Solutions to Koman, August 2008

Problem 3

(1) To see that $f(z) = z^3 \cos(1/z)$ is holomorphic in $\mathbb{C} \setminus \{0\}$, we use that $z^3$, $\cos(z)$ are holomorphic in $\mathbb{C}$ and $1/z$ is holomorphic in $\mathbb{C} \setminus \{0\}$. Then the composition $\cos(1/z)$ is holomorphic in $\mathbb{C} \setminus \{0\}$ and so is the product with $z^3$.

(2) In the power series for $\cos$ we insert the argument $1/z$ and find

$$\cos(1/z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{-2k}}{(2k)!},$$

valid for all $z \neq 0$. Multiplying this by $z^3$ gives

$$f(z) = z^3 - \frac{z}{2} + \frac{1}{4!z} - \frac{1}{6!z^3} + \ldots,$$

which is a Laurent series with centre 0, converging locally uniformly to $f(z)$. It is therefore the Laurent series for $f$ with centre 0. There are infinitely many negative powers present, so by Theorem 6.21 (iii) we get that $z = 0$ is an essential singularity.

(3) The coefficient to $z^{-1}$ is the residue. Therefore

$$\text{Res}(f, 0) = \frac{1}{24}.$$

Problem 4

(1) The numerator has a double zero at $z = -1$. The denominator can be factorized

$$z^4 + 5z^2 + 4 = (z^2 + 1)(z^2 + 4),$$

which shows that it has four simple zeros $\pm i, \pm 2i$. There are no common zeros between the numerator and the denominator, and therefore $f$ has a double zero at $z = -1$ and four simple poles, two at the upper half-plane $z = i, 2i$ and their conjugates in the lower half-plane. Since there are no poles on the real axis, we can apply Theorem 7.8.

(2) This gives the value $I$ for the integral $I = 2\pi i (\text{Res}(f, i) + \text{Res}(f, 2i))$.

To find the residues we know by 2o p. 139

$$\text{Res}(f, i) = \frac{(z + 1)^2}{4z^3 + 10z}|_{z=i} = \frac{1}{3}$$

and

$$\text{Res}(f, 2i) = \frac{(z + 1)^2}{4z^3 + 10z}|_{z=2i} = -\frac{i}{4} - \frac{1}{3},$$

hence

$$I = 2\pi i \left( \frac{1}{3} - \frac{i}{4} - \frac{1}{3} \right) = \frac{\pi}{2}.$$