Extra Exercises for Chapter 2

E 1.2 For $f \in \mathcal{L}^1(\mathbb{R})$ and $a > 0$ let f_a denote the function equal to f on $]-a, a]$ and extended to a periodic function on $\mathbb{R}$ with period $2a$.

Show that the Fourier series of f_a can be written as

$$f_a(x) \sim \sum_{n=-\infty}^{\infty} c_{a,n} e^{\mathrm{i}n\frac{\pi}{a}x}, \quad \text{where} \quad c_{a,n} = \frac{1}{2a}\int_{-a}^{a} f(x) e^{-\mathrm{i}n\frac{\pi}{a}x}\, dx$$

and as

$$f_a(x) \sim \sum_{n=-\infty}^{\infty} \frac{1}{2a} g_a\left(\frac{n}{2a}\right) e^{\mathrm{i}n\frac{\pi}{a}x}, \quad \text{where} \quad g_a(y) = \int_{-a}^{a} f(x) e^{-\mathrm{i}2\pi xy}\, dx. \tag{0.1}$$

Show that $g_a(y) \to \mathcal{F}f(y)$ for $a \to \infty$ for each $y \in \mathbb{R}$.

Show that the sum in (0.1) can be considered as an infinite Riemann sum and explain that formally it approaches

$$\int_{-\infty}^{\infty} \mathcal{F}f(y) e^{\mathrm{i}2\pi xy}\, dy,$$

so one is tempted to claim that this integral equals $f(x)$, i.e. that

$$f(x) = \int_{-\infty}^{\infty} \mathcal{F}f(y) e^{\mathrm{i}2\pi xy}\, dy. \tag{0.2}$$

A lot of research for 200 years has been spent trying to make this rigorous.

Explain that one can write the formula (0.2) in the following equivalent forms

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(y) e^{\mathrm{i}xy}\, dy, \quad \text{where} \quad \widehat{f}(y) = \int_{-\infty}^{\infty} f(x) e^{-\mathrm{i}xy}\, dx,$$

and

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \widehat{f}(y) e^{\mathrm{i}xy}\, dy, \quad \text{where} \quad \widehat{f}(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) e^{-\mathrm{i}xy}\, dx.$$

E 1.3 Let $f \in \mathcal{L}^1(\mathbb{R})$ and assume that $f(x) \geq 0$ for almost all x and $\int f(x)\,dx = 1$. (In other words f is density for a probability measure). Prove that $|\mathcal{F}f(t)| < 1$ for all $t \neq 0$. (Of course $\mathcal{F}f(0) = 1$.)

E 3.1 Let $f \in \mathcal{L}^1(\mathbb{R})$ and $h \in \mathbb{R}$. Prove that

$$\mathcal{F}(f(x)e^{2\pi i h x})(t) = \mathcal{F}(f)(t-h), \quad \mathcal{F}(\tau_h f)(t) = \mathcal{F}(f)(t)e^{-2\pi i h t}$$

E 3.2 From complex analysis it is known that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itx} e^{-\frac{1}{2}x^2}\,dx = e^{-\frac{1}{2}t^2}, \quad t \in \mathbb{R}.$$

Show that $e^{-\pi x^2}$ is a fixed point for $\mathcal{F}$.

E 3.3 Define $f(x) = 1/\cosh(\pi x),\ x \in \mathbb{R}$.

1°. Prove that $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

2°. Prove that

$$f * f(x) = \frac{2x}{\sinh(\pi x)},$$

where the right-hand side for $x = 0$ is to be understood as the limit for $x \to 0$.

3°. Prove that $\mathcal{F}(f)(t) = 1/\cosh(\pi t),\ t \in \mathbb{R}$, i.e., that $f(x) = 1/\cosh(\pi x)$ is a fixed point for $\mathcal{F}$ (or an eigenvector for $\mathcal{F}$ corresponding to the eigenvalue 1 e.g. in the space $\mathcal{S}$ to be introduced in section 8).

Hint: Use the residue theorem for the function

$$F(z) = \frac{e^{-2\pi i t z}}{\cosh(\pi z)},$$

which is meromorphic in $\mathbb{C}$. Integrate F along the sides of the rectangle with vertices $\pm R, \pm R + i$, where $R > 0$ is fixed and let $R \to \infty$.