University of Copenhagen
Exam at the Faculty of Science August 2009

Complex Analysis (KomAn)

Welcome to this three hours exam. You are allowed to use pencil and rubber if the writing is readable and you erase thoroughly. It is recommended to cross out if you do not want parts to be evaluated.

The exam consists of 4 problems and is formulated on two pages. If the problems are solved in a satisfactory way you get 100 points. Each problem has the value of 25 points as indicated below.

The exam has two parts. During the first 90 minutes you are not allowed to use any written material or other devices and you shall solve problems 1 and 2. For the next 90 minutes you are allowed to use your notes, books, pocket calculators, but not personal computers, and you shall deal with problems 3 and 4.

After the first 90 minutes your answers of the first part will be collected. You are of course welcome to think about the problems 3 and 4 during the first part.

First part without auxiliary materials

Problem 1 (25 points)

(1) Define a path $\gamma : [a,b] \to \mathbb{C}$ and the contour integral $\int_\gamma f$ of a continuous function f along γ.

(2) Define the concept of a primitive.

(3) Prove the following result: *Let $f : G \to \mathbb{C}$ be a continuous function on a domain $G \subseteq \mathbb{C}$ and assume that $\int_\gamma f = 0$ for every closed staircase line γ in G. Then f has a primitive in G.*

Problem 2 (25 points)

(1) Define a Möbius transformation and explain that $f(z) = az, a \neq 0$, $f(z) = z + b$ and $f(z) = 1/z$ are Möbius transformations.

(2) Define a generalized circle.

(3) Prove that every Möbius transformation maps a generalized circle into a generalized circle.

The problem set is continued on page 2

Second part with auxiliary materials

Problem 3 (25 points)

Let G denote the domain $G = \mathbb{C} \setminus [1, \infty[$ in $\mathbb{C}$.

(1) Explain that the function

$$f(z) = \begin{cases} \frac{\text{Log}(1-z)}{-z}, & z \in G,\ z \neq 0 \\ 1\,, & z = 0 \end{cases}$$

is holomorphic in G. Here Log denote the principal logarithm defined in $\mathbb{C} \setminus]-\infty, 0]$.

(2) Show that f has the power series

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n+1}, \quad |z| < 1.$$

(3) For each $z \in G$ define

$$D(z) = \int_{[0,z]} f(u)\, du,$$

where the path of integration is the segment $[0, z] = \{tz \mid t \in [0, 1]\}$ from 0 to z. Show that D is holomorphic in G and that D has the power series

$$D(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}, \quad |z| < 1.$$

Problem 4 (25 points)

Consider the meromorphic function in $\mathbb{C}$

$$f(z) = \frac{1}{1 + \sin z}.$$

(1) Find the poles of f and their order.

(2) Find the zeros and poles of the function

$$g(z) = \frac{\cos z}{1 + \sin z}$$

and prove that they are all simple.

(3) Prove that

$$\frac{1}{2\pi i} \int_{\partial K(0,\pi)} \frac{\cos z}{1 + \sin z}\, dz = 2,$$

where the indicated path is traversed counterclockwise.