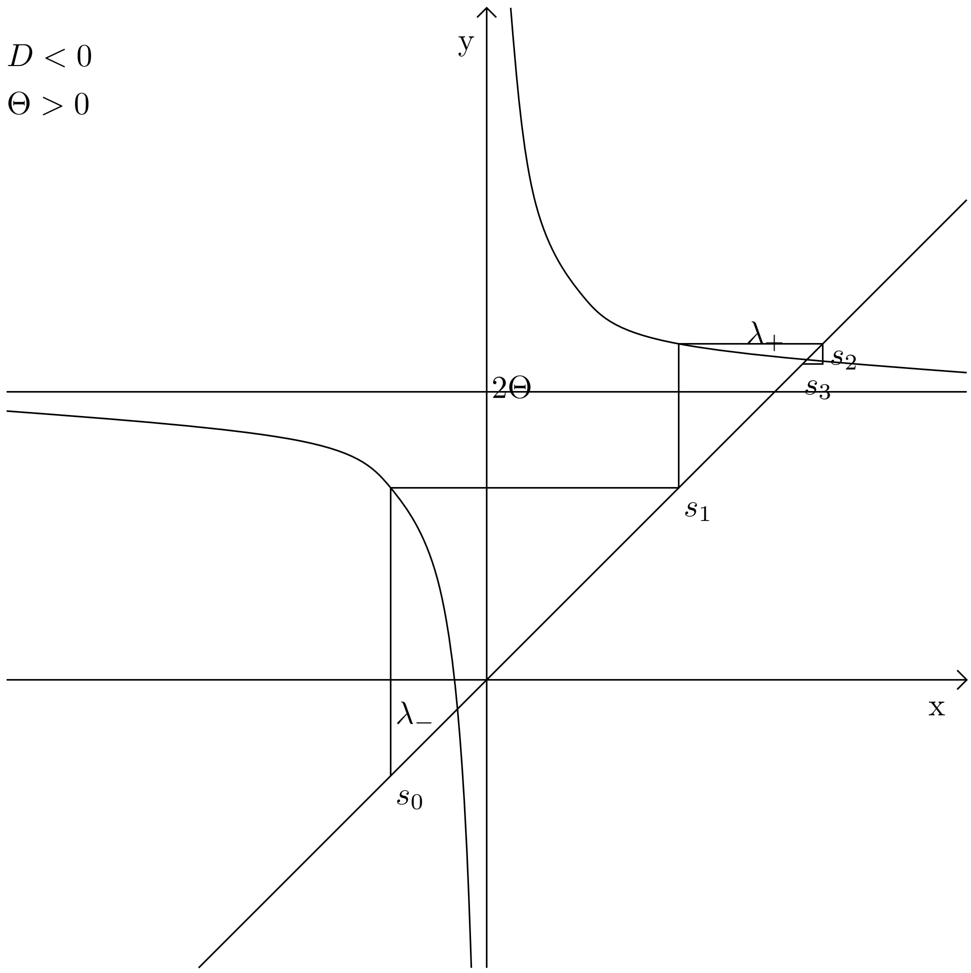
Hence the derivatives in the fixpoints are

$$f'(\lambda_\pm) = \frac{D}{\lambda_\pm^2} = \frac{\Theta^2 - \Delta}{\left(\Theta \pm \sqrt{\Delta}\right)^2} = \frac{\Theta \mp \sqrt{\Delta}}{\Theta \pm \sqrt{\Delta}} \tag{13.29}$$

with product equal to 1. So, one is smaller than one, the other is bigger. Hence one is an attractor, the other is a repeller. Which one is which depends simply on the signs of D and Θ.

In the case of $\Delta > 0$ and $D > 0$ there are two positive fixpoints, and if further $\Theta > 0$, the bigger fixpoint is the attractor.

In the case of $D < 0$, we must have $\Delta > 0$, and the two fixpoints have different signs. If further $\Theta > 0$, the bigger fixpoint is the attractor.



Attraction of the ratios by iteration of $y = 2\Theta - \frac{D}{x}$

With the discriminant, $\Delta < 0$, the equation (13.19) has no solutions, there are no fixpoints, and we have no lines available. But we shall nevertheless be able to solve the equations.

From chapter 5, theorem 1, we obtain

$$f_1(n+2) - (a+d)f_1(n+1) + (ad - bc)f_1(n) = 0 \tag{13.30}$$

equivalent to the equation (using (13.9–13.10))

$$f_1(n+2) - 2\Theta f_1(n+1) + Df_1(n) = 0 \tag{13.31}$$

In this equation we shall substitute

$$f_1(n) = \left(\sqrt{D}\right)^n \xi_n \tag{13.32}$$

What is allowed, because from (13.11) we know that

$$\Delta < 0 \Rightarrow 0 \leq \Theta^2 < D \tag{13.33}$$

From (13.31) we get

$$\left(\sqrt{D}\right)^{n+2}\xi_{n+2} - 2\Theta\left(\sqrt{D}\right)^{n+1}\xi_{n+1} + \left(\sqrt{D}\right)^{n+2}\xi_n = 0 \tag{13.34}$$

or rather,

$$\xi_{n+2} - 2\frac{\Theta}{\sqrt{D}}\xi_{n+1} + \xi_n = 0 \tag{13.35}$$

From (13.33) we get

$$\frac{|\Theta|}{\sqrt{D}} < 1 \tag{13.36}$$

Let ϕ and β be disposable. We shall define

$$\xi_n := \sin(n\phi + \beta) \tag{13.37}$$

Then we have the following easy computation:

(13.38)

$$\begin{aligned}
\xi_{n+2} + \xi_n &= \sin(n\phi + \beta + 2\phi) + \sin(n\phi + \beta) \\
&= \sin(n\phi + \beta)\cos(2\phi) + \cos(n\phi + \beta)\sin(2\phi) + \sin(n\phi + \beta) \\
&= \sin(n\phi + \beta)\left(2\cos^2\phi - 1 + 1\right) + 2\sin\phi\cos\phi\cos(n\phi + \beta) \\
&= 2\cos\phi\left(\cos(n\phi + \beta)\sin\phi + \sin(n\phi + \beta)\cos\phi\right) \\
&= 2\cos\phi\sin((n+1)\phi + \beta) \\
&= 2\cos\phi\,\xi_{n+1}
\end{aligned}$$

hence, ξ_n defined by (13.37) solves (13.35) if and only if

$$\cos\phi = \frac{\Theta}{\sqrt{D}} \tag{13.39}$$

and this equation has always infinitely many solutions, ϕ, due to (13.36).

With this choice of ϕ, we get the solutions from (13.32),

$$f_1(n) = \left(\sqrt{D}\right)^n (v_1\cos(n\phi) + w_1\sin(n\phi)) \tag{13.40}$$

$$f_2(n) = \left(\sqrt{D}\right)^n (v_2\cos(n\phi) + w_2\sin(n\phi)) \tag{13.41}$$

Now, choosing $n = 0$ we get

$$\begin{aligned}
f_1(0) &= v_1 \\
f_2(0) &= v_2
\end{aligned}$$

or

$$v_1 = x^0 \qquad (13.42)$$
$$v_2 = y^0$$

and with the remark that

$$\sin\phi = \pm\sqrt{1-\cos^2\phi} = \pm\sqrt{1-\frac{\Theta^2}{D}} = \pm\sqrt{\frac{D-\Theta^2}{D}} = \pm\frac{\sqrt{-\Delta}}{\sqrt{D}}$$

we get from (13.40–13.41) with $n = 1$

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \pm\frac{1}{\sqrt{-\Delta}}(\mathbf{A}-\Theta\mathbf{E})\begin{pmatrix} x^0 \\ y^0 \end{pmatrix} \qquad (13.43)$$

Substitution of (13.42–13.43) in (13.40–13.41) yields

$$\begin{pmatrix} f_1(n) \\ f_2(n) \end{pmatrix} = \left(\sqrt{D}\right)^n \left(\cos(n\phi)\mathbf{E} \pm \frac{\sin(n\phi)}{\sqrt{-\Delta}}(\mathbf{A}-\Theta\mathbf{E})\right)\begin{pmatrix} f_1(0) \\ f_2(0) \end{pmatrix} \qquad (13.44)$$

This explicit solution proves that the question of stability for these solutions is equivalent to the question of the size of D. The solutions shall go to zero if and only if $D < 1$. This corresponds to the conclusion (13.12) above.

In the case of $\Delta \geq 0$ we shall try to find the solutions similarly.

Let us define the difference operator,

$$\Delta_\Theta x_n := x_{n+1} - \Theta x_n \qquad (13.45)$$

For a given value of the discriminant, Δ, we consider the second order equation, with initial conditions:

$$\Delta_\Theta^2 \delta_n = \Delta\delta_n \qquad (13.46)$$
$$\delta_0 = 0 \quad ; \quad \Delta_\Theta\delta_0 = 1 \qquad (13.47)$$

The solution must be

$$\delta_n = \begin{cases} \frac{(\Theta+\sqrt{\Delta})^n-(\Theta-\sqrt{\Delta})^n}{2\sqrt{\Delta}} & \text{for } \Delta > 0 \\ n\Theta^{n-1} & \text{for } \Delta = 0 \end{cases} \qquad (13.48)$$

Hence we get

$$\Delta_\Theta\delta_n = \begin{cases} \frac{(\Theta+\sqrt{\Delta})^n+(\Theta-\sqrt{\Delta})^n}{2} & \text{for } \Delta > 0 \\ \Theta^n & \text{for } \Delta = 0 \end{cases} \qquad (13.49)$$

The system (13.7) is written

$$\begin{pmatrix} f_1(n+1) \\ f_2(n+1) \end{pmatrix} = \mathbf{A} \begin{pmatrix} f_1(n) \\ f_2(n) \end{pmatrix} \qquad \begin{pmatrix} f_1(0) \\ f_2(0) \end{pmatrix} = \begin{pmatrix} x^0 \\ y^0 \end{pmatrix} \tag{13.50}$$

and then transformed to

$$\Delta_\Theta \begin{pmatrix} f_1(n) \\ f_2(n) \end{pmatrix} = (\mathbf{A} - \Theta\mathbf{E}) \begin{pmatrix} f_1(n) \\ f_2(n) \end{pmatrix} \qquad \begin{pmatrix} f_1(0) \\ f_2(0) \end{pmatrix} = \begin{pmatrix} x^0 \\ y^0 \end{pmatrix} \tag{13.51}$$

but here the matrix $\mathbf{A} - \Theta\mathbf{E}$ has trace zero, so by iterating (13.51) we obtain

$$\Delta_\Theta^2 \begin{pmatrix} f_1(n) \\ f_2(n) \end{pmatrix} = (\mathbf{A} - \Theta\mathbf{E})^2 \begin{pmatrix} f_1(n) \\ f_2(n) \end{pmatrix} = \Delta\mathbf{E} \begin{pmatrix} f_1(n) \\ f_2(n) \end{pmatrix} \tag{13.52}$$

The equations splits in two of form (13.46) and may therefore be solved by (13.48–13.49), i.e.

$$\begin{pmatrix} f_1(n) \\ f_2(n) \end{pmatrix} = \Delta_\Theta \delta_n \mathbf{v} + \delta_n \mathbf{w} \qquad \begin{pmatrix} f_1(0) \\ f_2(0) \end{pmatrix} = \begin{pmatrix} x^0 \\ y^0 \end{pmatrix} \tag{13.53}$$

Trying $n = 0$ we see that $\mathbf{v} = \begin{pmatrix} x^0 \\ y^0 \end{pmatrix}$, and letting then $n = 1$, we see, that $\mathbf{w} = (\mathbf{A} - \Theta\mathbf{E}) \begin{pmatrix} x^0 \\ y^0 \end{pmatrix}$. Hence, (13.53) can be written precisely as

$$\begin{pmatrix} f_1(n) \\ f_2(n) \end{pmatrix} = (\Delta_\Theta \delta_n \mathbf{E} + \delta_n (\mathbf{A} - \Theta\mathbf{E})) \begin{pmatrix} x^0 \\ y^0 \end{pmatrix} \tag{13.54}$$

From (13.44) we get the idea, to be applied only when $D > 0$, to try to write

$$\delta_n = \begin{cases} \left(\sqrt{D}\right)^n \frac{1}{\sqrt{\Delta}} \frac{\left(\frac{\Theta+\sqrt{\Delta}}{\sqrt{D}}\right)^n - \left(\frac{\Theta-\sqrt{\Delta}}{\sqrt{D}}\right)^n}{2} & \text{for } \Delta > 0 \\ n\Theta^{n-1} = \Theta^n \cdot \frac{n}{\Theta} & \text{for } \Delta = 0 \end{cases} \tag{13.55}$$

In the case of $\Delta > 0$ we have obtained that the product

$$\frac{\Theta + \sqrt{\Delta}}{\sqrt{D}} \times \frac{\Theta - \sqrt{\Delta}}{\sqrt{D}} = \frac{\Theta^2 - \Delta}{D} = 1$$

according to (13.11).

This means that if we furthermore assume that $\Theta > 0$, we may introduce

$$\alpha = \log\left(\frac{\Theta + \sqrt{\Delta}}{\sqrt{D}}\right) \tag{13.56}$$

and obtain the formula for δ_n,

$$\delta_n = \begin{cases} \left(\sqrt{D}\right)^n \frac{1}{\sqrt{\Delta}} \frac{e^{\alpha n} - e^{-\alpha n}}{2} = e^{\frac{\log D}{2} n} \frac{1}{\sqrt{\Delta}} \sinh(\alpha n) & \text{for } \Delta > 0 \\ \Theta^n \cdot \frac{n}{\Theta} = \left(\sqrt{D}\right)^n \cdot \frac{n}{\Theta} = e^{\frac{\log D}{2} n} \cdot \frac{n}{\Theta} & \text{for } \Delta = 0 \end{cases} \tag{13.57}$$

Now in the case of $\Delta > 0$ we may rewrite (13.54) in a convenient form, as

$$\begin{pmatrix} f_1(n) \\ f_2(n) \end{pmatrix} = e^{\frac{\log D}{2} n} \left(\cosh(\alpha n)\mathbf{E} + \frac{\sinh(\alpha n)}{\sqrt{\Delta}} (\mathbf{A} - \Theta\mathbf{E}) \right) \begin{pmatrix} x^0 \\ y^0 \end{pmatrix} \tag{13.58}$$

In the case of $\Delta = 0$ we may write

$$\begin{pmatrix} f_1(n) \\ f_2(n) \end{pmatrix} = e^{\frac{\log D}{2} n} \left(\mathbf{E} + n \left(\frac{1}{\Theta}\mathbf{A} - \mathbf{E} \right) \right) \begin{pmatrix} x^0 \\ y^0 \end{pmatrix} \tag{13.59}$$

CHAPTER 14. The generalized logarithm of a matrix

In chapter 13 we found the solutions to the difference equations (13.7) on the forms (13.44) and (13.54), the last to be rewritten for $D > 0$, $\Theta > 0$ and $\Delta \geq 0$ in the forms (13.58–59).

