Problem set for October 29, 2009 (Corrected version)

Problem 1. Prove that $C_c^2(\mathbb{R}) \subset \mathcal{F}(\mathcal{L}^1(\mathbb{R}))$, i.e. that any C^2-function f with compact support is the Fourier transform of some integrable function.

Problem 2.

1°. Show that $f = 1_{[-1,1]} * 1_{[-1,1]}$ is given by

$$f(x) = \begin{cases} 2 - |x| & \text{if } |x| < 2 \\ 0 & \text{if } |x| \geq 2. \end{cases}$$

2°. Show that

$$\int_{-\infty}^{\infty} \left(\frac{\sin t}{t} \right)^2 dt = \pi.$$

3° Show that

$$\int_{-\infty}^{\infty} \left(\frac{\sin t}{t} \right)^4 dt = \frac{2\pi}{3}.$$

Problem 3. As usual $\mathbb{C}[x]$ denotes the vector space of polynomials in x with complex coefficients and $\mathbb{C}_n[x]$ denotes the subspace of polynomials of deegree $\leq n$. Define $V = \mathbb{C}[x]e^{-\pi x^2}$, i.e.,

$$V = \{p(x)e^{-\pi x^2} \mid p \in \mathbb{C}[x]\}$$

and similarly $V_n = \mathbb{C}_n[x]e^{-\pi x^2}$, $n = 0, 1, \ldots$. In the following you can use that $\mathcal{F}(e^{-\pi x^2})(t) = e^{-\pi t^2}$, cf. **E 3.2**, and since $V \subset \mathcal{S} \subset L^2(\mathbb{R})$, it is clear that V_n is a Hilbert space of dimension $n+1$ with the inner product from $L^2(\mathbb{R})$.

1°. Show that for $n = 0, 1, \ldots$

$$\mathcal{F}(x^n e^{-\pi x^2})(t) = P_n(t)e^{-\pi t^2},$$

where $P_n(t)$ is a polynomial of degree n satisfying

$$P_{n+1}(t) = \frac{i}{2\pi} P_n'(t) - itP_n(t).$$

Conclude that the leading coefficient of P_n is $(-i)^n$ and calculate P_n for $n = 0, 1, 2, 3$.

2°. Show that $\mathcal{F}(V_n) \subseteq V_n$ and conclude that $\mathcal{F}$ is a unitary transformation of V_n for each n. (Unitary means isomorphism of V_n preserving the inner product from $L^2(\mathbb{R})$.)

3°. Using the Gram-Schmidt procedure on the sequence

$$e^{-\pi x^2}, xe^{-\pi x^2}, x^2 e^{-\pi x^2}, \dots, x^n e^{-\pi x^2},$$

prove that there is a unique sequence $H_k, k = 0, 1, \dots, n$ of polynomials with real coefficients such that H_k is of degree k with positive leading coefficient and such that $H_k(x)e^{-\pi x^2}, k = 0, 1, \dots, n$ is an orthonormal system in V_n, i.e.,

$$\int_{-\infty}^{\infty} H_k(x) H_l(x) e^{-2\pi x^2}\, dx = \delta_{kl}.$$

4°. Prove (e.g. by induction) that

$$\mathcal{F}(H_k(x)e^{-\pi x^2})(t) = (-i)^k H_k(t) e^{-\pi t^2},\ k = 0, 1, \dots, n.$$

(Historical information: The polynomials H_k are scaled versions of the Hermite polynomials.)