Welcome to this three hours exam. You are allowed to use pencil and rubber if the writing is readable and you erase thoroughly. It is recommended to cross out if you do not want parts to be evaluated.

The exam consists of 4 problems and is formulated on two pages. If the problems are solved in a satisfactory way you get 100 points. Each problem has the value of 25 points as indicated below.

The exam has two parts. During the first 90 minutes you are not allowed to use any written material or other devices and you shall solve Problems 1 and 2. For the next 90 minutes you are allowed to use your notes, books, pocket calculators, but not personal computers and phones, and you shall deal with Problems 3 and 4.

After the first 90 minutes your answers of the first part will be collected. You are of course welcome to think about the Problems 3 and 4 during the first part.

First part without auxiliary materials

**Problem 1** (25 points)

1. Define a domain and a star-shaped domain in the complex plane.
2. Formulate Cauchy’s integral theorem for a star-shaped domain.
3. Cauchy’s integral formula is

\[ f(z_0) = \frac{1}{2\pi i} \oint_{\partial K(a,r)} \frac{f(z)}{z-z_0} \, dz, \quad z_0 \in K(a,r) \]

valid for \( f \in \mathcal{H}(G) \), where \( G \) is an open subset of \( \mathbb{C} \) and \( \overline{K(a,r)} \subseteq G \).

Prove this formula. (You can use Cauchy’s integral theorem without proof).

**Problem 2** (25 points)

Let \( f \in \mathcal{H}(G) \) be a holomorphic function in a domain \( G \) and assume that \( f \) is not identically zero.

1. Prove that if \( f(a) = 0 \) for some \( a \in G \), then there exists a unique natural number \( n \) and a unique \( g \in \mathcal{H}(G) \) with \( g(a) \neq 0 \) such that

\[ f(z) = (z-a)^n g(z), \quad z \in G. \]

Problem 2 is continued on page 2
(2) Prove that \( n = \text{ord}(f, a) \) is characterized by the equations
\[
    f(a) = f'(a) = \ldots = f^{(n-1)}(a) = 0, \quad f^{(n)}(a) \neq 0.
\]

(3) Let \( f(z) = (\sin z)^5 \). Find \( \text{ord}(f, 0) \).

Second part with auxiliary materials

**Problem 3** (25 points)

(1) Show that the function
\[
f(z) = \frac{(\sin z) \log(1 + z)}{z^2},
\]
(where \( \log \) denotes the principal logarithm \( \log : \mathbb{C}_\pi \to \mathbb{C} \)) has a removable singularity at \( z = 0 \) and find \( \lim_{z \to 0} f(z) \).

(2) Explain that \( f(z) \) has a power series expansion
\[
f(z) = \sum_{n=0}^{\infty} a_n z^n
\]
valid for \( |z| < 1 \), and find \( a_0, a_1 \).

(3) Find the value of
\[
\int_{\partial K(0, \frac{1}{2})} \frac{f(z)}{z^2} \, dz,
\]
where it is assumed, as usual, that the indicated path is traversed counterclockwise.

**Problem 4** (25 points)

(1) Find the zeros of the polynomial
\[
p(z) = (z^2 + 1)(z^2 - 2iz - 2).
\]

(2) Show that
\[
\int_{-\infty}^{\infty} \frac{dx}{p(x)} = -\frac{\pi}{5}.
\]
It shall be possible to follow your calculations step by step.