Complex Analysis (KomAn)

Welcome to this three hours exam. You are allowed to use pencil and rubber if the writing is readable and you erase thoroughly. It is recommended to cross out if you do not want parts to be evaluated.

The exam consists of 4 problems and is formulated on two pages. If the problems are solved in a satisfactory way you get 100 points. Each problem has the value of 25 points as indicated below.

The exam has two parts. During the first 90 minutes you are not allowed to use any written material or other devices and you shall solve problems 1 and 2. For the next 90 minutes you are allowed to use your notes, books, pocket calculators, but not personal computers, and you shall deal with problems 3 and 4.

After the first 90 minutes your answers of the first part will be collected. You are of course welcome to think about the problems 3 and 4 during the first part.

First part without auxiliary materials

Problem 1 (25 points)

(1) Formulate Cauchy’s integral formula for $f$ and $f'$, where $f$ is a holomorphic function.

(2) Prove the following theorem: Let $G \subseteq \mathbb{C}$ be open. If a sequence $f_1, f_2, \ldots$ from $\mathcal{H}(G)$ converges locally uniformly in $G$ to a function $f : G \rightarrow \mathbb{C}$, then $f$ is holomorphic and $f'_n \rightarrow f'$ locally uniformly in $G$.

(Morera’s Theorem can be used without proof).

Problem 2 (25 points)

The exponential function $\exp$ is defined by the power series

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad z \in \mathbb{C}.$$ 

(1) Explain why $\exp$ is a holomorphic function satisfying

$$\frac{d \exp(z)}{dz} = \exp(z).$$

(2) Prove that $\exp(z_1 + z_2) = \exp(z_1) \exp(z_2)$, $z_1, z_2 \in \mathbb{C}$.

(3) Determine the set

$$\{z \in \mathbb{C} \mid \exp(z) = i\}.$$
Problem 3 (25 points)

Consider the function

\[ f(z) = \frac{z - \log(1 + z)}{z^2}, \quad z \in G = \mathbb{C} \setminus (-\infty, -1], \]

where \( \log \) denotes the principal logarithm.

(1) Prove that \( z = 0 \) is a removable singularity of \( f \) and find the value \( \lim_{z \to 0} f(z) \).

(2) Find the power series of \( f \) with centre 0.

(3) Prove that

\[ g(z) = \frac{z - \log(1 + z)}{z^2 \sin(z)}, \]

has a pole at \( z = 0 \). Determine its order and calculate \( \text{Res}(g, 0) \).

Problem 4 (25 points)

Consider the rational function

\[ f(z) = \frac{1}{(1 + z^2)^3}. \]

(1) Find the poles of \( f \) and calculate the residue at poles in the upper half-plane.

(2) Verify the formula

\[ \int_{-\infty}^{\infty} \frac{dx}{(1 + x^2)^3} = \frac{3\pi}{8}. \]

It is important that you explain step by step what you are doing.