**Københavns Universitet**
**Eksamen ved Det naturvidenskabelige Fakultet August 2006**

# Complex Analysis (KomAn)

Wellcome to this three hours exam. You are allowed to use pencil and rubber if the writing is readable and you erase thoroughly. It is recommended to cross out if you do not want parts to be evaluated.

The exam consists of 4 problems (called opgaver 1-4) and is formulated on two pages. If the problems are solved in a satisfactory way you get 100 points. Each problem has the value of 25 points as indicated below.

The exam has two parts. During the first 90 minutes you are not allowed to use any written material or other devices and you shall solve problems 1 and 2. For the next 90 minutes you are allowed to use your notes, books, pocket calculators etc. (but not PC's) and you shall deal with problems 3 and 4.

After the first 90 minutes your answers of the first part will be collected. You are of course wellcome to think about the problems 3 and 4 during the first part.

**First part without auxiliary materials**

## Opgave 1 (25 points)

1) Explain that the exponential function $\exp(z)$ maps the set $\{z \in \mathbb{C} \mid -\pi < \operatorname{Im} z < \pi\}$ bijectively onto $\mathbb{C}_\pi = \mathbb{C} \setminus ]-\infty, 0]$.

2) Define the principal value of the logarithm denoted $\operatorname{Log} : \mathbb{C}_\pi \to \mathbb{C}$ and explain that it is holomorphic with

$$\frac{d \operatorname{Log} z}{dz} = \frac{1}{z}, \quad z \in \mathbb{C}_\pi.$$

3) The function $f(z) = \operatorname{Log}(1+z)$ is holomorphic in $G = \mathbb{C} \setminus ]-\infty, -1]$. Give a proof of the formula

$$\operatorname{Log}(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z^n, \quad |z| < 1.$$

## Opgave 2 (25 points)

Let $f$ be holomorphic in a domain $G \subseteq \mathbb{C}$, assume that $f$ is not identically 0 and that $f(a) = 0$ for a point $a \in G$.

1) Prove that there exist a uniquely determined natural number $n$ and a uniquely determined holomorphic function $g \in \mathcal{H}(G)$ with $g(a) \neq 0$, such that the following holds

$$f(z) = (z-a)^n g(z), \text{ for } z \in G.$$


**Opgave 2 fortsættes på side 2**

2) Prove that there exists $r > 0$ so that $f(z) \neq 0$ for $z$ satisfying $0 < |z - a| < r$.

3) Define the order denoted $\operatorname{ord}(f, a)$ of the zero $a$ for $f$.

**Second part with auxiliary materials**

## Opgave 3 (25 points)

Consider the function

$$f(z) = \frac{z^3 - 3z^2 + 2z}{\sin(\pi z)}.$$

1) Prove that $f$ has a removable singularity for $z = 0$ and determine $\lim_{z\to 0} f(z)$.

2) Determine the remaining isolated singularities of $f$ and decide for each of them if they are removable, poles or essential singularities.

3) Calculate the curve integrals

$$\int_{\partial K(0,\frac{\pi}{4})} f(z)\,dz, \quad \int_{\partial K(0,\frac{\pi}{2})} f(z)\,dz,$$

where the curves in question in both cases are supposed to be the indicated circles followed once with positive orientation.

## Opgave 4 (25 points)

Prove the formula

$$\int_{-\infty}^{\infty} \frac{\cos(2\lambda x)}{1 + x + x^2}\,dx = \frac{2\pi}{\sqrt{3}} e^{-|\lambda|\sqrt{3}} \cos\lambda \text{ for } \lambda \in \mathbb{R}.$$

It is important that each step in the calculation is clearly explained.
*Hint:* Calculate first the value of the integral

$$\int_{-\infty}^{\infty} \frac{e^{2i\lambda x}}{1 + x + x^2}\,dx.$$