Introduction to the classical orthogonal polynomials.

A weight-function on an interval \([a,b]\) is a non-negative integrable function \(w: [a,b] \to [0,\infty]\), such that the finite measure

\[\mu = w(x) \int_a^b (x) dx\]

has moments of any order and infinite support. The latter is of course true if \(w(x) > 0\) for Lebesgue almost all \(x \in [a,b]\).

A system of orthogonal polynomials associated with the weight-function is a sequence \((p_n)_{n=0}^\infty\) of polynomials with real coefficients such that

a) \(p_n\) is of degree \(n\)

b) \(\int p_n(x)p_m(x) w(x) dx = 0\) if \(n \neq m\).

This means that we only require orthogonality, not orthornormality, but there is of course a unique sequence \(k_m \to 0\) such that \(\frac{P_n(x)}{k_m p_n(x)}\) is the orthonormal polynomials in the sense of the notes p. 63.

The three main types of intervals

1) \(-\infty < a < b < \infty\)
2) \(a = -\infty\), \(b < \infty\) or \(-\infty < a\), \(b = \infty\)
3) \(a = -\infty\), \(b = \infty\)

lead to three different classes of orthogonal polynomials.
Using a suitable affine transformation $y = kx + b$, $k \neq 0$, $b \in \mathbb{R}$, these types of intervals can be transformed to the standard intervals $[-1, 1]$, $[0, \infty]$, $[-\infty, \infty]$, where we consider the following weight functions:

<table>
<thead>
<tr>
<th>Interval</th>
<th>Weight function $\omega(x)$</th>
<th>Name of polynomials</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[-1, 1]$</td>
<td>$\omega(x) = (1-x)\alpha (1+x)^{\beta}$, $\alpha, \beta &gt; -1$</td>
<td>Jacobi polynomials</td>
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<tr>
<td>$[0, \infty]$</td>
<td>$\omega(x) = x^\alpha e^{-x}$, $\alpha &gt; -1$</td>
<td>Laguerre polynomials</td>
</tr>
<tr>
<td>$[-\infty, \infty]$</td>
<td>$\omega(x) = e^{-\frac{1}{2}x^2}$</td>
<td>Hermite polynomials</td>
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Note that the conditions $\alpha, \beta > -1$ in the first case are necessary and sufficient in order that

$$\int_{-1}^{1} (1-x)\alpha (1+x)^{\beta} \, dx < \infty,$$

and similarly in the second case

$$\int_{0}^{\infty} x^\alpha e^{-x} \, dx < \infty \text{ if and only if } \alpha > -1.$$ Note that the weight function can tend to infinity at the end-points of the interval, say at

$-1$ if $-1 < \alpha < 0$ in the first case.

These three types of orthogonal polynomials are called the classical orthogonal polynomials. They are studied in many books.

The classical orthogonal polynomials can be defined by a type of formula called Rodrigues's formula, and they satisfy a second order differential equation.
Rodrigues' formula (O. Rodrigues 1816)

Let us assume that \( w \in C^\infty([a,b]) \) and \( w(x) > 0 \) for all \( x \in [a,b] \) and let us consider the expression

\[
F_n(x) = \frac{1}{w(x)} D^n \left[ w(x) X^n(x) \right], \quad n \geq 0, \quad x \in [a,b],
\]

where \( D \) is differentiation with respect to \( x \) and

\[
X(x) = \begin{cases} 
(b-x)(x-a), & \text{if } -\infty < a < b < \infty, \text{ case } 1 \\
x-a, & \text{if } -\infty < a < b = \infty, \text{ case } 2 \\
1, & \text{if } -\infty = a, b = \infty, \text{ case } 3.
\end{cases}
\]

We want to determine \( w \), so that \( F_n \) is a polynomial of degree \( n \) for all \( n \).

When \( n=0 \) we get \( F_0 = 1 \) from (1), and for \( n=1 \)

\[
F_1(x) = X'(x) + \frac{w(x)}{w(x)} X(x) = Ax + B, \quad A \neq 0
\]

hence

\[
\frac{w(x)}{w(x)} = \frac{Ax + B - X(x)}{X(x)} = \begin{cases} 
\frac{(A+2)x + B - a - 1}{(b-x)(x-a)}, & \text{case } 1 \\
\frac{Ax + B - 1}{x-a}, & \text{case } 2 \\
Ax + B, & \text{case } 3.
\end{cases}
\]

Case 1. Decomposing the right-hand side of (3) gives

\[
\frac{w'(x)}{w(x)} = \frac{\alpha}{x-b} + \frac{\beta}{x-a} \quad \text{for certain } \alpha, \beta \in \mathbb{R}
\]

hence

\[
\log w(x) = \alpha \log(b-x) + \beta \log(x-a) + \log C; \quad a < x < b
\]

for some constant \( C > 0 \), i.e.,

\[w(x) = C (b-x)^\alpha (x-a)^\beta.\]
Since we want \( W(x) \) to have finite integral over \([a,b]\), we must restrict \( \alpha, \beta > -1 \). In case \( \alpha = -1, \beta = 1 \), this leads to the weight-function for \( \text{Jacobi polynomials} \).

