

Matematik 3GE

ENGLISH TEXT

Written exam, 4 hours. All course material is allowed during the exam (alle sædvanlige hjælpemidler er tilladt).

There are 3 problems divided into 11 questions. All questions are given approximately the same weight.

Problem 1

Let S be the parametrised surface given by

$$\mathbb{R}^2 \ni (u, v) \rightarrow (u, v, uv) \in \mathbb{R}^3$$

- Prove* that S is a regular surface.
- Find* the Gauss curvature K and the mean curvature H of S expressed as functions of the parameters (u, v) .
- Find* all the asymptotic curves on S .
- Find* the principal curvatures of S at the point $P = (0, 0, 0)$ and *show* that the principal directions at P are given by vectors $(1, 1, 0)$ and $(1, -1, 0)$.

Problem 2

Let \mathcal{H} be the regular surface of revolution in \mathbb{R}^3 given by

$$\mathcal{H} = \left\{ ((2 - \cos t)\cos u, (2 - \cos t)\sin u, \sin t) \mid -\frac{\pi}{2} < t < \frac{\pi}{2} \text{ and } -\pi < u \leq \pi \right\}$$

- Prove* that the Gauss curvature K of \mathcal{H} is always negative
- Prove* that $\{(x, y, 0) \mid x^2 + y^2 = 1\}$ is the unique parallel of \mathcal{H} which is a closed geodesic.
- Let $s \rightarrow \gamma(s)$ be an arbitrary *closed* regular curve on \mathcal{H} . We denote by $r(s)$ the distance from z-axis to $\gamma(s)$ and by $\theta(s)$ the angle between the parallel through the point $\gamma(s)$ and the vector tangent to γ at $\gamma(s)$.
 - Prove* that the function $\theta(s)$ has value zero in at least one point s_0 .
 - Suppose that γ is a *closed geodesic*. *Prove* that $r(s_0) = 1$ or $r(s) > r(s_0)$ for all $s \neq s_0$.

HINT: You can use the Clairaut relation $r \cos \theta = c$.
 - Prove* that any closed geodesic γ on \mathcal{H} coincides with the parallel $r = 1$.

Problem 3

Let $\mathcal{F} \subset \mathbb{R}^3$ be a closed, connected, regular oriented surface with Gauss curvature K non-zero everywhere.

- (a) *Prove* that the image of the Gauss map, $N(\mathcal{F})$, is an open subset of the unit sphere S^2 of \mathbb{R}^3 .

HINT: Use the inverse function theorem for regular surfaces.

- (b) *Prove* that the Gauss map from \mathcal{F} to S^2 is surjective.
- (c) *Prove* that if the Gauss curvature K of \mathcal{F} is everywhere positive, then the mean curvature H of \mathcal{F} is never zero and has constant sign.

- (d) *FACT (which can be used without proof):*

Suppose that \mathcal{F} is an arbitrary closed, connected, regular surface in \mathbb{R}^3 . Then

$$\mathbb{R}^3 \setminus \mathcal{F} = A \cup B$$

where A and B are open, connected, disjoint subsets of \mathbb{R}^3 and A is bounded. \mathcal{F} can be oriented by choosing the unit normal vector N to point towards A everywhere on \mathcal{F} .

If the mean curvature H of \mathcal{F} has constant sign, then A is convex.

Prove that if \mathcal{F} is a closed connected regular surface in \mathbb{R}^3 such that its Gauss curvature is greater than zero at every point, then the Gauss map is injective and hence a diffeomorphism from \mathcal{F} to S^2 .

Good luck !

DANSK TEKST

EKSAMEN i 3GE - sommer 2000

Skriftlig eksamen, 4 timer. Alle sædvanlige hjælpemidler er tilladt.

Da er 3 opgaver fordelt i 11 underopgaver. Alle underopgaver vægtes tilnærmelsesvis ens.

Opgave 1

Lad S være fladen parametriseret ved

$$\mathbb{R}^2 \ni (u, v) \rightarrow (u, v, uv) \in \mathbb{R}^3$$

- Vis* at S er en regulær flade.
- Bestem* Gauss krumningen K og middelkrumningen H af S som funktioner af parametrene (u, v) .
- Bestem* samtlige asymptotiske kurver på S .
- Bestem* de principale krumninger af S i punktet $P = (0, 0, 0)$ og *vis* at de principale retninger i P er givet ved vektorerne henholdsvis $(1, 1, 0)$ og $(1, -1, 0)$.

Opgave 2

Lad \mathcal{H} være den regulære omdrejningsflade i \mathbb{R}^3 givet ved

$$\mathcal{H} = \left\{ ((2 - \cos t)\cos u, (2 - \cos t)\sin u, \sin t) \mid -\frac{\pi}{2} < t < \frac{\pi}{2} \text{ og } -\pi < u \leq \pi \right\}$$

- Vis* at Gauss krumningen K af \mathcal{H} er negativ overalt.
- Vis* at $\{(x, y, 0) \mid x^2 + y^2 = 1\}$ er den eneste parallelkurve på \mathcal{H} som er en lukket geodætisk kurve.
- Lad $s \rightarrow \gamma(s)$ være en vilkårlig *lukket* regulær kurve på \mathcal{H} . Vi betegner med $r(s)$ afstanden fra z-aksen til $\gamma(s)$, og med $\theta(s)$ vinklen mellem parallelkurven gennem punktet $\gamma(s)$ og tangentvektoren til γ i $\gamma(s)$.
 - Vis* at funktionen $\theta(s)$ antager værdien nul i mindst ét punkt s_0 .
 - Antag nu, at γ er en *lukket geodætisk kurve*. *Vis* at $r(s_0) = 1$ eller $r(s) > r(s_0)$ for alle $s \neq s_0$.

VINK: Man kan bruge Clairauts relation $r \cos \theta = c$.
 - Vis* at parallelkurven $r = 1$ er den eneste lukkede geodætiske kurve γ på \mathcal{H} .

Opgave 3

Lad $\mathcal{F} \subset \mathbb{R}^3$ være en kompakt, sammenhængende, orienteret, regulær overflade hvis Gauss krumning K overalt er forskellig fra nul.

- (a) *Vis* at billedet ved Gauss afbildingen, $N(\mathcal{F})$, er en åben delmængde af enhedskuglen S^2 i \mathbb{R}^3 .

VINK: Brug invers funktion sætningen for regulære overflader.

- (b) *Vis* at Gauss afbildingen $N : \mathcal{F} \rightarrow S^2$ er surjektiv.
- (c) *Vis* at hvis Gauss krumningen K overalt er positiv, da er middelkrumningen H af \mathcal{F} aldrig nul og har konstant fortegn.
- (d) *FAKTUM (kan benyttes uden bevis):*

Antag at \mathcal{F} er en kompakt, sammenhængende, orienteret, regulær overflade i \mathbb{R}^3 . Da er

$$\mathbb{R}^3 \setminus \mathcal{F} = A \cup B$$

hvor A og B er åbne disjunkte delmængder af \mathbb{R}^3 og A er begrænset. \mathcal{F} kan orienteres ved at vælge den normale enhedsvektor overalt til at pege ind i A .

Hvis middelkrumningen H af \mathcal{F} har konstant fortegn, da er A konvex.

Vis at hvis \mathcal{F} er en kompakt, sammenhængende, orienteret, regulær overflade i \mathbb{R}^3 , hvis Gauss krumning overalt er positiv, da er Gauss afbildingen en diffeomorfi af \mathcal{F} på S^2 .

God fornøjelse !