Mathematics 3 GE

This is a 4 hour written exam. All usual resources are allowed. There are a total of 12 questions distributed on 4 problems. Each question carries approximately the same weight but emphasis is also placed on the overall impression. A Danish version follows after the English. Solutions may be written in English or in Danish.

Problem 1

Let $f$ and $g$ be $C^\infty$ functions from $\mathbb{R}^2$ to $\mathbb{R}$. Consider the surfaces

$$S_f = \{(x, y, f(x, y)) \mid (x, y) \in \mathbb{R}^2 \}$$

and

$$S_g = \{(u, g(u, v), v) \mid (u, v) \in \mathbb{R}^2 \}.$$

1°: Prove that $S_f$ and $S_g$ are diffeomorphic.
2°: Prove that if $g = f + c_1$ or if $g = -f + c_2$, with constants $c_1, c_2 \in \mathbb{R}$, then $S_f$ and $S_g$ are isometric.
3°: Show by an example that also other functions $g$ than those mentioned in 2° may define surfaces $S_g$ that are isometric to $S_f$.
4°: Suppose now that $g$ only depends on $u$, in other words: $g(u, v) = \phi(u)$ for all $(u, v) \in \mathbb{R}^2$. The corresponding surface $S_g$ is now called $S_{\phi}$. Construct an isometry of $S_{\phi}$ onto $\mathbb{R}^2$. (Consider possibly first a reparametrization to arc length of the curve $u \mapsto (u, \phi(u))$.)

Problem 2

Two regular oriented surfaces $S_1$ and $S_2$ with Gauss maps $N_1$ and $N_2$, respectively, intersect each other along a curve $C$ in such a manner that they are never tangent to each other. It is assumed that $C = S_1 \cap S_2$ is the trace of a regular curve $\beta$, parametrized by arc length. Hence, the assumptions imply among other things that in each point $\beta(s)$ on the curve, $\{\beta'(s), N_1(\beta(s)), N_2(\beta(s))\}$ constitutes a basis for $\mathbb{R}^3$ (consisting of 3 unit vectors).

1°: Prove that if $C$ is a geodesic on both $S_1$ and $S_2$, then $C$ is a line segment.
2°: Prove that if $C$ is an asymptotic curve on both $S_1$ and $S_2$, then $C$ is a line segment.
Problem 3

Let \((X, U)\) be an orthogonal parametrization of a regular surface \(S\) and consider on \(X(U)\) the vector fields \(X_u\) and \(X_v\).

1°: Compute the covariant derivative

\[
(D_{X_v(p)} X_u)(p) = (\nabla_{X_v(p)} X_u)(p)
\]

of \(X_u\) relative to \(X_v(p)\) in an arbitrary point \(p\). The result should be expressed in the basis \(\{X_u(q), X_v(q)\}\), with \(p = X(q)\), by means of \(E\) and \(G\) together with derivatives of these.

2°: State necessary and sufficient conditions on \(E\) and \(G\) for \(X_u\) to be a parallel field along all coordinate curves (i.e. both the curves corresponding to \(u\) constant as well as those corresponding to \(v\) constant).

Problem 4

Let \(S\) be a regular oriented surface and let \((X, U)\) be a local parametrization of \(S\), compatible with the orientation, and such that

\[
U = \{(u, v) \mid u > 0 \text{ and } v > 0\}.
\]

Assume furthermore that the coefficients of the first fundamental form with respect to this parametrization are given by

\[
E(u, v) = \frac{1}{2}, \quad F(u, v) = 0, \quad \text{and} \quad G(u, v) = \frac{u^2}{8 \cdot v^2} \quad \text{for} \quad (u, v) \in U.
\]

1°: Consider the curve \(\alpha\) on \(X(U)\) given by \(\alpha(t) = X(t, \frac{t^2}{4})\) (with \(t > 0\)). Determine the angle of intersection between \(\alpha\) and the coordinate curve corresponding to \(u = 1\).

2°: Prove that the sum of angles in any geodesic triangle \(T\) contained in \(X(U)\) is equal to \(\pi\).

3°: Prove that the vector field \(w(t)\) along \(\alpha\) given by

\[
(3\sqrt{2} \cos(\ln t))X_u(t, \frac{t^2}{4}) + (\frac{-3 \cdot t}{\sqrt{2}} \sin(\ln t))X_v(t, \frac{t^2}{4})
\]

is parallel along \(\alpha\).

4°: Determine the geodesic curvature (up to sign) of \(\alpha\) in the point \(\alpha(t)\) corresponding to \(t = e^{\frac{\pi}{2}}\) (it is maybe of use to observe that the angle between the tangent to the curve and \(X_u\) is constant).

The Danish version follows on page 3