

## Solutions to Koman, August 2007

### Problem 3

(1) The power series obtained by differentiation of each term is the power series for  $\exp(-z^2)$  and since the power series for the exponential function is known to have infinite radius of convergence, we get by Lemma 1.11 that the given power series also has infinite radius of convergence. We have also established then that  $f'(z) = \exp(-z^2)$ .

(2) The entire function  $\exp(-z^2)$  has a primitive in the whole complex plane given by the path integral

$$F(z) = \int_{[0,z]} \exp(-w^2) dw$$

and clearly  $F(0) = 0$ . Since  $f$  and  $F$  are primitives of  $\exp(-z^2)$  with the same initial value  $f(0) = F(0) = 0$ , we know they are equal. Inserting the parametrization  $w = tz$ ,  $t \in [0, 1]$  in the path integral we find

$$f(z) = F(z) = \int_0^1 \exp(-t^2 z^2) z dt,$$

which proves the given formula.

(3) Since the given power series only contains odd powers we get  $f(-z) = -f(z)$ . By the usual rules for complex conjugation of algebraic expressions we get

$$\overline{\sum_{k=0}^n (-1)^k \frac{z^{2k+1}}{(2k+1)k!}} = \sum_{k=0}^n (-1)^k \frac{\bar{z}^{2k+1}}{(2k+1)k!},$$

and letting  $n \rightarrow \infty$  we get  $\overline{f(z)} = f(\bar{z})$ .

### Problem 4

(1) The poles are the solutions in  $\mathbb{C}$  to the equation  $\sin z = 1$ , where the solutions  $z = \pi/2 + 2p\pi$ ,  $p \in \mathbb{Z}$  are well known. There are not other solutions, because by Euler's formula  $\sin z = 1$  implies that  $w = e^{iz}$  is a zero of the polynomial  $w^2 - 2iw - 1 = 0$ . This polynomial has  $w = i$  as a double zero, so  $e^{iz} = i = e^{i\pi/2}$ , hence  $\exp(i(z - \pi/2)) = 1$ , which by Theorem 1.18 shows that  $z - \pi/2 = 2p\pi$ .

This shows that the poles of  $f$  are  $z = \pi/2 + 2p\pi$ ,  $p \in \mathbb{Z}$ . The derivative of  $1 - \sin z$  is  $-\cos z$  which is also zero at these points, while the second derivative  $\sin z$  is 1. Therefore all the poles are of order 2.

(2) The zeros of  $\cos$  are  $\pi/2 + p\pi$ ,  $p \in \mathbb{Z}$  and they are all simple. In the function  $g$  the zeros  $\pi/2 + 2p\pi$  of  $\cos$  matches the zeros of  $1 - \sin z$ , which are of order 2. Therefore the points  $z = \pi/2 + 2p\pi$  will be simple poles of  $g$ , while  $z = 3\pi/2 + 2p\pi$  will be simple zeros of  $g$ .

(3) Within the disc  $K(\pi/2, 1)$  there is only one of the poles of  $g$  namely  $z = \pi/2$ . By The Cauchy residue theorem the integral is therefore  $2\pi i \operatorname{Res}(g, \pi/2) = -4\pi i$ ,

because the residue is -2. To see this one can use either Exc. 7.3 (with  $n = 1$ ) or use the definition (the pole is simple):

$$\operatorname{Res}(g, \pi/2) = \lim_{z \rightarrow \pi/2} \frac{(z - \pi/2) \cos z}{1 - \sin z}.$$

This can be found using L'Hospital's rule twice, or one can put  $w = z - \pi/2$ , so we shall find the limit for  $w \rightarrow 0$  of

$$\frac{w \cos(w + \pi/2)}{1 - \sin(w + \pi/2)} = \frac{w \sin w}{\cos w - 1} = \frac{w^2(1 - w^2/3! + \dots)}{-w^2/2! + w^4/4! - \dots}$$

but this is clearly -2.