**Case 2.** In this case (5) can be written

\[
\frac{w'(x)}{w(x)} = \frac{x}{x-a} - k, \quad \alpha, k \in \mathbb{R}
\]

hence

\[
w(x) = C e^{-kx} (x-a)^{\alpha}, \quad C > 0.
\]

The integrability of \( w \) over \([a, \infty)\) requires \( k > 0, \alpha > -1 \).

The affine transformation

\[
x = a + \frac{1}{k} y, \quad y = k(x-a)
\]

leads to

\[
w(y) = C e^{-k(y+y)k} (y) = C e^{\frac{k^2}{2}y} y^{-\frac{\alpha}{k}}
\]

which is the weight function for the \( \text{Legendre polynomials} \).

**Case 3.** From (3) we now get

\[
w(x) = \exp \left( \frac{1}{2} Ax^2 + Bx + C \right)
\]

and the integrability of \( w \) over \( \mathbb{R} \) requires \( A < 0 \), i.e., \( A = -h^2 \), \( h > 0 \), and if \( y = x - \frac{B}{A} \) we get

\[
w(y) = w \left( \frac{y}{h} (y+\frac{B}{A}) \right) = C \exp \left( -\frac{1}{2} y^2 \right), \quad C > 0,
\]

which is the weight function for the \( \text{Jacobi polynomials} \).

This analysis shows that \( F_n \) given by (1) can only be a polynomial of degree 1 for \( n = 1 \) if \( w \) after a suitable affine transformation is the weight function for one of
the classical orthogonal polynomials.

We next show, that if $\alpha$ is one of the three classical weight functions, then $F_n$ given by (1) is a polynomial of degree $n$ and $(F_n)$ is an orthogonal system with respect to $w$.

Case 1. (Jacobi polynomials). $\alpha$ as (1) reads

$$F_n(x) = (-1)^n (1 + x)^n (1 - x)^n \sum_{\beta=0}^{\infty} \binom{n}{\beta} (-1)^{\beta} \binom{2n+\beta}{n} \left\{ (1 - x)^{\beta + n} (1 + x)^{\beta + n} \right\}$$

By Leibniz's formula for the $n$th derivative of a product, the right-hand side becomes

$$(1 - x)^{\beta + n} (1 + x)^{\beta + n} \sum_{\beta=0}^{\infty} \binom{n}{\beta} \binom{2n+\beta}{n} \frac{d^\beta}{dx^\beta} (1 - x)^{\beta + n} (1 + x)^{\beta + n}$$

which is a polynomial of degree $n$ in $x$, and the leading coefficient is

$$(-1)^n \sum_{\beta=0}^{\infty} \binom{n}{\beta} \binom{2n+\beta}{n} \frac{d^\beta}{dx^\beta} (1 - x)^{\beta + n} (1 + x)^{\beta + n}$$

and the sum is $> 0$. To see that

$$\int_{-1}^{1} F_n(x) F_n(x)(-x)^\alpha (1-x)^\beta dx = 0 \quad \text{for } m > n,$$

it suffices to show that

$$\int_{-1}^{1} F_n(x) p(x)(-x)^\alpha (1-x)^\beta dx = 0 \quad \text{for } p \in P_{n-1}, m > 1.$$

Inserting the expression for $F_n$ in (4) we get
\[ \int \left[ p(x) D^m \left\{ (1-x)^{\alpha+m} (1+x)^{\beta+m} \right\} \right] dx, \]

which by partial integration equals
\[ \left[ p(x) D^{m-1} \left\{ (1-x)^{\alpha+m} (1+x)^{\beta+m} \right\} \right] - \int p'(x) D^{m-1} \left\{ (1-x)^{\alpha+m} (1+x)^{\beta+m} \right\} dx. \]

The first term vanishes, because Leibniz' formula shows that all terms contain \( 1-x \) and \( 1+x \) to a positive power. Here it is important that \( \alpha, \beta > -1 \). After (m-1) partial integrations we get
\[ = (1)^{m} \int D^m p(x) (1-x)^{\alpha+m} (1+x)^{\beta+m} dx, \]

which is 0, because \( D^m p = 0 \).

