Problems for March 20, 2014

Problem 1: Solve E 2.6.2
Problem 2: Solve E 2.6.4

Problem 3: Let \((P_n)\) be the orthonormal polynomials associated with a non-degenerate normalized \(s \in \mathcal{P}\), i.e., \(s_0 = 1, D_n = \det \mathcal{H}_n > 0, n = 0, 1, \ldots\), and let \(a_n, b_n, n \geq 0\) be the coefficients in the three-term recurrence relation (2.6.1).

Let
\[
    p_n(x) = \sqrt{\frac{D_n}{D_{n-1}}} P_n(x)
\]
denote the corresponding sequence of monic orthogonal polynomials (i.e., \(p_n(x) = x^n + \text{terms of lower degree}\)).

(i) Show that
\[
    xp_n(x) = p_{n+1}(x) + \alpha_n p_n(x) + \beta_n p_{n-1}(x), \quad n \geq 1, \quad xp_0(x) = p_1(x) + \alpha_0 p_0(x),
\]
with \(\alpha_n = a_n, n \geq 0, \beta_n = b_n^2 - 1, n \geq 1\).

(ii) Let now \((p_n)\) denote a sequence of monic polynomials, \(p_n\) of degree \(n, n \geq 0\) (in particular \(p_0 = 1\)), and assume that we have a three-term recurrence relation of the form
\[
    xp_n(x) = p_{n+1}(x) + \alpha_n p_n(x) + \beta_n p_{n-1}(x), \quad n \geq 1, \quad xp_0(x) = p_1(x) + \alpha_0 p_0(x),
\]
with \(\alpha_n \in \mathbb{R}, \beta_n > 0\).

Prove that if we define
\[
    k_n = \frac{1}{\sqrt{\beta_1 \cdots \beta_n}}, n \geq 1, k_0 = 1,
\]
then \(P_n(x) = k_n p_n(x)\) are the orthonormal polynomials with respect to some \(s \in \mathcal{P}\), which is normalized and non-degenerate.