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Solutions to Problem set 3

Exc. 1. Let $f : [0, \infty[\to \mathbb{R}$ be either non-negative and increasing or integrable over $[0, R]$ for each $R > 0$ and define

$$F(x) = \int_0^x f(t)\, dt, \quad x \geq 0.$$

(a) If $f(x) = O(x^\alpha)$ as $x \to \infty$, where $\alpha \geq 0$, then for suitable $x_0, K > 0$ we have $|f(x)| \leq Kx^\alpha$ for $x \geq x_0$. Then for $x \geq x_0$

$$|F(x)| \leq \int_0^{x_0} |f(t)|\, dt + \int_{x_0}^x Kt^\alpha\, dt = C + \frac{K}{\alpha+1} x^{\alpha+1}$$

for a suitable constant C, independent of x. For $x \geq x_0$ we therefore get

$$\frac{|F(x)|}{x^{\alpha+1}} \leq \frac{C}{x_0^{\alpha+1}} + \frac{K}{\alpha+1},$$

i.e. $F(x) = O(x^{\alpha+1})$.

(b) If $f(x) = o(x^\alpha)$ as $x \to \infty$, where $\alpha \geq 0$, then for arbitrary $\varepsilon > 0$ there exists x_0 such that $|f(x)| \leq \varepsilon x^\alpha$ for $x \geq x_0$. Then for $x \geq x_0$

$$|F(x)| \leq \int_0^{x_0} |f(t)|\, dt + \int_{x_0}^x \varepsilon t^\alpha\, dt = C + \frac{\varepsilon}{\alpha+1} x^{\alpha+1},$$

where C is a suitable constant indenpendent of x, hence for $x \geq x_1 \geq x_0$, where x_1 is chosen so big that $C/x_1^{\alpha+1} \leq \varepsilon$,

$$\frac{|F(x)|}{x^{\alpha+1}} \leq \frac{C}{x^{\alpha+1}} + \frac{\varepsilon}{\alpha+1} \leq \varepsilon \left(1 + \frac{1}{\alpha+1}\right),$$

i.e. $F(x) = o(x^{\alpha+1})$.

(In the case of f being increasing one can simply use $F(x) \leq xf(x)$.)

(c) If $f(x) \sim Cx^\alpha$, then for arbitrary $\varepsilon > 0$ there exists x_0 such that for $t \geq x_0$

$$C - \varepsilon \leq \frac{f(t)}{t^\alpha} \leq C + \varepsilon.$$

Multiplying these inequalities with t^α and integrating with respect to t over $[x_0, x], x > x_0$ gives

$$\frac{C-\varepsilon}{\alpha+1}(x^{\alpha+1} - x_0^{\alpha+1}) \leq \int_{x_0}^x f(t)\, dt \leq \frac{C+\varepsilon}{\alpha+1}(x^{\alpha+1} - x_0^{\alpha+1}).$$

Adding the constant $K=\int_0^{x_0} f(t)\,dt$ to these inequalities and dividing by $x^{\alpha+1}$, we get for $x>x_0$:

$$\frac{K}{x^{\alpha+1}}+\frac{C-\varepsilon}{\alpha+1}(1-(x_0/x)^{\alpha+1})\le\frac{F(x)}{x^{\alpha+1}}\le\frac{K}{x^{\alpha+1}}+\frac{C+\varepsilon}{\alpha+1}(1-(x_0/x)^{\alpha+1}).$$

Taking lim inf and lim sup in these inequalities for $x\to\infty$ we get

$$\frac{C-\varepsilon}{\alpha+1}\le\liminf_{x\to\infty}\frac{F(x)}{x^{\alpha+1}}\le\limsup_{x\to\infty}\frac{F(x)}{x^{\alpha+1}}\le\frac{C+\varepsilon}{\alpha+1},$$

and since ε is arbitrary, this forces lim inf and lim sup, which are independent of ε, to be equal to $C/(\alpha+1)$, showing that the limit exists and is equal to this constant, i.e. $F(x)\sim(C/(\alpha+1))x^{\alpha+1}$.

