Elementary exercises about complex numbers

Exc. 1. Let \( z_1 = 2 - 3i, z_2 = 4 + 6i \). Write \( z_1/z_2 \) as \( x + iy, x, y \in \mathbb{R} \).

Exc. 2. Find \( r = |z| \) and \( \arg(z) \) for the following complex numbers

\[
z = 1, i, -i, 1 + i, 1 - i, -1 + i, -1 - i, z = 1 + i\sqrt{3}, z = -2 + i2\sqrt{3}
\]

and mark them on a drawing of the complex plane.

Exc. 3. Find the following powers of \( i \):

\[i^8, i^{42}, i^{11}, i^{105}\]

Exc. 4. Describe the sets

\[
\{ z \in \mathbb{C} \mid |z - i| \leq 1 \}, \quad \{ z \in \mathbb{C} \mid |z - i| = 1 \}, \\
\{ z \in \mathbb{C} \mid |z - i| < 1 \}, \quad \{ z \in \mathbb{C} \mid |z - i| > 1 \}.
\]

Exc. 5. For \( z, w \in \mathbb{C} \) prove

\[
|z + w|^2 + |z - w|^2 = 2|z|^2 + 2|w|^2.
\]

Why is this equation called the law of the parallelogram?

Exc. 6. Let \( a \in \mathbb{C} \setminus \{0\}, b \in \mathbb{R} \). Describe the following sets

\[
\{ z \in \mathbb{C} \mid \text{Re}(az) = b \}, \quad \{ z \in \mathbb{C} \mid \text{Im}(az) = b \}.
\]

Exc. 7. Given \( a, b \in \mathbb{R}, b \neq 0 \), show that the equation \( z^2 = a + ib \) has the solutions

\[
z = \pm \left( \sqrt{\frac{r + a}{2}} + i \sqrt{\frac{r - a}{2}} \right) \quad \text{when } b > 0
\]

\[
z = \pm \left( \sqrt{\frac{r + a}{2}} - i \sqrt{\frac{r - a}{2}} \right) \quad \text{when } b < 0,
\]

where \( r = |a + ib| = \sqrt{a^2 + b^2} \). (When \( c > 0 \) we will here use the convention that \( \sqrt{c} > 0 \).)

Solve the equations \( z^2 = 3 + i4, z^2 = -3 + i4, z^2 = 3 - i4, z^2 = -3 - i4 \).