These last forms, (13.44) and (13.58–59), allowed immediatly for a continously differentiable interpolation of the solutions to the difference equation. The question to be asked is, do these interpolating functions satisfy some differential equations of the form (5.1)?

If we rewrite the equations in the forms

$$x_{n+1} = ax_n + by_n \qquad x_0 = x^0 \tag{14.1}$$

$$y_{n+1} = cx_n + dy_n \qquad y_0 = y^0 \tag{14.2}$$

we may rewrite the solution (13.44) as

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = e^{\left(\frac{\log D}{2}\right)n} \left(\cos(n\phi)\mathbf{E} + \frac{\sin(n\phi)}{\phi} \left(\frac{\phi}{\sqrt{-\Delta}} (\mathbf{A} - \Theta\mathbf{E}) \right) \right) \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \tag{14.3}$$

reminding us of the solution (5.20) to some differential equation.

Now, the matrix $\mathbf{A} - \Theta\mathbf{E}$ has trace zero and discriminant equal to Δ. Hence the discriminant of $\frac{\phi}{\sqrt{-\Delta}} (\mathbf{A} - \Theta\mathbf{E})$ becomes equal to $\left(\frac{\phi}{\sqrt{-\Delta}}\right)^2 \Delta = -\phi^2$, as wanted.

Now it is quite easy to write down a useful matrix, we just have to correct the trace. It becomes

$$\begin{pmatrix} \frac{\phi}{\sqrt{-\Delta}} \frac{a-d}{2} + \frac{\log D}{2} & \frac{\phi b}{\sqrt{-\Delta}} \\ \frac{\phi c}{\sqrt{-\Delta}} & \frac{\phi}{\sqrt{-\Delta}} \frac{d-a}{2} + \frac{\log D}{2} \end{pmatrix} \tag{14.4}$$

This matrix represent in a sense the logarithm of the matrix $\mathbf{A}$, and if we take the exponential of it, as defined by (5.26), we obtain $\mathbf{A}$.

Similarly, (13.58) may be rewritten as

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = e^{\frac{\log D}{2} n} \left(\cosh(\alpha n)\mathbf{E} + \frac{\sinh(\alpha n)}{\alpha} \left(\frac{\alpha}{\sqrt{\Delta}} (\mathbf{A} - \Theta\mathbf{E}) \right) \right) \begin{pmatrix} x^0 \\ y^0 \end{pmatrix} \tag{14.5}$$

again having the right discriminant.

The matrix for the differential equation is obtained by correcting the trace, it becomes

$$\begin{pmatrix} \frac{\alpha}{\sqrt{\Delta}} \frac{a-d}{2} + \frac{\log D}{2} & \frac{\alpha b}{\sqrt{\Delta}} \\ \frac{\alpha c}{\sqrt{\Delta}} & \frac{\alpha}{\sqrt{\Delta}} \frac{d-a}{2} + \frac{\log D}{2} \end{pmatrix} \tag{14.6}$$

This matrix represents the logarithm of the matrix $\mathbf{A}$ in the special case of $\Delta > 0$, $D > 0$ and $\Theta > 0$.

In the case of $\Delta = 0$ we may rewrite the solution (13.59) as

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = e^{\frac{\log D}{2} n} \left(\mathbf{E} + n \left(\frac{1}{\Theta} \mathbf{A} - \mathbf{E} \right) \right) \begin{pmatrix} x^0 \\ y^0 \end{pmatrix} \tag{14.7}$$

This time the discriminant is already zero, so the matrix for the differential equation is obtained by correcting the trace. It becomes

$$\begin{pmatrix} \frac{a-d}{a+d} + \frac{\log D}{2} & \frac{2b}{a+d} \\ \frac{2c}{a+d} & \frac{d-a}{a+d} + \frac{\log D}{2} \end{pmatrix} \tag{14.8}$$

This matrix represents the logarithm of the matrix $\mathbf{A}$ in the special case of $\Delta = 0$, $D > 0$ and $\Theta > 0$.

If Θ or D are negative, the solutions (13.48–49) will be disturbed by a factor $(-1)^n$ somewhere. This leads to the idea to look at every second term of the solution to (14.1), treating the even numbers as (14.1) with stepsize 2, and the odd numbers similar, but with initial values

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \mathbf{A} \begin{pmatrix} x^0 \\ y^0 \end{pmatrix}$$

Both of these corresponds to the use of the coefficient matrix, $\mathbf{A}^2$, why we shall consider this matrix closer. It is simple to compute,

$$\mathbf{A}^2 = \begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & d^2 + bc \end{pmatrix} \tag{14.9}$$

But from the Cayley–Hamilton theorem in chapter 5 we have the formula,

$$(\mathbf{A} - \Theta\mathbf{E})^2 = \Delta\mathbf{E} \tag{14.10}$$

which may be rewritten by using (13.11) as

$$\mathbf{A}^2 = 2\Theta\mathbf{A} + (\Delta - \Theta^2)\mathbf{E} = 2\Theta\mathbf{A} - D\mathbf{E} \tag{14.11}$$

Hence, the half trace of $\mathbf{A}^2$, $\hat{\Theta}$, becomes

$$\hat{\Theta} = 2\Theta^2 - D = \Theta^2 + \Delta = D + 2\Delta \tag{14.12}$$

and the discriminant, $\hat{\Delta}$, which is independent of the term $D\mathbf{E}$, therefore must be

$$\hat{\Delta} = (2\Theta)^2\Delta \tag{14.13}$$

Hence we can deduce the value of the determinant, $\hat{D}$, by (13.11)

$$\hat{D} = \hat{\Theta}^2 - \hat{\Delta} = (\Theta^2 + \Delta)^2 - 4\Theta^2\Delta = (\Theta^2 - \Delta)^2 = D^2 \tag{14.14}$$

a formula, familiar to some students.

We remark, that for $\Delta > 0$ it is obvious, that $\hat{\Theta} > 0$ and $\hat{D} > 0$. So, we are now able to apply the formula for the logarithm, (13). We just have to compute the value of $\hat{\alpha}$, according to (13.56), i.e.

$$\begin{aligned} \hat{\alpha} &= \log\left(\frac{\hat{\Theta} + \sqrt{\hat{\Delta}}}{\sqrt{\hat{D}}}\right) = \log\left(\frac{\Theta^2 + \Delta + \sqrt{4\Delta\Theta^2}}{\sqrt{D^2}}\right) = \\ &= \log\left(\frac{\Theta^2 + \Delta + 2|\Theta|\sqrt{\Delta}}{|D|}\right) = 2\log\left(\frac{|\Theta| + \sqrt{\Delta}}{\sqrt{|D|}}\right) \end{aligned} \tag{14.15}$$

We are lead to want to generalize the definition of α, (13.56), to

$$\alpha = \log\left(\frac{|\Theta| + \sqrt{\Delta}}{\sqrt{|D|}}\right) \tag{14.16}$$

Now we are prepared to write down the logarithm of $\mathbf{A}^2$ according to (14.6):

(14.17)

$$\begin{aligned} &\begin{pmatrix} \frac{2\alpha}{\sqrt{4\Theta^2\Delta}}\frac{a^2-d^2}{2} + \frac{\log D^2}{2} & \frac{2\alpha 2\Theta b}{\sqrt{4\Theta^2\Delta}} \\ \frac{2\alpha 2\Theta c}{\sqrt{4\Theta^2\Delta}} & \frac{2\alpha}{\sqrt{4\Theta^2\Delta}}\frac{d^2-a^2}{2} + \frac{\log D^2}{2} \end{pmatrix} = \\ =&\begin{pmatrix} \frac{2\alpha}{2|\Theta|\sqrt{\Delta}}\frac{2\Theta(a-d)}{2} + \log|D| & \frac{2\alpha 2\Theta b}{2|\Theta|\sqrt{\Delta}} \\ \frac{2\alpha 2\Theta c}{2|\Theta|\sqrt{\Delta}} & \frac{2\alpha}{2|\Theta|\sqrt{\Delta}}\frac{2\Theta(d-a)}{2} + \log|D| \end{pmatrix} = \\ =2&\begin{pmatrix} \frac{\Theta}{|\Theta|}\frac{\alpha}{\sqrt{\Delta}}\frac{a-d}{2} + \frac{\log|D|}{2} & \frac{\Theta}{|\Theta|}\frac{\alpha b}{\sqrt{\Delta}} \\ \frac{\Theta}{|\Theta|}\frac{\alpha c}{\sqrt{\Delta}} & \frac{\Theta}{|\Theta|}\frac{\alpha}{\sqrt{\Delta}}\frac{d-a}{2} + \frac{\log|D|}{2} \end{pmatrix} \\ =2&\frac{\Theta}{|\Theta|}\frac{\alpha}{\sqrt{\Delta}}(\mathbf{A} - \Theta\mathbf{E}) + \log|D|\mathbf{E} \end{aligned}$$

We are now able to define a general logarithm to a 2×2–matrix. We shall define the value, α, by

$$\alpha = \begin{cases} \arccos\left(\frac{\Theta}{\sqrt{D}}\right) & \text{for } \Delta < 0 \\ \log\left(\frac{|\Theta| + \sqrt{\Delta}}{\sqrt{|D|}}\right) & \text{for } \Delta > 0 \end{cases} \tag{14.18}$$

with this definition of α we may define the general logarithm of a 2×2–matrix, $\mathbf{A}$, as

$$\log \mathbf{A} = \left(\frac{|\Theta|\alpha}{\sqrt{|\Delta|}}\right)\frac{1}{\Theta}(\mathbf{A} - \Theta\mathbf{E}) + \frac{\log|D|}{2}\mathbf{E} \tag{14.19}$$

where the first parenthesis shall be omitted in the case of $\Delta = 0$, in which case we have $\alpha = 0$ and $D > 0$.

CHAPTER 15. Economical examples of difference equations

EXAMPLE 1. This example is taken from [3]. Let us assume that the demand, D_t, and the supply, S_t, depend on the price p_t in such a way that the demand is a decreasing function of the price while the supply is an increasing function of the price. For simplicity, we shall assume the dependence to be linear functions. Furthermore, we shall assume the production to cause some delay, which means that the supply depends not on the actual price, but on yesterday's price, p_{t-1}. In formulas

$$D_t = a + bp_t \qquad \text{with} \qquad b < 0 \tag{15.1}$$
$$S_t = c + dp_{t-1} \qquad \text{with} \qquad d > 0 \tag{15.2}$$

If we introduce the assumption that the free market forces the price to change such that demand and supply become equal,

$$D_t = S_t \tag{15.3}$$

these three equations give rise to a dynamics of prices. We get obviously the difference equation

$$p_t - \frac{d}{b}\,p_{t-1} = \frac{c-a}{b} \tag{15.4}$$

with the solutions

$$p_t = k\Big(\frac{d}{b}\Big)^t + \frac{c-a}{b-d}\,. \tag{15.5}$$

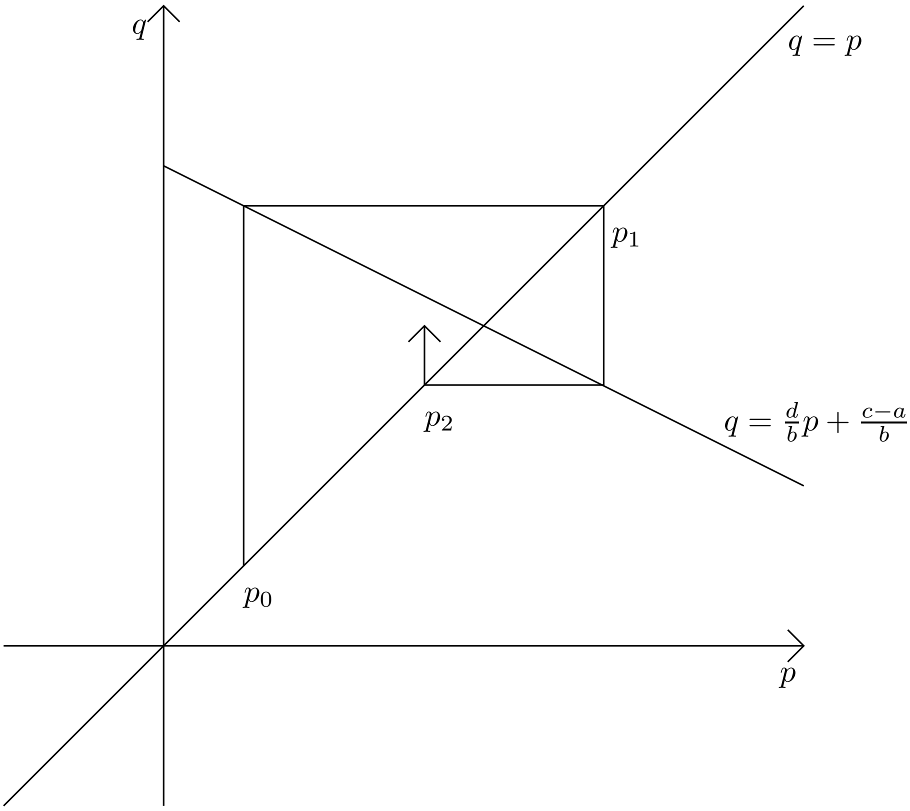
The constant term is interpreted as the equilibrium price, and it is stable, if and only if $|d| < |b|$.

