Problem 3

(1) The given function is quotient of two holomorphic functions \( \varphi(z) = \frac{\sin(z) \log(1 + z)}{\psi(z)} = \frac{\sin(z)}{z^2} \) in \( G = \mathbb{C} \setminus \{ -\infty, -1 \} \), and since we divide by \( \psi(z) \) which has a double zero at \( z = 0 \), we know that \( z = 0 \) is an isolated singularity for \( f \). Both \( \sin(z) \) and \( \log(1 + z) \) has a simple zero at \( z = 0 \), and therefore \( \varphi(z) \) has a double zero at \( z = 0 \) like \( \psi(z) \). This shows that \( z = 0 \) is a removable singularity. The limit of \( f \) for \( z \to 0 \) can be found using l'Hospital's rule:

\[
\lim_{z \to 0} f(z) = \frac{\varphi''(0)}{\psi''(0)},
\]

so we need

\[
\varphi''(z) = -\frac{2 \cos z}{1 + z} - \frac{\sin z}{(1 + z)^2}
\]

hence \( \varphi''(0) = 2 \) and finally \( \lim_{z \to 0} f(z) = 2/2 = 1 \).

Alternatively we can use the power series for \( \sin(z) \) and \( \log(1 + z) \) giving

\[
\sin(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots,
\]

\[
\log(1 + z) = 1 - \frac{z}{2} + \frac{z^2}{3} - \cdots
\]

(the first one for all \( z \in \mathbb{C} \), the second for \( |z| < 1 \)), and now it is clear that \( \lim_{z \to 0} f(z) = 1 \).

(2) Giving \( f \) the value 1 at \( z = 0 \), \( f \) becomes holomorphic in \( G \), so the power series for \( f \), valid in the largest open disc in \( G \) centered at 0, i.e. in \( K(0, 1) \), is

\[
f(z) = \sum_{n=0}^{\infty} a_n z^n = \left( 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots \right) \left( 1 - \frac{z}{2} + \frac{z^2}{3} - \cdots \right)
\]

showing that \( a_0 = 1, a_1 = -1/2 \). Notice that \( a_n = f^{(n)}(0)/n! \), so we already knew that \( a_0 = f(0) = 1 \), and now we know also that \( f'(0) = -1/2 \).

The product above has been calculated by what is called Cauchy multiplication.

(3) From the power series for \( f \) we get

\[
\frac{f(z)}{z^2} = \frac{1}{z^2} - \frac{1}{2z} + \frac{1}{6} + \cdots
\]

showing that \( \text{Res}(f(z)/z^2, 0) = -1/2 \), hence

\[
\int_{\partial K(0, 1/2)} \frac{f(z)}{z^2} \, dz = 2\pi i \text{Res}(f(z)/z^2, 0) = -\pi i
\]

by the residue theorem. Note that we could also argue that

\[
a_1 = f'(0) = \frac{1}{2\pi i} \int_{\partial K(0, 1/2)} \frac{f(z)}{z^2} \, dz,
\]
by Cauchy’s integral theorem for the derivative, and in this way we get the result.

**Problem 4**

(1) By solving the two second degree equations $z^2 + 1 = 0$, $z^2 - 2iz - 2 = 0$, we find the four zeros $z = \pm i$, $z = i \pm 1$ in a polynomial of degree 4, hence we have them all and they are all simple.

(2) The rational function $1/p$ has four simple poles, 3 in the upper half-plane ($z = i, z = i \pm 1$), one ($z = -i$) in the lower half-plane and no poles on the real axis. The degree of the denominator is $4 > 2 + 0$, so we can apply Theorem 7.8 and using Remark 7.9 we get

$$\int_{-\infty}^{\infty} \frac{dx}{p(x)} = -2\pi i \text{Res}(1/p, -i) = -\pi/5$$

because we find

$$\text{Res}(1/p, -i) = 1/p'(-i) = 1/(10i).$$