Problem 1: Systems of nonlinear equations and matrix norms.

a) Starting from \( x(1) \), perform a Newton iteration on the system of equations \( F(x) = 0 \) (that is find \( x(2) \)) where

\[
F(x) = \begin{pmatrix} x_1^3 - x_2 \\ x_2^2 \end{pmatrix}, \quad x(1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\]

b) Determine the condition number in the norm \( \| \cdot \|_\infty \) for the Jacobi-matrix for the vector function \( F \) in the point \( x(1) \) with the notation given in (a).

c) The following three errors are given

\[
e_1 = \| x(1) - x^* \|_2 = 1.41, \quad e_2 = \| x(2) - x^* \|_2 = 0.55, \quad e_3 = \| x(3) - x^* \|_2 = 0.10
\]

They represent the standard Euclidian norm of the error in three steps in a Newton iteration, \( x^* \) being the exact solution of \( F(x) = 0 \). Determine the order of convergence of the iteration based on these numbers.

Problem 2: Systems of linear equations.

a) Determine a Doolittle LU-decomposition for the matrix

\[
A = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 6 & 7 \\ 0 & 12 & 18 \end{pmatrix}.
\]

b) Solve using LU-decomposition and a forward and a backward substitution the equation

\[
Ax = \begin{pmatrix} 0 \\ 5 \\ 18 \end{pmatrix},
\]

for \( x \in \mathbb{R}^3 \), and with \( A \) given in (a).

Problem 3: Interpolation.

A third degree polynomial \( y = f(x) \) takes the following values: \( f(0) = 15, f(3) = f(5) = 0, f(4) = -13 \).

a) Determine by polynomial interpolation the function value in 1, \( f(1) \).

b) Determine the function expression in the subinterval \([0, 3]\) for the natural cubic spline interpolation \( S \) of the given data: \( S(0) = 15, S(3) = S(5) = 0, S(4) = -13 \).

c) Determine by interpolation with the natural cubic spline an approximation of the function value in 1, \( f(1) \), and explain why this number is different from the number from (a).

Problem 4: Numerical integration.
a) Compute with the composite trapezoidal rule with two equally large subintervals (3 nodalpoints) an approximation to
\[ \int_0^1 x^4 \, dx, \text{ hvor } x \in \mathbb{R}. \]

b) How many equally large subintervals were necessary in (a) for the absolute value of the error to be guaranteed less than 0.01?

(Exam finished).