

# Problems for October 27, 2008

CB

**Problem 1:** E 2.6.2 (p. 89).

**Problem 2:** Define the Legendre polynomials by

$$P_n(x) = \frac{(-1)^n}{2^n n!} D^n \{(1-x^2)^n\}, \quad (1)$$

where  $D^n$  denotes differentiation  $n$  times with respect to  $x$ .

Explain that  $P_n(x)$  is a polynomial of degree  $n$  with leading coefficient  $2^{-n} \binom{2n}{n}$  for each  $n$  and that it can also be given by the formula

$$P_n(x) = 2^{-n} \sum_{k=0}^n \binom{n}{k}^2 (x-1)^k (x+1)^{n-k}. \quad (2)$$

Show that  $P_n(1) = 1$  and prove the orthogonality relation

$$\frac{1}{2} \int_{-1}^1 P_n(x) P_m(x) dx = \frac{\delta_{nm}}{2n+1}. \quad (3)$$

Let  $p_n(x) = \sqrt{2n+1} P_n(x)$  denote the orthonormal Legendre polynomials (with respect to the probability measure  $\frac{1}{2} 1_{[-1,1]}(x) dx$ ). Prove the two equivalent formulas

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x) \quad (4)$$

$$xp_n(x) = b_n p_{n+1}(x) + b_{n-1} p_{n-1}(x), \quad b_n = \frac{n+1}{\sqrt{(2n+1)(2n+3)}}. \quad (5)$$

(Hint: Remember that the leading coefficient of  $p_n$  is  $1/(b_0 \dots b_{n-1})$ .)

Explain why  $a_n$  from the three term recurrence relation is 0.

**Problem 3:** E 2.6.4. (p. 90).

**Problem 4:** E 2.7.1. (p. 108).