Exc. 2. (a) By Euler's formulas the equation $\cos z=a$ is equivalent to solving $e^{2iz}-2ae^{iz}+1=0$, which has the solutions $e^{iz}=a\pm\sqrt{a^2-1}$. The product of the two roots is 1 so they are never zero, and if e^{iz} is a solution then the other is e^{-iz}. We then find

$$z=\pm(i\ln|a+\sqrt{a^2-1}|+\operatorname{Arg}(a+\sqrt{a^2-1}))+2p\pi,p\in\mathbb{Z}.$$

Here Arg is the principal argument belonging to $]-\pi,\pi]$, and there is a choice of the square root. We see that there are infinitely many solutions of the form

$$z_p=\pm z_0+2p\pi,p\in\mathbb{Z}$$

where $z_0=\operatorname{Arg}(a+\sqrt{a^2-1})+i\ln|a+\sqrt{a^2-1}|$. They lie on the two lines $y=\pm\ln|a+\sqrt{a^2-1}|$ with a horizontal spacing of 2π. Without loss of generality we can assume that the square root is chosen such that $0\le\operatorname{Re}z_0\le\pi$.

To examine if the equation $f(z)=\cos z-a=0$ has simple or multiple zeros, we differentiate and find $f'(z)=-\sin z$, which has only the zeros $p\pi$ and they are not zeros of $f''(z)=-\cos z$. Since $\cos(p\pi)=\pm1$ we see that $a=1$ has the solutions $z=2p\pi$ each of order 2, $a=-1$ has the solutions $z=\pi+2p\pi$ also of order 2 and for $a\ne\pm1$ the solutions are of order 1.

(b) We distinguish 3 cases:

(b1) If $\operatorname{Re}z_0=0$, i.e. $z_0=iy_0$ (so $a=\cosh(y_0)$), then the solutions are $z_k=\pm iy_0+2p\pi$. Defining $r_p=|z_0+2p\pi|,p=0,1,\ldots$, then

$$0\le r_0<r_1<r_2<\cdots$$

and $|\pm z_0\pm2p\pi|=r_p$. Therefore

$$n(r,a,\cos)=2+4p,\quad r_p\le r<r_{p+1},p=0,1,\ldots,\tag{1}$$

hence with r as above

$$\frac{2+4p}{r_{p+1}}\le\frac{n(r,a,\cos)}{r}\le\frac{2+4p}{r_p}.$$

Since $r_p = 2p\pi|1 + z_0/2p\pi|$, we see that the left and right expression above tend to $2/\pi$ for $p \to \infty$ and hence $n(r, a, \cos) \sim 2r/\pi$.

(b2) If $\operatorname{Re} z_0 = \pi$, i.e. $z_0 = \pi + iy_0$ (so $a = -\cosh(y_0)$), then defining $r_0 = |z_0|, r_p = |z_0 + 2p\pi|, p = 1, 2, \dots$ we have

$$n(r, a, \cos) = 4(p + 1), \quad r_p \le r < r_{p+1}, p = 0, 1, \dots, \tag{2}$$

hence with r as above

$$\frac{4 + 4p}{r_{p+1}} \le \frac{n(r, a, \cos)}{r} \le \frac{4 + 4p}{r_p}.$$

Since $r_p = 2p\pi|1 + z_0/2p\pi|$, we see as in (b1) that $n(r, a, \cos) \sim 2r/\pi$.

