

Problem set 3. Deadline June 21, 2007

Exc. 1. Let $f : [0, \infty[\rightarrow [0, \infty[$ be an increasing function, and define

$$F(x) = \int_0^x f(t) dt, \quad x \geq 0.$$

In the following use of the symbols O, o, \sim , we assume that $x \rightarrow \infty$.

- (a) Let $\alpha \geq 0$ be given. Prove that if $f(x) = O(x^\alpha)$ then $F(x) = O(x^{\alpha+1})$.
- (b) Let $\alpha > 0$ be given. Prove that if $f(x) = o(x^\alpha)$ then $F(x) = o(x^{\alpha+1})$.
- (c) Let $\alpha \geq 0$ be given. Prove that if $f(x) \sim Cx^\alpha$, where $C > 0$, then $F(x) \sim (C/(\alpha + 1))x^{\alpha+1}$.

Exc. 2. Consider the entire function $f(z) = \cos z$.

- (a) Find the solutions to the equation $\cos z = a$ for each $a \in \mathbb{C}$, and decide the multiplicity of each solution.
- (b) Prove that for $a \in \mathbb{C}$

$$n(r, a, \cos) \sim \frac{2r}{\pi}, \quad r \rightarrow \infty.$$

- (c) Prove that

$$N(r, a, \cos) \sim \frac{2r}{\pi}, \quad r \rightarrow \infty$$

for each $a \in \mathbb{C}$.

- (d) Calculate the deficiency $\delta(a, \cos)$ for all $a \in \mathbb{C}^*$.

Exc. 3. (a) Prove that \tan has two Picard exceptional values and find them.

- (b) Prove that \tan has order 1.
- (c) Prove that $m(r, \tan) = O(\log r)$ for $r \rightarrow \infty$. *Hint:* $\tan = -f'/f$ for $f = \cos$.