Københavns Universitet
Eksamen ved Det naturvidenskabelige Fakultet vinter 1999-2000

Matematik 3 GE

Written exam, 4 hours. All course material is allowed during the exam (alle sædvanlige hjælpemidler er tilladt).
There are 4 problems weighted approximately evenly (De fire problemer tillæges ens vægt).

Problem 1

1. Let S_1 and S_2 be two nonsingular surfaces in $\mathbb{R}^3$ and Φ an isometry from S_1 to S_2. Show that, if

$$\gamma : [0, 1] \to S_1$$

is a geodesic curve, then so is

$$\Phi \circ \gamma : [0, 1] \to S_2.$$

2. Prove that every geodesic curve on a two-sphere S in $\mathbb{R}^3$ is an arc with the center coinciding with the center of the sphere.

3. Show that, given an isometry $\mathcal{O}$ of the two-sphere

$$S = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3,$$

there exists a linear orthogonal transformation $A : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\mathcal{O}$ is the restriction of A to S.
HINT: Assume first that $\mathcal{O}$ has one fixpoint, f.ex. $\mathcal{O}((1, 0, 0)) = (1, 0, 0)$.

Problem 2

Let $\mathcal{C}$ denote the cylinder in $\mathbb{R}^3$ given by

$$\mathcal{C} = \{(x, y, z)|x^2 + y^2 = 1\}$$

1. Determine the first and second fundamental form of $\mathcal{C}$.

2. Prove that $\mathcal{C} \setminus \{(x, y, z) \mid x = 0\}$ is isometric to an open subset of $\mathbb{R}^2$ with its standard metric.

3. Find all geodesics on $\mathcal{C}$ passing through the point $(1, 0, 1)$.

Problem 3

 Opgavesættet fortsættes på side 2

Let $\mathcal{H}$ denote the hyperboloid

$$\mathcal{H} = \{(x, y, z) \mid x^2 + y^2 - z^2 = 1\}$$

1. Find the geodesic curvature of the curves (circles)

$$\gamma_z : \theta \to ((1 + z^2)^{\frac{1}{2}} \cos\theta, (1 + z^2)^{\frac{1}{2}} \sin\theta, z)$$

at the point $\theta = 0$.

2. Show that the geodesic curvature $k_g(\gamma_z)$ of γ_z and the Gauss curvature K of $\mathcal{H}$ at any point (x, y, z) depends only on z.

3. Let $b > 0$ be a fixed number. Use 1. and 2. in the Gauss-Bonnet theorem for the strip

$$A_b = \{(x, y, z) \in \mathcal{H} \mid 0 < z < b\}$$

to obtain an equation in which the only unknown quantity is the Gauss curvature K.

4. Find the Gauss curvature K of $\mathcal{H}$ by differentiating with respect to b the equation found in 3..

Problem 4

1. Let

$$F = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 < 1 \text{ and } z = f(x, y)\}$$

where f is a smooth function which satisfies

$$\forall(x, y) : \ 0 < x^2 + y^2 < 1 \Longrightarrow f(x, y) < f(0, 0)$$

(i.e. f attains local maximum at the origin). Show that $(0, 0, f(0, 0))$ is an elliptic point of F.

2. Let S be a closed (compact, without boundary) regular surface in $\mathbb{R}^3$. Show that S has at least two points with strictly positive curvature.
HINT: For example look at two points P and Q on S which satisfy

$$||P - Q|| = sup_{p,q \in S}||p - q||,$$

where $||x||$ stands for the length of a vector x in $\mathbb{R}^3$