University of Copenhagen
Exam at the Faculty of Science April 2010

Complex Analysis (KomAn)

Welcome to this three hours exam. You are allowed to use pencil and rubber if the writing is readable and you erase thoroughly. It is recommended to cross out if you do not want parts to be evaluated.

The exam consists of 4 problems and is formulated on two pages. If the problems are solved in a satisfactory way you get 100 points. Each problem has the value of 25 points as indicated below.

The exam has two parts. During the first 90 minutes you are not allowed to use any written material or other devices and you shall solve problems 1 and 2. For the next 90 minutes you are allowed to use your notes, books, pocket calculators, but not personal computers, and you shall deal with problems 3 and 4.

After the first 90 minutes your answers of the first part will be collected. You are of course welcome to think about the problems 3 and 4 during the first part.

First part without auxiliary materials

Problem 1 (25 points)

(1) Define the concept of a holomorphic function and explain the meaning of a simply connected domain in intuitive terms. Formulate Cauchy's integral theorem.

(2) Prove Goursat's Lemma: *Let $G \subseteq \mathbb{C}$ be open and let $f \in \mathcal{H}(G)$. Then*

$$\int_{\partial\triangle} f(z)\,dz = 0$$

for every solid triangle $\triangle \subseteq G$.

Problem 2 (25 points)

(1) Let f be a rational function

$$f(z) = \frac{p(z)}{q(z)} = \frac{a_0 + a_1 z + \cdots + a_m z^m}{b_0 + b_1 z + \cdots + b_n z^n}\,, \quad a_m \neq 0\,, b_n \neq 0\,.$$

Prove that f has at most n poles counted with multiplicity in the complex plane. You can use the fundamental theorem of algebra without proof.

(2) Assume now that $n \geq m+2$ and that f has no poles on the real axis. Prove that

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{j=1}^{k} \mathrm{Res}(f, z_j)\,,$$

Problem 2 is continued on page 2

where $z_1, \dots, z_k$ are the poles of f in the upper half-plane. You can use the residue theorem without proof.

Second part with auxiliary materials

Problem 3 (25 points)

Let G denote the domain $G = \mathbb{C} \setminus [1, \infty[$ in $\mathbb{C}$.

(1) Explain that the function

$$f(z) = \exp\left(-\frac{1}{2}\operatorname{Log}(1-z)\right)$$

is well-defined and holomorphic for $z \in G$. Here Log denotes the principal logarithm defined in $\mathbb{C}\setminus]-\infty, 0]$.

(2) Show that $(1-z)(f(z))^2 = 1$ for all $z \in G$.

(3) For $a \in \mathbb{C}$ and $n = 0, 1, 2, \dots$ define

$$(a)_0 = 1,\ \ (a)_1 = a,\ \ (a)_2 = a(a+1), \dots, (a)_n = a(a+1)\cdot \ldots \cdot (a+n-1), \dots .$$

Show that f has the power series expansion

$$f(z) = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n}{n!} z^n \quad \text{for} \quad z \in K(0,1).$$

Problem 4 (25 points)

Consider the meromorphic function in $\mathbb{C}$

$$f(z) = \frac{\sin(\pi z)}{z^2(z^2-1)}.$$

(1) Show that f has removable singularities for $z = \pm 1$, and find the values $f(1), f(-1)$ which make f differentiable for $z = \pm 1$.

(2) Find the zeros and poles of f and their order.

(3) Find the value of the integral

$$\frac{1}{2\pi i} \int_{\partial K(0,\pi)} \frac{f'(z)}{f(z)}\, dz,$$

where the indicated path is traversed counterclockwise.