# Matematik 3NA, Numerical Analysis. The Finite Element Method. 

Institute for Mathematical sciences, University of Copenhagen, January 31st 2006.

Open book exam. Programmable computers are not allowed. Pencils or any other legible writing utensils may be used. The answer may be in danish or english as preferred. All questions are weighted the same ( $10 \%$ ). A grade on the 13 scale is given only after approval of the programming projects.
Theorems must be referenced by name (or number in Brenner/Scott). They can be used without proof but the assumptions must be verified.

Part 1. Sobolev spaces and weak derivatives:
Consider the function

$$
f(x)= \begin{cases}x^{4 / 3} & \text { for } x>0  \tag{1}\\ x^{2} & \text { for } x \leq 0\end{cases}
$$

1: Show that $f$ is weakly differentiable on $[-1,1]$ and find $D_{w}^{1} f$.
2: Find the largest integer $m$ so that $f \in \mathcal{W}_{2}^{m}(-1,1)$.
3: Find all positive integers $p$ so that $f \in \mathcal{W}_{p}^{2}(-1,1)$.

## Part 2. Variational formulation of elliptic boundary value problems:

4: Show existence and uniqueness of solution to the following variational problem:

$$
\begin{equation*}
\text { Find } u \in \stackrel{\circ}{\mathcal{W}}_{2}^{1}(-1,1): \int_{-1}^{1}\left(u^{\prime} v^{\prime}+\frac{1}{2} u v\right) \mathrm{d} x=\int_{-1}^{1} f v \mathrm{~d} x \quad \forall v \in \stackrel{\circ}{\mathcal{W}}_{2}^{1}(-1,1) \tag{2}
\end{equation*}
$$

where $\stackrel{\circ}{\mathcal{W}}_{2}^{1}(-1,1)=\left\{v \in \mathcal{W}_{2}^{1}(-1,1): v(-1)=v(1)=0\right\}$.

## Part 3. Finite element discretization:

Consider the Reference finite element $(K, \mathcal{P}, \mathcal{N})$ where

- $K=[0,1]$
- $\mathcal{P}=P_{q}=\operatorname{span} B$ where $B=\left\{\phi_{1}, \phi_{2}, \ldots, \phi_{q+1}\right\}$ is a Lagrange basis corresponding to the set of degrees of freedom $\mathcal{N}$.

5: What is the minimal set of degrees of freedom $\mathcal{N}$ allowing $\mathcal{C}^{2}$ global smoothness of a finite element space based on elements affine equivalent to the reference element?
What is the smallest $q$ allowing $\mathcal{C}^{2}$ global smoothness of this space?
6: Write down the formula for $\phi_{1}(x)$ where $\phi_{1}$ is the Lagrange basis function with $\phi_{1}(0)=1$ and all other degrees of freedom having values 0. (Hint: Some algebra is required).

7: Make a drawing of $\phi_{1}$ emphasizing the behavior at $x=0$ and $x=1$.
Consider a subdivision of $(-1,1)$ into 3 subintervals $K_{1}=\left(-1,-\frac{1}{2}\right), K_{2}=\left(-\frac{1}{2}, 0\right), K_{3}=(0,1)$, and let there be given finite elements for each subinterval, affine equivalent to the reference element $(K, \mathcal{P}, \mathcal{N})$.

8: For all 3 subintervals, sketch the basis function corresponding to $\phi_{1}$ from the reference element. (You do not need to find the function expressions).

Consider the following discretization of (2):

$$
\begin{equation*}
\text { Find } u_{f e} \in \mathcal{V}_{f e}(-1,1): \int_{-1}^{1}\left(u_{f e}^{\prime} v^{\prime}+\frac{1}{2} u_{f e} v\right) \mathrm{d} x=\int_{-1}^{1} f v \mathrm{~d} x \quad \forall v \in \mathcal{V}_{f e}(-1,1) \tag{3}
\end{equation*}
$$

where $\mathcal{V}_{f e}$ is the finite element space induced by the definitions above, i.e. $\mathcal{V}_{f e} \subset \stackrel{\circ}{\mathcal{W}}_{2}^{1}, \mathcal{V}_{f e}\left(K_{j}\right)=$ $P_{q}\left(K_{j}\right), j=1,2,3$ and $\mathcal{V}_{f e} \subset \mathcal{C}^{2}$.
9: What is the dimension of $\mathcal{V}_{f e}(-1,1)$ ?
10: Sketch the 3 global basis functions $\psi_{1}, \psi_{2}$ and $\psi_{3}$, for $\mathcal{V}_{f e}(-1,1)$, constructed from the local basis functions $\left\{\phi_{i}\right\}_{i=1}^{q+1}$ for the reference element, such that $\psi_{1}\left(-\frac{1}{2}\right)=1, \psi_{2}^{\prime}\left(-\frac{1}{2}\right)=1$ and $\psi_{3}^{\prime \prime}\left(-\frac{1}{2}\right)=$ 1.
(You do not need to find the function expressions).

