Københavns Universitet
Naturvidenskabelig kandidateksamen, Spring Term 1998

Mathematics 3 GE

This is a 4 hour written exam. All usual resources are allowed. There are a total of 12 questions distributed on 4 problems. Each question carries approximately the same weight but emphasis is also placed on the overall impression. A Danish version follows after the English. Solutions may be written in English or in Danish.

Problem 1

Let f and g be C^∞ functions from $\mathbb{R}^2$ to $\mathbb{R}$. Consider the surfaces

$$S_f = \{(x, y, f(x, y)) \mid (x, y) \in \mathbb{R}^2\}$$

and

$$S_g = \{(u, g(u, v), v) \mid (u, v) \in \mathbb{R}^2\}.$$

1°: Prove that S_f and S_g are diffeomorphic.

2°: Prove that if $g = f + c_1$ or if $g = -f + c_2$, with constants $c_1, c_2 \in \mathbb{R}$, then S_f and S_g are isometric.

3°: Show by an example that also other functions g than those mentioned in 2° may define surfaces S_g that are isometric to S_f.

4°: Suppose now that g only depends on u, in other words: $g(u, v) = \phi(u)$ for all $(u, v) \in \mathbb{R}^2$. The corresponding surface S_g is now called S_ϕ. Construct an isometry of S_ϕ onto $\mathbb{R}^2$. (Consider possibly first a reparametrization to arc length of the curve $u \mapsto (u, \phi(u))$.)

Problem 2

Two regular oriented surfaces S_1 and S_2 with Gauss maps N_1 and N_2, respectively, intersect each other along a curve C in such a manner that they are never tangent to each other. It is assumed that $C = S_1 \cap S_2$ is the trace of a regular curve β, parametrized by arc length. Hence, the assumptions imply among other things that in each point $\beta(s)$ on the curve, $\{\beta'(s), N_1(\beta(s)), N_2(\beta(s))\}$ constitutes a basis for $\mathbb{R}^3$ (consisting of 3 unit vectors).

1°: Prove that if C is a geodesic on both S_1 and S_2, then C is a line segment.

2°: Prove that if C is an asymptotic curve on both S_1 and S_2, then C is a line segment.

The exam set continues on page 2

Problem 3

Let (X, U) be an orthogonal parametrization of a regular surface S and consider on $X(U)$ the vector fields X_u and X_v.

1°: Compute the covariant derivative

$$\left(D_{X_v(p)} X_u\right)(p) = \left(\nabla_{X_v(p)} X_u\right)(p)$$

of X_u relative to $X_v(p)$ in an arbitrary point p. The result should be expressed in the basis $\{X_u(q), X_v(q)\}$, with $p = X(q)$, by means of E and G together with derivatives of these.

2°: State necessary and sufficient conditions on E and G for X_u to be a parallel field along all coordinate curves (i.e. both the curves corresponding to u constant as well as those corresponding to v constant).

Problem 4

Let S be a regular oriented surface and let (X, U) be a local parametrization of S, compatible with the orientation, and such that

$$U = \{(u, v) \mid u > 0 \text{ and } v > 0\}.$$

Assume furthermore that the coefficients of the first fundamental form with respect to this parametrization are given by

$$E(u, v) = \frac{1}{2}\ , \ F(u, v) = 0\ , \ \text{and}\ \ G(u, v) = \frac{u^2}{8 \cdot v^2} \text{ for } (u, v) \in U.$$

1°: Consider the curve α on $X(U)$ given by $\alpha(t) = X(t, \frac{t^2}{4})$ (with $t > 0$). Determine the angle of intersection between α and the coordinate curve corresponding to $u = 1$.

2°: Prove that the sum of angles in any geodesic triangle T contained in $X(U)$ is equal to π.

3°: Prove that the vector field $w(t)$ along α given by

$$(3\sqrt{2}\cos(\ln t))X_u(t, \frac{t^2}{4}) + (\frac{-3 \cdot t}{\sqrt{2}} \sin(\ln t))X_v(t, \frac{t^2}{4})$$

is parallel along α.

4°: Determine the geodesic curvature (up to sign) of α in the point $\alpha(t)$ corresponding to $t = e^{\frac{\pi}{2}}$ (it is maybe of use to observe that the angle between the tangent to the curve and X_u is constant).

The Danish version follows on page 3