

Solutions to the exam in Complex Analysis (KomAn), April 2007

Problem 3

1+2) We know that $e^z = 1$ if and only if $z = 2p\pi i, p \in \mathbb{Z}$ and these numbers are simple zeros of $e^z - 1$. Therefore $z = 0$ is a double zero and $z = 2p\pi i, p \neq 0$ a simple zero of the denominator. Using that the numerator has no zeros, we get that these numbers are poles of f of order 2 and 1 respectively. There are no removable singularities.

To calculate the residue of f at $z = 0$ we look at

$$\varphi(z) = z^2 f(z) = \frac{z}{e^z - 1}$$

and know that $\text{Res}(f, 0) = \varphi'(0)$. The function φ is studied in Exercise 6.12 and we know from there that $\varphi'(0) = B_1 = -1/2$, hence $\text{Res}(f, 0) = -1/2$.

The function $z^2 + 4\pi^2$ has two simple zeros $z = \pm 2\pi i$, and therefore these two numbers become removable singularities for g . The limit can be found by l'Hospital's rule:

$$\lim_{z \rightarrow \pm 2\pi i} g(z) = \lim_{z \rightarrow \pm 2\pi i} \frac{2z}{e^z - 1 + ze^z} = \frac{\pm 4\pi i}{\pm 2\pi i} = 2.$$

Problem 4

1) We get $p(i) = 2 + 2i - 5 - 2i + 3 = 0$. Since p has real coefficients also $p(-i) = 0$, so $z^2 + 1$ divides p . By division we get

$$p(z) = (z^2 + 1)(2z^2 - 2z + 3)$$

and the second factor has the zeros $z = (1 \pm i\sqrt{5})/2$.

2) The integral can be calculated using Theorem 7.8. The numerator has degree 0 and the denominator degree 4 and it has no zeros on the real axis. There are two poles i and $(1 + i\sqrt{5})/2$ in the upper half-plane, so we get

$$\int_{-\infty}^{\infty} \frac{dx}{p(x)} = 2\pi i \left(\text{Res}\left(\frac{1}{p}, i\right) + \text{Res}\left(\frac{1}{p}, \frac{1 + i\sqrt{5}}{2}\right) \right).$$

We have $p'(z) = (z^2 + 1)(4z - 2) + 2z(2z^2 - 2z + 3)$, hence $p'(i) = 4 + 2i$ and

$$p'\left(\frac{1 + i\sqrt{5}}{2}\right) = \left(\left(\frac{1 + i\sqrt{5}}{2}\right)^2 + 1 \right) (2i\sqrt{5}) = -5,$$

hence

$$\begin{aligned} \text{Res}\left(\frac{1}{p}, i\right) &= \frac{1}{p'(i)} = \frac{1}{4 + 2i} = \frac{1}{5} - \frac{i}{10} \\ \text{Res}\left(\frac{1}{p}, \frac{1 + i\sqrt{5}}{2}\right) &= \frac{1}{p'\left(\frac{1 + i\sqrt{5}}{2}\right)} = -\frac{1}{5}. \end{aligned}$$

This gives

$$\int_{-\infty}^{\infty} \frac{dx}{p(x)} = 2\pi i \left(\frac{1}{5} - \frac{i}{10} - \frac{1}{5} \right) = \frac{\pi}{5}.$$