EXAMPLE 2. This example is taken from [11]. In a Keynesian model of national economics, we shall consider our consumptions from two sides: 1) We can afford to consume, C_t, corresponding to our production, Y_t, except the savings or investments, I_t. 2) We consume most of last years production. So, we have the equations

$$Y_t - I_t = C_t = a + b\,Y_{t-1} \tag{15.6}$$

with $a \geq 0$ and $0 < b < 1$.

$-1 < \frac{d}{b} < 0$



The demand–supply cycle

Under the additional assumptions, that the investment, I_t, is constant for a series of years, but then may be autonomously changed, we shall compare two situations,

$$I_t = I_0 \,, \tag{15.7}$$

$$I_t = I_0 + \Delta I \,. \tag{15.8}$$

The solutions are for $I = I_0$ $(+\Delta I)$

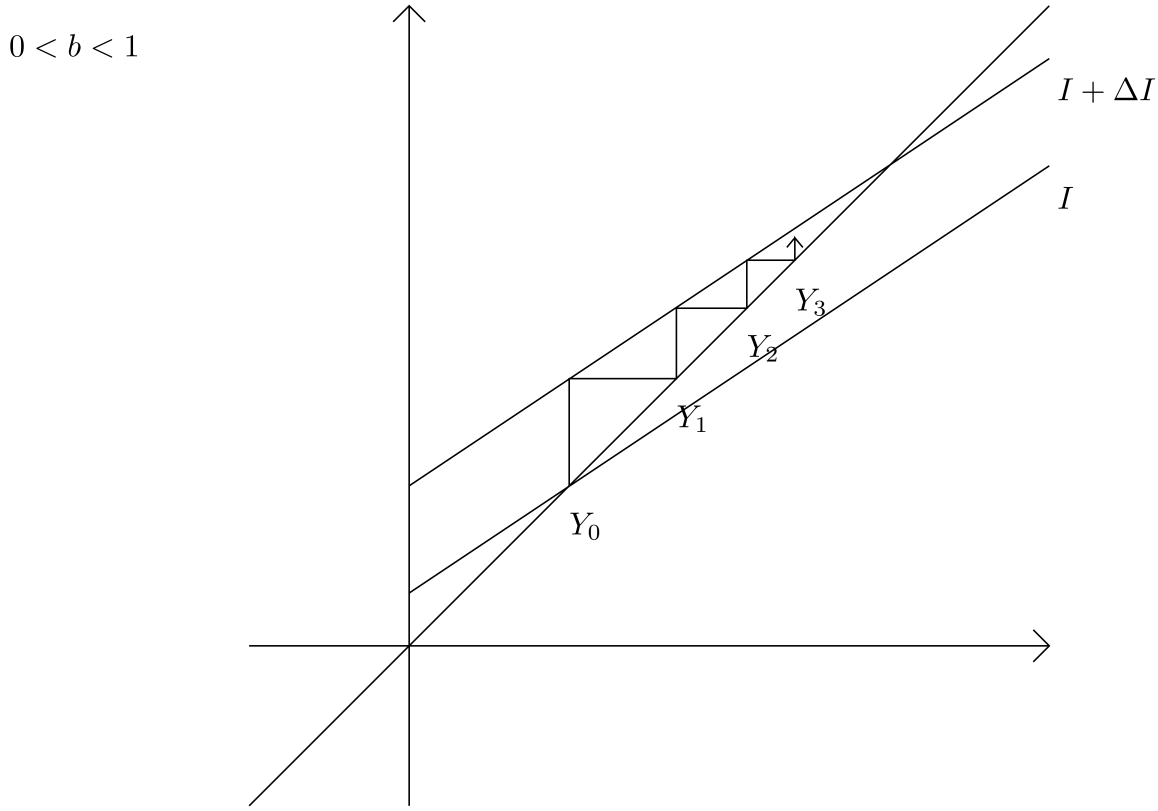
$$Y_t = k\, b^t + \frac{a + I}{1 - b} \tag{15.9}$$

stable and converging towards the equilibrium, because $0 < b < 1$.

This model is interpreted as follows: After many years we are in the equilibrium state of

$$Y_t = \frac{a + I_0}{1 - b} \,. \tag{15.10}$$

Then we change to (15.9) with $I = I_0 + \Delta I$ and hence, – starting a new counting of years – $k = -\Delta I/(1 - b)$.



Change of investments in a Keynesian model

Example 3. This example is taken from [7]. In Harrod's model we introduce $s = 1 - b$ as the propensity to save, and the idea, that the investment is governed by the increase in national income,

$$I_t = k\,\Delta\,Y_{t-1} = k(Y_t - Y_{t-1}) \tag{15.11}$$

such that the national income obeys the dynamics

$$sY_{t-1} = k(Y_t - Y_{t-1}) \tag{15.12}$$

or

$$Y_t - \frac{k+s}{k}\,Y_{t-1} = 0 \tag{15.13}$$

with the solutions

$$Y_t = c\Big(\frac{k+s}{k}\Big)^t \tag{15.14}$$

growing towards infinity rather than an equilibrium.

EXAMPLE 4. This example is taken from [18]. Paul Samuelson suggests that the investment be divided into two parts, one autonomous part, $I_t'' = G$, constant for some time, and another induced part, I_t'. These assumptions lead to the model

$$C_t = b\,Y_{t-1} \qquad 0 < b < 1 \tag{15.15}$$

$$I_t = I_t' + I_t'' \tag{15.16}$$

$$I_t'' = G \tag{15.17}$$

$$I_t' = k(C_t - C_{t-1}) \tag{15.18}$$

$$Y_t = C_t + I_t \tag{15.19}$$

We get this time a second order equation in Y_t, namely

$$Y_t - b(1+k)Y_{t-1} + bk\,Y_{t-2} = G \tag{15.20}$$

with the equilibrium solution

$$Y_e = \frac{G}{1-b}\,. \tag{15.21}$$

The question is, whether this solution is stable or not. A secondary question may be, whether the solutions are monotonic or oscillating.

We just have to apply the theory in Ch. 13. The conditions of stability is (15.5), i.e.

$$|b(1+k)| - 1 < bk < 1\,. \tag{15.22}$$

And the condition for oscillations is

$$b^2(1+k)^2 < 4bk\,. \tag{15.23}$$

In the b–k–plane we get the forms of (15.22)

$$b < 1 \;\wedge\; b < \frac{1}{k} \tag{15.24}$$

and of (15.23)

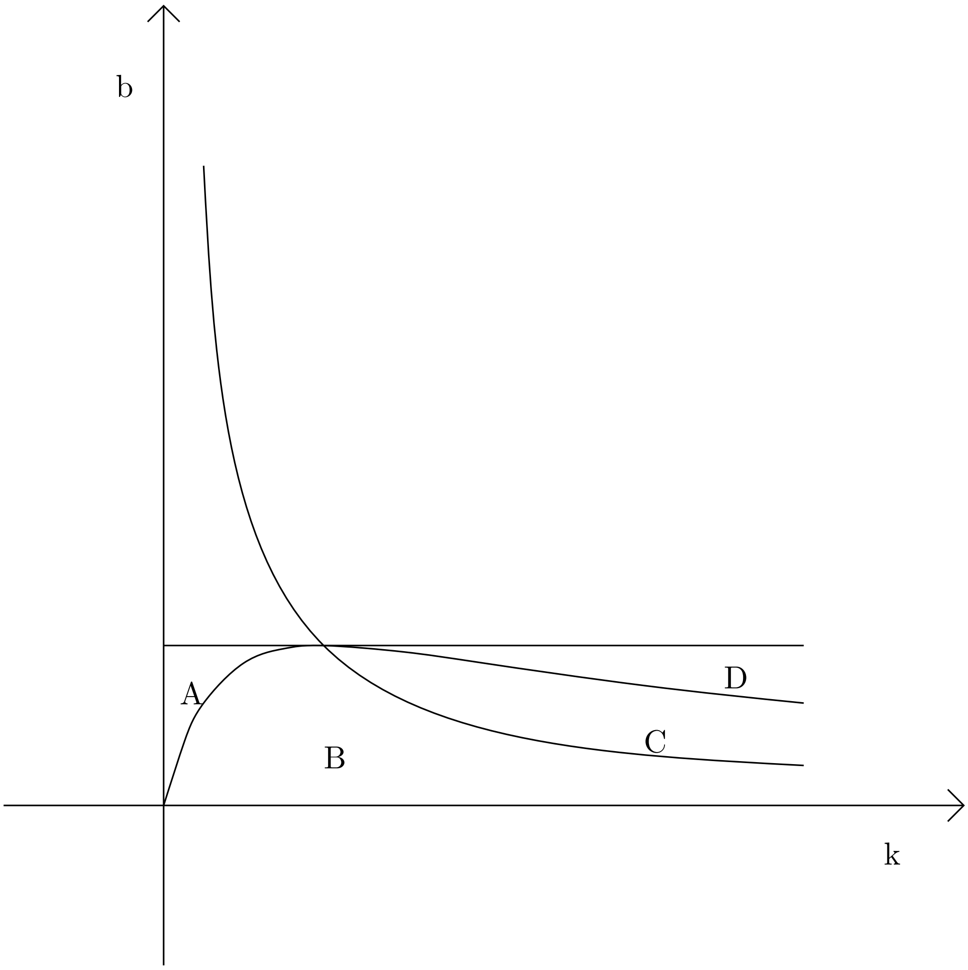
$$b < \frac{4k}{(1+k)^2} \tag{15.25}$$

We have stability in the areas A and B, with oscillations in B, while instability occurs in the areas C and D, oscillating in C.

EXAMPLE 5. Let π be salary and Y the national income. Then we assume the following two dynamic equations derived by the supply and demand, respectively,

$$\pi_t = \pi_{t-1} + \lambda(Y_t - Y^*) \tag{15.26}$$

$$\pi_t = m - \frac{1}{\phi}(Y_t - Y_{t-1}) \tag{15.27}$$



Stability of Samuelson's model

This means that the development has an equilibrium in the point $(\pi_t, Y_t) = (m, Y^*)$. These equations is conveniently rewritten in the form ignoring the equilibrium:

$$Y_t = \frac{1}{1+\lambda\phi} Y_{t-1} - \frac{\phi}{1+\lambda\phi} \pi_{t-1} \tag{15.28}$$

$$\pi_t = \frac{\lambda}{1+\lambda\phi} Y_{t-1} + \frac{1}{1+\lambda\phi} \pi_{t-1} \tag{15.29}$$

To figure out the behavior we just need to compute the product of the mixed coefficients, i.e.,

$$-1 < -\frac{\lambda\phi}{(1+\lambda\phi)^2} < 0 \tag{15.30}$$

And the look at the figure in chapter 13, to find the point

$$\left(\frac{1}{1+\lambda\phi}, \frac{1}{1+\lambda\phi}\right) \tag{15.31}$$

a point of alternating, but damped oscillations.

CHAPTER 16. Non–linear difference equations

A simple example of a non–linear difference equation is the following

$$f(n+1) = \lambda\, f(n)^2\,. \tag{16.1}$$

This is perhaps the only one we may solve explicitly, the solution is

$$f(n) = \frac{1}{\lambda} a^{2^n} \quad \text{with} \quad a = \lambda f(0)\,. \tag{16.2}$$

Already the complication of an added constant, $c \neq 0$, e.g.

$$f(n+1) = f(n)^2 + c \tag{16.3}$$

makes the equation unsolvable in explicit form for almost all values of c.

A simulation of the solution to (16.3) is obtained by successive computations of $f(1)$, $f(2)$, etc. We may consider the process as iteration of the function

$$y = x^2 + c \tag{16.4}$$

from different start points $x_0 = f(0)$. This is the simplest example of iteration of a non-linear function, depending on a parameter, c_0. In the general form is given a function

$$y = F_c(x) \tag{16.5}$$

and we consider the behavior of the sequences

$$x_{n+1} = F_c(x_n)\,, \qquad x_0 = a\,, \tag{16.6}$$

for different choices of a and c.

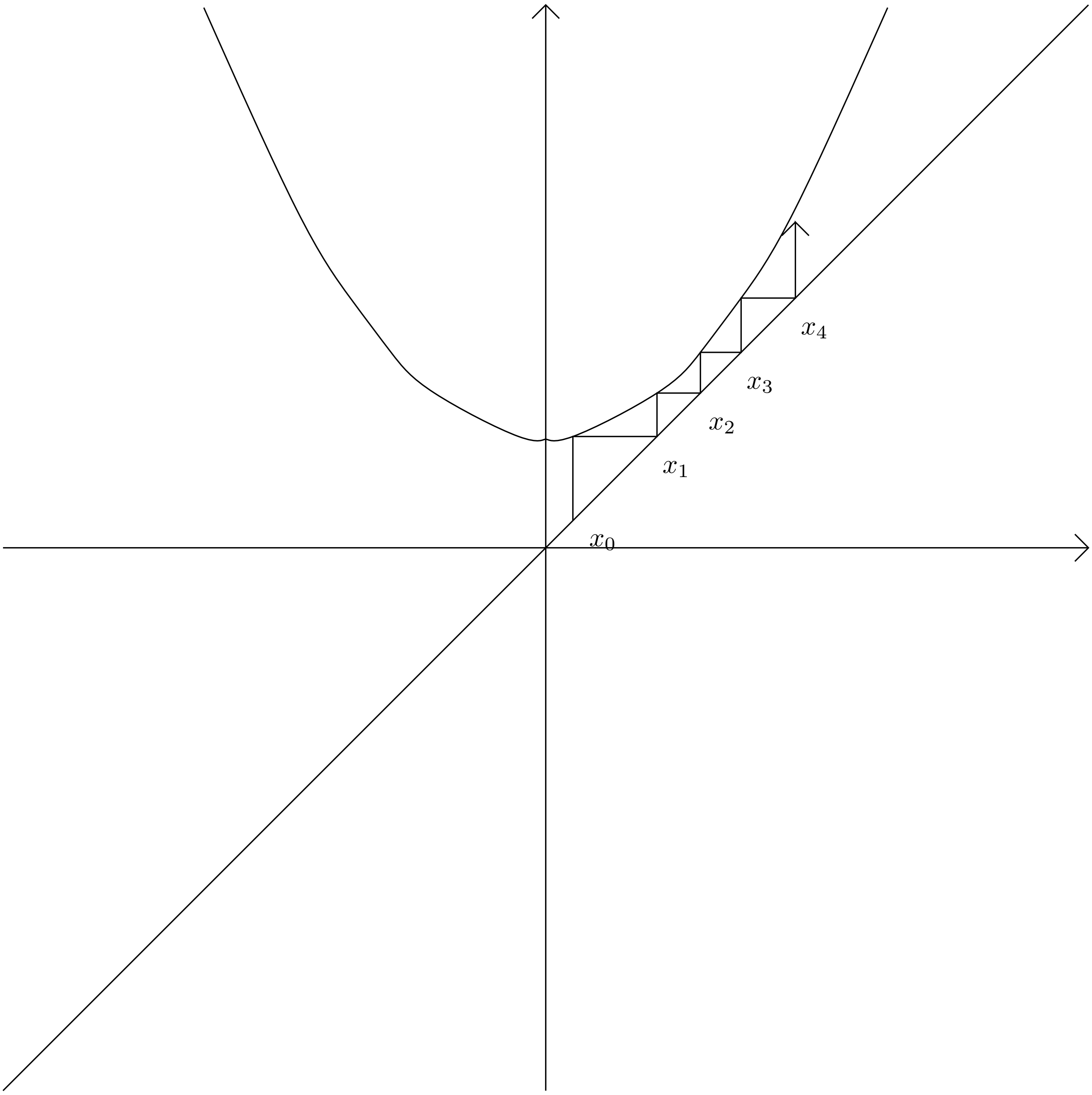
We shall consider the special case

$$F_c(x) = x^2 + c \tag{16.7}$$

in full detail.

