Matematik 3NA, numerisk analyse.
Finite element metoden.


Open book exam. Programmable computers are not allowed.
All questions are weighted the same (10%).
A grade on the 13 scale is given only after approval of the programming projects.

Theorems must be referenced by name (or number in Brenner/Scott). They can be used without proof but the assumptions must be verified.

Part 1. Sobolev spaces and weak derivatives:
Consider the function
\[ f(x) = \begin{cases} x^{1/3} & \text{for } x > 0 \\ x & \text{for } x \leq 0 \end{cases} \]

1: Show that \( f \) is weakly differentiable on \([-1, 1]\) and find \( D^1_w f \).

2: Find the largest integer \( m \) so that \( f \in W^m((-1, 1)) \).

3: Find all positive integers \( p \) so that \( f \in W^p_0((-1, 1)) \).

Part 2. Variational formulation of elliptic boundary value problems:
Find the largest integer \( m \) so that \( f \in W^m((-1, 1)) \).

4: Find existence and uniqueness of solution to the following variational problem:

\[ \int_{-1}^{1} u'v' \, dx = \int_{-1}^{1} f \, v \, dx \quad \forall v \in \overline{W}^{1, 1}_2((-1, 1)) \]

where \( \overline{W}^{1, 1}_2((-1, 1)) = \{ v \in W^{1, 1}_2((-1, 1)) : v(-1) = v(1) = 0 \} \).

Part 3. Finite element discretization:
Consider the Reference finite element \((K, \mathcal{P}, \mathcal{N})\) where

- \( K = [0, 1] \)
- \( \mathcal{P} = P_2 = \text{span} B \) for \( B = \{ \phi_1, \phi_2, \phi_3, \phi_4 \} \), where \( \phi_1(x) = (1+2x)(x-1)^2 \), \( \phi_2(x) = x(x-1)^2 \), \( \phi_3(x) = x^2(3 - 2x) \) and \( \phi_4(x) = x^2(2 - x) \).
- \( \mathcal{N} = \{ v \to v(0), v \to v(1), v \to v'(0), v \to v'(1) \} \), i.e. the degrees of freedom are the values and the first derivatives in the endpoints of the element.

You may take it for granted that \( B \) is a basis for \( P_3 \).

5: Show that \( B \) is a basis for \( \mathcal{P} \) dual to \( \mathcal{N} \), i.e. that \( B \) is a Lagrange basis corresponding to the degrees of freedom in \( \mathcal{N} \).

6: Make a drawing of \( \phi_1 \) emphasizing the behavior at \( x = 0 \) and \( x = 1 \).

Consider a subdivision of \((-1, 1)\) into 3 subintervals \( K_1 = (-1, -\frac{1}{2}), K_2 = (-\frac{1}{2}, 0), K_3 = (0, 1) \), and let there be given finite elements for each subinterval, affine equivalent to the reference element \((K, \mathcal{P}, \mathcal{N})\).

7: For all 3 subintervals, draw the basis function corresponding to \( \phi_1 \) from the reference element. (You do not need to find the function expressions).

Consider the following discretization of (2):

\[ \int_{-1}^{1} u_{fe}' v' \, dx = \int_{-1}^{1} f \, v \, dx \quad \forall v \in \mathcal{V}_{fe}(-1, 1) \]

where \( \mathcal{V}_{fe} \) is the finite element space induced by the definitions above, i.e. \( \mathcal{V}_{fe} \subset \mathcal{W}^{1, 1}_2 \), \( \mathcal{V}_{fe}(K_j) = P_2(K_j) \) and \( \mathcal{V}_{fe} \subset C^r \) where \( r \) is the continuity order of the global interpolant defined by the 3 finite elements above.

8: What is the value of \( r \)?

9: What is the dimension of \( \mathcal{V}_{fe}(-1, 1) \)?

10: Draw all the global basis functions for \( \mathcal{V}_{fe}(-1, 1) \), corresponding to the local basis functions \( \{ \phi_i \}_{i=1}^4 \) for the reference element.

(The problem set is finished).