Matematik 3NA, numerisk analyse. Finite element metoden.

Studienævn for Matematiske Fag, Københavns Universitet, den 23. Januar 2001.
Open book exam. Programmable computers are not allowed.
All questions are weighted the same (10%).
A grade on the 13 scale is given only after approval of the programming projects.

Theorems must be referenced by name (or number in Brenner/Scott). They can be used without proof but the assumptions must be verified.

Part 1. Sobolev spaces and weak derivatives:
Consider the function

$$f(x) = \begin{cases} x^{1/3} & \text{for } x > 0 \\ x & \text{for } x \le 0 \end{cases} \tag{1}$$

1: Show that f is weakly differentiable on $[-1,1]$ and find $D_w^1 f$.

2: Find the largest integer m so that $f \in \mathcal{W}_2^m(-1,1)$.

3: Find all positive integers p so that $f \in \mathcal{W}_p^1(-1,1)$.

Part 2. Variational formulation of elliptic boundary value problems:

4: Show existence and uniqueness of solution to the following variational problem:

$$\text{Find } u \in \overset{\circ}{\mathcal{W}}{}_2^1(-1,1): \int_{-1}^{1} u'v'dx = \int_{-1}^{1} fvdx \quad \forall\ v \in \overset{\circ}{\mathcal{W}}{}_2^1(-1,1) \tag{2}$$

where $\overset{\circ}{\mathcal{W}}{}_2^1(-1,1) = \{v \in \mathcal{W}_2^1(-1,1): \ v(-1) = v(1) = 0\}$.

Part 3. Finite element discretization:
Consider the *Reference* finite element $(K, \mathcal{P}, \mathcal{N})$ where

- $K = [0,1]$
- $\mathcal{P} = P_3 = \text{span}B$ for $B = \{\phi_1, \phi_2, \phi_3, \phi_4\}$, where $\phi_1(x) = (1+2x)(x-1)^2$, $\phi_2(x) = x(x-1)^2$, $\phi_3(x) = x^2(3-2x)$ and $\phi_4(x) = x^2(x-1)$.
- $\mathcal{N} = \{v \to v(0), v \to v(1), v \to v'(0), v \to v'(1)\}$, i.e. the degrees of freedom are the values and the first derivatives in the endpoints of the element.

You may take it for granted that B is a basis for P_3.

5: Show that B is a basis for $\mathcal{P}$ *dual to* $\mathcal{N}$, i.e. that B is a Lagrange basis corresponding to the degrees of freedom in $\mathcal{N}$.

6: Make a drawing of ϕ_1 emphasizing the behavior at $x = 0$ and $x = 1$.

Consider a subdivision of $(-1,1)$ into 3 subintervals $K_1 = (-1, -\frac{1}{2})$, $K_2 = (-\frac{1}{2}, 0)$, $K_3 = (0,1)$, and let there be given finite elements for each subinterval, affine equivalent to the reference element $(K, \mathcal{P}, \mathcal{N})$.

7: For all 3 subintervals, draw the basis function corresponding to ϕ_1 from the reference element. (You do **not** need to find the function expressions).

Consider the following discretization of (2):

$$\text{Find } u_{fe} \in \mathcal{V}_{fe}(-1,1): \int_{-1}^{1} u_{fe}'v'dx = \int_{-1}^{1} fvdx \quad \forall v \in \mathcal{V}_{fe}(-1,1) \tag{3}$$

where $\mathcal{V}_{fe}$ is the finite element space induced by the definitions above, i.e. $\mathcal{V}_{fe} \subset \overset{\circ}{\mathcal{W}}{}_2^1$, $\mathcal{V}_{fe}(K_j) = P_3(K_j)$ and $\mathcal{V}_{fe} \subset \mathcal{C}^r$ where r is the continuity order of the global interpolant defined by the 3 finite elements above.

8: What is the value of r?

9: What is the dimension of $\mathcal{V}_{fe}(-1,1)$?

10: Draw all the global basis functions for $\mathcal{V}_{fe}(-1,1)$, corresponding to the local basis functions $\{\phi_i\}_{i=1}^4$ for the reference element.

(The problem set is finished).