Welcome to this three hours exam. You are allowed to use pencil and rubber if the writing is readable and you erase thoroughly. It is recommended to cross out if you do not want parts to be evaluated.

The exam consists of 4 problems (called opgaver 1-4) and is formulated on two pages. If the problems are solved in a satisfactory way you get 100 points. Each problem has the value of 25 points as indicated below.

The exam has two parts. During the first 90 minutes you are not allowed to use any written material or other devices and you shall solve problems 1 and 2. For the next 90 minutes you are allowed to use your notes, books, pocket calculators etc. and you shall deal with problems 3 and 4.

After the first 90 minutes your answers of the first part will be collected. You are of course welcome to think about the problems 3 and 4 during the first part.

First part without help materials

Opgave 1 (25 points)

The exponential function $\exp: \mathbb{C} \to \mathbb{C}$ is defined by the power series in the usual way

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad z \in \mathbb{C}.$$ 

1) Explain that the exponential function is holomorphic in $\mathbb{C}$ and satisfies

$$\frac{d}{dz} \exp(z) = \exp(z).$$

2) Show the functional equation

$$\exp(z_1 + z_2) = \exp(z_1) \exp(z_2), \quad z_1, z_2 \in \mathbb{C}.$$ 

3) Define the number $e$ and show that $\exp(n) = e^n$ for $n \in \mathbb{Z}$.

4) Determine the set of solutions \( \{z \in \mathbb{C} \mid \exp(z) = -1\} \).

Opgave 2 (25 points)

Let $G \subseteq \mathbb{C}$ be an open set and $f: G \to \mathbb{C}$ a holomorphic function. Prove that $f$ is differentiable infinitely often and that the Taylor series with centre $a \in G$ is convergent.
with sum $f$ in the largest open disc $K(a, \rho) \subseteq G$:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z - a)^n, \quad z \in K(a, \rho).$$

(The Cauchy integral formula can be used without proof).

**Second part with help materials**

**Opgave 3** (25 points)

Consider the function $f(z) = \tan z$.

1) Find the zeros and poles of $f$, and determine their orders.

2) Show that $\frac{\tan z}{z}$ has a removable singularity for $z = 0$ and calculate

$$\lim_{z \to 0} \frac{\tan z}{z}.$$

3) Calculate the curve integral

$$\int_{\partial K(0, \pi)} \frac{\tan z}{z^2} dz,$$

where the curve in question is supposed to be the indicated circle followed once with positive orientation.

**Opgave 4** (25 points)

Consider the polynomial

$$p(z) = z^5 - 3z^3 + ez + 1.$$

1) Show that $p$ has 5 zeros (counted with multiplicity) in $K(0, 2)$.

2) Calculate the curve integral

$$\frac{1}{2\pi i} \int_{\partial K(0, 2)} \frac{5z^4 - 9z^2 + e}{z^5 - 3z^3 + ez + 1} dz,$$

where the curve in question is supposed to be the indicated circle followed once with positive orientation.

3) Calculate the curve integral

$$\frac{1}{2\pi i} \int_{\partial T} \frac{dz}{z^5 - 3z^3 + ez + 1},$$

where $T$ denotes the triangle with vertices $3, 2+i, 2-i$, and you are supposed to integrate along the boundary of $T$ once and using positive orientation.