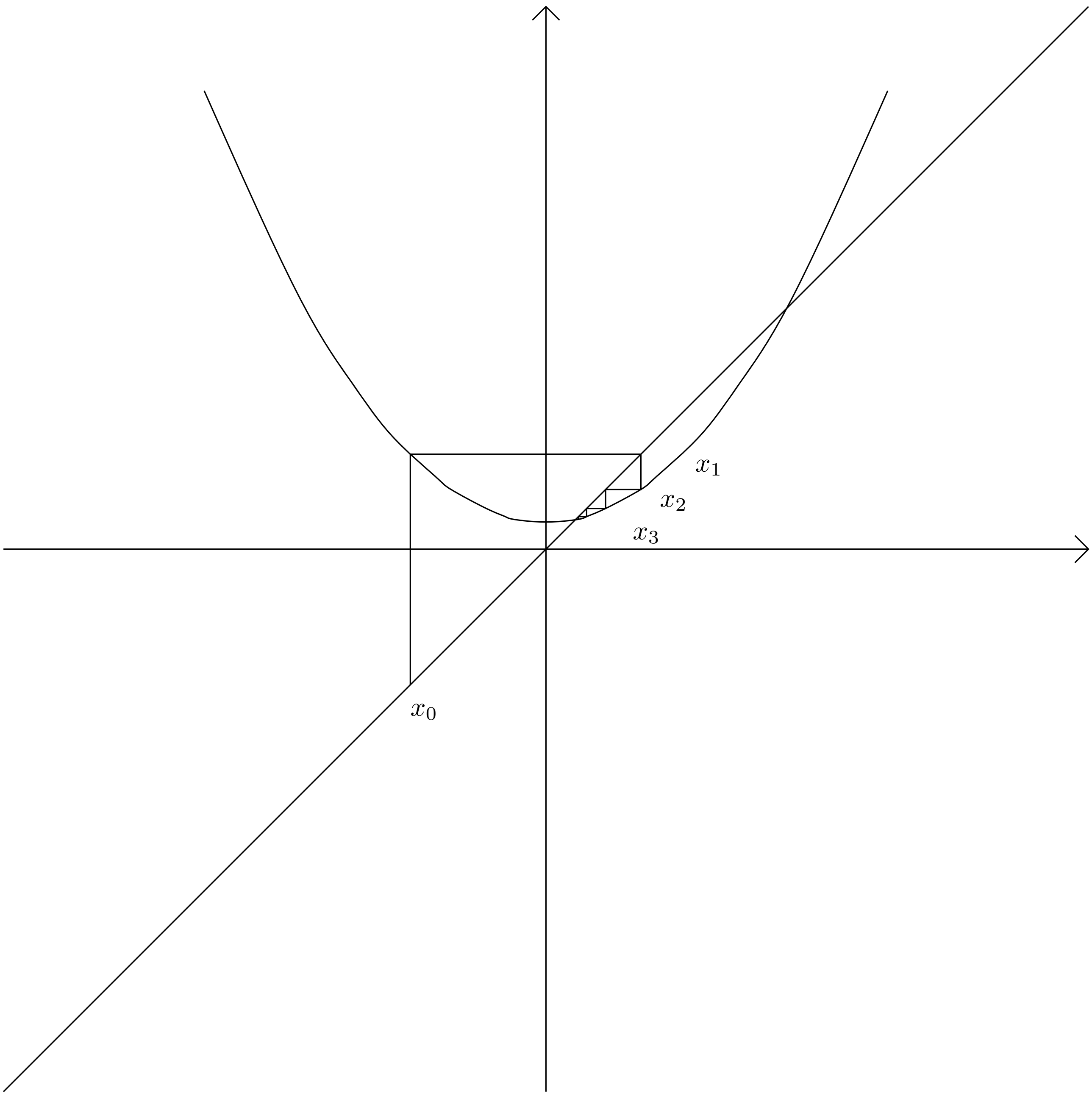
If $c > \frac{1}{4}$, then the parabola (16.4) does not cut the diagonal $y = x$, hence the iteration (16.6) has no fix point, a, with

$$a = F_c(a)\,. \tag{16.8}$$



Iteration of $x^2 + c \quad c > \frac{1}{4}$

Actually, every sequence, (16.6), diverges towards $+\infty$ in this case.
But for $-\frac{3}{4} < c < \frac{1}{4}$ the parabola cuts the diagonal twice:



Iteration of $x^2 + c$, $\quad -\frac{3}{4} < c < \frac{1}{4}$

The fix points solves the equation

$$x^2 + c = x \tag{16.9}$$

hence they are

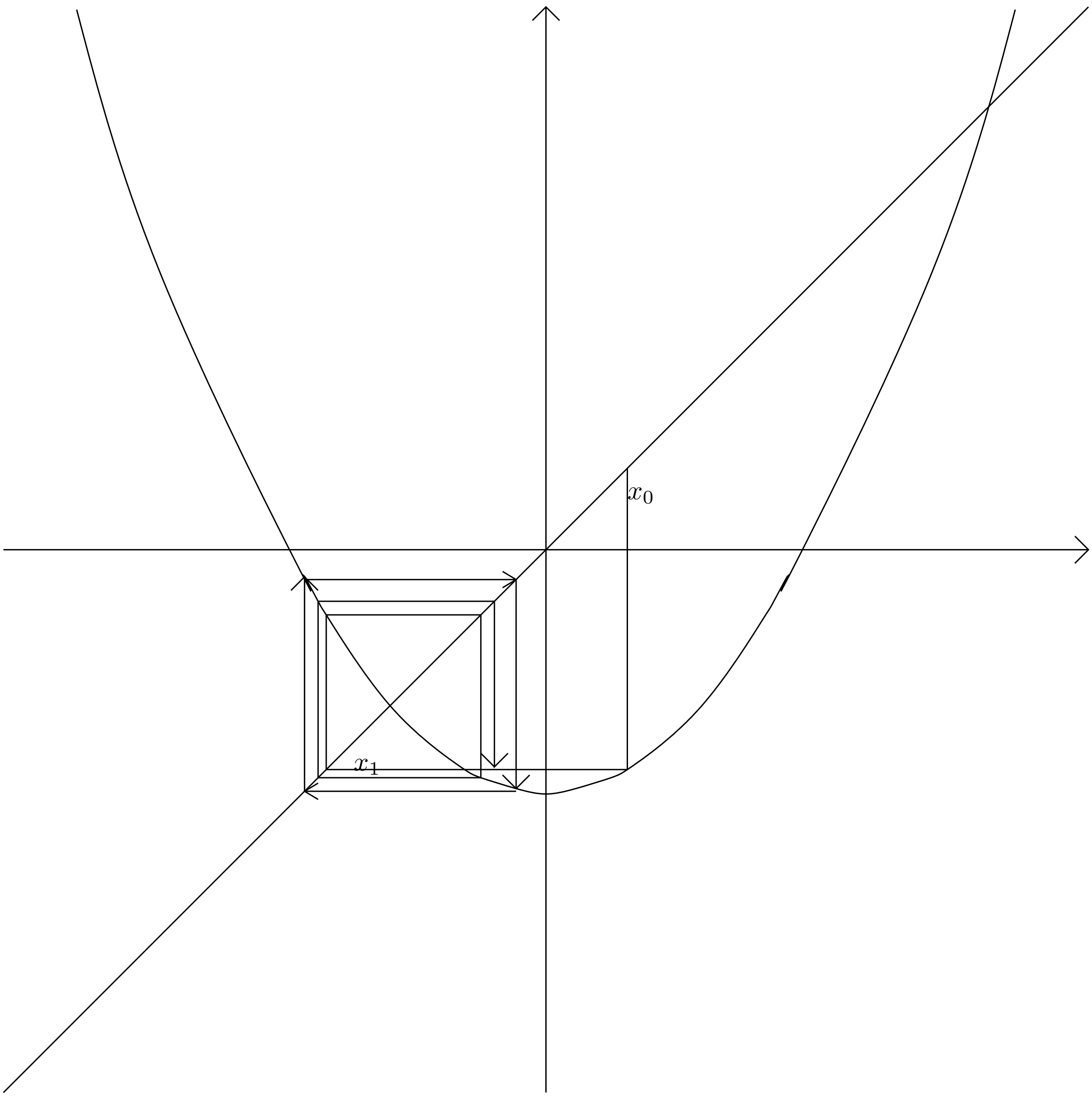
$$\alpha = \frac{1}{2} \pm \sqrt{\frac{1}{4} - c}\,. \tag{16.10}$$

Furthermore, the tangents to the parabola in these points have the slopes

$$2\alpha = 1 \pm \sqrt{1 - 4c} > -1 \tag{16.11}$$

making one of them attractive and the other repulsive, cf. Chapters 5–6.

But for $c < -\frac{3}{4}$, both slopes are numerically greater than 1, making both fix points repulsive. Where do the iterations then go?



Iteration of $x^2 + c \quad c < -\frac{3}{4}$

At least some of them will converge towards an attractor, consisting of a pair of points giving rise to a cycle of length 2. The iteration jumps back and forth, never going to rest.

This behavior is not so difficult to analyze. A point of period 2 will be a fixed point for the iterated function

$$y = F_c^2(x) = F_c(x^2+c) = (x^2+c)^2 + c = \\ = x^4 + 2c\,x^2 + c^2 + c\,. \tag{16.12}$$

To find a fix point for (16.12) is easier than one might expect. We shall solve

$$x^4 + 2c\,x^2 - x + c^2 + c = 0 \tag{16.13}$$

but this equation already has the zeros (16.10), hence we may write (16.13) as

$$(x^2 - x + c)(x^2 + x + c + 1) = 0\,. \tag{16.14}$$

The two new roots are simply

$$\beta = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - c - 1} = \frac{1}{2} \cdot \left(-1 \pm \sqrt{-4c-3}\right)\,. \tag{16.15}$$

Note that they are complex for $-\frac{3}{4} < c$, but appear in the real world for $c < -\frac{3}{4}$.

The next question is, if these points of period 2 make an attractive cycle or a repelling one ? This question is the same as to ask, if the fix points (16.15) are attractors or repellers for F_c^2 ? And this is a question, whether

$$\left|(F_c^2)'(\beta)\right| < 1 \tag{16.16}$$

or not.

Now, differentiating (16.12) yields

$$\left(F_c^2\right)'(x) = 4(x^2+c)x\,. \tag{16.17}$$

Because β satisfies the equation

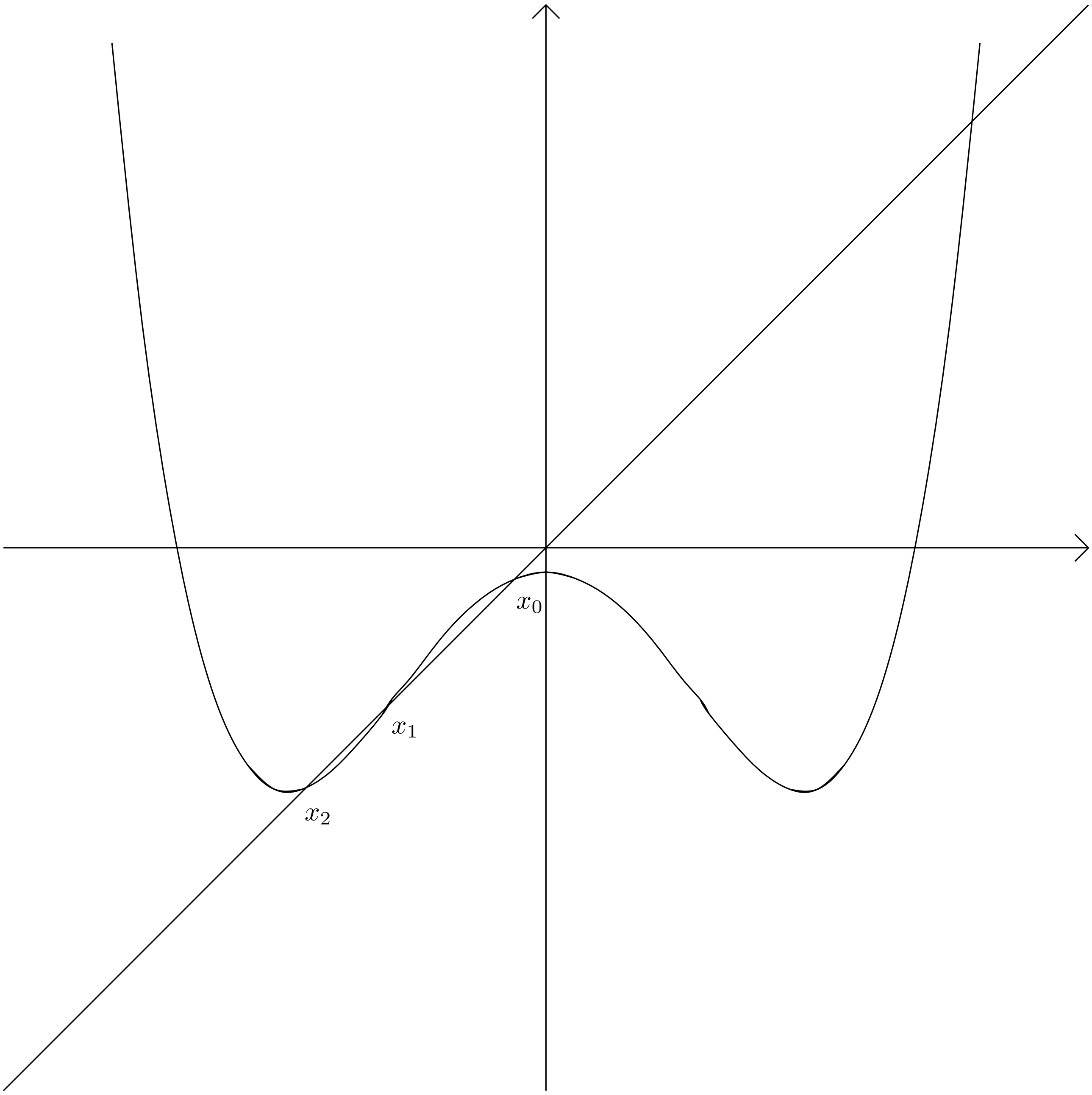
$$x^2 + x + c + 1 = 0 \tag{16.18}$$

we find

$$\left(F_c^2\right)'(\beta) = 4(-1-\beta)\beta = -4(\beta+\beta^2) = \\ = -4(-c-1) = 4(c+1)\,. \tag{16.19}$$

