Matematik 3NA, Numerical Analysis. The Finite Element Method.

Institute of Mathematical Sciences, University of Copenhagen, Januuary 6th 2005.
Open book exam. Programmable computers are not allowed. Pencils or any other legible writing utensils may be used. All questions are weighted the same (10%). A grade on the 13 scale is given only after approval of the programming projects.

Theorems must be referenced by name (or number in Brenner & Scott). They can be used without proof but the assumptions must be verified.

Part 1. Sobolev spaces and weak derivatives:

1: Show that the following function f is weakly differentiable on $[-1,1]$ and find $D_w^1 f$.

$$f(x) = \begin{cases} x^{4/3} & \text{for } x > 0 \\ x^2 & \text{for } x \leq 0 \end{cases} \tag{1}$$

2: Find the largest integer m so that $f \in \mathcal{W}_2^m(-1,1)$. Find also all positive integers p so that $f \in \mathcal{W}_p^2(-1,1)$.

3: Assume that the appropriate conditions are satisfied so that we have the following "Inverse Estimate":

$$\|v\|_{\mathcal{W}_p^\ell(K)} \leq Ch^{m-\ell+\frac{n}{p}-\frac{n}{q}}\|v\|_{\mathcal{W}_q^m(K)}.$$

What is the relation between ℓ, m, p and q for this to "earn" the title Inverse Estimate? In what sense is it "Inverse"? What makes the Inverse Estimates more interesting than any other "equivalence of norms" inequality?

Part 2. Variational formulation of elliptic boundary value problems:

4: Show existence and uniqueness of solution to the following variational problem:

$$\text{Find } u \in \overset{\circ}{\mathcal{W}}{}_2^1(-1,1) : \int_{-1}^1 \left(u'v' + \frac{1}{2}uv\right) dx = \int_{-1}^1 fv dx \quad \forall\, v \in \overset{\circ}{\mathcal{W}}{}_2^1(-1,1) \tag{2}$$

where $\overset{\circ}{\mathcal{W}}{}_2^1(-1,1) = \{v \in \mathcal{W}_2^1(-1,1) : \ v(-1) = v(1) = 0\}$ and f is the function from (1).

5: Write up a boundary value problem that (2) would be a generalization of. (You do not have to motivate this, just get it right).

Part 3. Finite element discretization:
Consider the *Reference* finite element $(K, \mathcal{P}, \mathcal{N})$ where

- $K = [0,1]$.
- $\mathcal{P} = P_q$ with Lagrange basis $\{\phi_1, \phi_2, \ldots, \phi_{q+1}\}$ corresponding to the dof $\mathcal{N}$.

6: What is the smallest q allowing $\mathcal{C}^3$ global smoothness of a finite element space based on elements affine equivalent to the reference element? (Give a motivation, not just a number).
5 is the smallest q allowing $\mathcal{C}^2$ global smoothness. Construct the set of degrees of freedom $\mathcal{N}$ giving $\mathcal{C}^2$ global smoothness with $q = 5$.

7: Make an explicit numbering of the degrees of freedom $\mathcal{N}$ found in problem 6 for $q = 5$ and find the formula for $\phi_1(x)$. Make a drawing of ϕ_1 emphasizing the behavior at $x = 0$ and $x = 1$.

Consider a subdivision of $(-1,1)$ into 3 subintervals $K_1 = (-1, -\frac{1}{2})$, $K_2 = (-\frac{1}{2}, 0)$, $K_3 = (0,1)$, and let there be given finite elements for each subinterval, affine equivalent to the reference element $(K, \mathcal{P}, \mathcal{N})$.

8: For all 3 subintervals, draw the local basis functions corresponding to ϕ_1 from the reference element. Draw (in a separate drawing) the 3 global Lagrange basis functions that has value, first derivative and second derivative respectively in $-\frac{1}{2}$ equal to 1. (Still $q = 5$, $\mathcal{C}^2$ global smoothness. You do **not** need to find the function expressions).

9: What is the dimension of the global space $\mathcal{V}_{fe} \subseteq \overset{\circ}{\mathcal{W}}{}_2^1(-1,1)$? (Still $q = 5$, $\mathcal{C}^2$ global smoothness. Give a motivation, not just a number).

10: Write up a discretization of (2) corresponding to a generic finite element space $\mathcal{V}_{fe}$. Write up explicitly the matrix problem corresponding to the case considered above ($q = 5$, $\mathcal{C}^2$ global smoothness). (Do not evaluate the integrals but define for example $a_{ij} = \int_{-1}^1 \left(\phi_i'(x)\phi_j'(x) + \frac{1}{2}\phi_i(x)\phi_j(x)\right) dx$. Make it clear where we get zeroes because of "non overlapping supports").

(The problem set is finished).