(b3) If $0 < \operatorname{Re} z_0 < \pi$ there is exactly one solution of $\cos z = a$ in every strip $p\pi < \operatorname{Re} z < (p + 1)\pi, p \in \mathbb{Z}$. Putting $r_{2p} = |z_0 + 2p\pi|, r_{2p+1} = |-z_0 + (2p + 2)\pi|$ for $p = 0, 1, \dots$, then $r_0 < r_1 < r_2 < r_3 < \cdots$ and

$$n(r, a, \cos) = 2 + 4p, \quad r_{2p} \le r < r_{2p+1}$$

$$n(r, a, \cos) = 4 + 4p, \quad r_{2p+1} \le r < r_{2p+2},$$

leading to

$$\frac{2 + 4p}{r_{2p+1}} \le \frac{n(r, a, \cos)}{r} \le \frac{2 + 4p}{r_{2p}}, \quad r_{2p} \le r < r_{2p+1}$$

respectively

$$\frac{4 + 4p}{r_{2p+2}} \le \frac{n(r, a, \cos)}{r} \le \frac{4 + 4p}{r_{2p+1}}, \quad r_{2p+1} \le r < r_{2p+2}.$$

Like in (b1) and (b2) this leads to $n(r, a, \cos) \sim 2r/\pi$.

(c) When $a = 1$ we have $n(0, 1, \cos) = 2$ and for $a \neq 1$ we have $n(0, a, \cos) = 0$. Therefore

$$N(r, 1, \cos) = \int_0^r (n(t, 1, \cos) - 2)\frac{dt}{t} + 2\ln r, \quad N(r, a, \cos) = \int_0^r n(t, a, \cos)\frac{dt}{t}, a \neq 1,$$

and using Exc. 1 (c) (assuming only that f is locally integrable) we see that $N(r, a, \cos) \sim 2r/\pi$.

(d) Clearly ∞ is a Picard exceptional value for cos so $\delta(\infty, \cos) = 1$. For $a \in \mathbb{C}$ we have

$$\delta(a, \cos) = 1 - \limsup_{r \to \infty} \frac{N(r, a, \cos)}{T(r, \cos)},$$

and we know $T(r, \cos) \sim 2r/\pi$ from Problem set 2, exc. 3. We have just found that $N(r, a, \cos)$ has the same asymptotic behaviour, hence

$$\frac{N(r, a, \cos)}{T(r, \cos)} \to 1,$$

showing that $\delta(a, \cos) = 0$. Notice that in this case the sum of all the deficiencies is 1, while the general upper bound is 2.

Exc. 3. (a) Using Euler's formulas we find

$$\tan z = \frac{e^{iz} - e^{-iz}}{2i} / \frac{e^{iz} + e^{-iz}}{2} = -i \frac{e^{2iz} - 1}{e^{2iz} + 1}$$

so $\tan z = i$ is equivalent with $1 - e^{2iz} = e^{2iz} + 1$, i.e. with $2e^{2iz} = 0$, which is impossible. Also $\tan z = -i$ is equivalent with $e^{2iz} - 1 = e^{2iz} + 1$, which means $-1 = 1$, again impossible. This shows that $z = \pm i$ are Picard exceptional values, and sincde there are at most 2, we have found them all.

(b) We know that $\rho(\tan) = \rho(\frac{\sin}{\cos}) \leq 1$, the latter because both sine and cosine has order 1 and the order of a quotient is at most the maximun of the orders of the numerator and the denominator.

We also know that $T(r, \tan) \geq N(r, \tan) = N(r, 0, \cos)$, hence

$$\frac{\ln T(r, \tan)}{\ln r} \geq \frac{\ln N(r, 0, \cos)}{\ln r}.$$

By problem 2(c) $\ln N(r, 0, \cos) - \ln r - \ln(2/\pi) \to 0$ and therefore

$$\frac{\ln N(r, 0, \cos)}{\ln r} \to 1,$$

and finally

$$\limsup_{r \to \infty} \frac{\ln T(r, \tan)}{\ln r} \geq 1,$$

which shows that $\rho(\tan) \geq 1$. Combined with the opposite inequality above we have found $\rho(\tan) = 1$.

(c) We know from Langley's notes page 28 that $m(r, f'/f) = O(\log r)$ if f is meromorphic of finite order. Applying this to $f = \cos$ of order 1 and using that $m(r, g) = m(r, -g)$ we get

$$m(r, \tan) = m(r, -f'/f) = O(\log r).$$