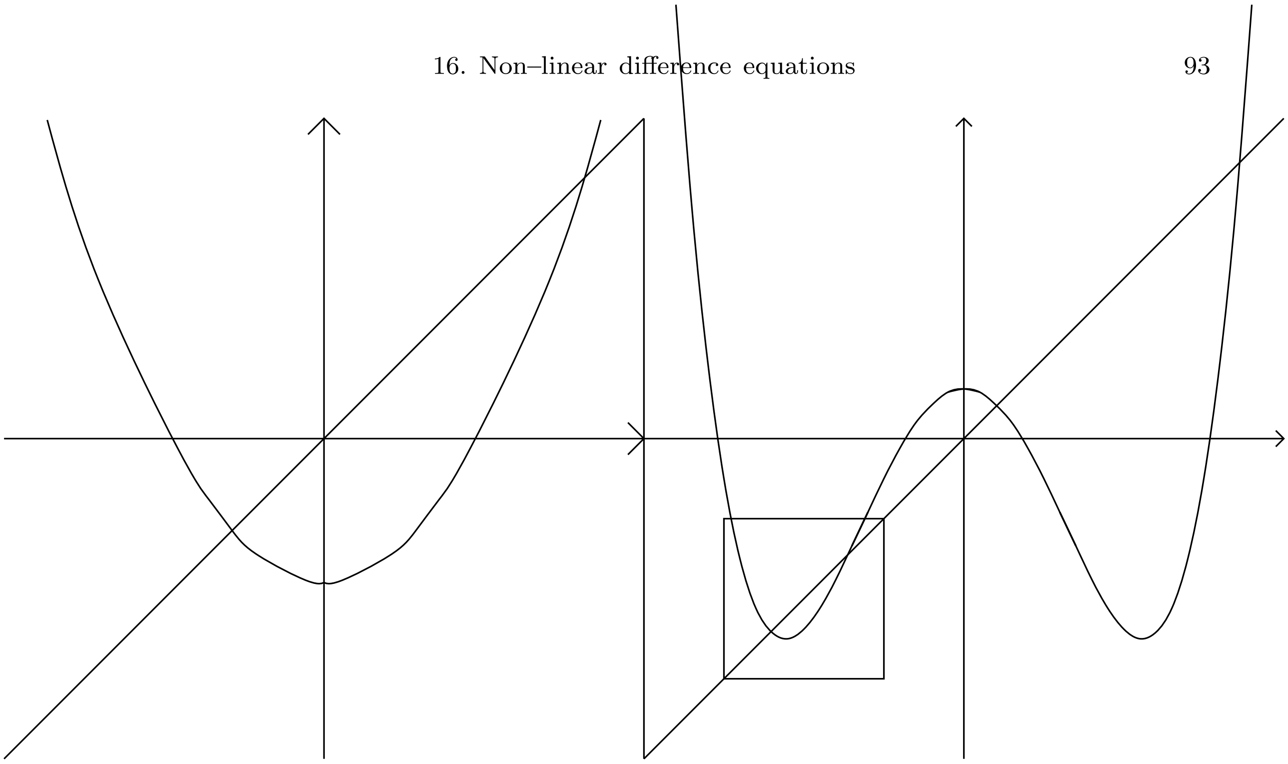
Hence (16.16) is satisfied if and only if

$$-\frac{5}{4} < c < -\frac{3}{4} \tag{16.20}$$



Graph of $(x^2+c)^2+c$

In particular, this means that as soon as c becomes smaller than $-\frac{5}{4}$, then the attractors β become repellers and F_c^2 gets an attracting cycle of length 2. This behavior is very similar to the behavior of F_c as c passed through $-\frac{3}{4}$.



$x^2 - \frac{3}{4}$ and $((x^2 - \frac{5}{4})^2 - \frac{5}{4}$

Looking at the graph of F_c^2 it is not so surprising; locally F_c^2 looks very much like F_c.

The process goes on, as c declines new periods of higher orders will appear, making the previous attractors into repellers. When c becomes smaller than ca. -1.5616 a point of period 3 appears. The dynamics becomes more and more "chaotic".

The case of $c = -2$ is of particular interest, but we shall postpone the discussion until later, in order to look at it from the other side, i.e. the behavior in the case of $c < -2$.

If we iterate

(16.21) $$F_2(x) = x^2 - 2$$

then for each x, $|x| \leq 2$, we have

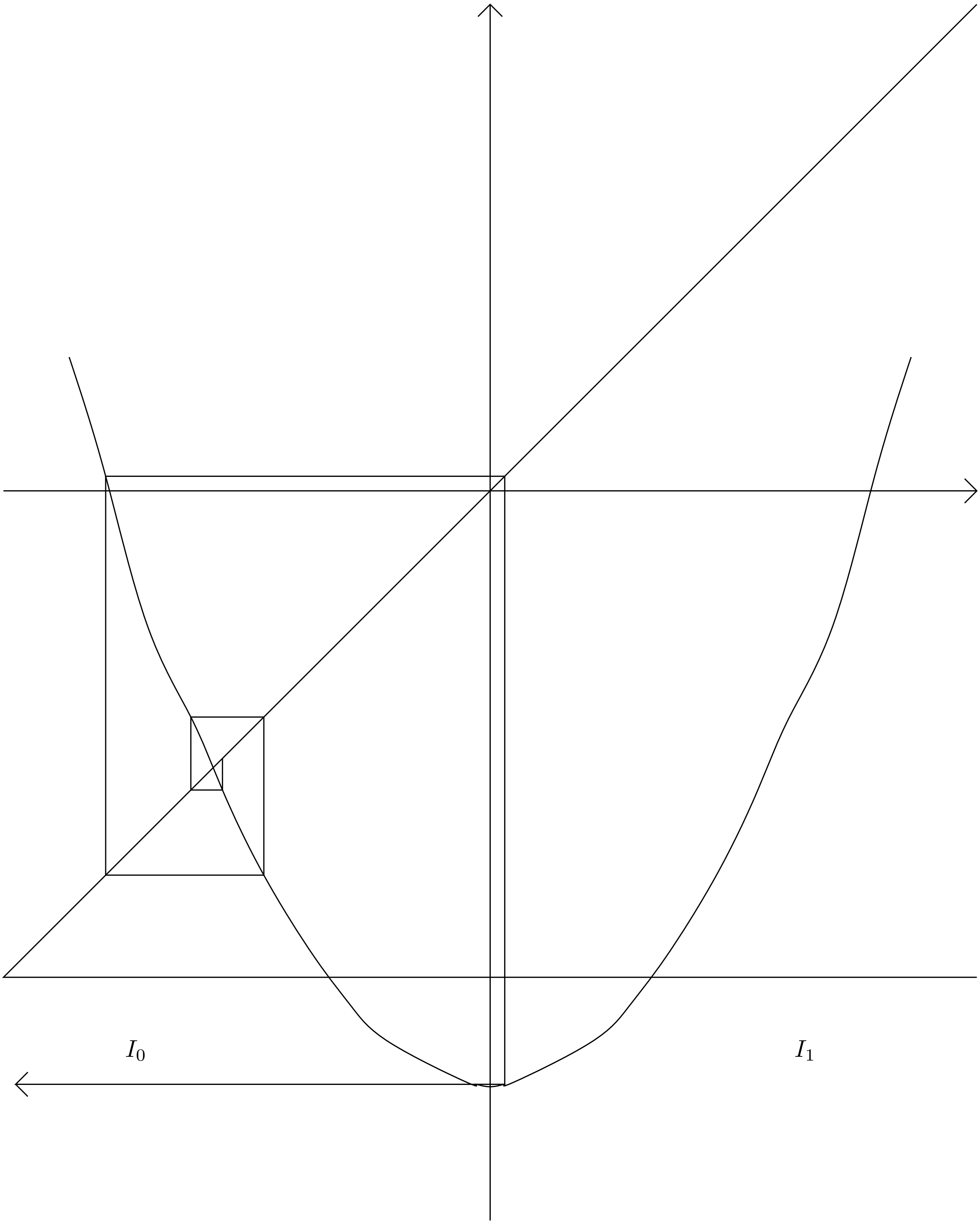
(16.22) $$|x| \leq 2 \Rightarrow |F_2(x)| \leq 2\,.$$

Hence, F_2 maps the interval $I = [-2, 2]$ into itself. But for $c < -2$, we may consider the iterated sequence starting from 0, i.e.

(16.23) $$0\,, \quad F_c(0) = c\,, \quad F_c(c) = c^2 + c$$

and note that for $c < -2$ we have

(16.24) $$c^2 + c > -c$$



$$x^2 + c, \quad c < -2$$

such that the sequence diverges towards infinity. This divergence also emerges from the neighborhood of 0.

This means that the interval

$$I = \left[-\frac{1}{2}(1+\sqrt{1-4c}) \,,\ \frac{1}{2}(1+\sqrt{1-4c}) \right] \tag{16.25}$$

is divided in three, of which the middle one around 0 diverges. But as is obvious from the figure, also the middle part of each of the two other intervals must eventually diverge.

We shall define the set of not diverging points of start,

$$\Lambda = \{ x \in I \mid F_c^k(x) \in I \,,\ k = 0, 1, \dots \} \,. \tag{16.26}$$

(This set is not empty, because it contains all points of finite period.)

We divided I in 3 subintervals,

$$I_0 = \left[-\frac{1}{2}(1+\sqrt{1-4c}),\ -\sqrt{-c-2} \right] \tag{16.27}$$

$$I_1 = \left[\sqrt{-c-2},\ \frac{1}{2}(1+\sqrt{1-4c}) \right] \tag{16.28}$$

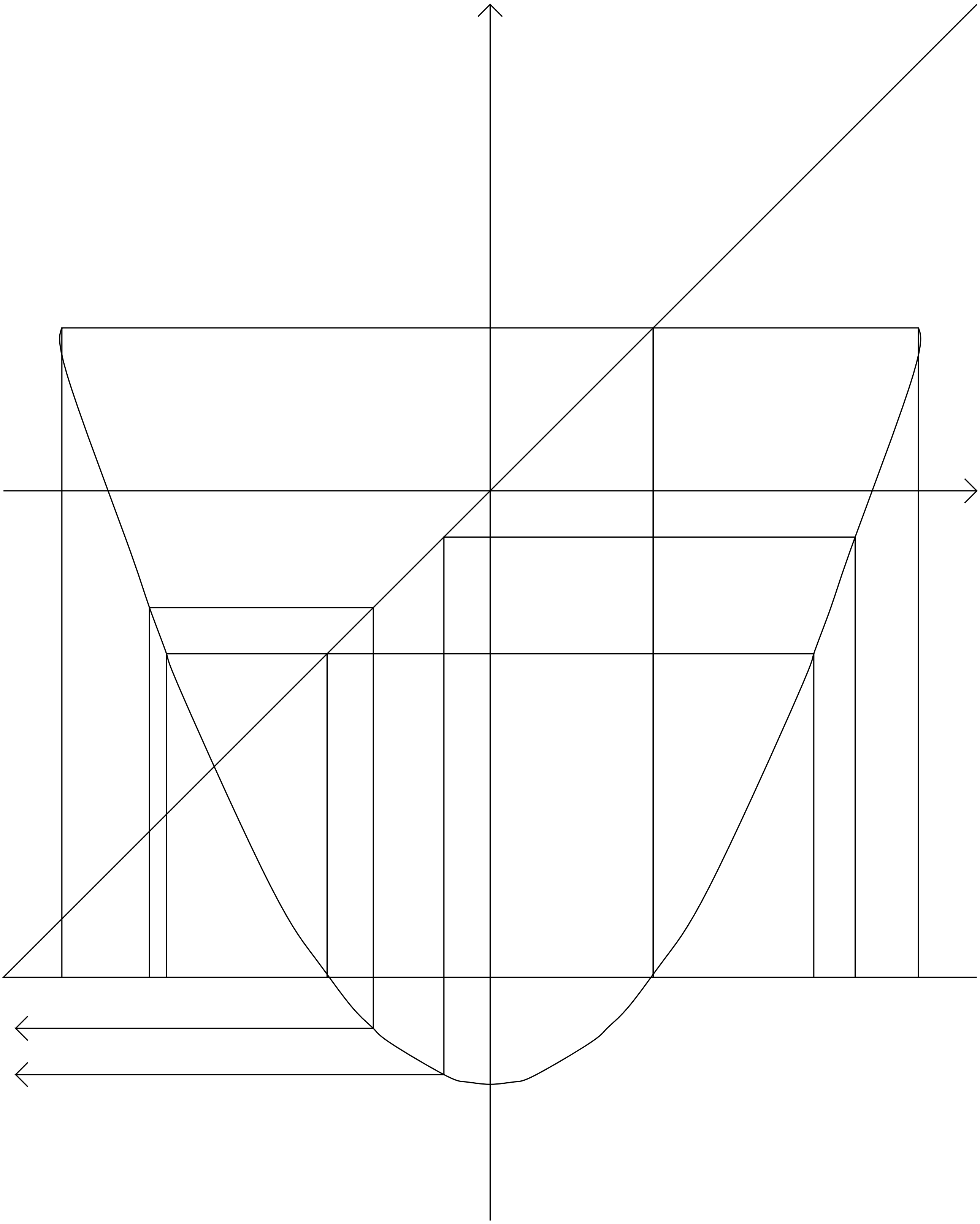
$$M = \left] -\sqrt{-c-2} \,,\ \sqrt{-c-2} \right[\tag{16.29}$$

