Problem 3

1) We have \( \log z = \ln |z| + i \arg(z) = 0 \) if and only if \( \ln |z| = 0 \) and \( \arg(z) = 0 \), i.e. if and only if \( |z| = 1 \) and \( z \) is real and positive. This shows that \( z = 1 \) is the only zero of \( \log \). Since \( \log'(z) = 1/z \) which is 1 for \( z = 1 \), we see that \( z = 1 \) is a simple zero.

2) By L’Hospital’s rule

\[
\lim_{z \to 0} \frac{z}{\log(1 + z)} = \lim_{z \to 0} \frac{1}{1/(1 + z)} = 1.
\]

The fraction \( z/\log(1 + z) \) of two functions, which are both holomorphic in \( \mathbb{C} \setminus (-\infty, -1] \), is holomorphic except at the zeros of the denominator, but \( z = 0 \) is the only such zero, and we have just proved that \( z = 0 \) is a removable singularity. Therefore the function in question is holomorphic.

3) Since \( K(0, 1) \) is the largest disc with center 0 contained in \( \mathbb{C} \setminus (-\infty, -1] \), we know by Theorem 4.8 that the function in question has a power series valid for \( |z| < 1 \) as the one described in the text. Putting \( f(z) = z/\log(1 + z) \) we know that \( b_n = f^{(n)}(0)/n! \), hence \( b_0 = 1 \) known from the limit in 2). We find

\[
f'(z) = \log(1 + z) - \frac{z}{z + 1} \log^2(1 + z).
\]

The numerator \( \varphi \) and denominator \( \psi \) become 0 for \( z = 0 \), so we use L’Hospital and calculate \( \frac{\varphi'}{\psi'} \):

\[
\frac{\frac{1}{1+z} - \frac{z+1-z}{(z+1)^2}}{2 \log(1 + z)} = \frac{1 - \frac{1}{z+1}}{2 \log(1 + z)}.
\]

Again the new numerator and denominator vanish for \( z = 0 \), so we use L’Hospital once more, leading to the expression:

\[
\frac{1}{2(z + 1)}
\]

This expression has the limit \( 1/2 \) for \( z \to 0 \), which shows that \( b_1 = 1/2 \).

Alternatively, using the known power series for \( \log(1 + z) \) we have

\[
z = \left( \sum_{n=0}^{\infty} b_n z^n \right) \left( \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n} \right) = z \left( \sum_{n=0}^{\infty} b_n z^n \right) \left( \sum_{n=0}^{\infty} (-1)^{n} \frac{z^n}{n+1} \right),
\]

hence

\[
1 = (b_0 + b_1 z + \cdots)(1 - z/2 + z^2/3 - + \cdots) = b_0 + z(b_1 - b_0/2) + \cdots,
\]

and by the identity theorem for power series (Theorem 1.14) we get \( b_0 = 1, b_1 - b_0/2 = 0 \), leading to \( b_1 = 1/2 \) (and we could easily find more \( b_n \)'s).
Problem 4

1) Since \( a^2 + z^2 = (z - ia)(z + ia) \), it is clear that \( (a^2 + z^2)^n \) has \( z = ia \) and \( z = -ia \) as zeros of order \( n \) and there are no other zeros. This means that \( f \) has poles of order \( n \) in \( z = ia \) in the upper half-plane and \( z = -ia \) in the lower half-plane. There are no poles on the real axis and the degree of the denominator is at least 2.

Defining \( \varphi(z) = (z - ia)^n f(z) = \frac{1}{(z + ia)^n} = (z + ia)^{-n} \),

we know by page 7.3 method 3º that

\[
\text{Res}(f, ia) = \frac{\varphi^{(n-1)}(ia)}{(n-1)!}.
\]

Clearly

\[
\varphi^{(k)}(z) = (-n)(-n-1) \cdots (-n-k+1)(z+ia)^{-n-k} = (-1)^k \frac{(n+k-1)!}{(n-1)!}(z+ia)^{-n-k},
\]

for \( k = 0, 1, \ldots \). This gives for \( k = n-1 \)

\[
\text{Res}(f, ia) = \frac{\varphi^{(n-1)}(ia)}{(n-1)!} = (-1)^{n-1} \frac{(2n-2)!}{((n-1)!)^2} (2ia)^{-2n+1} = \frac{(-1)^{n-1}}{i^{2n-1}(2a)^{2n-1}} \frac{(2n-2)!}{n-1}.
\]

This proves that

\[
\text{Res}(f, ia) = -i \frac{(2n-2)!}{n-1} \frac{1}{(2a)^{2n-1}}.
\]

2) By symmetry

\[
\int_0^\infty \frac{dx}{(a^2 + x^2)^n} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{(a^2 + x^2)^n},
\]

and the last integral equals

\[
2\pi i \text{Res}(f, ia)
\]

by Theorem 7.8. Putting this together, we find

\[
\int_0^\infty \frac{dx}{(a^2 + x^2)^n} = \left(\frac{2n-2}{n-1}\right) \frac{\pi}{(2a)^{2n-1}}.
\]