Welcome to this three hours exam. You are allowed to use pencil and rubber if the writing is readable and you erase thoroughly. It is recommended to cross out if you do not want parts to be evaluated.

The exam consists of 4 problems and is formulated on two pages. If the problems are solved in a satisfactory way you get 100 points. Each problem has the value of 25 points as indicated below.

The exam has two parts. During the first 90 minutes you are not allowed to use any written material or other devices and you shall solve problems 1 and 2. For the next 90 minutes you are allowed to use your notes, books, pocket calculators etc. and you shall deal with problems 3 and 4.

After the first 90 minutes your answers of the first part will be collected. You are of course welcome to think about the problems 3 and 4 during the first part.

First part without auxiliary materials

Problem 1 (25 points)

(1) Define the concept \( \text{Res}(f, a) \), i.e. the residue of a function \( f \) at a point \( z = a \).

(2) Formulate and prove Cauchy’s residue theorem.

Problem 2 (25 points)

(1) Let \( G \) be a domain in \( \mathbb{C} \) and let \( f : G \to \mathbb{C} \) be a holomorphic function. Assume that \( |f| \) has a local maximum at \( a \in G \). Prove that \( f \) is constant in \( G \).

(2) Make a drawing of the set \( K = \{ z \in \mathbb{C} \mid \frac{1}{e} \leq |z| \leq 1, \text{Re} \, z \geq 0 \} \).

Let \( \text{Log} \) denote the principal logarithm defined in \( \mathbb{C} \setminus (-\infty, 0] \).

(3) Determine the number

\[
M = \max \{ |\text{Log}(z)| \mid z \in K \}
\]

and find the points \( z \in K \) such that \( M = |\text{Log}(z)| \).

The problem set is continued on page 2.
Problem 3 (25 points)

Consider the function
\[ f(z) = \frac{z - \sin(z)}{z^3}. \]

(1) Prove that \( z = 0 \) is a removable singularity for \( f \) and find the value \( \lim_{z \to 0} f(z) \).

(2) Explain that it is possible to consider \( f \) as an entire function and find its power series expansion with centre 0.

(3) Find the poles of the function
\[ g(z) = \frac{z - \sin(z)}{z^5 \cos(z)}, \]
determine their orders and calculate \( \text{Res}(g, 0) \).

Problem 4 (25 points)

Let \( a \in \mathbb{C} \) satisfy \( |a| < 1 \). Show that
\[
\int_{0}^{2\pi} \frac{e^{2it}}{1 - 2a \cos(t) + a^2} \, dt = \frac{2\pi a^2}{1 - a^2}
\]
by transforming it to an integral along \( \partial K(0, 1) \). It is important that your calculations can be followed step by step.