Now,

$$F_c(M) \cap I = \varnothing \tag{16.30}$$

and points outside I shall never return to I by iteration of F_c. But

$$F_c(I_0) = F_c(I_1) = I \,. \tag{16.31}$$



$$x^2 + c, \quad c < -2$$

Hence there are fractal similarity between the intervals I_0, I_1. We construct

Λ by omitting the middle intervals successively

$$M_0 = I_0 \cap F_c^{-1}(M) \tag{16.32}$$

$$M_1 = I_1 \cap F_c^{-1}(M) \tag{16.33}$$

etc. The set Λ left over is a Cantor set. For $x \in \Lambda$ we define a sequence of 0 and 1

$$s(x) = s_0 s_1 s_2 \ldots \tag{16.34}$$

by the definition

$$s_j(x) = \begin{cases} 0 & \text{if} \quad F_c^j(x) \in I_0 \\ 1 & \text{if} \quad F_c^j(x) \in I_1 \end{cases} \tag{16.35}$$

which is an orbit description of the sequence $\left(F_c^j(x)\right)$.

The set Λ is closed, because the complement is open. All endpoints of the intervals belong to Λ and are characterized by the sequences ending with $111\ldots$. These points are dense in Λ, so every sequence defines a point in Λ.

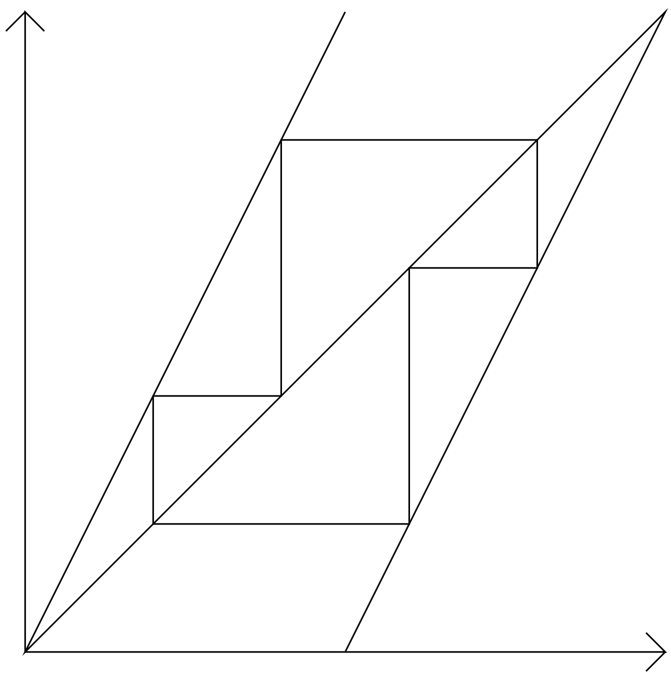
If we consider the sequences as binary numbers representing the interval $[0,1]$, then we may interpret F_c in a useful way. We have

$$F_c(s_0 s_1 \ldots) = s_1 s_2 \ldots \tag{16.36}$$

This function is multiplication with 2 and subtracting s_0, i.e.

$$g(x) = 2x \mod 1$$

This way of describing the dynamics is most convenient. If we iterate the function g, it is obvious, that the number of fixed points for g^n is 2^n. This is the number of points of period n for g, counting those points which period is a divisor in n.



$2x \mod 1$ med følgen fra $a = \frac{1}{5}$

It is obvious that all these points are repelling, having slopes steeper than one. This function (16.37) is the typical chaos.

Now, let us return to $c = -2$. The function

$$f(x) = x^2 - 2 \tag{16.38}$$

may be considered as a transformation of

$$g(\theta) = 2\theta \mod 1 \tag{16.39}$$

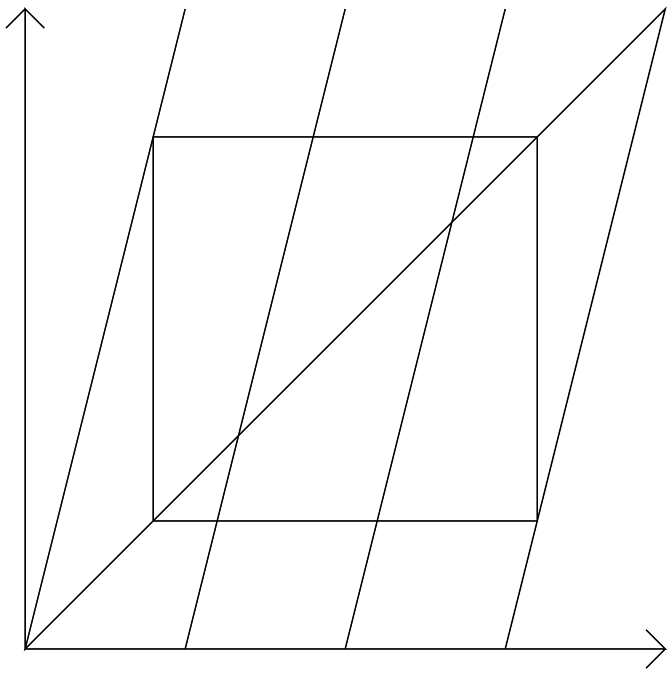
by the transform

$$x = c(\theta) = 2\cos(\theta 2\pi)\,. \tag{16.40}$$

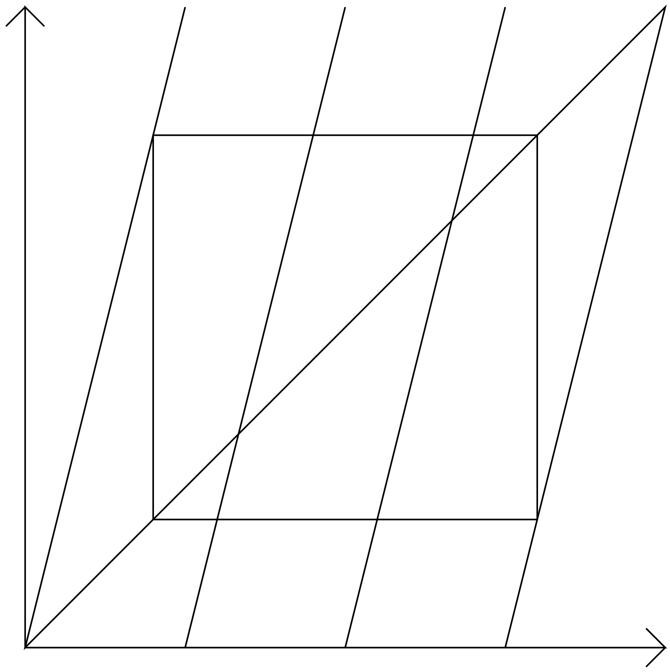
Actually we get

$$\begin{aligned} f(c(\theta)) = f(x) &= 4\cos^2\theta 2\pi - 2 \\ &= 2\cos 2\theta 2\pi \\ &= 2\cos(g(\theta)2\pi) \\ &= c(g(\theta)) \end{aligned} \tag{16.41}$$

Hence f exhibits exactly the same chaos as g does.



2(2x mod 1) mod 1



2(2x mod 1) mod 1

CHAPTER 17. Complex dynamics

We consider the function

$$F(z) = z^2 + c \tag{17.1}$$

almost as before, except that this time z and c are allowed to be complex numbers. This difference makes the orbit a sequence of points in the plane.

In the simple case of $c = 0$ we shall see three types of behavior

$$\begin{aligned} |z| < 1: &\qquad F^n(z) \to 0 \\ |z| = 1: &\qquad |F^n(z)| = 1 \\ |z| > 1: &\qquad |F^n(z)| \to \infty\,. \end{aligned} \tag{17.2}$$

This behavior is easy to describe; let $z = re^{i\theta}$, then $F(z) = z^2 = r^2 e^{i2\theta}$ and hence

$$F^n(z) = r^{2^n}\, e^{i2^n\theta}\,. \tag{17.3}$$

If $r = 1$, then it is crucial whether $\theta = \frac{p\pi}{2^m}$ for $p \in \mathbb{Z}$, $m \in \mathbb{N}_0$ or not. Hence the behavior is chaotic.

Let us further consider the transform

$$H(z) = z + \tfrac{1}{z}\,. \tag{17.4}$$

This transform changes the map above

$$z \to z^2 \tag{17.5}$$

into the map

$$z + \tfrac{1}{z} \to z^2 + \tfrac{1}{z^2} = (z + \tfrac{1}{z})^2 - 2\,. \tag{17.6}$$

Hence, the map

$$z \to z^2 - 2 \tag{17.7}$$

is dynamically similar to (17.5).

Definition. For a function $F(z)$ we define the *filled-in Julia set* as

$$K = \{z \mid F^n(z) \not\to \infty\} \tag{17.8}$$

and the *Julia set* as its boundary

$$J = \partial K\,. \tag{17.9}$$

Examples. For $F(z) = z^2$ we have

$$K = \{z \mid |z| \le 1\} \tag{17.10}$$

$$J = \partial K = \{z \mid |z| = 1\} \tag{17.11}$$

and for $F(z) = z^2 - 2$ we have

$$K = J = [-2, 2]\ . \tag{17.11}$$

(If $z_n \to 0$ then $H(z_n) \to \infty$).

Suppose $|c| > 2$. If we start in a point z with $|z| \ge |c|$, then $F^n(z) \to \infty$. Actually

$$\begin{aligned} |z^2 + c| &\ge |z|^2 - |c| \\ &\ge |z|^2 - |z| \\ &= |z|(|z| - 1) \\ &\ge (|c| - 1)|z| \end{aligned}$$

where $|c| - 1 > 1$. Hence

$$F^n(z) \ge (|c| - 1)^n |z| \to \infty\,.$$

Conclusion.

$$K \subseteq D(0, |c|)\,. \tag{17.12}$$

In order to find K, we must ask, which points in $D(0, |c|)$ are mapped outside the disc by F ?

So, we ask, what is F^{-1} of the circle ?

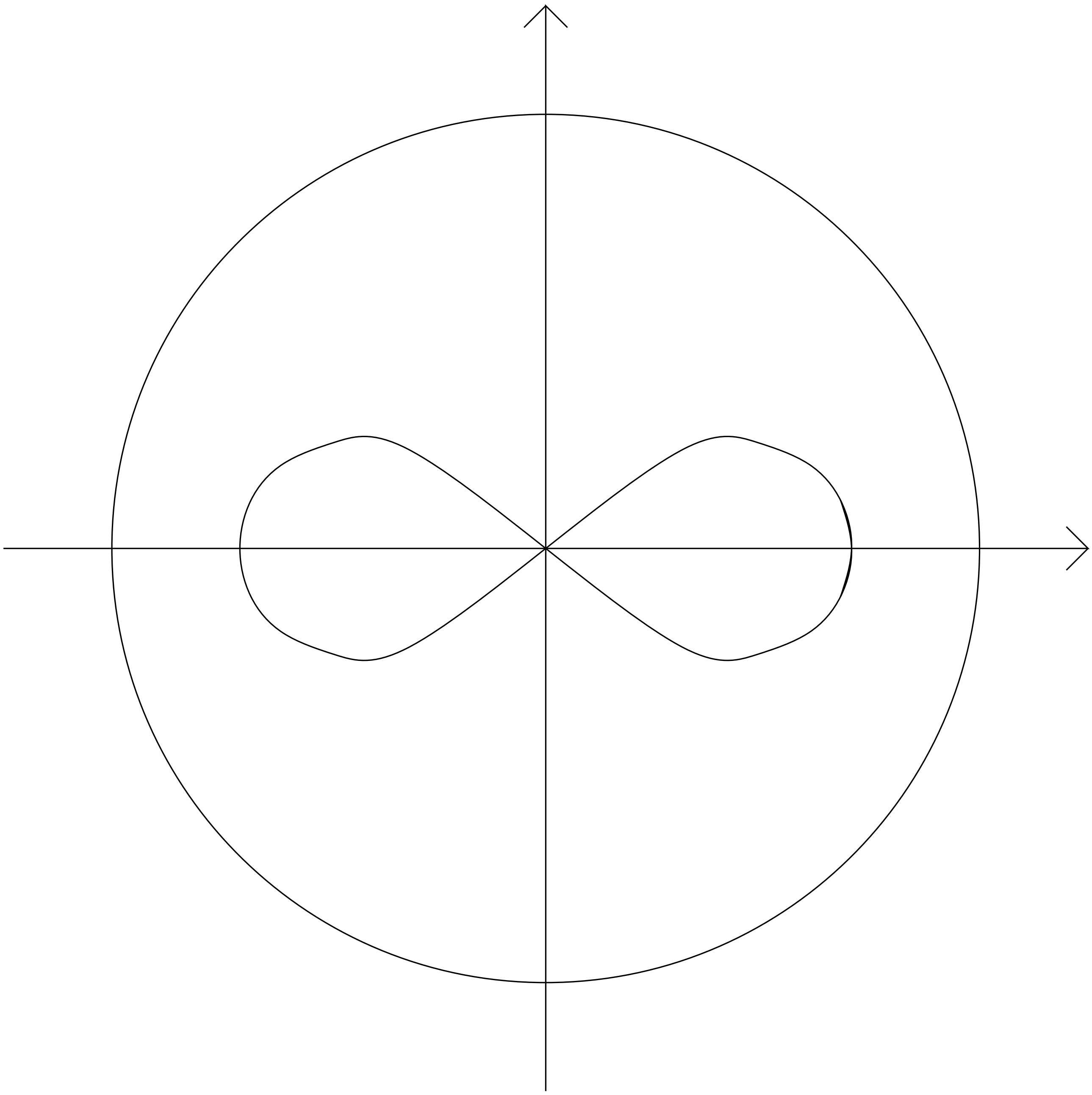
$$w = z^2 + c \iff z = \pm\sqrt{w - c}\,. \tag{17.13}$$

What is the curve of z, as $w = |c|e^{i\theta}$? Now, if $c = |c|e^{i\gamma}$, then we get

$$\begin{aligned} z &= \pm\sqrt{|c|} \cdot \sqrt{e^{i\theta} - e^{i\gamma}} \\ &= \pm\sqrt{|c|} \cdot \sqrt{e^{i(\theta-\gamma)} - 1} \cdot e^{i\frac{\gamma}{2}}\,. \end{aligned}$$

This gives a shape like the number "8" with 0 in the center and turned the angle $\frac{\gamma}{2}$. The diameter is $2\sqrt{2|c|} < 2|c|$, so it is inside the disc.

