

Solutions to Koman, August 2008

Problem 3

(1) To see that $f(z) = z^3 \cos(1/z)$ is holomorphic in $\mathbb{C} \setminus \{0\}$, we use that z^3 , $\cos(z)$ are holomorphic in \mathbb{C} and $1/z$ is holomorphic in $\mathbb{C} \setminus \{0\}$. Then the composition $\cos(1/z)$ is holomorphic in $\mathbb{C} \setminus \{0\}$ and so is the product with z^3 .

(2) In the power series for \cos we insert the argument $1/z$ and find

$$\cos(1/z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{-2k}}{(2k)!}$$

valid for all $z \neq 0$. Multiplying this by z^3 gives

$$f(z) = z^3 - \frac{z}{2!} + \frac{1}{4!z} - \frac{1}{6!z^3} + \dots,$$

which is a Laurent series with centre 0, converging locally uniformly to $f(z)$. It is therefore the Laurent series for f with centre 0. There are infinitely many negative powers present, so by Theorem 6.21 (iii) we get that $z = 0$ is an essential singularity.

(3) The coefficient to z^{-1} is the residue. Therefore

$$\operatorname{Res}(f, 0) = \frac{1}{24}.$$

Problem 4

(1) The numerator has a double zero at $z = -1$. The denominator can be factorized

$$z^4 + 5z^2 + 4 = (z^2 + 1)(z^2 + 4),$$

which shows that it has four simple zeros $\pm i, \pm 2i$. There are no common zeros between the numerator and the denominator, and therefore f has a double zero at $z = -1$ and four simple poles, two at the upper half-plane $z = i, 2i$ and their conjugates in the lower half-plane. Since there are no poles on the real axis, we can apply Theorem 7.8.

(2) This gives the value I for the integral

$$I = 2\pi i (\operatorname{Res}(f, i) + \operatorname{Res}(f, 2i)).$$

To find the residues we know by 2° p. 139

$$\operatorname{Res}(f, i) = \frac{(z+1)^2}{4z^3 + 10z} \Big|_{z=i} = \frac{1}{3}$$

and

$$\operatorname{Res}(f, 2i) = \frac{(z+1)^2}{4z^3 + 10z} \Big|_{z=2i} = -\frac{i}{4} - \frac{1}{3},$$

hence

$$I = 2\pi i \left(\frac{1}{3} - \frac{i}{4} - \frac{1}{3} \right) = \frac{\pi}{2}.$$