University of Copenhagen
Exam at the Faculty of Natural Sciences August 2007

Complex Analysis (KomAn)

Welcome to this three hours exam. You are allowed to use pencil and rubber if the writing is readable and you erase thoroughly. It is recommended to cross out if you do not want parts to be evaluated.

The exam consists of 4 problems and is formulated on two pages. If the problems are solved in a satisfactory way you get 100 points. Each problem has the value of 25 points as indicated below.

The exam has two parts. During the first 90 minutes you are not allowed to use any written material or other devices and you shall solve problems 1 and 2. For the next 90 minutes you are allowed to use your notes, books, pocket calculators etc. and you shall deal with problems 3 and 4.

After the first 90 minutes your answers of the first part will be collected. You are of course wellcome to think about the problems 3 and 4 during the first part.

First part without auxiliary materials

Problem 1 (25 points)

Let $G \subseteq \mathbb{C}$ be a non-empty open set.

(1) Define what it means that a function $f : G \to \mathbb{C}$ is holomorphic in G.

(2) Assume now that $f : G \to \mathbb{C}$ is holomorphic and that $\overline{K(a,r)} \subseteq G$. Prove Cauchy's integral formula

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial K(a,r)} \frac{f(z)}{z - z_0}\, dz \quad \text{for} \quad z_0 \in K(a,r).$$

(Cauchy's integral theorem can be used without proof.)

Problem 2 (25 points)

(1) Define the following concepts for a holomorphic function:
  (a) An isolated singularity
  (b) A removable singularity, a pole and an essential singularity.

(2) Assume now that $G \subseteq \mathbb{C}$ is an open set, $a \in G$ and that f is defined by the Laurent series

$$f(z) = \sum_{k=-\infty}^{\infty} c_k (z-a)^k$$

Problem 2 is continued on page 2

for $z \in K'(a,\rho) \subseteq G$.

Prove that f has a pole of order $n \geq 1$ at a if and only if $c_k = 0$ for $k < -n$ and $c_{-n} \neq 0$.

You can freely use the following formula for the coefficients in the Laurent series

$$c_k = \frac{1}{2\pi i} \int_{\partial K(a,r)} \frac{f(z)}{(z-a)^{k+1}}\, dz, \quad k \in \mathbb{Z},\ 0 < r < \rho,$$

where the indicated path is followed once counterclockwise.

Second part with auxiliary materials

Problem 3 (25 points)

Consider the power series

$$f(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)n!}, \quad z \in \mathbb{C}.$$

(1) Prove that the radius of convergence is ∞ and that $f'(z) = e^{-z^2}$ for all $z \in \mathbb{C}$.

(2) For each $z \in \mathbb{C}$ prove that

$$f(z) = z \int_0^1 e^{-t^2 z^2}\, dt.$$

(3) Prove that $f(-z) = -f(z), f(\overline{z}) = \overline{f(z)}$ for $z \in \mathbb{C}$.

Problem 4 (25 points)

Consider the meromorphic function

$$f(z) = \frac{1}{1 - \sin z},\ z \in \mathbb{C}.$$

(1) Find the poles of f and determine their orders.

(2) Consider the meromorphic function

$$g(z) = \frac{\cos z}{1 - \sin z},\ z \in \mathbb{C}.$$

Find the zeros and poles of g and prove that they are all simple.

(3) Find the value of the integral

$$\int_{\partial K(\pi/2,1)} \frac{\cos z}{1 - \sin z}\, dz,$$

where the indicated path is followed once counterclockwise.