This means, that K belongs to the interior of this "number", 8. Repetition of the process, divides K in two in each step making K a complex Counter set. On K the dynamics is chaotic.



$$\pm 2\sqrt{e^{i\phi}-1}, \quad \phi \in [0, 2\pi]$$

The structure of K depends on the value c. For $c \in \mathbb{R}$, $-2 \leq c \leq \frac{1}{4}$, the set K is connected. This is related to the question, how behaves the orbit of 0? The fact is that iff the orbit of 0 is bounded then the Julia set is connected.

Hence it is interesting to ask, what is

$$M = \{c \mid \text{ orbit of } 0 \text{ by } z^2 + c \text{ is bounded}\}\,. \tag{17.15}$$

Definition. The set M of (17.15) is called the *Mandelbrot* set.

An interesting question to ask is, whether the function F has an attracting fix point ? We know this to be the case for $c \in \mathbb{R}$, $-\frac{3}{4} < c < \frac{1}{4}$.

So, in general, a fix point solves

$$z^2 + c = z\,. \tag{17.16}$$

And it is attractive if the derivative of F, i.e. $2z$, is numerically less than 1. So, we must have

$$c = \tfrac{1}{2}(2z) - \tfrac{1}{4}(2z)^2\,, \qquad |2z| < 1\,. \tag{17.17}$$

This means that c lies inside a cardioid.

The next question is whether F has an attractive 2-cycle, as we know it has for $c \in \mathbb{R}$, $-\frac{5}{4} < c < -\frac{3}{4}$. This time z solves

$$(z^2 + c)^2 + c = z \tag{17.18}$$

and again, $|F'(z)| = |4z(z^2 + c)| < 1$.

This is the case, if z solves (17.18) but not (17.16), and therefore solves

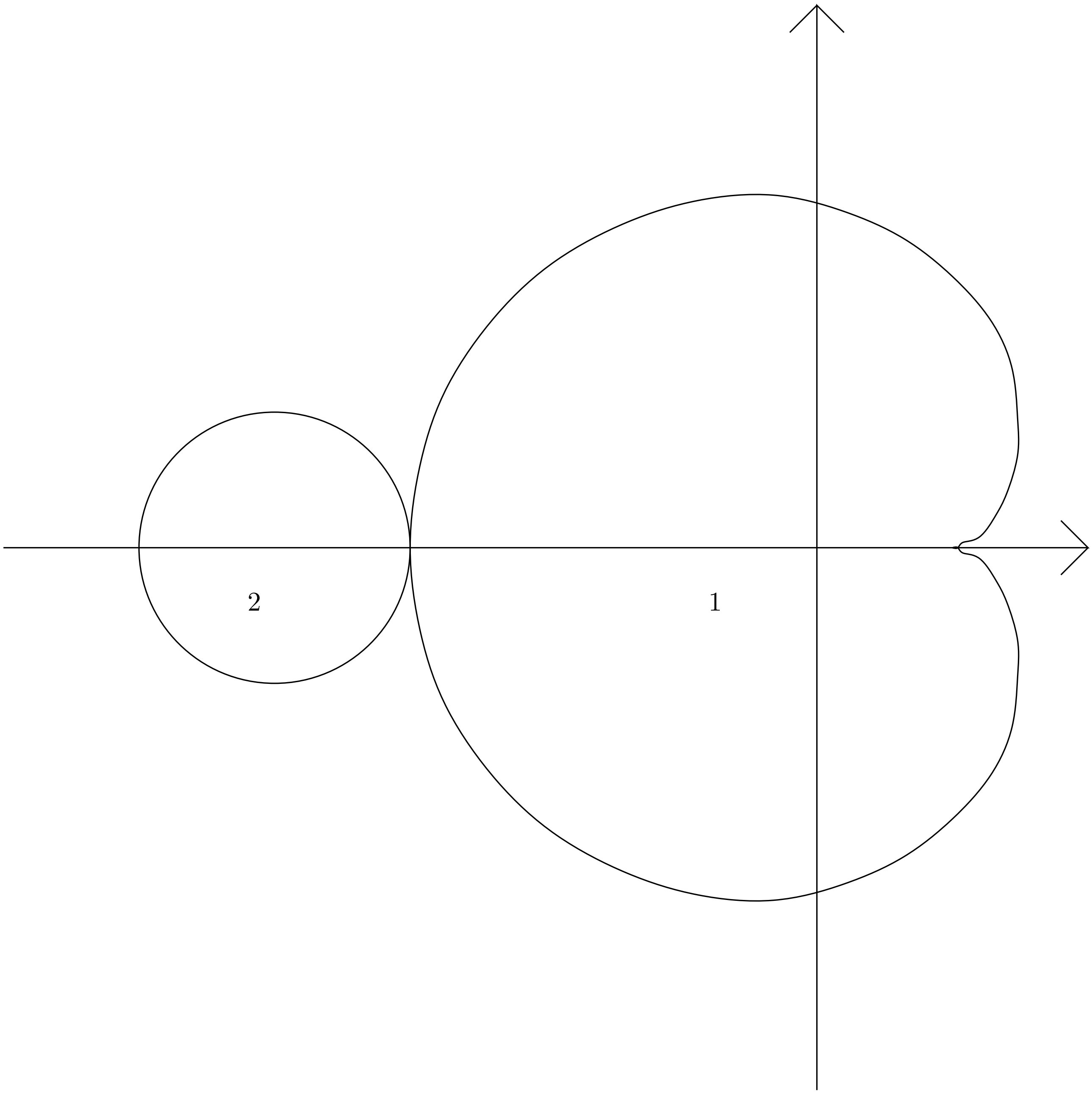
$$z^2 + z + c + 1 = 0 \tag{17.19}$$

which gives

$$|F'(z)| = |4z(-1 - z)| = |4(c + 1)| < 1 \tag{17.20}$$

or, c belongs to the disc

$$c \in D\left(-1, \tfrac{1}{4}\right)\,.$$



Cardioid and circel, the beginning of the Mandelbrot set

CHAPTER 18. Economical examples of non–linear equations

These examples are taken from [4].

EXAMPLE 1. Consider the example 1 of Chapter 6. Suppose that one of the two equations is quadratic, e.g., that the equations are

$$D_t = a + bp_t \qquad \text{with} \qquad b < 0 \tag{18.1}$$

$$S_t = c + dp_{t-1} + ep_{t-1}^2 \qquad \text{with} \qquad d > 0, e < 0 \tag{18.2}$$

Then the equality of demand and supply establishes the iteration

$$p_t = \frac{c-a}{b} + \frac{d}{b}\, p_{t-1} + \frac{e}{b}\, p_{t-1}^2 \tag{18.3}$$

which solution may approach limit cycles or be chaotic depending on the parameters, a, b, c, d, e.

EXAMPLE 2. Rößler suggests a model for three variables, v, the rate of employment, measured in deviations from 90%, u, the unit labor cost, and z, the public net income generating expenditure. The model is formulated in differential equations as

$$\begin{aligned} \dot{u} &= 0.5v \\ \dot{v} &= -0.5u + 0.15v - 0.3z \\ \dot{z} &= 0.01 + 85z(v - 0.05) \end{aligned} \tag{18.4}$$

The behavior of this may be considered as a mutual wave of u and v, disturbed by some slow changes in the size of z. We may consider the two first equations as approximately linear,

$$\begin{aligned} \dot{u} &= 0.5v \\ \dot{v} &= -0.5(u + 0.6z) + 0.15v \end{aligned} \tag{18.5}$$

The matrix of coefficients a is

$$A = \begin{pmatrix} 0 & .5 \\ -0.5 & 0.15 \end{pmatrix} \tag{18.6}$$

Of type $b > 0 > c$ and the point $(0, 0.15)$ inside the strip of negative discriminant, but with positive half trace, giving rise to a growing oscillation. During this process, the center of spiral will move with z.

CHAPTER 19. Difference equations for polynomials

In principle, the formula (1.9) allows the solution to all equations of the form

$$\Delta f(x) = g(x) \tag{19.1}$$

with $g(x)$ a polynomial in x. The polynomial solutions $f(x)$ are uniquely determined up to a constant. In order to use this formula, we must be able to write an arbitrary polynomial in terms of the basic polynomials, the factorials, rather than the usual monomials.

This is nothing but a change of basis, to be done with the appropriate matrix. We have simply:

$$\begin{pmatrix} x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \\ x^6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 \\ 1 & 7 & 6 & 1 & 0 & 0 \\ 1 & 15 & 25 & 10 & 1 & 0 \\ 1 & 31 & 90 & 65 & 15 & 1 \end{pmatrix} \cdot \begin{pmatrix} [x]_1 \\ [x]_2 \\ [x]_3 \\ [x]_4 \\ [x]_5 \\ [x]_6 \end{pmatrix} \tag{19.2}$$

This matrix can be extended infinitely, see. [1]. It simply says that e.g.,

$$x^5 = 1 \times [x]_1 + 15 \times [x]_2 + 25 \times [x]_3 + 10 \times [x]_4 + 0 \times [x]_5$$

Definition 1. The entries in the matrix in (19.2) are called Stirling numbers of the second kind, and are denoted as $\mathfrak{S}_n^{(k)}$ when they appear as coefficients in the formula:

$$x^n = \sum_{k=1}^{n} \mathfrak{S}_n^{(k)} [x]_k \tag{19.3}$$

From (19.3) we get by multiplying with x,

$$\begin{aligned} x^{n+1} &= \sum_{k=1}^{n} \mathfrak{S}_n^{(k)} x[x]_k \\ &= \sum_{k=1}^{n} \mathfrak{S}_n^{(k)} (x-k+k)[x]_k \\ &= \sum_{k=1}^{n} \mathfrak{S}_n^{(k)} [x]_{k+1} + \sum_{k=1}^{n} \mathfrak{S}_n^{(k)} k[x]_k \\ &= \sum_{k=1}^{n+1} \mathfrak{S}_n^{(k-1)} [x]_k + \sum_{k=1}^{n} \mathfrak{S}_n^{(k)} k[x]_k \\ &= \sum_{k=1}^{n+1} \left(\mathfrak{S}_n^{(k-1)} + k\mathfrak{S}_n^{(k)} \right) [x]_k \end{aligned}$$

From this we derive the recurrence formula:

$$\mathfrak{S}_{n+1}^{(k)} = \mathfrak{S}_n^{(k-1)} + k\mathfrak{S}_n^{(k)} \tag{19.4}$$

Definition 2. The Stirling numbers of the first kind are the solutions to the inverse problem, i.e., the coefficients to the expressions of the factorials, $[x]_n$, in terms of the monomials, x^k, i.e.

$$[x]_n = \sum_{k=1}^{n} S_n^{(k)} x^k \tag{19.5}$$

They may conveniently be arranged in a matrix too:

$$\begin{pmatrix} [x]_1 \\ [x]_2 \\ [x]_3 \\ [x]_4 \\ [x]_5 \\ [x]_6 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 2 & -3 & 1 & 0 & 0 & 0 \\ -6 & 11 & -6 & 1 & 0 & 0 \\ 24 & -50 & 35 & -10 & 1 & 0 \\ -120 & 274 & -225 & 85 & -15 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \\ x^6 \end{pmatrix} = \tag{19.6}$$

This matrix just states, that e.g.,

$$[x]_5 = x^5 - 10x^4 + 35x^3 - 50x^4 + 24x \tag{19.7}$$

The matrix is simply the inverse of the matrix in (19.2). These numbers may be found in [1] too.

If we multiply (19.5) with $x - n$, we obtain

$$\begin{aligned} [x]_{n+1} &= \sum_{k=1}^{n} S_n^{(k)} (x-n) x^k \\ &= \sum_{k=1}^{n} S_n^{(k)} x^{k+1} - \sum_{k=1}^{n} S_n^{(k)} n x^k \\ &= \sum_{k=1}^{n+1} \left(S_n^{(k-1)} - n S_n^{(k)} \right) x^k \end{aligned} \tag{19.8}$$

From this we derive the recurrence formula:

$$S_{n+1}^{(k)} = S_n^{(k-1)} - n S_n^{(k)} \tag{19.9}$$

However, it is a cumbersome task to solve (19.1) with the help of these transformations. Hence, it is tempting to ask for the basic solutions to the equations of form (19.1) with monomials on the right side. It proves convenient to choose them as: $g(x) = nx^{n-1}$.

The polynomial solutions to the equations,

$$\Delta f_n(x) = nx^{n-1}, \quad n \in \mathbb{N} \tag{19.10}$$

are uniquely determined up to the constant term. Differentiation of (19.10) with respect to x yields

$$\Delta f_n'(x) = n(n-1)x^{n-2} \tag{19.11}$$

proving that $\frac{1}{n} f_n'(x)$ solves (19.10) for $n-1$, hence that

$$\frac{1}{n} f_n'(x) - f_{n-1}(x) \tag{19.12}$$

is a constant.