**Case 2** (Laguerre polynomials) Now (1) reads
\[ F_m(x) = x^{-\alpha} D^m (x^{-\alpha} e^{-x}). \]

By Leibniz' formula we see that \( F_m \) is a polynomial of degree \( m \) and the system \( \{F_m\} \) is orthogonal with respect to \( x^{\alpha} e^{-x} \) on \( [0, \infty) \).

**Case 3** (Hermite polynomials) Now (1) reads
\[ F_m(x) = e^{\frac{1}{2} x^2} D^m (e^{-\frac{1}{2} x^2}). \]

Clearly \( F_0 = 1 \) and assume that \( F_m \) is a polynomial of
degree \( m \). Then
\[
F_{m+1}(x) = e^{\frac{1}{2}x^2} D(D^m e^{-\frac{1}{2}x^2}) = e^{\frac{1}{2}x^2} D(e^{-\frac{1}{2}x^2} F_m(x))
\]

\[= F_m'(x) - x F_m(x),\]
so \( F_{m+1} \) is a polynomial of degree \( m+1 \).

Again for \( p \in \mathbb{P}_m \), \( m \geq 1 \)

\[
\int_{-\infty}^{\infty} F_m(x) p(x) e^{-\frac{1}{2}x^2} dx = \int_{-\infty}^{\infty} p(x) D^{m-1}(e^{-\frac{1}{2}x^2}) dx
\]

\[=[p(\infty) D^{m-1}(e^{-\frac{1}{2}x^2})]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} p'(x) D^{m-1}(e^{-\frac{1}{2}x^2}) dx\]

\[= \ldots = (-1)^m \int_{-\infty}^{\infty} D^m p(x) e^{-\frac{1}{2}x^2} dx = 0,\]
so \( \{F_m\} \) is an orthogonal system. We have used that

\[
\lim_{x \to \pm\infty} p(x) e^{-\frac{1}{2}x^2} = 0
\]

for any polynomial \( p(x) \).

**Summing up:**

**Rodrigues' Formula**

(5) \[ F_m(x) = \frac{1}{\omega(x)} D^m \{\omega(x) X^m(x)\}, \quad m \geq 0, \quad x \in \mathbb{R}, \quad \beta \leq 0 \]

with

\[
\begin{align*}
\mathbb{R} & : \quad ]-1, 1[ & \quad ]0, \infty[ & \quad ]-\infty, 0[ \n
\omega(x) : & \quad (1-x)(1+x) & x & e^{-x} & e^{-\frac{1}{2}x^2} \\
X(x) : & \quad (1-x)^\alpha (1+x)^\beta & x^\beta e^{-x} & \frac{1}{x!} & H_m \\
F_m : & \quad \mathcal{P}_m(\beta) & \frac{1}{\Gamma(\beta)} & L^{(\alpha)}_m & H_m
\end{align*}
\]
defines a system of orthogonal polynomials for the weight function \( w(x) \), namely the classical orthogonal polynomials: Jacobi, Laguerre, Hermite.

Rodriguez showed the formula in 1876 for the special case of Jacobi polynomials where \( \alpha = \beta = 0 \) called Legendre polynomials \( P_n(x) \)

\[
\int_{-1}^{1} P_n(x) P_m(x) \, dx = 0 \quad \text{for} \quad n \neq m.
\]

**Normalization** Let \( \ell_n \) be the leading coefficient of \( F_n \) given by (5). Then

\[
\int_{-1}^{1} F_n^2(x) w(x) \, dx = (-1)^m m! \ell_m \int_{-1}^{1} x^m(w(x) \, dx.
\]

**Proof.** By (5) we have after partial integration

\[
\int_{-1}^{1} F_n^2(x) w(x) \, dx = \int_{-1}^{1} F_n(x) D^m \{ w(x) x^m \} \, dx = (-1)^m \int_{-1}^{1} D^m \{ F_n(x) \} w(x) x^m \, dx
\]

Because

\[
\left[ D^k \{ w(x) x^m \} \right]_{x=a} = 0 \quad \text{for} \quad k = 0, 1, \ldots, m-1.
\]

Therefore

\[
\int_{-1}^{1} F_n^2(x) w(x) \, dx = (-1)^m m! \ell_m \int_{-1}^{1} x^m w(x) \, dx.
\]

**Remark.** One can of course find an expression for (6) in the three cases, but the calculation is somewhat long.
Theorem (Differential equation for the classical orthogonal polynomials). With the notation from Rodrigues' formula,\[ y = F_0(x) \] satisfies the second order linear differential equation
\[ (7) \quad x(x) y'' + F_1(x) y' - n \left( F'_1 + \frac{m-1}{2} x'' \right) y = 0, \quad a < x < b \]
Note that \[ F'_1 + \frac{m-1}{2} x'' \] is a constant.

