Problem set for March 18, 2010

Problem 3.1 (i) Show that the function

$$f(z) = \frac{\exp(z) - 1 - z}{z^2}$$

has a removable singularity for $z = 0$ and find $\lim_{z \to 0} f(z)$.

(ii) Explain that f can be considered as an entire holomorphic function and determine the power series of f and its radius of convergence.

(iii) Explain that the number

$$\frac{1}{2\pi i} \int_{\partial K(0,r)} \frac{\exp(z) - 1 - z}{z^3} \, dz$$

is independent of $r > 0$ and find this number.

Problem 3.2 (i) Let G be a domain in $\mathbb{C}$ and assume that $f, g \in \mathcal{H}(G)$. Assume that $z = a \in G$ is a zero of f of order p and a zero of g of order q, i.e. $\operatorname{ord}(f, a) = p, \operatorname{ord}(g, a) = q$.

Show that $h = fg$ has a zero of order $p + q$ for $z = a$, i.e. that

$$\operatorname{ord}(fg, a) = \operatorname{ord}(f, a) + \operatorname{ord}(g, a).$$

(ii) Find the zeros of $f(z) = z(\sin z)^2 \in \mathcal{H}(\mathbb{C})$ and determine the order of these zeros.

(iii) With f as in (ii) determine $n \in \mathbb{Z}$ such that $f(z)z^n$ has a pole of order 2 for $z = 0$.