Definition 3. The choice of polynomial solutions to (19.10) for which the constant term in (19.12) is zero, are called the Bernoulli polynomials and are denoted as $B_n(x)$.

Suppose we have written the Bernoulli polynomials on the form

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B^n_{n-k} x^k \tag{19.13}$$

for suitable constants B^n_{n-k}. The index is chosen to have B^n_0 as coefficient to the leading term, x^n, and B^n_n as the constant term.

Differentiation of B_n yields

$$\begin{aligned} B'_n(x) &= \sum_{k=1}^{n} \binom{n}{k} B^n_{n-k} k x^{k-1} \\ &= \sum_{k=1}^{n} n \binom{n-1}{k-1} B^n_{n-k} x^{k-1} \\ &= n \sum_{k=0}^{n-1} \binom{n-}{k} B^n_{n-1-k} x^k \end{aligned}$$

From (19.12) this is known to be equal to

$$nB_{n-1}(x) = n \sum_{k=0}^{n-1} \binom{n-1}{k} B^{n-1}_{n-1-k} x^k$$

The conclusion from comparing the coefficients is, that

$$B^n_{n-1-k} = B^{n-1}_{n-1-k}, \qquad \text{for } k = 0, 1, \cdots, n-1 \tag{19.14}$$

Definition 4. The common value in (19.14) is called the Bernoulli numbers, and are denoted as B_{n-1-k}, omitting the superflous superscript.

Hence we may rewrite (19.13) as

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} x^k \tag{19.15}$$

such that the Bernoulli numbers take their appropriate part of the description of the Bernoulli polynomials. The fact that they satisfy (19.10) may lead to a computation of their coefficients. We compute

$$
\begin{aligned}
\Delta B_n(x) &= \sum_{k=0}^{n} \binom{n}{k} B_{n-k}((x+1)^k - x^k) \\
&= \sum_{k=0}^{n} \binom{n}{k} B_{n-k} \sum_{j=0}^{k-1} \binom{k}{j} x^j \\
&= \sum_{j=0}^{n-1} x^j \sum_{k=j+1}^{n} \binom{n}{k}\binom{k}{j} B_{n-k} \\
&= \sum_{j=0}^{n-1} x^j \sum_{k=j+1}^{n} \binom{n}{j}\binom{n-j}{k-j} B_{n-k} \\
&= \sum_{j=0}^{n-1} \binom{n}{j} x^j \sum_{k=j+1}^{n} \binom{n-j}{k-j} B_{n-k} \\
&= \sum_{j=0}^{n-1} \binom{n}{j} x^j \sum_{k=1}^{n-j} \binom{n-j}{k} B_{n-j-k} \\
&= \sum_{i=1}^{n} \binom{n}{i} x^n - i \sum_{k=1}^{i} \binom{i}{k} B_{i-k} \\
&= \sum_{i=1}^{n} \binom{n}{i} x^n - i \sum_{j=0}^{i-1} \binom{i}{j} B_j
\end{aligned}
\tag{19.16}
$$

where we have applied the formula

$$
\binom{x}{n}\binom{n}{m} = \binom{x}{m}\binom{x-m}{n-m} \qquad n, m \in \mathbb{N}_0 \tag{19.17}
$$

and reversed summationvariables, e.g., $i = n - j$ and $j = i - k$.

Now we know from (19.10) that this final polynomial equals nx^{n-1}. This means that the coefficients are n for $i = 1$ and 0 else. This gives the formulas for the Bernoulli numbers:

$$
\begin{aligned}
B_0 &= 1 \\
\sum_{j=0}^{i-1} \binom{i}{j} B_j &= 0 \text{ for } i > 1
\end{aligned}
\tag{19.18}
$$

Sometimes people like to confuse the reader by adding the number B_i to the last sum to get the "implicit" recursion formula

$$
\sum_{j=0}^{i} \binom{i}{j} B_j = B_i \tag{19.19}
$$

maybe it looks nicer.

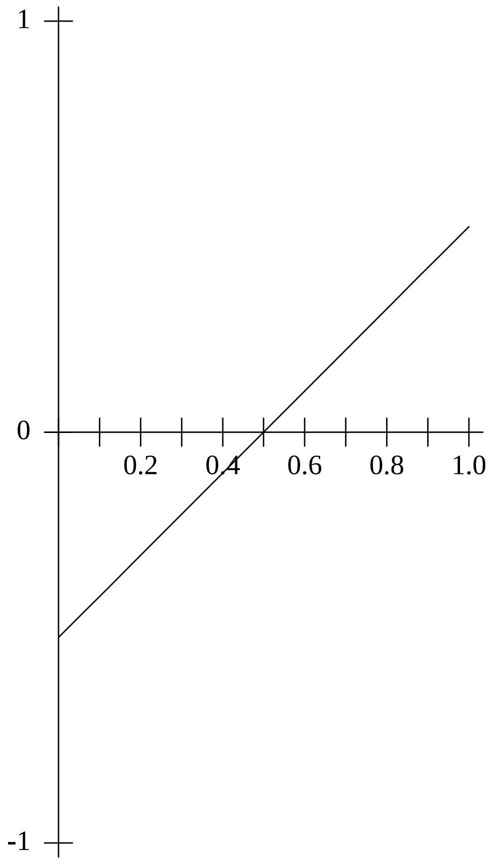
The formulas (19.18) or (19.19) allows the computing of the Bernoulli numbers, we get for the first few (the odd inedexed are 0 from 3 on)

$$
\begin{aligned}
&B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_4 = -\frac{1}{30},\ B_6 = \frac{1}{42} \\
&B_8 = -\frac{1}{30},\ B_{10} = \frac{5}{66},\ B_{12} = -\frac{691}{2730}
\end{aligned}
\tag{19.20}
$$

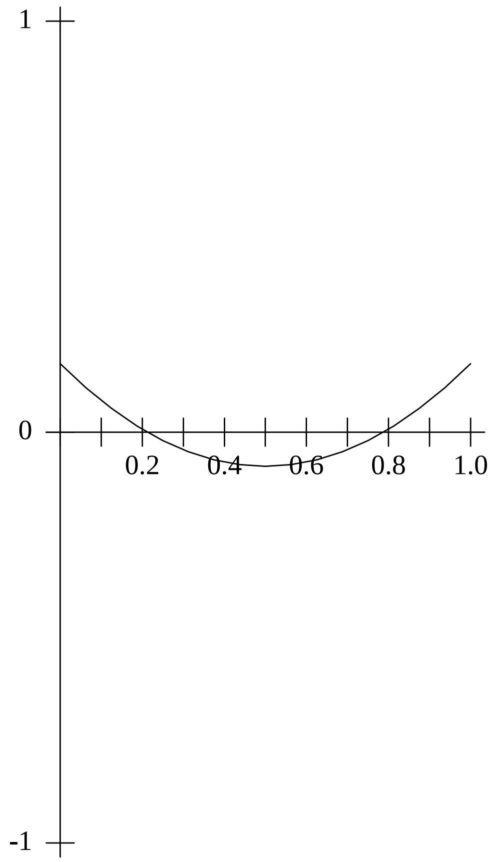
As soon as we have the numbers, we get the polynomials straightaway

$$
\begin{aligned}
B_0 &= 1 \\
B_1 &= x - \frac{1}{2} \\
B_2 &= x^2 - x + \frac{1}{6} \\
B_3 &= x(x-1)\left(x - \frac{1}{2}\right) \\
B_4 &= x^4 - 2x^3 + x^2 - \frac{1}{30} \\
B_5 &= x(x-1)\left(x - \frac{1}{2}\right)\left(x^2 - x - \frac{1}{3}\right)
\end{aligned}
\tag{19.21}
$$

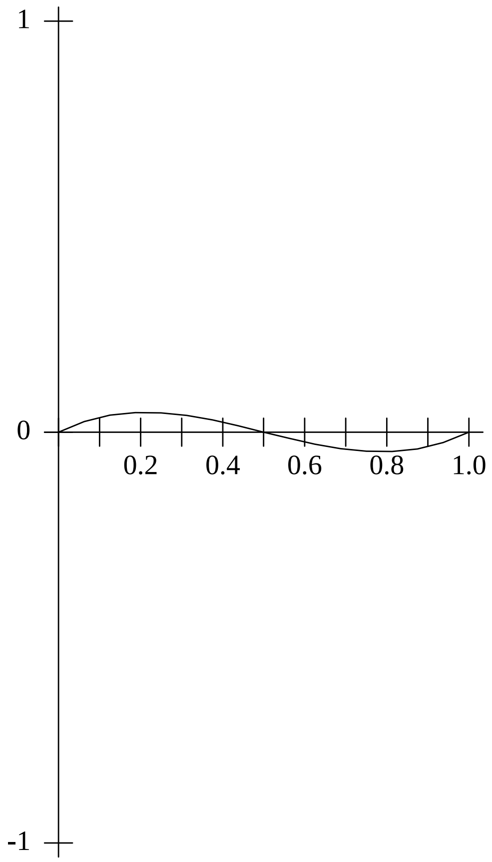
The first four of them look like the following graphs:



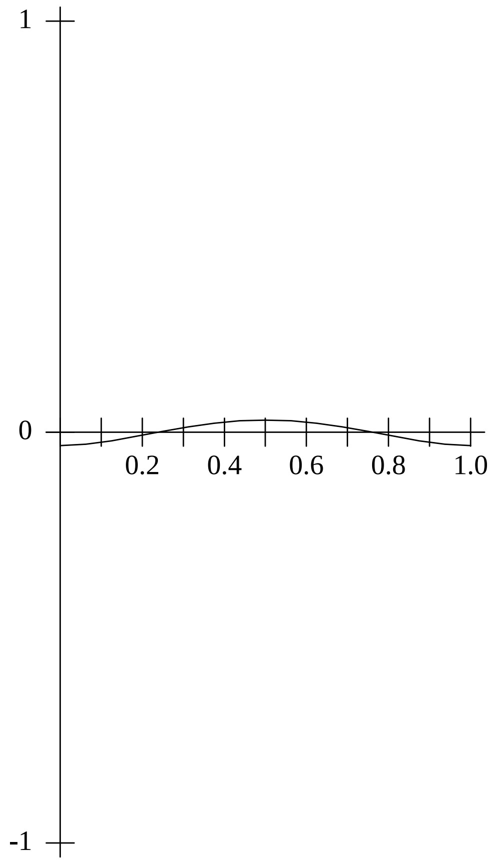
$B_1(x)$



$B_2(x)$



$B_3(x)$



$B_4(x)$

Actually, all odd Bernoulli polynomials look like $\pm B_3$ and the even ones look like $\pm B_4$.

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Chapter 20. REFERENCES

1. Milton Abramowitz and Irene A. Stegun (ed.), *Handbook of Mathematical Functions,* Dover Publications, Inc., New York, 1965, p. 1046.
2. Louis Brand, A Sequence Defined by a Difference Equation, *Am. Math. Monthly* 62 (1955) 489–492.
3. Giancarlo Gandolfo, *Mathematical Methods and Models in Economic Dynamics,* North–Holland Publishing Company, Amsterdam, 1971, 511 p.
4. Richard M. Goodwin, *Chaotic Economic Dynamics,* Clarendon Press, Oxford, 1990, 137 p.
5. Ronald Lewis Graham, Donald Knuth and Oren Patashnik, *Concrete Mathematics,* Addison–Wesley Publishing Company, Reading, Mass., 1989, 625 p.
6. R. F. Harrod, An Essay in Dynamic Theory, *Econ. J.* 49 (1939) 14–33.
7. Bjarne Sloth Jensen, *The Dynamic Systems of Basic Economic Growth Models,* Kluwer Academic Publishers, Dordrecht, 1994, 355 p.
8. Bjarne Sloth Jensen, Mogens Esrom Larsen, Growth and long–run stability, *Acta Applicandae Math.* 9 (1987) 219–237.
9. Bjarne Sloth Jensen, Mogens Esrom Larsen, Coupled linear differential equations with real coefficients, *Am. Math. Monthly* 96 (1989) 727–731.
10. Bjarne Sloth Jensen, Mogens Esrom Larsen, Homogeneous dynamics, (contained in Bjarne Sloth Jensen, *Studies in the Economic Dynamics of Growing Economics,* Dissertation, Uppsala universtitet, Uppsala) (1992) 1–43.
11. John Maynard Keynes, *The General Theory of Employment Interest and Money,* Macmillan, London, 1936, 403 p.
12. Mogens Esrom Larsen, En ualgebraisk analyse af et par lineære differentialligninger med konstante koefficienter, *Normat* 36 (1988) 104–113.
13. Mogens Esrom Larsen, The Eternal Triangle – A History of a Counting Problem, *Coll. J. Math.* 20, No. 5 November (1989) 370–384.
14. W. Leontief *et al.*, *Studies in the Structure of the American Economy,* Oxford University Press, Oxford, 1953.
15. N. E. Nörlund, *Vorlesungen über Differenzenrechnung,* Springer Verlag, Berlin, 1924, 551.
16. Paul A. Samuelson, *Foundation of Economic Analysis,* Harward University Press, Cambridge, Mass., 1983, 604 p.
17. Y. Shinkai, On Equilibrium Growth of Capital and Labor, *Int. Econ. Rev.* 1 (1960) 107–111.
18. R. M. Solow and Paul A. Samuelson, Balanced Growth under Constant Return to Scale, *Econometrica* 21 (1953) 412–424.
19. R. M. Solow, A Contribution to the Theory of Economic Growth, *Q. J. Econ.* 70 (1956) 65–94.

20. R. M. Solow, Note on Uzawa's Two–Sector Model of Economic Growth, *Rev. Econ. Stud.* 29 (1961) 48–50.

21. H. Uzawa, On a Two–Sector Model of Economic Growth, *Rev. Econ. Stud.* 29 (1961) 40–47.

22. H. Uzawa, On a Two–Sector Model of Economic Growth II, *Rev. Econ. Stud.* 30 (1963) 105–118.

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