Proof: We first calculate \[ D^{m+1} \{ XD(x^n) \} \] using Leibniz' formula, remembering that \( X \) is a polynomial of degree \( m \).

\[
D^{m+1} \{ XD(x^n) \} = XD^{m+2}(x^n) + \binom{m+1}{1} X'D^{m+1}(x^n) + \binom{m+1}{2} X'' D^{m}(x^n)
\]
\[
= XD^2(\omega x) + (m+1) X' D(\omega x) + \frac{m(m+1)}{2} X''(\omega x).
\]

Next, we write

\[
D^{m+1} \{ XD(x^n) \} = D^{m+1} \{ XD(x X^{n-1}) \} = D^{m+1} \{ X^m D(\omega x) + \omega x^2 \left( D(x^{n-1}) \right) \}
\]
\[
= D^{m+1} \{ x \omega F_1 + (n-1) \omega x x^{n-1} \}' = D^{m+1} \{ \omega x^{n} (F'_1 + (n-1) x') \}'
\]
and using Leibniz' formula, the last expression equals

\[
(F'_1 + (n-1) x') D^{m+1}(\omega x^n) + (n+1) (F'_1 + (n-1) x') D^m(\omega x^n)
\]
\[
= (F'_1 + (n-1) x') D(\omega x^n) + (n+1) (F'_1 + (n-1) x') \omega x^n.
\]

Comparing the two expressions for \[ D^{m+1} \{ XD(x^n) \} \] yields:

\[
(8) \quad XD^2(\omega x) + (2 X-x') D(\omega x) - (n+1) \left[ F'_1 + \frac{m-2}{2} x'' \right] (\omega x) = 0,
\]
which is a second order differential equation for \( F_m \). Inserting 
\[(F_m w)" = F_m' w + F_m w'^{\prime}\] and 
\[(F_m w)"" = F_m w'^{\prime \prime} + 2F_m' w'^{\prime} + F_m w'^{\prime \prime}\]
we can rearrange (8) to

\[\omega X F_m" + \left[ 2X \omega' + (2X' - F_1) \omega \right] F_m' \]

\[+ \left[ X \omega'^{\prime \prime} + (2X' - F_1) \omega' - n(n+1)(F_1' + \frac{m^2}{2} X') \omega \right] F_m = 0.\]

Finally, we want to eliminate \( \omega' \) and \( \omega'' \) by using
\[\omega F_1 = (\omega X)' = \omega X' + \omega' X,\]

hence
\[\omega' X = \omega (F_1' - X')\]

and hence
\[\omega'' X = \omega' (F_1' - X')' + \omega (F_1' - X'') = \omega' (F_1' - 2X') + \omega (F_1' - X'').\]

This gives
\[2X \omega' + (2X' - F_1) \omega = F_1 \omega\]

and
\[X \omega'' + (2X' - F_1) \omega' = \omega (F_1' - X'')\]

so finally

\[\omega \left\{ X F_m" + F_1 F_m' - n(F_1' + \frac{m^2}{2} X') F_m \right\} = 0\]

and (7) follows.

\[\square\]

Remark. Put \( \lambda_m = n(F_1' + \frac{m^2}{2} X'') \), which is a constant. Then (7) shows that \( F_m \) is an eigenvalue corresponding to the eigenvalue \( \lambda_m \) for the differential operator

\[X \gamma'' + F_1 \gamma'.\]
Case 1. The Jacobi polynomials $P_n^{(\alpha, \beta)}$ satisfy
$$(1-x^2)y'' + (\beta - x - (\alpha+\beta + 2))y' + n(n+\alpha+\beta+1)y = 0.$$ 

Case 2. The Laguerre polynomials $L_n^{(\alpha)}$ satisfy
$$xy'' + (\alpha + 1 - x)y' + ny = 0.$$ 

Case 3. The Hermite polynomials $H_n$ satisfy
$$y'' - xy' + ny = 0.$$ 

All three cases are examples of a Sturm-Liouville differential equation.